Overview about hydrodynamics, fluctuations and universality of interacting particle systems

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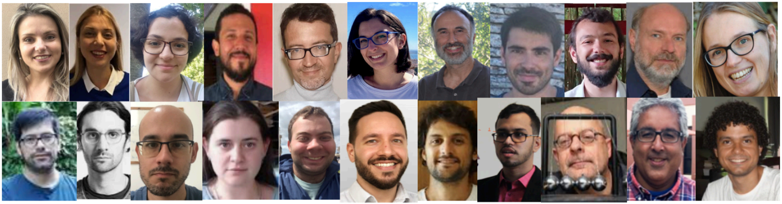
Dedicated to

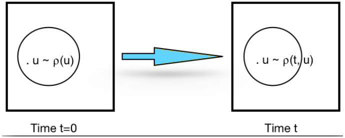
Claudio Landim and Milton Jara,
all my professors from IMPA,
all my friends from IMPA and from Rio,
all the staff from IMPA,
for making of IMPA such a pleasant place
to study and to visit!

Collaborators, students and friends:



What is the motivation?

Goal: analyse the evolution of a gas (the number of components is huge, no precise description of the microscopic state of the system).

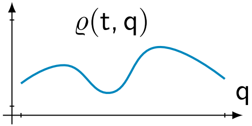


- **Find the invariant states of a system.**
- **Characterize them by a quantity** $\varrho(\cdot)$**.**
- **Fix** $u \in \Lambda$ **and a neighbourhood** Λ_u **(microscopically big). Due to interaction, the system reaches a local equilibrium** $\varrho(u)$.
- **Let time evolve. Now the equilibrium close to** u **is given by** $\varrho(t, u)$**. How does** $\varrho(t, u)$ **evolve?**

How do we approach this mathematically?

- Discretize Λ according to a parameter N and get Λ_N.
- Two scales for space/time.
- At each cell we put a certain number of particles.
- The dynamics conserves some quantity of interest.
- Waiting times are given by independent Poisson processes; the particle system is a Markov chain.

QUESTION: What is the macroscopic law describing the evolution of the conserved quantity of the system?

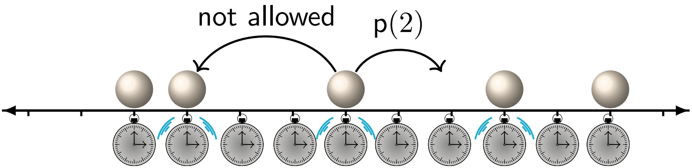


Hydrodynamic limit: one conservation law

The exclusion process

- N: scaling parameter;
- **Space**:
  - microscopic (discrete);
  - macroscopic (continuous);
- **Time**:
  - microscopic $t\theta(N)$;
  - macroscopic t;
- Independent Poissonian clocks;
- Transition probability $p(\cdot)$;
- $\eta_t^N(x)$ = number of particles at site x;
- Infinitesimal generator
  $(\mathcal{S}f)(\eta) = \sum_{x,y} p(y-x)(f(\eta^{x,y}) - f(\eta))$.
- **Markov processes**; (continuous time)
- Density $\sum_x \eta_t^N(x)$ is conserved.

Exclusion: After one ring of a clock a particle jumps from x to y at rate $p(y-x)$.



The hydrodynamics for exclusion?

- For η, let $\pi_t^N(\eta, dq) = \frac{1}{N}\sum_x \eta_{t\theta(N)}(x)\delta_{x/N}(dq)$, be the **empirical measure. (Time scale $\theta(N)$)**
- Assumption: fix $\mathfrak{g} : \Lambda \to [0,1]$ measurable and probability measures $\{\mu_N\}_{N\geq 1}$ such that for every $H \in C(\Lambda)$,

$$\frac{1}{N}\sum_x H(\tfrac{x}{N})\,\eta(x) \to_{N\to+\infty} \int_\Lambda H(q)\,\mathfrak{g}(q)dq,$$

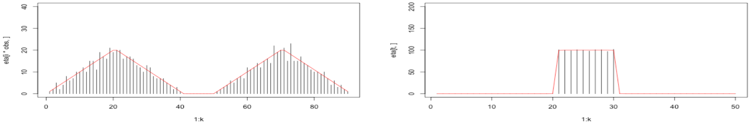
wrt μ_N. (μ_N is associated to $g(\cdot)$)

- Then: for any $t > 0$,

$$\pi_t^N(\eta, dq) \to_{N\to+\infty} \varrho(t,q)dq,$$

wrt $\mu_N(t)$, where $\varrho(t,q)$ evolves according to the hydrodynamic equation.

