Instituto Nacional de Matemática Pura e Aplicada

Doctoral Thesis

MINIMAL SURFACES OF FINITE TOTAL CURVATURE IN $\mathbb{M}^2 \times \mathbb{R}$

Rafael Ponte

Rio de Janeiro
Friday 12th July, 2019

Instituto Nacional de Matemática Pura e Aplicada

Rafael Ponte

Thesis presented to the Post-graduate Program in Mathematics at Instituto Nacional de Matemática Pura e Aplicada as partial fulfillment of the requirements for the degree of Doctor in Mathematics.

Advisor: Harold Rosenberg

Rio de Janeiro
2019

CHAPTER 1

Acknowledgments

First and foremost, I would like to thank God for giving me all the strength and hope I needed through the time dedicated to the Doctorate. I also thank my parents and my sisters for the support and love during these four years (and my whole life).

I also express my deep gratitude to Professor Harold Rosenberg, my advisor. His patience, encouragement and guidance were crucial for my mathematical education. His vast knowledge and talent, along with his gentle personality, makes him the great man he is.

I would like to thank all my friends in Rio, for the support in the toughest moments, and for my most sincere laughs in this period. To name a few, I thank Alcides Júnior, Cayo Dória, Eduardo Garcez, Ermerson Araújo, Gregory Cosac, Jamerson Bezerra, Makson Sales, Marlon Lopez, Mateus Melo, Miguel Ibieta, Renan Santos, Sandoel Vieira, Tiecheng Xu, Vitor Alves and Walner Santos. Specially, I would like to thank Vanderson Lima, a former professor of mine, who introduced me to Professor Rosenberg and gave me some precious advices in the beginning of my Doctorate; Ivan Passoni, my academic brother, for all the insightful conversations, for the support in the hard situations and for telling me some of the funniest jokes I have ever heard; Rodrigo Matos, a friend from the undergraduate times, for all the good moments. His mathematical enthusiasm and talent are always a tremendous source of inspiration to me.

I thank my former professors from Universidade Federal of Ceará (UFC), for giving me a solid knowledge of mathematics to face the challenge of the Doctorate. I particularly name Antonio Caminha Muniz Neto, my Masters' advisor, for kindly sharing his experience and knowledge of mathematics and life with me, even after the end of my Masters.

I also thank the pofessors Pierre Bérard, Laurent Mazet, Magdalena Rodríguez and Marcos Cavalcante for the interest in my work and for the valuable suggestions about it.

I would like to thank IMPA staff for being always available and for the efficiency to handle bureaucracies.

Last but not least, I thank Capes for the financial support.

Abstract

This thesis deals with minimal surfaces in product spaces of the form $\mathbb{M} \times \mathbb{R}$, where $\mathbb{M}$ is a Hadamard surface with pinched sectional curvature, that is, the sectional curvature of $\mathbb{M}$ is contained between two negative constants.

In the first part, we construct minimal annuli embedded in $\mathbb{M} \times \mathbb{R}$ whose ends are asymptotic to totally geodesic vertical planes (here, the metric of $\mathbb{M}$ must be analytic). These annuli are the generalization of the horizontal catenoids which were previously constructed in some Thurston geometries.

In the second part, we study minimal surfaces of finite total curvature in $\mathbb{M} \times \mathbb{R}$. In particular, we proved that, when the total curvature of a minimal surface in $\mathbb{M} \times \mathbb{R}$ is finite, it must be an integer multiple of 2π. Besides, we have listed some examples of minimal surfaces of finite total curvature in these spaces. Moreover, we conclude that the surfaces constructed in the previous chapter have finite total curvature, and its value is -4π. We also proved that these surfaces have bounded stability index.

Keywords: Minimal surfaces, Hadamard manifolds, Finite total curvature

Contents

CHAPTER 2

Introduction

The study of minimal surfaces is a classical field in mathematics, which remains very active nowadays. Started in the eighteenth century by Euler and Lagrange, they were initially focused on understanding such surfaces in $\mathbb{R}^3$. Throughout the eighteenth and nineteenth centuries, several of these examples were found. Besides, in this period, the concept of minimal surface gained a mathematically rigorous definition.

In 1860, Weierstrass obtained a way to represent minimal surfaces in $\mathbb{R}^3$ from meromorphic data on a Riemann surface, now called the *Weierstrass Representation*. In the second half of the twentieth century, Osserman resumed this part of the theory, obtaining several theorems; especially with respect to minimal surfaces of finite total curvature in $\mathbb{R}^3$. We also have, in 1983, the celebrated Jorge-Meeks formula, which calculates the total curvature in terms of geometric and topological data of the surface.

In recent decades, the interest in understanding minimal surfaces in homogeneous three-dimensional Riemannian manifolds (Thurston's geometries) has become more intense. We highlight here the pioneering article by Harold Rosenberg [38], where several examples of minimal surfaces were obtained in $\mathbb{S}^2 \times \mathbb{R}$ and in $\mathbb{H}^2 \times \mathbb{R}$, and also in other ambient spaces. Among the geometries of Thurston, the case where the ambient manifold is $\mathbb{H}^2 \times \mathbb{R}$ had a particularly strong development. During this time, many examples were constructed (for example, in the works [16], [30], [9] and [10]). In [36] and

[28], the conjugate surface method was used to construct minimal annuli in slabs of $\mathbb{H}^2 \times \mathbb{R}$. In [31], properly embedded minimal annuli are constructed in vertical slabs of $\widetilde{PSL}_2(\mathbb{R})$, using variational methods.

Another object commonly studied in differential geometry are the minimal surfaces with finite total curvature in three-dimensional spaces. A classical result in this subject states that, if $\Sigma \subset \mathbb{R}^3$ is a complete immersed minimal surface of finite total curvature, then Σ is conformally equivalent to a compact Riemann surface with a finite number of points removed. Moreover, its Weierstrass data can be extended meromorphically to the punctures and its total curvature is an integral multiple of 4π (see [32] for those results). Other references for finite total curvature minimal surfaces in $\mathbb{R}^3$ are [11], [20] and [43]. In [22], the authors obtain a formula for the total curvature of a minimal surface Σ in terms of topological and geometrical invariants (see also [11] for a discussion of these results).

In 2006, Laurent Hauswirth and Harold Rosenberg discussed, in the article [19], the minimal surfaces of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$. In this work, a model to represent minimal surfaces, which is similar to Weierstrass Representation, is presented. This model allowed them to extend some results of Osserman to this new context, as well as to prove a Jorge-Meeks type formula for such surfaces.

Recently, the geometry of minimal surfaces on $\mathbb{M} \times \mathbb{R}$, where $\mathbb{M}$ is a Hadamard surface, has been studied quite frequently. Among the articles, we mention [14], by José Gálvez and Harold Rosenberg, which proves a Jenkins-Serrin type theorem for domains in $\mathbb{M}$, and [13], by José Gálvez and Victorino Lozano, which constructs convex barriers (with respect to the mean curvature) in $\mathbb{M} \times \mathbb{R}$, allowing the extension of results already known for $\mathbb{H}^2 \times \mathbb{R}$.

In Chapter 3 of this thesis, we study minimal surfaces in $\mathbb{M}^2 \times \mathbb{R}$, where $(\mathbb{M}, g)$ is a Hadamard manifold with analytic metric satisfying $-1 \leq K_{sect} \leq -k^2$, for a positive number k. Strongly influenced by [31], we prove the following theorem:

Theorem 3.21. For two complete and disjoint geodesics γ_1 and γ_2 whose distance is smaller than $2ln(\sqrt{2}+1)$, there exists a complete embedded minimal annulus in $\mathbb{M} \times \mathbb{R}$ whose boundary at infinity is the union of the four vertical lines passing through the endpoints (at infinity) of γ_1 and γ_2 and, for each geodesic γ that is ultraparallel to both γ_1 and γ_2, the intersection of this annulus with $\gamma \times \mathbb{R}$ is compact. This surface is a bigraph which is symmetric with respect to the horizontal slice $\mathbb{M} \times \{0\}$.

Let Ω be the geodesic ideal quadrilateral having γ_1 and γ_2 as sides. In $\mathbb{H}^2$, it is easy to see that there exists a nested sequence $(\Omega^n)_n$ of bounded geodesic quadrilaterals such that $\bigcup_{i=1}^{\infty} \Omega^i = \Omega$. The sides of Ω^n are γ_1^n, γ_2^n, η_1^n and η_2^n; moreover, γ_i^n is contained in γ_i, for $i = 1, 2$, and the inequality $l(\alpha_1^n)+l(\alpha_2^n) > l(\eta_1^n) + l(\eta_2^n)$ holds for all n. For a Hadamard surface with pinched curvature (i.e., $-1 \leq K_{sect} \leq -k^2$, for $k > 0$), we will apply comparison theorems to obtain such a sequence.

The most crucial step to prove the result is the following intermediate theorem:

Theorem 3.1. Let Ω^* be a bounded geodesic convex quadrilateral whose sides are α_1^*, α_2^*, η_1^* and η_2^* such that $l(\alpha_1^*)+l(\alpha_2^*) > l(\eta_1^*)+l(\eta_2^*)$. There exists a proper minimal annulus Σ^* in $\mathbb{M}\times\mathbb{R}$ asymptotic to $\alpha_i^* \times \mathbb{R}$, $i = 1, 2$, whose boundary is formed by the vertical lines along the vertices of Ω^*. Moreover, for each complete geodesic α intersecting the geodesics η_i^*, the set $\Sigma^*\cap(\alpha\times\mathbb{R})$ is compact. Moreover, it is a bigraph which is symmetric with respect to the horizontal slice $\mathbb{M} \times \{0\}$.

In [31], a crucial fact for some arguments is the existence of a uniform bound for the height of vertical minimal annuli in $\widetilde{PSL}_2(\mathbb{R}, \tau)$. It is used to characterize the intersection of horizontal slices with certain types of horizontal minimal annuli. As an alternative idea, we will use the Alexandrov Reflection Principle to guarantee that those horizontal anulli are always symmetric with respect to some horizontal plane, and this allows us to prove the same results. Furthermore, this invariance under some vertical reflection also gives us the same symmetry for the annuli constructed in the main theorem.

In this chapter, we can show the usefulness of variational methods in the study of minimal surfaces, as well as that of the mean curvature comparison theorems, mainly used in the construction of barriers. Here, we also give new proofs of some auxiliary results which have analogous versions in [31], either to clarify or to simplify them.

In Chapter 4, we generalize [19] to the case of $\mathbb{M} \times \mathbb{R}$; here, $\mathbb{M}$ is a Hadamard surface whose sectional curvature satisfies the inequalities $-a^2 \leq K_{\mathbb{M}} \leq -b^2$, where a and b are positive constants. Inspired by [37], we add a refinement to the generalization. We also present some examples of minimal surfaces with finite total curvature in $\mathbb{M} \times \mathbb{R}$.

Here, the main tools are the comparison theorems, which allow us to construct complete mean convex barriers. Moreover, the analysis of harmonic maps taking values in a Hadamard surface plays a relevant role in the proof of the results.

CHAPTER 3

Construction of minimal annuli in $\mathbb{M}^2 \times \mathbb{R}$

In this chapter, our objective is proving that, for sufficiently close complete geodesics γ_1 and γ_2 in $\mathbb{M}$, there exists a complete, properly embedded minimal annulus in $\mathbb{M} \times \mathbb{R}$ asymptotic to $(\gamma_1 \cup \gamma_2) \times \mathbb{R}$ whose boundary at infinity consists of the four vertical lines at infinity passing through the endpoints of the two geodesics.

3.1 Minimal annuli in bounded domains

Let $\Omega \subset \mathbb{M}$ be a convex bounded domain whose boundary is given by closed geodesic arcs γ_1, η_1, γ_2 and η_2. Denote by $\widetilde{\gamma}_i$ the completions of γ_i and by $\widetilde{\eta}_i$ the complete geodesics which form a convex ideal quadrilateral $\widetilde{\Omega}$ with $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$, the curves $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ being disjoint up to infinity. Suppose that

$$l(\gamma_1) + l(\gamma_2) > l(\eta_1) + l(\eta_2). \tag{3.1}$$

The main result of this section is the following:

Theorem 3.1. *There exists a proper minimal annulus Σ in $\mathbb{M}\times\mathbb{R}$ asymptotic to $\gamma_i \times \mathbb{R}$, $i = 1, 2$, whose boundary is formed by the vertical lines along the vertices of Ω such that, for each complete geodesic α intersecting the geodesics η_i, the set $\Sigma \cap (\alpha \times \mathbb{R})$ is compact. Moreover, it is a bigraph with respect to the horizontal slice $\mathbb{M} \times \{0\}$, and Σ and $\mathbb{M} \times \{0\}$ meet orthogonally.*

The proposition below shows that the main result does not hold if we change slightly the hypothesis (3.1).

Proposition 3.2. *If Ω satisfies*

$$l(\eta_1) + l(\eta_2) > l(\gamma_1) + l(\gamma_2),$$

there is no annulus Σ satisfying the above conditions.

Proof. Assume that the proposition is not true, so there exists such an annulus Σ. After a small perturbation, we can take a convex bounded domain Ω' whose boundary is given by the geodesic arcs γ_1', η_1', γ_2' and η_2' satisfying $l(\gamma_1') + l(\gamma_2') < l(\eta_1') + l(\eta_2')$ (see Figure 3.1). By Theorem 3.3 of [24], there exists a minimal graph S over Ω' assuming the values $+\infty$ on γ_i' and 0 on η_j'. Then, we notice that, for large $h > 0$, the image of the vertical translation of S by h (call it $T_h(S)$) is disjoint from Σ. Choosing $h^* := inf\{h \in \mathbb{R}; T_{h'}(S) \cap \Sigma = \emptyset \text{ for } h' > h\}$, we see that T_{h^*} and Σ have a first point of contact, and it must be in the interior of both surfaces, contradicting the Maximum Principle. $\square$

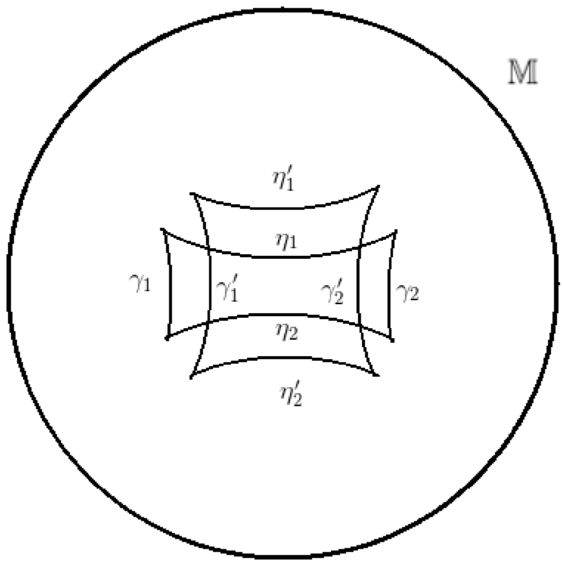

Figure 3.1: Proposition 3.2

3.1.1 Compact minimal annuli

Let $\gamma_1 : [0,1] \to \mathbb{M}$ be a parametrization of $\bar{\gamma}_1$ with constant speed. Also let $\{G_n^1 \subset \bar{\gamma}_1 \times [-n,n]\}_{n\in\mathbb{N}}$ be a family of smooth closed convex curves satisfying, for all n, the properties:

1. G_n^1 is symmetric with respect to $\mathbb{M} \times \{0\}$;

2. For some $\epsilon \in (0, \frac{1}{2})$, the following equalities hold:

   - $G_n^1 \cap (\mathbb{M} \times [-n+\epsilon, n-\epsilon]) = \{\gamma_1(0), \gamma_1(1)\} \times [-n+\epsilon, n-\epsilon]$;
   - $G_n^1 \cap (\gamma_1([\epsilon, 1-\epsilon]) \times [-n,n]) = \gamma_1([\epsilon, 1-\epsilon]) \times \{-n, n\}$;
   - $G_{n+1}^1 \cap (\mathbb{M} \times [n+1-\epsilon, n+1]) = \{(x, t+1); (x,t) \in G_n^1 \cap (\mathbb{M} \times [n-\epsilon, n])\}$;

3. The set $G_n^1 \cap (\bar{\gamma}_1 \times (n-\epsilon, n))$ consists of two connected components, each one being a smooth curve smoothing the upper corners of $\partial(\bar{\gamma}_i \times [-n,n])$, and those components are not tangent to vertical or horizontal directions at any point.

It is simple to construct a smooth horizontal vector field V along $\bar{\gamma}_1$ such that $V(i)$ is tangent to η_{i+1}, $i = 0,1$ and, for each $t \in [0,1]$, the map $s \mapsto exp_{\gamma_1(t)}(sV(t))$ is a nondegenerate geodesic and all of them foliate the region between the geodesics η_j. Extending the vector field V to $\bar{\gamma}_1 \times \mathbb{R}$ by parallel transport, and denoting by $F : \bar{\gamma}_1 \times \mathbb{R} \times \mathbb{R} \to \mathbb{M} \times \mathbb{R}$ the map

$$F(\gamma_1(t), s, u) = (exp_{\gamma_1(t)}(uV(t)), s),$$

we can define G_n^2 as $(\bar{\gamma}_2 \times \mathbb{R}) \cap F(G_n^1 \times \mathbb{R})$. It is easily verified that the curves G_n^2 satisfies similar properties as the ones already stated for G_n^1. In that situation, concerning the parametrization of γ_2, for each $t \in [0,1]$, $\gamma_2(t)$ is the point $\bar{\gamma}_2 \cap F(\{\gamma_1(t)\} \times \{0\} \times \mathbb{R})$.

