On the Matrix-Valued Bispectral Problem

by

Brian David Vasquez Campos

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY
MATHEMATICS
IMPA

impa

Instituto de Matemática
Pura e Aplicada

March, 2021

On the Matrix-Valued Bispectral Problem

by

Brian David Vasquez Campos

B.Sc. Mathematics (University of Tolima, Colombia) 2015

M.Sc. Mathematics (IMPA, Brazil) 2016

Abstract

In this dissertation, we study algebraic properties of full rank 1 algebras in a general framework and derive a method to verify if one such matrix polynomial sub-algebra is bispectral. By a full rank 1 algebra we mean a sub-algebra of a graded algebra which contains an ideal generated by a monomial with an invertible coefficient. Furthermore, we give a presentation in terms of generators and relations for some finitely presented algebras. In the former example we put forth a Pierce decomposition of that algebra. As a byproduct, we answer positively a conjecture of F. A. Grünbaum concerning certain noncommutative matrix algebras associated to the bispectral problem. Additionally, we prove the bispectrality of some class of matrix Schrödinger operators with polynomial potentials which satisfy a second-order matrix autonomous differential equation. The physical equation is constructed using the formal theory of the Laurent series and after that obtaining local solutions using estimations in the Frobenius norm. Furthermore, the characterization of the algebra of polynomial eigenvalues in the spectral variable is given using some family of functions $\mathcal{P} = \{P_k\}_{k\in\mathbb{N}}$ with the remarkable property of satisfying a general version of the Leibniz rule.

Contents

Dedication.

to my parents Jorge and Damaris, my brother Roger and my sister Katherin.

Acknowledgments

I would like to thank God, my family, and the Pedraza Salguero family. I want to thank Nataly Pedraza for her love, understanding and support which were essential to complete this project. I would like to thank my thesis advisor, Jorge Passamani Zubelli, for his mentorship, patience, and support. I would also like to thank Alberto Grünbaum, Emanuel Carneiro, Hossein Movasati, Partha Guha and Henrique Bursztyn for serving on my dissertation committee as well as for their valuable comments about the topics discussed here. Furthermore, I am very grateful to the people at the IMPA because of their support since I came to Rio de Janeiro. The most amount of mathematical knowledge I have today I obtained from IMPA. Especially, I would like to thank professors Roberto Imbuzeiro, Emanuel Carneiro, Felipe Linares, Carolina Araujo, and Karl-Otto Stöhr for the exceptional courses I took from them. These courses and others I took at IMPA gave me enough background to research the topics presented here. I thank my friends at IMPA. Especially, to Oscar and Erik. I thank my friends at the University of Tolima, Jaime, Juan Gabriel, Miguel, and Daniel. I acknowledge the support and funding from CAPES-Brazil.

Introduction

Classical orthogonal polynomials as many important special functions satisfy remarkable relations both in the physical as well as in the spectral variables [8]. More precisely, they are eigenfunctions of an operator in the physical variable (say x) with eigenvalues depending on the spectral variable (say z) as well as the other way around, eigenfunctions of an operator in z with x–dependent eigenvalues. Such bispectral property was explored in the scalar case in the work of J. J. Duistermaat and F. A. Grünbaum [7]. It turned out to have deep connections with many problems in Mathematical Physics. Indeed, it could be arranged in suitable manifolds which were naturally parameterized by the flows of the Korteweg de-Vries (KdV) hierarchy or its master-symmetries [20, 25]. It led to generalizations associated with the Kadomtsev-Petviashvili (KP) hierarchy [22, 18, 12, 13]. A good reference for historical remarks about the bispectral problem is [24].

The Bispectral Problem was originally posed by J. J. Duistermaat and F. A. Grünbaum [7], it consists of finding all the bispectral triples (L, ψ, B) that satisfy systems of equations

$$L\psi(x,z) = \psi(x,z)F(z) \qquad (\psi B)(x,z) = \theta(x)\psi(x,z) \tag{1}$$

with $L = L(x, \partial_x)$, $B = B(z, \partial_z)$ linear scalar differential operators, i.e., $L\psi = \sum_{i=0}^{l} a_i(x) \cdot \partial_x^i \psi$, $\psi B = \sum_{j=0}^{m} \partial_z^j \psi \cdot b_j(z)$. The functions $a_i : U \subset \mathbb{C} \to \mathbb{C}, b_j : V \subset \mathbb{C} \to \mathbb{C}, F : V \subset \mathbb{C} \to \mathbb{C}, \theta : U \subset \mathbb{C} \to \mathbb{C}$ and the nontrivial common eigenfunction $\psi : U \times V \subset \mathbb{C}^2 \to \mathbb{C}$ are in principle compatible sized meromorphic scalar valued functions defined in suitable open subsets $U, V \subset \mathbb{C}$.

The bispectral problem was completely solved in the scalar case for Schrödinger operators $L = -\partial_x^2 + U(x)$ and the potentials $U(x)$ for which bispectrality follows were characterized in [7]. The noncommutative (or matrix) version of the bispectral problem was first studied in [19, 21, 23] for the situation where both the physical and spectral operators were acting on the same side of the eigenfunction and the eigenvalues are scalar valued. Later on, several generalizations were considered. See [15, 14, 3, 10, 11, 9, 4] and references therein. This thesis follows up on the possibility of having the physical and the spectral operators acting on different sides. We also follow the suggestion in [11] of considering both eigenvalues as matrix valued.

The noncommutative bispectral problem which we shall study consists of finding all the bispectral triples (L, ψ, B) that satisfy systems of equations (1) with $L = L(x, \partial_x)$, $B = B(z, \partial_z)$ linear matrix differential operators, i.e., $L\psi = \sum_{i=0}^{l} a_i(x) \cdot \partial_x^i \psi$, $\psi B = \sum_{j=0}^{m} \partial_z^j \psi \cdot b_j(z)$. The functions $a_i : U \subset \mathbb{C} \to \mathbb{C}, b_j : V \subset \mathbb{C} \to \mathbb{C}, F : V \subset \mathbb{C} \to \mathbb{C}, \theta : U \subset \mathbb{C} \to \mathbb{C}$ and the nontrivial common eigenfunction $\psi : U \times V \subset \mathbb{C}^2 \to \mathbb{C}$ are in principle compatible sized meromorphic matrix valued functions defined in suitable open subsets $U, V \subset \mathbb{C}$. We remark that all the differential operators are considered in a neighborhood of an arbitrary given point and following [7] we assume that the functions are smooth enough so that all the derivatives considered make sense. In [6] were posed a few conjectures about some algebras of differential operators associated with orthogonal matrix polynomials and in [17] was proved one of these conjectures.

The impetus for the research presented here are three conjectures proposed in [11] about bispectral algebras and their challenging presentations in terms of generators and relations for which we present answers.

The conjectures are:

First Conjecture: Consider the matrix valued function

$$\psi(x,z) = e^{xz}\begin{pmatrix} z - x^{-1} & x^{-2} \\ 0 & z - x^{-1} \end{pmatrix}$$

and observe that $L\psi = -z^2\psi$ for the operator

$$L = -\partial_x^2 + 2\begin{pmatrix} x^{-2} & -2x^{-3} \\ 0 & x^{-2} \end{pmatrix}.$$

Conjecture 1. *The algebra of all matrix valued polynomials $\theta(x)$ for which there exists some operator B such that*

$$(\psi B)(x,z) = \theta(x)\psi(x,z)$$

is the algebra of all polynomials of the form

$$\begin{pmatrix} r_0^{11} & r_0^{12} \\ 0 & r_0^{11} \end{pmatrix} + \begin{pmatrix} r_1^{11} & r_1^{12} \\ 0 & r_1^{11} \end{pmatrix} x + \begin{pmatrix} r_2^{11} & r_2^{12} \\ r_1^{11} & r_2^{22} \end{pmatrix} x^2 + \begin{pmatrix} r_3^{11} & r_3^{12} \\ r_2^{22} + r_2^{11} - r_1^{12} & r_3^{22} \end{pmatrix} x^3 + x^4 p(x),$$

where $p \in M_2(\mathbb{C})[x]$ and all the variables $r_0^{11}, r_0^{12}, r_1^{11}, r_1^{12}, r_2^{11}, r_2^{22}, r_3^{11}, r_3^{12}, r_3^{22} \in \mathbb{C}$ are arbitrary.

Second Conjecture: Consider the matrix valued function

$$\psi(x,z) = e^{xz}\begin{pmatrix} z - x^{-1} & x^{-2} & -x^{-3} \\ 0 & z - x^{-1} & x^{-2} \\ 0 & 0 & z - x^{-1} \end{pmatrix}$$

and observe that $L\psi = -z^2\psi$ for the operator

$$L = -\partial_x^2 + 2\begin{pmatrix} x^{-2} & -2x^{-3} & 3x^{-4} \\ 0 & x^{-2} & -2x^{-3} \\ 0 & 0 & x^{-2} \end{pmatrix}.$$

Conjecture 2. *The algebra of all matrix valued polynomials $\theta(x)$ for which there exists some operator B such that*

$$(\psi B)(x,z) = \theta(x)\psi(x,z)$$

is the algebra of all polynomials of the form

$$\begin{pmatrix} r_0^{11} & r_0^{12} & r_0^{13} \\ 0 & r_0^{22} & r_0^{23} \\ 0 & 0 & r_0^{11} \end{pmatrix} + \begin{pmatrix} r_1^{11} & r_1^{12} & r_1^{13} \\ r_0^{22} - r_0^{11} & r_1^{22} & r_1^{23} \\ 0 & r_0^{22} - r_0^{11} & r_1^{11} + r_0^{23} - r_0^{12} \end{pmatrix} x$$

$$+ \begin{pmatrix} r_2^{11} & r_2^{12} & r_2^{13} \\ r_1^{22} - r_1^{11} - r_0^{23} + r_0^{12} & r_2^{22} & r_2^{23} \\ r_0^{22} - r_0^{11} & r_1^{22} - r_1^{11} & r_2^{11} + r_1^{23} - r_1^{12} \end{pmatrix} x^2 + \begin{pmatrix} r_3^{11} & r_3^{12} & r_3^{13} \\ r_3^{21} & r_3^{22} & r_3^{23} \\ r_1^{22} - 2r_1^{11} - r_0^{23} + r_0^{12} & r_3^{32} & r_3^{33} \end{pmatrix} x^3$$

$$+ \begin{pmatrix} r_4^{11} & r_4^{12} & r_4^{13} \\ r_4^{21} & r_4^{22} & r_4^{23} \\ r_3^{32} + r_3^{21} - r_2^{22} - r_2^{11} + r_1^{12} & r_4^{22} & r_4^{33} \end{pmatrix} x^4$$

$$+ \begin{pmatrix} r_5^{11} & r_5^{12} & r_5^{13} \\ r_5^{21} & r_5^{22} & r_5^{23} \\ r_4^{32} + r_4^{21} - r_3^{33} - r_3^{22} - r_3^{11} + r_2^{23} + r_2^{12} - r_1^{13} & r_5^{32} & r_5^{33} \end{pmatrix} x^5 + x^6 p(x),$$

where $p \in M_3(\mathbb{C})[x]$ *and all the variables* $r_0^{11}, r_0^{12}, \ldots, r_5^{33} \in \mathbb{C}$ *are arbitrary.*

Third Conjecture: Consider the matrix valued function

$$\psi(x,z) = \frac{e^{xz}}{(x-2)xz}\begin{pmatrix} \frac{x^3z^2-2x^2z^2-2x^2z+3xz+2x-2}{xz} & \frac{1}{x} \\ \frac{xz-2}{z} & x^2z-2xz-x+1 \end{pmatrix}$$

it is easy to check that $\psi\mathcal{B} = \theta\psi$ for

$$\mathcal{B} = \partial_z^3 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \partial_z^2 \cdot \begin{pmatrix} 0 & 0 \\ -\frac{2z+1}{z} & 0 \end{pmatrix} + \partial_z \cdot \begin{pmatrix} 1 & 0 \\ \frac{2(z-1)}{z^2} & 1 \end{pmatrix} + \begin{pmatrix} -z^{-1} & 0 \\ 6z^{-3} & z^{-1} \end{pmatrix}$$

and

$$\theta(x) = \begin{pmatrix} x & 0 \\ x^2(x-2) & x \end{pmatrix}.$$

Conjecture 3. *The algebra of all matrix valued polynomials* $F(z)$ *for which there exists some operator* L *such that*

$$(L\psi)(x,z) = \psi(x,z)F(z)$$

is the algebra of all polynomials of the form

$$\begin{pmatrix} a & 0 \\ b-a & b \end{pmatrix} + \begin{pmatrix} c & c \\ a-b-c & -c \end{pmatrix} z + \begin{pmatrix} a-b-c & c+a-b \\ d & e \end{pmatrix} \frac{z^2}{2} + z^3 p(z),$$

where $p \in M_2(\mathbb{C})[z]$ *and all the variables* a, b, c, d, e *are arbitrary.*

The conjectures are addressed by a method to verify the bispectrality of some remarkable algebras of matrix valued polynomials and by a general theorem to obtain presentations for finitely generated algebras.

We start with a definition

Definition 1. *Let* $\mathbb{K}$ *be a field of characteristic zero,* C *be a* $\mathbb{K}$*-algebra and* $S \subset C$*. We define*

$$\mathbb{K} \cdot < S >= span\left\{ \prod_{j=1}^{n} s_j \mid s_1, \ldots, s_n \in S, n \in \mathbb{N} \right\}.$$

This definition enable us to state the following theorems:

Theorem 1 (Full Rank One Algebras). *Let* C *be a graded* $\mathbb{K}$*-algebra,* $\Gamma \subset C$ *and* $A \subset C$ *two* $\mathbb{K}$*-algebras with the following properties:*

1. Γ *is a full rank* 1 *algebra with decomposition* $\Gamma = E \oplus \bigoplus_{j=k_0}^{\infty} C_j$*, for some* $k_0 \in \mathbb{N}$.
2. $\mathbb{K} \cdot \langle E \rangle = \Gamma$.
3. $A \bigcap \oplus_{k=0}^{k_0-1} C_k = E$.

Then, $\Gamma = A$.

Theorem 2 (Presentation of finitely generated algebras). *Let* A *be a finitely generated* $\mathbb{K}$*-algebra by* $\beta_1, \beta_2, \ldots, \beta_n$ *such that:*

- *There exist an ideal* I *of* $\mathbb{K} \cdot \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ *and an epimorphism of algebras*

$$f : \mathbb{K} \cdot \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle / I \longrightarrow A,$$

$$f(\overline{\alpha_j}) = \beta_j$$

- *There exists a subalgebra $\mathbb{K} \subset R \subset \mathbb{K} \cdot \langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle / I$ such that $\mathbb{K} \cdot \langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle / I$ is a free left R-module generated by $\{x_j\}_{j=0}^{\infty}$, i.e.,*

$$\mathbb{K} \cdot \langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle / I = \bigoplus_{j=0}^{\infty} R x_j .$$

- *$f|_R \colon R \longrightarrow A$ is a monomorphism.*
- *The set $\{f(x_j)\}_{j=0}^{\infty}$ is a basis for A as a left $f(R)$-module.*

Then, f is an isomorphism.

With these tools we give positive answers to the three conjectures and give nice presentations for the foregoing algebras. To tackle the Conjectures 1 and 2 we consider the following generalization:

Consider a nilpotent element $S \in M_N(\mathbb{K})$ of degree $D \geq 2$, consider the matrix valued function

$$\psi(x,z) = e^{xz} \left(Iz + \sum_{m=1}^{D} (-1)^m S^{m-1} x^{-m} \right) ,$$

and note that $L\psi(x,z) = -z^2 \psi(x,z)$ for the ordinary differential operator

$$L = -\partial_x^2 + 2 \sum_{m=1}^{D} (-1)^{m+1} m S^{m-1} x^{-m-1}.$$

Now we define a family of maps $\mathcal{P} = \{P_k\}_{k \in \mathbb{N}}$.

Definition 2. *For $k \in \mathbb{N}$ and $\theta \in M_N(\mathbb{K}[x])$, we define*

$$P_k(\theta) = \frac{\theta^{(k+1)}(0)}{k!} - \sum_{j=k+2}^{k+D} (-1)^{k-j} \left[\frac{\theta^{(j)}(0)}{j!}, S^{j-k-1} \right] . \tag{2}$$

This family $\mathcal{P}$ allows us to describe the algebra

$$\mathbb{A} = \{\theta \in M_N(\mathbb{K}[x]) | \exists \mathcal{B} = \mathcal{B}(z, \partial_z), (\psi\mathcal{B})(x,z) = \theta(x)\psi(x,z)\} .$$

Theorem 3. *Let*

$$\Gamma := \Big\{ \theta \in M_N(\mathbb{K}[x]) \mid P_0(x\theta(x)) = P_0(\theta(0)x), P_0(x^j\theta(x)) = 0, j \geq 2, \\ \sum_{k=0}^{q} (-1)^k S^{k+D-q-1} P_k(\theta) = 0, 0 \leq q \leq D-1 \Big\}.$$

Then, $\Gamma = \mathbb{A}$.

Moreover, for each θ we have an explicit expression for the operator $\mathcal{B}$.

To prove Theorem 3 we use Theorem 1. It remains to give the nice presentations in terms of generators as well as relations. The following corollaries gives the answer to this question.

Corollary 1. *Let Γ be the sub-algebra of $M_2(\mathbb{C})[x]$ of the form*

$$\begin{pmatrix} r_0^{11} & r_0^{12} \\ 0 & r_0^{11} \end{pmatrix} + \begin{pmatrix} r_1^{11} & r_1^{12} \\ 0 & r_1^{11} \end{pmatrix} x + \begin{pmatrix} r_2^{11} & r_2^{12} \\ r_1^{11} & r_2^{22} \end{pmatrix} x^2 + \begin{pmatrix} r_3^{11} & r_3^{12} \\ r_2^{22} + r_2^{11} - r_1^{12} & r_3^{22} \end{pmatrix} x^3 + x^4 p(x),$$

where $p \in M_2(\mathbb{C})[x]$ *and all the variables* $r_0^{11}, r_0^{12}, r_1^{11}, r_1^{12}, r_2^{11}, r_2^{22}, r_3^{11}, r_3^{12}, r_3^{22} \in \mathbb{C}$ *are arbitrary. Then* $\Gamma = \mathbb{A}$*. Moreover, for each* θ *we have an explicit expression for the operator* $\mathcal{B}$*.*

Furthermore, we have the presentation $\mathbb{A} = \mathbb{C} \cdot \langle \alpha_0, \alpha_1 \mid I = 0 \rangle$ *with the ideal* I *given by*

$$I := \langle \alpha_0^2, \alpha_1^3 + \alpha_0\alpha_1\alpha_0 - 3\alpha_1\alpha_0\alpha_1 + \alpha_0\alpha_1^2 + \alpha_1^2\alpha_0 \rangle .$$

Corollary 2. *Let* Γ *the sub-algebra of* $M_3(\mathbb{C})[x]$ *of the form*

$$\begin{pmatrix} r_0^{11} & r_0^{12} & r_0^{13} \\ 0 & r_0^{22} & r_0^{23} \\ 0 & 0 & r_0^{11} \end{pmatrix} + \begin{pmatrix} r_1^{11} & r_1^{12} & r_1^{13} \\ r_0^{22} - r_0^{11} & r_1^{22} & r_1^{23} \\ 0 & r_0^{22} - r_0^{11} & r_1^{11} + r_0^{23} - r_0^{12} \end{pmatrix} x$$

$$+ \begin{pmatrix} r_2^{11} & r_2^{12} & r_2^{13} \\ r_1^{22} - r_1^{11} - r_0^{23} + r_0^{12} & r_2^{22} & r_2^{23} \\ r_0^{22} - r_0^{11} & r_1^{22} - r_1^{11} & r_2^{11} + r_1^{23} - r_1^{12} \end{pmatrix} x^2 + \begin{pmatrix} r_3^{11} & r_3^{12} & r_3^{13} \\ r_3^{21} & r_3^{22} & r_3^{23} \\ r_1^{22} - 2r_1^{11} - r_0^{23} + r_0^{12} & r_3^{32} & r_3^{33} \end{pmatrix} x^3$$

$$+ \begin{pmatrix} r_4^{11} & r_4^{12} & r_4^{13} \\ r_4^{21} & r_4^{22} & r_4^{23} \\ r_3^{32} + r_3^{21} - r_2^{22} - r_2^{11} + r_1^{12} & r_4^{22} & r_4^{33} \end{pmatrix} x^4$$

$$+ \begin{pmatrix} r_5^{11} & r_5^{12} & r_5^{13} \\ r_5^{21} & r_5^{22} & r_5^{23} \\ r_4^{32} + r_4^{21} - r_3^{33} - r_3^{22} - r_3^{11} + r_2^{23} + r_2^{12} - r_1^{13} & r_5^{32} & r_5^{33} \end{pmatrix} x^5 + x^6 p(x) ,$$

where $p \in M_3(\mathbb{C})[x]$ *and all the variables* $r_0^{11}, r_0^{12}, \ldots, r_5^{33} \in \mathbb{C}$ *are arbitrary.*

Then, $\Gamma = \mathbb{A}$ *and for each* θ *we have an explicit expression for the operator* $\mathcal{B}$*.*

Furthermore, we have the presentation $\mathbb{A} = \mathbb{C} \cdot \langle \alpha_2, \alpha_3 \mid I = 0 \rangle$ *with*

$$I = \langle \alpha_2^3, \alpha_3^2 - \alpha_3, (\alpha_3\alpha_2)^2\alpha_3 - 4\alpha_3\alpha_2^2\alpha_3 \rangle .$$

On the other hand, the answer to the Conjecture 3 is given by the following:

Theorem 4. *Let* Γ *be the sub-algebra of* $M_2(\mathbb{C})[z]$ *of the form*

$$\begin{pmatrix} a & 0 \\ b - a & b \end{pmatrix} + \begin{pmatrix} c & c \\ a - b - c & -c \end{pmatrix} z + \begin{pmatrix} a - b - c & c + a - b \\ d & e \end{pmatrix} \frac{z^2}{2} + z^3 p(z),$$

where $p \in M_2(\mathbb{C})[z]$ *and all the variables* a, b, c, d, e *are arbitrary. Then* $\Gamma = \mathbb{A}$*.*

Furthermore, we have the presentation $\mathbb{A} = \mathbb{C} \cdot \langle \theta_1, \theta_3, \theta_4, \theta_5 \mid I = 0 \rangle$ *with*

$$I = \langle \theta_1^2 - \theta_1, \theta_4^2, \theta_4\theta_5, \theta_4\theta_1 + \theta_4\theta_3 - 2\theta_4 - \theta_5\theta_4 - \theta_5^2, \theta_3^2 - \theta_3 + \theta_5 - 3\theta_3\theta_4\theta_3\theta_5 - \theta_1\theta_4 - \theta_5\theta_1,$$

$$\theta_3\theta_1 - \theta_1 - \theta_4 - \frac{1}{2}\theta_4\theta_1 + \frac{1}{2}\theta_4\theta_3 + \theta_5\theta_1 - \frac{1}{2}\theta_5\theta_4 + \frac{1}{2}\theta_5^2 + \theta_3\theta_4 - \theta_1\theta_5 - \theta_3\theta_5,$$

$$\theta_1\theta_3 - \theta_3 + \theta_4 + \theta_5 - \frac{3}{2}\theta_4\theta_1 + \frac{3}{2}\theta_4\theta_3 - 2\theta_5\theta_1 - \frac{3}{2}\theta_5\theta_4 + \frac{3}{2}\theta_5^2 + 3\theta_3\theta_4 + \theta_3\theta_5,$$

$$\theta_5\theta_3 - \theta_4\theta_1 + \theta_4\theta_3 - \theta_5\theta_1 - \theta_5\theta_4 + \theta_5^2, \theta_5\theta_1\theta_5 - \theta_5^2\theta_1 - \theta_5\theta_4, \theta_5\theta_4\theta_1 - \theta_5^3 + \theta_5\theta_1\theta_4 + \theta_5^2\theta_1,$$

$$\theta_4\theta_1\theta_5 + \theta_4\theta_3\theta_5 - \theta_3^3, \theta_5\theta_3\theta_4 + \theta_5\theta_1\theta_4 \rangle$$

The first part of the previous theorem is proved using Theorem 1. The presentations of these results are achieved with the Theorem 2.

These are the main goals for the Chapters 1 and 2. In Chapter 3 we seek conditions on the potentials V for matrix Schrödinger operators of the form $L = -\partial_x^2 + V(x)$ that ensure bispectrality of their eigenfunctions. We begin with the definition of the family $\mathcal{P} = \{P_k\}_{k\in\mathbb{N}}$ which will be used to describe the map $\theta \mapsto \mathcal{B}$ such that $(\psi\mathcal{B})(x,z) = \theta(x)\psi(x,z)$ and the bispectral algebra

$$\mathbb{A} = \{\theta \in M_N(\mathbb{C}[x]) \mid \exists B = B(z,\partial_z), (\psi B)(x,z) = \theta(x)\psi(x,z)\}.$$

Definition 3. *For a meromorphic matrix valued function Laurent expansion $V = \sum_{j=-1}^{\infty} V_j x^j$, $k \in \mathbb{N}$, and $\theta \in M_N(\mathbb{C}[x])$, we define*

$$P_k(\theta) = \frac{\theta^{(k)}(0)}{(k-1)!} - \frac{1}{2}\sum_{j=0}^{k}\left[\frac{\theta^{(j)}(0)}{j!}, V_{k-1-j}\right].$$

Furthermore, we consider some important block matrix functions.

Definition 4. *Under the same notation of Definition 3, for $m \in \mathbb{N}$ define*

$$A_1^{[m]} = \begin{pmatrix} \frac{V_0}{2} & \frac{1}{2}V_{-1}+I_N & 0 & \dots & 0 & 0 & 0 \\ \frac{V_1}{2} & \frac{V_0}{2} & \frac{1}{2}V_{-1}+2I_N & \dots & 0 & 0 & 0 \\ . & . & \dots & . \ \dots & . & . \\ . & . & \dots & . \ \dots & . & . \\ . & . & \dots & . \ \dots & . & . \\ \frac{V_{m-2}}{2} & \frac{V_{m-3}}{2} & \frac{V_{m-4}}{2} & \dots & \frac{V_0}{2} & \frac{1}{2}V_{-1}+(m-1)I_N & 0 \\ \frac{V_{m-1}}{2} & \frac{V_{m-2}}{2} & \frac{V_{m-3}}{2} & \dots & \frac{V_1}{2} & \frac{V_0}{2} & \frac{1}{2}V_{-1}+mI_N \\ \frac{V_m}{2} & \frac{V_{m-1}}{2} & \frac{V_{m-2}}{2} & \dots & \frac{V_2}{2} & \frac{V_1}{2} & \frac{V_0}{2} \end{pmatrix},$$

$$A_2^{[m]} = \begin{pmatrix} V_{m+1} & V_m & \dots & V_1 \\ V_{m+2} & V_{m+1} & \dots & V_2 \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \end{pmatrix},$$

and for $\theta \in M_N(\mathbb{C}[x])$ we define $P_1^{m+1}(\theta) = (P_1(\theta), P_2(\theta), \dots, P_m(\theta), P_{m+1}(\theta))^T$ and $P_{m+2}^{\infty}(\theta) = (P_{m+2}(\theta), P_{m+3}(\theta), \dots)^T$.

We make use of the machinery defined above to state the following general result:

Theorem 5. *Let $\Gamma = \Big\{\theta \in M_N(\mathbb{C}[x]) \mid P_0(\theta) = 0, V_{-1}e_1(A_1^{[m]})^k P_1^{m+1}(\theta) = 0, A_2^{[m]}(A_1^{[m]})^k P_1^{m+1}(\theta) = 0, k \geq 0,$ $P_{m+2}^{\infty}(\theta) = 0, m = \deg(\theta)\Big\}$ then $\Gamma = \mathbb{A}$. Moreover, for each θ we have an explicit expression for the operator B such that*

$$(\psi B)(x,z) = \theta(x)\psi(x,z).$$

A remarkable class of potentials with the bispectral property is given by the:

Theorem 6 (Bispectrality of a Class of Polynomial Potentials). *If $V(V_0, V_1, x)$ is a polynomial of degree n in x such that $V''(x) = V'(x)V(x)$ and $(V_0, V_1) \in M_N(\mathbb{C})^2$ satisfy*

$$V_1^{i_1}V_0^{i_2}\dots V_1^{i_n}V_0^{i_{n+1}} = 0, \tag{3}$$

for any $i_1 \geq 1$, $i_1 + \dots + i_{n+1} \leq n+1$, and $n+2 \leq \deg_{1,2}(V_1^{i_1}V_0^{i_2}\dots V_1^{i_n}V_0^{i_{n+1}})$, then $V \in \mathbb{A}$. In particular, the operator $L = -\partial_x^2 + V(x)$ is bispectral.

If $n = 1$ the equations (3) turns out to be $V_1V_0 = V_1^2 = 0$. A nontrivial example for $N = 2$ is given by

$$V_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$V_0 = \begin{pmatrix} V_{011} & V_{012} \\ 0 & 0 \end{pmatrix}.$$

To obtain the potential $V(x) = V_0 + V_1 x$.

Here $V_0 = V(0)$ and $V_1 = V'(0)$. The role of the autonomous matrix equation $V''(x) = V'(x)V(x)$ is very important, because using Laurent series with simple pole at the origin we obtain a sequence $\{V_k(V_0, V_1, V_2)\}_{k\in\mathbb{N}}$ of algebraic morphisms

$$V_{-1} = \begin{pmatrix} -2I_m & 0 \\ 0 & 0 \end{pmatrix},$$

for $0 \leq m \leq N$.

$$V_0 = \begin{pmatrix} 0 & 0 \\ V_{021} & V_{022} \end{pmatrix},$$

$$V_1 = \begin{pmatrix} 0 & 0 \\ V_{121} & V_{122} \end{pmatrix},$$

$$V_2 = \begin{pmatrix} 0 & V_{212} \\ \frac{V_{122}V_{021}}{6} & \frac{V_{122}V_{022}}{2} \end{pmatrix}.$$

If $k \geq 3$ we can write

$$V_k = \sum_{j=1}^{k-1} jT_k^{-1}(V_jV_{k-1-j}). \tag{4}$$

with $T_k : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ defined by $T_k(a) = k(k-1)a + V_{-1}a - kaV_{-1}$.

If we define the grading $\deg_{1,2,3}$ on the ring $\mathbb{C}\langle V_0, V_1, V_2\rangle$ to be $\deg_{1,2,3}(V_0) = 1$, $\deg_{1,2,3}(V_1) = 2$, $\deg_{1,2,3}(V_2) = 3$ we obtain that $\deg_{1,2,3}(V_k) = k + 1$.

In this dissertation, we consider from now on $\mathbb{K}$ to be a field with characteristic zero. Furthermore, some computations were performed with the software Singular and Maxima.

Both for practical as well as for purely mathematical reasons it is desirable to look at the corresponding integral operator in more complicated situations than the real line, or equivalently in the case when Fourier analysis is replaced by the decomposition in terms of eigenfunctions of a general second order differential operator on the line.

J. J. Duistermaat 1 and F. A. Grünbaum [7]

1

Matrix Bispectrality of Full Rank One Algebras

1.1 Introduction

The main goal of this chapter is to establish a method to verify whether an algebra of matrix polynomials is bispectral or not. We apply this method to some family of algebras parametrized by the size N of the matrix and a nilpotent element $S \in M_N(\mathbb{C})$. Furthermore, the isomorphism between the matrix eigenvalues and the corresponding operator is given explicitly using some family of maps $\mathcal{P}$. This family of algebras has a remarkable algebraic property: to have a Pierce decomposition.

1.2 General Results

We consider the triples (L, ψ, B) satisfying systems of equations

$$L\psi(x,z) = \psi(x,z)F(z) \qquad (\psi B)(x,z) = \theta(x)\psi(x,z) \tag{1.1}$$

with $L = L(x, \partial_x)$, $B = B(z, \partial_z)$ linear matrix differential operators, i.e., $L\psi = \sum_{i=0}^{l} a_i(x) \cdot \partial_x^i \psi$, $\psi B = \sum_{j=0}^{m} \partial_z^j \psi \cdot b_j(z)$. The functions $a_i : U \subset \mathbb{C} \to \mathbb{C}, b_j : V \subset \mathbb{C} \to \mathbb{C}, F : V \subset \mathbb{C} \to \mathbb{C}, \theta : U \subset \mathbb{C} \to \mathbb{C}$ and the nontrivial common eigenfunction $\psi : U \times V \subset \mathbb{C}^2 \to \mathbb{C}$ are in principle compatible sized meromorphic matrix valued functions defined in suitable open subsets $U, V \subset \mathbb{C}$.

A triple (L, ψ, B) satisfying (1.1) is called a bispectral triple.

Now we fix the normalized* operator L and the eigenfunctions $\psi(\cdot, z)$. We are interested in the bispectral pairs associated to $L = L(x, \partial_x)$, i.e., operators $B = B(z, \partial_z)$ such that $(\psi B)(x, z) = \theta(x)\psi(x, z)$ for some function $\theta = \theta(x)$. It is not hard to verify that the set of operators $B = B(z, \partial_z)$ satisfying (1.1) generates a noncommutative algebra of operators.

We first note that θ satisfying Equation (1.1) has to be an element of the algebra of polynomials with $N \times N$ matrix coefficients, which we denote by $M_N(\mathbb{C})[x]$. The proof follows closely an argument in the original paper of [7]. See the Appendix .1.

Clearly the set

$$\mathbb{A} = \left\{\theta \in M_N(\mathbb{C})[x] \,\middle|\, \exists \mathcal{B} = \mathcal{B}(z, \partial_z), (\psi \mathcal{B})(x, z) = \theta(x)\psi(x, z)\right\} \tag{1.2}$$

is a noncommutative $\mathbb{C}$-algebra.

We shall start with some theoretical results.

Definition 5. *Define $Bp(z, L)$ to be the set of* bispectral partners *to L, i.e.,*

$$Bp(z, L) = \{B = B(z, \partial_z) | \exists \theta \in M_N(\mathbb{C}[x]), (\psi \mathcal{B})(x, z) = \theta(x)\psi(x, z)\} .$$

A straightforward consequence of the definition is the following.

Lemma 1. *The set $Bp(z, L)$ is a $\mathbb{C}$-algebra.*

However, much more can be said about the properties of the algebra $Bp(z, L)$ in the case that will be studied in the sequel. For that we have to consider the following important class of algebras:

Definition 6. *Let $\mathbb{K}$ be a field, C be a graded $\mathbb{K}$-algebra, we define a full rank 1 algebra to be a subalgebra $A \subset C$ such that*

$$A = E \oplus \bigoplus_{j=k_0}^{\infty} C_j$$

*If $L = L(x, \partial_x)$, $L = \sum_{i=0}^{l} a_i(x)\partial_x^i$ with a_l constant and scalar, $a_{l-1} = 0$, then L is called normalized.

for some finite dimensional $\mathbb{K}$-vector space E and $k_0 \in \mathbb{N}$. Furthermore, we denote by $\left\{e_i^{[j]}\right\}_{1\leq i\leq N, j\geq 0}$ some basis for C_j. See [5].

Remark 1. *Note that for a full rank 1 algebra $A \subset C$. We consider k_0 the smallest positive integer such that $C_j \subset A$, for all $j \geq k_0$. For this k_0 we can write $E = \left(\bigoplus_{j=0}^{k_0-1} C_j\right) \cap A$.*

The results in Theorems 7, 8, 9, and 14 will be used in the sequel to provide a positive answer to the conjectures of Grunbaum [11]. They are of interest on their own.

Theorem 7. *Let C be a graded $\mathbb{K}$-algebra where $\dim_{\mathbb{K}} C_j = N < \infty$, $C_j = \sum_{i=1}^{N} \mathbb{K} \cdot e_i^{[j]}$. Suppose that for every t, $1 \leq t \leq N$, $i, j \in \mathbb{N}$, there exist $1 \leq r, s \leq N$ such that $e_t^{[i+j]} = e_r^{[i]} e_s^{[j]}$ and $A \subset C$ is a full rank 1 algebra. Then, A is a finitely generated $\mathbb{K}$-algebra.*

Proof. *We write*

$$A = E \oplus \bigoplus_{j=k_0}^{\infty} C_j.$$

Since E is a finite dimensional $\mathbb{K}$-vector space E, we can consider a basis $\{\alpha_1, ..., \alpha_m\}$ for E and write $E = \sum_{s=1}^{m} \mathbb{K} \cdot \alpha_s$. Define

$$A_0 := \mathbb{K} \cdot \langle e_i^{[k]}, \alpha_s \mid 1 \leq i \leq N, k_0 \leq k \leq 2k_0 - 1, 1 \leq s \leq m \rangle.$$

We claim that $A = A_0$.

First of all, we prove that for every $q \geq 2$, $(q-1)k_0 \leq k \leq qk_0 - 1$, $e_i^{[k]} \in A_0$. The initial step is clear for $q = 2$. Assume that $e_i^{[p]} \in A_0$ for $(q-1)k_0 \leq p \leq qk_0 - 1, 1 \leq i \leq N$ and note that $qk_0 \leq k \leq (q+1)k_0 - 1$ implies $(q-1)k_0 \leq k - k_0 \leq qk_0 - 1$ and $e_i^{[k-k_0]} \in A_0$ for $1 \leq i \leq N$. Consider $1 \leq i \leq N$, by hypothesis there exists $1 \leq r, s \leq N$ such that $e_i^{[k]} = e_r^{[k-k_0]} \cdot e_s^{k_0} \in A_0$. This proves the inductive step. The assertion follows by induction.

Since $k_0 + \mathbb{N} = \bigcup_{q=2}^{\infty} \{k \in \mathbb{N} \mid (q-1)k_0 \leq k \leq qk_0 - 1\}$. We have that $e_i^{[k]} \in A_0$ for $1 \leq i \leq N$, $k \geq k_0$ then $\bigoplus_{j=k_0}^{\infty} C_j \subset A_0$. But $E = \sum_{s=1}^{m} \mathbb{K} \cdot \alpha_s \subset A_0$. Thus, $A = A_0$ and A is a finitely generated k-algebra.

Remark 2. *The converse is not true. Consider for example the graded algebra $C = M_N(\mathbb{K}[x])$ and $A = \mathbb{K}[x]$, then A is a finitely generated $\mathbb{K}$-algebra which is not of full rank 1.*

Now we use the following theorem whose proof may be found in [16].

Theorem 8 (Stafford). *Let $R \subset S$ be algebras over a central field $\mathbb{K}$ such that S is Noetherian and S/R is a finite dimensional $\mathbb{K}$-vector space. Then, R is Noetherian.*

Corollary 3. *Let C be a Noetherian graded algebra and $A \subset C$ be a full rank 1 $\mathbb{K}$-algebra over a central field $\mathbb{K}$. Then, A is Noetherian.*

Proof. *Since $A = E \oplus \bigoplus_{j=k_0}^{\infty} C_j$ for some finite dimensional vector space E, we can consider the complement of any subspace F with respect to $\bigoplus_{j=0}^{k_0-1} C_j$ and obtain $C = F \oplus A$ then $\dim_{\mathbb{K}}(C/A) = \dim_{\mathbb{K}}(F) < \infty$. Since C is Noetherian the previous theorem implies the assertion.* □

Definition 7. *Let $\mathbb{K}$ be a field, C be a $\mathbb{K}$-algebra and $S \subset C$. We define*

$$\mathbb{K}\cdot < S >= span \left\{ \prod_{j=1}^{n} s_j \mid s_1, \dots, s_n \in S, n \in \mathbb{N} \right\}.$$

The following theorem connects the bispectral property to full rank 1 algebras.

Theorem 9 (Full Rank One Algebras). *Let C be a graded $\mathbb{K}$-algebra, $\Gamma \subset C$ and $A \subset C$ two $\mathbb{K}$-algebras with the following properties:*

1. *Γ is a full rank 1 algebra, with decomposition $\Gamma = E \oplus \bigoplus_{j=k_0}^{\infty} C_j$, for some $k_0 \in \mathbb{N}$.*
2. *$\mathbb{K} \cdot \langle E \rangle = \Gamma$.*
3. *$A \bigcap \oplus_{k=0}^{k_0-1} C_k = E$.*

Then, $\Gamma = A$.

