A Combinatorial Presentation for Branched Coverings of the 2-Sphere

Arcelino Lobato

Universidade Federal do Maranhão

bruno.arcelino@ufma.br

Abstract

William Thurston (1946-2012) obtained an combinatorial characterization for orientation preserving generic branched self-coverings of the 2-sphere by associating a planar graph to them [4]. Here, the *Thurston* result is generalized to any orientation preserving branched covering of the 2-sphere. Thurston's definition of *local balance* is generalized in order to capture the *branched coverings* combinatorial essence in high genera.

Background

Definition 1 (Branched Covering)**.** *Let S_g be an oriented closed genus g surface. A surjective continuous map $f : S_g \to \mathbb{S}^2$ is a degree d branched covering if it is* orientation preserving *and for each $z \in \mathbb{S}^2$ it becomes the map $z \mapsto z^k$ in local charts about z and $f(z)$ (sending z and $f(z)$ to 0) for some positive integer k. The integer k is called the local* degree *of f at z. $z \in S_g$ is a critical point for f if $deg_z(f) > 1$, C_f is the set of critical points of f and $R_f := f(C_f)$. The degree of f is $d := |f^{-1}(\zeta)|$ for any $\zeta \in \mathbb{S}^2 - R_f$.*

Definition 2 (Cell Graph)**.** *A* cell graph *G into an oriented topological surface S is the 1-skeleton (i.e., the union of the 0 and 1 cells) of a cellular decomposition of the surface S. A face of G is a 2-cell of a cell decomposition.*

Definition 3 (A-B alternating face coloring)**.** *Let A and B be two colours. An A-B alternating face coloring for a* cell graph *G is an assingnement of the colours A and B to the faces of G in a manner that:*

- *adjacents faces possesses different colour attached to it;*
- *the faces left on the left side of its boundary are all of the colour A.*

If the number of faces of each colors are igual, G is said to be globally balanced.

Definition 4 (positivity of a cycle)**.** *A cycle into Γ, a cell graph with an A-B alternating face coloring, is* positive *if it keeps only A faces on its left side.*

Definition 5 (cobordant multicycle)**.** *Let Γ be a cell graph that admits an A-B alternating face coloring. We say that a collection of disjoint cycles $L := \{\gamma_1, \cdots, \gamma_k\}$ of Γ are a* cobordant multicycle of Γ *if:*

i. $S - \{\gamma_1, \cdots, \gamma_k\}$ is disconnected;

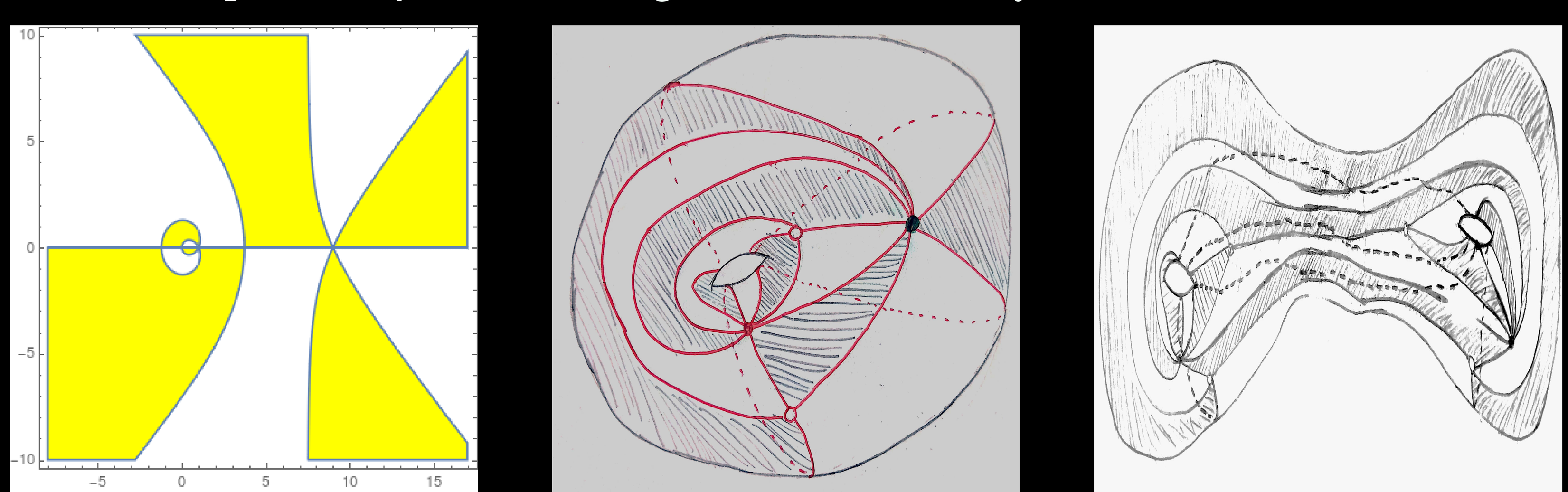
ii. there is a connected component R of $S - \{\gamma_1, \cdots, \gamma_k\}$ such that $\partial R = \bigsqcup_{n=1}^{k} \gamma_n$.