Let N be the innerwise pointing unit normal vector field along G_n^1. Define, for each n, a smooth function $f_n : G_n^1 \to \mathbb{R}$ satisfying the following properties:

1. The exponential graph of f_n (i.e., the set $\{exp_x(f(x)N(x)), x \in G_n^1\}$, and denoted by $Exp(f_n)$) is a closed convex curve which is a vertical bigraph over $\bar{\gamma}_1$, and symmetric with respect to $\mathbb{M} \times \{0\}$;

2. The set $Exp(f_n)$ is contained in the disc determined by $Exp(f_{n+1})$;

3. Concerning the sign of f_n, we have:

- $f_n \geq 0$;
- $f_n(x) = 0$ if $x \in G_n^1 \cap (\mathbb{M} \times \{0\})$ or if $x \in G_n^1 \cap (\bar{\gamma}_i \times [n - \epsilon_n, n])$, where $(\epsilon_n)_n$ is a strictly increasing sequence which converges to ϵ;

4. $lim_{n\to\infty} ||f_n||_{C^n(G_n^1)} = 0$.

We define Γ_n^1 as the exponential graph of f_n. As in the case of the curves G_n^i, we define Γ_n^2 as $(\bar{\gamma}_2 \times \mathbb{R}) \cap F(\Gamma_n^1 \times \mathbb{R})$. By definition, it is clear that, for $i = 1, 2$, the curve Γ_n^i have bounded geometry and converges smoothly to the boundary of $\gamma_i \times \mathbb{R}$.

Proposition 3.3. *Let Ω be the bounded quadrilateral domain as before. If $l(\gamma_1) + l(\gamma_2) > l(\eta_1) + l(\eta_2)$, then, for sufficiently large n, there exists a minimal area annulus in $\mathbb{M} \times \mathbb{R}$ whose boundary is $\Gamma_n^1 \cup \Gamma_n^2$.*

Proof. By the choice of Γ_n^i, we can construct annuli whose boundary is $\Gamma_n^1 \cup \Gamma_n^2$ and whose area differ from $2((l(\eta_1)+l(\eta_2))n+Area(\Omega))$ by a number bounded from above by a constant independent on n, say, C_1. Analogously, the sum of the areas of the minimal discs bounded by Γ_n^i differ from $2(l(\gamma_1)+l(\gamma_2))n$ by a number bounded from below by a constant independent on n, say, C_2. It is enough, by Theorem 1 of [25], to verify that the inequality

$$2((l(\eta_1) + l(\eta_2))n + Area(\Omega)) + C_1 < 2(l(\gamma_1) + l(\gamma_2))n + C_2$$

holds when n is sufficiently large, which is obviously true, given the hypotheses. □

Remark. We point out that, in Proposition 3.3, we consider the ambient space to be $\Omega \times [-n, n]$. The notion of mean convex manifold used in [25] includes this space; the proof is a slight modification of the one shown in the reference in the case of Euclidean space (in fact, it suffices to consider the bounds for the sectional curvatures and the comparison theorems of the Hessian and Laplacian).

Denote by $\mathcal{A}$ the set of minimal annuli whose boundary is $\Gamma_n^1 \cup \Gamma_n^2$ and by $\mathcal{A}^s$ the subset of $\mathcal{A}$ consisting of the stable ones.

Proposition 3.4. *For sufficiently large n, there exists an element Σ_n^s of $\mathcal{A}^s$ such that, if V is the open region of $\mathbb{M}$ between γ_1 and γ_2, all the elements of $\mathcal{A}$ are contained in the closure of the bounded component of $(\bar{V} \times \mathbb{R})\backslash\Sigma_n^s$.*

Proof. Proposition 3.3 assures that $\mathcal{A}^s$ is nonempty. Moreover, by Theorem 5 of [26], given any element A of $\mathcal{A}$, we can obtain an element of $\mathcal{A}^s$ which is contained in the closure of the unbounded component of $(\bar{V} \times \mathbb{R})\backslash A$. It suffices, then, to prove the proposition for $\mathcal{A}^s$, instead of $\mathcal{A}$.

If $A_1, A_2 \in \mathcal{A}^s$, we say that $A_1 \preceq A_2$ if A_1 is contained in the closure of $(\bar{V} \times \mathbb{R})\backslash A_2$. This defines an order relation on $\mathcal{A}^s$. If $\mathcal{B}$ is a totally ordered subset of $\mathcal{A}^s$, we will obtain an upper bound for it. Now, if p is a point of V, define $f : \mathcal{B} \to \mathbb{R}$ given by $f(A) = max\{t \in \mathbb{R}; (p,t) \in A\}$. Clearly, $A_1 \preceq A_2$ in $\mathcal{B}$ if and only if $f(A_1) \leq f(A_2)$. This implies that, in order to find an upper bound for $\mathcal{A}^s$, we just need to take a countable subset (say, the image of a sequence $(A_n)_n$ in $\mathcal{A}^s$ such that $(f(A_n))_n$ is a monotone increasing sequence converging to $sup f(\mathcal{A})$). Considering such a sequence, for each natural n, Theorem 5 of [26] gives a minimal surface B_n contained in the closure of the unbounded component of $(\bar{V} \times \mathbb{R})\backslash A_n$ with the same boundary as A_n, minimizing area among the annuli of this region. By the area-minimizing property, we have area and curvature estimates for the surfaces B_n, then it has a subsequence which converges to $B \in \mathcal{A}^s$ (in fact, the area of $\partial(\Omega \times [0,1])$ is an upper bound for the areas of B_n). It is easy to see that B is an upper bound for $(A_n)_n$, and by Zorn's Lemma, $\mathcal{A}^s$ has maximal elements. If R_1 and R_2 are two maximal annuli on $\mathcal{A}^s$, we can find a stable annulus in the unbounded component of $(\bar{V} \times \mathbb{R})\backslash(R_1 \cup R_2)$ (again, by Theorem 5 of [26]), contradicting the maximality of R_1 and R_2, so the maximal element is unique. $\square$

We now are going to consider another annulus Σ_n^u having $\Gamma_n^1 \cup \Gamma_n^2$ as boundary. If the annulus Σ_n^s is semi-stable, define Σ_n^u as Σ_n^s. If not, we obtain an unstable annulus by a reasoning similar to Proposition 2.2.7 of [31]. We only remark that it is possible to find a pair of curves $(\beta_1, \beta_2) \subset (\widetilde{\gamma}_1 \times \mathbb{R}) \times (\widetilde{\gamma}_2 \times \mathbb{R})$ which don't span a minimal annulus. In fact, if (β_1, β_2) in $(\widetilde{\gamma}_1 \times \mathbb{R}) \times (\widetilde{\gamma}_2 \times \mathbb{R})$ lie inside a sufficiently small tubular neighborhood of a horizontal geodesic tranversal to both planes, any minimal annulus spanned by this pair of curves would be contained in this neighborhood (by the Maximum Principle), and we obtain a contradiction proceeding in a similar way to the Proposition 3.2.

We finish this subsection with a simple proposition, which describes the shape of the elements of $\mathcal{A}$.

Lemma 3.5. *If $A \in \mathcal{A}$, then A is a bigraph over a domain contained in Ω. Moreover, A is symmetric with respect to $\mathbb{M} \times \{0\}$, and those surfaces meet* *orthogonally.*

Proof. We are going to use the Alexandrov Reflection Principle. For each $s \in \mathbb{R}$, define $A_s^+ := A \cap (\mathbb{M} \times [s, \infty))$ and $A_s^- := A \cap (\mathbb{M} \times (-\infty, s])$. For two subsets $S_1, S_2 \subset \Omega \times \mathbb{R}$, we say that S_1 *is above* S_2 if, for any two points $(x, h_1) \in S_1$ and $(x, h_2) \in S_2$ having the same projection over Ω, we have that $h_1 \geq h_2$. By the choice of Γ_n^i and the Boundary Maximum Principle, we have that A_n^+ consists of two horizontal segments and their points do not have vertical tangent planes. So, for $\delta > 0$ sufficiently small, the points of $A_{n-\delta}^+$ do not have vertical tangent planes. Moreover, the set $A_{n-\delta}^+$ is a graph over a subset of Ω. Indeed, if it is not true, there exists a sequence $(\delta_k)_k$ of positive numbers converging to zero such that the points $p_k, \tilde{p}_k \in A_{n-\delta_k}^+$ have the same projection on $\mathbb{M}$. Then, the sequences $(p_k)_k$ and $(\tilde{p}_k)_k$ converge, up to a subsequence, to the same point $p \in A_n^+$. But, since the points of A_n^+ do not have vertical tangent planes, the surface A is a graph in a neighborhood of p, and the projection over $\mathbb{M}$ is injective in this neighborhood, a contradiction. Therefore $A_{n-\delta}^+$ is a graph over a subset of Ω. Denoting by $r(A_s^+)$ (resp. $r(A_s^-)$) the reflection of $r(A_s^+)$ (resp. $r(A_s^-)$) by $\mathbb{M} \times \{s\}$, we have that $r(A_{n-\delta}^+)$ is above $A_{n-\delta}^-$, provided $A_{n-2\delta}^+$ is a vertical graph over a subset of Ω.

If $i := inf\{t \in [0, n], r(A_t^+)$ is above A_t^- and A_t^+ is a graph over a subset of $\Omega\}$, we need to prove that $i = 0$. Since we proved that $n - \delta$ is in the set for small δ, i is well-defined. Moreover, it is clear that $r(A_i^+)$ is above A_i^-, and if we had $i > 0$, all the points of $(\mathbb{M} \times \{i\}) \cap A$ would not have vertical tangent planes. Otherwise, if this were the case for some point in $(\mathbb{M} \times \{i\}) \cap A$, this point should be in $Int(A)$, and, by the Maximum Principle, $r(A_i^+) = A_i^-$, which is not true. So, for small $\delta' > 0$, the set $A_{i-\delta'}^+$ has no points with vertical tangent planes, then it is a graph over a subset of Ω. In fact, if this were not true, as in the previous paragraph, we could take sequences $(p_k)_k$ and $(\tilde{p}_k)_k$ converging to $p \in A_i^+$ and $\tilde{p} \in A \cap (\mathbb{M} \times \{i\})$, respectively, such that p_k and $\tilde{p}_k$ have the same projection over $\mathbb{M}$, and so do the points p and $\tilde{p}$. Both points cannot be equal, otherwise it would contradict the fact that the points of $A \cap (\mathbb{M} \times \{i\})$ do not have vertical tangent planes. Therefore, we can find disjoint neighborhoods V and $\tilde{V}$ in A containing p and $\tilde{p}$, respectively, such that both project bijectively onto the same open set of $\mathbb{M}$. In that case, it is possible to find two points in A_i^+ which project over the same point in $\mathbb{M}$, a contradiction. By a similar reasoning and the Interior Maximum Principle, the set $r(A_{i-\delta'}^+)$ is above $A_{i-\delta'}^-$, so i is not the infimum if it is positive. So $i = 0$, $r(A_0^+)$ is above A_0^- and $A \cap (\mathbb{M} \times (0, \infty))$ is a vertical graph over a subdomain of Ω, both properties being true because they hold for a sequence of positive numbers converging to zero.

Proceeding analogously, we have that $r(A_0^-)$ is above A_0^+, and consequently $r(A_0^+) = A_0^-$, and the symmetry is proved. For the orthogonality part, notice that, by the symmetry of A with respect to $\mathbb{M} \times \{0\}$, the tangent planes of points in $A \cap (\mathbb{M} \times \{0\})$ are invariant by reflection on $\mathbb{M} \times \{0\}$, so those planes must be vertical or horizontal. If A and $\mathbb{M} \times \{0\}$ intersect transversally at q, the tangent plane at q is vertical, and the surfaces intersect orthogonally at q. If the surfaces are tangent at q, on one hand, the tangent plane of A at q is horizontal. On the other hand, it is true that q is the limit of a sequence $(q_n)_n$ in $A \cap (\mathbb{M} \times \{0\})$ of points where the intersection is transverse, so the tangent plane of A at q is vertical, a contradiction. So the orthogonality is proved. $\square$

From now on, we are going to denote by Σ_n a minimal annulus in $\mathbb{M} \times \mathbb{R}$ whose boundary is $\Gamma_n^1 \cup \Gamma_n^2$, for all n, unless otherwise stated.

3.1.2 Foliations

In this section, we study the intersection of Σ_n with a leaf of $\mathcal{F}$ different from the planes $\widetilde{\gamma}_i \times \mathbb{R}$. We are going to state some results concerning the topology of the intersection of both surfaces, as well as estimates on the number of points tangent to one of the foliations and the number of the leaves that are tangent to the surface. There are equivalent lemmas and propositions in [31] but, although the reference [31] only treats the case when the ambient manifold is $\widetilde{PSL}_2(\mathbb{R}, \tau)$, the proofs will mostly follow the same reasoning. However, remarks will be added when necessary.

We highlight here three classes of minimal foliations of domains of $\mathbb{M} \times \mathbb{R}$:

1. The foliation $\mathcal{F}^h$ given by the slices $\mathbb{M} \times \{t\}$, for each $t \in \mathbb{R}$;

2. Given a minimal graph of a function w (denoted by $Gr(w)$) over a domain Λ of $\mathbb{M}$, the family $\{T_h(Gr(w))\}_{h \in \mathbb{R}}$ of its vertical translations defines a foliation of $\Lambda \times \mathbb{R}$. For our purposes, we are going to suppose that Λ is the ideal convex quadrilateral whose sides are $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$, $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$. Besides, w assumes the smooth value f on $\widetilde{\gamma}_1 \cup \widetilde{\gamma}_2$ and a constant value on $\widetilde{\eta}_1 \cup \widetilde{\eta}_2$. Moreover, we suppose that, for each n, there are two real numbers $a < b$ depending on n such that $T_h(Gr(f))$ doesn't

intersect Γ_i^n if $h \notin [a,b]$, $T_h(Gr(f)) \cap \Gamma_i^n$ is nonempty and connected if $h = a, b$ and, for $h \in (a,b)$, the intersection of both curves is transverse and consists of two points in each component. We will refer to this foliation as $\mathcal{F}^{Gr(w)}$.

3. Fix a horizontal geodesic α and let $\mathcal{F}^\alpha$ be the foliation given by the planes of the form $\beta \times \mathbb{R}$, where β varies through the horizontal geodesics which are perpendicular to α. We call α the *defining geodesic* for $\mathcal{F}^\alpha$. For simplicity, we denote by $\mathcal{F}^\gamma$ and $\mathcal{F}^\eta$ the foliations whose defining geodesics are the perpendicular to the γ_i-curves and the perpendicular to the η_i-curves, respectively, $i = 1, 2$.

We denote by $\mathcal{F}$ a foliation among $\mathcal{F}^h$, $\mathcal{F}^{Gr(w)}$, $\mathcal{F}^\gamma$, $\mathcal{F}^\eta$ or $\mathcal{F}^{\tilde{\eta}_i}$, $i = 1, 2$.

An immediate conclusion from the definition of $\mathcal{F}$ is that the intersection of a leaf Φ of $\mathcal{F}$ with Γ_n^i is empty or is composed of two points or is a connected subset of Γ_n^i. In the case where the intersection is given by two points, Φ intersects Σ_n and Γ_n^i transversely. We will refer to that as property (B), as in [31].

Lemma 3.6. *Let Φ be a leaf of the foliation $\mathcal{F}$ such that $\omega := \Phi \cap \Sigma_n \neq \emptyset$ and $\mathcal{T}$ the set of points in Σ_n which are tangent to Φ.*

1. ω contains at most one cycle, and it must be nontrivial.

2. When ω does not have cycles:

(a) If $Int(\Sigma_n) \cap \mathcal{T}$ is non-empty, it is one point. Furthermore, Φ meets $\partial\Sigma_n$ at four distinct points and each component of $\omega \backslash (\mathcal{T} \cup \partial\Sigma_n)$ is diffeomorphic to $\mathbb{R}$ and joins $\mathcal{T} \cap Int(\Sigma_n)$ with a point of $\Phi \cap \partial\Sigma_n$.

(b) If $Int(\Sigma_n) \cap \mathcal{T}$ is empty, each component of $\omega \cap Int(\Sigma_n)$ is diffeomorphic to $\mathbb{R}$ and joins two distinct components of $\Phi \cap \partial\Sigma_n$.

3. When ω has exactly one cycle C, the curve separates Σ_n in two components. Denote by A the closure of a component, thus $\partial\mathcal{A} \subset C \cup \Gamma$, with $\Gamma = \Gamma_n^1$ or Γ_n^2.

(a) If $C \cap \Gamma = \emptyset$, then $\mathcal{A}$ is an annulus with no horizontal points in its interior. Moreover, each component of $\omega \cap Int(\mathcal{A})$ is diffeomorphic to $\mathbb{R}$ and joins two distinct components of $C \cup (\Phi \cap \Gamma)$.

(b) If $C \cap \Gamma \neq \emptyset$, then $\Phi \cap \Gamma$ is connected. Moreover, $Int(\mathcal{A})$ is a disc and $\omega \cap Int(\mathcal{A}) = \emptyset$.

Proof. We are going to prove only the assertion 2(a). All the others follow by the same ideas of the proof of Lemma 2.2.9 in [31].