Proof. *We shall break the proof in 2 steps.*

Step 1: The inclusion $\Gamma \subset A$.

Using (2) and (3) we have $E \subset A$ and Γ is the algebra generated by E, since A is an algebra we obtain the inclusion $\Gamma \subset A$.

Step 2: The inclusion $A \subset \Gamma$.

Consider $\theta \in A$ and write $\theta = \theta_1 + \theta_2$ with $\theta_1 \in \oplus_{k=0}^{k_0-1} C_k$ and $\theta_2 \in \oplus_{k=k_0}^{\infty} C_k$, since $\Gamma \supset \oplus_{k=k_0}^{\infty} C_k$ we have that $\theta_2 \in \Gamma \subset A$. In particular $\theta_1 = \theta - \theta_2 \in A \bigcap \oplus_{k=0}^{k_0-1} C_k = E \subset \Gamma$, then $\theta = \theta_1 + \theta_2 \in \Gamma$. □

Definition 8. *The shift operator $S_N \in M_N(\mathbb{K}[x])$ is defined by*

$$S_N = \sum_{s=1}^{N-1} e_{s,s+1}$$

for $N \geq 2$, where as usual $e_{r,s}$ denotes the matrix with 1 at entry (r,s) and zeros elsewhere.

We recall that for $N \geq 2$

$$S_N^j = \begin{cases} \sum_{s=1}^{N-j} e_{s,s+j} & \text{if } 0 \leq j \leq N-1, \\ 0 & \text{if } j \geq N. \end{cases}$$

In particular S_N is nilpotent of degree N.

The following theorem give us a concrete example of a nontrivial full rank 1 algebra.

Theorem 10. *Let $N \in \mathbb{Z}_+$ and the following elements in $M_N(\mathbb{K}[x])$:*

$\alpha_0 = S_N$,

$\alpha_1 = Ix + (-1)^N e_{N1} x^N$,

$\alpha_k = e_{1N} x^k$, if $2 \leq k \leq N-1$,

$\beta_k = e_{kk} x^N + (-1)^N e_{N1} x^{2N-1}$, if $1 \leq k \leq N$.

Then, $x^{2N} M_N(\mathbb{K}[x])$ is contained in the subalgebra A of $M_N(\mathbb{K}[x])$ that is generated by α_j, β_k, $0 \leq j \leq N-1, 1 \leq k \leq N$.

Proof. *Since $\beta_k^2 = e_{kk} x^{2N}$ for $2 \leq k \leq N-1$ and $\alpha_1^n = Ix^n + (-1)^n n e_{N1} x^{n+N-1}$ for $n \geq 1$ we have $\beta_k \alpha_1^n = e_{kk} x^{n+2N} \in A$ for $n \geq 1$, in other words $e_{kk} x^n \in A$ for $n \geq 2N$. On the other hand, $\alpha_0^{N-1} \alpha_1^n \alpha_0^{N-1} = (-1)^N n e_{1N} x^{n+N-1} \in A$ for $n \geq 1$, hence $e_{1N} x^n \in A$ for $n \geq N$. However, $e_{1N} x^k \in A$ for $2 \leq k \leq N-1$, therefore $e_{1N} x^n \in A$ for $n \geq 2$.*

Note that $\alpha_2 \alpha_1^n = e_{1N} x^n + (-1)^N n e_{11} x^{n+N-1} \in A$, $\alpha_1^n \alpha_2 = e_{1N} x^n + (-1)^N n e_{NN} x^{n+N-1} \in A$. Then $e_{11} x^n \in A$ and $e_{NN} x^n \in A$ for $n \geq N+1$. This implies that $\beta_1 \alpha_1^n = (e_{11} x^N + (-1)^N e_{N1} x^{2N-1})(Ix^n + (-1)^N n e_{N1} x^{n+N-1}) = e_{11} x^{n+N} + (-1)^N e_{N1} x^{2N+n-1} \in A$ for $n \geq 1$. Thus, $e_{N1} x^n \in A$ for $n \geq 2N$.

The previous proposition implies $\alpha_0^j = \sum_{s=1}^{N-j} e_{s,s+j}$ for $0 \leq j \leq N-1$ then $\alpha_0^{N-i}(e_{N1} x^n) \alpha_0^{j-1} = e_{ij} x^n \in A$ for $1 \leq i,j \leq N$, $n \geq 2N$ and this proves the assertion. □

Corollary 4. *The algebra A is full rank 1*

Proof. *Note that $A = E \oplus x^{2N} M_N(\mathbb{K}[x])$ with $E = A \cap \oplus_{j=0}^{2N-1} M_N(\mathbb{K}[x])_j$.* □

In the next section we give a family of algebras whose bispectrality can be obtained using the Theorem 9.

1.3 The Examples

We begin with the example of the matrix algebra given in the paper [11]. There the algebra considered is the set of polynomials of the form

$$\theta(x) = \begin{pmatrix} r_0^{11} & r_0^{12} \\ 0 & r_0^{11} \end{pmatrix} + \begin{pmatrix} r_1^{11} & r_1^{12} \\ 0 & r_1^{11} \end{pmatrix} x + \begin{pmatrix} r_2^{11} & r_2^{12} \\ r_1^{11} & r_2^{22} \end{pmatrix} x^2 + \begin{pmatrix} r_3^{11} & r_3^{12} \\ r_2^{22} + r_2^{11} - r_1^{12} & r_3^{22} \end{pmatrix} x^3 + x^4 p(x) \quad (1.3)$$

where $p \in M_2(\mathbb{C})[x]$ and all the variables $r_0^{11}, r_0^{12}, r_1^{11}, r_1^{12}, r_2^{11}, r_2^{22}, r_3^{11}, r_3^{12}, r_3^{22} \in \mathbb{C}$. Note that this algebra is full rank one and the relations that must be determined to obtain the complete description of the algebra are in the monomials of degree less than or equal to three. In the following subsection, we generalize this algebra to an arbitrary size of matrix N and find the relations that determine them.

1.3.1 Family of Algebras Linked to a Nilpotent Element in $M_N(\mathbb{K})$

As a particular example, we consider a nilpotent element $S \in M_N(\mathbb{K})$ of degree $D \geq 2$, consider the matrix valued function

$$\psi(x,z) = e^{xz}\left(Iz + \sum_{m=1}^{D} (-1)^m S^{m-1} x^{-m} \right),$$

and note that $L\psi(x,z) = -z^2\psi(x,z)$ for the ordinary differential operator

$$L = -\partial_x^2 + 2\sum_{m=1}^{D} (-1)^{m+1} m S^{m-1} x^{-m-1}.$$

Moreover, if m is even we have:

$$(adL)^m(\theta)\psi = (-1)^{m/2} 2^m (-z^2)^{m/2} \psi b_m = (-1)^{m/2} 2^m L^{m/2} \psi b_m = \left((-1)^{m/2} 2^m (L^{m/2}) \cdot b_m\right)\psi.$$

Therefore,

$$\left((adL)^m(\theta) - \left((-1)^{m/2} 2^m (L^{m/2}) \cdot b_m\right)\right)\psi = 0.$$

However, the operator $(adL)^m(\theta) - \left((-1)^{m/2}2^m(L^{m/2}) \cdot b_m\right)$ is independent of z and its kernel contains the infinite dimensional linearly independent set $\{\psi(\cdot, z)\}_{z\in\mathbb{C}}$. Thus, the operator is zero and $(adL)^m(\theta) = (-1)^{m/2}2^m(L^{m/2}) \cdot b_m$.

Now we characterize the algebra $\mathbb{A} = \{\theta \in M_N(\mathbb{K}[x]) | \exists \mathcal{B} = \mathcal{B}(z, \partial_z), (\psi\mathcal{B})(x, z) = \theta(x)\psi(x, z)\}$ for this particular example. We begin with the definition of the family $\mathcal{P} = \{P_k\}_{k\in\mathbb{N}}$ which will be used to describe the map $\theta \mapsto \mathcal{B}$ such that $(\psi\mathcal{B})(x, z) = \theta(x)\psi(x, z)$.

Definition 9. *For $k \in \mathbb{N}$ and $\theta \in M_N(\mathbb{K}[x])$, we define*

$$P_k(\theta) = \frac{\theta^{(k+1)}(0)}{k!} - \sum_{j=k+2}^{k+D} (-1)^{k-j} \left[\frac{\theta^{(j)}(0)}{j!}, S^{j-k-1}\right]. \tag{1.4}$$

The family $\mathcal{P} = \{P_k\}_{k\in\mathbb{N}}$ can be used to describe the algebra prescribed by Equation (1.3). If $\theta(x) = \sum_{k=0}^{m} a_k x^k \in M_2(\mathbb{C}[x])$ for $m \geq 4$ satisfies $P_0(x\theta(x)) = P_0(\theta(0)x)$, $P_0(x^j\theta(x)) = 0$ for $j \geq 2$,

$$\sum_{k=0}^{q}(-1)^k S_2^{k-q+1} P_k(\theta) = 0, \text{for } 0 \leq q \leq 1.$$

Then, $[S_2, a_0] = 0$, $[S_2, a_1] = [S_2, a_0]$, $S_2 a_2 S_2 = S_2 a_1$, and $S_2 a_3 S_2 = a_2 S_2 + S_2 a_2 - a_1$. Writing

$$a_k = \begin{pmatrix} r_k^{11} & r_k^{12} \\ r_k^{21} & r_k^{22} \end{pmatrix},$$

we have that $r_0^{21} = 0$, $r_0^{22} = r_0^{11}$, $r_1^{21} = 0$, $r_1^{22} = r_1^{11} = r_2^{21}$ and $r_3^{21} = r_2^{22} + r_2^{11} - r_1^{12}$. These equations are exactly those that describe the algebra of the form of Equation (1.3) as a sub-algebra of $M_2(\mathbb{C}[x])$.

We show now some properties of the family $\mathcal{P} := \{P_k\}_{k\in\mathbb{N}}$.

Lemma 2. *For every $\theta \in M_N(\mathbb{K}[x])$,*

$$P_k(\theta) = P_0\left(\frac{\theta^{(k+1)}(0)}{k!}x - \sum_{r=2}^{D} \frac{\theta^{(r+k)}(0)}{(r+k)!}x^r\right). \tag{1.5}$$

Proof. *In fact,*

$$P_0\left(\frac{\theta^{(k+1)}(0)}{k!}x - \sum_{r=2}^{D} \frac{\theta^{(r+k)}(0)}{(r+k)!}x^r\right) = \frac{\theta^{(k+1)}(0)}{k!} - \sum_{r=2}^{D}(-1)^r \left[\frac{\theta^{(r+k)}(0)}{(r+k)!}, S^{r-1}\right]$$

$$= \frac{\theta^{(k+1)}(0)}{k!} - \sum_{j=k+2}^{k+D} (-1)^{k-j} \left[\frac{\theta^{(j)}(0)}{j!}, S^{j-k-1} \right] = P_k(\theta).$$

The previous lemma allows us to study the properties of the family $\mathcal{P} = \{P_k\}_{k\in\mathbb{N}}$ through P_0.

Lemma 3 (Product Formula for P_0). *If $\theta_1, \theta_2 \in M_N(\mathbb{K}[x])$ then,*

$$P_0(\theta_1\theta_2) = \sum_{s=0}^{D} \left\{ P_0(x^s\theta_1(x)) \frac{\theta_2^{(s)}(0)}{s!} + \frac{\theta_1^{(s)}(0)}{s!} P_0(x^s\theta_2(x)) \right\} - (\theta_1\theta_2)'(0).$$

Proof. *By the definition of P_0,*

$$P_0(\theta_1\theta_2) = (\theta_1\theta_2)'(0) - \sum_{r=2}^{D} (-1)^r \left[\frac{(\theta_1\theta_2)^r(0)}{r!}, S^{r-1} \right]$$

$$= \theta_1'(0)\theta_2(0) + \theta_1(0)\theta_2'(0) - \sum_{r=2}^{D} (-1)^r \left[\sum_{t=0}^{r} \frac{\theta_1^{(t)}(0)}{t!} \frac{\theta_2^{(r-t)}(0)}{(r-t)!}, S^{r-1} \right]$$

$$= \theta_1'(0)\theta_2(0) + \theta_1(0)\theta_2'(0) - \sum_{r=2}^{D} \sum_{t=0}^{r} (-1)^r \left[\frac{\theta_1^{(t)}(0)}{t!}, S^{r-1} \right] \frac{\theta_2^{(r-t)}(0)}{(r-t)!}$$

$$- \sum_{r=2}^{D} \sum_{t=0}^{r} (-1)^r \frac{\theta_1^{(r-t)}(0)}{(r-t)!} \left[\frac{\theta_2^{(t)}(0)}{t!}, S^{r-1} \right] = \theta_1'(0)\theta_2(0) + \theta_1(0)\theta_2'(0) - \sum_{r=2}^{D} (-1)^r \left[\frac{\theta_1^{(r)}(0)}{r!}, S^{r-1} \right] \theta_2(0)$$

$$- \sum_{r=2}^{D} \sum_{t=0}^{r-1} (-1)^r \left[\frac{\theta_1^{(t)}(0)}{t!}, S^{r-1} \right] \frac{\theta_2^{(r-t)}(0)}{(r-t)!}$$

$$- \sum_{r=2}^{D} (-1)^r \theta_1(0) \left[\frac{\theta_2^{(r)}(0)}{r!}, S^{r-1} \right] - \sum_{r=2}^{D} \sum_{t=0}^{r-1} (-1)^r \frac{\theta_1^{(r-t)}(0)}{(r-t)!} \left[\frac{\theta_2^{(t)}(0)}{t!}, S^{r-1} \right]$$

$$= \left(\theta_1'(0) - \sum_{r=2}^{D} (-1)^r \left[\frac{\theta_1^{(r)}(0)}{r!}, S^{r-1} \right] \right) \theta_2(0) + \theta_1(0) \left(\theta_2'(0) - \sum_{r=2}^{D} (-1)^r \left[\frac{\theta_2^{(r)}(0)}{r!}, S^{r-1} \right] \right)$$

$$- \sum_{r=2}^{D} \sum_{t=0}^{r-1} (-1)^r \left[\frac{\theta_1^{(t)}(0)}{t!}, S^{r-1} \right] \frac{\theta_2^{(r-t)}(0)}{(r-t)!} - \sum_{r=2}^{D} \sum_{t=0}^{r-1} (-1)^r \frac{\theta_1^{(r-t)}(0)}{(r-t)!} \left[\frac{\theta_2^{(t)}(0)}{t!}, S^{r-1} \right]$$

$$= P_0(\theta_1)\theta_2(0) + \theta_1(0)P_0(\theta_2) - \sum_{r=2}^{D} (-1)^r \left[\theta_1(0), S^{r-1} \right] \frac{\theta_2^{(r)}(0)}{r!} - \sum_{r=2}^{D} (-1)^r \frac{\theta_1^{(r)}(0)}{r!} \left[\theta_2(0), S^{r-1} \right]$$

$$- \sum_{r=2}^{D} \sum_{t=1}^{r-1} (-1)^r \left[\frac{\theta_1^{(t)}(0)}{t!}, S^{r-1} \right] \frac{\theta_2^{(r-t)}(0)}{(r-t)!} - \sum_{r=2}^{D} \sum_{t=1}^{r-1} (-1)^r \frac{\theta_1^{(r-t)}(0)}{(r-t)!} \left[\frac{\theta_2^{(t)}(0)}{t!}, S^{r-1} \right]$$

$$= P_0(\theta_1)\theta_2(0) + \theta_1(0)P_0(\theta_2) - \sum_{r=2}^{D} (-1)^r \left(\left[\theta_1(0), S^{r-1} \right] \frac{\theta_2^{(r)}(0)}{r!} + \frac{\theta_1^{(r)}(0)}{r!} \left[\theta_2(0), S^{r-1} \right] \right)$$

$$-\sum_{r=2}^{D}\sum_{t=1}^{r-1}(-1)^r\left(\left[\frac{\theta_1^{(t)}(0)}{t!},S^{r-1}\right]\frac{\theta_2^{(r-t)}(0)}{(r-t)!}+\frac{\theta_1^{(r-t)}(0)}{(r-t)!}\left[\frac{\theta_2^{(t)}(0)}{t!},S^{r-1}\right]\right).$$

However,

$$\sum_{r=2}^{D}\sum_{t=1}^{r-1}(-1)^r\left(\left[\frac{\theta_1^{(t)}(0)}{t!},S^{r-1}\right]\frac{\theta_2^{(r-t)}(0)}{(r-t)!}+\frac{\theta_1^{(r-t)}(0)}{(r-t)!}\left[\frac{\theta_2^{(t)}(0)}{t!},S^{r-1}\right]\right)$$

$$=\sum_{t=1}^{D-1}\sum_{r=t+1}^{D}(-1)^r\left(\left[\frac{\theta_1^{(t)}(0)}{t!},S^{r-1}\right]\frac{\theta_2^{(r-t)}(0)}{(r-t)!}+\frac{\theta_1^{(r-t)}(0)}{(r-t)!}\left[\frac{\theta_2^{(t)}(0)}{t!},S^{r-1}\right]\right)$$

$$=\sum_{t=1}^{D-1}\sum_{s=1}^{D-t}(-1)^{s+t}\left(\left[\frac{\theta_1^{(t)}(0)}{t!},S^{s+t-1}\right]\frac{\theta_2^{(s)}(0)}{(s)!}+\frac{\theta_1^{(s)}(0)}{(s)!}\left[\frac{\theta_2^{(t)}(0)}{t!},S^{s+t-1}\right]\right)$$

$$=\sum_{s=1}^{D-1}\sum_{t=1}^{D-s}(-1)^{s+t}\left(\left[\frac{\theta_1^{(t)}(0)}{t!},S^{s+t-1}\right]\frac{\theta_2^{(s)}(0)}{(s)!}+\frac{\theta_1^{(s)}(0)}{(s)!}\left[\frac{\theta_2^{(t)}(0)}{t!},S^{s+t-1}\right]\right)$$

$$=\sum_{s=1}^{D-1}\sum_{u=s+1}^{D}(-1)^{u}\left(\left[\frac{\theta_1^{(u-s)}(0)}{(u-s)!},S^{u-1}\right]\frac{\theta_2^{(s)}(0)}{(s)!}+\frac{\theta_1^{(s)}(0)}{(s)!}\left[\frac{\theta_2^{(u-s)}(0)}{(u-s)!},S^{u-1}\right]\right)$$

$$=-\sum_{s=1}^{D-1}\left\{P_0\left(\sum_{u=s+1}^{D}\frac{\theta_1^{(u-s)}(0)}{(u-s)!}x^u\right)\frac{\theta_2^{(s)}(0)}{(s)!}+\frac{\theta_1^{(s)}(0)}{(s)!}P_0\left(\sum_{u=s+1}^{D}\frac{\theta_2^{(u-s)}(0)}{(u-s)!}x^u\right)\right\}.$$

On the other hand,

$$\sum_{r=2}^{D}(-1)^r\left(\left[\theta_1(0),S^{r-1}\right]\frac{\theta_2^{(r)}(0)}{r!}+\frac{\theta_1^{(r)}(0)}{r!}\left[\theta_2(0),S^{r-1}\right]\right)$$

$$=(-1)^D\left(\left[\theta_1(0),S^{D-1}\right]\frac{\theta_2^{(D)}(0)}{D!}+\frac{\theta_1^{(D)}(0)}{D!}\left[\theta_2(0),S^{D-1}\right]\right)$$

$$+\sum_{r=2}^{D-1}(-1)^r\left(\left[\theta_1(0),S^{r-1}\right]\frac{\theta_2^{(r)}(0)}{r!}+\frac{\theta_1^{(r)}(0)}{r!}\left[\theta_2(0),S^{r-1}\right]\right)$$

$$=(-1)^D\left(\left[\theta_1(0),S^{D-1}\right]\frac{\theta_2^{(D)}(0)}{r!}+\frac{\theta_1^{(D)}(0)}{D!}\left[\theta_2(0),S^{D-1}\right]\right)$$

$$-\sum_{s=1}^{D-1}\left(P_0(\theta_1(0)x^s)\frac{\theta_2^{(s)}(0)}{s!}+\frac{\theta_1^{(s)}(0)}{s!}P_0(\theta_2(0)x^s)\right)+(\theta_1\theta_2)'(0).$$

Then,

$$P_0(\theta_1\theta_2)=P_0(\theta_1)\theta_2(0)+\theta_1(0)P_0(\theta_2)-(\theta_1\theta_2)'(0)$$

$$+\sum_{s=1}^{D-1}\left\{P_0\left(\sum_{u=s}^{D}\frac{\theta_1^{(u-s)}(0)}{(u-s)!}x^u\right)\frac{\theta_2^{(s)}(0)}{(s)!}+\frac{\theta_1^{(s)}(0)}{(s)!}P_0\left(\sum_{u=s}^{D}\frac{\theta_2^{(u-s)}(0)}{(u-s)!}x^u\right)\right\}$$

$$-(-1)^D\left(\left[\theta_1(0),S^{D-1}\right]\frac{\theta_2^{(D)}(0)}{D!}+\frac{\theta_1^{(D)}(0)}{D!}\left[\theta_2(0),S^{D-1}\right]\right)$$

$$= \sum_{s=0}^{D} \left\{ P_0(x^s\theta_1(x))\frac{\theta_2^{(s)}(0)}{s!} + \frac{\theta_1^{(s)}(0)}{s!}P_0(x^s\theta_2(x)) \right\} - (\theta_1\theta_2)'(0).$$

□

Some remarkable cases of the Lemma 3 are stated in the following corollaries.

Corollary 5. *If* $\theta_1, \theta_2 \in M_N(\mathbb{K}[x])$ *with* $\theta_1 = c \in M_N(\mathbb{K})$ *is a constant, then*

$$P_0(c\theta_2) = cP_0(\theta_2) + \sum_{s=2}^{D} P_0(cx^s).\frac{\theta_2^{(s)}(0)}{s!}.$$

Corollary 6. *If* $\theta_1(0) = 0$*, then*

$$P_0(\theta_1\theta_2) = P_0(\theta_1)\theta_2(0) - \theta_1'(0)P_0(\theta_2(0)x) + \sum_{s=1}^{D} \left\{ P_0(x^s\theta_1(x))\frac{\theta_2^{(s)}(0)}{s!} + \frac{\theta_1^{(s)}(0)}{s!}P_0(x^s\theta_2(x)) \right\}.$$

Corollary 7. *If* $\theta_1(0) = \theta_2(0) = 0$*, then*

$$P_0(\theta_1\theta_2) = \sum_{s=1}^{D} \left\{ P_0(x^s\theta_1(x))\frac{\theta_2^{(s)}(0)}{s!} + \frac{\theta_1^{(s)}(0)}{s!}P_0(x^s\theta_2(x)) \right\}.$$

The next lemma tells us that knowledge of any P_0 determines the family $\mathcal{P} = \{P_k\}_{k\in\mathbb{N}}$.

Lemma 4. *For every* $\theta \in M_N(\mathbb{K}[x])$*, we have that*

$$P_0(\theta) = -\frac{k}{k+1}P_k(\theta'(0)x^{k+1}) + P_k(x^k(\theta(x) - \theta(0))).$$

Theorem 11 (Product Formula for P_k)**.** *If* $\theta_1, \theta_2 \in M_N(\mathbb{K}[x])$*, then*

$$P_k(\theta_1\theta_2) = \sum_{t=0}^{k+D} \left\{ P_k(x^t\theta_1(x))\frac{\theta_2^{(t)}(0)}{t!} + \frac{\theta_1^{(t)}(0)}{t!}P_k(x^t\theta_2(x)) \right\} - \frac{(\theta_1\theta_2)^{(k+1)}(0)}{k!}.$$

Proof. *It is an application of the Lemma 2 and the Lemma 3 .*

Lemma 5 (Translation)**.** *For every* $\theta \in M_N(\mathbb{K}[x])$*,* $k \geq 0$*, we have that*

$$P_k(\theta) = \begin{cases} P_k(\frac{\theta^{(k+1-t)}(0)}{(k+1-t)!}x^{k+1}) + P_{k-t}(\theta) - \frac{\theta^{(k-t+1)}(0)}{(k-t)!} & \text{if } 0 \leq t \leq k, \\ P_k(\theta(0)x^{k+1}) + P_0(x(\theta(x) - \theta(0))) & \text{if } t = k+1, \\ P_0(x^{t-k}\theta(x)) & \text{if } t \geq k+2. \end{cases} \quad (1.6)$$

Proof. *The proof is a straightforward computation.*

We shall now provide the aforementioned description of the Algebra $\mathbb{A}$.

Theorem 12. *Let*

$$\Gamma := \Big\{\theta \in M_N(\mathbb{K}[x]) \mid P_0(x\theta(x)) = P_0(\theta(0)x), P_0(x^j\theta(x)) = 0, j \geq 2, \\ \sum_{k=0}^{q}(-1)^k S^{k+D-q-1}P_k(\theta) = 0, 0 \leq q \leq D-1\Big\}.$$

Then, $\Gamma = \mathbb{A}$.

Moreover, for each θ *we have an explicit expression for the operator* $\mathcal{B}$.

Before proving the theorem, we study the relations defining the algebra Γ. They are given in the following result:

Proposition 1. *The algebra* Γ *is the subset of* $\theta \in M_N(\mathbb{K}[x])$ *such that*

$$\sum_{j=0}^{q}(-1)^{q-j-D}\left[S^{D-q+j-1}, \frac{\theta^{(j)}(0)}{j!}\right] = 0, \tag{1.7}$$

$$\sum_{j=0}^{q}(-1)^{q-j-D+1}S^{j+D-q-1}P_j(\theta) = 0, \tag{1.8}$$

for $0 \leq q \leq D-1$.

Proof. *We notice that* Γ *is defined by two relations:*

$$P_0(x\theta(x)) = P_0(\theta(0)x), P_0(x^j\theta(x)) = 0,\ j \geq 2 \tag{1.9}$$

and Equation (1.8) *(after a trivial change of the summation variable). Equation* (1.9) *is equivalent to*

$$\sum_{r=D-q}^{D}(-1)^r\left[S^{r-1}, \frac{\theta^{(r-D+q)}(0)}{(r-D+q)!}\right] = 0,$$

for $0 \leq q \leq D-1$. *If* $q = D-1$, *then*

$$0 = \sum_{r=2}^{D}(-1)^r\left[S^{r-1}, \frac{\theta^{(r-1)}(0)}{(r-1)!}\right] = P_0\left(\sum_{r=2}^{D}\frac{\theta^{(r-1)}(0)}{(r-1)!}x^r\right)$$

$$= P_0\left(\sum_{r=1}^{D-1} \frac{\theta^{(r)}(0)}{r!}x^{r+1}\right) = P_0(x(\theta(x) - \theta(0))).$$

In other words, $P_0(x\theta(x)) = P_0(x\theta(0))$.

If $0 \leq q \leq D-2$, *then* $2 \leq D-q$ *and*

$$0 = \sum_{r=D-q}^{D} (-1)^r \left[S^{r-1}, \frac{\theta^{(r-D+q)}(0)}{(r-D+q)!}\right] = P_0\left(\sum_{r=D-q}^{D} \frac{\theta^{(r-D-q)}(0)}{(r-D-q)!}x^r\right)$$

$$= P_0\left(\sum_{j=0}^{q} \frac{\theta^{(j)}(0)}{j!}x^{j+D-q}\right) = P_0\left(x^{D-q}\left(\sum_{j=0}^{q} \frac{\theta^{(j)}(0)}{j!}x^j\right)\right) = P_0(x^{D-q}\theta(x))$$

for $0 \leq q \leq D-2$. *Thus,* $P_0(x^j\theta(x)) = 0$ *for* $j \geq 2$.

Now let us prove the theorem.

Proof. *We shall break the proof in different steps.*

Step 1: The set Γ *is an algebra.*

Clearly, Γ *is a vector space since* P_k *is linear for all* $0 \leq k \leq D-1$.

If $\theta_1, \theta_2 \in \Gamma$, *then*

$$P_0(x\theta_i(x)) = P_0(\theta_i(0)x),\ P_0(x^j\theta_i(x)) = 0,\ \sum_{k=0}^{q}(-1)^k S^{k+D-q-1}P_k(\theta_i) = 0,$$

for $j \geq 2,\ 0 \leq q \leq D-1,\ i = 1, 2$.

Note that, using Corollary 6 and $P_0(x\theta_1(x))(0) = 0$, *we obtain*

$$P_0(x\theta_1(x)\theta_2(x)) = P_0((x\theta_1(x))\theta_2(x)) = P_0(\theta_1(0)x)\theta_2(0) - \theta_1(0)P_0(\theta_2(0)x) + (x\theta_1)'(0)P_0(x\theta_2(x))$$

$$= P_0(\theta_1(0)x)\theta_2(0) - \theta_1(0)P_0(\theta_2(0)x) + \theta_1(0)P_0(\theta_2(0)x) = \theta_1(0)\theta_2(0) = P_0(\theta_1(0)\theta_2(0)x).$$

If $j \geq 2$, *then* $P_0(x^{j-1}\theta_1(x))(0) = 0, P_0(x\theta_2(x))(0) = 0$. *Using Corollary 7 we obtain*

$$P_0(x^j\theta_1(x)\theta_2(x)) = P_0((x^{j-1}\theta_1(x))(x\theta_2(x)))$$

$$= \sum_{s=1}^{D}\left\{P_0(x^{j+s-1}\theta_1(x))\frac{(x\theta_2)^{(s)}(0)}{s!} + \frac{(x^{j-1}\theta_1)^{(s)}(0)}{s!}P_0(x^{s+1}\theta_2(x))\right\} = 0$$

Lemma 5 (Translation) implies

$$P_k(x^{k+1}\theta_i(x)) = P_k(\theta_i(0)x^{k+1}), P_k(x^t\theta_i(x)) = P_0(x^{t-k}\theta_i(x)) = 0,$$

for $t \geq k+2,\ i = 1, 2.$

Using Theorem 11 (Product Formula for P_k*) we have:*

$$P_k(\theta_1\theta_2) = \sum_{t=0}^{k+D}\left\{P_k(x^t\theta_1(x))\frac{\theta_2^{(t)}(0)}{t!} + \frac{\theta_1^{(t)}(0)}{t!}P_k(x^t\theta_2(x))\right\} - \frac{(\theta_1\theta_2)^{(k+1)}(0)}{k!}$$

$$= \sum_{t=0}^{k}\left\{P_k(x^t\theta_1(x))\frac{\theta_2^{(t)}(0)}{t!} + \frac{\theta_1^{(t)}(0)}{t!}P_k(x^t\theta_2(x))\right\} + P_k(\theta_1(0)x^{k+1})\frac{\theta_2^{(k+1)}(0)}{(k+1)!}$$

$$+\frac{\theta_1^{(k+1)}(0)}{(k+1)!}P_k(\theta_2(0)x^{k+1}) - \frac{(\theta_1\theta_2)^{(k+1)}(0)}{k!}$$

$$= P_k(\theta_1)\theta_2(0) + \theta_2(0)P_k(\theta_2)$$

$$+\sum_{t=0}^{k}\left\{P_k(x^t\theta_1(x))\frac{\theta_2^{(t)}(0)}{t!} + \frac{\theta_1^{(t)}(0)}{t!}P_k(x^t\theta_2(x)) - (k+1)\frac{\theta_1^{(t)}(0)}{t!}\frac{\theta_2^{(k+1-t)}(0)}{(k+1-t)!}\right\}.$$

Thus, for $0 \leq q \leq D-1$ *we have*

$$\sum_{k=0}^{q}(-1)^k S^{k+D-q-1}P_k(\theta_1\theta_2) = \sum_{k=0}^{q}(-1)^k S^{k+D-q-1}\Bigg[P_k(\theta_1)\theta_2(0) + \theta_2(0)P_k(\theta_2)$$

$$+\sum_{t=0}^{k}\left\{P_k(x^t\theta_1(x))\frac{\theta_2^{(t)}(0)}{t!} + \frac{\theta_1^{(t)}(0)}{t!}P_k(x^t\theta_2(x)) - (k+1)\frac{\theta_1^{(t)}(0)}{t!}\frac{\theta_2^{(k+1-t)}(0)}{(k+1-t)!}\right\}\Bigg]$$

$$= \sum_{k=0}^{q}(-1)^k\left[S^{k+D-q-1}, \theta_1(0)\right]P_k(\theta_2)$$

$$+\sum_{k=0}^{q}\sum_{t=1}^{k}(-1)^k S^{k+D-q-1}\left\{(k+1)\frac{\theta_1^{(k+1-t)}(0)}{(k+1-t)!} + P_{k-t}(\theta_1) - \frac{\theta_1^{(k+1-t)}(0)}{(k-t)!}\right\}\frac{\theta_2^{(t)}(0)}{t!}$$

$$+\sum_{k=0}^{q}\sum_{t=1}^{k}(-1)^k S^{k+D-q-1}\frac{\theta_1^{(t)}(0)}{t!}\left\{(k+1)\frac{\theta_2^{(k+1-t)}(0)}{(k+1-t)!} + P_{k-t}(\theta_2) - \frac{\theta_2^{(k+1-t)}(0)}{(k-t)!}\right\}$$

$$-\sum_{k=0}^{q}\sum_{t=1}^{k}(k+1)(-1)^k S^{k+D-q-1}\frac{\theta_1^{(t)}(0)}{t!}\frac{\theta_2^{(k+1-t)}(0)}{(k-t+1)!}$$

$$= \sum_{k=0}^{q}(-1)^k\left[S^{k+D-q-1}, \theta_1(0)\right]P_k(\theta_2)$$

$$+\sum_{k=0}^{q}\sum_{t=1}^{k}(-1)^k S^{k+D-q-1}\left\{P_{k-t}(\theta_1)-\frac{\theta_1^{(k+1-t)}(0)}{(k-t)!}\right\}\frac{\theta_2^{(t)}(0)}{t!}$$

$$+\sum_{k=0}^{q}\sum_{t=1}^{k}(-1)^k S^{k+D-q-1}\frac{\theta_1^{(t)}(0)}{t!}\left\{P_{k-t}(\theta_2)-\frac{\theta_2^{(k+1-t)}(0)}{(k-t)!}\right\}$$

$$+\sum_{k=0}^{q}\sum_{t=1}^{k}(k+1)(-1)^k S^{k+D-q-1}\frac{\theta_1^{(t)}(0)}{t!}\frac{\theta_2^{(k+1-t)}(0)}{(k-t+1)!}.$$

However,

$$\sum_{k=0}^{q}\sum_{t=1}^{k}(-1)^k S^{k+D-q-1}\frac{\theta_1^{(t)}(0)}{t!}\left\{P_{k-t}(\theta_2)-\frac{\theta_2^{(k+1-t)}(0)}{(k-t)!}\right\}$$

$$=\sum_{s=0}^{q-1}\sum_{k=s+1}^{q}(-1)^k S^{k+D-q-1}\frac{\theta_1^{(k-s)}(0)}{(k-s)!}\left\{P_s(\theta_2)-\frac{\theta_2^{(s+1)}(0)}{s!}\right\}$$

$$=\sum_{s=0}^{q-1}(-1)^s\left(\sum_{j=1}^{q-s}(-1)^j S^{j+s+D-q-1}\frac{\theta_1^{j}(0)}{j!}\right)\left\{P_s(\theta_2)-\frac{\theta_2^{(s+1)}(0)}{s!}\right\}$$

$$=\sum_{s=0}^{q-1}(-1)^s\left(\sum_{j=1}^{q-s}(-1)^j\frac{\theta_1^{j}(0)}{j!}S^{j+s+D-q-1}-\left[S^{D-q+s-1},\theta_1(0)\right]\right)\left\{P_s(\theta_2)-\frac{\theta_2^{(s+1)}(0)}{s!}\right\}.$$

After a few simple calculations this term is equal to:

$$-\sum_{s=0}^{q-1}(-1)^s\left[S^{D-q+s-1},\theta_1(0)\right]P_s(\theta_2)-\sum_{k=0}^{q}\sum_{s=0}^{k-1}(-1)^k S^{k+D-q-1}\frac{\theta_1^{(s+1)}(0)}{(s+1)!}\frac{\theta_2^{(k-s)}(0)}{(k-s-1)!}.$$

Similarly, we can see that

$$\sum_{k=0}^{q}\sum_{t=1}^{k}(-1)^k S^{k+D-q-1}\left\{P_{k-t}(\theta_1)-\frac{\theta_1^{(k+1-t)}(0)}{(k-t)!}\right\}\frac{\theta_2^{(t)}(0)}{t!}$$

$$=-\sum_{k=0}^{q}\sum_{s=0}^{k-1}(-1)^k S^{k+D-q-1}\frac{\theta_1^{(s+1)}(0)}{s!}\frac{\theta_2^{(k-s)}(0)}{(k-s)!}.$$

Thus,

$$\sum_{k=0}^{q}(-1)^k S^{k+D-q-1}P_k(\theta_1\theta_2)=\sum_{s=0}^{q}(-1)^s\left[S^{D-q+s-1},\theta_1(0)\right]P_s(\theta_2)$$

$$-\sum_{k=0}^{q}\sum_{s=0}^{k-1}(-1)^k S^{k+D-q-1}\frac{\theta_1^{(s+1)}(0)}{s!}\frac{\theta_2^{(k-s)}(0)}{(k-s)!}$$

$$
-\sum_{s=0}^{q-1}(-1)^s\left[S^{D-q+s-1},\theta_1(0)\right]P_s(\theta_2)-\sum_{k=0}^{q}\sum_{s=0}^{k-1}(-1)^k S^{k+D-q-1}\frac{\theta_1^{(s+1)}(0)}{(s+1)!}\frac{\theta_2^{(k-s)}(0)}{(k-s-1)!}
$$

$$
+\sum_{k=0}^{q}\sum_{s=0}^{k-1}(-1)^k(k+1)S^{k+D-q-1}\frac{\theta_1^{(s+1)}(0)}{(s+1)!}\frac{\theta_2^{(k-s)}(0)}{(k-s)!}
$$

$$
=(-1)^q\left[S^{D-1},\theta_1(0)\right]P_k(\theta_2)-\sum_{k=0}^{q}\sum_{s=0}^{k-1}(-1)^k(s+1)S^{k+D-q-1}\frac{\theta_1^{(s+1)}(0)}{s!}\frac{\theta_2^{(k-s)}(0)}{(k-s)!}
$$

$$
-\sum_{k=0}^{q}\sum_{s=0}^{k-1}(-1)^k(k-s)S^{k+D-q-1}\frac{\theta_1^{(s+1)}(0)}{(s+1)!}\frac{\theta_2^{(k-s)}(0)}{(k-s)!}
$$

$$
+\sum_{k=0}^{q}\sum_{s=0}^{k-1}(-1)^k(k+1)S^{k+D-q-1}\frac{\theta_1^{(s+1)}(0)}{(s+1)!}\frac{\theta_2^{(k-s)}(0)}{(k-s)!}
$$

$$
=(-1)^q\left[S^{D-1},\theta_1(0)\right]P_q(\theta_2).
$$

Nevertheless, $[S^{D-1},\theta_1(0)] = 0$ *and we obtain* $\sum_{k=0}^{q}(-1)^k S^{k+D-q-1}P_k(\theta_1\theta_2) = 0$ *for* $0 \le q \le D-1$. *Therefore,* $\theta_1\theta_2 \in \Gamma$. *Thus,* Γ *is an algebra.*

Step 2: Since Γ *contains the ideal* $x^{2D}M_N(\mathbb{K}[x])$*, if we define* $E = \oplus_{j=0}^{2D-1}M_N(\mathbb{K}[x])_j \cap \Gamma$*, then we have this step.*

Step 3: The algebra Γ *is generated by E, i.e.,* $\mathbb{K}\cdot\langle E\rangle = \Gamma$.