Some hydrodynamic equations



Possible hydrodynamic equations

Heat Equation (HE): $\partial_t \varrho_t = \Delta \varrho_t$ (p symmetric, tN^2)
Viscous Burgers: $\partial_t \varrho_t = \Delta \varrho_t + \nabla \varrho_t(1 - \varrho_t)$ (p w. asym, tN^2)
Inviscid Burgers: $\partial_t \varrho_t = \nabla \varrho_t(1 - \varrho_t)$ (p asymmetric, tN)
Porous Medium (PM): $\partial_t \varrho_t = \Delta \varrho_t^m, 2 \leq m \in \mathbb{N}$ (p sym, tN^2)
Fractional HE: $\partial_t \varrho_t = -(-\Delta)^{\gamma/2} \varrho_t$ ($p(\cdot) = \frac{c}{|\cdot|^{1+\gamma}}$, tN^γ, $\gamma \in (0,2)$)
Fract. PM: $\partial_t \varrho_t = -(-\Delta)^{\gamma/2} \varrho_t^2$ ($p(\cdot) = \frac{c}{|\cdot|^{1+\gamma}}$, tN^γ, $\gamma \in (0,2)$)

Adding boundary conditions

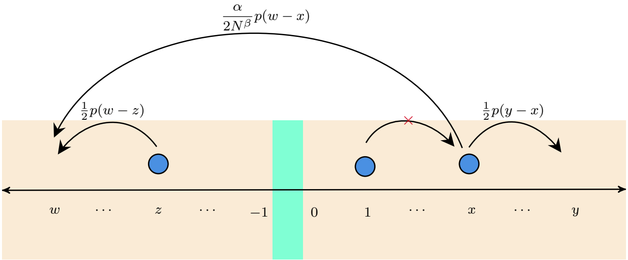
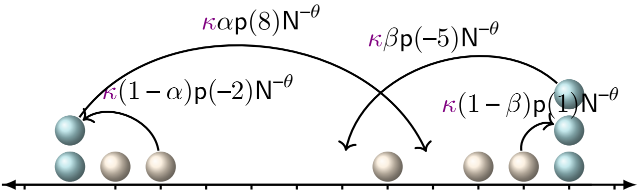


Figure: Symmetric exclusion with long jumps and a slow barrier.

Exclusion with a slow/fast boundary

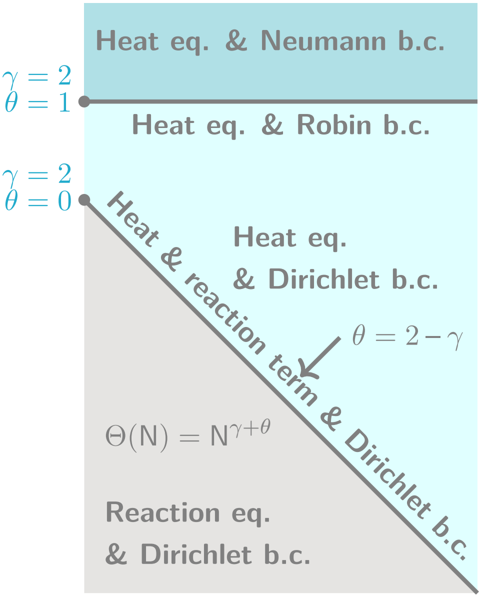


Let $p(\cdot)$ be a transition probability given at $z \in \mathbb{Z}$ by

$$p(z) = \mathbf{1}_{z \neq 0} \frac{c_\gamma}{|z|^{\gamma+1}}$$

where c_γ is a normalizing constant. Note that $p(\cdot)$ has mean zero and for $\gamma > 2$ its variance is finite:

$$\sigma_\gamma^2 = \sum_{z \in \mathbb{Z}} z^2 p(z) < \infty.$$



- **Heat equation:**
  $\partial_t \varrho_t = \frac{\sigma^2}{2}\Delta \varrho_t$
  - $\theta = 1$ Robin b.c.:
    $\partial_q \varrho_t(0) = \frac{2m\kappa}{\sigma^2}(\varrho_t(0) - \alpha)$,
    $\partial_q \varrho_t(1) = \frac{2m\kappa}{\sigma^2}(\beta - \varrho_t(1))$,
- **Reaction-diffusion equation:**

$$\partial_t \varrho_t = \frac{\sigma^2}{2}\Delta \varrho_t + \kappa(V_0 - V_1 \varrho_t)$$