In fact, $\mathcal{T} \cap \Sigma_n$ consists of isolated points, and since this set is compact, it must be finite. So, consider the combinatorial graph G whose set of vertices V is given by $\mathcal{T} \cap Int(\Sigma_n)$ and, for each component of $\omega \backslash (\mathcal{T} \cup \partial\Sigma_n)$ connecting $p_1, p_2 \in \mathcal{T} \cap Int(\Sigma_n)$, we have an edge connecting those vertices.

We know that $\Phi \cap \partial\Sigma_n$ has at most four components, which means that if there are $2g$ arcs coming out of a point $p \in \mathcal{T} \cap Int(\Sigma_n)$, the degree of vertex p in G is at least $2g - 4$. So, if a vertex has degree zero, $\Phi \cap \partial\Sigma_n$ has four components, and by property (B), all the other vertices must have degree $2g \geq 4$, because they can not be connected to the boundary. If a vertex has degree 1, then for some $i = 1, 2$, the set $\Gamma_n^i \cap \Phi$ consists of two points, and again by property (B), all the other vertices must have degree $2g - 1 \geq 3$. If none of those cases occur, all the vertices have degree at least 2. Then, by an elementary result of graph theory, the graph has a cycle in the three cases, so ω has a cycle, a contradiction. Then $\mathcal{T} \cap Int(\Sigma_n)$ has exactly one point. Furthermore, by the absence of cycles in ω, we know that the components of $\omega \backslash (\mathcal{T} \cup \partial\Sigma_n)$ join the point in $\mathcal{T} \cap Int(\Sigma_n)$ and the components of $\Phi \cap \partial\Sigma_n$. Therefore, since the set $\omega \backslash (\mathcal{T} \cup \partial\Sigma_n)$ has at least four components and $\Phi \cap \partial\Sigma_n$ has at most four components, we have that $\Phi \cap \partial\Sigma_n$ has precisely four points. □

This lemma allows us to describe precisely the possible intersections between Φ and Σ_n. This analysis is carefully done in [31], after Corollary 2.2.10.

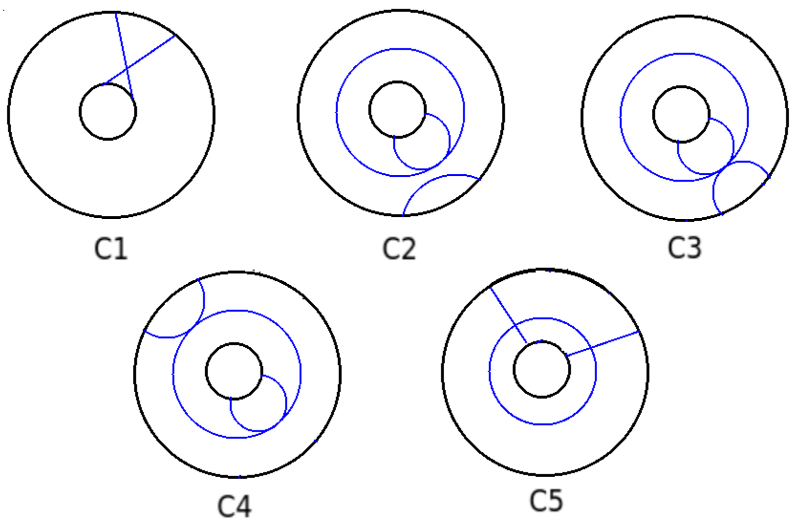

Figure 3.2: Intersections of Φ and Σ_n (1)

We now give a few definitions. For each $t \in (n, n)$, let $\omega(t)$ be the intersection of Σ_n with the plane $\{z = t\}$. A point $p \in \Sigma_n$ is called a *horizontal point* if Σ_n is tangent to the plane $\{z = z(p)\}$ at p. The set of horizontal points is denoted by $\mathcal{H}$ and $\mathcal{H}(t) := \mathcal{H} \cap \omega(t)$. Denote by h_n^+ (resp. h_n^-) the maximum value (resp. the minimum value) of the restriction $z : \mathcal{H} \to \mathbb{R}$ of the height function. Although we have the relation $h_n^+ = -h_n^-$, the definition of both quantities is useful when we have curves in more general positions. For each $t \in (n, n)$, define $\Sigma_n^+(t) = \Sigma_n \cap \{z \geq t\}$ and $\Sigma_n^-(t) = \Sigma_n \cap \{z \leq t\}$.

Proposition 3.7. *The following properties for Σ_n holds:*

1. *Σ_n has exactly two horizontal points, and they are symmetric with respect to $\mathbb{M} \times \{0\}$.*

2. *If $t > h_n^+$ (resp. $t < h_n^-$), then $\Sigma_n^+(t)$ (resp. $\Sigma_n^-(t)$) consists of two simply connected components. Then, $\omega(t)$ consists of two components, both diffeomorphic to $[0, 1]$ and joining two points in a same component of $\partial\Sigma_n$.*

3. *For each $t \in (h_n^-, h_n^+)$ (in particular, for $t = 0$), the sets $\Sigma_n^+(t)$ and $\Sigma_n^-(t)$ are simply connected. Moreover, $\omega(t)$ consists of two components, both diffeomorphic to $[0, 1]$ and joining two points in two distinct components of $\partial\Sigma_n$.*

4. *The set $\Sigma_n \cap \{h_n^- < z < h_n^+\}$ consists of two simply connected components.*

Proof. Clearly, $\omega(\pm n)$ is composed of two horizontal segments, and by the Boundary Maximum Principle, the intersection between Σ_n and $\{z = \pm n\}$ is transverse. If $t \in (-n, n)$, it is clear that $\omega(t) \cap \partial\Sigma_n$ is composed of four points. Besides, by Lemma 3.5, there are no horizontal points in $\omega(0)$, so if Σ_n has a finite number of horizontal points, this number must have even parity.

By Morse theory, it is known that, for a sufficiently small $\epsilon > 0$, the sets $\Sigma_n^+(n - \epsilon)$ and $\Sigma_n^-(-n + \epsilon)$ consist of two simply connected components. Besides, if $-n < t < s < n$ are such that there is no horizontal point whose height is in the interval $[t, s]$, then $\Sigma_n^-(t)$ and $\Sigma_n^+(t)$ are diffeomorphic to $\Sigma_n^-(s)$ and $\Sigma_n^+(s)$, respectively. So, for $t \in (h_n^+, n)$ (respectively $t \in (-n, h_n^-)$), the set $\Sigma_n^+(t)$ (resp. $\Sigma_n^-(t)$) is given by two simply connected components, and $\omega(t)$ is formed by two arcs which connect two points of the same component of $\partial\Sigma_n$. We can conclude that $\omega(h_n^+)$ must be of one of the forms C1,..., C5 (see Figure 3.2) using the previous lemma (the details can be found in [31]). We are going to analyse those cases.

Case 1. $\omega(h_n^+)$ is of the type C1. In that case, by symmetry, the set $\omega(h_n^-)$ must be of the tye C1. If $t \in (h_n^-, h_n^+)$, then $\{z = t\}$ does not intersect Σ_n tangentially (at any point), otherwise it would separate the sets $\omega(h_n^-)$ and $\omega(h_n^+)$ in Σ_n and it would be of type C1, which leads to a contradiction. Then $\omega(t)$ consists of two disjoint arcs, both joining different components of $\partial\Sigma_n$, for $t \in (h_n^-, h_n^+)$.

Case 2. $\omega(h_n^+)$ is of the type C2. Using symmetry again, we have that $\omega(h_n^-)$ must be of type C2. If $t \in (h_n^-, h_n^+)$, then $\{z = t\}$ does not intersect Σ_n tangentially, otherwise it would intersect the three components of $\Sigma_n \cap \{h_n^- < z < h_n^+\}$ (two topological discs and one topological annulus) without crossing its boundaries, but none of the configurations C1,..., C5 would satisfy this property. This gives us, for $t \in (h_n^-, h_n^+)$, that $\omega(t)$ consists of three components, two of them being diffeomorphic to $[0, 1]$ and connecting two points of the same component of $\partial\Sigma_n$, and the other being a nontrivial cycle. Therefore, one of the components of $\Sigma_n^+(0)$ is a topological disc D whose boundary is composed of $\Gamma_n^j \cap \mathbb{M} \times [0, +\infty)$, for some $j \in \{1, 2\}$, and a component of $\omega(0)$ which is not the closed curve. Notice now that the union of D with its reflection by $\mathbb{M} \times \{0\}$ is a minimal disc contained in Σ_n and spanned by Γ_n^j, a contradiction. So Case 2 is not possible.

Case 3. $\omega(h_n^+)$ is of the type C3, C4 or C5. In that case, $\Sigma_n^-(h_n^+)\backslash\omega(h_n^+)$ is formed by two simply connected components $\mathcal{A}_1$ and $\mathcal{A}_2$, and for each $i = 1, 2$, there exists $j \in \{1, 2\}$ satisfying $\partial A_i \cap \Gamma_n^j = \emptyset$. This leads to the conclusion that $h_n^- = h_n^+$. Since this equality can not happen, the mentioned patterns of intersection can not occur.

Now we will prove the result. Indeed, the first two items follow from the above analysis. For the others, it is enough to observe that only **Case 1** can occur, and the conclusion follows immediately from what was exposed above. □

Proposition 3.8. *The annulus Σ_n is not tangent to any leaf of $\mathcal{F}^\gamma$.*

Proof. Let Φ be a leaf of the mentioned foliation. Suppose the intersection $\Phi \cap \Sigma_n$ is non-empty. If $\Phi = \tilde{\gamma}^i \times \mathbb{R}$, for $i = 1, 2$, the intersection is transverse by the Boundary Maximum Principle. If not, $\Phi \cap \partial\Sigma_n = \emptyset$. By the Lemma above, $\Phi \cap Int(\Sigma_n)$ has to contain a cycle, and only this curve. Thus, Φ is not tangent to Σ_n. $\square$

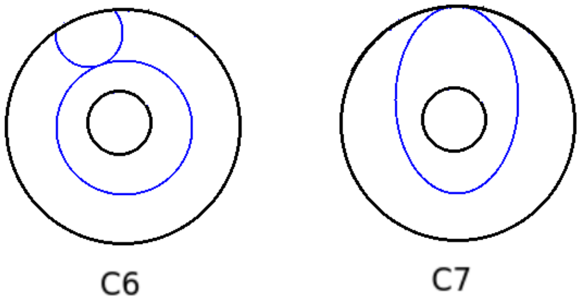

Figure 3.3: Intersections of Φ and Σ_n (2)

Proposition 3.9. *The minimal annulus Σ_n is tangent to the foliation $\mathcal{F}^\eta$ at most at two points.*

Proof. Let Φ be a leaf of the mentioned foliation. Suppose the intersection $\Phi \cap \Sigma_n$ is non-empty. If $\Phi = \alpha^i \times \mathbb{R}$, where α^i is the complete geodesic containing η_i, $i = 1, 2$, the Boundary Maximum Principle guarantees that the intersection is transverse. If that is not the case, we have that Φ intersects $\partial\Sigma_n$ in four points. Then, by the lemma above, the intersection $\Phi \cap \partial\Sigma_n$ is given by one of the pictures of the Figure 1.

If $\Phi \cap \partial\Sigma_n$ is of the type C3, C4 or C5, given another leaf Φ' of the foliation which intersects Σ_n, the intersection of this leaf is transverse. In fact, if it were tangent at some point, it would be of one of the types shown in Figure 1. It can not be of type C1, because the curves of $\Phi' \cap \partial\Sigma_n$ would connect the two components of the boundary and then they would cross the cycle of $\Phi \cap \partial\Sigma_n$. Besides, it can not be any of the other types, because the cycle of $\Phi' \cap \partial\Sigma_n$ would intersect one of the curves connecting $\partial\Sigma_n$ and the cycle of $\Phi \cap \partial\Sigma_n$, and we obtain a contradiction. Therefore, when one of the tangent leaves have this pattern of tangency, the foliation is tangent to Σ_n in a set of two points, at most.

If $\Phi \cap \partial\Sigma_n$ is of the type C1 or C2, we can not have two leaves Φ_1 and Φ_2 of the foliation such that Φ, Φ_1 and Φ_2 are pairwise different and tangent to Σ_n. Indeed, assuming the opposite, if $\Phi \cap \partial\Sigma_n$ is of type C1, $\Phi_i \cap \partial\Sigma_n$ is of type C1, because the segments of $\Phi \cap \partial\Sigma_n$ would intersect any nontrivial cycle in Σ_n. We then observe that one of the leaves (say, Φ) separates $\mathbb{M} \times \mathbb{R}$ in two components, each of them containing only one of the other leaves, and consequently Φ would separate the intersections $\Phi_i \cap \partial\Sigma_n$, which cannot happen. By a similar analysis, $\Phi \cap \partial\Sigma_n$ is of type C2, the other intersections will also be of the same type. In that case, there would be a leaf (say, Φ) whose cycle of the intersection lies in the annulus bounded by the cycles of $\Phi_i \cap \partial\Sigma_n$. But there is an arc in $\Phi \cap \partial\Sigma_n$ which connects the cycle to a component of $\partial\Sigma_n$ (see Figure 1), but it would cross the cycle of one of the other intersections, a contradiction. Then, in this situation, Σ_n is tangent to $\mathcal{F}^{\eta}$ in at most two points. □

Proposition 3.10. *The annulus Σ_n is tangent to the foliation $\mathcal{F}^{Gr(w)}$ at most at two points, for all n.*

Proof. It is enough to use the hypotheses stated in the definition of $\mathcal{F}^{Gr(w)}$ and proceed in a similar way of the previous proposition. □

Proposition 3.11. *The surface Σ_n is tangent to the foliation $\mathcal{F}^{\tilde{\eta}_i}$ at most at two points, for $i = 1, 2$ and for all n.*

Proof. As a consequence of Lemma 3.6, the possible configurations of $\omega(t)$ containing a tangent point are shown in Figure 3.3. Precisely, we have one tangent point in both situations. Taking two leaves Φ_1, Φ_2 of $\mathcal{F}^{\tilde{\eta}_i}$ which are tangent to Σ_n, we can notice that the set $Int(\Sigma_n)\backslash(\Phi_1 \cup \Phi_2)$ has one component which is a topological annulus whose boundary does not intersect $\partial\Sigma_n\backslash(\Phi_1 \cup \Phi_2)$. Since every leaf Φ which intersects Σ_n tangentially must have a non-trivial cycle in the intersection and also intersect the boundary of Σ_n, we obtain a contradiction. So Φ_1 and Φ_2 are the only tangent leaves, and the number of the tangent points is at most two in these cases. □

3.1.3 Curvature estimates

For any n sufficiently large, denote by Σ_n a minimal annulus whose boundary is $\Gamma_n^1 \cup \Gamma_n^2$. The main goal of this subsection is the following proposition:

Proposition 3.12. *The sequence $(sup_{x\in\Sigma_n}||A_n(x)||)_{n\in\mathbb{N}}$ is bounded.*

Proof. If this were not true, we have that, defining λ_n as $sup_{x\in\Sigma_n}||A_n(x)||$, then $lim_{n\to\infty}\lambda_n = \infty$. Denote by p_n a point in Σ_n satisfying $||A_n(p_n)|| = \lambda_n$. We then apply a blow-up process, which will be explained in the following.

Let $\lambda_n : T_{p_n}(\mathbb{M} \times \mathbb{R}) \to T_{p_n}(\mathbb{M} \times \mathbb{R})$ be scalar multiplication by λ_n in $T_{p_n}(\mathbb{M} \times \mathbb{R})$. Define U_n as the space $T_{p_n}(\mathbb{M} \times \mathbb{R})$ endowed with the metric

$(exp_{p_n} \circ \lambda_n)^*(g + dt^2)$ (for simplicity, we will denote the map $exp_{p_n} \circ \lambda_n$ by ϕ_n). Since the curvature of $\mathbb{M}$ is pinched between between two constants, the sequence of Riemannian manifolds $(U_n)_n$ converges smoothly to the space $\mathbb{R}^3$ with the Euclidean metric. We also define $\widetilde{\Sigma}_n$ as $\phi_n^{-1}(\Sigma_n) \subset U_n$. Clearly, the surfaces $\widetilde{\Sigma}_n$ are minimal in U_n.

Remark. Taking one of the foliations $\mathcal{F}^h$, $\mathcal{F}^{Gr(w)}$, $\mathcal{F}^\gamma$, $\mathcal{F}^\eta$ or $\mathcal{F}^{\tilde{\eta}_i}$, $i = 1, 2$ (see Subsection 3.1.2 for the notation) and denoting it by $\mathcal{F}$, we see that $\mathcal{F}_n := (\phi_n)^*(\mathcal{F})$ is a foliation of U_n by minimal surfaces. Since the curvature of the leaves of $\mathcal{F}$ is bounded, we have that, up to a subsequence, the sequence $(\mathcal{F}_n)_n$ converges to a foliation $\mathcal{F}_\infty$ in $\mathbb{R}^3$ whose leaves are Euclidean planes.