This step follows applying Theorem 10 and since E contains the elements mentioned in that theorem.

Step 4: The inclusion $\mathbb{A}\cap\oplus_{j=0}^{2D-1}M_N(\mathbb{K}[x])_j \subset E$.

Let $\theta \in \mathbb{A}\cap\oplus_{j=0}^{2D-1}M_N(\mathbb{K}[x])_j$ *then there exists* $\mathcal{B}=\mathcal{B}(z,\partial_z)$ *such that* $(\psi\mathcal{B})(x,z)=\theta(x)\psi(x,z)$. *We write* $\theta(x)=\sum_{j=0}^{2D-1}a_jx^j$. *After a few simple computations we obtain that:*

$$
\mathcal{B}=\sum_{j=0}^{2D-1}\partial_z^j\cdot\left(a_k+\sum_{l=1}^{2D-1-j}\frac{(-1)^l}{z^l}\sum_{r=j+l-1}^{2D-1}(\mu^{l-1})_{jr}S^{r-j-l+1}P_r(\theta)\right). \tag{1.10}
$$

With $\mu \in M_{2D}(\mathbb{K}[x])$ *given by*

$$
\mu_{rj}=\begin{cases}(-1)^{r-j} & \text{if } r+2\le j\le \min\{r+D,2D-1\},\\ r & \text{if } j=r+1,\\ 0 & \text{if otherwise.}\end{cases}
$$

Furthermore,

$$e^{-xz}(\psi\mathcal{B}-\theta\psi)=x^{-D}\left(\sum_{q=0}^{D-1}\left\{\sum_{j=0}^{q}(-1)^{q-j-D}\left[S^{D-q+j-1},a_j\right]\right.\right.$$

$$\left.\left.+\sum_{j=0}^{q}\sum_{l=1}^{2D-1-j}\frac{(-1)^{q-j-D+l}}{z^l}\sum_{r=j+l-1}^{2D-1}(\mu^{l-1})_{jr}S^{r+D-q-l}P_r(\theta)\right\}\right)x^q$$

$$=x^{-D}\left(\sum_{q=0}^{D-1}\left\{\sum_{j=0}^{q}(-1)^{q-j-D}\left[S^{D-q+j-1},a_j\right]\right.\right.$$

$$+\sum_{l=1}^{2D-2-q}\left(\sum_{j=0}^{q}(-1)^{q-j-D+l-1}\sum_{r=j+l-1}^{2D-1}(\mu^{l-1})_{jr}S^{r+D-q-l}P_r(\theta)\right)\frac{1}{z^l}$$

$$\left.\left.+\sum_{l=2D-1-q}^{2D-1}\left(\sum_{j=0}^{2D-l-1}(-1)^{q-j-D+l}\sum_{r=j+l-1}^{2D-1}(\mu^{l-1})_{jr}S^{r+D-q-l}P_r(\theta)\right)\frac{1}{z^l}\right\}x^q\right).$$

However,

$$\sum_{j=0}^{q}(-1)^{q-j-D+l}\sum_{r=j+l-1}^{2D-1}(\mu^{l-1})_{jr}S^{r+D-q-l}P_r(\theta)$$

$$=\sum_{r=l-1}^{q+l-1}(-1)^{q-D+l}\left(\sum_{j=0}^{r-l+1}(-1)^j(\mu^{l-1})_{jr}\right)S^{r+D-q-l}P_r(\theta),$$

for $1\le l\le 2D-2-q$, *and*

$$\sum_{j=0}^{2D-l-1}(-1)^{q-j-D+l}\sum_{r=j+l-1}^{2D-1}(\mu^{l-1})_{jr}S^{r+D-q-l}P_r(\theta)$$

$$=\sum_{r=l-1}^{2D-2}(-1)^{q-D+l}\left(\sum_{j=0}^{r-l+1}(-1)^j(\mu^{l-1})_{jr}\right)S^{r+D-q-l}P_r(\theta),$$

for $2D-1-q\le l\le 2D-1$.

Note that r-th component of $v\mu$ *is given by* $(v\mu)_r=\sum_{j=0}^{2D-1}(-1)^j\mu_{jr}=\sum_{j=0}^{2D-1}(-1)^j\mu_{jr}=(-1)^{r-1}(r-1)+\sum_{j=0}^{r-2}(-1)^j(-1)^{r-j}=0$ *for* $v\in\mathbb{K}^{2D}$ *defined by* $v_j=(-1)^j$ *and* $0\le j,r\le 2D-1$. *Therefore,* $v\mu=0$. *Clearly this implies* $\sum_{j=0}^{r-l+1}(-1)^j(\mu^{l-1})_{jr}=(v\mu^{l-1})_r=0$ *for* $l\ge 2$.

Therefore,

$$e^{-xz}(\psi\mathcal{B}-\theta\psi)=x^{-D}\left(\sum_{q=0}^{D-1}\left\{\sum_{j=0}^{q}(-1)^{q-j-D}\left[S^{D-q+j-1},a_j\right]\right.\right.$$

$$+\frac{1}{z}\sum_{j=0}^{q}(-1)^{q-j-D+1}S^{j+D-q-1}P_j(\theta)\Bigg\}x^q\Bigg).$$

Since $\theta \in \mathbb{A}$ we have

$$\sum_{j=0}^{q}(-1)^{q-j-D}\left[S^{D-q+j-1}, \frac{\theta^{(j)}(0)}{j!}\right] = 0,$$

$$\sum_{j=0}^{q}(-1)^{q-j-D+1}S^{j+D-q-1}P_j(\theta) = 0,$$

for $0 \leq q \leq D-1$. *Thus,* $\theta \in E$.

Step 5: The inclusion $E \subset \mathbb{A} \cap \oplus_{j=0}^{2D-1} M_N(\mathbb{K}[x])_j$.

By the previous step we have Equation (1.10) *valid for every* $\theta \in E$ *and using Proposition 1 we obtain that* $(\psi\mathcal{B})(x,z) = \theta(x)\psi(x,z)$. *Then,* $\theta \in \mathbb{A} \cap \oplus_{j=0}^{2D-1} M_N(\mathbb{K}[x])_j$.

Furthermore, we have an explicit expression for the operator $\mathcal{B}$.

If $\theta(x) = \sum_{j=0}^{M} a_j x^j \in \Gamma$, *then*

$$\mathcal{B} = \sum_{j=0}^{M}\partial_z^j \cdot \left(a_k + \sum_{l=1}^{M-j}\frac{(-1)^l}{z^l}\sum_{r=j+l-1}^{M}(\mu^{l-1})_{jr}S^{r-j-l+1}P_r(\theta)\right) \tag{1.11}$$

with $\mu \in M_{M+1}(\mathbb{K}[x])$ *given by*

$$\mu_{rj} = \begin{cases} (-1)^{r-j} & \text{if } r+2 \leq j \leq \min\{r+D, M\}, \\ r & \text{if } j = r+1, \\ 0 & \text{if otherwise.} \end{cases} \tag{1.12}$$

satisfies $(\psi\mathcal{B})(x,z) = \theta(x)\psi(x,z)$. □

In particular, Theorem 12 implies that the $\mathbb{K}$-algebra Γ is not trivial.

A remarkable property of this family of algebras is the existence of a Pierce decomposition whose definition we shall now recall.

Definition 10. *Let R be a noncommutative ring with unit. We say that a set of elements $r_1, \ldots, r_n \in R$ is Pierce decomposition of R if $1 = \sum_{j=1}^{n} r_j$ and $r_i r_j = \delta_{ij}$ for all $1 \leq i, j \leq n$.*

See [1] for more information on the Pierce decomposition. The next definition presents a Pierce decomposition of the algebra $\mathbb{A}$.

Definition 11. *Define* $\alpha_k(x) = e_{kk} + \sum_{j=1}^{N-1} a_{kj}x^j \in M_N(\mathbb{K}[x])$ *for* $2 \leq k \leq N-1$, $N \geq 3$ *with*

$$a_{kj} = (-1)^{j+1}(\delta_{k,j+1}e_{k1} + \delta_{k,N-j}e_{Nk}) + (-1)^j\delta_{N,j+1}e_{N1} = (-1)^{j+1}(\delta_{j,k-1}e_{k1} + \delta_{j,N-k}e_{Nk}) + (-1)^j\delta_{j,N-1}e_{N1}$$

for $1 \leq j \leq N-1$.

In the previous definition we have two cases:

- If N is even and the numbers $k-1, N-k, N-1$ are different, then

$$a_{kj} = \begin{cases} (-1)^k e_{k1} & \text{if } j = k-1, \\ (-1)^{N-k+1}e_{Nk} & \text{if } j = N-k, \\ (-1)^{N-1}e_{N1} & \text{if } j = N-1, \\ e_{kk} & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.13}$$

- If N is odd
  - If $k \neq \frac{N+1}{2}$, then $k-1, N-k, N-1$ are different:

$$a_{kj} = \begin{cases} (-1)^k e_{k1} & \text{if } j = k-1, \\ (-1)^{N-k+1}e_{Nk} & \text{if } j = N-k, \\ (-1)^{N-1}e_{N1} & \text{if } j = N-1, \\ e_{kk} & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.14}$$

  - If $k = \frac{N+1}{2}$, then

$$a_{\frac{N+1}{2}j} = \begin{cases} (-1)^{\frac{N+1}{2}}\left(e_{\frac{N+1}{2},1} + e_{N,\frac{N+1}{2}}\right) & \text{if } j = \frac{N-1}{2}, \\ (-1)^{N-1}e_{N1} & \text{if } j = N-1, \\ e_{kk} & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.15}$$

The following lemma relates these elements with the family $\{P_k\}_{k\in\mathbb{N}}$. Its importance is that it shall be used to prove that the elements $\alpha_k' s$ satisfy the second family of relations that defines $\mathbb{A}$.

Lemma 6. • *If $k < \frac{N+1}{2}$, then $k-1 < N-k < N-1$ and*

$$P_l(\alpha_k) = \begin{cases} (-1)^l\left(e_{k,k-l-1} - e_{k+l+1,k}\right) & \text{if } 0 \le l \le k-3, \\ (-1)^k k e_{k1} + (-1)^{k+1} e_{2k-1,k} & \text{if } l = k-2, \\ (-1)^l\left(e_{l+2,1} - e_{k+l+1,k}\right) & \text{if } k-1 \le l \le N-k-2, \\ (N-k-1)(-1)^{N-k-1} e_{Nk} + (-1)^{N-k+1} e_{N-k+1,1} & \text{if } l = N-k-1, \\ (-1)^{l+1}\left(e_{N,N-l-1} - e_{l+2,1}\right) & \text{if } N-k \le l \le N-3, \\ (N-1)(-1)^{N-1} e_{N1} & \text{if } N-k \le l \le N-3. \end{cases} \tag{1.16}$$

• *If $k = \frac{N+1}{2}$, then $N-k = k-1 = \frac{N-1}{2} < N-1$ and*

$$P_l(\alpha_k) = \begin{cases} (-1)^l\left(e_{\frac{N+1}{2},\frac{N-1}{2}-l} - e_{\frac{N+3}{2}+l,\frac{N+1}{2}}\right) & \text{if } 0 \le l \le k-3 = N-k-2 = \frac{N-5}{2}, \\ \frac{N-3}{2}(-1)^{\frac{N+1}{2}} e_{N,\frac{N+1}{2}} + (-1)^{\frac{N+1}{2}} e_{\frac{N+1}{2},1} & \text{if } l = k-2 = N-k-1 = \frac{N-3}{2}, \\ (-1)^{l+1}\left(e_{N,N-l-1} - e_{l+2,1}\right) & \text{if } N-k \le l \le N-3, \\ (N-1)(-1)^{N-1} e_{N1} & \text{if } l = N-2. \end{cases} \tag{1.17}$$

• *If $k > \frac{N+1}{2}$, then $N-k < k-1 < N-1$ and*

$$P_l(\alpha_k) = \begin{cases} (-1)^l\left(e_{k,k-l-1} - e_{k+l+1,k}\right) & \text{if } 0 \le l \le N-k-2, \\ (N-k-1)(-1)^{N-k+1} e_{Nk} + (-1)^{N-k+1} e_{k,2k-N} & \text{if } l = N-k-1, \\ (-1)^l\left(e_{k,k-l-1} - e_{N,N-l-1}\right) & \text{if } N-k \le l \le k-3, \\ k(-1)^k e_{k1} + (-1)^{k+1} e_{N,N-k+1} & \text{if } l = k-2, \\ (-1)^{l+1}\left(e_{N,N-l-1} - e_{l+2,1}\right) & \text{if } k-1 \le l \le N-3, \\ (N-1)(-1)^{N-1} e_{N1} & \text{if } l = N-2. \end{cases} \tag{1.18}$$

Proof. *The proof is a straightforward.* □

Now we prove that this family is contained in $\mathbb{A}$.

Theorem 13. *For $N \ge 3$ we have $\{\alpha_k\}_{1 \le k \le N-1} \subset \mathbb{A}$.*

Before proving this theorem we have a handy remark.

Remark 3. *We shall adopt the convenient convention that $\sum_{i \in \emptyset} x_i = 0$. Define $e_{ij} = 0$ if i or j is outside the set $\{1, \ldots, N\}$.*

Proof. *We first verify the first family of relations. Pick $2 \leq k \leq N-1$, then*

$$P_0(x\alpha_k(x)) = e_{kk} - (-1)^k \left[(-1)^k e_{k1}, S_N^{k-1}\right] - (-1)^{N-k+1}\left[(-1)^{N-k+1} e_{Nk}, S_N^{N-k}\right] - (-1)^N \left[(-1)^{N-1} e_{N1}, S_N^{N-1}\right]$$

$$= e_{kk} - \left[e_{k1}, S_N^{k-1}\right] - \left[e_{Nk}, S_N^{N-k}\right] - \left[e_{N1}, S_N^{N-1}\right] = e_{kk} - (e_{kk} - e_{11}) - (e_{NN} - e_{kk}) + (e_{NN} - e_{11}) = e_{kk} = P_0(\alpha_k(0)x).$$

If N is odd

$$P_0\left(x\alpha_{\frac{N+1}{2}}(x)\right) = e_{\frac{N+1}{2},\frac{N+1}{2}} - (-1)^{\frac{N+1}{2}}\left[(-1)^{\frac{N+1}{2}}\left(e_{\frac{N+1}{2},1} + e_{N,\frac{N+1}{2}}\right), S_N^{\frac{N-1}{2}}\right] - (-1)^N\left[(-1)^{N-1} e_{N1}, S_N^{N-1}\right]$$

$$= e_{\frac{N+1}{2},\frac{N+1}{2}} - \left(e_{\frac{N+1}{2},\frac{N+1}{2}} + e_{NN} - \left(e_{11} + e_{\frac{N+1}{2},\frac{N+1}{2}}\right)\right) + e_{NN} - e_{11} = e_{\frac{N+1}{2},\frac{N+1}{2}} = P_0\left(\alpha_{\frac{N+1}{2}}(0)x\right).$$

If $r \geq 2$, then

$$P_0(x^r\alpha_k(x)) = -(-1)^r\left[e_{kk}, S_N^{r-1}\right] - (-1)^{k+r-1}\left[(-1)^k e_{k1}, S_N^{k+r-2}\right] - (-1)^{N-k+r}\left[(-1)^{N-k+1} e_{N1}, S_N^{N-k+r-1}\right]$$

$$-(-1)^r\left(e_{k,k+r-1} - e_{k-r+1,k}\right) - (-1)^{r-1}\left(e_{k,k+r-1} - 0\right) - (-1)^{r+1}\left(0 - e_{k-r+1,k}\right) = 0.$$

If N is odd

$$P_0\left(x^r\alpha_{\frac{N+1}{2}}(x)\right) = -(-1)^r\left[e_{\frac{N+1}{2},\frac{N+1}{2}}, S_N^{r-1}\right] - (-1)^{\frac{N-1}{2}+r}\left[(-1)^{\frac{N+1}{2}}\left(e_{\frac{N+1}{2},1} + e_{N,\frac{N+1}{2}}\right), S_N^{\frac{N-3}{2}+r}\right]$$

$$= -(-1)^r\left(e_{\frac{N+1}{2},\frac{N-1}{2}+r} - e_{\frac{N+3}{2}-r,\frac{N+1}{2}}\right) + (-1)^r\left(e_{\frac{N+1}{2},\frac{N-1}{2}+r} - e_{\frac{N+3}{2}-r,\frac{N+1}{2}}\right) = 0.$$

The second family of relations has a number of cases which we will check.

Using Lemma 6:

- *If $q < \frac{N-3}{2}$,*
  - *If $2 \leq k < q+2$, then $q < N-k-1$. Since, $k-N+q+1 < 0$ and $q+3 < N$ we have*

$$\sum_{j=0}^{q}(-1)^j S_N^{N+j-q-1} P_j(\alpha_k) = \sum_{j=0}^{k-3}(-1)^j S_N^{N+j-q-1}(-1)^j\left(e_{k,k-j-1} - e_{k+j+1,k}\right)$$

$$+(-1)^k S_N^{N-q+k-3}\left((-1)^k k e_{k1} + (-1)^{k+1} e_{2k-1}\right) + \sum_{j=k-1}^{q}(-1)^j S_N^{N+j-q-1}(-1)^j\left(e_{j+2,1} - e_{k+j+1,k}\right)$$

$$= \sum_{j=0}^{k-3} e_{k-N-j+q+1,k-j-1} - \sum_{j=0}^{k-3} e_{k-N+q+2,k} + ke_{q+3-N,1} - e_{k-N+q+2,k}$$

$$+ \sum_{j=k-1}^{q} e_{q+3-N,1} - \sum_{j=k-1}^{q} e_{k-N+q+2,k} = 0.$$

– *If* $k = q + 2$*. Since,* $2q - N + 3 < 0$*.*

$$\sum_{j=0}^{q} (-1)^j S_N^{N+j-q-1} P_j(\alpha_k) = \sum_{j=0}^{q-1} (-1)^j S_N^{N+j-q-1} (-1)^j \left(e_{q+2,q+1-j} - e_{q+j+3,q+2}\right)$$

$$+(-1)^q e_{1N} \left((-1)^q (q+2) e_{q+2} + (-1)^{q+1} e_{2q+3,q+2}\right) = \sum_{j=0}^{q-1} (-1)^j S_N^{N+j-q-1} (-1)^j e_{2q-N-j+3,q+1-j}$$

$$- \sum_{j=0}^{q-1} (-1)^j S_N^{N+j-q-1} (-1)^j e_{2q-N+4,q+2} = 0.$$

– *If* $q + 2 < k < N - q - 1$*, then* $q < k + 2$*. Since,* $k - N + q + 1 < 0$*.*

$$\sum_{j=0}^{q} (-1)^j S_N^{N+j-q-1} P_j(\alpha_k) = \sum_{j=0}^{q} (-1)^j S_N^{N+j-q-1} (-1)^j \left(e_{k,k-j-1} - e_{k+j+1,k}\right)$$

$$= \sum_{j=0}^{q} (-1)^j S_N^{N+j-q-1} (-1)^j e_{k-N-j+q+1,k-j-1} - \sum_{j=0}^{q} (-1)^j S_N^{N+j-q-1} (-1)^j e_{k-N+q+2,k} = 0.$$

– *If* $k = N - q - 1$*, then* $N - k = q + 1$*. Since,* $k - N + q + 1 < 0$*.*

$$\sum_{j=0}^{q} (-1)^j S_N^{N+j-q-1} P_j(\alpha_k) = \sum_{j=0}^{q-1} (-1)^j S_N^{N+j-q-1} P_j(\alpha_k) + (-1)^q S_N^{N-1} P_q(\alpha_k)$$

$$= \sum_{j=0}^{q-1} (-1)^j S_N^{N+j-q-1} \left(e_{k,k-j-1} - e_{k+j+1,k}\right)$$

$$+(-1)^q e_{1N} \left((N-k)(-1)^{N-k+1} e_{Nk} + (-1)^q (e_{k,k-q-1} - e_{N,N-q-1})\right)$$

$$= \sum_{j=0}^{q-1} e_{k-N-j+q+1,k-j-1} - \sum_{j=0}^{q-1} e_{k-N+q+2,k} + (N-k) e_{1k} - e_{1,N-q-1}$$

$$= -(N-k-1) e_{1k} + (N-k) e_{1k} - e_{1k} = 0.$$

– *If $k > N-q-1$, then $q+1 > k$. Since, $j \geq k-N+q+1$ implies $k-N-j+q+1 \leq 0 < 1$.*

$$\sum_{j=0}^{q}(-1)^j S_N^{N+j-q-1} P_j(\alpha_k) = \sum_{j=0}^{N-k-2}(-1)^j S_N^{N+j-q-1}(-1)^j\left(e_{k,k-j-1} - e_{k+j+1,k}\right) +$$

$$(-1)^{N-k+1} S_N^{2N-k-q-2}\left((N-k-1)(-1)^{N-k+1} e_{Nk} + (-1)^{N-k+1} e_{k,2k-N}\right) +$$

$$\sum_{j=N-k}^{q}(-1)^j S_N^{N+j-q-1}(-1)^j\left(e_{k,k-j-1} - e_{N,N-j-1}\right) = \sum_{j=0}^{N-k-2} e_{k-N-j+q+1,k-j-1} - \sum_{j=0}^{N-k-2} e_{k-N+q+2,k}$$

$$+(N-k-1)e_{k-N+q+2,k} + e_{2k-2N+q+2,2k-N} + \sum_{j=N-k}^{q} e_{k-N-j+q+1,k-j-1} + \sum_{j=N-k}^{q} e_{q+1-j,N-j-1}$$

$$= \sum_{j=0}^{q} e_{k-N-j+q+1,k-j-1} - \sum_{j=N-k}^{q} e_{q+1-j,k-j-1} = \sum_{j=q-N+k+1}^{q} e_{k-N-j+q+1,k-j-1} = 0.$$

- *If $q = \frac{N-3}{2}$ (for N odd) then,*

  – *The case $2 \leq k < q+2 = \frac{N+1}{2}$ is similar to the case $q < \frac{N-3}{2}$ and $2 \leq k < q+2$.*

  – *If $k = q+2 = \frac{N+1}{2}$ then,*

$$\sum_{j=0}^{q}(-1)^j S_N^{N+j-q-1} P_j\left(\alpha_{\frac{N+1}{2}}\right) = \sum_{j=0}^{q-1}(-1)^j S_N^{N+j-q-1}(-1)^j\left(e_{\frac{N+1}{2},\frac{N-1}{2}+j} - e_{\frac{N+3}{2}+j,\frac{N+1}{2}}\right)$$

$$+(-1)^{\frac{N-3}{2}} S_N^{N-1}(-1)^{\frac{N+1}{2}}\frac{N+1}{2} e_{\frac{N+1}{2},1} + (-1)^{\frac{N-3}{2}} S_N^{N-1}(-1)^{\frac{N+1}{2}}\frac{N-3}{2} e_{N,\frac{N+1}{2}}$$

$$= \sum_{j=0}^{q-1} e_{\frac{N+1}{2}-N-j+q+1,\frac{N-1}{2}-j} - \sum_{j=0}^{q-1} e_{\frac{N+3}{2}-N+q+1,\frac{N+1}{2}} + \frac{N-3}{2} e_{1,\frac{N+1}{2}}$$

$$= \sum_{j=0}^{q-1} e_{-j,\frac{N-1}{2}-j} - \sum_{j=0}^{q-1} e_{1,\frac{N+1}{2}} + \frac{N-3}{2} e_{1,\frac{N+1}{2}} = -q e_{1,\frac{N+1}{2}} + \frac{N-3}{2} e_{1,\frac{N+1}{2}} = 0.$$

  – *The case $q+2 = \frac{N+1}{2} = N-q-1 < k$ is similar to the case $q < \frac{N-3}{2}$ and $k > N-q-1$.*

- *If $\frac{N-3}{2} < q \leq N-3$,*

  – *If $0 \leq k \leq N-q-2$, then $q < N-k-1$. Since, $k-N+q+1 < 0$ and $q+3 \leq N$. Therefore,*

$$\sum_{j=0}^{q}(-1)^j S_N^{N+j-q-1} P_j(\alpha_k) = \sum_{j=0}^{k-3}(-1)^j S_N^{N+j-q-1}(-1)^j\left(e_{k,k-j-1} - e_{k+j+1,k}\right)$$

$$+(-1)^k S_N^{N-q+k-3}\left((-1)^k k e_{k1} + (-1)^{k+1} e_{2k-1,k}\right) + \sum_{j=k-1}^{q} (-1)^j S_N^{N+j-q-1}(-1)^j \left(e_{j+2,1} - e_{k+j+1,k}\right)$$

$$= \sum_{j=0}^{k-3} e_{k-N-j+q+1,k-j-1} - \sum_{j=0}^{k-3} e_{k-N+q+2,k} + k e_{q+3-N,1} - e_{k-N+q+2,k}$$

$$+ \sum_{j=k-1}^{q} e_{q+3-N,1} - \sum_{j=k-1}^{q} e_{k-N+q+2,k} = 0.$$

– *If* $N-q-1 \leq k < q+2$, *then* $N-k-1 < q$, $k-2 < q$. *Since* $q < N-k-1$ *and* $q+3 \leq N$. *Therefore,*

$$\sum_{j=0}^{q}(-1)^j S_N^{N+j-q-1} P_j(\alpha_k) = \sum_{j=0}^{k-3}(-1)^j S_N^{N+j-q-1}(-1)^j \left(e_{k,k-j-1} - e_{k+j+1,k}\right)$$

$$+(-1)^k S_N^{N-q+k-3}\left((-1)^k k e_{k1} + (-1)^{k+1} e_{2k-1,k}\right) + \sum_{j=k-1}^{q} (-1)^j S_N^{N+j-q-1}(-1)^j \left(e_{j+2,1} - e_{k+j+1,k}\right)$$

$$= \sum_{j=0}^{k-3} e_{k-N-j+q+1,k-j-1} - \sum_{j=0}^{k-3} e_{k-N+q+2,k} + k e_{q+3-N,1} - e_{k-N+q+2,k} + \sum_{j=k-1}^{N-k-2} e_{q+3-N,1} - \sum_{j=k-1}^{q} e_{k-N+q+2,k}$$

$$+(-1)^{N-k-1} S_N^{2N-k-q-2}\left((-1)^{N-k-1}(N-k-1) e_{Nk} + (-1)^{N-k+1} e_{N-k+1,1}\right)$$

$$+ \sum_{j=N-k}^{q} (-1)^j S_N^{N+j-q-1}(-1)^j \left(e_{N,N-j-1} - e_{j+2,1}\right)$$

$$= \sum_{j=0}^{k-3} e_{k-N-j+q+1,k-j-1} - \sum_{j=0}^{k-3} e_{k-N+q+2,k} + k e_{q+3-N,1} - e_{k-N+q+2,k} + \sum_{j=k-1}^{N-k-2} e_{q+3-N,1} - \sum_{j=k-1}^{q} e_{k-N+q+2,k}$$

$$+(N-k-1) e_{k-N+q+2,k} + e_{q+3-N,1} + \sum_{j=N-k}^{q} e_{q+1-j,N-j-1} - \sum_{j=N-k}^{q} e_{q+3-N,1}$$

$$= \sum_{j=0}^{k-3} e_{k-N-j+q+1,k-j-1} + \sum_{j=N-k}^{q} e_{q+1-j,N-j-1} = 0.$$

– *If* $k = N-q-1$, *then* $N-k-1 = q$. *Since* $q+3 \leq N$.

$$\sum_{j=0}^{q}(-1)^j S_N^{N+j-q-1} P_j(\alpha_k) = \sum_{j=0}^{k-3}(-1)^j S_N^{N+j-q-1}(-1)^j \left(e_{k,k-j-1} - e_{k+j+1,k}\right)$$

$$+(-1)^k S_N^{N-q+k-3}\left((-1)^k k e_{k1} + (-1)^{k+1} e_{2k-1,k}\right) + \sum_{j=k-1}^{N-k-2} (-1)^j S_N^{N+j-q-1}(-1)^j \left(e_{j+2,1} - e_{k+j+1,k}\right)$$

$$+(-1)^q S_N^{N-1}\left((N-k-1)(-1)^{N-k-1}e_{Nk}+(-1)^{N-k+1}e_{N-k+1,1}\right)$$

$$=\sum_{j=0}^{k-3} e_{k-N-j+q+1,k-j-1}-\sum_{j=0}^{k-3} e_{k-N+q+2,k}+ke_{q+3-N,1}-e_{k-N+q+2,k}$$

$$+\sum_{j=k-1}^{N-k-2} e_{q+3-N,1}-\sum_{j=k-1}^{N-k-2} e_{k-N+q+2,k}+(N-k-1)e_{1k}$$

$$=\sum_{j=0}^{k-3} e_{-j,k-j-1}-\sum_{j=0}^{k-3} e_{1k}-e_{1k}-\sum_{j=k-1}^{N-k-2} e_{1k}+(N-k-1)e_{1k}=0.$$

– *If* $k=q+2$*, then* $k-2=q$*. Since* $q<N-k-1$*.*

$$\sum_{j=0}^{q}(-1)^j S_N^{N+j-q-1}P_j(\alpha_k)=\sum_{j=0}^{k-3}(-1)^j S_N^{N+j-q-1}(-1)^j\left(e_{k,k-j-1}-e_{k+j+1,k}\right)$$

$$+(-1)^k S_N^{N-q+k-3}\left((-1)^k k e_{k1}+(-1)^{k+1}e_{2k-1,k}\right)=\sum_{j=0}^{k-3} e_{k-N-j+q+1,k-j-1}-\sum_{j=0}^{k-3} e_{k-N+q+2,k}$$

$$+ke_{q+3-N,1}-e_{k-N+q+2,k}=0.$$

– *If* $k>q+2$*, then* $k>q>N-q-1$ *and* $N-k-1<q$

$$\sum_{j=0}^{q}(-1)^j S_N^{N+j-q-1}P_j(\alpha_k)=\sum_{j=0}^{N-k-2}(-1)^j S_N^{N+j-q-1}(-1)^j\left(e_{k,k-j-1}-e_{k+j+1,k}\right)$$

$$+(-1)^{N-k-1}S_N^{2N-k-q-2}\left((-1)^{N-k+1}(N-k)e_{Nk}+(-1)^{N-k+1}\left(e_{k,2k-N}-e_{Nk}\right)\right)$$

$$+\sum_{j=N-k}^{q}(-1)^j S_N^{N+j-q-1}(-1)^j\left(e_{k,k-j-1}-e_{N,N-j-1}\right)$$

$$=\sum_{j=0}^{N-k-2} e_{k-N-j+q+1,k-j-1}-\sum_{j=0}^{N-k-2} e_{k-N+q+2,k}+(N-k)e_{k-N+q+2,k}+e_{2k-2N+q+2,2k-N}-e_{k-N+q+2,k}$$

$$+\sum_{j=N-k}^{q} e_{k-N-j+q+1,k-j-1}-\sum_{j=N-k}^{q} e_{q+1-j,N-j-1}=\sum_{j=0}^{q} e_{k-N-j+q+1,k-j-1}-\sum_{j=N-k}^{q} e_{q+1-j,N-j-1}$$

$$=\sum_{j=k-N+q+1}^{q} e_{k-N-j+q+1,k-j-1}=0.$$

- *If* $q = N-2$ *and* $k < \frac{N+1}{2}$ *then* $k-1 < N-k \leq N-1$. *Therefore,*

$$\sum_{j=0}^{N-2}(-1)^j S_N^{N+j-q-1} P_j(\alpha_k) = \sum_{j=0}^{N-2}(-1)^j S_N^{j+1} P_j(\alpha_k) = \sum_{j=0}^{k-3}(-1)^j S_N^{j+1}(-1)^j(e_{k,k-j-1} - e_{k+j+1,k})$$

$$+(-1)^k S_N^{k-1}\left\{(-1)^k k e_{k1} + (-1)^{k+1} e_{2k-1,k}\right\} + \sum_{j=k-1}^{N-k-2}(-1)^j S_N^{j+1}(-1)^j(e_{j+2,1} - e_{k+j+1,k})$$

$$+(-1)^{N-k+1} S_N^{N-k}\left\{(N-k-1)(-1)^{N-k-1} e_{Nk} + (-1)^{N-k+1} e_{N-k+1,1}\right\}$$

$$+\sum_{j=N-k}^{N-3}(-1)^j S_N^{j+1}(-1)^{j+1}(e_{N,N-j-1} - e_{j+2,1}) + (-1)^N S_N^{N-1}(N-1)(-1)^{N-1} e_{N1}$$

$$= \sum_{j=0}^{k-3} e_{k-j-1,k-j-1} - \sum_{j=0}^{k-3} e_{k,k} + k e_{11} - ekk + \sum_{j=k-1}^{N-k-2} e_{11} - \sum_{j=k-1}^{N-k-2} e_{kk} + (N-k-1)e_{kk}$$

$$+e_{11} - \sum_{j=N-k}^{N-3} e_{N-j-1,N-j-1} + \sum_{j=N-k}^{N-3} e_{11} - (N-1)e_{11}$$

$$= \sum_{j=0}^{k-3} e_{k-j-1,k-j-1} - (k-2)e_{kk} + k e_{11} - e_{kk} + (N-k)e_{11} - (N-2k)e_{kk} + e_{11}$$

$$-\sum_{r=0}^{k-3} e_{k-r-1,k-r-1} + (k-2)e_{11} - (N-1)e_{11} = 0.$$

- *If* $q = N-1$ *and* $k < \frac{N+1}{2}$ *then*

$$\sum_{j=0}^{N-1}(-1)^j S_N^{N+j-q-1} P_j(\alpha_k) = \sum_{j=0}^{N-1}(-1)^j S_N^{j} P_j(\alpha_k) = \sum_{j=0}^{k-3}(-1)^j S_N^{j}(-1)^j(e_{k,k-j-1} - e_{k+j+1,k})$$

$$+(-1)^k S_N^{k-2}\left\{(-1)^k k e_{k1} + (-1)^{k+1} e_{2k-1,k}\right\} + \sum_{j=k-1}^{N-k-2}(-1)^j S_N^{j}(-1)^j(e_{j+2,1} - e_{k+j+1,k})$$

$$+(-1)^{N-k-1} S_N^{N-k-1}\left\{(N-k-1)(-1)^{N-k-1} e_{Nk} + (-1)^{N-k+1} e_{N-k+1,1}\right\}$$

$$+\sum_{j=N-k}^{N-3}(-1)^j S_N^{j}(-1)^{j+1}(e_{N,N-j-1} - e_{j+2,1}) + (-1)^{N-2} S_N^{N-2}(N-1)(-1)^{N-1} e_{N1}$$

$$= \sum_{j=0}^{k-3} e_{k-j,k-j-1} - \sum_{j=0}^{k-3} e_{k+1,k} + k e_{21} - e_{k+1,k} + \sum_{j=k-1}^{N-k-2} e_{21} + \sum_{j=k-1}^{N-k-2} e_{k+1,k} + (N-k-1)e_{k+1,k}$$

$$+e_{21} - \sum_{j=N-k}^{N-3} e_{N-j,N-j-1} + \sum_{j=N-k}^{N-3} e_{21} - (N-1)e_{21}$$

$$= \sum_{j=0}^{k-3} e_{k-j,k-j-1} - (k-2)e_{k+1,k} + ke_{21} - e_{k+1,k} + (N-2k)e_{21} - (N-2k)e_{k+1,k} + (N-k-1)e_{k+1,k}$$

$$+e_{21} - \sum_{r=0}^{k-3} e_{k-r,k-r-1} + (k-2)e_{21} - (N-1)e_{21} = 0.$$

- *If* $q = N-2$ *and* $K > \frac{N+1}{2}$ *then* $N-k < k-1 \leq N-1$ *and*

$$\sum_{j=0}^{N-2}(-1)^j S_N^{N+j-q-1} P_j(\alpha_k) = \sum_{j=0}^{N-2}(-1)^j S_N^{j+1} P_j(\alpha_k) = \sum_{j=0}^{N-k-2} (-1)^j S_N^{j+1} (-1)^j (e_{k,k-j-1} - e_{k+j+1,k})$$

$$+(-1)^{N-k-1} S_N^{N-k} \left\{(N-k-1)(-1)^{N-k-1} e_{Nk} + (-1)^{N-k+1} e_{k,2k-N}\right\}$$

$$+\sum_{j=N-k}^{k-3} (-1)^j S_N^{j+1} (-1)^j (e_{k,k-j-1} - e_{N,N-j-1}) + (-1)^k S_N^{k-1} \left\{k(-1)^k e_{k1} + (-1)^{k+1} e_{N,N-k+1}\right\}$$

$$+\sum_{j=k-1}^{N-3} (-1)^j S_N^{j+1} (-1)^{j+1} (e_{N,N-j-1} - e_{j+2,1}) + (-1)^N S_N^{N-1} (N-1)(-1)^{N-1} e_{N1}.$$

$$= \sum_{j=0}^{N-k-2} e_{k-j-1,k-j-1} - \sum_{j=0}^{N-k-2} e_{kk} + (N-k-1)e_{kk} + e_{2k-n,2k-N} + \sum_{j=N-k}^{k-3} e_{k-j-1,k-j-1}$$

$$- \sum_{j=N-k}^{k-3} e_{N-j-1,N-j-1} + ke_{11} - e_{N-k-1,N-k-1} - \sum_{j=k-1}^{N-3} e_{N-j-1,N-j-1} + \sum_{j=k-1}^{N-3} e_{11} - (N-1)e_{11}$$

$$= \sum_{j=0}^{k-3} e_{k-j-1,k-j-1} - (N-k-1)e_{kk} + (N-k-1)e_{kk} - \sum_{j=N-k}^{N-3} e_{N-j-1,N-j-1} + ke_{11} + (N-k-1)e_{11} - (N-1)e_{11}$$

$$= \sum_{j=0}^{k-3} e_{k-j-1,k-j-1} - \sum_{r=0}^{k-3} e_{k-r-1,k-r-1} = 0.$$

- *If* $q = N-1$ *and* $k > \frac{N+1}{2}$ *then* $N-k < k-1 \leq N-1$ *and*

$$\sum_{j=0}^{N-1}(-1)^j S_N^{N+j-q-1} P_j(\alpha_k) = \sum_{j=0}^{N-1}(-1)^j S_N^{j} P_j(\alpha_k) = \sum_{j=0}^{N-k-2} (-1)^j S_N^{j} (-1)^j (e_{k,k-j-1} - e_{k+j+1,k})$$

$$+(-1)^{N-k-1} S_N^{N-k-1} \left\{(N-k-1)(-1)^{N-k-1} e_{Nk} + (-1)^{N-k-1} e_{k,2k-N}\right\}$$

$$+\sum_{j=N-k}^{k-3}(-1)^j S_N^j(-1)^j(e_{k,k-j-1}-e_{N,N-j-1})$$

$$+(-1)^k S_N^{k-2}\left\{k(-1)^k e_{k1}+(-1)^{k+1}e_{N,N-k+1}\right\}+\sum_{j=k-1}^{N-3}(-1)^j S_N^j(-1)^{j+1}(e_{N,N-j-1}-e_{j+2,1})$$

$$+(-1)^N S_N^{N-2}(N-1)(-1)^{N-1}e_{N1}=\sum_{j=0}^{N-k-2}e_{k-j,k-j-1}-\sum_{j=0}^{N-k-2}e_{k+1,k}+(N-k-1)e_{k+1,k}+e_{2k-N+1,2k-N}$$

$$+\sum_{j=N-k}^{k-3}e_{k-j,k-j-1}-\sum_{j=N-k}^{k-3}e_{N-j,N-j-1}+ke_{21}-e_{N-k+2,N-k+1}-\sum_{j=k-1}^{N-3}e_{N-j,N-j-1}+\sum_{j=k-1}^{N-3}e_{21}-(N-1)e_{21}$$

$$=\sum_{j=0}^{k-3}e_{k-j,k-j-1}-(N-k-1)e_{k+1,k}+(N-k-1)e_{k-1,k}-\sum_{j=N-k}^{N-3}e_{N-j,N-j-1}+ke_{21}+(N-k-1)e_{21}$$

$$-(N-1)e_{21}=\sum_{j=0}^{k-3}e_{k-j,k-j-1}-\sum_{r=0}^{k-3}e_{k-r,k-r-1}=0.$$

- *If $q=N-2$ and $k=\frac{N+1}{2}$ for N odd we have $N-k=k-1<N-1$, then*

$$\sum_{j=0}^{N-2}(-1)^j S_N^{N+j-q-1}P_j(\alpha_k)=\sum_{j=0}^{N-2}(-1)^j S_N^{j+1}P_j(\alpha_k)=\sum_{j=0}^{\frac{N-5}{2}}(-1)^j S_N^{j+1}(-1)^j(e_{\frac{N+1}{2},\frac{N-1}{2}-j}-e_{\frac{N+3}{2}+j,\frac{N+1}{2}})$$

$$+(-1)^{\frac{N-3}{2}}S_N^{\frac{N-1}{2}}\left\{\left(\frac{N-3}{2}\right)(-1)^{\frac{N+1}{2}}e_{N,\frac{N+1}{2}}+(-1)^{\frac{N+1}{2}}\left(\frac{N+1}{2}\right)e_{\frac{N+1}{2},1}\right\}$$

$$+\sum_{j=\frac{N-1}{2}}^{N-3}(-1)^j S_N^{j+1}(-1)^{j+1}(e_{N,N-j-1}-e_{j+2,1})+(-1)^{N-2}S_N^{N-1}(N-1)(-1)^{N+1}e_{N1}$$

$$=\sum_{j=0}^{\frac{N-5}{2}}e_{\frac{N-1}{2}-j,\frac{N-1}{2}-j}-\sum_{j=0}^{\frac{N-5}{2}}e_{\frac{N+1}{2},\frac{N+1}{2}}+\left(\frac{N-3}{2}\right)e_{\frac{N+1}{2},\frac{N+1}{2}}+\left(\frac{N+1}{2}\right)e_{\frac{N+1}{2},1}-\sum_{j=\frac{N-1}{2}}^{N-3}e_{N-j-1,N-j-1}$$

$$+\sum_{j=\frac{N-1}{2}}^{N-3}e_{11}-(N-1)e_{11}=\sum_{j=0}^{\frac{N-5}{2}}e_{\frac{N-1}{2}-j,\frac{N-1}{2}-j}-\left(\frac{N-3}{2}\right)e_{\frac{N+1}{2},\frac{N+1}{2}}+\left(\frac{N-3}{2}\right)e_{\frac{N+1}{2},\frac{N+1}{2}}+\left(\frac{N+1}{2}\right)e_{11}$$

$$-\sum_{r=0}^{\frac{N-5}{2}}e_{\frac{N-1}{2}-r,\frac{N-1}{2}-r}+\left(\frac{N-3}{2}\right)e_{11}-(N-1)e_{11}=0.$$

- *If $q = N-1$ and $k = \frac{N+1}{2}$ then $N-k = k-1 < N-1$ and*

$$\sum_{j=0}^{N-1}(-1)^j S_N^{N+j-q-1} P_j(\alpha_k) = \sum_{j=0}^{N-1}(-1)^j S_N^j P_j(\alpha_k) = \sum_{j=0}^{\frac{N-5}{2}}(-1)^j S_N^j (-1)^j (e_{\frac{N+1}{2},\frac{N-1}{2}-j} - e_{\frac{N+3}{2}+j,\frac{N+1}{2}})$$

$$+(-1)^{\frac{N-3}{2}} S_N^{\frac{N-3}{2}} \left\{ \binom{\frac{N-3}{2}}{} (-1)^{\frac{N+1}{2}} e_{N,\frac{N+1}{2}} + (-1)^{\frac{N+1}{2}} \left(\frac{N+1}{2}\right) e_{\frac{N+1}{2},1} \right\}$$

$$+\sum_{j=\frac{N-1}{2}}^{N-3}(-1)^j S_N^j (-1)^{j+1}(e_{N,N-j-1} - e_{j+2,1}) + (-1)^{N-2} S_N^{N-2}(N-1)(-1)^{N+1} e_{N1}$$

$$= \sum_{j=0}^{\frac{N-5}{2}} e_{\frac{N+1}{2}-j,\frac{N-1}{2}-j} - \sum_{j=0}^{\frac{N-5}{2}} e_{\frac{N+3}{2},\frac{N+1}{2}} + \left(\frac{N-3}{2}\right) e_{\frac{N+3}{2},\frac{N+1}{2}} + \left(\frac{N+1}{2}\right) e_{21} - \sum_{j=\frac{N-1}{2}}^{N-3} e_{N-j,N-j-1}$$

$$+\sum_{j=\frac{N-1}{2}}^{N-3} e_{21} - (N-1)e_{21} = \sum_{j=0}^{\frac{N-5}{2}} e_{\frac{N+1}{2}-j,\frac{N-1}{2}-j} - \left(\frac{N-3}{2}\right) e_{\frac{N+3}{2},\frac{N+1}{2}} + \left(\frac{N-3}{2}\right) e_{\frac{N+3}{2},\frac{N+1}{2}}$$

$$+\left(\frac{N+1}{2}\right) e_{21} - \sum_{r=0}^{\frac{N-5}{2}} e_{\frac{N+1}{2}-r,\frac{N-1}{2}-r} + \left(\frac{N-3}{2}\right) e_{21} - (N-1)e_{21} = 0.$$

Recall that a Pierce decomposition of a noncommutative ring R with unit 1 is a finite set of elements $r_1, \ldots, r_n \in R$ such that $1 = \sum_{j=1}^{n} r_j$ and $r_i r_j = \delta_{ij}$ for all $1 \leq i, j \leq n$. Now we state the Pierce decomposition of $\mathbb{A}$.