If each cycle $\gamma_n \in L$ is positive then L is said to be a positive cobordant multicycle of Γ. *R is the* interior of *L.*

Definition 6 (local balancedness)**.** *Let Γ be cell graph with an alternating A-B face coloring.We say that Γ is* locally balanced *if for any A-B alternating face coloring and for any* positive cobordant multicycle of Γ *the number of A faces inside it (i.e, on the interior of that multicycle) is strictly greater than the number of B faces thering.*

Proposition 1 (meaningfulness of Definiton 6)**.** *For* planar *globally balanced graphs the (generalized) definition of local balancedness is equivalent to the Thurston's definition [4].*

Definition 7 (Balanced Graph (generalized Grothendiek's *Dessins d'Enfant*))**.** *A* balanced graph *is a cellularly embedded graph Γ into an oriented compact surface that is* global *and* locally balanced.



(a) balanced graph of type (0,4,6) (b) balanced graph of type (0,4,6) (c) balanced graph of type (2,4,6)

Definition 8 (Pullback graph)**.** *Let Σ be an isotopy class relative to R_f of a* Jordan curve *containig R_f. The isotopy class relative to C_f of $\Gamma := f^{-1}(\Sigma)$ is called the* pullback graph *of f with respect to Σ, or simply, Σ-*pullback graph *of f. (In order to use balanced graphs to study* Thurston maps *we impose the* postcritical curve *Σ to containg the postcritical set of that map.)*

The vertex set of Γ is the set $f^{-1}(R_f)$.

Main Result

Theorem 1 (General version of a theorem by Thurston)**.** *A cell graph Γ into a genus g oriented compact surface S_g is a pullback graph if and only if it is a balanced graph.*

about the proof: The crucial insight for promoting a pullback graph to a balanced graph, and vice versa,consists of recognizing its combinatorial structure as stemming from a perfect matching on an adjacent bipartite graph. The *balance conditions* than correspond to necessary and sufficient condition of the *Hall's Marriage Theorem* are satisfied.

An application: Proving $B.$ & $M.$ Shapiros Conjecture.

Definition 9 (generic real globally balanced graph)**.** *A* planar globally balanced graph *is real if it have a cycle that contains all corners of the graph. A balanced graph with* $2d$ *faces that have* $2d-2$ *corners is called* generic degree d real balanced graphs.