Claim. *There exists a subsequence $(\widetilde{\Sigma}_k)_k$ of the sequence $(\widetilde{\Sigma}_n)_n$ and a minimal surface $\widetilde{\Sigma}_\infty$ in $\mathbb{R}^3$ satisfying the following properties:*

1. *$\widetilde{\Sigma}_\infty$ is embedded in $\mathbb{R}^3$;*
2. *$\widetilde{\Sigma}_\infty$ is contained in the accumulation set of the subsequence;*
3. *$O \in \widetilde{\Sigma}_\infty$ and $||A_{\widetilde{\Sigma}_\infty}||(O) = lim_{k\to\infty}||A_{\widetilde{\Sigma}_k}||(O) = 1$;*
4. *If the boundary of $\widetilde{\Sigma}_\infty$ is nonempty, it is a straight line;*
5. *The surface $\widetilde{\Sigma}_\infty$ is complete;*
6. *The surface $\widetilde{\Sigma}_\infty$ has finite total curvature.*

Proof. The proofs of the first five items follow the same ideas of Lemma 2.2.21 of [31]. For the last one, we denote by $\widehat{\Sigma}_\infty$ the union of $\widetilde{\Sigma}_\infty$ with its image by the reflection through the straight line which is the boundary of $\widetilde{\Sigma}_\infty$ when the boundary of $\widetilde{\Sigma}_\infty$ is nonempty, and $\widetilde{\Sigma}_\infty$ otherwise. Clearly, $\widehat{\Sigma}_\infty$ is a minimal surface of $\mathbb{R}^3$ without boundary. It is enough to prove that the Gauss map of $\widehat{\Sigma}_\infty$ takes on five different values a finite number of times, because the surface $\widehat{\Sigma}_\infty$ will have finite total curvature in this case, by the Mo-Osserman's theorem (see [27] for the theorem).

We divide the proof in two cases:

Case 1. Suppose $\widetilde{\Sigma}_\infty$ has no boundary. If $\mathcal{F}$ is one of the foliations $\mathcal{F}^h$, $\mathcal{F}^\gamma$ or $\mathcal{F}^\eta$ of $\mathbb{M} \times \mathbb{R}$, we know that Σ_n is tangent to $\mathcal{F}$ at most two points, and since ϕ_n preserves angles, $\mathcal{F}_n := (\phi_n)^*(\mathcal{F})$ is tangent to $\widetilde{\Sigma}_n$ at most two points.

By the above remark, the sequence $(\mathcal{F}_n)_n$ converges to a foliation $\mathcal{F}_\infty$ of $\mathbb{R}^3$ by Euclidean planes, up to a subsequence, and by Lemma 2.2.20 of

[31], $\mathcal{F}_\infty$ is tangent to $\widetilde{\Sigma}_\infty$ at most two points. It is easy to see that the angles between the leaves of the foliations $\mathcal{F}^h$, $\mathcal{F}^\gamma$ and $\mathcal{F}^\eta$ are bounded away from 0 and π at points of $\Omega \times \mathbb{R}$, so the limit foliations defined by them are nonparallel. This means that there are 6 values on $\mathbb{S}^2$ whose inverse image by the Gauss map of $\widetilde{\Sigma}_n$ has a finite number of elements, so $\widetilde{\Sigma}_n$ has finite total curvature.

Case 2. Suppose $\widetilde{\Sigma}_\infty$ has nonempty boundary. In that case, the set $\partial\widetilde{\Sigma}_\infty$ is a straight line L. Moreover, we have that $(\lambda_n d_{\mathbb{M}\times\mathbb{R}}(p_n, \partial\Sigma_n))_n$ is bounded from above (at least for a subsequence). Then, up to a subsequence, the sequence $(\breve{p}_n)_n$ converges to $p \in \bar{\gamma}_1 \cup \bar{\gamma}_2$ and $(\partial\breve{\Sigma}_n)_n$ converges to a curve $\Gamma \subset \mathbb{M} \times \mathbb{R}$ (since the elements of $(\partial\breve{\Sigma}_n)_n$ have bounded geometry and they accumulate around p). Let us assume that p is contained in $\bar{\gamma}_1$. Furthermore, for each n, we can choose $q_n \in \partial\breve{\Sigma}_n$ such that $d_{\mathbb{M}\times\mathbb{R}}(\breve{p}_n, q_n) = d_{\mathbb{M}\times\mathbb{R}}(\breve{p}_n, \partial\breve{\Sigma})$ and, in this case, $(q_n)_n$ converges to p. It is true that $((\phi_n)^{-1}(q_n))_n$ converges to a point $\tilde{p} \in L$, since $d_{U_n}(0, \partial\widetilde{\Sigma}_n) = \lambda_n d_{\mathbb{M}\times\mathbb{R}}(p_n, \partial\Sigma_n)$.

We consider three foliations $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ in $\mathbb{M} \times \mathbb{R}$ in the following way. If the tangent space $T_p\Gamma$ is vertical, define $\mathcal{F}_1 := \mathcal{F}^h$, $\mathcal{F}_2 := \mathcal{F}^\gamma$ and $\mathcal{F}_3 := \mathcal{F}^\eta$. If not, consider a function $f : \widetilde{\gamma}_1 \cup \widetilde{\gamma}_2 \to \mathbb{R}$ such that the graph of f is tangent to Γ at p. For each $i = 1, 2, 3$, define $u_i : \widetilde{\Omega} \to \mathbb{R}$ as a function such that $Gr(u_i)$ is a minimal graph over $\widetilde{\Omega}$ and $u_i = f$ along $\widetilde{\gamma}_1 \cup \widetilde{\gamma}_2$ and $u_i = i$ along $\widetilde{\eta}_1 \cup \widetilde{\eta}_2$. We define $\mathcal{F}_i$ to be the foliation $\mathcal{F}^{Gr(u_i)}$ of $\widetilde{\Omega} \times \mathbb{R}$. We also assume f satisfies the properties stated in the definition of $\mathcal{F}^{Gr(u_i)}$ (recall Subsection 3.1.2). By Maximum Principle, we have that $u_1 < u_2 < u_3$ in Ω and the tangent planes at p of $Gr(u_i)$ are distinct.

Regardless the position of $T_p\Gamma$ in $T_p(\mathbb{M}\times\mathbb{R})$, we have that the curvature of the leaves of $\mathcal{F}_i$ are uniformly bounded, for $i = 1, 2, 3$. Moreover, when $T_p\Gamma$ is vertical, the sequence of foliations $((\phi_n)^*(\mathcal{F}_i))_n$ converge to a foliation of $\mathbb{R}^3$ by Euclidean planes, and when $T_p\Gamma$ is not vertical, the sequence of foliations $((\phi_n)^*(\mathcal{F}_i))_n$ converge to a foliation of the half-space of $\mathbb{R}^3$ determined by the limit of $(\Lambda_i^n := (\phi_n)^*(\widetilde{\gamma}_1 \times \mathbb{R}))_n$, and all of its leaves are Euclidean half-planes. In both cases, we will call by $\widetilde{\mathcal{F}}_i$ the limit foliation induced by $(\mathcal{F}_i^n)_n$; moreover, if $T_p\Gamma$ is not vertical, we will call by $\widehat{\mathcal{F}}_i$ the foliation obtained by reflection along the boundary of the foliated half-space, and if $T_p\Gamma$ is vertical, we make $\widehat{\mathcal{F}}_i := \widetilde{\mathcal{F}}_i$. If $\angle_r(A, B)$ is the angle at r between the curve A and the leaf of the foliation B passing through r, we know that

$$\angle_{\tilde{p}}(L, \widetilde{\mathcal{F}}_i) = lim_n \angle_{\phi_n{}^{-1}(q_n)}(\partial\widetilde{\Sigma}_n, \mathcal{F}_i^n) = lim_n \angle_{q_n}(\partial\breve{\Sigma}_n, \mathcal{F}_i) = \angle_p(\Gamma, \mathcal{F}_i),$$

then, the limit foliations we obtain are either parallel or perpendicular to L, and consequently they are invariant by the symmetry about that line.

We now prove that $\widetilde{\Sigma}_\infty$ is not tangent to any foliation $\widetilde{\mathcal{F}}_i$ on L. Suppose that $T_p\Gamma$ is not vertical. Then, if $\widetilde{\Sigma}_\infty$ is tangent to $\widetilde{\mathcal{F}}_i$ on L, let Λ be the leaf containing L in its boundary. We can suitably choose f so that its graph is tangent to $\partial\breve{\Sigma}_n$ at q_n, for large n. Clearly, Λ must be the limit of a sequence of leaves $\Lambda_n \in \mathcal{F}_i^n$, $\partial\Lambda_n$ tangent to $\partial\widetilde{\Sigma}_n$ at q_n for each n (up to a subsequence). But, for all n, we have that $\widetilde{\Sigma}_\infty$ is in one side of Λ_n, so $\widetilde{\Sigma}_\infty$ is in one side of Λ. The Maximum Principle would imply that $\widetilde{\Sigma}_\infty = \Lambda$, a contradiction to item 3 of Claim 3.1.3. The case when $T_p\Gamma$ is vertical is analogous; see the proof of Assertion 2.2.1 of [31] for the ideas.

As in Case 1, we have that $\widetilde{\Sigma}_\infty$ is tangent to the foliation $\widetilde{\mathcal{F}}_i$ at most at two points. Then $\widehat{\Sigma}_\infty$ is tangent to $\widehat{\mathcal{F}}_i$ at most at 4 points. Furthermore, the three foliations $\widehat{\mathcal{F}}_i$ are non-parallel, so there are 6 values on $\mathbb{S}^2$ whose inverse image by the Gauss map of $\widehat{\Sigma}_n$ has a finite number of elements, so $\widehat{\Sigma}_n$ has finite total curvature. □

Claim. *The minimal surface $\widetilde{\Sigma}_\infty$ has empty boundary.*

Proof. If this is not the case, we have $\partial\widetilde{\Sigma}_\infty$ is a straight line L. Moreover, $\partial\widetilde{\Sigma}_\infty$ is contained in a region Υ bounded by two planes which intersect along L forming an angle smaller than π. Then we reflect $\widetilde{\Sigma}_\infty$ and Υ along L and we call the union of the set with its reflection by $\widehat{\Sigma}_\infty$ and $\widehat{\Upsilon}$. By Claim 3.1.3, the minimal surface $\widehat{\Sigma}_\infty$ is complete, embedded and has finite total curvature. By [Sc1], each end of $\partial\widetilde{\Sigma}_\infty$ is asymptotic to a plane or a catenoid. If all of its ends are planar, the planes must contain L, there is an unique tangent plane to all the ends of $\partial\widetilde{\Sigma}_\infty$ (Theorem 6 of [6]), and, by the Half-space theorem, it should be a flat plane, a contradiction, because $||A_{\widetilde{\Sigma}_\infty}||(O) = 1$. So, there must be a catenoidal end. Since a catenoid can not be contained in $\widehat{\Upsilon}$, we obtain a contradiction, because $\widetilde{\Sigma}_\infty \subset \widehat{\Upsilon}$. □

By all the above discussion, the surface $\widetilde{\Sigma}_\infty \subset \mathbb{R}^3$ is complete, embedded, non-flat, minimal surface without boundary of finite total curvature. We will prove that this surface can not arise from the reasoning above, and the proposition is then proved.

Since $\widetilde{\Sigma}_\infty \subset \mathbb{R}^3$ is not a flat plane, by Theorem 3.1 of [20], it must have at least two ends. Let ν be a Jordan curve which is the boundary of such an end. This curve is homotopically nontrivial and it separates the surface in two noncompact parts. Let $(\tilde{\nu}_n)_{n\in\mathbb{N}}, \tilde{\nu}_n \subset \widetilde{\Sigma}_n$ be a sequence of Jordan curves converging to ν. It guarantees that $\tilde{\nu}_n$ is homotopically nontrivial for n sufficiently large. Otherwise, we would have a subsequence $(\tilde{\nu}_{n_k})_{k\in\mathbb{N}}$ of homotopically trivial curves. In that case, each curve $\tilde{\nu}_{n_k}$ bounds a disc D_{n_k} in $\widetilde{\Sigma}_{n_k}$, and since we have the isoperimetric inequality $L(\tilde{\nu}_{n_k})^2 \geq 4\pi A(D_{n_k})$, by Theorem 1.1 of [42], the sequence $(D_{n_k})_k$ converges (up to a subsequence) to a disc in $\widetilde{\Sigma}_\infty$ bounded by ν, which is a contradiction.

Define as ν_n the curve $\phi_n(\tilde{\nu}_n) \subset \Sigma_n$. Clearly, $l_{U_n}(\tilde{\nu}_n) = \lambda_n l_{\mathbb{M}\times\mathbb{R}}(\nu_n)$ and $(l_{U_n}(\tilde{\nu}_n))_n$ converges to $l_{\mathbb{R}^3}(\nu)$, so $(l_{\mathbb{M}\times\mathbb{R}}(\nu_n))_n$ converges to 0. If $\pi : \mathbb{M}\times\mathbb{R} \to \mathbb{M}$ is the projection onto the first factor, we obtain that $lim_{n\to\infty} l_{\mathbb{M}}(\pi(\nu_n)) = 0$. Clearly the sequence of curves $\pi(\nu_n)$ converges, up to a subsequence, to a point $p \in \Omega$, so $lim_{n\to\infty}(d(\gamma_1, \pi(\nu_n)) + d(\pi(\nu_n)), \gamma_2)) = d(\gamma_1, p) + d(p, \gamma_2)$. We can suppose that there exist a positive c such that $d(\gamma_1, p) \geq c$, since the geodesics are ultraparallel.

Let $\mathcal{A}_n$ be the sub-annulus of Σ_n bounded by Γ_1^n and ν_n. Let q_1, q_2 be two points such that γ_1 is properly contained in the geodesic connecting q_1 and q_2. Since the curves $\pi(\nu_n)$ are contained in Ω and they converge to p, there exists a point ξ_n such that the geodesic triangle whose vertices are q_1, q_2 and ξ_n is the smallest triangular domain containing $\pi(\nu_n)$ and whose set of vertices contains q_1 and q_2 (we call the geodesic triangle T^n). We have that the angle of T^n at the vertex ξ_n (call it θ_n) is such that $\theta_n < \pi - \theta$, for some $\theta > 0$, because $d_{\mathbb{M}}(\xi_n, \gamma_1) > d_{\mathbb{M}}(\pi(\nu_n), \gamma_1) \geq c$. We can also suppose $\theta_n \geq \theta$.

By the maximum principle, $\mathcal{A}_n \subset T^n \times \mathbb{R}$. Moreover, the sequence $(\tilde{\nu}_n)_n$ converges to ν. We then conclude there is a subsequence of $\phi_n^{-1}(T^n \times \mathbb{R}) \subset U_n$ converging to a region R in $\mathbb{R}^3$ bounded by two half-planes whose angle lies in the interval $[\theta, \pi - \theta]$. In fact, for $i = 1, 2$, if β_i^n are the complete geodesics of $\mathbb{M}$ such that β_i^n connects the points q_i and ξ_n, $n \in \mathbb{N}$, the sequence of totally geodesic planes $(\phi_n^{-1}(\beta_i^n \times \mathbb{R}))_n$ converge to planes in $\mathbb{R}^3$, since $\phi_n^{-1}(\beta_i^n \times \mathbb{R})$ contains a point of $\tilde{\nu}_n$. So, by the range of variation of θ_n, the limit planes can not be parallel, so, up to a subsequence, the sequence $(\phi_n^{-1}(T^n \times \mathbb{R}))_n$ converges to a region as described before.

We also have that $\phi_n^{-1}(\mathcal{A}_n)$ converges to a part of $\widetilde{\Sigma}_\infty$ having ν as boundary, so the set R must contain an end of $\widetilde{\Sigma}_\infty$, which is impossible, since this end must be asymptotic to a plane or a catenoid, but the ends of such surfaces can not be contained in R. □

3.1.4 Convergence of the sequence $(\Sigma_n)_{n\in\mathbb{N}}$

It was already proved that, for n sufficiently large, there is a stable annulus Σ_n^s and another annulus Σ_n^u whose boundary is $\Gamma_1^n \cup \Gamma_2^n$. The sequences $(\Sigma_n^s)_n$ and $(\Sigma_n^u)_n$ have uniform curvature bounds. For simplicity, we are going to denote the surfaces Σ_n^s and Σ_n^u by Σ_n. It was also proved that there exist two horizontal points in Σ_n (call them $p^+(\Sigma_n)$ and $p^-(\Sigma_n)$) satisfying $h_n^+ = h^+(\Sigma_n) = h(p^+(\Sigma_n)) \geq h(p^-(\Sigma_n)) = h^-(\Sigma_n) = h_n^-$.

Henceforth, we denote by $\breve{\Sigma}_n$ a vertical translation of Σ_n (although the notation is ambiguous, in each situation, the translation will be specified). We will denote by $p^+(\breve{\Sigma}_n)$ (resp. $p^-(\breve{\Sigma}_n)$) the image of $p^+(\Sigma_n)$ (resp. $p^-(\Sigma_n)$) by a vertical translation. The objective of the rest of the section is to choose an appropriated sequence $(\breve{\Sigma}_n)_n$ of translation and obtain a subsequence which converges to a minimal annulus satisfying the hypotheses of Theorem 3.1.

By Theorem 3.3 of [24], there exists, for $i = 1, 2$, a solution u_i^+ to the minimal surface equation on Ω such that $u_i^+ = +\infty$ on γ_i and $u_i^+ = 0$ on $\eta_1 \cup \eta_2 \cup \gamma_j$, when $\{i, j\} = \{1, 2\}$. Consider the function $u^+ = sup(u_1^+, u_2^+)$.