Corollary 8. *If $\alpha_1 = I - \sum_{k=2}^{N-1} \alpha_k$, then $\{\alpha_k\}_{1\leq k\leq N-1}$ is a Pierce decomposition of $\mathbb{A}$.*

Proof. *In fact, if $2 \leq k < l \leq N-1$, then*

$$\alpha_k \alpha_l = \left(e_{kk} + \sum_{j=1}^{N-1} a_{kj} x^j\right)\left(e_{ll} + \sum_{r=1}^{N-1} a_{lr} x^r\right) = \sum_{j,r=1}^{N-1} a_{kj} a_{lr} x^{j+r}.$$

However,

$$a_{kj} a_{lr} = \left((-1)^{j+1}(\delta_{k,j+1} e_{k1} + \delta_{k,N-j} e_{Nk}) + (-1)^j \delta_{N,j+1} e_{N1}\right)\left((-1)^{r+1}(\delta_{l,r+1} e_{l1} + \delta_{l,N-r} e_{Nl}) + (-1)^j \delta_{N,r+1} e_{N1}\right) = 0$$

implies $\alpha_k \alpha_l = 0$.

On the other hand,

$$\alpha_k^2 = \left(e_{kk} + \sum_{j=1}^{N-1} a_{kj}x^j\right)\left(e_{kk} + \sum_{j=1}^{N-1} a_{kj}x^j\right) = e_{kk} + \sum_{j=1}^{N-1}(e_{kk}a_{kj} + a_{kj}e_{kk})x^j + \sum_{j,r=1}^{N-1} a_{kj}a_{kr}x^{j+r}.$$

However,

$$e_{kk}a_{kj} + a_{kj}e_{kk} = (-1)^{j+1}\left(\delta_{k,j+1}e_{k1} + \delta_{k,N-j}e_{Nk}\right)$$

and

$$a_{kj}a_{kr} = (-1)^{j+r}\delta_{k,N-j}\delta_{k,r+1}e_{N1} = (-1)^{N-1}\delta_{j,N-k}\delta_{r,N-k}e_{N1}$$

imply

$$\alpha_k^2 = e_{kk} + \sum_{j=1}^{N-1}(-1)^{j+1}\left(\delta_{k,j+1}e_{k1} + \delta_{k,N-j}e_{Nk}\right)x^j + (-1)^{N-1}e_{N1}x^{N-1} = e_{kk} + \sum_{j=1}^{N-1} a_{kj}x^j = \alpha_k.$$

Thus, $\alpha_k\alpha_l = \delta_{kl}\alpha_k$ *for* $2 \leq k, l \leq N-1$. *By the definition of* α_1 *and the previous properties we can extend this to* $\alpha_k\alpha_l = \delta_{kl}\alpha_k$ *for* $1 \leq k, l \leq N-1$. *The assertion follows by the Theorem 13.*

Moreover, the duality of the classical particle systems can also be manifested through bispectrality in that the dynamics of the two operators in a bispectral triple under some integrable hierarchy can be seen to display the particle motion of the two dual systems respectively.

Alex Kasman [14]

2

Bispectral Algebras and their Presentations

2.1 Introduction

The main goal of this chapter is to give a presentation of each (bispectral) algebra using its generators and some relations among them. Thus, describing the ideal of relations. We give three examples of bispectral algebras to ilustrate a general theorem of presentations of finitely generated algebras. In the latter case, the eigenvalue $F(z)$ is scalar valued and $\theta(x)$ is matrix valued. For a given scalar eigenvalue function the corresponding algebra of matrix eigenvalues is characterized. These results give positive answers to the three conjectures in [11]. In this chapter, we use the software Singular and Maxima to obtain a set of generators and nice relations among them and after that, we prove that in fact, this set of nice relations are enough to give presentations for these algebras.

The following theorem may be used to obtain presentations for full rank 1 algebras. It is inspired by the work presented in [26].

Theorem 14 (Presentation of finitely generated algebras). *Let A be a finitely generated $\mathbb{K}$-algebra by $\beta_1, \beta_2, \ldots, \beta_n$ such that:*

- *There exist an ideal I of $\mathbb{K} \cdot \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ and an epimorphism of algebras*

$$f : \mathbb{K} \cdot \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle / I \longrightarrow A,$$

$$f(\overline{\alpha_j}) = \beta_j$$

- *There exists a subalgebra $\mathbb{K} \subset R \subset \mathbb{K} \cdot \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle / I$ such that $\mathbb{K} \cdot \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle / I$ is a free left R-module generated by $\{x_j\}_{j=0}^{\infty}$, i.e.,*

$$\mathbb{K} \cdot \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle / I = \bigoplus_{j=0}^{\infty} R x_j .$$

- *$f\,|_R \colon R \longrightarrow A$ is a monomorphism.*

- *The set $\{f(x_j)\}_{j=0}^{\infty}$ is a basis for A as a left $f(R)$-module.*

  Then, f is an isomorphism.

Before proving the theorem we have the following remark.

Remark 4. *The theorem guarantees a presentation of A in terms of generators and relations through the isomorphism f, i.e.,*

$$A = \mathbb{K} \cdot \langle \beta_1, \beta_2, \ldots, \beta_n \mid P(\beta_1, \beta_2, \ldots, \beta_n) = 0, \forall P \in I \rangle$$

Proof. *It is enough to prove that f is injective. Pick $x \in \ker(f)$ and write $x = \sum_{j=0}^{m} r_j x_j$, then $0 = f(x) = \sum_{j=0}^{m} f(r_j) f(x_j)$. However, since $\{f(x_j)\}_{j=0}^{\infty}$ is a basis for A as a left $f(R)$-module we have $f(r_j) = 0$ for $0 \leq j \leq m$. Here we use that $f\,|_R \colon R \longrightarrow A$ is an monomorphism to conclude $r_j = 0$ for $0 \leq j \leq m$ and $x = 0$.* $\square$

Corollaries 9, 10, and Proposition 15 give an answer to the Conjectures 1, 2 and 3 of [11] about three bispectral full rank 1 algebras as we shall describe in the following sections. Moreover, these algebras are Noetherian and finitely generated because they are contained in the $N \times N$ matrix polynomial ring $M_N(\mathbb{K}[x])$.

Corollary 9. *Let Γ be the sub-algebra of $M_2(\mathbb{C})[x]$ of the form*

$$\begin{pmatrix} r_0^{11} & r_0^{12} \\ 0 & r_0^{11} \end{pmatrix} + \begin{pmatrix} r_1^{11} & r_1^{12} \\ 0 & r_1^{11} \end{pmatrix} x + \begin{pmatrix} r_2^{11} & r_2^{12} \\ r_1^{11} & r_2^{22} \end{pmatrix} x^2 + \begin{pmatrix} r_3^{11} & r_3^{12} \\ r_2^{22} + r_2^{11} - r_1^{12} & r_3^{22} \end{pmatrix} x^3 + x^4 p(x),$$

where $p \in M_2(\mathbb{C})[x]$ and all the variables $r_0^{11}, r_0^{12}, r_1^{11}, r_1^{12}, r_2^{11}, r_2^{22}, r_3^{11}, r_3^{12}, r_3^{22} \in \mathbb{C}$. Then $\Gamma = \mathbb{A}$. Moreover, for each θ we have an explicit expression for the operator $\mathcal{B}$.

Furthermore, we have the presentation $\mathbb{A} = \mathbb{C} \cdot \langle \alpha_0, \alpha_1 \mid I = 0 \rangle$ with the ideal I given by

$$I := \langle \alpha_0^2, \alpha_1^3 + \alpha_0\alpha_1\alpha_0 - 3\alpha_1\alpha_0\alpha_1 + \alpha_0\alpha_1^2 + \alpha_1^2\alpha_0 \rangle .$$

Proof. *The first part of the proof is given by the Theorem 12. We will give a proof of the existence of the presentation. $\mathbb{A}$ is generated by $\beta_0 = e_{12}$, $\beta_1 = Ix + e_{21}x^2$, $\beta_2 = e_{12}x + e_{11}x^2$, $\beta_3 = e_{12}x + e_{22}x^2$, $\beta_4 = e_{12}x^2$, $\beta_5 = e_{12}x - e_{21}x^3$, $\beta_6 = e_{11}x^3$, $\beta_7 = e_{12}x^3$, $\beta_8 = e_{22}x^3$.*

Moreover, we can eliminate the variables β_j for $2 \leq j \leq 8$. In fact, $\beta_2 = \beta_0\beta_1$, $\beta_3 = \beta_1\beta_0$, $\beta_4 = \beta_0\beta_1\beta_0$, $\beta_5 = \frac{\beta_0\beta_1 + \beta_1\beta_0 - \beta_1^2}{2}$, $\beta_6 = \frac{\beta_0\beta_1\beta_0 - \beta_0\beta_1^2}{2}$, $\beta_7 = \frac{\beta_0\beta_1^2\beta_0}{2}$, $\beta_8 = \frac{\beta_0\beta_1\beta_0 - \beta_1^2\beta_0}{2}$.

Furthermore, we are going to check the presentation using Theorem 14. We begin with some general results:

Proposition 2. *Let A be a $\mathbb{K}$-algebra. Suppose that $\beta_0 \in A$ is a nilpotent element of degree 2, then*

$$\left\{\beta_1^j \mid j \geq 0\right\} \cup \left\{\beta_1^j\beta_0 \mid j \geq 0\right\} \cup \left\{\beta_1^j\beta_0\beta_1 \mid j \geq 0\right\} \cup \left\{\beta_1^j\beta_0\beta_1\beta_0 \mid j \geq 0\right\}$$

is a linearly independent set over $\mathbb{K}$ if and only if

$$\left\{\beta_1^j\beta_0 \mid j \geq 0\right\} \cup \left\{\beta_1^j\beta_0\beta_1\beta_0 \mid j \geq 0\right\}$$

is a linearly independent set over $\mathbb{K}$.

Proof. *Clearly the condition is sufficient. We consider the expression:*

$$\sum_{j=0}^{n} a_j\beta_1^j + \sum_{j=0}^{n} b_j\beta_1^j\beta_0 + \sum_{j=0}^{n} c_j\beta_1^j\beta_0\beta_1 + \sum_{j=0}^{n} d_j\beta_1^j\beta_0\beta_1\beta_0 = 0 \tag{2.1}$$

for $a_j, b_j, c_j, d_j \in \mathbb{K}$, $n \in \mathbb{N}$.

Multiply by β_0 *on the right and using that* $\beta_0^2 = 0$ *we obtain:*

$$\sum_{j=0}^{n} a_j\beta_1^j\beta_0 + \sum_{j=0}^{n} c_j\beta_1^j\beta_0\beta_1\beta_0 = 0.$$

If we assume that $\left\{\beta_1^j\beta_0 \mid j \geq 0\right\} \cup \left\{\beta_1^j\beta_0\beta_1\beta_0 \mid j \geq 0\right\}$ *is linearly independent we have* $a_j = c_j = 0$ *and* (2.1) *reduces to:*

$$\sum_{j=0}^{n} b_j\beta_1^j\beta_0 + \sum_{j=0}^{n} d_j\beta_1^j\beta_0\beta_1\beta_0 = 0.$$

Again, using this assumption we have $b_j = d_j = 0$. *With this fact we obtain the necessity.*

Proposition 3. *Taking the elements* β_0 *and* β_1 *in* $\mathbb{A}$ *we obtain that*

$$\left\{\beta_1^j\beta_0 \mid j \geq 0\right\} \cup \left\{\beta_1^j\beta_0\beta_1\beta_0 \mid j \geq 0\right\}$$

is a linearly independent set.

Proof. *Note that* $\beta_1^j\beta_0 = e_{12}x^j + je_{22}x^{j+1}$ *and* $\beta_1^j\beta_0\beta_1\beta_0 = e_{12}x^{j+2} + je_{22}x^{j+3}$. *Consider the expression:*

$$\sum_{j=0}^{n} a_j\beta_1^j\beta_0 + \sum_{j=0}^{n} b_j\beta_1^j\beta_0\beta_1\beta_0 = 0.$$

Replacing the previous relations we obtain:

$$\sum_{j=0}^{n} a_j(e_{12}x^j + je_{22}x^{j+1}) + \sum_{j=0}^{n} b_j\beta_1^j(e_{12}x^{j+2} + je_{22}x^{j+3}) = 0.$$

Using the entries of the matrix we obtain:

$$\sum_{j=0}^{n} a_jx^j + \sum_{j=0}^{n} b_jx^{j+2} = 0 \text{ and } \sum_{j=0}^{n} ja_jx^{j+1} + \sum_{j=0}^{n} jb_jx^{j+3} = 0.$$

Equivalently,

$$\sum_{j=0}^{n} a_j x^j + \sum_{j=2}^{n+2} b_{j-2} x^j = 0 \text{ and } \sum_{j=0}^{n} j a_j x^j + \sum_{j=2}^{n+2} (j-2) b_{j-2} x^j = 0.$$

Hence,

$$a_0 + a_1 x + \sum_{j=2}^{n} (a_j + b_{j-2}) x^j + b_{n-1} x^{n+1} + b_n x^{n+2} = 0 \text{ and}$$

$$a_1 x + \sum_{j=2}^{n} (j a_j + (j-2) b_{j-2}) x^j + (n-1) b_{n-1} x^{n+1} + n b_n x^{n+2} = 0.$$

Therefore,

$$a_0 = a_1 = b_{n-1} = b_n = 0, \begin{pmatrix} 1 & 1 \\ j & j-2 \end{pmatrix} \begin{pmatrix} a_j \\ b_{j-2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 2 \leq j \leq n.$$

Since $\det \begin{pmatrix} 1 & 1 \\ j & j-2 \end{pmatrix} = -2 \neq 0$ *we have* $a_j = b_{j-2} = 0,\ 2 \leq j \leq n$ *and*

$$\left\{ \beta_1^j \beta_0 \mid j \geq 0 \right\} \cup \left\{ \beta_1^j \beta_0 \beta_1 \beta_0 \mid j \geq 0 \right\}$$

is linearly independent.

Lemma 7. *Consider the algebra* $\mathbb{K} \cdot \langle \alpha_0, \alpha_1 \rangle / I$ *with*

$$I = < \alpha_0^2, \alpha_1^3 + \alpha_0 \alpha_1 \alpha_0 - 3 \alpha_1 \alpha_0 \alpha_1 + \alpha_0 \alpha_1^2 + \alpha_1^2 \alpha_0 >$$

then $\{1, \alpha_0, \alpha_0\alpha_1, \alpha_0\alpha_1\alpha_0\}$ *is a system of generators for* $\mathbb{K} \cdot \langle \alpha_0, \alpha_1 \rangle / I$ *as a free left R-module, with* $R = \mathbb{K} \cdot \langle \alpha_1 \rangle / I$*.*

Proof. *Define* $M = R \oplus R \cdot \alpha_0 \oplus R \cdot \alpha_0\alpha_1 \oplus R \cdot \alpha_0\alpha_1\alpha_0$*. We have to see that* $\mathbb{K} \cdot \langle \alpha_0, \alpha_1 \rangle / I = M$*. It is enough to show that* M *is invariant under left and right multiplication by* α_0 *and* α_1*.*

- $\alpha_1 M \subset M$.

  Since $\alpha_1 \in R$*.*

- $M\alpha_0 \subset M$*. In fact,* $M\alpha_0 \subset R \cdot \alpha_0 \oplus R \cdot \alpha_0\alpha_1\alpha_0 \subset M$*.*

- $M\alpha_1 \subset M$*.*

Since $\alpha_0\alpha_1^2 = -\alpha_1^3 - \alpha_1^2\alpha_0 + 3\alpha_1\alpha_0\alpha_1 - \alpha_0\alpha_1\alpha_0$ *we have* $\alpha_0\alpha_1^2\alpha_0 = -\alpha_1^3\alpha_0 + 3(\alpha_1\alpha_0)^2$ *and* $0 = -\alpha_0\alpha_1^3\alpha_0 - \alpha_0\alpha_1^2\alpha_0 + 3(\alpha_0\alpha_1)^2$.

Furthermore,

$$\alpha_0\alpha_1^3 = -\alpha_1^4 - \alpha_1^2(\alpha_0\alpha_1) + 3\alpha_1(\alpha_0\alpha_1^2) - (\alpha_0\alpha_1)^2.$$

Hence,

$$3(\alpha_0\alpha_1)^2 = \alpha_0\alpha_1^3 + \alpha_0\alpha_1^2\alpha_0 = \alpha_0\alpha_1^3 - \alpha_1^3\alpha_0 + 3(\alpha_1\alpha_0)^2$$

$$= -\alpha_1^4 - \alpha_1^2(\alpha_0\alpha_1) + 3\alpha_1(\alpha_0\alpha_1^2) - (\alpha_0\alpha_1)^2 - \alpha_1^3\alpha_0 + 3(\alpha_1\alpha_0)^2.$$

Equivalently,

$$4(\alpha_0\alpha_1)^2 = -\alpha_1^4 - \alpha_1^3\alpha_0 - \alpha_1^2(\alpha_0\alpha_1) + 3\alpha_1(\alpha_0\alpha_1^2) + 3(\alpha_1\alpha_0)^2.$$

However,

$$\alpha_1\alpha_0\alpha_1^2 = -\alpha_1^4 - \alpha_1^3\alpha_0 + 3\alpha_1^2(\alpha_0\alpha_1) - (\alpha_1\alpha_0)^2.$$

Thus,

$$4(\alpha_0\alpha_1)^2 = -\alpha_1^4 - \alpha_1^3\alpha_0 - \alpha_1^2(\alpha_0\alpha_1) + 3\alpha_1^4 - 3\alpha_1^3\alpha_0 + 9\alpha_1^2(\alpha_0\alpha_1) - 3(\alpha_1\alpha_0)^2 + 3(\alpha_1\alpha_0)^2 = -4\alpha_1^4 - 4\alpha_1^3\alpha_0 + 8\alpha_1^2(\alpha_0\alpha_1).$$

Therefore,

$$(\alpha_0\alpha_1)^2 = -\alpha_1^4 - \alpha_1^3\alpha_0 + 2\alpha_1^2(\alpha_0\alpha_1).$$

This implies that $(\alpha_0\alpha_1)^2 \in M$, $\alpha_0\alpha_1^2 \in M$. *Since* M *is a left* R*-module we have* $M\alpha_1 \subset R\alpha_1 \oplus R\alpha_0\alpha_1 \oplus R\alpha_0\alpha_1^2 \oplus R(\alpha_0\alpha_1)^2 \subset M$.

- $\alpha_0 M \subset M$.

  We claim that $\alpha_0\alpha_1^n \in M$ *for every* $n \in \mathbb{N}$. *For* $n = 0$ *is clear. Assume this for some* $n \in \mathbb{N}$ *and note that* $\alpha_0\alpha_1^{n+1} = (\alpha_0\alpha_1^n)\alpha_1 \in M\alpha_1 \subset M$. *The claim follows by induction.*

  In particular, $\alpha_0 R \subset M$. *Thus,* $\alpha_0 M \subset \alpha_0 R \oplus \alpha_0 R\alpha_0 \oplus \alpha_0 R\alpha_0\alpha_1 \oplus \alpha_0 R\alpha_0\alpha_1\alpha_0 \subset R \oplus R\cdot\alpha_0 \oplus R\cdot\alpha_0\alpha_1 \oplus R\cdot\alpha_0\alpha_1\alpha_0 \subset M$.

□

Finally, we conclude with the proof of the nice presentation. Define

$$f\colon \mathbb{C}\cdot\langle\alpha_0,\alpha_1\rangle/I \longrightarrow \mathbb{A},$$

$$f(\overline{\alpha_j}) = \beta_j$$

the previous lemma guarantees the existence of a subalgebra $R = \mathbb{C}\cdot\langle\alpha_1\rangle/I$ *and a system of generators* $\{1,\alpha_0,\alpha_0\alpha_1,\alpha_0\alpha_1\alpha_0\}$ *for* $\mathbb{C}\cdot\langle\alpha_0,\alpha_1\rangle/I$ *as a free left R-module. Furthermore,* $f\mid_R\colon R \longrightarrow A$ *is a monomorphism.*

The Proposition 3 implies that $\{1,\beta_0,\beta_0\beta_1,\beta_0\beta_1\beta_0\}$ *is a linearly independent set over* $\mathbb{C}$*.*

Thus, we are under the hypothesis of Theorem 14 and f is an isomorphism. □

This conclude the proof of the assertion. □

Corollary 10. *Let* Γ *the sub-algebra of* $M_3(\mathbb{C})[x]$ *of the form*

$$\begin{pmatrix} r_0^{11} & r_0^{12} & r_0^{13} \\ 0 & r_0^{22} & r_0^{23} \\ 0 & 0 & r_0^{11} \end{pmatrix} + \begin{pmatrix} r_1^{11} & r_1^{12} & r_1^{13} \\ r_0^{22}-r_0^{11} & r_1^{22} & r_1^{23} \\ 0 & r_0^{22}-r_0^{11} & r_1^{11}+r_0^{23}-r_0^{12} \end{pmatrix} x$$

$$+\begin{pmatrix} r_2^{11} & r_2^{12} & r_2^{13} \\ r_1^{22}-r_1^{11}-r_0^{23}+r_0^{12} & r_2^{22} & r_2^{23} \\ r_0^{22}-r_0^{11} & r_1^{22}-r_1^{11} & r_2^{11}+r_1^{23}-r_1^{12} \end{pmatrix} x^2 + \begin{pmatrix} r_3^{11} & r_3^{12} & r_3^{13} \\ r_3^{21} & r_3^{22} & r_3^{23} \\ r_1^{22}-2r_1^{11}-r_0^{23}+r_0^{12} & r_3^{32} & r_3^{33} \end{pmatrix} x^3$$

$$+\begin{pmatrix} r_4^{11} & r_4^{12} & r_4^{13} \\ r_4^{21} & r_4^{22} & r_4^{23} \\ r_3^{32}+r_3^{21}-r_2^{22}-r_2^{11}+r_1^{12} & r_4^{22} & r_4^{33} \end{pmatrix} x^4$$

$$+\begin{pmatrix} r_5^{11} & r_5^{12} & r_5^{13} \\ r_5^{21} & r_5^{22} & r_5^{23} \\ r_4^{32}+r_4^{21}-r_3^{33}-r_3^{22}-r_3^{11}+r_2^{23}+r_2^{12}-r_1^{13} & r_5^{32} & r_5^{33} \end{pmatrix} x^5 + x^6 p(x),$$

where $p \in M_3(\mathbb{C})[x]$ *and all the variables* $r_0^{11}, r_0^{12}, \ldots, r_5^{33} \in \mathbb{C}$ *are arbitrary.*

Then, $\Gamma = \mathbb{A}$ *and for each* θ *we have an explicit expression for the operator* $\mathcal{B}$*.*

Furthermore, we have the presentation $\mathbb{A} = \mathbb{C}\cdot\langle\alpha_2,\alpha_3 \mid I = 0\rangle$ *with*

$$I = \langle \alpha_2^3, \alpha_3^2 - \alpha_3, (\alpha_3\alpha_2)^2\alpha_3 - 4\alpha_3\alpha_2^2\alpha_3 \rangle \, .$$

Proof. *The proof is a straightforward check of the relations given in the Proposition 1 for the case $S = S_N$ the shift operator which is nilpotent of degree N. We will give a proof of the presentation.*

Note that $\mathbb{A}$ is generated by $\beta_0 = e_{13}, \beta_1 = e_{12} - e_{33}x + e_{21}x^2 + e_{31}x^3, \beta_2 = e_{12} + e_{23}, \beta_3 = e_{22} + (e_{21} + e_{32})x + e_{31}x^2, \beta_4 = e_{22}x + S_3x^2 + e_{31}x^3, \beta_5 = Ix - e_{31}x^3, \beta_6 = e_{13}x - e_{11}x^3, \beta_7 = e_{13}x - e_{22}x^3, \beta_8 = e_{13}x - e_{33}x^3, \beta_9 = e_{13}x^2, \beta_{10} = e_{13}x - e_{31}x^5, \beta_{11} = e_{23}x - e_{33}x^2, \beta_{12} = S_3x + e_{31}x^4, \beta_{13} = Ix^2 - 2e_{31}x^4, \beta_{14} = e_{12}x^2 + e_{31}x^5, \beta_{15} = e_{22}x^2 - e_{31}x^4, \beta_{16} = e_{23}x^2 - e_{31}x^5, \beta_{17} = e_{12}x^3, \beta_{18} = e_{13}x^3, \beta_{19} = e_{21}x^3 + e_{31}x^4, \beta_{20} = e_{23}x^3, \beta_{21} = e_{32}x^3 + e_{31}x^4, \beta_{22} = e_{11}x^4, \beta_{23} = e_{12}x^4, \beta_{24} = e_{13}x^4, \beta_{25} = e_{21}x^4 + e_{31}x^5, \beta_{26} = e_{22}x^4, \beta_{27} = e_{23}x^4, \beta_{28} = e_{32}x^4 + e_{31}x^5, \beta_{29} = e_{33}x^4, \beta_{30} = e_{11}x^5, \beta_{31} = e_{12}x^5, \beta_{32} = e_{13}x^5, \beta_{33} = e_{21}x^5, \beta_{34} = e_{22}x^5, \beta_{35} = e_{23}x^5, \beta_{36} = e_{32}x^5, \beta_{37} = e_{33}x^5.$

However, we can eliminate the variables β_j for $j \neq 2, 3$. In fact,

$\beta_0 = \beta_2^2,\ \beta_1 = 1/2\beta_3\beta_2\beta_3 - \beta_3\beta_2 + \beta_2,\ \beta_4 = 1/2\beta_3\beta_2\beta_3,\ \beta_5 = -1/2\beta_3\beta_2\beta_3 + \beta_2\beta_3 + \beta_3\beta_2 - \beta_2,\ \beta_6 = -1/2\beta_2^2\beta_3\beta_2\beta_3 + \beta_2^2\beta_3\beta_2,\ \beta_7 = -1/2\beta_2(\beta_3\beta_2)^2 + \beta_2^2\beta_3\beta_2 + \beta_2\beta_3\beta_2^2,\ \beta_8 = -1/2(\beta_3\beta_2)^2\beta_2 + \beta_2\beta_3\beta_2^2,\ \beta_9 = \beta_2^2\beta_3\beta_2^2,\ \beta_{10} = -1/2(\beta_3\beta_2)^2\beta_2\beta_3 - 1/2\beta_2(\beta_2\beta_3)^2 + \beta_2\beta_3\beta_2^2\beta_3 - 1/2\beta_2(\beta_3\beta_2)^2 + \beta_3\beta_2^2\beta_3\beta_2 - 1/2(\beta_3\beta_2)^2\beta_2 + \beta_2^2\beta_3\beta_2 + \beta_2\beta_3\beta_2^2,\ \ \beta_{11} = \beta_3\beta_2^2,\ \beta_{12} = -1/2(\beta_2\beta_3)^2 + \beta_3\beta_2^2\beta_3 - 1/2(\beta_3\beta_2)^2 + \beta_2^2\beta_3 + \beta_2\beta_3\beta_2 + \beta_3\beta_2^2 - \beta_2^2,\ \beta_{13} = (\beta_2\beta_3)^2 - 2\beta_3\beta_2^2\beta_3 + (\beta_3\beta_2)^2 - \beta_2^2\beta_3 - \beta_2\beta_3\beta_2 - \beta_3\beta_2^2 + \beta_2^2,\ \beta_{14} = 1/2\beta_3\beta_2\beta_3\beta_2\beta_2\beta_3 + 1/2\beta_2\beta_2\beta_3\beta_2\beta_3 - \beta_2\beta_3\beta_2\beta_2\beta_3 + 1/2\beta_2\beta_3\beta_2\beta_3\beta_2 - \beta_3\beta_2\beta_2\beta_3\beta_2 + 1/2\beta_3\beta_2\beta_3\beta_2\beta_2 - \beta_2\beta_3\beta_2\beta_2,\ \beta_{15} = 1/2(\beta_2\beta_3)^2 - \beta_3\beta_2^2\beta_3 + 1/2(\beta_3\beta_2)^2 - \beta_2^2\beta_3 - \beta_3\beta_2^2,\ \beta_{16} = 1/2\beta_3\beta_2\beta_3\beta_2^2\beta_3 + 1/2\beta_2(\beta_2\beta_3)^2 - \beta_2\beta_3\beta_2^2\beta_3 + 1/2\beta_2(\beta_3\beta_2)^2 - \beta_3\beta_2^2\beta_3\beta_2 + 1/2(\beta_3\beta_2)^2\beta_2 - \beta_2^2\beta_3\beta_2,\ \beta_{22} = \beta_2^2\beta_3\beta_2^2\beta_3 - 1/2\beta_2^2(\beta_3\beta_2)^2 + \beta_2^2\beta_3\beta_2^2,\ \beta_{17} = 1/2\beta_2^2(\beta_3\beta_2)^2 - \beta_2^2\beta_3\beta_2^2,\ \beta_{18} = 1/2\beta_2(\beta_2\beta_3)^2\beta_2^2,\ \beta_{19} = \beta_3\beta_2^2\beta_3 - 1/2(\beta_3\beta_2)^2 + \beta_3\beta_2^2,\ \beta_{29} = -1/2(\beta_2\beta_3)^2\beta_2^2 + \beta_3\beta_2^2\beta_3\beta_2^2 + \beta_2^2\beta_3\beta_2^2,\ \beta_{20} = 1/2(\beta_2\beta_3)^2\beta_2^2 - \beta_2^2\beta_3\beta_2^2,\ \beta_{21} = -1/2(\beta_2\beta_3)^2 + \beta_3\beta_2^2\beta_3 + \beta_2^2\beta_3,\ \beta_{23} = \beta_2^2\beta_3\beta_2^2\beta_3\beta_2 - 1/2\beta_2^2\beta_3\beta_2\beta_3\beta_2^2,\ \beta_{24} = -\beta_2^2\beta_3\beta_2^2\beta_3\beta_2^2,\ \beta_{25} = 1/2\beta_3\beta_2\beta_3\beta_2^2\beta_3 - \beta_3\beta_2^2\beta_3\beta_2 + 1/2\beta_3\beta_2\beta_3\beta_2^2,\ \beta_{26} = -1/2\beta_2^2\beta_3\beta_2\beta_3\beta_2 + \beta_2\beta_3\beta_2^2\beta_3\beta_2 - 1/2\beta_2\beta_3\beta_2\beta_3\beta_2^2 + \beta_2^2\beta_3\beta_2^2,\ \beta_{37} = 1/2\beta_3\beta_2\beta_3\beta_2^2\beta_3\beta_2^2 + 1/2\beta_2^2\beta_3\beta_2\beta_3\beta_2^2 - \beta_2\beta_3\beta_2^2\beta_3\beta_2^2,\ \beta_{27} = -1/2\beta_2^2\beta_3\beta_2\beta_3\beta_2^2 + \beta_2\beta_3\beta_2^2\beta_3\beta_2^2,\ \beta_{28} = 1/2\beta_3\beta_2\beta_3\beta_2^2\beta_3 + 1/2\beta_2^2\beta_3\beta_2\beta_3 - \beta_2\beta_3\beta_2^2\beta_3,\ \beta_{30} = 1/2\beta_2^2\beta_3\beta_2\beta_3\beta_2^2\beta_3 - \beta_2^2\beta_3\beta_2^2\beta_3\beta_2 + 1/2\beta_2^2\beta_3\beta_2\beta_3\beta_2^2,\ \beta_{36} = 1/2\beta_3\beta_2\beta_3\beta_2^2\beta_3\beta_2 + 1/2\beta_2^2(\beta_3\beta_2)^2 - \beta_2\beta_3\beta_2^2\beta_3\beta_2 + 1/2(\beta_2\beta_3)^2\beta_2^2 - \beta_3\beta_2^2\beta_3\beta_2^2 - \beta_2^2\beta_3\beta_2^2,\ \beta_{31} = 1/2\beta_2^2\beta_3\beta_2\beta_3\beta_2^2\beta_3\beta_2 - \beta_2^2\beta_3\beta_2^2\beta_3\beta_2^2,\ \beta_{32} = 1/2\beta_2^2\beta_3\beta_2\beta_3\beta_2^2\beta_3\beta_2^2,\ \beta_{33} = 1/2(\beta_2\beta_3)^2\beta_2^2\beta_3 - \beta_2^2\beta_3\beta_2^2\beta_3 + 1/2\beta_2^2(\beta_3\beta_2)^2 - \beta_2\beta_3\beta_2^2\beta_3\beta_2 + 1/2(\beta_2\beta_3)^2\beta_2^2 - \beta_2^2\beta_3\beta_2^2,\ \beta_{34} = 1/2(\beta_2\beta_3)^2\beta_2^2\beta_3\beta_2 - \beta_2^2\beta_3\beta_2^2\beta_3\beta_2 + 1/2\beta_2^2(\beta_3\beta_2)^2\beta_2 - \beta_2\beta_3\beta_2^2\beta_3\beta_2^2,\ \beta_{35} = 1/2\beta_2\beta_3\beta_2\beta_3\beta_2^2\beta_3\beta_2^2 - \beta_2^2\beta_3\beta_2^2\beta_3\beta_2^2.$

Furthermore, we are going to check the presentation using Theorem 14. We begin with some general results:

Lemma 8. *Let A be a $\mathbb{K}$-algebra. Suppose that $\beta_2 \in A$ is a nilpotent element of degree $D \geq 3$. Suppose that*

$$\left\{\beta_2^{D-1}(\beta_3\beta_2)^j\beta_2^{D-2} \mid j \geq 0\right\}$$

is a linearly independent set over $\mathbb{K}$. Then, $\left\{\beta_2^{D-1}(\beta_3\beta_2)^j\beta_2^k \mid j \geq 0, 1 \leq k \leq D-2\right\}$ is linearly independent over $\mathbb{K}$.

Proof. *Consider the expression*

$$\sum_{j=1}^{n}\sum_{k=1}^{D-2} c_{jk}\beta_2^{D-2}(\beta_3\beta_2)^j\beta_2^k = 0. \tag{2.2}$$

Multiplying by β_2^{D-3} on the right:

$$\sum_{j=1}^{n}\sum_{k=1}^{D-2} c_{j1}\beta_2^{D-2}(\beta_3\beta_2)^j\beta_2^{D-2} = 0. \tag{2.3}$$

However, $\left\{\beta_2^{D-1}(\beta_3\beta_2)^j\beta_2^{D-2} \mid j \geq 0\right\}$ is linearly independent over $\mathbb{K}$. Thus $c_{j1} = 0$ for $0 \leq j \leq n$.