- **Reaction equation:**
  $\partial_t \varrho_t = \kappa(V_0 - V_1 \varrho_t)$

$$V_1(q) = \frac{c_\gamma}{\gamma}\left(\frac{1}{q^\gamma} + \frac{1}{(1-q)^\gamma}\right)$$

$$V_0(q) = \frac{c_\gamma}{\gamma}\left(\frac{\alpha}{q^\gamma} + \frac{\beta}{(1-q)^\gamma}\right).$$

When $\gamma < 2$?

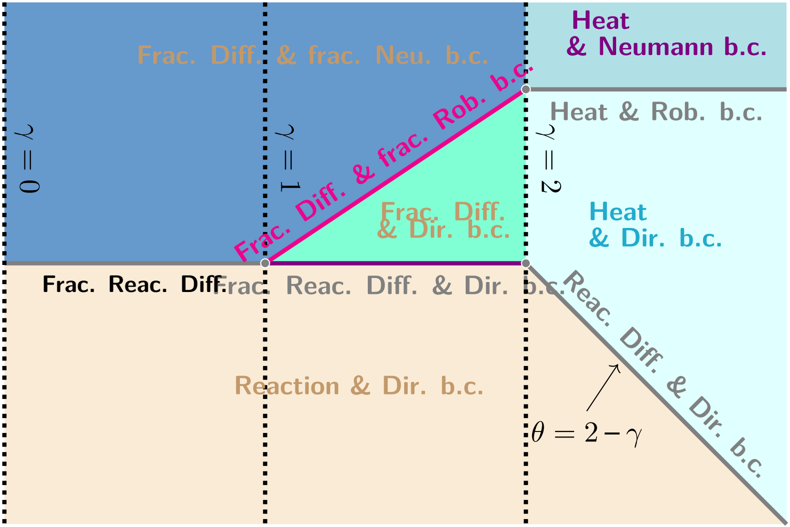
We will get a collection of fractional reaction-diffusion equations

$$\partial_t \varrho_t(q) = \mathbb{L}_\kappa \varrho_t(q) + \kappa V_0(q).$$

where the operator $\mathbb{L}_\kappa = \mathbb{L} - \kappa V_1$, $\mathbb{L}$ is the regional fractional laplacian on $[0,1]$, whose action on functions $H \in C_c^\infty(0,1)$ is given by

$$\begin{aligned}(\mathbb{L}H)(q) &= -(-\Delta)^{\gamma/2} H(q) + V_1(q) H(q) \\ &= c_\gamma \lim_{\varepsilon \to 0} \int_0^1 \mathbf{1}_{|u-q| \geq \varepsilon} \frac{H(u) - H(q)}{|u-q|^{1+\gamma}} du, \quad q \in (0,1).\end{aligned}$$

The whole picture:



Hydrodynamic Limit: more conservation laws

Particle exchange models

- In this case we have three types of particles A, B and C.
- At each site only ONE particle and $\eta(x) \in \{A, B, C\}$.
- Invariant measures (NESS) are NOT known.
- The rates $r_\alpha, \tilde{r}_\alpha$ satisfy $\sum_\alpha r_\alpha = 1$ and $\sum_\alpha \tilde{r}_\alpha = 1$ that represent the density reservoir of each type.

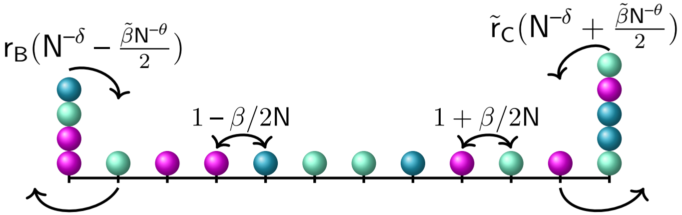


Figure: Dynamics of the model with particles of species A, B and C/holes.

And the hydrodynamic limit in this case?