Lemma 3.13. *For n sufficiently large, the surface Σ_n is below the graph of $u^+ + h_n^+$ and above the graph of $-u^+ + h_n^-$.*

Proof. It is enough to prove that Σ_n is below the graph of $u^+ + h_n^+$, and if $\breve{\Sigma}_n$ is the vertical translation such that $h^+(\breve{\Sigma}_n) = 0$, it suffices to prove that $\breve{\Sigma}_n$ lies below u^+. By Proposition 3.7, the region $\breve{\Sigma}_n \cap \{z > 0\}$ is composed of two simply connected components D_1 and D_2, whose boundaries are clearly in $(\gamma_i \times \mathbb{R}) \cup \{z = 0\}$. We now prove that D_i lies below the graph of u_i^+. Take a bounded convex quadrilateral Ω' containing Ω whose boundary is formed by the geodesics γ_1', η_1', γ_2' and η_2', cyclically mentioned; besides, $\gamma_i \Subset \gamma_i'$, $i = 1, 2$. Again, by Theorem 3.3 of [24], there is a minimal solution v_i on Ω' satisfying $v_i = \infty$ on γ_i' and $u_i^\pm = 0$ on $\eta_1' \cup \eta_2' \cup \gamma_j'$, when $\{i, j\} = \{1, 2\}$. Furthermore, the graph of v_i lies above the graph of u_i^+, as a consequence of the proof of the theorem. When Ω' converges to Ω, v_i converges to u_i^+, so we only need to prove that D_i lies below the graph of v_i. For this, notice that D_i does not intersect $T_h(Gr(v_i))$ if h is large enough. Then, we can take $h_0 := inf\{h \in \mathbb{R}, T_{h'}(D_i) \cap D_i = \emptyset, \forall h' > h\}$. We must have that $T_{h_0}(Gr(v_i))$ and D_i have a first point of contact and it can not be in neither of its boundaries, and it contradicts the Maximum Principle. $\square$

Lemma 3.14. *Let Δ be an wedge in $\mathbb{M}$ bounded by two half-geodesics starting at O forming an angle smaller than π. Let $S \subset \Delta \times \mathbb{R}$ be a minimal surface* *in $\mathbb{M} \times \mathbb{R}$ and a point $p \in S$ satisfying $||A_S|| \leq C$ and $d_S(p, \partial S) \geq \delta$. Then, there exists $\epsilon > 0$, depending only on C and δ, such that p is not in the cylinder $D_\epsilon(O) \times \mathbb{R}$, where $D_\epsilon(O)$ is the open disc in $\mathbb{M}$ centered at O of radius ϵ.*

Proof. Suppose that there exist a sequence $(S_n)_n$ of minimal surfaces and points $p_n \in S_n$ satisfying $d_{S_n}(p_n, \partial S_n) \geq \delta$ and $p_n \in D_{1/n}(O) \times \mathbb{R}$. We then apply a blow-up process using the sequence of points $(p_n)_n$ and the sequence of constants $(\lambda_n := n)_n$. Using the notation of Proposition 3.12, we write $\tilde{p}_n := \phi_n^{-1}(p_n)$ and $\widetilde{S}_n := \phi_n^{-1}(S_n)$. Considering the sequence of ambient manifolds U_n, we have that $d_{U_n}(\tilde{p}_n, \phi_n^{-1}(\{O\} \times \mathbb{R})) \leq 1$, $||A_{\widetilde{S}_n}|| \leq C/n$ and $d_{\widetilde{S}_n}(\tilde{p}_n, \partial \widetilde{S}_n) \geq n\delta$. The sequence $(U_n)_n$ converges to the Euclidean space $\mathbb{R}^3$ and $(\widetilde{S}_n)_n$ converges to a minimal surface S. This limit surface must be contained in a wedge determined by two half-planes whose angle is smaller than π. On the other hand, the surface S must be a complete plane, which leads to a contradiction. □

Lemma 3.15. *Assume that the sequence $(\breve{p}_n := p^+(\breve{\Sigma}_n))_n$ converges to $\breve{p}_\infty$. Then there exists a subsequence of $\breve{\Sigma}_n$ converging to a minimal surface $\breve{\Sigma}_\infty$ in a neighborhood of $\breve{p}_\infty$ with multiplicity one.*

Proof. We know that the sequence $(\breve{\Sigma}_n)_n$ has uniform curvature bounds and $\breve{p}_\infty$ is an accumulation point of $\bigcup_{i=1}^{\infty} \breve{\Sigma}_i$. By the Appendix B of [7], there exists a subsequence of $(\breve{\Sigma}_n)_n$ converging to a minimal lamination $\mathcal{L}$ containing $\breve{p}_\infty$ (for simplicity, we suppose the subsequence is actually the whole sequence). Let $\breve{\Sigma}_\infty$ be the leaf of $\mathcal{L}$ passing through $\breve{p}_\infty$.

Assume the lemma is not true. Then, there exists a sequence of leaves $(L_n)_n$ of $\mathcal{L}$ converging to $\breve{\Sigma}_\infty$ in a neighborhood of $\breve{p}_\infty$. Since $\breve{p}_\infty$ is a horizontal point of $\breve{\Sigma}_\infty$, by Lemma 2.2.20 of [31], there are horizontal points in L_n near $\breve{\Sigma}_\infty$, and by the same lemma, we obtain that, for large n, there are at least three horizontal points in $\breve{\Sigma}_n$ near $\breve{p}_\infty$, contradicting Proposition 3.7. So, there exists a neighborhood V of $\breve{p}_\infty$ in $\mathbb{M} \times \mathbb{R}$ such that $V \cap \mathcal{L} = V \cap \breve{\Sigma}_\infty$. If the sequence of surfaces $V \cap \breve{\Sigma}_n$ converges to $V \cap \breve{\Sigma}_\infty$ with multiplicity at least 3, by Lemma 2.2.20 of [31], there must be at least three horizontal points in $\breve{\Sigma}_n$ for large n, contradicting Proposition 3.7.

Suppose the convergence around $\breve{p}_\infty$ happens with multiplicity two. It implies that there is an open set U containing $\breve{p}_\infty$ such that, for all $V \subset U$ containing $\breve{p}_\infty$, there exists $n_V \in \mathbb{N}$ depending on V such that $\{p^+(\Sigma_n), p^-(\Sigma_n)\} \subset$

V for $n \geq n_V$. It implies that $(p^+(\breve{\Sigma}_n))_n$ and $(p^-(\breve{\Sigma}_n))_n$ converge to $\breve{p}_\infty$, and since the surface $\breve{\Sigma}_n$ is symmetric with respect to some horizontal slice between $h(p^+(\breve{\Sigma}_n))$ and $h(p^-(\breve{\Sigma}_n))$, the limit surface $\breve{\Sigma}_\infty$ has its boundary formed by four vertical lines passing through the vertices of Ω, so $\breve{p}_\infty \in Int(\breve{\Sigma}_\infty)$, and $\mathbb{M} \times \{h(\breve{p}_\infty)\}$ is a plane of symmetry for $\breve{\Sigma}_\infty$. Then, since the tangent plane of $\breve{\Sigma}_\infty$ at $\breve{p}_\infty$ is horizontal, the only way to have symmetry near $\breve{p}_\infty$ with respect to $\mathbb{M} \times \{h(\breve{p}_\infty)\}$ is having the coincidence of $\breve{\Sigma}_\infty$ and $\mathbb{M} \times \{h(\breve{p}_\infty)\}$ in a neighborhood of $\breve{p}_\infty$, then $\breve{\Sigma}_\infty$ is a subset of $\Omega \times \{h(\breve{p}_\infty)\}$, a contradiction. So the multiplicity must be one. □

Lemma 3.16. *Let $\breve{\Sigma}_n$ be a vertical translation of Σ_n satisfying that the set $\{h^+(\breve{\Sigma}_n); n \in \mathbb{N}\}$ is bounded and the sequence $(h^+(\breve{\Sigma}_n))_n$ goes to $-\infty$. Then there is a subsequence of $(\breve{\Sigma}_n)_n$ which converges to a minimal surface $\breve{\Sigma}_\infty$. This limit surface is simply connected and it is a vertical graph in $\mathbb{M} \times \mathbb{R}$.*

Proof. Following the idea of [31], we are going to prove this lemma in three steps. First, we prove the existence of a subsequence of $(\breve{\Sigma}_n)$ which converges with multiplicity one to a surface $\breve{\Sigma}_\infty$ with boundary. Then we prove that $\breve{\Sigma}_\infty$ is simply connected and finally, we prove that the limit surface is a graph over a subdomain of Ω.

It is known that $(\breve{\Sigma}_n)_n$ has uniform curvature bound. Moreover, since $\{p^+(\breve{\Sigma}_n)\}$ is bounded, it has an accumulation point, so does $(\breve{\Sigma}_n)_n$. It implies, by Lemma 3.15, that there exists a subsequence of $(\breve{\Sigma}_n)_n$ which converges to a minimal surface $\breve{\Sigma}_\infty$ with multiplicity one.

Now, we prove that there is an $\epsilon > 0$ such that $(D_\epsilon(p) \times \mathbb{R}) \cap \breve{\Sigma}_n$ contains only one component of $\breve{\Sigma}_n$, for all sufficiently large n, where p is a vertex of Ω. In fact, suppose that there exists a subsequence $(k_n)_{n\in\mathbb{N}}$ such that the set $(D_{n^{-1}}(p) \times \mathbb{R}) \cap \breve{\Sigma}_{k_n}$ contains at least two components of $\breve{\Sigma}_{k_n}$ (without loss of generality, suppose $k_n = n$). Define h_n as $(h^+(\breve{\Sigma}_n) + h^-(\breve{\Sigma}_n))/2$. If C_n is a component of $(D_{n^{-1}}(p) \times \mathbb{R}) \cap \breve{\Sigma}_n$ which does not contain points of $\partial\breve{\Sigma}_n$, take a point q_n of $C_n \cap \{z = h_n\}$ that minimizes the distance (in $\mathbb{M}$) to $p_n := (p, h_n)$.

Consider the maps $exp_p \times exp_{h_n} : T_p\mathbb{M} \times T_{h_n}\mathbb{R} \to \mathbb{M} \times \mathbb{R}$ and, for $\lambda_n \in \mathbb{R}$, the map $\lambda_n : T_p\mathbb{M} \times T_{h_n}\mathbb{R} \to T_p\mathbb{M} \times T_{h_n}\mathbb{R}$ which is the multiplication by λ_n. Denoting by ϕ_n the map $(exp_p \times exp_{h_n}) \circ \lambda_n$, we consider the ambient spaces $U_n := (T_p\mathbb{M} \times T_{h_n}\mathbb{R}, \phi_n^*(g + dt^2))$. If we blow-up the sequence of spaces $(U_n)_n$ using the constants $\lambda_n := d_{\mathbb{M}}(q_n, p_n)^{-1}$ around the points $(p_n)_n$, we obtain that $\{\phi_n^{-1}(C_n)\}_{n\in\mathbb{N}}$ has a subsequence converging to a plane or a half-plane

P in $\mathbb{R}^3$ that is tangent to $D_1(O) \times \mathbb{R}$, where O is the fixed point of the blow-up. If P were a half-plane, then its boundary should be $\{O\} \times \mathbb{R}$, which contradicts the tangency to $D_1(O) \times \mathbb{R}$. Hence we obtain a complete plane inside $W \times \mathbb{R}$, where W is an wedge of $\mathbb{R}^2$ whose angle is smaller than π, contradiction. We conclude from this argument that the sequence $(\breve{\Sigma}_n)_{n\in\mathbb{N}}$ converges with multiplicity 1 in small cyllindric neighborhoods of $\{p\} \times \mathbb{R}$, where p is a vertex of Ω, and also the whole sequence converges to a surface $\breve{\Sigma}_\infty$ with multiplicity 1.

For the second part, we clearly have that $\breve{\Sigma}_\infty$ is the limit of the sequence $\mathcal{A}_n := (\breve{\Sigma}_n \cap \{(x, y, z) \in \mathbb{M} \times \mathbb{R}; z > \frac{p^+(\breve{\Sigma}_n)+p^-(\breve{\Sigma}_n)}{2}\})_n$. Given a loop α in $\breve{\Sigma}_\infty$, let $(\alpha_n)_n$ be a sucession of closed curves $\alpha_n \subset \breve{\Sigma}_n$ converging to α. For large n, α_n is contained in $\mathcal{A}_n$, and since $\mathcal{A}_n$ is simply connected, there is a disc D_n in $\mathcal{A}_n$ whose boundary is α_n. Using the Theorem 1.1 of [42], the sequence $(D_{n_k})_k$ converges (up to a subsequence) to a disc in $\breve{\Sigma}_\infty$ bounded by α, so α is homotopically trivial.

For the third part, we shall prove that $Int(\breve{\Sigma}_\infty)$ has no points with vertical tangent plane. Indeed, if that is not the case, take a point q in $Int(\breve{\Sigma}_\infty)$ whose tangent plane (say, Q) is vertical. Then we consider a foliation of $\mathbb{M} \times \mathbb{R}$ containing Q made of vertical totally geodesic planes. Using Lemma 2.2.20 of [31], we obtain that, for large n, $(\breve{\Sigma}_n)$ has vertical tangent planes in points contained in a neighborhood of q, but this is not possible. So the tangent planes of the points of $Int(\breve{\Sigma}_\infty)$ are not vertical.

Finally, we are going to prove that the projection $\pi : Int(\breve{\Sigma}_\infty) \to \mathbb{M}$ is injective. Assume the contrary, then we can find an open set $O \subset \mathbb{M}$ and functions $f_i : O \to \mathbb{R}, i = 1, 2$, such that the graph of f_i (notation: $Gr(f_i)$) is an open subset of $Int(\breve{\Sigma}_\infty)$. Choosing O small enough, we can suppose that $Gr(f_1)$ and $Gr(f_2)$ are disjoint and that, for sufficiently large n, there exists functions $f_i^n : Gr(f_i) \to \mathbb{R}$ such that the exponential graph of f_i^n is an open subset of $\breve{\Sigma}_n$. So, we define the maps $g_i^n : O \to \mathbb{M}$ given by $g_i^n = \pi \circ Gr(f_i^n) \circ Gr(f_i)$. Clearly, those maps converge uniformly to the inclusion $i : O \to \mathbb{M}$. So, choosing a point $o \in O$, we can find, for large n, two different points o_1^n and o_2^n in O satisfying $g_i^n(o_i^n) = o$, and it leads to the fact that the projection $\pi : \mathcal{A}_n \to \mathbb{M}$ is not injective, a contradiction. □

Proposition 3.17. *If $h^+(\Sigma_n) - h^-(\Sigma_n)$ goes to $+\infty$, then the sequence $(n - h^+(\Sigma_n))_n = (h^-(\Sigma_n) + n)_n$ is bounded.*

Proof. It suffices to show that $n - h^+(\Sigma_n)$ is bounded. If it does not happen, then we have a subsequence of $(n - h^+(\Sigma_n))_n$ which goes to $+\infty$, and we suppose it is $(n - h^+(\Sigma_n))_n$, without loss of generality.

Let $\breve{\Sigma}_n$ be the vertical translation of Σ_n satisfying $h^+(\breve{\Sigma}_n) = 0$. It is true that $(h^-(\breve{\Sigma}_n))_n$ tends to $-\infty$, and we can apply the Lemma 3.16 and conclude that there exists a subsequence of $(\breve{\Sigma}_n)_n$ converging to the minimal graph $\breve{\Sigma}_\infty$ of a function $u : U \to \mathbb{R}$ defined over a subdomain of Ω. Moreover, since $n - h^+(\Sigma_n) \to \infty$, the boundary of $\breve{\Sigma}_\infty$ consists of four vertical lines passing through the vertices of Ω.

Our goal is to prove that $U = \Omega$ and that u assumes values $+\infty$ in $\gamma_1 \cup \gamma_2$ and $-\infty$ in $\eta_1 \cup \eta_2$. Notice that $\breve{\Sigma}_\infty$ has only one horizontal point p, and its height is 0. The set $I := \breve{\Sigma}_\infty \cap \{z = 0\}$ consists of four arcs connecting p and the vertices of Ω. It separates Ω in four components, and let P_i (resp. Q_i) the component bounded by I and γ_i (resp. I and η_i), $i = 1, 2$. Clearly the surface $\breve{\Sigma}_\infty$ is also divided in four components, given by $Gr(u|_{P_i \cap U})$ and $Gr(u|_{Q_i \cap U})$. Then, by Lemma 3.13, $Gr(u|_{P_1 \cap U}) \cup Gr(u|_{P_2 \cap U})$ lies below the graph of u^+. Since $n - h^+(\Sigma_n)$ goes to $+\infty$ and $u|_{(P_1 \cup P_2) \cap U}$ does not change sign, we conclude that $P_1 \cup P_2 \subset U$ and, by construction, the function assumes the value $+\infty$ in $\gamma_1 \cup \gamma_2$.