Thus (2.2) *reduces to*

$$\sum_{j=1}^{n}\sum_{k=2}^{D-2} c_{jk}\beta_2^{D-2}(\beta_3\beta_2)^j\beta_2^k = 0. \tag{2.4}$$

Assume that

$$\sum_{j=1}^{n}\sum_{k=k_0}^{D-2} c_{jk}\beta_2^{D-2}(\beta_3\beta_2)^j\beta_2^k = 0. \tag{2.5}$$

Multiplying by $\beta_2^{D-2-k_0}$ on the right:

$$\sum_{j=1}^{n} c_{jk_0}\beta_2^{D-2}(\beta_3\beta_2)^j\beta_2^{D-2} = 0. \tag{2.6}$$

However, $\left\{\beta_2^{D-1}(\beta_3\beta_2)^j\beta_2^{D-2} \mid j \geq 0\right\}$ is linearly independent over k. Thus $c_{jk_0} = 0$ for $1 \leq j \leq n$. Thus

$$\sum_{j=1}^{n}\sum_{k=k_0+1}^{D-2} c_{jk}\beta_2^{D-2}(\beta_3\beta_2)^j\beta_2^k = 0. \tag{2.7}$$

Since the case $k_0 = 1 \Rightarrow k_0 = 2$ was seen we have that $c_{jk} = 0$ for $1 \leq j \leq n, 1 \leq k \leq D-2$. □

Proposition 4. *Let A be a $\mathbb{K}$-algebra. Suppose that $\beta_2 \in A$ is a nilpotent element of degree $D \geq 3$, then*

$$\left\{\beta_2^i(\beta_3\beta_2)^j\beta_3 \mid 0 \leq i \leq D-1, j \geq 0\right\} \cup \left\{\beta_2^i(\beta_3\beta_2)^j\beta_2^k \mid 0 \leq i \leq D-1, j \geq 1, 1 \leq k \leq D-2\right\}$$

$$\cup \left\{ \beta_2^i (\beta_3 \beta_2)^j \mid 0 \leq i \leq D-1, j \geq 0 \right\}$$

is a linearly independent set over $\mathbb{K}$ *if and only if*

$$\left\{ \beta_2^{D-1} (\beta_3 \beta_2)^j \beta_2^{D-2} \mid j \geq 0 \right\}$$

is a linearly independent set over $\mathbb{K}$.

Proof. *The sufficiency of the statement is clear. To show the necessity we consider the expresion*

$$\sum_{j=0}^{n} \sum_{i=0}^{D-1} a_{ij} \beta_2^i (\beta_3 \beta_2)^j + \sum_{j=0}^{n} \sum_{i=0}^{D-1} b_{ij} \beta_2^i (\beta_3 \beta_2)^j \beta_3 + \sum_{j=1}^{n} \sum_{i=0}^{D-1} \sum_{k=1}^{D-2} c_{ijk} \beta_2^i (\beta_3 \beta_2)^j \beta_2^k = 0 \tag{2.8}$$

$a_{ij}, b_{ij}, c_{ijk} \in \mathbb{K}, n \geq 0.$

We have to see that $a_{ij} = b_{ij} = c_{ijk} = 0.$

We are going to see that

$$\sum_{j=0}^{n} \sum_{i=l}^{D-1} a_{ij} \beta_2^i (\beta_3 \beta_2)^j + \sum_{j=0}^{n} \sum_{i=l}^{D-1} b_{ij} \beta_2^i (\beta_3 \beta_2)^j \beta_3 + \sum_{j=1}^{n} \sum_{i=l}^{D-1} \sum_{k=1}^{D-2} c_{ijk} \beta_2^i (\beta_3 \beta_2)^j \beta_2^k = 0 \tag{2.9}$$

for some $0 \leq l \leq D-1$ *implies that* $a_{lj} = b_{lj} = c_{ljk} = 0.$

For $l = 0$ *we have the equation* (2.8). *Multiplying by* β_2^{D-1} *on the left and on the right:*

$$\sum_{j=0}^{n} \sum_{i=0}^{D-1} b_{ij} \beta_2^{D-1} (\beta_3 \beta_2)^j \beta_3 \beta_2^{D-1} = \sum_{j=0}^{n} \sum_{i=0}^{D-1} b_{ij} \beta_2^{D-1} (\beta_3 \beta_2)^{j+1} \beta_2^{D-2} = 0. \tag{2.10}$$

However, $\left\{ \beta_2^{D-1} (\beta_3 \beta_2)^j \beta_2^{D-2} \mid j \geq 0 \right\}$ *is linearly independent over* $\mathbb{K}$. *Thus,* $b_{0j} = 0$ *for* $0 \leq j \leq n$.

This reduces (2.8) *to*

$$\sum_{j=0}^{n} \sum_{i=0}^{D-1} a_{ij} \beta_2^i (\beta_3 \beta_2)^j + \sum_{j=1}^{n} \sum_{i=0}^{D-1} \sum_{k=1}^{D-2} c_{ijk} \beta_2^i (\beta_3 \beta_2)^j \beta_2^k = 0. \tag{2.11}$$

Multiplying by β_2^{D-1} *on the left:*

$$\sum_{j=0}^{n} a_{0j} \beta_2^{D-1} (\beta_3 \beta_2)^j + \sum_{j=1}^{n} \sum_{k=1}^{D-2} c_{0jk} \beta_2^{D-2} (\beta_3 \beta_2)^j \beta_2^k = 0. \tag{2.12}$$

Multiplying by β_2^{D-2} on the right:

$$\sum_{j=0}^{n} a_{0j}\beta_2^{D-1}(\beta_3\beta_2)^j\beta_2^{D-2} = 0. \tag{2.13}$$

Thus, $a_{0j} = 0$ for $0 \leq j \leq n$. Since $\{\beta_2^{D-1}(\beta_3\beta_2)^j\beta_2^{D-2} \mid j \geq 0\}$ is linearly independent over $\mathbb{K}$.

This reduces (2.12) *to*

$$\sum_{j=0}^{n}\sum_{k=1}^{D-2} c_{0jk}\beta_2^{D-2}(\beta_3\beta_2)^j\beta_2^k = 0. \tag{2.14}$$

However, by Lemma 8, $\{\beta_2^{D-1}(\beta_3\beta_2)^j\beta_2^k \mid j \geq 0, 1 \leq k \leq D-2\}$ is linearly independent over $\mathbb{K}$. Thus $c_{0jk} = 0$ for $1 \leq j \leq n, 1 \leq k \leq D-2$.

Assume (2.9) *for l and multiply this by β_2^{D-l-1} on the left:*

$$\sum_{j=0}^{n} a_{lj}\beta_2^{D-1}(\beta_3\beta_2)^j + \sum_{j=0}^{n} b_{lj}\beta_2^{D-1}(\beta_3\beta_2)^j\beta_3 + \sum_{j=1}^{n}\sum_{k=1}^{D-2} c_{ljk}\beta_2^{D-1}(\beta_3\beta_2)^j\beta_2^k = 0. \tag{2.15}$$

Multiplying by β_2^{D-1} on the right:

$$\sum_{j=1}^{n} b_{lj}\beta_2^{D-1}(\beta_3\beta_2)^j\beta_3\beta_2^{D-1} = \sum_{j=1}^{n}\sum_{k=1}^{D-2} b_{lj}\beta_2^{D-1}(\beta_3\beta_2)^{j+1}\beta_2^{D-2} = 0. \tag{2.16}$$

However, $\{\beta_2^{D-1}(\beta_3\beta_2)^j\beta_2^{D-2} \mid j \geq 0\}$ is linearly independent over $\mathbb{K}$. Thus, $b_{lj} = 0$ for $0 \leq j \leq n$.

Therefore, (2.15) *reduces to:*

$$\sum_{j=0}^{n} a_{lj}\beta_2^{D-1}(\beta_3\beta_2)^j + \sum_{j=1}^{n}\sum_{k=1}^{D-2} c_{ljk}\beta_2^{D-1}(\beta_3\beta_2)^j\beta_2^k = 0. \tag{2.17}$$

Multiplying by β_2^{D-2} on the right:

$$\sum_{j=0}^{n} a_{lj}\beta_2^{D-1}(\beta_3\beta_2)^j\beta_2^{D-2} = 0. \tag{2.18}$$

However, $\{\beta_2^{D-1}(\beta_3\beta_2)^j\beta_2^{D-2} \mid j \geq 0\}$ is linearly independent over $\mathbb{K}$. Thus, $a_{lj} = 0$ for $0 \leq j \leq n$.

Therefore,

$$\sum_{j=1}^{n}\sum_{k=1}^{D-2} c_{ljk}\beta_2^{D-1}(\beta_3\beta_2)^j\beta_2^k = 0. \tag{2.19}$$

However, by Lemma 8, $\left\{\beta_2^{D-1}(\beta_3\beta_2)^j\beta_2^k \mid j \geq 0, 1 \leq k \leq D-2\right\}$ *is linearly independent over* $\mathbb{K}$. *Thus,* $c_{ljk} = 0$ *for* $1 \leq j \leq n, 1 \leq k \leq D-2$.

Thus, we obtain (2.19) *for* $l+1$. *Then* (2.19) *is valid for* $0 \leq l \leq D-1$, *i.e,* $a_{ij} = b_{ij} = c_{ijk} = 0$. $\square$

Lemma 9. *Consider the algebra* $\mathbb{K} \cdot \langle \alpha_2, \alpha_3 \rangle / I$ *with*

$$I = < \alpha_2^3, \alpha_3^2 - \alpha_3, (\alpha_3\alpha_2)^2\alpha_3 - 4\alpha_3\alpha_2^2\alpha_3 >$$

then $\{(\alpha_3\alpha_2)^n \mid n \geq 0\} \cup \{(\alpha_3\alpha_2)^n\alpha_3 \mid n \geq 0\} \cup \{(\alpha_3\alpha_2)^n\alpha_2 \mid n \geq 0\}$ *is a system of generators for* $\mathbb{K} \cdot \langle \alpha_2, \alpha_3 \rangle / I$ *as a free left R-module, with* $R = \mathbb{K} \cdot \langle \alpha_2 \rangle / I$.

Proof. *Define* $M = \bigoplus_{n=0}^{\infty} R \cdot (\alpha_3\alpha_2)^n \oplus \bigoplus_{n=0}^{\infty} R \cdot (\alpha_3\alpha_2)^n\alpha_3 \oplus \bigoplus_{n=1}^{\infty} R \cdot (\alpha_3\alpha_2)^n\alpha_2$. *We have to see that* $\mathbb{K} \cdot \langle \alpha_2, \alpha_3 \rangle / I = M$. *It is enough to show that* M *is invariant under left and right multiplication by* α_2 *and* α_3.

- $\alpha_2 M \subset M$.

  Since $\alpha_2 \in R$.

- $M\alpha_2 \subset M$.

  Since $R(\alpha_3\alpha_2)^n\alpha_2 \subset M$, $[(\alpha_3\alpha_2)^n\alpha_3]\,\alpha_2 = (\alpha_3\alpha_2)^{n+1} \in M$, *for* $n \geq 0$ *and* $[(\alpha_3\alpha_2)^n\alpha_2]\,\alpha_2 = 0 \in M$ *for* $n \geq 1$. *Then* $M\alpha_2 \subset \bigoplus_{n=0}^{\infty} R \cdot (\alpha_3\alpha_2)^{n+1} \oplus \bigoplus_{n=0}^{\infty} R \cdot (\alpha_3\alpha_2)^n\alpha_2 \subset M$.

- $M\alpha_3 \subset M$.

  Note that $[(\alpha_3\alpha_2)^n\alpha_2]\,\alpha_3 = (\alpha_3\alpha_2)^n\alpha_2\alpha_3 = (\alpha_3\alpha_2)^{n-1}\alpha_3\alpha_2^2\alpha_3 = \frac{1}{4}(\alpha_3\alpha_2)^{n-1}(\alpha_3\alpha_2)^2\alpha_3 = \frac{1}{4}(\alpha_3\alpha_2)^{n+1}\alpha_3$ *for every* $n \geq 1$, *then* $M\alpha_3 \subset \bigoplus_{n=0}^{\infty} R \cdot (\alpha_3\alpha_2)^n\alpha_3 \oplus \bigoplus_{n=0}^{\infty} R \cdot (\alpha_3\alpha_2)^n\alpha_3 \oplus \bigoplus_{n=1}^{\infty} R \cdot (\alpha_3\alpha_2)^n\alpha_2\alpha_3 \subset \bigoplus_{n=0}^{\infty} R \cdot (\alpha_3\alpha_2)^n\alpha_3 \oplus \bigoplus_{n=1}^{\infty} R \cdot (\alpha_3\alpha_2)^{n+1}\alpha_3 \subset M$.

- $\alpha_3 M \subset M$.

  Note that $\alpha_3\alpha_2^2(\alpha_3\alpha_2)^n = (\alpha_3\alpha_2^2\alpha_3)\alpha_2(\alpha_3\alpha_2)^{n-1} = \frac{1}{4}\left[(\alpha_3\alpha_2)^2\alpha_3\right]\alpha_2(\alpha_3\alpha_2)^{n-1} = \frac{1}{4}(\alpha_3\alpha_2)^2\alpha_3\alpha_2(\alpha_3\alpha_2)^{n-1} = \frac{1}{4}(\alpha_3\alpha_2)^{n+2} \in M$ *for* $n \geq 1$ *and* $\alpha_3\alpha_2^2 = (\alpha_3\alpha_2)\alpha_2 \in M$. *Then* $\alpha_3\alpha_2^2(\alpha_3\alpha_2)^n \in M$ *for every* $n \geq 0$.

  On the other hand $\alpha_3\alpha_2(\alpha_3\alpha_2)^n = (\alpha_3\alpha_2)^{n+1} \in M$, $\alpha_3(\alpha_3\alpha_2)^n = (\alpha_3\alpha_2)^n \in M$ *for all* $n \geq 0$.

  Furthermore, $\alpha_3(\alpha_3\alpha_2)^n\alpha_3 = (\alpha_3\alpha_2)^n\alpha_3 \in M$, $\alpha_3(\alpha_3\alpha_2)^n\alpha_2 = (\alpha_3\alpha_2)^n\alpha_2 \in M$, *for all* $n \geq 0$. $(\alpha_3\alpha_2)(\alpha_3\alpha_2)^n\alpha_3 = (\alpha_3\alpha_2)^{n+1}\alpha_3 \in M$, $(\alpha_3\alpha_2)(\alpha_3\alpha_2)^n\alpha_2 = (\alpha_3\alpha_2)^{n+1}\alpha_2 \in M$ *for all* $n \geq 0$.

On the other hand $(\alpha_3\alpha_2^2)(\alpha_3\alpha_2)^n\alpha_3 = \frac{1}{4}(\alpha_3\alpha_2)^{n+2}\alpha_3 \in M$, $(\alpha_3\alpha_2^2)(\alpha_3\alpha_2)^n\alpha_2 = \frac{1}{4}(\alpha_3\alpha_2)^{n+2}\alpha_2 \in M$ *for all* $n \geq 0$. *In particular* $\alpha_3 M \subset M$. □

Finally, we conclude with the proof of the nice presentation. Define

$$f\colon \mathbb{C}\cdot\langle\alpha_2,\alpha_3\rangle/I \longrightarrow \mathbb{A},$$

$$f(\overline{\alpha_j}) = \beta_j$$

the previous lemma guarantees the existence of a subalgebra $R = \mathbb{C}\cdot\langle\alpha_2\rangle/I$ *and a system of generators* $\{(\alpha_3\alpha_2)^n \mid n \geq 0\} \cup \{(\alpha_3\alpha_2)^n\alpha_3 \mid n \geq 0\} \cup \{(\alpha_3\alpha_2)^n\alpha_2 \mid n \geq 0\}$ *for* $\mathbb{C}\cdot\langle\alpha_2,\alpha_3\rangle/I$ *as a free left R-module. Furthermore* $f|_R\colon R \longrightarrow A$ *is a monomorphism.*

Since $\beta_2^2(\beta_3\beta_2)^n\beta_2 = 2^{n-1}e_{13}x^{n+1}$ *for* $n \geq 1$ *applying the Proposition 4 with* $D = 3$ *we obtain*

$$\left\{\beta_2^i(\beta_3\beta_2)^j\beta_3 \mid 0 \leq i \leq D-1, j \geq 0\right\} \cup \left\{\beta_2^i(\beta_3\beta_2)^j\beta_2 \mid 0 \leq i \leq D-1, j \geq 1\right\}$$

$$\cup \left\{\beta_2^i(\beta_3\beta_2)^j \mid 0 \leq i \leq D-1, j \geq 0\right\}$$

is a linearly independent set over $\mathbb{C}$.

Thus, we are under the hypothesis of Theorem 14 and f is an isomorphism. □

2.3.1 An Example linked to the Spin Calogero Systems

This example is linked to the spin Calogero systems whose relation with bispectrality can be found in [2]. We consider the case when both "eigenvalues" F and θ are matrix valued. Let

$$\psi(x,z) = \frac{e^{xz}}{(x-2)xz}\begin{pmatrix} \frac{x^3z^2-2x^2z^2-2x^2z+3xz+2x-2}{xz} & \frac{1}{x} \\ \frac{xz-2}{z} & x^2z-2xz-x+1 \end{pmatrix}$$

and

$$\mathcal{L} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.\partial_x^2 + \begin{pmatrix} 0 & \frac{1}{(x-2)x^2} \\ -\frac{1}{x-2} & 0 \end{pmatrix}.\partial_x + \begin{pmatrix} -\frac{1}{x^2(x-2)^2} & \frac{x-1}{x^3(x-2)^2} \\ \frac{2x-1}{x(x-2)^2} & -\frac{2x^2-4x+3}{x^2(x-2)^2} \end{pmatrix},$$

then $\mathcal{L}\psi = \psi F$ with

$$F(z) = \begin{pmatrix} 0 & 0 \\ 0 & z^2 \end{pmatrix}.$$

On the other hand, it is easy to check that $\psi\mathcal{B} = \theta\psi$ for

$$\mathcal{B} = \partial_z^3 . \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \partial_z^2 . \begin{pmatrix} 0 & 0 \\ -\frac{2z+1}{z} & 0 \end{pmatrix} + \partial_z . \begin{pmatrix} 1 & 0 \\ \frac{2(z-1)}{z^2} & 1 \end{pmatrix} + \begin{pmatrix} -z^{-1} & 0 \\ 6z^{-3} & z^{-1} \end{pmatrix}$$

and

$$\theta(x) = \begin{pmatrix} x & 0 \\ x^2(x-2) & x \end{pmatrix}.$$

The following theorem characterizes the algebra $\mathbb{A}$ of all polynomial F such that there exist $\mathcal{L} = \mathcal{L}(x, \partial_x)$ with $\mathcal{L}\psi = \psi F$.

Theorem 15. *Let Γ be the sub-algebra of $M_2(\mathbb{C})[z]$ of the form*

$$\begin{pmatrix} a & 0 \\ b-a & b \end{pmatrix} + \begin{pmatrix} c & c \\ a-b-c & -c \end{pmatrix} z + \begin{pmatrix} a-b-c & c+a-b \\ d & e \end{pmatrix} \frac{z^2}{2} + z^3 p(z),$$

where $p \in M_2(\mathbb{C})[z]$ and all the variables a, b, c, d, e are arbitrary. Then $\Gamma = \mathbb{A}$.

Furthermore, we have the presentation $\mathbb{A} = \mathbb{C} \cdot \langle \theta_1, \theta_3, \theta_4, \theta_5 \mid I = 0 \rangle$ with

$$I = \langle \theta_1^2 - \theta_1, \theta_4^2, \theta_4\theta_5, \theta_4\theta_1 + \theta_4\theta_3 - 2\theta_4 - \theta_5\theta_4 - \theta_5^2, \theta_3^2 - \theta_3 + \theta_5 - 3\theta_3\theta_4\theta_3\theta_5 - \theta_1\theta_4 - \theta_5\theta_1,$$

$$\theta_3\theta_1 - \theta_1 - \theta_4 - \frac{1}{2}\theta_4\theta_1 + \frac{1}{2}\theta_4\theta_3 + \theta_5\theta_1 - \frac{1}{2}\theta_5\theta_4 + \frac{1}{2}\theta_5^2 + \theta_3\theta_4 - \theta_1\theta_5 - \theta_3\theta_5,$$

$$\theta_1\theta_3 - \theta_3 + \theta_4 + \theta_5 - \frac{3}{2}\theta_4\theta_1 + \frac{3}{2}\theta_4\theta_3 - 2\theta_5\theta_1 - \frac{3}{2}\theta_5\theta_4 + \frac{3}{2}\theta_5^2 + 3\theta_3\theta_4 + \theta_3\theta_5,$$

$$\theta_5\theta_3 - \theta_4\theta_1 + \theta_4\theta_3 - \theta_5\theta_1 - \theta_5\theta_4 + \theta_5^2, \theta_5\theta_1\theta_5 - \theta_5^2\theta_1 - \theta_5\theta_4, \theta_5\theta_4\theta_1 - \theta_5^3 + \theta_5\theta_1\theta_4 + \theta_5^2\theta_1,$$

$$\theta_4\theta_1\theta_5 + \theta_4\theta_3\theta_5 - \theta_3^3, \theta_5\theta_3\theta_4 + \theta_5\theta_1\theta_4\rangle$$

Proof. *We shall break the proof in different steps.*

Step 1: The set Γ is an algebra. Clearly if $F_1, F_2 \in \Gamma$, then $F_1 + F_2 \in \Gamma$ and $\alpha F_1 \in \Gamma$ if $\alpha \in \mathbb{C}$,

$$F_1(z) = \begin{pmatrix} a_1 & 0 \\ b_1 - a_1 & b_1 \end{pmatrix} + \begin{pmatrix} c_1 & c_1 \\ a_1 - b_1 - c_1 & -c_1 \end{pmatrix} z + \begin{pmatrix} a_1 - b_1 - c_1 & c_1 + a_1 - b_1 \\ d_1 & e_1 \end{pmatrix} \frac{z^2}{2} + z^3 p_1(z),$$

$$F_2(z) = \begin{pmatrix} a_2 & 0 \\ b_2 - a_2 & b_2 \end{pmatrix} + \begin{pmatrix} c_2 & c_2 \\ a_2 - b_2 - c_2 & -c_2 \end{pmatrix} z + \begin{pmatrix} a_2 - b_2 - c_2 & c_2 + a_2 - b_2 \\ d_2 & e_2 \end{pmatrix} \frac{z^2}{2} + z^3 p_2(z).$$

Thus,

$$F_1(z)F_2(z) = \begin{pmatrix} a_1a_2 & 0 \\ b_1b_2 - a_1a_2 & b_1b_2 \end{pmatrix} + \begin{pmatrix} a_1c_2 + b_2c_1 & a_1c_2 + b_2c_1 \\ a_1a_2 - b_1b_2 - a_1c_2 - b_2c_1 & -(a_1c_2 + b_2c_1) \end{pmatrix} z +$$

$$\begin{pmatrix} a_1a_2 - b_1b_2 - a_1c_2 - b_2c_1 & a_1c_2 + b_2c_1 + a_1a_2 - b_1b_2 \\ b_2e_1 - a_2e_1 + b_1d_2 + a_2d_1 - 3b_1c_2 + 3a_1c_2 + 2b_2c_1 - 2a_2c_1 - b_1b_2 + a_1b_2 + a_2b_1 - a_1a_2 & b_1e_2 + b_2e_1 - b_1c_2 + a_1c_2 - b_1b_2 + a_1b_2 + a_2b_1 - a_1a_2 \end{pmatrix} \frac{z^2}{2} + z^3 p(z).$$

For some polynomial $p \in M_2(\mathbb{C})[z]$. *In particular* $F_1F_2 \in \Gamma$. *Since* $M_2(\mathbb{C})[z]$ *is an algebra and* Γ *is closed for operations induced by* $M_2(\mathbb{C})[z]$ *we have that* Γ *is an algebra.*

Step 2: There exists a finite dimensional vector space E such that $\Gamma = E \oplus z^3 M_2(\mathbb{C})[z]$.

Consider $F \in \Gamma$, *then*

$$F(z) = \begin{pmatrix} a & 0 \\ b - a & b \end{pmatrix} + \begin{pmatrix} c & c \\ a - b - c & -c \end{pmatrix} z + \begin{pmatrix} a - b - c & c + a - b \\ d & e \end{pmatrix} \frac{z^2}{2} + z^3 p(z)$$

$$= a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 + e\alpha_5 + z^3 p(z),$$

with $\alpha_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \frac{z^2}{2}$, $\alpha_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} z + \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \frac{z^2}{2}$, $\alpha_3 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} z + \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \frac{z^2}{2}$, $\alpha_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{z^2}{2}$, $\alpha_5 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{z^2}{2}$. *If* $E = span\ \{\alpha_i | 1 \leq i \leq 5\}$ *we obtain this step.*

Step 3: The algebra Γ *is generated by E,* $\mathbb{C} \cdot \langle E \rangle = \Gamma$.

Since $\alpha_1 + \alpha_2 = I$ *we have that* $E = span\ \{I, \alpha_1, \alpha_3, \alpha_4, \alpha_5\}$. *On the other hand,*

$$\alpha_1^2 = \frac{4 + 2z^2 + 2z^3 + z^4}{4} e_{11} + \frac{2z^2 + z^4}{4} e_{12} + \frac{-2 + 2z - z^2 + z^3}{2} e_{21} + \frac{z^3 - z^2}{2} e_{22}$$

$$\alpha_1\alpha_3 = -\frac{-4z+2z^2+z^4}{4}e_{11} + \frac{4z+2z^2+z^4}{4}e_{12} - \frac{2z-3z^2+z^3}{2}e_{21} + \frac{-2z+z^2+z^3}{2}e_{22}$$

$$\alpha_1\alpha_4 = \frac{e_{11}z^4}{4},\ \alpha_1\alpha_5 = \frac{e_{12}z^4}{4},\ \alpha_3\alpha_1 = -\frac{z^4-4z^3}{4}e_{11} - \frac{z^4-2z^3}{4}e_{12} - \frac{2z^2+z^3}{2}e_{21} - \frac{z^3}{2}e_{22}$$

$$\alpha_3^2 = \frac{z^4-6z^3}{4}e_{11} - \frac{2z^3+z^4}{4}e_{12} + \frac{z^3}{2}e_{21} - \frac{z^3}{2}e_{22}$$

$$\alpha_3\alpha_4 = \frac{2z^3+z^4}{4}e_{11} - \frac{z^3}{2}e_{21},\ \alpha_3\alpha_5 = \frac{2z^3+z^4}{4}e_{12} - \frac{z^3}{2}e_{22}$$

$$\alpha_4\alpha_1 = \frac{2z^2+z^4}{4}e_{21} + \frac{z^4}{4}e_{22},\ \alpha_4\alpha_3 = -\frac{z^4-2z^3}{4}e_{21} + \frac{2z^3+z^4}{4}e_{22}$$

$$\alpha_4^2 = 0,\ \alpha_4\alpha_5 = 0,\ \alpha_5\alpha_1 = \frac{z^3-z^2}{2}e_{21},\ \alpha_5\alpha_3 = -\frac{z^3}{2}e_{21} - \frac{z^3}{2}e_{22},\ \alpha_5\alpha_4 = \frac{e_{21}z^4}{4},\ \alpha_5^2 = \frac{e_{22}z^4}{4}.$$

Therefore, $e_{ij}z^3 \in \mathbb{C}\cdot\langle E\rangle$ *and* $e_{ij}z^4 \in \mathbb{C}\cdot\langle E\rangle$ *for* $1 \le i,j \le 2$ *and using* α_4, α_5 *we obtain that* $e_{ij}z^k \in \mathbb{C}\cdot\langle E\rangle$ *for* $1 \le i,j \le 2$ *and* $k \ge 3$*. In particular* $\mathbb{C}\cdot\langle E\rangle = \Gamma$.

Step 4: The inclusion $\mathbb{A} \cap \oplus_{k=0}^{2} M_2(\mathbb{C})[z]_k \subset E$.

Let $F \in \mathbb{A} \cap \oplus_{k=0}^{2} M_2(\mathbb{C})[z]_k$ *then there exists* $\mathcal{L} = \mathcal{L}(x, \partial_x)$ *such that* $\mathcal{L}\psi = \psi F$*. We write*

$$F(z) = \begin{pmatrix} s_0^{11} & s_0^{12} \\ s_0^{21} & s_0^{22} \end{pmatrix} + \begin{pmatrix} s_1^{11} & s_1^{12} \\ s_1^{21} & s_1^{22} \end{pmatrix} z + \begin{pmatrix} s_2^{11} & s_2^{12} \\ s_2^{21} & s_2^{22} \end{pmatrix} z^2.$$

After a computation we obtain that

$$\mathcal{L} = \begin{pmatrix} \frac{s_0^{11}x^5+(-s_1^{12}-4s_0^{11})x^4+(3s_2^{12}-4s_2^{11}+2s_1^{12}+4s_0^{11})x^3+(-3s_2^{12}+12s_2^{11}+s_1^{21})x^2+(-s_2^{22}+2s_2^{21}-11s_2^{11}-2s_1^{21})x-3s_2^{21}}{4x^3-4x^4+x^5} & \frac{s_0^{12}x^6+(-s_1^{12}-4s_0^{12})x^5+(-s_2^{12}+4s_1^{12}+4s_0^{12})x^4+(s_2^{12}+s_1^{22}-4s_1^{12}-s_1^{11})x^3+(s_2^{22}-s_2^{12}+5s_2^{11}-2s_1^{22}+2s_1^{11})x^2+(-s_2^{22}-7s_2^{11})x-s_2^{21}}{4x^4-4x^5+x^6} \\ \frac{s_0^{21}x^4+(-s_1^{22}+s_1^{21}+s_1^{11}-4s_0^{21})x^3+(2s_2^{22}-4s_2^{21}-s_2^{12}+2s_1^{22}-4s_1^{21}-2s_1^{11}+4s_0^{21})x^2+(-s_2^{22}+11s_2^{21}+s_2^{11}+4s_1^{21})x-9s_2^{21}}{4x^2-4x^3+x^4} & \frac{s_0^{22}x^5+(s_1^{12}-4s_0^{22})x^4+(-2s_2^{22}-s_2^{12}-2s_1^{12}+4s_0^{22})*x^3+(4s_2^{22}+3s_2^{12}-s_1^{21})x^2+(-3s_2^{22}+4s_2^{21}-s_2^{11}+2s_1^{21})x-5s_2^{21}}{4x^3-4x^4+x^5} \end{pmatrix} \tag{2.20}$$

$$+ \begin{pmatrix} \frac{s_1^{11}x^3+(-s_2^{12}-2s_1^{11})x^2+s_2^{21}}{x^3-2x^2} & \frac{s_1^{12}x^3+(-s_2^{12}-2s_1^{12})x^2+2s_2^{12}x+s_2^{22}-s_2^{11}}{x^3-2x^2} \\ \frac{s_1^{21}x^2+(-s_2^{22}+s_2^{21}+s_2^{11}-2s_1^{21})x-2s_2^{21}}{x^2-2x} & \frac{s_1^{22}x^3+(s_2^{12}-2s_1^{22})x^2-s_2^{21}}{x^3-2x^2} \end{pmatrix} \partial_x + \begin{pmatrix} s_2^{11} & s_2^{12} \\ s_2^{21} & s_2^{22} \end{pmatrix} \partial_x^2$$

and $s_0^{21} = s_0^{22} - s_0^{11}, s_0^{12} = 0, s_1^{12} = s_1^{11}, s_1^{22} = -s_1^{11}, s_1^{21} = s_0^{11} - s_0^{22} - s_1^{11}, s_2^{11} = \frac{s_0^{11}-s_0^{22}-s_1^{11}}{2}, s_2^{12} = \frac{s_1^{12}+s_0^{11}-s_0^{22}}{2}$*, then*

$F \in E$.

Step 5: The inclusion $E \subset \mathbb{A} \cap \oplus_{k=0}^{2} M_2(\mathbb{C})[z]_k$

By the previous step we have Equation 2.20 valid for every $F \in E$ and $(\mathcal{L}\psi)(x,z) = \psi(x,z)F(z)$, then $F \in \mathbb{A} \cap \oplus_{k=0}^{2} M_2(\mathbb{C})[z]_k$.

Furthermore, we are going to check the presentation using Theorem 14.

Lemma 10. *Consider the $\mathbb{K}$-algebra $\mathbb{K} \cdot \langle \theta_1, \theta_3, \theta_4, \theta_5 \rangle / I$ with $\mathbb{K}$ a central field of characteristic 0 and*

$$I = \langle \theta_1^2 - \theta_1, \theta_4^2, \theta_4\theta_5, \theta_4\theta_1 + \theta_4\theta_3 - 2\theta_4 - \theta_5\theta_4 - \theta_5^2, \theta_3^2 - \theta_3 + \theta_5 - 3\theta_3\theta_4 - \theta_3\theta_5 - \theta_1\theta_4 - \theta_5\theta_1,$$

$$\theta_3\theta_1 - \theta_1 - \theta_4 - \frac{1}{2}\theta_4\theta_1 + \frac{1}{2}\theta_4\theta_3 + \theta_5\theta_1 - \frac{1}{2}\theta_5\theta_4 + \frac{1}{2}\theta_5^2 + \theta_3\theta_4 - \theta_1\theta_5 - \theta_3\theta_5,$$

$$\theta_1\theta_3 - \theta_3 + \theta_4 + \theta_5 - \frac{3}{2}\theta_4\theta_1 + \frac{3}{2}\theta_4\theta_3 - 2\theta_5\theta_1 - \frac{3}{2}\theta_5\theta_4 + \frac{3}{2}\theta_5^2 + 3\theta_3\theta_4 + \theta_3\theta_5,$$

$$\theta_5\theta_3 - \theta_4\theta_1 + \theta_4\theta_3 - \theta_5\theta_1 - \theta_5\theta_4 + \theta_5^2, \theta_5\theta_1\theta_5 - \theta_5^2\theta_1 - \theta_5\theta_4, \theta_5\theta_4\theta_1 - \theta_5^3 + \theta_5\theta_1\theta_4 + \theta_5^2\theta_1,$$

$$\theta_4\theta_1\theta_5 + \theta_5^2\theta_1 + \theta_5\theta_4 - \theta_5^3, \theta_5\theta_3\theta_4 + \theta_5\theta_1\theta_4, \theta_3\theta_5\theta_1 - \theta_1\theta_5 - \theta_3\theta_5 + \theta_3\theta_4 + \theta_5^2,$$

$$\theta_4\theta_1\theta_5 + \theta_4\theta_3\theta_5 - \theta_5^3, \theta_4\theta_3\theta_5 - \theta_5\theta_1\theta_5, \theta_5\theta_3\theta_4 + \theta_5\theta_1\theta_4,$$

$$\theta_1\theta_5\theta_1 + \theta_1\theta_5 + \theta_3\theta_5 - \theta_5^2 + \theta_1\theta_4 - \theta_5^2\theta_1 - \theta_5\theta_4 - \theta_3\theta_5^2,$$

$$\theta_1\theta_5\theta_3 - 2\theta_3\theta_4 - \theta_1\theta_5 - \theta_3\theta_5 + \theta_5^2 + 3\theta_5\theta_4 - \theta_1\theta_4 + \theta_5^2\theta_1 + \theta_3\theta_5^2 + 2\theta_5\theta_1\theta_4 + 2\theta_3\theta_5\theta_4,$$

$$\theta_3\theta_4\theta_1 + \theta_1\theta_4 + \theta_1\theta_5 + \theta_3\theta_5 - \theta_5\theta_4 - \theta_3\theta_5^2 - \theta_5^2\rangle.$$

Then, $\{\theta_4\theta_1, \theta_3, \theta_1\} \cup \{\theta_5^n \mid n \geq 0\} \cup \{\theta_5^n\theta_4 \mid n \geq 0\} \cup \{\theta_5^n\theta_1\theta_4 \mid n \geq 0\} \cup \{\theta_5^n\theta_1 \mid n \geq 1\} \cup \{\theta_3\theta_5^n \mid n \geq 1\} \cup \{\theta_1\theta_5^n \mid n \geq 1\} \cup \{\theta_3\theta_5^n\theta_4 \mid n \geq 0\} \cup \{\theta_1\theta_5^n\theta_4 \mid n \geq 1\}$ is a system of generators for $\mathbb{K} \cdot \langle \theta_1, \theta_3, \theta_4, \theta_5 \rangle / I$ as a free $\mathbb{K}$-vector space.

Proof. *Define $M = \mathbb{K} \cdot \theta_1 \oplus \mathbb{K} \cdot \theta_3 \oplus \mathbb{K} \cdot \theta_4\theta_1 \oplus \bigoplus_{n=0}^{\infty} \mathbb{K} \cdot \theta_5^n \oplus \bigoplus_{n=0}^{\infty} \mathbb{K} \cdot \theta_5^n\theta_4 \oplus \bigoplus_{n=0}^{\infty} \mathbb{K} \cdot \theta_5^n\theta_1\theta_4 \oplus \bigoplus_{n=1}^{\infty} \mathbb{K} \cdot \theta_5^n\theta_1 \oplus \bigoplus_{n=1}^{\infty} \mathbb{K} \cdot \theta_3\theta_5^n \oplus \bigoplus_{n=1}^{\infty} \mathbb{K} \cdot \theta_1\theta_5^n \oplus \bigoplus_{n=1}^{\infty} \mathbb{K} \cdot \theta_3\theta_5^n\theta_4 \oplus \bigoplus_{n=1}^{\infty} \mathbb{K} \cdot \theta_1\theta_5^n\theta_4$. We have to see that $\mathbb{K} \cdot \langle \theta_1, \theta_3, \theta_4, \theta_5 \rangle / I = M$. It is enough to show that M is invariant under left and right multiplication by $\theta_1, \theta_3, \theta_4$ and θ_5.*

- *$M\theta_5 \subset M$.*

  Note that $\theta_3\theta_4\theta_5 = 0 \in M$, $\theta_4\theta_1\theta_5 = -\theta_5^2\theta_1 - \theta_5\theta_4 + \theta_5^3 \in M$. On the other hand $\theta_1\theta_4\theta_5 \in M$,

$\theta_3\theta_5 \in M$, $\theta_1\theta_5 \in M$, $\theta_5^n \in M$ for every $n \geq 1$, $\theta_5^n\theta_4\theta_5 = 0 \in M$, for every $n \geq 0$, $\theta_5^n\theta_1\theta_4\theta_5 = 0 \in M$, for every $n \geq 1$.

Furthermore, $\theta_5^n\theta_1\theta_5 = \theta_5^{n+1}\theta_1 + \theta_5^n\theta_4 \in M$ for every $n \geq 1$, $(\theta_3\theta_5^n)\theta_5 = \theta_3\theta_5^{n+1} \in M$ for every $n \geq 1$, $(\theta_1\theta_5^n)\theta_5 = \theta_1\theta_5^{n+1} \in M$ for every $n \geq 1$, $(\theta_3\theta_5^n\theta_4)\theta_5 = 0 \in M$ for every $n \geq 1$, $(\theta_1\theta_5^n\theta_4)\theta_5 = 0 \in M$ for every $n \geq 1$. In particular $M\theta_5 \subset M$.

- $M\theta_4 \subset M$.

  Note that $\theta_4 \in M$, $(\theta_3\theta_4)\theta_4 = 0 \in M$, $\theta_4\theta_1 = \theta_5\theta_3 + \theta_4\theta_3 - 0\theta_5\theta_1 ? \theta_5\theta_4 + \theta_5^2$ and $\theta_4\theta_3 = 2\theta_4 + \theta_5\theta_4 + \theta_5^2 - \theta_4\theta_1$ imply $\theta_4\theta_1 = \frac{1}{2}\theta_5\theta_3 + \theta_4 + \theta_5^2 - \frac{1}{2}\theta_5\theta_1$, hence $\theta_4\theta_1\theta_4 = \theta_5^2\theta_4 - \theta_5\theta_1\theta_4 \in M$.