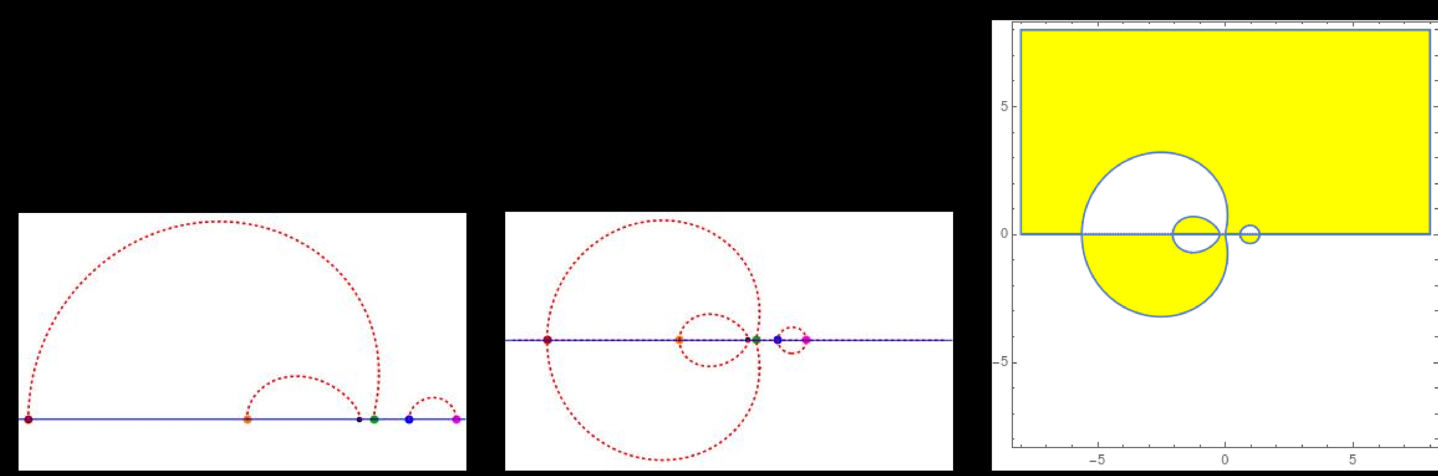


Figure 1: Condtruction of a balanced graph from a noncrossing pairing of 6 real points

Leaving the vertices fixed, the counting of noncrossing pairing of $2d-2$ distinct points in $\mathbb{RP}^1$ is $\rho_d = \frac{1}{d}\binom{2d-2}{d-1}$ (the d-Catalan number). Then the number of *real globally balanced graphs* with $2d$ faces for prescribed $2d-2$ vertices into $\mathbb{RP}^1$ is ρ_d.

Theorem 2. *A real balanced graph is locally balanced.*

Than there exists at least ρ_d rational maps counted up to post-composition with a *Möbius map.*

Lisa Goldberg in [3] showd that up to post-composition with a automorphism of $\mathbb{CP}^1$ there exists, for a generic choice of $2d-2$ points in $\mathbb{CP}^1$, $\rho(d)$ generic degree d rational function with its critical points being that $2d-2$ preescribed points. Then we have proved the following

Theorem 3 (Eremenko-Gabrielov-Mukhin-Tarasov-Varchenko-[2],[5]+A.[1])**.** *Let $R : \mathbb{CP}^1 \to \mathbb{CP}^1$ be a rational function with only simple critical points. If $C_R \subset \mathbb{RP}^1$ than* up to *a post-composition with a* Möbius Transformation, *R is a real rational function, i.e.*

$$\exists\, \sigma \in Aut(\mathbb{CP}^1), \exists\, F \in \mathbb{R}(z); F = \sigma \circ R$$

References

[1] *Arcelino B. L. do Nascimento. Branched coverings of the two sphere, phd thesis, 2021 - usp, 2015.*

[2] *A. Eremenko and A. Gabrielov. Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry.* Ann. of Math. (2), *155(1):105–129, 2002.*

[3] *Lisa R. Goldberg. Catalan numbers and branched coverings by the Riemann sphere.* Adv. Math., *85(2):129–144, 1991.*

[4] *Sarah Koch and Tan Lei. On balanced planar graphs, following w. thurston, arxiv: 1502.04760, 2015.*

[5] *Evgeny Mukhin, Vitaly Tarasov, and Alexander Varchenko. The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz.* Ann. of Math. (2), *170(2):863–881, 2009.*