For simplicity, we denote $r(u|_{Q_i})$ by S_i. Let B_i be the part of the boundary of $Q_i \cap U$ that is not contained in I. We can see that $u(x)$ tends to $-\infty$ when x approaches $B_1 \cup B_2$, and by the boundedness of the curvature of $\breve{\Sigma}_\infty$, the sequence of surfaces $T_n(S_1 \cup S_2)$, the vertical translation of $S_1 \cup S_2$ by n, converges to $(B_1 \cup B_2) \times \mathbb{R}$. Clearly the tangent planes of $(B_1 \cup B_2) \times \mathbb{R}$ and B_1 and B_2 are smooth curves, and since $(B_1 \cup B_2) \times \mathbb{R}$ is a minimal surface, B_1 and B_2 are geodesics. Consequently, $B_i = \gamma_i, i = 1, 2$, and the goal is proved. Finally, by Theorem 3.3 of [24], the equality $l(\gamma_1) + l(\gamma_2) = l(\eta_1) + l(\eta_2)$ holds, which contradicts the hypotheses about Ω. □

Proposition 3.18. *There does not exist simultaneously a sequence of minimal surfaces $(\Sigma_n^s)_n$ and a sequence of minimal surfaces $(\Sigma_n^u)_n$ satisfying $h^+(\Sigma_n^s) - h^-(\Sigma_n^s) \to \infty$, $h^+(\Sigma_n^u) - h^-(\Sigma_n^u) \to \infty$.*

In order to prove this proposition, we are going to proceed in three steps. First, we are going to describe in Lemma 3.19 three possible limits for the sequence Σ_n when $h^+(\Sigma_n) - h^-(\Sigma_n) \to \infty$. We then construct a Jacobi field on those limits, and this will be carried out in Lemma 3.20. The third step is to prove that such Jacobi fields can not exist on the limits obtained in Lemma 3.19.

Lemma 3.19. *Let $(\Sigma_n)_n$ be a sequence of surfaces satisfying the divergence property $h^+(\Sigma_n) - h^-(\Sigma_n) \to \infty$.*

1. *Let $\breve{\Sigma}_n$ a vertical translation of Σ_n such that the following identities $lim_{n\to\infty} h^+(\breve{\Sigma}_n) = +\infty$ and $lim_{n\to\infty} h^-(\breve{\Sigma}_n) = -\infty$ are satisfied. Then the sequence $(\breve{\Sigma}_n)_n$ has a subsequence converging to the minimal surface $(\eta_1 \times \mathbb{R}) \cup (\eta_2 \times \mathbb{R})$.*

2. *The sequence $(\breve{\Sigma}_n := T_n(\Sigma_n))_n$ has a subsequence converging to a minimal surface $\breve{\Sigma}_\infty$, a vertical graph of Scherk type on Ω assuming the continuous data on $\gamma_1 \cup \gamma_2$ and $-\infty$ on $\eta_1 \cup \eta_2$.*

3. *The sequence $(\breve{\Sigma}_n := T_{-n}(\Sigma_n))_n$ has a subsequence converging to a minimal surface $\breve{\Sigma}_\infty$, a vertical graph of Scherk type on Ω assuming the continuous data on $\gamma_1 \cup \gamma_2$ and $+\infty$ on $\eta_1 \cup \eta_2$.*

Proof. 1. For $m, n \in \mathbb{N}$, define $A_{n,m} := \breve{\Sigma}_n \cap (\mathbb{M} \times [-m, m])$. It is clear that

$$A_{n,1} \subset A_{n,2} \subset \cdots \subset A_{n,n-1} \subset A_{n,n} = A_{n,n+1} = \cdots = \breve{\Sigma}_n.$$

It is clear that, by Proposition 3.7, $A_{n,m}$ consists of two components $A^1_{n,m}$ and $A^2_{n,m}$ satisfying $A^i_{n,m} \subset A^i_{n,m+1}$, for $m, n \in \mathbb{N}, i = 1, 2$. Since there exists an uniform bound on the curvature for the sequence $\breve{\Sigma}_n$ and $A^i_{n,m}$ is contained in a compact region, then there is a subsequence $(A^i_{k^1_n,1})_n$ of $(A^i_{n,1})_n$ converging to a minimal surface containing the vertical segments of $\partial(\eta_i \times [-1, 1])$ in its boundary. Inductively, for $j > 1$, there exists a convergent subsequence $(A^i_{k^j_n,j})_n$ of $(A^i_{k^{j-1}_n,j})_n$. A diagonal argument implies that the sequence $(\breve{\Sigma}_{k^n_n})_n$ converges to a minimal surface $\breve{\Sigma}_\infty$. This limit surface is composed of two components $\mathcal{A}_1$ and $\mathcal{A}_2$, where $\mathcal{A}_i$ is a minimal surface whose boundary is the same of $\eta_i \times \mathbb{R}$. It is true that the projection of $\mathcal{A}_1$ and $\mathcal{A}_2$ is contained in Ω. Choosing $i \in \{1, 2\}$, let η^* the horizontal geodesic such that $\eta^* \times \mathbb{R} \in \mathcal{F}^\eta$ and the region of $\mathbb{M}$ bounded by η_i and η^* is the smallest one containing the projection of $\mathcal{A}_i$ in $\mathbb{M}$. If $\eta' \times \mathbb{R}$ is tangent to $\mathcal{A}_i$, they coincide by the maximum principle, then $\mathcal{A}_i = \eta_i \times \mathbb{R}$, as we wanted. If it does not happen, take a point $p \in \partial_\infty \mathcal{A}_i \cap (\eta' \times \{\pm\infty\})$ (for $B \subset \mathbb{M} \times \mathbb{R}$, $\partial_\infty B$ is the asymptotic boundary of B) and let $(p_n)_n$ be a sequence of points in $\mathcal{A}_i$ converging to p in the compactification. If we take vertical translations T_{k_n} such that $T_{k_n}(p_n)$ has height zero, then the sequence $(T_{k_n}(\mathcal{A}_i))_n$ has a subsequence converging to a minimal surface L_i which is tangent to $\eta^* \times \mathbb{R}$ and lies in a side of that plane, so $L_i = \eta^* \times \mathbb{R}$. On the other hand, L_i must contain the boundary of $\eta_i \times \mathbb{R}$, so $\eta^* = \eta_i$, and we conclude that $\mathcal{A}_i = \eta_i \times \mathbb{R}$, and finally that $\breve{\Sigma}_\infty = (\eta_1 \cup \eta_2) \times \mathbb{R}$.

2. By Lemma 3.16, the sequence $(\breve{\Sigma}_n)_n$ converges, up to a subsequence, to a minimal graph $\breve{\Sigma}_\infty$ defined by a function $u : U \to \mathbb{R}$ over a simply connected subdomain of Ω. Since $\partial\breve{\Sigma}_\infty$ is the limit of $(\partial\breve{\Sigma}_n)_n$, the boundary of $\breve{\Sigma}_\infty$ consists of two connected smooth curves C_1 and C_2, where the curves C_i satify $C_i \cap (0,\infty) = \emptyset$, $C_i \cap [-n, 0] = T_{-n}(G_n^i \cap [0, n])$ and $C_i \cap (-\infty, -n) = \partial(\bar{\gamma}_i \times [-\infty, -n))$ (recall the definition of G_n^i in Subsection 3.1.1). We then conclude that u assumes continuous data in $\gamma_1 \cup \gamma_2$. It remains to prove that $U = \Omega$ and that u assumes the value $-\infty$ in $\eta_1 \cup \eta_2$. The proof of those facts follows a similar procedure to the one used in Proposition 3.17.

3. By symmetry, $T_n(\Sigma_n)$ is the reflection of $T_n(\Sigma_n)$ by $\mathbb{M} \times \{0\}$, so, by the previous item, there is a subsequence of $(T_n(\Sigma_n))_n$ which converges to the reflection of $\breve{\Sigma}_\infty$ by $\mathbb{M} \times \{0\}$, and the conclusion is immediate.

$\square$

For each $n \in \mathbb{N}$, choose a point $p_n \in \Sigma_n^s$ as follows. If Σ_n^s is stable-unstable, take a nonnegative eigenfunction u_n of the Jacobi operator of Σ_n^s associated to 0. Let p_n be a point of Σ_n^s where u_n attains its maximum. If Σ_n^s is strictly stable, let q_n be the point in Σ_n^u which maximizes the distance to Σ_n^s, and let $p_n \in \Sigma_n^s$ such that $d_{\mathbb{M}\times\mathbb{R}}(p_n, q_n) = d_{\mathbb{M}\times\mathbb{R}}(\Sigma_n^s, q_n)$. We can suppose $(\Sigma_n^s)_n$ is composed only by stable-unstable surfaces or only by strictly stable ones, up to taking a subsequence. Regardless the case, we will denote by d_n the maximum value of the function $q \in \Sigma_n^u \mapsto d_{\mathbb{M}\times\mathbb{R}}(\Sigma_n^s, q)$.

Take the two sequences $(n - z(p_n))_n$ and $(n + z(p_n))_n$ of nonnegative numbers. Since $(n - z(p_n)) + (n + z(p_n)) \to +\infty$, up to a subsequence, we must have one of the three following possibilities:

1. Both sequences $(n - z(p_n))_n$ and $(n + z(p_n))_n$ go to $+\infty$;

2. The sequence $(n - z(p_n))_n$ is bounded and the sequence $(n + z(p_n))_n$ goes to $+\infty$;

3. The sequence $(n - z(p_n))_n$ goes to $+\infty$ and the sequence $(n + z(p_n))_n$ is bounded.

Denote the images of Σ_n^s, Σ_n^u, Γ_n^1, Γ_n^2 and p_n under the translation T_h by, respectively, $\breve{\Sigma}_n^s$, $\breve{\Sigma}_n^u$, $\breve{\Gamma}_n^1$, $\breve{\Gamma}_n^2$ and $\breve{p}_n$, being $h = -z(p_n)$ in the first case, $h = -n$ in the second and $h = n$ in the third.

By Proposition 3.17 and Lemma 3.19, each of the sequences $(\breve{\Sigma}_n^s)_n$ and $(\breve{\Sigma}_n^u)_n$ have a subsequence converging to the same minimal surface $\breve{\Sigma}_\infty$, where $\breve{\Sigma}_\infty$ is $(\eta_1 \times \mathbb{R}) \cup (\eta_2 \times \mathbb{R})$ for the first case or a vertical graph of Scherk type over Ω for the other ones. In any case, the sequence $(z(\breve{p}_n))_n$ is bounded. Hence, taking a subsequence, we can suppose that $(\breve{p}_n)_n$ is convergent, and $\breve{p}_\infty \in \breve{\Sigma}_\infty$ is its limit. Moreover, $d_n \to 0$ as n goes to ∞. Since the curvature of the sequence Σ_n^u is uniformly bounded, this surface is, locally, a graph over Σ_n^s of a function u_n.

Lemma 3.20. *Assuming that Proposition 3.18 is not true, there exists a Jacobi field w_∞ on $\breve{\Sigma}_\infty$ satisfying $0 \leq w_\infty \leq 1$ on $\breve{\Sigma}_\infty$, $w_\infty = 0$ on $\partial\breve{\Sigma}_\infty$ and $w_\infty(\breve{p}_\infty) = 1$.*

Proof. It follows the same ideas of the proof of Lemma 2.2.29 in [31]. □

Now, we are going to prove the Proposition 3.18:

Proof. Assuming that $h^+(\Sigma_n) - h^-(\Sigma_n)$ goes to $+\infty$, we have that, by the above argumentation, there exists a minimal surface $\breve{\Sigma}_\infty$ in $\mathbb{M} \times \mathbb{R}$ and a Jacobi field w_∞ over $\breve{\Sigma}_\infty$ such that

- $\breve{\Sigma}_\infty$ is $(\eta_1 \times \mathbb{R}) \cup (\eta_2 \times \mathbb{R})$ or a minimal graph of type Scherk of a function defined on Ω which assumes continuous data on $\gamma_1 \cup \gamma_2$ and $\pm\infty$ on $\eta_1 \cup \eta_2$;
- $0 \leq w_\infty \leq 1$, $w_\infty = 0$ on $\partial\breve{\Sigma}_\infty$ and $w_\infty \neq 0$.

We are going to prove that such Jacobi field can not exist, obtaining a contradiction, and therefore proving the proposition.

1. Suppose that $\breve{\Sigma}_\infty = (\eta_1 \times \mathbb{R}) \cup (\eta_2 \times \mathbb{R})$. We have that w_∞ satisfies the equation

$$\Delta w_\infty + (||A||^2 + Ric(N, N))w_\infty = 0.$$

Notice that $A = 0$ and $Ric(N,N) \leq 0$, and since w_∞ is nonnegative, it is true that $\Delta w_\infty \geq 0$. Since w_∞ attains its maximum in a point on the interior of $\breve{\Sigma}_\infty$ and this maximum is 1, the Maximum principle guarantees that $w_\infty \equiv 1$, contradicting the fact that w_∞ is zero on $\partial\breve{\Sigma}_\infty$.

2. Suppose that $\breve{\Sigma}_\infty$ is a Scherk type graph of a function u defined on Ω. It suffices to consider the case where $u = -\infty$ on $\eta_1 \cup \eta_2$ and $u = f_i$ on γ_i, the function f_i continuous. Since $\breve{\Sigma}_\infty$ is a graph, the function $N_3 := \langle N, \frac{\partial}{\partial z}\rangle$ is a Jacobi field for $\breve{\Sigma}_\infty$. We can then consider $X := N_3\nabla w_\infty - w_\infty \nabla N_3$, and this is a vector field over $\breve{\Sigma}_\infty$ whose divergence is zero.

For large n, denote $\breve{\Sigma}^n_\infty = \breve{\Sigma}_\infty \cap \{z \geq -n\}$. The surface $\breve{\Sigma}^n_\infty$ is the graph of a restriction of u to a subdomain Ω_n, whose boundary consists of the geodesics γ_1,γ_2 and two curves η^n_1 and η^n_2. In the last curves, u assumes the value $-n$. Using the divergence theorem, we have:

$$0 = \int_{\breve{\Sigma}^n_\infty} divX = \int_{I^n_1 \cup I^n_2} \langle N_3\nabla w_\infty - w_\infty\nabla N_3, \nu\rangle + \int_{J_1\cup J_2} \langle N_3 \nabla w_\infty, \nu\rangle, \quad (3.2)$$

where ν is the conormal vector field on $\partial\breve{\Sigma}^n_\infty$, I^n_i is the graph of u over η^n_i and J_i the graph of u over γ_i, $i = 1, 2$. We used the fact that w_∞ (resp. N_3) is zero along $\partial\breve{\Sigma}^n_\infty$ (resp. along the vertical part of $\partial\breve{\Sigma}^n_\infty$). It is true that w_∞ and $||\nabla w_\infty||$ are bounded and the restrictions of N_3 and $\langle\nabla N_3, \nu\rangle$ go to zero along $I^n_1 \cup I^n_2$ as n goes to ∞. Hence, we have that $\int_{I^n_1\cup I^n_2}\langle N_3\nabla w_\infty - w_\infty\nabla N_3, \nu\rangle$ goes to zero as n goes to ∞. On the other hand, by the maximum principle, the inequalities $\langle N_3\nabla w_\infty, \nu\rangle < 0$ and $N_3 > 0$ hold along $J_1 \cup J_2$ (for an appropriate choice of orientation), then $\int_{J_1\cup J_2}\langle N_3\nabla w_\infty, \nu\rangle < 0$, which gives us a contradiction, since the value of this integral does not depend on n. □

Remark. In order to guarantee that the functions N_3 and $\langle\nabla N_3, \nu\rangle$ converge to 0 as $n \to \infty$, consider the sequence $(T_n(\breve{\Sigma}_\infty))_{n\in\mathbb{N}}$ of minimal surfaces. Since these surfaces have uniformly bounded curvature, it is easy to see that $(T_n(\breve{\Sigma}_\infty))_{n\in\mathbb{N}}$ converges to $(\eta_1 \times \mathbb{R}) \cup (\eta_2 \times \mathbb{R})$ in $C^{2,\alpha}$ topology, so the sequence of Gauss maps converge in $C^{1,\alpha}$ topology to the Gauss map of $(\eta_1 \times \mathbb{R}) \cup (\eta_2 \times \mathbb{R})$, and the conclusion follows.

From the Proposition 3.18, we conclude that there exists a sequence of minimal surfaces $(\Sigma_n)_n$ such that $(h^+(\Sigma_n) - h^-(\Sigma_n))_n$ is a bounded sequence. We finish this section studying the convergence of this sequence and proving its main theorem.

We then prove the Theorem 3.1:

Proof. The surface Σ_n has two horizontal points $p_n := p^+(\Sigma_n)$ and $p^-(\Sigma_n)$, which are symmetric with respect to $\mathbb{M}\times\{0\}$. Then there is a subsequence of (p_n) which converges to p_∞. And, by Lemma 3.15, there is a subsequence of $(\Sigma_n)_n$ converging to the minimal surface Σ_∞ containing p_∞. This convergence is of multiplicity one, and this is proved in a similar way as in the proof of Lemma 3.16.

We have that the surface Σ_∞ is not simply connected. In fact, it lies between the graphs of $u^+ + c$ and $u^- - c$, where c is an upper bound for the heights of horizontal points of Σ_n, so, take a plane P in $\mathcal{F}^\gamma$ different from $\gamma_i \times \mathbb{R}$ that intersects Σ_∞. Since it intersects transversely the surfaces Σ_n, it also intersects Σ_∞ transversely. So, there is a cycle C in the intersection of P and Σ_∞. We can conclude, using the Maximum Principle, that C is nontrivial.