  On the other hand, $(\theta_1\theta_4)\theta_4 = \theta_1\theta_4^2 = 0 \in M$, $\theta_3\theta_4 \in M$, $\theta_1\theta_4 \in M$, $\theta_5^n\theta_4 \in M$ for every $n \geq 0$, $(\theta_5^n\theta_4)\theta_4 = 0 \in M$ for every $n \geq 0$, $(\theta_5^n\theta_1\theta_4)\theta_4 = 0 \in M$ for every $n \geq 1$, $(\theta_5^n\theta_1)\theta_4 = \theta_5^n\theta_1\theta_4 \in M$ for every $n \geq 1$, $(\theta_3\theta_5^n)\theta_4 = \theta_3\theta_5^n\theta_4 \in M$ for every $n \geq 1$, $(\theta_1\theta_5^n)\theta_4 = \theta_1\theta_5^n\theta_4 \in M$ for every $n \geq 1$, $(\theta_3\theta_5^n\theta_4)\theta_4 = 0 \in M$ for every $n \geq 1$, $(\theta_1\theta_5^n\theta_4)\theta_4 = 0 \in M$ for every $n \geq 1$. In particular $M\theta_4 \subset M$.

- $\theta_1 M \subset M$.

  Note that $\theta_1 \in M$. Since $\theta_1\theta_3 = \theta_3 - \theta_4 - \theta_5 + \frac{3}{2}\theta_4\theta_1 - \frac{3}{2}\theta_4\theta_3 + 2\theta_5\theta_1 + \frac{3}{2}\theta_5\theta_4 - \frac{3}{2}\theta_5^2 - 3\theta_3\theta_4 - \theta_3\theta_5$ multiplying by θ_4 on the right we obtain $\theta_1\theta_3\theta_4 = \theta_3\theta_4 - \theta_5\theta_4 - \theta_5\theta_1\theta_4 - \theta_3\theta_5\theta_4 \in M$.

  On the other hand $\theta_1\theta_4 - \theta_3\theta_4 - 2\theta_1\theta_4 - \theta_1\theta_5 - \theta_3\theta_5 + 2\theta_5\theta_4 + \theta_5\theta_1\theta_4 + \theta_3\theta_5\theta_4 + \theta_5^2\theta_1 + \theta_3\theta_5^2 - \theta_1\theta_5^2 + \theta_5^2 = 0$ implies $\theta_1\theta_4\theta_1 \in M$. Moreover, $\theta_1\theta_5\theta_1 + \theta_1\theta_4 + \theta_1\theta_5 + \theta_3\theta_5 - \theta_5^2\theta_1 - \theta_5\theta_4 - \theta_3\theta_5^2 - \theta_5^2 = 0$ implies $\theta_1\theta_5\theta_1 \in M$.

  However, multiplying $\theta_4\theta_1 + \theta_4\theta_3 - 2\theta_4 - \theta_5\theta_4 - \theta_5^2 = 0$ by θ_1 on the left we have $\theta_1\theta_4\theta_1 + \theta_1\theta_4\theta_3 - 2\theta_1\theta_4 - \theta_1\theta_5\theta_4 - \theta_1\theta_5^2 = 0$. Hence, $\theta_1\theta_4\theta_3 = \theta_5^2 - \theta_3\theta_4 - \theta_1\theta_5 - \theta_3\theta_5 + 2\theta_5\theta_4 + \theta_5\theta_1\theta_4 + \theta_3\theta_5\theta_4 + \theta_5^2\theta_1 + \theta_3\theta_5^2 + \theta_1\theta_5\theta_4 \in M$.

  Moreover, $\theta_1(\theta_1\theta_4) = \theta_1\theta_4 \in M$ and $\theta_1\theta_3 = \theta_3 - \theta_4 - \theta_5 + \frac{3}{2}\theta_4\theta_1 - \frac{3}{2}\theta_4\theta_3 + 2\theta_5\theta_1 + \frac{3}{2}\theta_5\theta_4 - \frac{3}{2}\theta_5^2 - 3\theta_3\theta_4 - \theta_3\theta_5 \in M$. On the other hand, $\theta_1^2 = \theta_1 \in M$, $\theta_5^n \in M$ for every $n \geq 1$, $\theta_1\theta_5^n\theta_4 \in M$ for every $n \geq 0$. Note that $\theta_1\theta_5^n\theta_1 = -\theta_1\theta_5^n - \theta_3\theta_5^n - \theta_1\theta_5^{n-1}\theta_4 + \theta_5^{n+1}\theta_1 + \theta_5^n\theta_4 + \theta_3\theta_5^{n+1} + \theta_5^{n+1} \in M$ for every $n \geq 2$ and $\theta_1\theta_5\theta_1 \in M$ imply $\theta_1\theta_5^n\theta_1 \in M$ for every $n \geq 1$.

  Since $\theta_1\theta_3 \in M$ we have $\theta_1\theta_3\theta_5^n \in M\theta_5^n \subset M$ for every $n \geq 1$. Furthermore, $\theta_1(\theta_1\theta_5^n) = \theta_1\theta_5^n \in M$ for every $n \geq 1$ and $\theta_1(\theta_3\theta_5^n\theta_4) = (\theta_1\theta_3\theta_5^n)\theta_4 \in M\theta_4 \subset M$ for every $n \geq 1$. However, $\theta_1\theta_3\theta_4 \in M$

then $\theta_1(\theta_3\theta_5^n\theta_4) \in M$ for every $n \geq 0$. Note that $\theta_1(\theta_1\theta_5^n\theta_4) = \theta_1\theta_5^n\theta_4 \in M$ for every $n \geq 1$. Thus $\theta_1 M \subset M$.

- $\theta_4 M \subset M$.

  Note that $\theta_4(\theta_4\theta_1) = 0 \in M$. Furthermore, $\theta_4\theta_3 = 2\theta_4 + \theta_5\theta_4 + \theta_5^2 - \theta_4\theta_1 \in M$ and $\theta_4\theta_1 \in M$, $\theta_4\theta_5^n = 0 \in M$ for every $n \geq 1$ and $\theta_4 \in M$. Moreover, $\theta_4\theta_5^n\theta_4 = 0 \in M$ for every $n \geq 0$, $\theta_4(\theta_5^n\theta_1\theta_4) = 0 \in M$ for every $n \geq 1$. Since $\theta_4(\theta_1\theta_4) = (\theta_4\theta_1)\theta_4 \in M\theta_4 \subset M$ we have $\theta_4\theta_5^n\theta_1\theta_4 \in M$ for every $n \geq 0$.

  On the other hand, $\theta_4(\theta_5\theta_1) = 0 \in M$ for every $n \geq 1$. Since $\theta_4\theta_3 \in M$ we have $\theta_4(\theta_3\theta_5^n) \in M\theta_5^n \subset M$ for every $n \geq 1$. Since $\theta_4\theta_1$ we have $\theta_4(\theta_1\theta_5^n) \in M\theta_5^n \subset M$ for every $n \geq 1$. Using that $\theta_4\theta_3\theta_5^n \in M$ we have $\theta_4(\theta_3\theta_5^n\theta_4) \in M\theta_4 \subset M$. Since $\theta_4\theta_1\theta_5^n \in M$ we have $\theta_4(\theta_1\theta_5^n\theta_4) = (\theta_4\theta_1\theta_5^n)\theta_4 \in M\theta_4 \subset M$. Thus, $\theta_4 M \subset M$.

- $M\theta_1 \subset M$.

  Note that $(\theta_4\theta_1)\theta_1 = \theta_4\theta_1 \in M$. Since $\theta_3\theta_1 = \theta_1 + \theta_4 + \frac{1}{2}\theta_4\theta_1 - \frac{1}{2}\theta_4\theta_3 - \theta_5\theta_1 + \frac{1}{2}\theta_5\theta_4 - \frac{1}{2}\theta_5^2 - \theta_3\theta_4 + \theta_1\theta_5 + \theta_3\theta_5$ and $\theta_4\theta_3 = 2\theta_4 + \theta_5\theta_4 + \theta_5^2 - \theta_4\theta_1$ we have $\theta_3\theta_1 = \theta_1 + \theta_4\theta_1 - \theta_5^2 - \theta_5\theta_1 + \theta_1\theta_5 + \theta_3\theta_5 - \theta_3\theta_4 \in M$.

  On the other hand, $\theta_1^2 = \theta_1 \in M$, $\theta_5^n\theta_1 \in M$ for every $n \geq 0$. Since $\theta_5^n\theta_4\theta_1 = \theta_5^{n+2} - \theta_5^n\theta_1\theta_4 - \theta_5^{n+1}\theta_1 \in M$ for every $n \geq 1$ and $\theta_4\theta_1 \in M$ we have $\theta_5^n\theta_4\theta_1 \in M$ for every $n \geq 0$. Since $\theta_5^n\theta_4\theta_1 = \theta_5^{n+2} - \theta_5^n\theta_1\theta_4 - \theta_5^{n+1}\theta_1$, for every $n \geq 1$, multiplying this equation by θ_1 on the right $\theta_5^n\theta_4\theta_1 = \theta_5^{n+2}\theta_1 - \theta_5^n\theta_1\theta_4\theta_1 - \theta_5^{n+1}\theta_1$, for every $n \geq 1$. Then, $(\theta_5^n\theta_1\theta_4)\theta_1 = \theta_5^{n+2}\theta_1 - \theta_5^n\theta_4\theta_1 - \theta_5^{n+1}\theta_1 \in M$. Since $(\theta_5^n\theta_1)\theta_1 = \theta_5^n\theta_1 \in M$ for every $n \geq 1$, $\theta_3\theta_5^n\theta_1 = \theta_1\theta_5^n + \theta_3\theta_5^n - \theta_3\theta_5^{n-1}\theta_4 - \theta_5^{n+1} \in M$ for every $n \geq 1$.

  Furthermore, $(\theta_1\theta_5^n)\theta_1 = \theta_1(\theta_5^n\theta_1) \in \theta_1 M \subset M$ for every $n \geq 1$ and $(\theta_1\theta_5^n\theta_4)\theta_1 = \theta_1(\theta_5^n\theta_4\theta_1) \in \theta_1 M \subset M$ for every $n \geq 1$. Since $\theta_3\theta_5^n\theta_4\theta_1 = -\theta_1\theta_5^n\theta_4 - \theta_1\theta_5^{n+1} - \theta_3\theta_5^{n+1} + \theta_5^{n+1}\theta_4 + \theta_3\theta_5^{n+2} + \theta_5^{n+2} \in M$ for every $n \geq 0$ we have $M\theta_1 \subset M$.

- $\theta_5 M \subset M$.

  Note that $\theta_5(\theta_4\theta_1) = (\theta_5\theta_4)\theta_1 \in M\theta_1 \subset M$ since $\theta_5\theta_4 \in M$. On the other hand, $\theta_5\theta_3 = 2\theta_4\theta_1 - 2\theta_4 - 2\theta_5^2 + \theta_5\theta_1 \in M$. Moreover, $\theta_5\theta_1 \in M$, $\theta_5(\theta_5^n) = \theta_5^{n+1} \in M$ for every $n \geq 0$ and $\theta_5(\theta_5^n\theta_4) = \theta_5^{n+1}\theta_4 \in M$ for every $n \geq 0$.

  However, $\theta_5(\theta_5^n\theta_1\theta_4) = \theta_5^{n+1}\theta_1\theta_4 \in M$ for every $n \geq 0$ and $\theta_5(\theta_5^n\theta_1) = \theta_5^{n+1}\theta_1 \in M$ for $n \geq 1$. Furthermore, $\theta_5(\theta_3\theta_5^n) = (\theta_5\theta_3)\theta_5^n \in M\theta_5^n \subset M$ for every $n \geq 1$, $\theta_5(\theta_1\theta_5^n) = (\theta_5\theta_1)\theta_5^n \in M\theta_5^n \subset M$

for every $n \geq 1$, $\theta_5(\theta_3\theta_5^n\theta_4) = (\theta_5\theta_3\theta_5^n)\theta_4 \in M\theta_4 \subset M$ for every $n \geq 0$, $\theta_5(\theta_1\theta_5^n\theta_4) = (\theta_5\theta_1\theta_5^n)\theta_4 \in M\theta_4 \subset M$ for every $n \geq 1$. Thus, $\theta_5 M \subset M$.

- $\theta_3 M \subset M$.

  Since $\theta_3\theta_4 \in M$ we have that $\theta_3(\theta_4\theta_1) = (\theta_3\theta_4)\theta_1 \in M\theta_1 \subset M$. Since $\theta_3^2 = \theta_3 - \theta_5 + 3\theta_3\theta_4 + \theta_3\theta_5 + \theta_1\theta_4 + \theta_5\theta_1 \in M$ and $\theta_3\theta_1 \in M$. Furthermore, $\theta_3\theta_5^n \in M$ for every $n \geq 0$, $\theta_3(\theta_5^n\theta_4) = (\theta_3\theta_5^n)\theta_4 \in M\theta_4 \subset M$ for every $n \geq 0$. Since $\theta_3(\theta_5^n\theta_1\theta_4) = (\theta_3\theta_5^n)\theta_1\theta_4 \in M\theta_1\theta_4 \subset M\theta_4 \subset M$ for every $n \geq 0$, $\theta_3(\theta_5^n\theta_1) = (\theta_3\theta_5^n)\theta_1 \in M\theta_1 \subset M$ for every $n \geq 1$.

  On the other hand, $\theta_3(\theta_3\theta_5) = \theta_3^2\theta_5^n \in M\theta_5^n \subset M$ for every $n \geq 1$ and $\theta_3(\theta_1\theta_5^n) = (\theta_3\theta_1)\theta_5^n \in M\theta_5^n \subset M$ for every $n \geq 1$. Since $\theta_3(\theta_3\theta_5^n\theta_4) = \theta_3^2\theta_5^n\theta_4 \in M\theta_5^n\theta_4 \subset M\theta_4 \subset M$ for every $n \geq 0$ and $\theta_3(\theta_1\theta_5^n\theta_4) = (\theta_3\theta_1)(\theta_5^n\theta_4) \in M\theta_5^n\theta_4 \subset M\theta_4 \subset M$ for every $n \geq 1$ we have that $\theta_3 M \subset M$.

- $M\theta_3 \subset M$.

  Note that $(\theta_4\theta_1)\theta_3 = \theta_4(\theta_1\theta_3) \in \theta_4 M \subset M$ and $\theta_3^2 \in M$. Since $\theta_3 \in M$ we have that $\theta_1\theta_3 \in \theta_1 M \subset M$. On the other hand $\theta_3 \in M$ implies $\theta_5^n\theta_3 \in \theta_5^n M \subset M$ for every $n \geq 0$ and $(\theta_5^n\theta_4)\theta_3 = \theta_5^n(\theta_4\theta_3) \in \theta_5^n M \subset M$ for every $n \geq 0$, since $\theta_4\theta_3 \in M$. Note that $(\theta_5^n\theta_1\theta_4)\theta_3 = \theta_5^n\theta_1\theta_4\theta_3 \in \theta_5^n\theta_1\theta_4 M \subset \theta_5^n\theta_1 M \subset \theta_5^n M \subset M$ for every $n \geq 0$ and $(\theta_5^n\theta_1)\theta_3 = \theta_5^n\theta_1\theta_3 \in \theta_5^n\theta_1 M \subset \theta_5^n M \subset M$ for every $n \geq 0$. Furthermore, since $\theta_5^n\theta_3 \in M$ we have $(\theta_3\theta_5^n)\theta_3 = \theta_3(\theta_5^n\theta_3) \in \theta_3 M \subset M$ for every $n \geq 1$ and $(\theta_1\theta_5^n)\theta_3 = \theta_1(\theta_5^n\theta_3) \in \theta_1 M \subset M$ for every $n \geq 1$.

  On the other hand, $\theta_3\theta_5^n\theta_4\theta_3 = 2\theta_3\theta_5^n\theta_4 + \theta_1\theta_5^n\theta_4 + \theta_1\theta_5^{n+1} + \theta_3\theta_5^{n+1} + \theta_3\theta_5^{n+1}\theta_4 - \theta_5^{n+1}\theta_4 - \theta_5^{n+2} \in M$ for every $n \geq 0$ and $\theta_1\theta_5^n\theta_4 = -\theta_3\theta_5^n\theta_4 + 2\theta_5^{n+1}\theta_4 + \theta_5^{n+2} + \theta_5^{n+1}\theta_1\theta_4 + \theta_3\theta_5^{n+1}\theta_4 + \theta_5^{n+2}\theta_1 + \theta_3\theta_5^{n+2} + \theta_1\theta_5^{n+1}\theta_4 - \theta_1\theta_5^{n+1} - \theta_3\theta_5^{n+1} \in M$ for every $n \geq 0$. Thus, $M\theta_3 \subset M$. □

Proposition 5. *Define $\beta_1 = \alpha_1 + \alpha_3$, $\beta_3 = \alpha_1 - \alpha_3$, $\beta_4 = 2\alpha_4$, $\beta_5 = 2\alpha_5$ then $\{\beta_4\beta_1, \beta_3, \beta_1\} \cup \{\beta_5^n \mid n \geq 0\} \cup \{\beta_5^n\beta_4 \mid n \geq 0\} \cup \{\beta_5^n\beta_1\beta_4 \mid n \geq 0\} \cup \{\beta_5^n\beta_1 \mid n \geq 1\} \cup \{\beta_3\beta_5^n \mid n \geq 1\} \cup \{\beta_1\beta_5^n \mid n \geq 1\} \cup \{\beta_3\beta_5^n\beta_4 \mid n \geq 0\} \cup \{\beta_1\beta_5^n\beta_4 \mid n \geq 1\}$ is a linearly independent set over k.*

Proof. *Note that*

$$\mathbb{A} = \mathbb{K}\cdot\beta_1 \oplus \mathbb{K}\cdot\beta_3 \oplus \mathbb{K}\cdot\beta_4\beta_1 \oplus \bigoplus_{n=0}^{\infty} \mathbb{K}\cdot\beta_5^n \oplus \bigoplus_{n=0}^{\infty} \mathbb{K}\cdot\beta_5^n\beta_4 \oplus \bigoplus_{n=0}^{\infty} \mathbb{K}\cdot\beta_5^n\beta_1\beta_4 \oplus \bigoplus_{n=1}^{\infty} \mathbb{K}\cdot\beta_5^n\beta_1 \oplus \bigoplus_{n=1}^{\infty} \mathbb{K}\cdot\beta_3\beta_5^n \oplus \bigoplus_{n=1}^{\infty} \mathbb{K}\cdot\beta_1\beta_5^n \oplus$$

$$\bigoplus_{n=1}^{\infty}\mathbb{K}\cdot\beta_3\beta_5^n\beta_4\oplus\bigoplus_{n=1}^{\infty}\mathbb{K}\cdot\beta_1\beta_5^n\beta_4 = \mathbb{K}\oplus\mathbb{K}\cdot\beta_1\oplus\mathbb{K}\cdot\beta_3\oplus\mathbb{K}\cdot\left(\frac{1}{2}\beta_4\beta_1-\frac{1}{2}\beta_4\beta_3+\beta_4+\beta_5\beta_1+\frac{1}{2}\beta_5\beta_4-\frac{1}{2}\beta_5^2-\beta_3\beta_4\right)$$

$$\oplus\mathbb{K}\cdot\left(\frac{1}{2}\beta_4\beta_1-\frac{1}{2}\beta_4\beta_3+\beta_4+\beta_5\beta_1+\frac{1}{2}\beta_5\beta_4-\frac{1}{2}\beta_5^2\right)\oplus\bigoplus_{n=0}^{\infty}\mathbb{K}\cdot(\beta_5^n-\beta_5^{n-1}\beta_4-\beta_3\beta_5^n)\oplus\bigoplus_{n=0}^{\infty}\mathbb{K}\cdot\beta_5^n\beta_4$$

$$\oplus\bigoplus_{n=0}^{\infty}\mathbb{K}\cdot(\beta_5^n\beta_1\beta_4+\beta_3\beta_5^n\beta_4)\oplus\bigoplus_{n=1}^{\infty}\mathbb{K}\cdot\beta_5^n\beta_1+\beta_5^{n-1}\beta_4\oplus\bigoplus_{n=1}^{\infty}\mathbb{K}\cdot\beta_5^n\beta_1\beta_4$$

$$\oplus\bigoplus_{n=1}^{\infty}\mathbb{K}\cdot(\beta_1\beta_5^n+\beta_3\beta_5^n)\oplus\bigoplus_{n=1}^{\infty}\mathbb{K}\cdot\beta_5^n\oplus\bigoplus_{n=1}^{\infty}\mathbb{K}\cdot(\beta_1\beta_5^n\beta_4+\beta_3\beta_5^n\beta_4).$$

The second equality is given by an isomorphism of $\mathbb{K}$ vector spaces sending $\{\beta_4\beta_1,\beta_3,\beta_1\}\cup\{\beta_5^n\mid n\geq 0\}\cup\{\beta_5^n\beta_4\mid n\geq 0\}\cup\{\beta_5^n\beta_1\beta_4\mid n\geq 0\}\cup\{\beta_5^n\beta_1\mid n\geq 1\}\cup\{\beta_3\beta_5^n\mid n\geq 1\}\cup\{\beta_1\beta_5^n\mid n\geq 1\}\cup\{\beta_3\beta_5^n\beta_4\mid n\geq 0\}\cup\{\beta_1\beta_5^n\beta_4\mid n\geq 1\}$ *to the set*

$$\left\{1,\beta_1,\beta_3,\frac{1}{2}\beta_4\beta_1-\frac{1}{2}\beta_4\beta_3+\beta_4+\beta_5\beta_1+\frac{1}{2}\beta_5\beta_4-\frac{1}{2}\beta_5^2-\beta_3\beta_4,\frac{1}{2}\beta_4\beta_1-\frac{1}{2}\beta_4\beta_3+\beta_4+\beta_5\beta_1+\frac{1}{2}\beta_5\beta_4-\frac{1}{2}\beta_5^2\right\}$$

$$\cup\{\beta_5^n-\beta_5^{n-1}\beta_4-\beta_3\beta_5^n\mid n\geq 0\}\cup\{\beta_5^n\beta_4\mid n\geq 0\}\cup\{\beta_5^n\beta_1\beta_4+\beta_3\beta_5^n\beta_4\mid n\geq 0\}\cup\{\beta_5^n\beta_1+\beta_5^{n-1}\beta_4\mid n\geq 1\}$$

$$\cup\{\beta_5^n\beta_1\beta_4\mid n\geq 1\}\cup\{\beta_1\beta_5^n+\beta_3\beta_5^n\mid n\geq 1\}\cup\{\beta_5^n\mid n\geq 1\}\cup\{\beta_1\beta_5^n\beta_4+\beta_3\beta_5^n\beta_4\mid n\geq 1\}$$

which is linearly independent because is exactly $\{1,\beta_1,\beta_3,\beta_4,\beta_5\}\cup\{e_{ij}x^k\mid 1\leq i,j\leq 2,k\geq 3\}$. □

Finally, we conclude with the proof of the presentation. Define

$$f:\mathbb{C}\cdot\langle\theta_1,\theta_3,\theta_4,\theta_5\rangle/I\longrightarrow\mathbb{A},$$

$$f(\overline{\theta_j})=\beta_j$$

The Lemma 10 guarantees the existence of the system of generators $\{\theta_4\theta_1,\theta_3,\theta_1\}\cup\{\theta_5^n\mid n\geq 0\}\cup\{\theta_5^n\theta_4\mid n\geq 0\}\cup\{\theta_5^n\theta_1\theta_4\mid n\geq 0\}\cup\{\theta_5^n\theta_1\mid n\geq 1\}\cup\{\theta_3\theta_5^n\mid n\geq 1\}\cup\{\theta_1\theta_5^n\mid n\geq 1\}\cup\{\theta_3\theta_5^n\theta_4\mid n\geq 0\}\cup\{\theta_1\theta_5^n\theta_4\mid n\geq 1\}$ *for* $\mathbb{C}\cdot\langle\theta_1,\theta_3,\theta_4,\theta_5\rangle/I$ *as a free $\mathbb{C}$-vector space. Furthermore* $f|_{\mathbb{C}}:\mathbb{C}\longrightarrow A$ *is a monomorphism.*

The Proposition 5 implies that $\{\beta_4\beta_1,\beta_3,\beta_1\}\cup\{\beta_5^n\mid n\geq 0\}\cup\{\beta_5^n\beta_4\mid n\geq 0\}\cup\{\beta_5^n\beta_1\beta_4\mid n\geq 0\}\cup\{\beta_5^n\beta_1\mid n\geq 1\}\cup\{\beta_3\beta_5^n\mid n\geq 1\}\cup\{\beta_1\beta_5^n\mid n\geq 1\}\cup\{\beta_3\beta_5^n\beta_4\mid n\geq 0\}\cup\{\beta_1\beta_5^n\beta_4\mid n\geq 1\}$ *is a linearly independent set over $\mathbb{C}$.*

Thus, we are under the hypothesis of Theorem 14 and f is an isomorphism. □

These Hamiltonian vector fields restrict in a natural way to certain invariant manifolds, like the N-soliton and the rational manifolds. We call the manifold of rational solutions the set of all rational functions, decaying at infinity, that stay rational by the flow of the KdV equation. It is in fact the union of infinitely many finite dimensional manifolds each one also invariant by the KdV flows.

Jorge P. Zubelli[23]

3

A Class of Matrix Schrödinger Bispectral Operators

3.1 Introduction

We are interested in constructing families of matrix bispectral Schrödinger operators. In order to do that, we start off by constructing solutions to the equation $L\psi = -z^2\psi$ with the matrix Schrödinger operator $L = -\partial_x^2 + V'(x)$ and the eigenfunction

$$\psi(x,z) = \left(Iz + \frac{1}{2}V(x)\right)e^{xz}.$$

We shall refer to this as the *physical equation*, in contradistinction with the equation in the spectral parameter. The necessary and sufficient condition for ψ to satisfy the physical equation in this particular case is

$$V''(x) = V'(x)V(x). \tag{3.1}$$

The main goals of this chapter are: Firstly, to obtain meromorphic solutions of the physical equation using the theory of the Laurent series. Secondly, to give a characterization of the algebra of polynomial eigenvalues $\theta(x)$ satisfying the differential equation $\psi B(z, \partial_z) = \theta(x)\psi$ and look for conditions on the function V to obtain that this algebra is not trivial. The plan of this chapter is as follows: In Section 3.2, we study the matrix autonomous differential equation (3.1) using Laurent series with a simple pole at the origin $V(x) = \sum_{k=-1}^{\infty} V_k x^k$ and obtaining conditions on the coefficients which gives rise to some remarkable properties such as $V_k(V_0, V_1, V_2)$ is quasihomogeneous of type $(1, 2, 3)$ and degree $k+1$. After that, we obtain estimates in the Frobenius norm to assure the existence of local meromorphic solutions of the equation (3.1). An important property of working in the matrix case is the existence of nonconstant polynomial solutions of this autonomous equation. In Section 3.3, we give a complete characterization of the algebra

$$\mathbb{A} = \{\theta \in M_N(\mathbb{C}[x]) \mid \exists B = B(z, \partial_z), (\psi B)(x, z) = \theta(x)\psi(x, z)\}$$

using the family of functions $\mathcal{P} = \{P_k\}_{k\in\mathbb{N}}$ defined by

$$P_k(\theta) = \frac{\theta^{(k)}(0)}{(k-1)!} - \frac{1}{2}\sum_{j=0}^{k}\left[\frac{\theta^{(j)}(0)}{j!}, V_{k-1-j}\right].$$

where $k \in \mathbb{N}$, $\theta \in M_N(\mathbb{C}[x])$. Furthermore, we prove the bispectral property for some class of polynomial potentials satisfying $V''(x) = V'(x)V(x)$.

3.2 Algebraic Morphisms Arising from the Matrix Equation $V''(x) = V'(x)V(x)$

We try to find meromorphic solutions for the matrix equation $V''(x) = V'(x)V(x)$ with a simple pole at $x = 0$. This will allow us to obtain some type of solutions of the bispectral problem. Leading to obtain formal solutions of the autonomous equation we see that the Taylor coefficients in the expansion $V(x) = \sum_{k=-1}^{\infty} V_k x^k$ turns out to be affine algebraic morphims. Furthermore, if the residue $V_{-1} = Res(V, 0) = 0$, then the holomorphic solution V has Taylor coefficients which are quasihomogeneous in the noncommutative variables V_0, V_1 and using some grading we obtain the bispectrality in the case of polynomial potentials.

3.2.1 The Matrix Equation $V''(x) = V'(x)V(x)$.

Let $V(x) = \sum_{k=-1}^{\infty} V_k x^k$, then $V'(x) = \sum_{k=-1}^{\infty} kV_k x^{k-1}$ and $V''(x) = \sum_{k=-1}^{\infty} k(k-1)V_k x^{k-2}$. Therefore, $V''(x) = V'(x)V(x)$ if, and only if,

$$k(k-1)V_k = \sum_{j=-1}^{k} jV_jV_{k-1-j}$$

for $k = -1, 0, 1, \cdots$.

- If $k = -1$, then $-V_{-1}^2 = 2V_{-1}$ and hence $V_{-1}(V_{-1} + 2I_N) = 0$.
- If $k = 0$, then $V_{-1}V_0 = 0$.
- If $k = 1$, then $V_{-1}V_1 = 0$.
- If $k \geq 2$, then $k(k-1)V_k = -V_{-1}V_k + kV_kV_{-1} + \sum_{j=1}^{k-1} jV_jV_{k-1-j}$. Thus,

$$T_k(V_k) = \sum_{j=1}^{k-1} jV_jV_{k-1-j}.$$

  where the operator $T_k : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ defined by $T_k(a) = k(k-1)a + V_{-1}a - kaV_{-1}$,

Since $V_{-1}(V_{-1} + 2I_N) = 0$, we have that 0 and -2 are the only eigenvalues of V_{-1}. The Jordan Canonical Form Theorem implies that V_{-1} has the form $diag(-2, 0, \cdots, 0, -2)$. After a change of coordinates, we may assume without loss of generality that

$$V_{-1} = \begin{pmatrix} -2I_m & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $V_{-1}V_0 = 0$ we have that

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -2I_m & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_{011} & V_{012} \\ V_{021} & V_{022} \end{pmatrix} = \begin{pmatrix} -2V_{011} & -2V_{012} \\ 0 & 0 \end{pmatrix}.$$

Then, $V_{011} = V_{012} = 0$. Thus,

$$V_0 = \begin{pmatrix} 0 & 0 \\ V_{021} & V_{022} \end{pmatrix}.$$

In the same way, $V_{-1}V_1 = 0$ implies that

$$V_1 = \begin{pmatrix} 0 & 0 \\ V_{121} & V_{122} \end{pmatrix}.$$

Now we write

$$V_k = \begin{pmatrix} V_{k11} & V_{k12} \\ V_{k21} & V_{k22} \end{pmatrix},$$

to obtain

$$T_k(V_k) = k(k-1)V_k + V_{-1}V_k - kV_kV_{-1} = k(k-1)\begin{pmatrix} V_{k11} & V_{k12} \\ V_{k21} & V_{k22} \end{pmatrix} + \begin{pmatrix} -2I_m & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} V_{k11} & V_{k12} \\ V_{k21} & V_{k22} \end{pmatrix}$$

$$-k\begin{pmatrix} V_{k11} & V_{k12} \\ V_{k21} & V_{k22} \end{pmatrix}\begin{pmatrix} -2I_m & 0 \\ 0 & 0 \end{pmatrix} = k(k-1)\begin{pmatrix} V_{k11} & V_{k12} \\ V_{k21} & V_{k22} \end{pmatrix} + \begin{pmatrix} -2V_{k11} & -2V_{k12} \\ 0 & 0 \end{pmatrix} + k\begin{pmatrix} 2V_{k11} & 0 \\ 2V_{k21} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (k(k-1)+2k-2)V_{k11} & (k(k-1)-2)V_{k12} \\ (k(k-1)+2k)V_{k21} & k(k-1)V_{k22} \end{pmatrix}$$

$$= \begin{pmatrix} (k-1)(k+2)V_{k11} & (k-2)(k+1)V_{k12} \\ k(k+1)V_{k21} & k(k-1)V_{k22} \end{pmatrix}.$$

For $k = 2$, we have

$$V_1V_0 = \begin{pmatrix} 0 & 0 \\ V_{121} & V_{122} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ V_{021} & V_{022} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ V_{122}V_{021} & V_{122}V_{022} \end{pmatrix} = T_2(V_2) = \begin{pmatrix} 4V_{211} & 0 \\ 6V_{221} & 2V_{222} \end{pmatrix}.$$

Therefore, $V_{211} = 0$, $V_{221} = \frac{V_{122}V_{021}}{6}$, $V_{222} = \frac{V_{122}V_{022}}{2}$. Thus,

$$V_2 = \begin{pmatrix} 0 & V_{212} \\ \frac{V_{122}V_{021}}{6} & \frac{V_{122}V_{022}}{2} \end{pmatrix}$$

Remember that

$$T_k(V_k) = k(k-1)V_k + V_{-1}V_k - kV_kV_{-1}$$

for $k \geq 2$. If $k \geq 3$, we have that T_k is invertible and

$$V_k = \sum_{j=1}^{k-1} j T_k^{-1}(V_j V_{k-1-j}). \tag{3.2}$$

Definition 12. *Fix an element $A \in M_N(\mathbb{C})$, define the multiplication operators $L_A, R_A : M_N(\mathbb{C}) \to M_N(\mathbb{C})$, $L_A(X) = AX$, $R_A(X) = XA$.*

We now look for the elements $A \in M_N(\mathbb{C})$ such that L_A and R_A commutes with T_k for $k \geq 1$.

Lemma 11. *L_A and R_A commutes with T_k for $k \geq 1$ if, and only if, $A_{12} = 0 \in M_{m\times(N-m)}(\mathbb{C})$ and $A_{21} = 0 \in M_{(N-m)\times m}(\mathbb{C})$.*

Proof. *Note that*

$$T_k R_A(X) = T_k(XA) = k(k-1)XA + V_{-1}XA - k(XA)V_{-1} = k(k-1)XA + V_{-1}XA - kXV_{-1} - kX[A, V_{-1}]$$

$$= T_k(X)A - kX[A, V_{-1}] = R_A T_k(X) - kX[A, V_{-1}],\ T_k L_A(X) = T_k(AX) = k(k-1)AX + V_{-1}AX - k(AX)V_{-1}$$

$$= k(k-1)AX + AV_{-1}X - k(AX)V_{-1} - [A, V_{-1}]X = AT_k(X) - [A, V_{-1}]X.$$

Then, L_A and R_A commutes with T_k for $k \geq 1$ if, and only if, $[A, V_{-1}] = 0$, but this condition says that $A_{12} = 0 \in M_{m\times(N-m)}(\mathbb{C})$ and $A_{21} = 0 \in M_{(N-m)\times m}(\mathbb{C})$. □

The following lemma gives us an interesting property of the operator T_k when we consider $M_N(\mathbb{C})$ with the Frobenius norm.

Lemma 12. *The operator $T_k^{-1} : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ satisfies*

$$\left\| T_k^{-1}(a) \right\|_F \leq \frac{4(k^2-3)}{(k-2)(k-1)(k+1)(k+2)} \left\| a \right\|_F$$

for $k \geq 3$.

Proof. *Since,*

$$T_k(a) = \begin{pmatrix} (k-1)(k+2)a_{11} & (k-2)(k+1)a_{12} \\ k(k+1)a_{21} & k(k-1)a_{22} \end{pmatrix}$$

for $k \geq 3$. We have,

$$T_k^{-1}(a) = \begin{pmatrix} \frac{1}{(k-1)(k+2)}a_{11} & \frac{1}{(k-2)(k+1)}a_{12} \\ \frac{1}{k(k+1)}a_{21} & \frac{1}{k(k-1)}a_{22} \end{pmatrix}.$$

Applying the Frobenius norm,

$$\left\| T_k^{-1}(a) \right\|_F^2 = \left(\frac{1}{(k-1)(k+2)} \right)^2 \|a_{11}\|_F^2 + \left(\frac{1}{(k-2)(k+1)} \right)^2 \|a_{12}\|_F^2 + \left(\frac{1}{k(k+1)} \right)^2 \|a_{21}\|_F^2$$

$$+ \left(\frac{1}{k(k-1)} \right)^2 \|a_{22}\|_F^2 \leq \left\{ \left(\frac{1}{(k-1)(k+2)} \right)^2 + \left(\frac{1}{(k-2)(k+1)} \right)^2 + \left(\frac{1}{k(k+1)} \right)^2 + \left(\frac{1}{k(k-1)} \right)^2 \right\} \|a\|_F^2.$$

Therefore,

$$\left\| T_k^{-1}(a) \right\|_F \leq \left\| \left(\frac{1}{(k-1)(k+2)}, \frac{1}{(k-2)(k+1)}, \frac{1}{k(k+1)}, \frac{1}{k(k-1)} \right) \right\|_2 \|a\|_F$$

$$= 2 \frac{\sqrt{k^6 - 5k^4 + 6k^2 + 8}}{(k-2)(k-1)k(k+1)(k+2)} \|a\|_F \leq \frac{4(k^2-3)}{(k-2)(k-1)(k+1)(k+2)} \|a\|_F,$$

for $k \geq 3$. To obtain the last inequality observe that

$$\frac{\sqrt{k^6 - 5k^4 + 6k^2 + 8}}{(k-2)(k-1)k(k+1)(k+2)} \leq \frac{2(k^2-3)}{(k-2)(k-1)(k+1)(k+2)}$$

$$\iff \sqrt{k^6 - 5k^4 + 6k^2 + 8} \leq 2k(k^2-3)$$

$$\iff k^6 - 5k^4 + 6k^2 + 8 \leq 4k^2(k^2-3)^2 = 4k^2(k^4 - 6k^2 + 9) = 4k^6 - 24k^4 + 36k^2$$

$$\iff f(k) := 3k^6 - 19k^4 + 30k^2 - 8 \geq 0.$$

However, we have the factorization of the polynomial $f \in \mathbb{Q}[x]$, $f(x) = (x^2-4)(x^2-2)(3x^2-1)$, and $f(x) \geq 0$ for $|x| \geq 2$. □

Remark 5. *Note that the inequality in the previous lemma implies that T_k^{-1} is a contraction for $k \geq 3$.*

We can use this result to estimate the norm of the sequence $\{V_j\}_{j \in \mathbb{N}}$.

Theorem 16. *If $\|V_0\|_F \leq \frac{1}{4}$, $\|V_1\|_F \leq \frac{1}{8}$, $\|V_2\|_F \leq \frac{1}{16}$, then*

$$\|V_k\|_F \leq \frac{1}{2^{k+2}}$$

for every $k \geq 3$.

Proof. *The proof is by induction. By hypothesis we have* $\|V_k\|_F \leq \frac{1}{2^{k+2}}$ *for* $k = 0, 1, 2$. *Assume the claim for some* $0 \leq j \leq k-1$ *and note that*

$$\|V_k\|_F \leq \sum_{j=1}^{k-1} j \left\| T_k^{-1}(V_j V_{k-1-j}) \right\|_F \leq \sum_{j=1}^{k-1} j \frac{4(k^2-3)}{(k-2)(k-1)(k+1)(k+2)} \left\| V_j V_{k-1-j} \right\|_F$$

$$\leq \sum_{j=1}^{k-1} j \frac{4(k^2-3)}{(k-2)(k-1)(k+1)(k+2)} \left\| V_j \right\|_F \left\| V_{k-1-j} \right\|_F$$

$$\leq \sum_{j=1}^{k-1} j \frac{4(k^2-3)}{(k-2)(k-1)(k+1)(k+2)} \left(\frac{1}{2^{j+2}}\right) \left(\frac{1}{2^{k-j+1}}\right)$$

$$= \frac{k(k^2-3)}{(k-2)(k+1)(k+2)} \left(\frac{1}{2^{k+2}}\right) \leq \frac{1}{2^{k+2}}.$$

Therefore, the claim follows by induction. □

This theorem allows us to give meromorphic solutions to the matrix equation $V''(x) = V'(x)V(x)$ in a punctured neighborhood of the origin. To do this, we consider the set $K \subset M_N(\mathbb{C})^3 \times \mathbb{C}$ defined by the relations

$$V_{-1}V_0 = V_{-1}V_1 = 0, V_1 V_0 = T_2(V_2), \|V_0\|_F \leq \frac{1}{4}, \|V_1\|_F \leq \frac{1}{8}, \|V_2\|_F \leq \frac{1}{16}, 0 < |x| \leq 1.$$

Corollary 11. *The formal power series* $V = V(V_0, V_1, V_2, x) = \sum_{k=-1}^{\infty} V_k(V_0, V_1, V_2)x^k$ *is meromorphic for* $(V_0, V_1, V_2, x) \in K$.