- For each α and $\xi_x^\alpha(\eta) = \mathbf{1}_{\eta(x)=\alpha}$, we consider $\pi^{N,\alpha}(\eta, dq) = \frac{1}{N}\sum_x \xi_x^\alpha(\eta)\delta_{\frac{x}{N}}(dq)$.
- We assume that for a measurable profile $\mathfrak{g} : [0,1] \to [0,1]^3$ with $\sum_\alpha \mathfrak{g}^\alpha = 1$ it holds

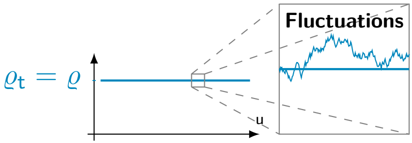
$$\lim_{N\to\infty} \mu_N\Big(\Big|\frac{1}{N}\sum_x H\left(\tfrac{x}{N}\right)\xi_x^\alpha(\eta) - \int_0^1 H(q)\mathfrak{g}^\alpha(q)dq\Big| > \epsilon\Big) = 0.$$

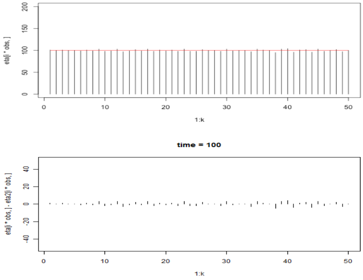
- Now we get a system of equations

$$\begin{cases} \partial_t \varrho^\alpha = \Delta\, \varrho^\alpha + \beta\nabla[\varrho^\alpha(\varrho^{\alpha+1} - \varrho^{\alpha+2})], & \\ \varrho_t^\alpha(0) = r_\alpha, \quad \varrho_t^\alpha(1) = \tilde{r}_\alpha, \quad t \in (0,T], & \alpha \in \{A,B,C\}, \\ \rho_0^\alpha(q) = \mathfrak{g}^\alpha(q), \quad q \in [0,1], & \end{cases}$$

but can also be obtained with other boundary conditions.

Fluctuations and universality





From the stationary state

- For the exclusion process the invariant measures are known (in special cases):

$$\nu_\varrho(d\eta) = \prod_x \varrho^{\eta_x}(1-\varrho)^{1-\eta_x}.$$

When there are boundaries not much is known.

- The density fluctuation field $\mathcal{Y}_\cdot^N$ is the time-trajectory of linear functionals acting on H as

$$\mathcal{Y}_t^N(H) \; = \; \frac{1}{\sqrt{N}} \sum_x H\left(\tfrac{x}{N}\right)\Big(\eta_{tN^2}(x) - \mathbb{E}_{\nu_\varrho}[\eta_{tN^2}(x)]\Big)\,.$$

Possible limiting equations

- SSEP in diffusive scaling, the Ornstein-Uhlenbeck eq. (OU)

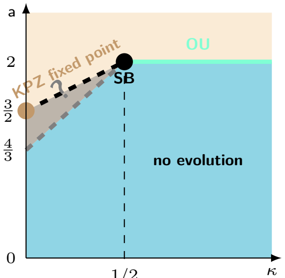
$$d\mathscr{Y}_t = \Delta \mathscr{Y}_t dt + \sqrt{\varrho(1-\varrho)} \nabla \dot{\mathscr{W}}_t.$$

- WASEP in diffusive scaling, the stochastic Burgers eq. (SB)

$$d\mathscr{Y}_t = \Delta \mathscr{Y}_t dt + \nabla \mathscr{Y}_t^2 dt + \sqrt{\varrho(1-\varrho)} \nabla \dot{\mathscr{W}}_t.$$

Above $\dot{\mathscr{W}}_t$ is the space-time white-noise.

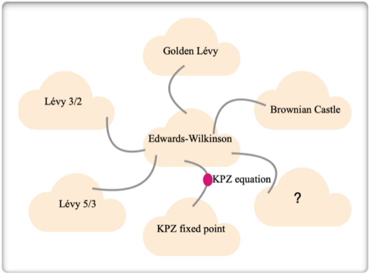
- ASEP in hyperbolic scaling, $d\mathscr{Y}_t = (1-2\varrho)(1-2p)\nabla \mathscr{Y}_t dt$.



Fluctuations for multi-component systems

Difficulties:

- Which fields to consider?
- If we have two conserved quantities then any linear combination is again conserved.
- Different time scales might co-exist.
- Correlations between the conserved quantities.
- Non-linear fluctuating hydrodynamics gives predictions (not rigorous).
- How to obtain these limiting processes?



Define fluctuation fields and velocities (Dynkin's formula) ✓.
Difficulty: How to close the equations!?