To prove that Σ_∞ is an annulus, we must prove that, for any two smooth Jordan curves noninstersecting and homotopically nontrivial, there exists an annulus A bounded by those two curves. In fact, if α and β are curves in Σ_∞ as described above, there are two sequences of nonintersecting curves $(\alpha_n)_n$ and $(\beta_n)_n$, $\alpha_n, \beta_n \subset \Sigma_n$, converging to α and β, respectively. For large n, α_n and β_n are nontrivial. In that case, the curves α_n and β_n bound an annulus $\mathcal{A}_n$ in Σ_n. Then, since α and β, together with the annuli $\mathcal{A}_n$, are contained in a convex compact set, there is a subsequence of the sequence $(\mathcal{A}_n)$ which converges to an annulus bounded by α and β, so Σ_∞ is an annulus.

Since the surface Σ_∞ is a limit of surfaces which are symmetric with respect to $\mathbb{M}\times\{0\}$, it is also symmetric with respect to this horizontal slice. The proof that Σ_∞ meets $\mathbb{M}\times\{0\}$ uses an argument similar to the one presented in Lemma 3.5. By Lemma 2.2.20 of [31], the surface $\Sigma_\infty \cap (\mathbb{M}\times(0,\infty))$ does not have points with vertical tangent plane. Finally, if two points in $\Sigma_\infty \cap (\mathbb{M}\times(0,\infty))$ had the same projection in $\mathbb{M}$, the same would be true for two points in $\Sigma_n \cap (\mathbb{M}\times(0,\infty))$, for large n, a contradiction. It finishes the proof of the theorem. $\square$

3.2 Minimal annulus in unbounded domains

Let γ_1 and γ_2 be ultraparallel geodesics in $\mathbb{M}$ whose distance is smaller than $2ln(\sqrt{2}+1)$. The main goal of this section is to prove the following theorem:

Theorem 3.21. *For two geodesics γ_1 and γ_2 satisfying the above condition,* *there exists a complete embedded minimal annulus in $\mathbb{M}\times\mathbb{R}$ whose boundary at infinity is the union of the four vertical lines passing through the endpoints of γ_1 and γ_2 and, for each geodesic γ that is ultraparallel to both γ_1 and γ_2, the intersection of this annulus with $\gamma\times\mathbb{R}$ is compact. This surface is a bigraph which is symmetric with respect to the horizontal slice $\mathbb{M}\times\{0\}$, and both surfaces meet orthogonally.*

First, we consider the proposition below:

Proposition 3.22. *There exists a bounded convex quadrilateral in $\mathbb{M}$ whose sides are geodesic segments (denoted by γ_1^*, γ_2^*, η_1^* and η_2^*), the geodesic arcs γ_i^* are contained in γ_i, for $i=1,2$, and the inequality $l(\gamma_1^*)+l(\gamma_2^*)>l(\eta_1^*)+l(\eta_2^*)$ holds.*

Proof. In a Hadamard surface, we can consider a standard coordinate system by choosing a geodesic α and noticing that the function $\phi_\alpha:\mathbb{R}^2\to\mathbb{M}$ such that $\phi_\alpha(s,t)=exp_{\alpha(t)}(sJ\alpha'(t))$, where J is the almost complex structure of $\mathbb{M}$, is a diffeomorphism. In these coordinates, we write the metric as $ds^2+G(s,t)dt^2$.

We know that, if $\tilde\gamma_1$ and $\tilde\gamma_2$ are two complete geodesics of $\mathbb{H}^2$ which are less than $2ln(\sqrt{2}+1)$ apart from each other, there is a convex quadrilateral $\widetilde\Lambda$ satisfying the required properties (replacing γ_i by $\tilde\gamma_i$). If $\tilde\gamma$ is the geodesic in $\mathbb{H}^2$ which is orthogonal to the $\tilde\gamma_i$, $i=1,2$, we can consider the coordinate system given by $\phi_{\tilde\gamma}$ such that $\phi_{\tilde\gamma}^{-1}(\tilde\gamma_i)=\{(-1)^ia\}\times\mathbb{R}$, for some $a>0$ and for $i=1,2$. In the space $\mathbb{M}$, we proceed in a similar way, writing as γ the geodesic which is orthogonal to the γ_i, $i=1,2$ and using the coordinate system given by ϕ_γ such that $\phi_\gamma^{-1}(\gamma_i)=\{(-1)^ia\}\times\mathbb{R}$, for the same $a>0$ as before and for $i=1,2$. If $d\tilde s^2+\tilde G d\tilde t^2$ and ds^2+Gdt^2 are the expressions of the metrics of $\mathbb{H}^2$ and $\mathbb{M}$, respectively, we can conclude, by Proposition 2 of [13], the inequality $\tilde G\geq G$, and for a curve $c:[0,1]\to\mathbb{R}^2$, we have $l(\phi_{\tilde\gamma}\circ c)\geq l(\phi_\gamma\circ c)$, so $d_{\mathbb{H}^2}(\phi_{\tilde\gamma}(p),\phi_{\tilde\gamma}(q))\geq d_{\mathbb{M}}(\phi_\gamma(p),\phi_\gamma(q))$, for $p,q\in\mathbb{R}^2$. It is possible, therefore, to construct a quadrilateral Λ in $\mathbb{M}$ satisfying the conditions of the statement, simply choosing the vertices of Λ to be the same, in coordinates, as the ones of $\widetilde\Lambda$. This finishes the proof of the proposition. □

Moreover, we have the following lemma:

Lemma 3.23. *For $i=1,2$, let Λ_i be a bounded convex quadrilateral whose sides are the geodesics γ_1^i, γ_2^i, η_1^i and η_2^i such that $\gamma_j^1\subset\gamma_j^2\subset\gamma_j$, for $i,j=1,2$. Then, the following inequality holds:*

$$l(\gamma_1^2) + l(\gamma_2^2) - l(\eta_1^2) - l(\eta_2^2) \geq l(\gamma_1^1) + l(\gamma_2^1) - l(\eta_1^1) - l(\eta_2^1),$$

and the equality holds if and only if $\Lambda_1 = \Lambda_2$*.*

Proof. It is a simple consequence of the triangle inequality. □

As a consequence of the two previous results, we have that there exists a sequence $(\Omega_n)_{n\in\mathbb{N}}$ of bounded convex quadrilaterals in $\mathbb{M}$ satisfying the following properties:

1. Its sides are geodesic segments, and we denote them by γ_1^n, γ_2^n, η_1^n and η_2^n;
2. The geodesic arc $\bar{\gamma}_i^n$ is contained in the interior of γ_i^{n+1} (with respect to its intrinsic topology), and all of those arcs are contained in γ_i;
3. The inequality $l(\gamma_1^n) + l(\gamma_2^n) > l(\eta_1^n) + l(\eta_2^n)$ holds for all n;
4. $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$, where Ω is the geodesic ideal quadrilateral whose vertices are the endpoints of the geodesics γ_i (see Figure 3.4).

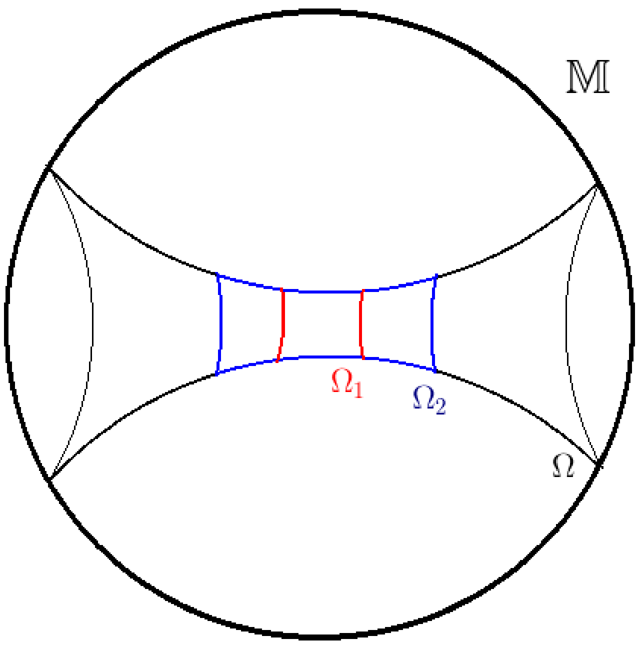

Figure 3.4: Part of the exhaustion

Using the ideas of comparison geometry that were already presented, together with ideas of the proof of Proposition 3.2, we can prove the following result:

Proposition 3.24. *If the distance of γ_1 and γ_2 is larger than $2k^{-1}ln(\sqrt{2}+1)$, then there is no complete embedded minimal annulus in $\mathbb{M} \times \mathbb{R}$ with the properties stated in Theorem 3.21.*

Taking a sequence $(\Omega_n)_{n\in\mathbb{N}}$ as before, we know that there is a minimal solution $u_{n,i}$ on Ω_n such that $u_{n,i} = \infty$ on γ_i^n and $u_{n,i} = 0$ on $\gamma_j^n \cup \eta_1^n \cup \eta_2^n$ for $\{i, j\} = \{1, 2\}$. Define $u_n = sup\{u_{n,1}, u_{n,2}\}$. Moreover, we can apply directly the Theorem 3.1 for each Ω_n, and we state below the conclusion:

Corollary 3.25. *For each n, there exists a minimal annulus Σ_n in $\mathbb{M} \times \mathbb{R}$ whose boundary is given by the four vertical lines passing through the vertices of Ω_n such that, for each complete geodesic α intersecting the geodesics η_1^n and η_2^n, the set $\Sigma_n \cap (\alpha \times \mathbb{R})$ is compact. Moreover, the surface Σ_n has uniform bounded curvature and the surface lies below the graph of $u_n + h^+(\Sigma_n)$ and above the graph of $-u_n + h^-(\Sigma_n)$.*

The desired annulus of Theorem 3.21 will be constructed by taking the limit of a sequence of vertical translations of Σ_n. This will be carried out in the rest of the section.

3.2.1 Foliation and curvature estimates

As in the Subsection 3.1.2, define, for each $t \in (-n, n)$, the set $\omega(t)$ as the intersection of Σ_n and $\{z = t\}$. A point $p \in \Sigma_n$ is called a *horizontal point* Σ_n is tangent to the plane $\{z = z(p)\}$ at p. The set of horizontal points is denoted by $\mathcal{H}$ and $\mathcal{H}(t) := \mathcal{H} \cap \omega(t)$. Denote by h_n^+ (resp. h_n^-) the maximum value (resp. the minimum value) of the restriction $z : \mathcal{H} \to \mathbb{R}$ of the height function. Although we have the relation $h_n^+ = -h_n^-$, the definition of both quantities is useful when we have curves in more general positions. For each $t \in (n, n)$, define $\Sigma_n^+(t) = \Sigma_n \cap \{z \geq t\}$ and $\Sigma_n^-(t) = \Sigma_n \cap \{z \leq t\}$.

Proposition 3.26. *The following properties for Σ_n holds:*

1. *Σ_n has exactly two horizontal points, and they are symmetric with respect to $\mathbb{M} \times \{0\}$.*

2. *If $t > h_n^+$ (resp. $t < h_n^-$), then $\Sigma_n^+(t)$ (resp. $\Sigma_n^-(t)$) consists of two simply connected components. Then, $\omega(t)$ consists of two components, both diffeomorphic to $[0,1]$ and joining the two vertical lines passing through the endpoints of γ_i^n.*

3. *For each $t \in (h_n, h_n^+)$ (in particular, for $t = 0$), the sets $\Sigma_n^+(t)$ and $\Sigma_n^-(t)$ are simply connected. Moreover, $\omega(t)$ consists of two components, both diffeomorphic to $[0,1]$ and joining the two vertical lines passing through the endpoints of η_i^n.*

4. *The set $\Sigma_n \cap \{h_n < z < h_n^+\}$ consists of two simply connected components.*

Proof. It follows from the Proposition 3.7 and the fact that all the stated properties still hold under convergence processes. □

In an analogous form, we can extend more results of Subsection 3.1.2 to the current situation.

Proposition 3.27.

1. *The annulus Σ_n is tangent to the foliation $\mathcal{F}^\eta$ of $\mathbb{M} \times \mathbb{R}$ at most at two points.*

2. *The annulus Σ_n is not tangent to any leaf of $\mathcal{F}^\gamma$.*

3. *The annulus Σ_n is tangent to the foliation $\mathcal{F}^{\eta_i}$ of $\mathbb{M} \times \mathbb{R}$ at most at two points for each $i = 1, 2$, where the curves η_i are the sides of Ω which are different from the γ_j, $j = 1, 2$.*

Proof. Again, by the Propositions 3.8, 3.9 and 3.11, along with the fact that the tangency properties remain valid under convergence (see Lemma 2.2.20 of [31]). □

Proposition 3.28. *The sequence of minimal annuli $(\Sigma_n)_n$ has a uniformly bounded curvature.*

Proof. This proof follows the same ideas of Proposition 3.12. Assuming the contrary, let $\lambda_n := sup_{\Sigma_n} ||A_{\Sigma_n}||$ and suppose that $lim_{n\to\infty} \lambda_n = \infty$. Let p_n be a point in Σ_n satisfying $||A_{\Sigma_n}(p_n)|| \geq \frac{\lambda}{2}$. Then, we consider the blow-up of the sequence $\mathbb{M} \times \mathbb{R}$ around the sequence $(p_n)_n$ of points using the constants λ_n, i.e., we look at the sequence of surfaces $(\widetilde{\Sigma}_n)_n := \phi_n^{-1}(\Sigma_n)$ and the sequence of ambient spaces $(U_n := (T_{p_n}(\mathbb{M}\times\mathbb{R}), \phi_n^*(g+dt^2)))_n$ (here, the map ϕ_n is defined as in 3.16, so that U_n is endowed with a product metric). Clearly, $\widetilde{\Sigma}_n$ is a minimal surface of U_n, and as $n\to\infty$, the ambient spaces converge to $\mathbb{R}^3$ with the Euclidean metric and the surfaces converge to a minimal immersion $\widetilde{\Sigma}_\infty$. Indeed, $\widetilde{\Sigma}_\infty$ is complete, embedded and has finite total curvature (for details, see Proposition 2.3.7 of [31]. We also point out that the Proposition 3.27 is used to prove the finiteness of the total curvature). Also as in the bounded domain case, the surface $\widetilde{\Sigma}_\infty$ has no boundary; the proof can be done in the same way of Claim 3.1.3.

Clearly, $||A_{\widetilde{\Sigma}_\infty}(O)||\geq\frac{1}{2}$, so $\widetilde{\Sigma}_\infty$ is not a flat plane. Then, $\widetilde{\Sigma}_\infty$ has at least two ends, by Theorem 3.1 of [20]. Take $\nu\subset\widetilde{\Sigma}_\infty$ to be a smooth Jordan curve which is the boundary of an end of $\widetilde{\Sigma}_\infty$. This curve is homotopically nontrivial and it separates the surface in two noncompact parts. Let $(\tilde{\nu}_n)_{n\in\mathbb{N}}, \tilde{\nu}_n\subset\widetilde{\Sigma}_n$ be a sequence of Jordan curves converging to ν. It guarantees that $\tilde{\nu}_n$ is homotopically nontrivial for n sufficiently large (the proof is the same as the one shown in 3.12). Define as ν_n the curve in Σ_n whose image by ϕ_n^{-1} is $\tilde{\nu}_n$. Clearly, $l_{U_n}(\tilde{\nu}_n)=\lambda_n l_{\mathbb{M}\times\mathbb{R}}(\nu_n)$ and $(l_{U_n}(\tilde{\nu}_n))_n$ converges to $l_{\mathbb{R}^3}(\nu)$, so $(l_{\mathbb{M}\times\mathbb{R}}(\nu_n))_n$ converges to 0. If $\pi:\mathbb{M}\times\mathbb{R}\to\mathbb{M}$ is the projection onto the first factor, we obtain that $lim_{n\to\infty}l_{\mathbb{M}}(\pi(\nu_n))=0$. Clearly the sequence of curves $\pi(\nu_n)$ converges, up to a subsequence, to a point $p\in\Omega$ (possibly one of its vertices), so $lim_{n\to\infty}(d(\gamma_1,\pi(\nu_n))+d(\pi(\nu_n)),\gamma_2))=d(\gamma_1,p)+d(p,\gamma_2)$. We can suppose that there exist a positive c such that $d(\gamma_1,p)\geq c$, since the geodesics are ultraparallel.

Since ν_n is nontrivial for large n, this curve separates the surface Σ_n in two components. Let $\mathcal{A}_n$ the sub-annulus of Σ_n bounded by ν_n and the two vertical lines passing through the endpoints of γ_1.

We will consider separatedly the cases where the sequence $(\pi(\nu_n))_n$, up to taking a subsequence, is contained in a compact subset of Ω (so it converges to a point inside Ω) or it converges to a point in $\partial^2\Omega$, the set of vertices of Ω.

1. Suppose the sequence $(\pi(\nu_n))_n$, up to taking a subsequence, is contained in a compact subset of Ω. Let q_1 and q_2 the two endpoints of γ_i. It is true that, for each n, there exists a point ξ_n in $\mathbb{M}$ such that the geodesic triangle whose vertices are q_1, q_2 and ξ_n is the smallest triangular domain containing $\pi(\nu_n)$ and whose set of vertices contains q_1 and q_2 (we call the geodesic triangle T^n). We have that the angle of T^n at the vertex ξ_n (call it θ_n) is such that $\theta_n<\pi-\theta$, for some $\theta>0$,

because $d_{\mathbb{M}}(\xi_n, \gamma_1) > d_{\mathbb{M}}(\pi(\nu_n), \gamma_1) \geq c$. We can also suppose $\theta_n \geq \theta$.