Proof. *If* $(V_0, V_1, V_2, x) \in K$, *then Theorem 16 implies that* $\|V_k(V_0, V_1, V_2)\|_F \leq \frac{1}{2^{k+2}}$ *and therefore* $\left\| V_k(V_0, V_1, V_2)x^k \right\|_F \leq \frac{1}{2^{k+2}}$. *Thus, the Weiertrass Theorem implies that the series* $\sum_{k=-1}^{\infty} V_k(V_0, V_1, V_2)x^k$ *converges absolutely and uniformly in compact subsets of K. Since the functions* $V_k(V_0, V_1, V_2)x^k$ *are meromorphic in K we obtain the same for V.* □

3.2.2 Some Properties of the Sequence $\{V_k(V_0, V_1, V_2)\}_{k\in\mathbb{N}}$

The sequence $\{V_k(V_0, V_1, V_2)\}_{k\in\mathbb{N}}$ has important properties which are given in the following results.

Proposition 6. *The function* $V_k(V_{021}, V_{022}, V_{121}, V_{122}, V_{212})$ *has polynomial coordinates for every* $k \in \mathbb{N}$.

Proof. *Note that*

$$V_0 = \begin{pmatrix} 0 & 0 \\ V_{021} & V_{022} \end{pmatrix}$$

$$V_1 = \begin{pmatrix} 0 & 0 \\ V_{121} & V_{122} \end{pmatrix}$$

and

$$V_2 = \begin{pmatrix} 0 & V_{212} \\ \frac{V_{122}V_{021}}{6} & \frac{V_{122}V_{022}}{2} \end{pmatrix}$$

has polynomial coordinates in $V_{021}, V_{022}, V_{121}, V_{122}, V_{212}$. *Assume that* $V_j(V_{021}, V_{022}, V_{121}, V_{122}, V_{212})$ *has polynomial coordinates for* $1 \leq j \leq k-1$, *since*

$$V_k = \sum_{j=1}^{k-1} jT_k^{-1}(V_j V_{k-1-j})$$

and the product of matrices $V_j V_{k-1-j}$ *is a polynomial in the block entries of* V_j *and* V_{k-1-j}. *Since* T_k^{-1} *is linear we obtain that* $V_k(V_{021}, V_{022}, V_{121}, V_{122}, V_{212})$ *has polynomial coordinates.* □

Corollary 12. $V_k(V_{021}, V_{022}, V_{121}, V_{122}, V_{212})$ *is an algebraic morphism for every* $k \in \mathbb{N}$.

Theorem 17. *If* $A \in M_N(\mathbb{C})$ *and* $A_{22}V_{j,21} = V_{j,21}A_{11}$, *for* $j = 0, 1, 2$, $A_{11}V_{12} = V_{12}A_{22}$, $[A_{22}, V_{0,22}] = [A_{22}, V_{1,22}] = 0$, *then*

$$V_j(V_{j21}A_{11}, V_{j22}A_{22}) = A^{j+1}V_j(V_{j21}, V_{j22}) = V_j(V_{j21}, V_{j22})A^{j+1},$$

for $j = 0, 1$.

$$V_2(AV_0, A^2V_1, A_{11}^3V_{212}) = A^3V_2(V_0, V_1, V_{212}) = V_2(V_0, V_1, V_{212})A^3$$

$$V_k(AV_0, A^2V_1, A^3V_2) = A^{k+1}V_k(V_0, V_1, V_2) = V_k(V_0, V_1, V_2)A^{k+1},$$

for every $k \geq 3$.

Proof. *In fact,*

$$AV_0(V_{021}, V_{022}) = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ V_{021} & V_{022} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ A_{22}V_{021} & A_{22}V_{022} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ V_{021}A_{11} & V_{022}A_{22} \end{pmatrix} = V_0(V_{021}A_{11}, V_{022}A_{22}) = V_0(V_{021}, V_{022})A,$$

$$A^2V_1(V_{121}, V_{122}) = \begin{pmatrix} A_{11}^2 & 0 \\ 0 & A_{22}^2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ V_{021} & V_{022} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ A_{22}^2V_{021} & A_{22}^2V_{022} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ V_{021}A_{11}^2 & V_{022}A_{22}^2 \end{pmatrix} = V_0(V_{021}A_{11}^2, V_{022}A_{22}^2) = V_1(V_{121}, V_{122})A^2,$$

$$A^3V_2(V_0, V_1, V_{212}) = \begin{pmatrix} A_{11}^3 & 0 \\ 0 & A_{22}^3 \end{pmatrix} \begin{pmatrix} 0 & V_{212} \\ \frac{V_{122}V_{021}}{6} & \frac{V_{122}V_{022}}{2} \end{pmatrix} = \begin{pmatrix} 0 & A_{11}^3V_{212} \\ \frac{A_{22}^3V_{122}V_{021}}{6} & \frac{A_{22}^3V_{122}V_{022}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & A_{11}^3V_{212} \\ \frac{(A_{22}^2V_{122})(A_{22}V_{021})}{6} & \frac{(A_{22}^2V_{122})(A_{22}V_{022})}{2} \end{pmatrix} = V_2(AV_0, A^2V_1, A_{11}^3V_{212}) = V_2(V_0, V_1, V_{212})A^3.$$

On the other hand, using the Lemma 11 we obtain

$$V_3(AV_0, A^2V_1, A^3V_2) = \sum_{j=1}^{2} jT_3^{-1}((A^{j+1}V_j)(A^{3-j}V_{2-j})) = T_3^{-1}((A^2V_1)^2 + 2(A^3V_2)(AV_0))$$

$$= T_3^{-1}(A^4(V_1^2 + 2V_2V_0)) = T_3^{-1}L_A^4(V_1^2 + 2V_2V_0) = L_A^4T_3^{-1}(V_1^2 + 2V_2V_0) = A^4V_3(V_0, V_1, V_2).$$

Similarly,

$$V_3(AV_0, A^2V_1, A^3V_2) = \sum_{j=1}^{2} jT_3^{-1}((A^{j+1}V_j)(A^{3-j}V_{2-j})) = T_3^{-1}((A^2V_1)^2 + 2(A^3V_2)(AV_0))$$

$$= T_3^{-1}((V_1^2 + 2V_2V_0)A^4) = T_3^{-1}R_A^4(V_1^2 + 2V_2V_0) = R_A^4T_3^{-1}(V_1^2 + 2V_2V_0) = V_3(V_0, V_1, V_2)A^4.$$

Assume the claim is true for $3 \leq j \leq k-1$ *and use again the Lemma 11 to obtain*

$$V_k(AV_0, A^2V_1, A^3V_2) = \sum_{j=1}^{k-1} jT_k^{-1}((A^{j+1}V_j)(A^{k-j}V_{k-1-j}))$$

$$= \sum_{j=1}^{k-1} jT_k^{-1}(A^{k+1}V_jV_{k-1-j}) = \sum_{j=1}^{k-1} jT_k^{-1}L_A^{k+1}(V_jV_{k-1-j}) = L_A^{k+1}\sum_{j=1}^{k-1} jT_k^{-1}(V_jV_{k-1-j}) = A^{k+1}V_k(V_0, V_1, V_2).$$

Similarly,

$$V_k(AV_0, A^2V_1, A^3V_2) = \sum_{j=1}^{k-1} jT_k^{-1}((A^{j+1}V_j)(A^{k-j}V_{k-1-j}))$$

$$= \sum_{j=1}^{k-1} jT_k^{-1}(V_jV_{k-1-j}A^{k+1}) = \sum_{j=1}^{k-1} jT_k^{-1}R_A^{k+1}(V_jV_{k-1-j}) = R_A^{k+1}\sum_{j=1}^{k-1} jT_k^{-1}(V_jV_{k-1-j}) = V_k(V_0, V_1, V_2)A^{k+1}.$$

Thus, the claim follows by induction. □

Corollary 13. $V_k(\lambda V_0, \lambda^2 V_1, \lambda^3 V_2) = \lambda^{k+1} V_k(V_0, V_1, V_2)$ *for every* $\lambda \in \mathbb{C}$, $k \in \mathbb{N}$, *i.e the function* $V_k(V_0, V_1, V_2)$ *is quasihomogeneous of type* $(1, 2, 3)$ *and degree* $k+1$.

Proof. *It is enough to consider* $A = \lambda I_N$. □

Two remarkable cases of the sequence of functions $\{V_k(V_0, V_1, V_2)\}_{k\in\mathbb{N}}$ are

- If $V_{-1} = -2I_N$, then the equations $V_{-1}V_0 = V_{-1}V_1 = 0$ implies that $V_0 = V_1 = 0$. On the other hand, $T_k^{-1}(a) = \frac{1}{(k-1)(k+2)}a$ and (3.2) imply that $V_k = 0$ for $k \geq 2$. In this case we have $V(x) = -\frac{2I_N}{x}$.

- If $V_{-1} = 0$, then V_0 and V_1 are arbitrary. Furthermore, $T_k^{-1}(a) = \frac{1}{k(k-1)}a$ and (3.2) imply that

  $$V_k = \frac{1}{k(k-1)} \sum_{j=1}^{k-1} jV_jV_{k-1-j},$$

  for $k \geq 2$. In particular, $V_k = V_k(V_0, V_1)$ is a noncommutative polynomial in the variables V_0 and V_1.

In the last case we have an interesting result

Proposition 7. *If* $V_{-1} = 0$, *then*

- V_1 *is a left divisor of* V_k *for* $k \geq 1$. *In particular* $V_k(V_0, 0) = 0$ *for* $k \geq 1$.

- $V_k(0, V_1) = 0$ *for k even,* $V_k(0, V_1) = r_{4k-1}V_1^{2k}$ *for* $k \geq 1$ *and* $V_k(0, V_1) = r_{4k+1}V_1^{2k+1}$ *for* $k \geq 0$ *for some coefficients* $r_{4k-1}, r_{4k+1} \in [0, 1]$.

3.2.3 Polynomial Solutions of the matrix equation $V''(x) = V'(x)V(x)$

If we want a polynomial solution of degree $\leq n$ for the equation $V'' = V'V$ we have to solve the system of matrix equations

$$V_s = \frac{1}{s(s-1)}\sum_{j=1}^{s-1} jV_jV_{s-1-j}, \quad \sum_{j=\max\{k-1-n,1\}}^{n} jV_jV_{k-1-j} = 0$$

for $2 \leq s \leq n, n+1 \leq k \leq 2n+1$.

Using the Proposition 7 we have one class of solutions to this problem.

Theorem 18. *Let V_1 to be a nilpotent matrix of degree $n+1 \leq N$,*

- *If $n = 2k$, then $V(x) = \sum_{j=1}^{k} r_{4j-1}V_1^{2j}x^j$ is a solution of $V'' = V'V$ for some $\left\{r_{4j-1}\right\}_{1\leq j\leq k} \subset \mathbb{C}$.*
- *If $n = 2k+1$, then $V(x) = \sum_{j=1}^{k} r_{4j+1}V_1^{2j+1}x^j$ is a solution of $V'' = V'V$ for some $\left\{r_{4j+1}\right\}_{1\leq j\leq k} \subset \mathbb{C}$.*

Remark 6. *In the scalar case we have the integral domain $\mathbb{C}[x]$, if V is a polynomial such that $\deg(V) \geq 2$ we have that V'' is a nonzero polynomial. Applying the function* deg *to the equation*

$$\deg(V'') = \deg(V) - 2 = \deg(V'V) = \deg(V') + \deg(V) = 2\deg(V) - 1$$

and therefore $\deg(V) = -1$, contradiction. Therefore, $\deg(V) \leq 1$, in the case $\deg(V) = 1$ there is no solution of the equation, in fact $V'' = 0$ and $V'V$ is a nonzero polynomial. Thus, we have the trivial constant solution $V(x) = V_0$.

3.3 Bispectrality of the Matrix Schrödinger Bispectral Operators for polynomial potentials

We begin with the definition of the family $\mathcal{P} = \{P_k\}_{k\in\mathbb{N}}$ which will be used to describe the map $\theta \mapsto \mathcal{B}$ such that $(\psi\mathcal{B})(x,z) = \theta(x)\psi(x,z)$ and the bispectral algebra

$$\mathbb{A} = \{\theta \in M_N(\mathbb{C}[x]) \mid \exists B = B(z,\partial_z), (\psi B)(x,z) = \theta(x)\psi(x,z)\}.$$

Definition 13. *For $k \in \mathbb{N}$ and $\theta \in M_N(\mathbb{C}[x])$, we define*

$$P_k(\theta) = \frac{\theta^{(k)}(0)}{(k-1)!} - \frac{1}{2}\sum_{j=0}^{k}\left[\frac{\theta^{(j)}(0)}{j!}, V_{k-1-j}\right].$$

Now we study some properties of the sequence $\{P_k\}_{k\in\mathbb{N}}$.

Lemma 13 (Product Formula for P_k). *If $\theta_1, \theta_2 \in M_N(\mathbb{C}[x])$, then*

$$P_k(\theta_1\theta_2) = \sum_{s=0}^{k}\left\{P_{k-s}(\theta_1)\frac{\theta_2^{(s)}(0)}{s!} + \frac{\theta_1^{(s)}(0)}{s!}P_{k-s}(\theta_2)\right\}$$

Proof. *By definition,*

$$P_k(\theta_1\theta_2) = \frac{(\theta_1\theta_2)^{(k)}(0)}{(k-1)!} - \frac{1}{2}\sum_{j=0}^{k}\left[\frac{(\theta_1\theta_2)^{(j)}(0)}{j!}, V_{k-1-j}\right]$$

$$= \frac{1}{(k-1)!}\sum_{j=0}^{k}\binom{k}{j}\theta_1^{(j)}(0)\theta_2^{(k-j)}(0) - \frac{1}{2}\sum_{j=0}^{k}\left[\frac{1}{j!}\sum_{r=0}^{j}\binom{j}{r}\theta_1^{(r)}(0)\theta_2^{(j-r)}(0), V_{k-1-j}\right]$$

$$= k\sum_{j=0}^{k}\frac{\theta_1^{(j)}(0)}{j!}\frac{\theta_2^{(k-j)}(0)}{(k-j)!} - \frac{1}{2}\sum_{j=0}^{k}\sum_{r=0}^{j}\left[\frac{\theta_1^{(r)}(0)}{r!}\frac{\theta_2^{(j-r)}(0)}{(j-r)!}, V_{k-1-j}\right]$$

$$= k\sum_{j=0}^{k}\frac{\theta_1^{(j)}(0)}{j!}\frac{\theta_2^{(k-j)}(0)}{(k-j)!} - \frac{1}{2}\sum_{j=0}^{k}\sum_{r=0}^{j}\left(\left[\frac{\theta_1^{(r)}(0)}{r!}, V_{k-1-j}\right]\frac{\theta_2^{(j-r)}(0)}{(j-r)!} + \frac{\theta_1^{(r)}(0)}{r!}\left[\frac{\theta_2^{(j-r)}(0)}{(j-r)!}, V_{k-1-j}\right]\right).$$

However,

$$P_k(\theta) - \frac{\theta^{(k)}(0)}{(k-1)!} = -\frac{1}{2}\sum_{j=0}^{k}\left[\frac{\theta^{(j)}(0)}{j!}, V_{k-1-j}\right];$$

for every $\theta \in M_N(\mathbb{C}[x])$.

Therefore,

$$\sum_{j=0}^{k}\sum_{r=0}^{j}\left(\left[\frac{\theta_1^{(r)}(0)}{r!}, V_{k-1-j}\right]\frac{\theta_2^{(j-r)}(0)}{(j-r)!} + \frac{\theta_1^{(r)}(0)}{r!}\left[\frac{\theta_2^{(j-r)}(0)}{(j-r)!}, V_{k-1-j}\right]\right)$$

$$= \sum_{r=0}^{k}\sum_{j=r}^{k}\left(\left[\frac{\theta_1^{(r)}(0)}{r!}, V_{k-1-j}\right]\frac{\theta_2^{(j-r)}(0)}{(j-r)!} + \frac{\theta_1^{(j-r)}(0)}{(j-r)!}\left[\frac{\theta_2^{(r)}(0)}{r!}, V_{k-1-j}\right]\right)$$

$$= \sum_{r=0}^{k}\sum_{s=0}^{k-r}\left(\left[\frac{\theta_1^{(r)}(0)}{r!}, V_{k-1-s-r}\right]\frac{\theta_2^{(s)}(0)}{s!} + \frac{\theta_1^{(s)}(0)}{s!}\left[\frac{\theta_2^{(r)}(0)}{r!}, V_{k-1-j}\right]\right)$$

$$= \sum_{s=0}^{k} \sum_{r=0}^{k-s} \left(\left[\frac{\theta_1^{(r)}(0)}{r!}, V_{k-1-s-r} \right] \frac{\theta_2^{(s)}(0)}{s!} + \frac{\theta_1^{(s)}(0)}{s!} \left[\frac{\theta_2^{(r)}(0)}{r!}, V_{k-1-j} \right] \right)$$

$$= \sum_{s=0}^{k} \left\{ \left(\sum_{r=0}^{k-s} \left[\frac{\theta_1^{(r)}(0)}{r!}, V_{k-1-s-r} \right] \right) \frac{\theta_2^{(s)}(0)}{s!} + \frac{\theta_1^{(s)}(0)}{s!} \left(\sum_{r=0}^{k-s} \left[\frac{\theta_2^{(r)}(0)}{r!}, V_{k-1-j} \right] \right) \right\}.$$

This implies that

$$P_k(\theta_1\theta_2) = k \sum_{j=0}^{k} \frac{\theta_1^{(j)}(0)}{j!} \frac{\theta_2^{(k-j)}(0)}{(k-j)!}$$

$$-\frac{1}{2} \sum_{s=0}^{k} \left\{ \left(\sum_{r=0}^{k-s} \left[\frac{\theta_1^{(r)}(0)}{r!}, V_{k-1-s-r} \right] \right) \frac{\theta_2^{(s)}(0)}{s!} + \frac{\theta_1^{(s)}(0)}{s!} \left(\sum_{r=0}^{k-s} \left[\frac{\theta_2^{(r)}(0)}{r!}, V_{k-1-j} \right] \right) \right\}$$

$$= k \sum_{j=0}^{k} \frac{\theta_1^{(j)}(0)}{j!} \frac{\theta_2^{(k-j)}(0)}{(k-j)!}$$

$$+ \sum_{s=0}^{k} \left\{ \left(P_{k-s}(\theta_1) - \frac{\theta_1^{(k-s)}(0)}{(k-s-1)!} \right) \frac{\theta_2^{(s)}(0)}{s!} + \frac{\theta_1^{(s)}(0)}{s!} \left(P_{k-s}(\theta_2) - \frac{\theta_2^{(k-s)}(0)}{(k-s-1)!} \right) \right\}$$

$$= k \sum_{j=0}^{k} \frac{\theta_1^{(j)}(0)}{j!} \frac{\theta_2^{(k-j)}(0)}{(k-j)!}$$

$$+ \sum_{s=0}^{k} P_{k-s}(\theta_1) \frac{\theta_2^{(s)}(0)}{s!} - \sum_{s=0}^{k} \frac{\theta_1^{(k-s)}(0)}{(k-s-1)!} \frac{\theta_2^{(s)}(0)}{s!}$$

$$+ \sum_{s=0}^{k} \frac{\theta_1^{(s)}(0)}{s!} P_{k-s}(\theta_2) - \sum_{s=0}^{k} \frac{\theta_1^{(s)}(0)}{s!} \frac{\theta_1^{(k-s)}(0)}{(k-s-1)!}$$

$$= k \sum_{j=0}^{k} \frac{\theta_1^{(j)}(0)}{j!} \frac{\theta_2^{(k-j)}(0)}{(k-j)!} + \sum_{s=0}^{k} \left\{ P_{k-s}(\theta_1) \frac{\theta_2^{(s)}(0)}{s!} - \frac{\theta_1^{(s)}(0)}{s!} P_{k-s}(\theta_2) \right\}$$

$$- \sum_{s=0}^{k} (k-s) \frac{\theta_1^{(k-s)}(0)}{(k-s)!} \frac{\theta_2^{(s)}(0)}{s!} - \sum_{s=0}^{k} (k-s) \frac{\theta_1^{(s)}(0)}{s!} \frac{\theta_1^{(k-s)}(0)}{(k-s)!}$$

$$= k \sum_{j=0}^{k} \frac{\theta_1^{(j)}(0)}{j!} \frac{\theta_2^{(k-j)}(0)}{(k-j)!} + \sum_{s=0}^{k} \left\{ P_{k-s}(\theta_1) \frac{\theta_2^{(s)}(0)}{s!} - \frac{\theta_1^{(s)}(0)}{s!} P_{k-s}(\theta_2) \right\}$$

$$- \sum_{s=0}^{k} s \frac{\theta_1^{(s)}(0)}{s!} \frac{\theta_2^{(k-s)}(0)}{(k-s)!} - \sum_{s=0}^{k} (k-s) \frac{\theta_1^{(s)}(0)}{s!} \frac{\theta_1^{(k-s)}(0)}{(k-s)!}$$

$$= \sum_{s=0}^{k} \left\{ P_{k-s}(\theta_1) \frac{\theta_2^{(s)}(0)}{s!} - \frac{\theta_1^{(s)}(0)}{s!} P_{k-s}(\theta_2) \right\}.$$

Thus,

$$P_k(\theta_1\theta_2) = \sum_{s=0}^{k} \left\{ P_{k-s}(\theta_1)\frac{\theta_2^{(s)}(0)}{s!} + \frac{\theta_1^{(s)}(0)}{s!}P_{k-s}(\theta_2) \right\}.$$

□

Remark 7. *If $V_j = 0$ for every $j \in \mathbb{N}$ the product formula specializes into the Leibniz rule*

$$(\theta_1\theta_2)^{(k)}(x) = \sum_{s=0}^{k} \binom{k}{s} \theta_1^{(k-s)}(x)\theta_2^{(s)}(x).$$

In this case, $P_k(\theta) = \frac{\theta^{(k)}(0)}{(k-1)!} = k\frac{\theta^{(k)}(0)}{k!}$ and applying the Product Formula (13) turns out

$$P_k(\theta_1\theta_2) = \sum_{s=0}^{k} \left\{ (k-s)\frac{\theta_1^{(k-s)}(0)}{(k-s)!}\frac{\theta_2^{(s)}(0)}{s!} + \frac{\theta_1^{(s)}(0)}{s!}(k-s)\frac{\theta_2^{(k-s)}(0)}{(k-s)!} \right\}$$

$$= \sum_{s=0}^{k} \left\{ (k-s)\frac{\theta_1^{(k-s)}(0)}{(k-s)!}\frac{\theta_2^{(s)}(0)}{s!} + s\frac{\theta_1^{(k-s)}(0)}{(k-s)!}\frac{\theta_2^{(s)}(0)}{s!} \right\}$$

$$= k\sum_{s=0}^{k} \frac{\theta_1^{(k-s)}(0)}{(k-s)!}\frac{\theta_2^{(s)}(0)}{s!} = k\frac{(\theta_1\theta_2)^{(k)}(0)}{k!},$$

in the words $(\theta_1\theta_2)^{(k)}(0) = \sum_{s=0}^{k}\binom{k}{s}\theta_1^{(k-s)}(0)\theta_2^{(s)}(0)$. Since θ_1 and θ_2 were arbitrary we can change them by their translations $\theta_1(x+\cdot)$ and $\theta_2(x+\cdot)$ to obtain $(\theta_1\theta_2)^{(k)}(x) = \sum_{s=0}^{k}\binom{k}{s}\theta_1^{(k-s)}(x)\theta_2^{(s)}(x)$, i.e., the Leibniz rule.

If we consider the formal power series $V(x) = \sum_{j=-1}^{\infty} V_j x^j$ we can write the family in a nice form as stated in the following theorem.

Theorem 19. *For every $k \in \mathbb{N}$ we have*

$$P_k = \frac{1}{k!}\frac{d^k}{dx^k}\Big|_{x=0}\left(kI + \frac{1}{2}x\, ad(V)\right)$$

Proof. *Since $V(x) = \sum_{j=-1}^{\infty} V_j x^j$ we have $xV(x) = \sum_{j=-1}^{\infty} V_j x^{j+1} = \sum_{l=0}^{\infty} V_{l-1}x^l$ and $V_{l-1} = \frac{1}{l!}\frac{d^l}{dx^l}\Big|_{x=0}(xV(x))$. Therefore,*

$$\sum_{j=0}^{k}\left[\frac{\theta^{(j)}(0)}{j!}, V_{k-1-j}\right] = \sum_{j=0}^{k}\left[\frac{\theta^{(j)}(0)}{j!}, \frac{1}{(k-j)!}\frac{d^{k-j}}{dx^{k-j}}\Big|_{x=0}(xV(x))\right] = \frac{1}{k!}\frac{d^k}{dx^k}\Big|_{x=0}(x[\theta, V]).$$

Thus,

$$P_k(\theta) = \frac{\theta^{(k)}(0)}{(k-1)!} - \frac{1}{2}\sum_{j=0}^{k}\left[\frac{\theta^{(j)}(0)}{j!}, V_{k-1-j}\right] = \frac{\theta^{(k)}(0)}{(k-1)!} - \frac{1}{2}\frac{1}{k!}\frac{d^k}{dx^k}\Big|_{x=0}(x\,[\theta, V])$$

$$= \frac{\theta^{(k)}(0)}{(k-1)!} + \frac{1}{2}\frac{1}{k!}\frac{d^k}{dx^k}\Big|_{x=0}(xad(V)(\theta)) = \frac{1}{k!}\frac{d^k}{dx^k}\Big|_{x=0}\left(kI + \frac{1}{2}xad(V)\right)(\theta).$$

Since θ is arbitrary we have the assertion. □

Corollary 14. *For every $k \in \mathbb{N}$, $P_k(V) = \frac{V^{(k-1)}(0)}{(k-1)!}$.*

Definition 14. *For $m \in \mathbb{N}$ define*

$$A_1^{[m]} = \begin{pmatrix} \frac{V_0}{2} & \frac{1}{2}V_{-1}+I_N & 0 & \cdots & 0 & 0 & 0 \\ \frac{V_1}{2} & \frac{V_0}{2} & \frac{1}{2}V_{-1}+2I_N & \cdots & 0 & 0 & 0 \\ . & . & \cdots & . & \cdots & . & . \\ . & . & \cdots & . & \cdots & . & . \\ . & . & \cdots & . & \cdots & . & . \\ \frac{V_{m-2}}{2} & \frac{V_{m-3}}{2} & \frac{V_{m-4}}{2} & \cdots & \frac{V_0}{2} & \frac{1}{2}V_{-1}+(m-1)I_N & 0 \\ \frac{V_{m-1}}{2} & \frac{V_{m-2}}{2} & \frac{V_{m-3}}{2} & \cdots & \frac{V_1}{2} & \frac{V_0}{2} & \frac{1}{2}V_{-1}+mI_N \\ \frac{V_m}{2} & \frac{V_{m-1}}{2} & \frac{V_{m-2}}{2} & \cdots & \frac{V_2}{2} & \frac{V_1}{2} & \frac{V_0}{2} \end{pmatrix},$$

$$A_2^{[m]} = \begin{pmatrix} V_{m+1} & V_m & \cdots & V_1 \\ V_{m+2} & V_{m+1} & \cdots & V_2 \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \end{pmatrix},$$

and for $\theta \in M_N(\mathbb{C}[x])$ we define $P_1^{m+1}(\theta) = (P_1(\theta), P_2(\theta), \cdots, P_m(\theta), P_{m+1}(\theta))^T$ and $P_{m+2}^{\infty}(\theta) = (P_{m+2}(\theta), P_{m+3}(\theta), \cdots)^T$.

Note that $A_1^{[m]}, A_2^{[m]}$ depend on V and $P_1^{m+1}(\theta), P_{m+2}^{\infty}(\theta)$ depend on θ. The following lemma gives a simplification of these matrices for m large enough when V is a polynomial.

Lemma 14. *If V is a polynomial of degree n and $m+1 = nq + r$, $q \geq 1$, $0 \leq r < n$, then*

$$A_1^{[m]} = \begin{pmatrix} A_1^{[r-1]} & rT_r^{r-1}\,,0_{r\times(n-r)} & 0_{n\times n} & \cdots & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ \frac{1}{2}(A_2^{[n-1]})^{1,2,\cdots,r} & A_1^{[n-1]} + rS_n & (n+r)T_n^{n-1} & \cdots & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \frac{1}{2}A_2^{[n-1]} & A_1^{[n-1]} + (n+r)S_n & \cdots & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ . & . & \cdots & . & \cdots & . & . \\ . & . & \cdots & . & \cdots & . & . \\ . & . & \cdots & . & \cdots & . & . \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \cdots & \frac{1}{2}A_2^{[n-1]} & A_1^{[n-1]} + (n(q-2)+r)S_n & (n(q-1)+r)T_n^{n-1} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} & \frac{1}{2}A_2^{[n-1]} & A_1^{[n-1]} + (n(q-1)+r)S_n \end{pmatrix},$$

and

$$A_2^{[m]} = \begin{pmatrix} 0_{n\times(m+1-n)} & A_2^{[n-1]} \\ 0_{\infty\times(m+1-n)} & 0_{\infty\times n} \end{pmatrix},$$

i.e., $A_1^{[m]}$ is a block tridiagonal matrix and $A_2^{[m]}$ is a block upper triangular matrix.

Proof. *If V is a polynomial of degree n the assertion about $A_2^{[m]}$ is clear. On the other hand, note that we can write*

$$A_1^{[m]} = \begin{pmatrix} \begin{matrix} & \\ A_1^{[m-n]} & \\ & \end{matrix} & \begin{matrix} & 0_{(m+1-2n)\times n} \\ & \\ & (m+1-n)T_n^{n-1} \end{matrix} \\ \begin{matrix} 0_{n\times(m+1-2n)} & \frac{1}{2}A_2^{[n-1]} \end{matrix} & A_1^{[n-1]} + (m+1-n)S_n \end{pmatrix} \tag{3.3}$$

Since the $A_1^{[m]}$ is a block matrix of size $(q+1)\times(q+1)$ we can use induction over q. Notice that the assertion is clear for $q = 1$ because in this case $m+1 = n+r$ and

$$A_1^{[m]} = \begin{pmatrix} A_1^{[r-1]} & rT_r^{r-1}\,,0_{r\times(n-r)} \\ \frac{1}{2}(A_2^{[n-1]})^{1,2,\cdots,r} & A_1^{[n-1]} + rS_n \end{pmatrix}.$$

Now let $m+1 = nq + r$ and assume the assertion for $m - n$ or equivalently for a matrix of size $q \times q$. Therefore,

$$A_1^{[m-n]} = \begin{pmatrix} A_1^{[r-1]} & rT_r^{r-1}\ ,0_{r\times(n-r)} & 0_{n\times n} & \cdots & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ \frac{1}{2}(A_2^{[n-1]})^{1,2,\cdots,r} & A_1^{[n-1]} + rS_n & (n+r)T_n^{n-1} & \cdots & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \frac{1}{2}A_2^{[n-1]} & A_1^{[n-1]} + (n+r)S_n & \cdots & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ . & . & \cdots & . & \cdots & . & . \\ . & . & \cdots & . & \cdots & . & . \\ . & . & \cdots & . & \cdots & . & . \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \cdots & \frac{1}{2}A_2^{[n-1]} & A_1^{[n-1]} + (n(q-3)+r)S_n & (n(q-2)+r)T_n^{n-1} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} & \frac{1}{2}A_2^{[n-1]} & A_1^{[n-1]} + (n(q-2)+r)S_n \end{pmatrix}.$$

If we replace this in (3.3) we obtain the claim. Thus, the assertion follows by induction. □

The following theorem characterizes bispectrality using the family $\{P_k\}_{k\in\mathbb{N}}$.

Theorem 20. *Let*

$$\Gamma = \left\{\theta \in M_N(\mathbb{C}[x]) \mid P_0(\theta) = 0,\, V_{-1}e_1(A_1^{[m]})^k P_1^{m+1}(\theta) = 0, A_2^{[m]}(A_1^{[m]})^k P_1^{m+1}(\theta) = 0, k \geq 0, P_{m+2}^{\infty}(\theta) = 0, m = \deg(\theta)\right\}$$

then $\Gamma = \mathbb{A}$. Moreover, for each θ we have an explicit expression for the operator B such that

$$(\psi B)(x,z) = \theta(x)\psi(x,z).$$

Remark 8. *Before proving the Theorem 20 we observe that since $A_1^{[m]} \in M_{(m+1)N}(\mathbb{C})$ the Cayley-Hamilton Theorem implies that we can assume that*

$\Gamma = \Big\{\theta \in M_N(\mathbb{C}[x]) \mid P_0(\theta) = 0,\, V_{-1}e_1(A_1^{[m]})^k P_1^{m+1}(\theta) = 0, A_2^{[m]}(A_1^{[m]})^k P_1^{m+1}(\theta) = 0, 0 \leq k \leq (m+1)N-1,$
$P_{m+2}^{\infty}(\theta) = 0, m = \deg(\theta)\}$.

Proof. *If we consider $\theta(x) = \sum_{j=0}^{m} a_j x^j$ and $B(z,\partial_z) = \sum_{j=0}^{m} \partial_z^j \cdot b_j(z)$ then,*

$$\Lambda(x,z) = e^{-xz}((\psi B)(x,z) - \theta(x)\psi(x,z))$$

$$= e^{-xz}\left(\sum_{j=0}^{m} \partial_z^j\left(\left(Iz + \frac{1}{2}V(x)\right)e^{xz}\right)\cdot b_j(z) - \sum_{j=0}^{m} a_j x^j\left(Iz + \frac{1}{2}V(x)\right)e^{xz}\right)$$

$$= e^{-xz}\left(\sum_{j=0}^{m}\left(\sum_{l=0}^{j}\binom{j}{l}\partial_z^l\left(Iz + \frac{1}{2}V(x)\right)\partial_z^{j-l}(e^{xz})\right)b_j(z) - \sum_{j=0}^{m} a_j x^j z e^{xz} - \sum_{j=0}^{m}\frac{a_j}{2}x^j V(x)e^{xz}\right)$$

$$= e^{-xz}\left(\sum_{j=0}^{m}\left(\left(Iz + \frac{1}{2}V(x)\right)x^j e^{xz} + jx^{j-1}e^{xz}\right)b_j(z) - \sum_{j=0}^{m} a_j x^j z e^{xz} - \sum_{j=0}^{m}\frac{a_j}{2}x^j V(x)e^{xz}\right)$$

$$= \sum_{j=0}^{m} b_j(z)x^j z + \sum_{j=0}^{m} \frac{1}{2} V(x)x^j b_j(z) + \sum_{j=0}^{m} jx^{j-1} b_j(z) - \sum_{j=0}^{m} a_j x^j z - \sum_{j=0}^{m} \frac{a_j}{2} x^j V(x).$$

Writing $V(x) = \sum_{j=-1}^{\infty} V_j x^j$ *then,*

$$\Lambda(x,z) = \sum_{j=0}^{m} b_j(z)x^j z + \sum_{j=0}^{m} \frac{1}{2} \left(\sum_{k=-1}^{\infty} V_j x^j \right) x^j b_j(z) + \sum_{j=0}^{m} jx^{j-1} b_j(z) - \sum_{j=0}^{m} a_j x^j z - \sum_{j=0}^{m} \frac{a_j}{2} x^j \left(\sum_{k=-1}^{\infty} V_j x^j \right)$$

$$= \sum_{j=0}^{m} b_j(z)x^j z + \sum_{j=0}^{m} \sum_{k=-1}^{\infty} \frac{1}{2} V_k b_j(z) x^{k+j} + \sum_{j=0}^{m} jx^{j-1} b_j(z) - \sum_{j=0}^{m} a_j x^j z - \sum_{j=0}^{m} \sum_{k=-1}^{\infty} \frac{a_j}{2} V_k x^{k+j}.$$

Let $s = k + j$ *then* s *varies from* -1 *to* ∞.

$$\Lambda(x,z) = \sum_{j=0}^{m} b_j(z)x^j z + \sum_{s=-1}^{\infty} \sum_{j=0}^{s+1} \frac{1}{2} V_{s-j} b_j(z) x^s + \sum_{j=1}^{m} jx^{j-1} b_j(z) - \sum_{j=0}^{m} a_j x^j z - \sum_{s=-1}^{\infty} \sum_{j=0}^{s+1} \frac{1}{2} a_j V_{s-j} x^s$$

$$= \sum_{j=0}^{m} b_j(z) z x^j + \sum_{s=-1}^{\infty} \left(\sum_{j=0}^{s+1} \frac{1}{2} V_{s-j} b_j(z) \right) x^s + \sum_{s=0}^{m-1} (s+1) b_{s+1}(z) x^s - \sum_{j=0}^{m} a_j z x^j - \sum_{s=-1}^{\infty} \left(\sum_{j=0}^{s+1} \frac{1}{2} a_j V_{s-j} \right) x^s$$

$$= \frac{1}{2}(V_{-1} b_0(z) - a_0 V_{-1}) + \sum_{s=0}^{m-1} \left(\sum_{j=0}^{s+1} \frac{1}{2} V_{s-j} b_j(z) + b_s(z) z + (s+1) b_{s+1}(z) - a_s z - \sum_{j=0}^{s+1} \frac{1}{2} a_j V_{s-j} \right) x^s$$

$$+ \left(b_m(z) z + \sum_{j=0}^{m+1} \frac{1}{2} V_{m-j} b_j(z) - a_m z - \sum_{j=0}^{m+1} \frac{1}{2} a_j V_{m-j} \right) x^m$$

$$+ \sum_{s=m+1}^{\infty} \left(\sum_{j=0}^{s+1} \frac{1}{2} (V_{s-j} b_j(z) - a_j V_{s-j}) \right) x^s$$

if, and only if,

$$V_{-1} b_0 - a_0 V_{-1} = 0,$$

$$(b_s(z) - a_s) z + (s+1) b_{s+1}(z) + \frac{1}{2} \sum_{k=0}^{s} (V_{s-k} b_k - a_k V_{s-k}) + \frac{1}{2}(V_{-1} b_{s+1} - a_{s+1} V_{-1}) = 0,$$

for $0 \leq s \leq m-1$.

$$(b_m(z) - a_m) z + \frac{1}{2} \sum_{k=0}^{m} (V_{m-k} b_k - a_k V_{m-k}) = 0,$$

$$\sum_{k=0}^{m} (V_{s-k} b_k - a_k V_{s-k}) = 0,$$

for $s \geq m+1$.