Fluctuations for particle exchange models

- Each site of $\mathbb{T}_N$ has only ONE particle.
- *Weakly asymmetric rates*: a configuration (α, β) on the bond $x, x+1$ is exchanged to (β, α) with rate $c_{x,x+1}^{\alpha\beta} = 1 + \frac{E_\alpha - E_\beta}{2N^\gamma}$.
- The total number of particles of each species N_α, $\alpha \in \{A, B, C\}$, is conserved and $N_A + N_B + N_C = N$.
- The stationary measure ν_ρ is product with parameters ρ_A, ρ_B and $\rho_C = 1 - \rho_A - \rho_B$ corresponding each species.
- Evolution in diffusive time scale tN^2.

I. Case $E_\alpha - E_{\alpha+2} = E_{\alpha+1} - E_{\alpha+2} = E$

Fields:

$$\mathcal{Z}_t^N = \mathcal{Y}_t^{N,\alpha} + \mathcal{Y}_t^{N,\alpha+1} \text{(SB)}$$
$$\tilde{\mathcal{Z}}_t^N = \mathcal{Y}_t^{N,\alpha} - \mathcal{Y}_t^{N,\alpha+1} \text{(OU)}$$

velocities $v = \pm \frac{E}{6N^\gamma}$.

II. Case $E_{\alpha+1} - E_\alpha = E_{\alpha+2} - E_\alpha = E$

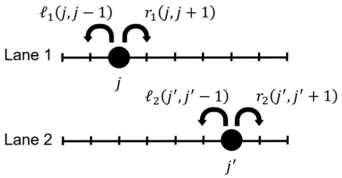
Fields:

$$\mathcal{Z}_t^N = \mathcal{Y}_t^{N,\alpha} \text{ (SB)}$$
$$\tilde{\mathcal{Z}}_t^N = \mathcal{Y}_t^{N,\alpha} + 2\mathcal{Y}_t^{N,\alpha+1} \text{(OU)}$$

velocities $v = \pm \frac{E}{6N^\gamma}$.

Fluctuations for two lane exclusion

The dynamics is:



The rates satisfy:

$$r_1(j, j+1) = p_1 + b_1\eta_j^{(2)} + c_1\eta_{j+1}^{(2)} + d_1\eta_j^{(2)}\eta_{j+1}^{(2)},$$
$$\ell_1(j+1, j) = q_1 + e_1\eta_j^{(2)} + f_1\eta_{j+1}^{(2)} + g_1\eta_j^{(2)}\eta_{j+1}^{(2)},$$
$$r_2(j, j+1) = p_2 + b_2\eta_j^{(1)} + c_2\eta_{j+1}^{(1)} + d_2\eta_j^{(1)}\eta_{j+1}^{(1)},$$
$$\ell_2(j+1, j) = q_2 + e_2\eta_j^{(1)} + f_2\eta_{j+1}^{(1)} + g_2\eta_j^{(1)}\eta_{j+1}^{(1)}.$$

The predictions give:

$G^2 \backslash G^1$	$\begin{pmatrix}\star & \\ & \bullet\end{pmatrix}$	$\begin{pmatrix}\star & \\ & 0\end{pmatrix}$	$\begin{pmatrix}\mathbf{0} & \\ & \bullet\end{pmatrix}$	$\begin{pmatrix}\mathbf{0} & \\ & 0\end{pmatrix}$
$\begin{pmatrix}\bullet & \\ & \star\end{pmatrix}$	(KPZ,KPZ)	(KPZ,KPZ)	($\frac{5}{3}$L,KPZ)	(D,KPZ')
$\begin{pmatrix}0 & \\ & \star\end{pmatrix}$	(KPZ,KPZ)	(KPZ,KPZ)	($\frac{5}{3}$L,KPZ)	(D,KPZ)
$\begin{pmatrix}\bullet & \\ & \mathbf{0}\end{pmatrix}$	(KPZ,$\frac{5}{3}L$)	(KPZ,$\frac{5}{3}L$)	(GM,GM)	(D,$\frac{3}{2}L$)
$\begin{pmatrix}0 & \\ & \mathbf{0}\end{pmatrix}$	(KPZ',D)	(KPZ,D)	($\frac{3}{2}$L,D)	(D,D)

GM is a Lévy process with parameter Gold mean $\alpha = \frac{1+\sqrt{5}}{2}$.

Thank you very much!!!

HAPPY BIRTHDAY!