If we define $c_j(z) = b_j(z) - a_j$ *we have*

$$\frac{1}{2}V_{-1}c_0(z) = -P_0(\theta),$$

$$\begin{pmatrix} z+\frac{V_0}{2} & \frac{1}{2}V_{-1}+I_N & 0 & \cdots & 0 & 0 & 0 \\ \frac{V_1}{2} & z+\frac{V_0}{2} & \frac{1}{2}V_{-1}+2I_N & \cdots & 0 & 0 & 0 \\ . & . & \cdots & . & \cdots & . & . \\ . & . & \cdots & . & \cdots & . & . \\ . & . & \cdots & . & \cdots & . & . \\ \frac{V_{m-2}}{2} & \frac{V_{m-3}}{2} & \frac{V_{m-4}}{2} & \cdots & z+\frac{V_0}{2} & \frac{1}{2}V_{-1}+(m-1)I_N & 0 \\ \frac{V_{m-1}}{2} & \frac{V_{m-2}}{2} & \frac{V_{m-3}}{2} & \cdots & \frac{V_1}{2} & z+\frac{V_0}{2} & \frac{1}{2}V_{-1}+mI_N \\ \frac{V_m}{2} & \frac{V_{m-1}}{2} & \frac{V_{m-2}}{2} & \cdots & \frac{V_2}{2} & \frac{V_1}{2} & z+\frac{V_0}{2} \end{pmatrix} \begin{pmatrix} c_0(z) \\ c_1(z) \\ . \\ . \\ . \\ c_{m-2}(z) \\ c_{m-1}(z) \\ c_m(z) \end{pmatrix} = \begin{pmatrix} -P_1(\theta) \\ -P_2(\theta) \\ . \\ . \\ . \\ -P_{m-1}(\theta) \\ -P_m(\theta) \\ -P_{m+1}(\theta) \end{pmatrix}$$

and

$$\begin{pmatrix} V_{m+1} & V_m & \cdots & V_1 \\ V_{m+2} & V_{m+1} & \cdots & V_2 \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \end{pmatrix} \begin{pmatrix} c_0(z) \\ c_1(z) \\ . \\ . \\ . \\ c_{m-2}(z) \\ c_{m-1}(z) \\ c_m(z) \end{pmatrix} = \begin{pmatrix} -P_{m+2}(\theta) \\ -P_{m+3}(\theta) \\ . \\ . \\ . \end{pmatrix}.$$

Using the notation defined above

$$\frac{1}{2}V_{-1}c_0(z) = -P_0(\theta), (A_1^{[m]}+z)c(z) = -P_1^{m+1}(\theta), A_2^{[m]}c(z) = -P_{m+2}^{\infty}(\theta).$$

However, $(A_1^{[m]}+z)^{-1} = \sum_{k=0}^{\infty}\frac{(-A_1^{[m]})^k}{z^{k+1}}$ *implies that* $c(z) = -\sum_{k=0}^{\infty}\frac{(-A_1^{[m]})^k}{z^{k+1}}P_1^{m+1}(\theta)$ *and* $P_{m+2}^{\infty}(\theta) = A_2^{[m]}\sum_{k=0}^{\infty}\frac{(-A_1^{[m]})^k}{z^{k+1}}P_1^{m+1}(\theta)$*, using z as variable we obtain* $P_0(\theta) = 0, A_2^{[m]}(A_1^{[m]})^kP_1^{m+1}(\theta) = 0, k \geq 0$ *and* $P_{m+2}^{\infty}(\theta) = 0$*. Furthermore,* $c_s(z) = -e_{s+1}(A_1^{[m]}+z)^{-1}P_1^{m+1}(\theta)$ *for* $0 \leq s \leq m$*. In particular* $c_0(z) = -e_1(A_1^{[m]}+z)^{-1}P_1^{m+1}(\theta)$ *then,* $V_{-1}e_1(A_1^{[m]}+z)^{-1}P_1^{m+1}(\theta) = \sum_{k=0}^{\infty}V_{-1}e_1\frac{(-A_1^{[m]})^k}{z^{k+1}}P_1^{m+1}(\theta) = 0$*. Using z as variable we obtain* $V_{-1}e_1(A_1^{[m]})^kP_1^{m+1}(\theta) = 0$ *for every* $k \in \mathbb{N}$*. We shall now use this remark to conclude the* *proof of the theorem.*

If $\theta \in \mathbb{A}$, then there exists $B = \sum_{j=0}^{m} \partial_z^j \cdot b_j(z)$ such that

$$\Lambda(x,z) = e^{-xz}((\psi B)(x,z) - \theta(x)\psi(x,z)) = 0.$$

But this is equivalent to $P_0(\theta) = 0$, $V_{-1}e_1(A_1^{[m]} + z)^{-1}P_1^{m+1}(\theta) = 0, (A_1^{[m]} + z)c(z) = P_1^{m+1}(\theta)$, $A_2^{[m]}c(z) = P_{m+2}^{\infty}(\theta)$ with $c(z) = b(z) - a$, $b = (b_0, \cdots, b_m)$, $a = (a_0, \cdots, a_m)$.

By the previous remark we have

$$P_0(\theta) = 0,\, V_{-1}e_1(A_1^{[m]} + z)^{-1}P_1^{m+1}(\theta) = 0, A_2^{[m]}(A_1^{[m]})^k P_1^{m+1}(\theta) = 0, k \geq 0 \text{ and } P_{m+2}^{\infty}(\theta) = 0.$$

Then $\theta \in \Gamma$. Since $\theta \in \mathbb{A}$ was arbitrary we have $\mathbb{A} \subset \Gamma$.

On the other hand, if $\theta \in \Gamma$, then $P_0(\theta) = 0$, $V_{-1}e_1(A_1^{[m]} + z)^{-1}P_1^{m+1}(\theta) = 0, A_2^{[m]}(A_1^{[m]})^k P_1^{m+1}(\theta) = 0, k \geq 0$ and $P_{m+2}^{\infty}(\theta) = 0$.

Taking

$$b_j(z) = a_j + e_j(A_1^{[m]} + z)^{-1}P_1^{m+1}(\theta),$$

for $0 \leq j \leq m$.

We have $c(z) = -\sum_{k=0}^{\infty} \frac{(-A_1^{[m]})^k}{z^{k+1}} P_1^{m+1}(\theta) = (A_1^{[m]} + z)^{-1}P_1^{m+1}(\theta)$ and therefore

$$\frac{1}{2}V_{-1}c_0(z) = -P_0(\theta), (A_1^{[m]} + z)c(z) = -P_1^{m+1}(\theta), A_2^{[m]}c(z) = -P_{m+2}^{\infty}(\theta).$$

By the previous arguments we obtain that

$$\Lambda(x,z) = e^{-xz}((\psi B)(x,z) - \theta(x)\psi(x,z)) = 0,$$

with $B = \sum_{j=0}^{m} b_j(z) \cdot \partial_z^j$. This implies that $\theta \in \mathbb{A}$. Since $\theta \in \Gamma$ was arbitrary we have $\Gamma \subset \mathbb{A}$.

Thus, $\Gamma = \mathbb{A}$ and for every $\theta \in \mathbb{A}$ there exists a unique operator $B = \sum_{j=0}^{m} \partial_z^j \cdot b_j(z)$ given by

$$b_j(z) = a_j - e_j(A_1^{[m]} + z)^{-1}P_1^{m+1}(\theta),$$

for $0 \leq j \leq m$, *such that*

$$(\psi B)(x,z) = \theta(x)\psi(x,z).$$

This concludes the proof of the assertion.

Corollary 15. *For* $\theta \in \mathbb{A}$ *the operator* $B = \sum_{j=0}^{m} \partial_z^j \cdot b_j(z)$ *such that*

$$(\psi B)(x,z) = \theta(x)\psi(x,z).$$

satisfies $\lim_{z\to\infty} b_j(z) = a_j$ *for* $0 \leq j \leq m$.

In the following result we rewrite the expressions defining the algebra Γ for another more simple to remind.

Lemma 15. *The algebra* Γ *is exactly the set of all polynomial* $\theta \in M_N(\mathbb{C})[x]$, $m = \deg(\theta)$ *such that* $[\theta, V]$ *is a polynomial of degree* $\leq m$ *and*

$$V_{-1}e_1(A_1^{[m]})^k P_1^{m+1}(\theta) = 0, A_2^{[m]}(A_1^{[m]})^k P_1^{m+1}(\theta) = 0,$$

for $0 \leq k \leq (m+1)N - 1$.

Proof. *Note that* $k \geq m+1$ *implies* $P_k(\theta) = \frac{\theta^{(k)}(0)}{(k-1)!} + \frac{1}{2}\frac{1}{k!}\frac{d^k}{dx^k}\Big|_{x=0}(xad(V)(\theta)) = \frac{1}{2}\frac{1}{k!}\frac{d^k}{dx^k}\Big|_{x=0}(xad(V)(\theta))$. *Since* $\frac{d^k}{dx^k}\Big|_{x=0}(xad(V)(\theta)) = P_k(\theta) = 0$ *for* $k \geq m+2$ *we have that* $x[\theta, V]$ *is a polynomial of degree* $\leq m+1$. *Furthermore,* $P_0(\theta) = \frac{1}{2}(xad(V))(\theta)\Big|_{x=0} = 0$ *we have that* $x[\theta, V]\Big|_{x=0} = 0$. *However, since* $x[\theta, V]$ *is a polynomial we have that* $[\theta, V]$ *is a polynomial of degree* $\leq m$. *Moreover, we have the restrictions* $V_{-1}e_1(A_1^{[m]})^k P_1^{m+1}(\theta) = 0, A_2^{[m]}(A_1^{[m]})^k P_1^{m+1}(\theta) = 0,$ *for* $0 \leq k \leq (m+1)N - 1$. □

Now we try to find some solutions of these equations. To do this we put restrictions on the matrix V_0 and V_1 to obtain $V \in \mathbb{A}$. We begin with a definition

Definition 15. *We define the grading* $\deg_{1,2}$ *on the ring* $\mathbb{C}\langle V_0, V_1\rangle$ *to be* $\deg_{1,2}(V_0) = 1$, $\deg_{1,2}(V_1) = 2$.

With this definition we can obtain interesting results.

Proposition 8. *If* $(V_0, V_1) \in M_N(\mathbb{C})^2$ *satisfies*

$$V_1^{i_1}V_0^{i_2}\cdots .V_1^{i_n}V_0^{i_{n+1}} = 0, \tag{3.4}$$

for any $i_1 \geq 1$, $i_1 + \cdots + i_{n+1} \leq n+1$, *and* $n+2 \leq \deg_{1,2}(V_1^{i_1} V_0^{i_2} \cdots . V_1^{i_n} V_0^{i_{n+1}})$, *then*

$$V_{j_1} \cdots V_{j_k} = 0,\ j_1 + \cdots + j_k \geq n, j_1 \geq 1, k \geq 2.$$

Proof. *Note that* (3.4) *implies that the monomials of degree* $\geq n+1$ *that begins with* V_1 *are zero. Since* $j_1 \geq 1$ *we have that every monomial in* $V_{j_1} \cdots V_{j_k}$ *begins with* V_1. *Furthermore, this polynomial is quasihomogenenous of degree* $\deg_{1,2}(V_{j_1} \cdots V_{j_k}) = (j_1+1) + \cdots + (j_k+1) = j_1 + \cdots + j_k + k \geq n+k \geq n+2$. *In particular, the polynomial* $V_{j_1} \cdots V_{j_k}$ *is a linear combination of monomials of the form* (3.4) *which are zero.* ☐

Remark 9. *Two important elements satisfying* (3.4) *are* $V_1^{i_1}$ *and* $V_1 V_0^n$ *with*

$$i_1 = \begin{cases} \frac{n+2}{2} & \text{if } n \text{ is even} \\ \left[\frac{n}{2}\right] + 2 & \text{if } n \text{ is odd} \end{cases}$$

Lemma 16. *For every* $k \geq 0$, $((A_1^{[m]})^k P_1^{n+1}(V))_i$ *is a polynomial in* $V_0, V_1, \cdots, V_n$ *such that the sum of the subindices in its monomials is* $\geq i$, $1 \leq i \leq n+1$.

Proof. *The proof is by induction over k. For* $k = 0$ *we are okay since*

$$(P_1^{n+1}(V))_i = P_i(V) = iV_i.$$

Assume the claim for $k \geq 0$ *and consider the case* $k+1$

$$((A_1^{[m]})^{k+1} P_1^{n+1}(V))_i = \sum_{j=1}^{n+1} (A_1^{[n]})_{ij} ((A_1^{[m]})^k P_1^{n+1}(V))_j$$

$$= \sum_{j=1}^{i} \frac{V_{i-j}}{2} ((A_1^{[m]})^k P_1^{n+1}(V))_j + i((A_1^{[m]})^k P_1^{n+1}(V))_{i+1}.$$

Since the sum of the subindices of the monomials of $((A_1^{[m]})^k P_1^{n+1}(V))_j$ *is* $\geq j$ *we obtain that the sum of the subindices in the monomials of* $((A_1^{[m]})^{k+1} P_1^{n+1}(V))_i$ *is* $\geq \min\{(i-j)+j, i+1\} = i$. ☐

Theorem 21 (Bispectral Property for a Class of Polynomial Potentials). *If* $V(V_0, V_1, x)$ *is a polynomial of degree n such that* $V''(x) = V'(x)V(x)$ *and* $(V_0, V_1) \in M_N(\mathbb{C})^2$ *satisfy* (3.4), *then* $V \in \mathbb{A}$. *In particular, the operator* $L = -\partial_x^2 + V'(x)$ *is bispectral.*

Proof. *Since $V''(x) = V'(x)V(x)$ we have that $L = -\partial_x^2 + V'(x)$ satisfies $(L\psi)(x,z) = -z^2\psi(x,z)$ with*

$$\psi(x,z) = \left(Iz + \frac{1}{2}V(x)\right)e^{xz}.$$

On the other hand, from the Theorem 1.1 we have $\mathbb{A} = \Gamma$. Furthermore, by the Lemma 15 we have that the right bispectral algebra is the set of all $\theta \in M_N(\mathbb{C})[x]$ such that $[\theta, V]$ is a polynomial of degree $\leq m$ and

$$V_{-1}e_1(A_1^{[m]})^k P_1^{m+1}(\theta) = 0, A_2^{[m]}(A_1^{[m]})^k P_1^{m+1}(\theta) = 0, \text{ for } 0 \leq k \leq (m+1)N - 1, \tag{3.5}$$

with $m = \deg(\theta)$.

However, since $V(x)$ is a polynomial we have $V_{-1} = 0$ and $[V, V] = 0$ is a polynomial of degree $\leq n := \deg(V)$.

Note that

$$(A_2^{[n]}(A_1^{[m]})^k P_1^{n+1}(V))_i = \sum_{j=1}^{n+1}(A_2^{[n]})_{ij}((A_1^{[m]})^k P_1^{n+1}(V))_j = \sum_{j=1}^{n+1} V_{i+n+1-j}((A_1^{[m]})^k P_1^{n+1}(V))_j.$$

By the Lemma 16 the sum of the subindices in the monomials of the polynomials $((A_1^{[m]})^k P_1^{n+1}(V))_j$ is $\geq j$. Therefore, the sum of the subindices of the monomials in the polynomial $(A_2^{[n]}(A_1^{[m]})^k P_1^{n+1}(V))_i$ is $\geq n$, $1 \leq i \leq n+1$. Thus, the Proposition 8 implies

$$A_2^{[n]}(A_1^{[m]})^k P_1^{n+1}(V) = 0, \ k \in \mathbb{N}.$$

Then, $V \in \mathbb{A}$. □

We conclude this chapter with some examples applying the previous theorems.

3.4 Illustrative Examples

In this section we give some ilustrative examples of bispectral operators $L = -\partial_x^2 + V'(x)$ with polynomial potentials V through the Theorem 21.

3.4.1 The Bispectral Algebra Associated to the Potential with Invertible Residue at $x = 0$

If $V_{-1} = -2I_N$, then $V_j = 0$ for every $j \in \mathbb{N}$ and $V(x) = \frac{-2I_N}{x}$. This implies that for every $m \in \mathbb{N}$,

$$A_1^{[m]} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I_N & \cdots & 0 & 0 & 0 \\ . & . & \cdots & . & \cdots & . & . \\ . & . & \cdots & . & \cdots & . & . \\ . & . & \cdots & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 0 & (m-2)I_N & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & (m-1)I_N \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

and $A_2^{[m]} = 0$.

Note that $A_1^{[m]} = \sum_{j=2}^{m}(j-1)I_N e_{j,j+1}$. We claim that $(A_1^{[m]})^k = \sum_{j=2}^{m-k+1}(j-1)j\cdots(j+k-2)I_N e_{j,j+k}$. We prove the claim by induction. The initial step $k=1$ is clear. Assume $k \geq 1$ and note that

$$(A_1^{[m]})^{k+1} = (A_1^{[m]})^k A_1^{[m]} = \left(\sum_{j=2}^{m-k+1}(j-1)j\cdots(j-k+2)I_N e_{j,j+k}\right)\left(\sum_{j=2}^{m}(j-1)I_N e_{j,j+1}\right)$$

$$= \sum_{j=2}^{m-k+1}\sum_{l=2}^{m}(j-1)j\cdots(j+k-2)(l-1)I_N e_{j,j+k}e_{l,l+1}$$

$$= \sum_{j=2}^{m-k}(j-1)j\cdots(j+k-2)(j+k-1)I_N e_{j,j+k}.$$

The claim follows by induction. We can write $(A_1^{[m]})^k = \sum_{j=2}^{m-k+1}\frac{(j+k-2)!}{(j-2)!}I_N e_{j,j+k}$. In particular, $(A_1^{[m]})^{m-1} = (m-1)!e_{2,m+1}$ and $(A_1^{[m]})^k = 0$ for every $k \geq m$.

This implies that,

$$(A_1^{[m]} + z)^{-1} = \sum_{k=0}^{\infty}\frac{(-1)^k(A_1^{[m]})^k}{z^{k+1}} = \frac{I_N}{z} + \sum_{k=1}^{m-1}\sum_{j=2}^{m-k+1}(-1)^k\frac{(j+k-2)!I_N}{(j-2)!z^{k+1}}e_{j,j+k}.$$

Therefore, if $\theta(x) = \sum_{l=0}^{m} a_l x^l$ we have $P_l(\theta) = la_l$ and

$$c(z) = -(A_1^{[m]} + z)^{-1} P_1^{m+1}(\theta) = -\left(\frac{I_N}{z} + \sum_{k=1}^{m-1} \sum_{j=2}^{m-k+1} (-1)^k \frac{(j+k-2)! I_N}{(j-2)! z^{k+1}} e_{j,j+k}\right) \left(\sum_{l=1}^{m+1} P_l(\theta) e_l\right)$$

$$= \sum_{k=1}^{m-1} \sum_{l=k+2}^{m+1} \frac{(l-2)!(-1)^{k+1}}{(l-k-2)! z^{k+1}} P_l(\theta) e_{l-k} - \sum_{l=1}^{m+1} \frac{P_l(\theta)}{z} e_l$$

$$= \sum_{k=1}^{m-1} \sum_{l=k+2}^{m+1} \frac{(l-2)!(-1)^{k+1}}{(l-k-2)! z^{k+1}} la_l e_{l-k} - \sum_{l=1}^{m+1} \frac{la_l}{z} e_l$$

$$= \sum_{k=1}^{m-1} \sum_{s=2}^{m-k+1} \frac{(s+k-2)!(s+k)(-1)^{k+1}}{(s-2)! z^{k+1}} a_{s+k} e_s - \sum_{l=1}^{m+1} \frac{la_l}{z} e_l$$

$$= \sum_{s=2}^{m} \left(\sum_{k=1}^{m-s+1} \frac{(s+k-2)!(s+k)(-1)^{k+1}}{(s-2)! z^{k+1}} a_{s+k}\right) e_s - \sum_{l=1}^{m+1} \frac{la_l}{z} e_l$$

$$= \sum_{s=2}^{m} \left(\sum_{j=s+1}^{m} \frac{(j-2)! j (-1)^{j-s+1}}{(s-2)! z^{j-s+1}} a_j\right) e_s - \sum_{l=1}^{m+1} \frac{la_l}{z} e_l$$

$$= -\frac{a_1}{z} e_l + \sum_{s=2}^{m} \left(\sum_{j=s+1}^{m} \frac{(j-2)! j (-1)^{j-s+1}}{(s-2)! z^{j-s+1}} a_j - \frac{sa_s}{z}\right) e_s$$

$$= -\frac{a_1}{z} e_l + \sum_{s=2}^{m} \left(\sum_{j=s}^{m} \frac{(j-2)! j (-1)^{j-s+1}}{(s-2)! z^{j-s+1}} a_j\right) e_s = \sum_{s=1}^{m+1} c_{s-1}(z) e_s.$$

Thus, $c_0(z) = b_0(z) - a_0 = -\frac{a_1}{z}$, $b_0(z) = a_0 - \frac{a_1}{z}$. Furthermore,

$$c_{s-1}(z) = \sum_{j=s}^{m} \frac{(j-2)! j (-1)^{j-s+1}}{(s-2)! z^{j-s+1}} a_j, 2 \leq s \leq m+1.$$

In other words,

$$c_s(z) = \sum_{j=s+1}^{m} \frac{(j-2)! j (-1)^{j-s}}{(s-1)! z^{j-s}} a_j, 1 \leq s \leq m.$$

On the other hand, we have the restrictions $\frac{1}{2} V_{-1} c_0(z) = 0$ and $P_0(\theta) = 0$. In this case the former restriction says that $c_0(z) = 0$ and the last is redundant. Therefore, $b_0(z) = a_0$ and $a_1 = 0$.

We conclude that, $\mathbb{A} = \left\{\theta \in M_N(\mathbb{C}[x]) \mid \theta'(0) = 0\right\}$ and for every $\theta \in \mathbb{A}$, $\theta(x) = \sum_{l=0}^{m} a_l x^l$ there exists $B(z, \partial_z) = \sum_{k=0}^{m} \partial_z^k \cdot b_k(z) = a_0 + \partial_z \cdot \sum_{j=2}^{m} \frac{(j-2)! j (-1)^{j-1}}{z^{j-1}} a_j + \sum_{k=2}^{m} \partial_z^k \cdot \left\{a_k + \sum_{j=k+1}^{m} \frac{(j-2)! j (-1)^{j-k}}{(k-1)! z^{j-k}} a_j\right\}$ such that $(\psi B)(x, z) = \theta(x) \psi(x, z)$.

3.4.2 Examples of Polynomial Potentials of degree $n = 1, 2, 3$

In this subsection we use Maxima to perform explicit examples of Theorem 21.

- For $n = 1$ the equations (3.4) turns out to be $V_1 V_0 = V_1^2 = 0$. For $N = 2$ we consider

$$V_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$V_0 = \begin{pmatrix} V_{011} & V_{012} \\ 0 & 0 \end{pmatrix}.$$

To obtain the potential $V(x) = V_0 + V_1 x$.

- For $n = 2$ the equations (3.4) turns out to be $V_1 V_0 V_1 = V_1 V_0^2 = V_1^2 = 0$. For $N = 4$ we consider

$$V_0 = \begin{pmatrix} V_{011} & V_{012} & V_{013} & V_{014} \\ 0 & 0 & V_{023} & V_{024} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$V_1 = \begin{pmatrix} 0 & V_{112} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To obtain

$$V_2 = \begin{pmatrix} 0 & 0 & \frac{V_{023} V_{112}}{2} & \frac{V_{024} V_{112}}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the potential $V(x) = V_0 + V_1 x + V_2 x^2$.

- For $n = 3$ the equations (3.4) turns out to be $V_1^3 = V_1 V_0 V_1 = V_1 V_0^3 = V_1^2 V_0 = V_1 V_0^2 V_1$. For

$N = 4$ we consider

$$V_0 = \begin{pmatrix} V_{011} & V_{012} & V_{013} & V_{014} \\ 0 & 0 & V_{023} & V_{024} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$V_1 = \begin{pmatrix} V_{111} & V_{112} & V_{113} & V_{114} \\ 0 & 0 & V_{123} & V_{124} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To obtain

$$V_2 = \begin{pmatrix} 0 & 0 & \frac{V_{023}V_{112}}{2} & \frac{V_{024}V_{112}}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$V_3 = \begin{pmatrix} 0 & 0 & \frac{V_{112}V_{123}}{2} & \frac{V_{112}V_{124}}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the potential $V(x) = V_0 + V_1 x + V_2 x^2 + V_3 x^3$.

Appendix

.1 The Ad-Condition and Polynomial Eigenvalues

We close this thesis with a generalization of a key lemma from the work of [7] to the noncommutative case.

Proposition 9. *Let* $\mathcal{L} = \mathcal{L}(x, \partial_x) = \sum_{j=0}^{l} L_j(x)\partial_x^j$, $\Theta = \Theta(x, \partial_x) = \sum_{s=0}^{m} \theta_s(x)\partial_x^s$ *then* $[L_l, \theta_m] = 0$ *implies* $\deg_{\partial_x}(ad(\mathcal{L})(\Theta)) \leq m + l - 1$ *and* $[L_l, \theta_m] \neq 0$ *implies* $\deg_{\partial_x}(ad(\mathcal{L})(\Theta)) = m + l$.

Proof. *By definition* $(ad\ \mathcal{L})(\Theta) = [\mathcal{L}, \Theta] = \mathcal{L}\Theta - \Theta\mathcal{L}$ *then*

$$(ad\mathcal{L})(\Theta) = \mathcal{L}\Theta - \Theta\mathcal{L} = \left(\sum_{j=0}^{l} L_j(x)\partial_x^j\right)\left(\sum_{s=0}^{m} \theta_s(x)\partial_x^s\right) - \left(\sum_{s=0}^{m} \theta_s(x)\partial_x^s\right)\left(\sum_{j=0}^{l} L_j(x)\partial_x^j\right)$$

$$= \sum_{j=0}^{l}\sum_{s=0}^{m} L_j\partial_x^j(\theta_s\partial_x^s) - \sum_{s=0}^{m}\sum_{j=0}^{l} \theta_s\partial_x^s(L_j\partial_x^j) = \sum_{j=0}^{l}\sum_{s=0}^{m} L_j \sum_{k=0}^{j} \begin{pmatrix} j \\ k \end{pmatrix} \theta_s^{(k)}\partial_x^{j-k+s} - \sum_{s=0}^{m}\sum_{j=0}^{l} \theta_s \sum_{r=0}^{s} \begin{pmatrix} s \\ r \end{pmatrix} L_j^{(r)}\partial_x^{s-r+j}$$

$$= \sum_{k=0}^{l}\sum_{s=0}^{m}\sum_{j=k}^{l} L_j \begin{pmatrix} j \\ k \end{pmatrix} \theta_s^{(j-k)}\partial_x^{k+s} - \sum_{s=0}^{m}\sum_{j=0}^{l}\sum_{j=s}^{m} \theta_j \begin{pmatrix} j \\ s \end{pmatrix} L_k^{(j-s)}\partial_x^{k+s}$$

$$= \sum_{k=0}^{l}\sum_{s=0}^{m} \left(\sum_{j=k}^{l} L_j \begin{pmatrix} j \\ k \end{pmatrix} \theta_s^{(j-k)} - \sum_{j=s}^{m} \theta_j \begin{pmatrix} j \\ s \end{pmatrix} L_k^{(j-s)}\right)\partial_x^{k+s}$$

$$= \sum_{r=0}^{m+l} \left(\sum_{k+s=r, 0\leq k\leq l, 0\leq s\leq m} \left(\sum_{j=k}^{l} L_j \begin{pmatrix} j \\ k \end{pmatrix} \theta_s^{(j-k)} - \sum_{j=s}^{m} \theta_j \begin{pmatrix} j \\ s \end{pmatrix} L_k^{(j-k+r)}\right)\right)\partial_x^r$$

$$= \sum_{r=0}^{m+l} a_r\partial_x^r$$

with

$$a_r = \sum_{k+s=r, 0\le k\le l, 0\le s\le m} \left(\sum_{j=k}^{l} L_j \begin{pmatrix} j \\ k \end{pmatrix} \theta_s^{(j-k)} - \sum_{j=s}^{m} \theta_j \begin{pmatrix} j \\ s \end{pmatrix} L_k^{(j-k+r)} \right)$$

in particular $a_{m+l} = L_l\theta_m - \theta_m L_l = [L_k, \theta_m]$, *hence* $[L_k, \theta_m] = 0$ *implies* $\deg_{\partial_x}(ad(\mathcal{L})(\theta)) \le m + l - 1$ *and* $[L_k, \theta_m] \ne 0$ *implies* $\deg_{\partial_x}(ad(\mathcal{L})(\theta)) = m + l$. □

The previous proposition implies that if $k+s = m+l-1,\ \ 0 \le k \le l,\ \ 0 \le s \le m$ then $(k,s) = (l-1, m)$ or $(k,s) = (l, m-1)$, therefore

$$a_{m+l-1} = \sum_{j=l-1}^{l} L_j \begin{pmatrix} j \\ j-1 \end{pmatrix} \theta_m^{(j-l+1)} - \theta_m L_{l-1} + L_l\theta_{m-1} - \sum_{j=m-1}^{m} \theta_j \begin{pmatrix} j \\ m-1 \end{pmatrix} L_l^{(j-m+1)}$$

$$= L_{l-1}\theta_m + lL_l\theta_m^{'} - \theta_m L_{l-1} + L_l\theta_{m-1} - \theta_{m-1}L_l - m\theta_m L_l^{'} = [L_{l-1}, \theta_m] + [L_l, \theta_{m-1}] + lL_l\theta_m^{'} - m\theta_m L_l^{'}.$$

If $[L_{l-1}, \theta_m] = 0$, $[L_l, \theta_{m-1}] = 0$, then

$$a_{m+l-1} = lL_l\theta_m^{'} - m\theta_m L_l^{'}.$$

In particular if $m = 0$, $\Theta = \theta_0$ and $[L_{l-1}, \theta_0] = 0$, then

$$a_{l-1} = lL_l\theta_0^{'}.$$

If we assume the system of equations 1.1 we obtain:

$$(ad\ \mathcal{L})(\theta)\psi = [\mathcal{L}, \theta]\,\psi = (\mathcal{L}\theta - \theta\mathcal{L})\,\psi = \mathcal{L}(\theta\psi) - \theta\mathcal{L}\psi = \mathcal{L}(\psi\mathcal{B}) - \theta\psi F$$

$$= (\mathcal{L}\psi)\mathcal{B} - (\psi\mathcal{B})F = (\psi F)\mathcal{B} - \psi\mathcal{B}F = \psi\,[F, \mathcal{B}] = \psi(ad\ F)(\mathcal{B}).$$

Now we prove by induction that

$$(ad\mathcal{L})^r(\theta)\psi = \psi(ad\ F)^r(\mathcal{B}),$$

for all $r \in \mathbb{Z}_+$.

The claim is clear for $r = 1$. Assume the condition for r and consider the case $r + 1$, then

$$\psi(ad\ F)^{r+1}(\mathcal{B}) = \psi(ad\ F)(ad\ F)^r(\mathcal{B}) = (F\psi)(ad\ F)^r(\mathcal{B}) - F(\psi)(ad\ F)^r(\mathcal{B})$$

$$= (F\psi)(ad\,F)^r(\mathcal{B}) - F(ad\mathcal{L})^r(\theta)\psi = (\mathcal{L}\psi)(ad\,F)^r(\mathcal{B}) - F(ad\mathcal{B})^r(\theta)\psi$$

$$= \mathcal{L}(\psi(ad\,F)^r(\mathcal{B})) - (ad\mathcal{L})^r(\theta)(F\psi) = \mathcal{L}(ad\mathcal{L})^r(\theta)\psi - (ad\mathcal{L})^r(\theta)(\mathcal{L}\psi)$$

$$= (\mathcal{L}(ad\mathcal{L})^r(\theta) - (ad\mathcal{L})^r(\theta)\mathcal{L})\psi = (ad\mathcal{L})(ad\mathcal{L})^r(\theta)\psi = (ad\mathcal{L})^{r+1}(\theta)\psi.$$

If $\deg_{\partial_z}\mathcal{B} = m$ and F is scalar we use the Proposition 9 to conclude $(ad\,\mathcal{L})^{m+1}(\theta)\psi = \psi(ad\,F)^{m+1}(\mathcal{B}) = 0$, similarly if $\deg\mathcal{L} = l$ then $\deg_{\partial_x}(ad\mathcal{L})^{m+1}(\theta) \leq (m+1)(l-1)$, in our case $\deg_{\partial_x}(ad\mathcal{L})^{m+1}(\theta) \leq m+1 < \infty$ since $\psi(\cdot,z) \in \ker((ad\mathcal{L})^{m+1}(\theta))$ for every $z \in \mathbb{C}$ and $\{\psi(\cdot,z)\}_{z\in\mathbb{C}}$ is a linearly independent set and $\dim\ker((ad\mathcal{L})^{m+1}(\theta)) \leq \deg_{\partial_x}(ad\mathcal{L})^{m+1}(\theta)$ if $(ad\mathcal{L})^{m+1}(\theta) \neq 0$ we have that $(ad\mathcal{L})^{m+1}(\theta) = 0$.

Finally we claim that if $\mathcal{L} = \sum_{j=0}^{l} L_j\partial_x^j$ with $L_l \in \mathbb{C}\setminus\{0\}$ and $L_{l-1} = 0$, then

$$\mathbf{coeff}((ad\mathcal{L})^{k+1}(\theta), \partial_x, (k+1)(l-1)) = (lL_l)^{k+1}\theta^{(k+1)}$$

for every $k \in \mathbb{N}$.

The claim is obvious for $k = 0$. If we assume that the claim is valid for k, then

$$\mathbf{coeff}((ad\mathcal{L})^{k+2}(\theta), \partial_x, (k+2)(l-1)) = \mathbf{coeff}((ad\mathcal{L})(ad\mathcal{L})^{k+1}(\theta), \partial_x, (k+2)(l-1))$$

$$= lL_l\partial_x((lL_l)^{k+1}\theta^{(k+1)}) = (lL_l)(lL_l)^{k+1}\theta^{(k+2)} = (lL_l)^{k+2}\theta^{(k+2)}$$

because L_l is constant and scalar.

Since $(ad\mathcal{L})^{m+1}(\theta) = 0$ we have that $\theta^{(m+1)} = 0$ and θ has to be a polynomial with $\deg\theta \leq m$.

Conclusions and Future Directions

In this dissertation we characterized the bispectral triples associated to a certain class of matrix-valued eigenfuctions. Furthermore, we established important properties of the full rank 1 algebras as a model of some bispectral algebras. These properties include the fact that they are Noetherian and finitely generated. An important role was played by the Ad-condition due to the fact that the matrix-valued operators were acting from opposite directions. Additionally, we characterized the bispectral algebra associated with some type of matrix Schrödinger operators with polynomial potential. This characterization was achieved using the family of matrix valued functions $\mathcal{P} = \{P_k\}_{k\in\mathbb{N}}$.

Clearly, there are many open directions to investigate related to the work developed here. To cite a few:

1. Motivated by the quest for bispectral partners to the operators in Section 1.3 we find a family of maps $\mathcal{P} = \{P_k\}_{k\in\mathbb{N}}$ with the translation and product properties of Theorem 11 and Lemma 5 generating the algebra in Example 1.3.1. Is it possible to do this for the general case? Or, would this be possible at least for the bispectral partners of a given Schrödinger operator?

2. To investigate the presentations of the full rank 1 algebras which by Theorem 7 are finitely generated. As we saw, the examples given in [11]and worked out here, are finitely presented. However, this is not necessarily true for general non-commutative rings.

3. To generalize Theorem 21 for analytic matrix-valued potentials with Laurent series with simple poles.

4. To study deeply the generating function for the family of noncommutative polynomials $\{V_k\}_{k\in\mathbb{N}}$ using function theoretic methods.

5. To look for a generalization of the Spin-Calogero examples in Section 2.3.1 to arbitrarily sized matrices

and characterize the associated bispectral algebra using a family of generators as well as the relations among them.

6. To define a matrix-valued inner product that orthogonalizes the family of noncommutative polynomials $\{V_k(V_0, V_1)\}_{k\in\mathbb{N}}$ in two noncommutative variables.

7. To analyze the bispectrality of linear matrix differential operators $L = \sum_{i=0}^{l} a_i(x) \cdot \partial_x^i$ of order greater than 2 whose bispectral eigenfunction is parametrized by the coefficients $a_i(x)$, $0 \leq i \leq l$.

Bibliography

[1] Ánh, P. & Birkenmeier, G. (2017). Peirce decompositions, idempotents and rings.

[2] Bergvelt, M., Gekhtman, M., & Kasman, A. (2008). Spin calogero particles and bispectral solutions of the matrix kp hierarchy. *Mathematical Physics Analysis and Geometry*, 12.

[3] Boyallian, C. & Liberati, J. (2008). Matrix-valued bispectral operators and quasideterminants. *Journal of Physics A: Mathematical and Theoretical*, 41, 365209.

[4] Casper, W. (2015). Elementary examples of solutions to bochner's problem for matrix differential operators. *Journal of Approximation Theory*, 229.

[5] Casper, W. R. (2017). *Bispectral Operator Algebras*. University of Washington. Thesis (Ph.D.)–University of Washington, 2017-06.

[6] Castro, M. M. & Grünbaum, F. A. (2006). The algebra of differential operators associated to a family of matrix-valued orthogonal polynomials: Five instructive examples. *International Mathematics Research Notices*, 2006. 47602.

[7] Duistermaat, J. & Grünbaum, F. (1986). Differential equations in the spectral parameter. *Communications in Mathematical Physics*, 103(2), 177–240.

[8] Durán, A. J. & Grünbaum, F. A. (2004). Orthogonal matrix polynomials satisfying second-order differential equations. *Int. Math. Res. Not.*, 2004(10).

[9] Geiger, J., Horozov, E., & Yakimov, M. (2015). Noncommutative bispectral darboux transformations. *Transactions of the American Mathematical Society*, 369.

[10] Grünbaum, A. & Iliev, P. (2003). A noncommutative version of the bispectral problem. *J. Comput. Appl. Math.*, 161, 99–118.

[11] Grünbaum, F. A. (2014). Some noncommutative matrix algebras arising in the bispectral problem. *Symmetry, Integrability and Geometry: Methods and Applications*, 10, 078.

[12] Horozov, E. (2002). Bispectral operators of prime order. *Communications in Mathematical Physics*, 231(2), 287–308.

[13] Iliev, P. (1999). *Discrete versions of the Kadomtsev-Petviashvili hierarchy and the bispectral problem*. PhD thesis, Dissertation.

[14] Kasman, A. (2015). Bispectrality of n-component kp wave functions: A study in non-commutativity. *Symmetry, Integrability and Geometry: Methods and Applications*, 11.

[15] Sakhnovich, A. & Zubelli, J. (2001). Bundle bispectrality for matrix differential equations. *Integral Eguations Operator Theory*, 41(4), 472.

[16] Stafford, J. T. (1985). On the ideals of a noetherian ring. *Transactions of the American Mathematical Society*, 289(1), 381–392.

[17] Tirao, J. (2011). The algebra of differential operators associated to a weight matrix: a first example. *XVIII Latin American Algebra Colloquium, in: Contem. Math*, (pp. 291–324).

[18] Wilson, G. (1993). Bispectral commutative ordinary differential operators. *J. Reine Angew. Math.*, 442, 177–204.

[19] Zubelli, J. (1990). Differential equations in the spectral parameter for matrix differential operators. *Physica D: Nonlinear Phenomena*, 43(2-3), 269–287.

[20] Zubelli, J. & Magri, F. (1991). Differential equations in the spectral parameter, Darboux transformations and a hierarchy of master symmetries for KdV. *Communications in Mathematical Physics*, 141(2), 329–351.

[21] Zubelli, J. P. (1992a). On a zero curvature problem related to the ZS-AKNS operator. *J. Math. Phys.*, 33(11), 3666–3675.

[22] Zubelli, J. P. (1992b). On the polynomial τ-functions for the KP hierarchy and the bispectral property. *Lett. Math. Phys.*, 24(1), 41–48.

[23] Zubelli, J. P. (1992c). Rational solutions of nonlinear evolution equations, vertex operators, and bispectrality. *J. Differential Equations*, 97(1), 71–98.

[24] Zubelli, J. P. (1997). *Topics on wave propagation and Huygens' principle*. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro.

[25] Zubelli, J. P. & Valerio Silva, D. S. (2000). Rational solutions of the master symmetries of the KdV equation. *Communications in Mathematical Physics*, 211(1), 85–109.

[26] Zurrián, I. (2016). The Algebra of Differential Operators for a Matrix Weight: An Ultraspherical Example. *International Mathematics Research Notices*, 2017(8), 2402–2430.