By the maximum principle, $\mathcal{A}_n \subset T^n \times \mathbb{R}$. Moreover, the sequence $(\tilde{\nu}_n)_n$ converges to ν. Moreover, there is a subsequence of $\phi_n^{-1}(T^n \times \mathbb{R}) \subset U_n$ converging to a region R bounded by two vertical half-planes whose angle lies in the interval $[\theta, \pi - \theta]$. By construction, $\phi_n^{-1}(\mathcal{A}_n) \subset U_n$ has a subsequence converging to a noncompact part of $\widetilde{\Sigma}_\infty$ bounded by ν. So, R contains an end of $\widetilde{\Sigma}_\infty$, which contradicts the fact that such ends must be asymptotic to the end of a plane or the end of a catenoid.

2. Suppose that $(\pi(\nu_n))_n$ converges to a point in $\partial^2\Omega$. This vertex must be an endpoint of the geodesic γ_2. Denoting again the endpoints of γ_1 by q_1 and q_2, and noticing that $lim_{n\to\infty} l_{\mathbb{M}}(\pi(\nu_n)) = 0$, we have that there exists points ξ_n and ζ_n in $\mathbb{M}$ such that the geodesic joining ξ_n and ζ_n is perpendicular to γ_2 and the geodesic quadrilateral Q^n whose vertices are q_1, q_2, ξ_n and ζ_n is convex, contains $\pi(\nu_n)$ and this is the smallest quadrilateral satisfying those properties.

   By the maximum principle, $\mathcal{A}_n \subset Q^n \times \mathbb{R}$. Moreover, there is a subsequence of $\phi_n^{-1}(Q^n \times \mathbb{R}) \subset U_n$ converging to a region R bounded by two parallel vertical planes and another vertical plane intersecting them. By construction, $\phi_n^{-1}(\mathcal{A}_n) \subset U_n$ has a subsequence converging to a noncompact part of $\widetilde{\Sigma}_\infty$ bounded by ν. So, R contains an end of $\widetilde{\Sigma}_\infty$, which contradicts the fact that such ends must be asymptotic to the end of a plane or the end of a catenoid.

□

3.2.2 Convergence of $(\Sigma_n)_{n\in\mathbb{N}}$ for the case of unbounded domains

Let $\breve{\Sigma}_n$ be the vertical translation of Σ_n such that $h^+(\Sigma_n) = 0$ and $\breve{p}_n$ the image of $p_n^+ := p^+(\Sigma_n)$ under this translation. In this section, we are going to prove that there is a subsequence of $(\breve{\Sigma}_n)_n$ which converges to the minimal annulus described in Theorem 3.21.

Lemma 3.29. *There exists a compact subset of $\mathbb{M}$ containing the set*

$$\{\pi(p_n^+); n \in \mathbb{N}\}.$$

Proof. By Proposition 3.28, the sequence $(\Sigma_n)_n$ has uniformly bounded curvature, say $sup_n sup_{\Sigma_n}||A_{\Sigma_n}(p)|| \leq C$. Then, the Uniform Graph Lemma (see Lemma 4.35 of [33]), there is a neighborhood of p_n^+ in Σ_n which is a graph over the disc $D_{r_n}(p_n^+) \subset T_{p_n^+}\Sigma_n$ (the last notation stands for the tangent plane of Σ_n at p_n^+), where $r_n := min\{\frac{1}{4C}, dist_{\Sigma_n}(p_n^+, \partial\Sigma_n)\}$. Furthermore, since the normal vector of Σ_n at p_n^+ is vertical, it is clear that $D_{r_n}^{\mathbb{M}}(\pi(p_n^+)) \subset \Omega_n$. In particular, $dist_{\Sigma_n}(p_n^+, \partial\Sigma_n) \geq dist_{\mathbb{M}}(\pi(p_n^+), \partial^2\Omega_n)$, then $D_{r_n'}(\pi(p_n^+)) \subset \Omega_n$, $r_n' := min\{\frac{1}{4C}, dist_{\mathbb{M}}(\pi(p_n^+), \partial^2\Omega_n)\}$. So, if the lemma is not true, there is a subsequence of $(\pi(p_n^+))_n$ converging to a point of $\partial^2\Omega$. But the interior angles of Ω_n converge to zero as n goes to ∞, and this is a contradiction. Indeed, if $\{r_n'; n \in \mathbb{N}\}$ is bounded from below by a positive number, the sequence $(\pi(p_n^+))_n$ can not have a subsequence converging to a vertex, because the distance between the geodesics meeting at this vertex goes to zero as those geodesics approach the vertex. So, the sequence $(r_n')_n$ must converge to zero, then $r_n' = dist_{\mathbb{M}}(\pi(p_n^+), \partial^2\Omega_n)$ for large n, then $D_{r_n'}(\pi(p_n^+))$ contains a point of $\partial^2\Omega_n$ (say, q_n) and, by convexity of the disc, it contains small segments of the geodesics which constitutes the sides of Ω_n, so the disc can not be contained in Ω_n. □

For $i = 1, 2$, by Theorem 3.2 of [14], there are minimal solutions u_i^* on Ω satisfying $u_i^* = \infty$ on γ_i and $u_i^* = 0$ on $\eta_1 \cup \eta_2 \cup \gamma_j$ with $\{i, j\} = \{1, 2\}$. Define $u^* = sup\{u_1^*, u_2^*\}$.

Lemma 3.30. *For all n, Σ_n is below the graph of $u^* + h^+(\Sigma_n)$ and above the graph of $-u^* + h^+(\Sigma_n)$.*

Proof. By Corollary 3.25, Σ_n is below the graph of $u_n + h^+(\Sigma_n)$ and above the graph of $-u_n + h^-(\Sigma_n)$. Moreover, by Maximum principle we have $u^* \geq u_n$ for all n, which proves the corollary. □

Lemma 3.31. *Using the notation above, there is a positive K sufficiently large such that $Int(\breve{\Sigma}_n^+(K))$ consists of two components, each of them being a graph over $\gamma_i^n \times (K, \infty)$, for $i = 1, 2$.*

Proof. For i fixed, let N the normal vector field of $\gamma_i^n \times \mathbb{R}$. Extend the vector field N to $\mathbb{M} \times \mathbb{R}$ by parallel transport along the geodesics which are normal to $\gamma_i^n \times \mathbb{R}$. We are going to prove that, for large K, there is no point $q \in Int(\breve{\Sigma}_n^+(K))$ such that $N_{\breve{\Sigma}_n}(q) \perp N$. If it does not hold, there exist a sequence $(t_n)_n$ of real numbers converging to $+\infty$ and $q_n \in \breve{\Sigma}_n \cap \{z = t_n\}$ satisfying $N_{\breve{\Sigma}_n}(q_n) \perp N$. We can proceed with a standard blow-up argument using the sequence of points $(q_n)_n$ and the sequence of scaling constants $(\lambda_n)_n$, where

$$\lambda_n^{-1} := sup_{q \in \breve{\Sigma}_n \cap \{-1+z(q_n) \leq z \leq 1+z(q_n)\}} d_{\mathbb{M}\times\mathbb{R}}(q, \breve{\Sigma}_n).$$

We have that $\lambda_n \to 0$ when $n \to \infty$, because $\breve{\Sigma}_n$ is below the graph of u^* and the graph of this function is asymptotic to $\gamma_i^n \times \mathbb{R}$. Using this argument, we conclude that the sequence defined by $\gamma_i^n \times \mathbb{R}$ converges, up to a subsequence, to a plane $Q \subset \mathbb{R}^3$. Similarly, the sequence of minimal surfaces defined by $\breve{\Sigma}_n \cap \{-1 + z(q_n) \leq z \leq 1 + z(q_n)\}$ converges to a plane or half-plane passing through $O \in \mathbb{R}^3$ (the fixed point of the blow-up) and their normal vectors N_P and N_Q, respectively, are orthogonal. Moreover, the distance between P and Q is at most 1, which shows that they are parallel, contradicting the orthogonality between N_Q and N_P. Therefore, there exist $K > 0$ such that the surface $Int(\breve{\Sigma}_n^+(K))$ is transverse to all the horizontal geodesics of $\mathbb{M} \times \mathbb{R}$ which are orthogonal to γ_i^n.

In order to finish the proof of the lemma, denote by $\breve{\Sigma}_n^+(K, i)$ the simply connected component of $\breve{\Sigma}_n^+(K)$ containing $\partial\bar{\gamma}_i^n \times [K, +\infty)$ in its ideal boundary. Along the flow of N, the surface $\breve{\Sigma}_n^+(K, i)$ projects onto $\partial\bar{\gamma}_i^n \times [K, +\infty)$, and we denote by $\pi_2 : Int(\breve{\Sigma}_n^+(K, i)) \to \gamma_i^n \times (K, +\infty)$ this projection. Clearly, π_2 is a local diffeomorphism. Moreover, given any point p in $\gamma_i^n \times (K, +\infty)$, the geodesic passing through p which is orthogonal to this plane intersect $Int(\breve{\Sigma}_n^+(K, i))$ only in a finite number of points, by transversality. Consequently, π_2 is a covering map, and since $\gamma_i^n \times (K, +\infty)$ is simply connected, π_2 is a diffeomorphism, hence $Int(\breve{\Sigma}_n^+(K, i))$ is a graph over $\gamma_i^n \times (K, +\infty)$. □

Given a complete geodesic ξ in $\mathbb{M}$, if q is a point in the asymptotic boundary of $\mathbb{M}$ disjoint from the closure of ξ, we call by $\mathcal{M}_{\alpha,q}$ the half-plane determined by ξ containing q in its asymptotic boundary. Moreover, if $(\chi_n)_n$ is a sequence of geodesics, each of them orthogonal to ξ, we say that this sequence *converges to* $q \in \mathbb{M} \cup \partial\mathbb{M}$ if $(\chi_n \cap \xi)_n$ converges to q in the closure topology.

Lemma 3.32. *If p is an ideal vertex of Ω which is an endpoint of γ_i, there is a geodesic $\gamma_i^\perp$ orthogonal to γ_i such that, for all sufficiently large n, the set $\Sigma_n \cap (\mathcal{M}_{\gamma_i^\perp, p} \times \mathbb{R})$ is a normal graph over a subdomain of $\gamma_i \times \mathbb{R}$.*

Proof. Suppose that p is an endpoint of the geodesic γ_1 and η_1. As before, extend N, the unit normal vector field of $\gamma_1 \times \mathbb{R}$, to $\mathbb{M} \times \mathbb{R}$ by parallel transport along the geodesics which are normal to $\gamma_1 \times \mathbb{R}$.

First, we prove that there is a complete geodesic $\gamma_1^\perp$ orthogonal to γ_1 such that, for all sufficiently large n, the vector field N is tranverse to Σ_n in the region $\mathcal{M}_{\gamma_1^\perp,p} \times \mathbb{R}$.

Suppose the previous claim is not true. Then, there is a sequence of geodesics $(\chi_n)_n$ orthogonal to γ_1 converging to p, a sequence $(k_n)_n$ in $\mathbb{N}$ and points $q_n \in \Sigma_{k_n}$ such that the vectors $N_{\Sigma_{k_n}}(q_n)$ and N are orthogonal at q_n (for simplicity, we assume $k_n = n$). We then use a blow-up argument with the sequences $(q_n)_n$ and $(\lambda_n)_n$, where $\lambda_n^{-1} := d_{\mathbb{M}\times\mathbb{R}}(q_n, \gamma_1 \times \mathbb{R})$. Using the notation of the Proposition 3.12, we have that the sequence of minimal surfaces

$$\phi_n^{-1}(\Sigma_n) \subset U_n := (T_{q_n}(\mathbb{M} \times \mathbb{R}), \phi_n^*(g + dt^2))$$

passes through a fixed point O, $N_{\phi_n^{-1}(\Sigma_n)}(O) \perp N$, has uniformly bounded curvature and it is located in one side of $\phi_n^{-1}(\gamma_1 \times \mathbb{R})$. Since the sequence $(\lambda_n)_n$ converges to zero as n goes to ∞, then the sequence $\phi_n^{-1}(\Sigma_n) \subset U_n$ converges to a vertical plane P in $\mathbb{R}^3$ passing through O satisfying $d_{\mathbb{R}^3}(O, P) = 1$.

Notice that $d_{\Sigma_n}(q_n, \partial\Sigma_n) \geq d_{\mathbb{M}\times\mathbb{R}}(q_n, \partial\Sigma_n) \geq d_{\mathbb{M}}(\pi(q_n), p_n)$, where p_n is the vertex of Ω_n contained in $\mathcal{M}_{\chi_n,p}$. Since the angle of Ω_n at p_n converges to 0 as $n \to \infty$, it is true that $\frac{dist_{\mathbb{M}}(\pi(q_n),p_n)}{dist_{\mathbb{M}}(\pi(q_n),\gamma_1)} \to \infty$, then $\lambda_n dist_{\Sigma_n}(q_n, \partial\Sigma_n) \to \infty$. This relation implies that the sequence $\phi_n^{-1}(\Sigma_n)$ converges to a complete plane Q in $\mathbb{R}^3$, since all of them passes through O and their curvatures converge uniformly to zero (the product $\lambda_n dist_{\Sigma_n}(q_n, \partial\Sigma_n)$ is the distance of O to $\phi_n^{-1}(\Sigma_n)$ in U_n). We have that Q is contained in one side of P, and it contradicts the fact that their normal vectors are orthogonal.

Now, we prove that the set $\Sigma_n \cap (\mathcal{M}_{\gamma_1^\perp,p} \times \mathbb{R})$ is connected. We can choose $\gamma_1^\perp$ such that it intersects η_1 in $\mathbb{M}$. We know that, for all n, Σ_n is a vertical bigraph over a subdomain of Ω_n, bounded by γ_1^n, γ_2^n and two strictly concave arcs ξ_1^n and ξ_2^n, each of them connecting the vertices of η_1^n and η_2^n. We say that p_n is one of the endpoints of ξ_1^n. By concavity, the curve ξ_1^n must intersect only once, and it is clear that $\Sigma_n \cap (\mathcal{M}_{\gamma_i^\perp,p} \times \mathbb{R})$ must be a bigraph over the region bounded by γ_1, $\gamma_1^\perp$ and ξ_1^n, which is connected, hence $\Sigma_n \cap (\mathcal{M}_{\gamma_i^\perp,p} \times \mathbb{R})$ is connected. We finish the proof proceeding as in Lemma 3.31. □

Proposition 3.33. *The sequence $h^+(\Sigma_n) - h^-(\Sigma_n)$ is bounded.*

Proof. Assuming the contrary, we can suppose that $(h^+(\Sigma_n) - h^-(\Sigma_n))_n$ goes to $+\infty$ as $n \to \infty$, and consider $\breve{\Sigma}_n$ as in the beginning of the subsection.

By Lemma 3.29, the points $\pi(\breve{p}_n)$ are contained in a compact subset of $\mathbb{M}$, so they have a subsequence converging to a point $\breve{p}_\infty$. Since the sequence of minimal surfaces $\breve{\Sigma}_n$ has bounded curvature and it has an accumulation point, by the Appendix B of [7], there is a subsequence of $(\breve{\Sigma}_n)_n$ converging to a minimal lamination $\mathcal{L}$ of $\mathbb{M}\times\mathbb{R}$. Let $\breve{\Sigma}_\infty$ the leaf of $\mathcal{L}$ passing through $\breve{p}_\infty$. Proceeding analogously as in Lemma 3.15, we have that there is a neighborhood U of $\breve{p}_\infty$ in $\mathbb{M}\times\mathbb{R}$ such that $\mathcal{L}\cap U = \breve{\Sigma}_\infty \cap U$ and $\breve{\Sigma}_n\cap U$ converges to $\breve{\Sigma}_\infty\cap U$ with multiplicity 1.

By Lemma 3.31, for M large enough, each minimal surface $\breve{\Sigma}_n\cap\{z>M\}$ is a normal graph over $\gamma_i^n\times\mathbb{R}$ lying below the graph of u^*. Passing the limit, the surface $\breve{\Sigma}_\infty\cap\{z>M\}$ is also a normal graph over $\gamma_i^n\times\mathbb{R}$ which lies below the graph of u^* (consequently, it is asymptotic to $(\gamma_1\cup\gamma_2)\times\mathbb{R}$ as $z\to\infty$), and it is a limit of multiplicity 1.

If q is a vertex of Ω, say, the common endpoint of γ_1 and η_1. By Lemma 3.32, there exists a geodesic $\gamma_1^\perp$ orthogonal to γ_1 such that $\Sigma_n\cap(\mathcal{M}_{\gamma_1^\perp,p}\times\mathbb{R})$ is a normal graph on a slab of $\gamma_1\times\mathbb{R}$. Consequently, $\breve{\Sigma}_n\cap(\mathcal{M}_{\gamma_1^\perp,p}\times\mathbb{R})$ is a normal graph over a subdomain of $\gamma_1^n\times\mathbb{R}$. Thus, $\breve{\Sigma}_\infty\cap(\mathcal{M}_{\gamma_1^\perp,p}\times\mathbb{R})$ is the limit with multiplicity 1 and is a normal graph over a region contained in $\gamma_1\times\mathbb{R}$. In particular, the boundary at infinity of $\breve{\Sigma}_\infty$ consists of the four vertical lines passing through the vertices of Ω.

We can conclude, using Proposition 3.26, that $\breve{\Sigma}_n^+(\frac{1}{2}h^-(\breve{\Sigma}_n))$ is simply connected, and since $\breve{\Sigma}_\infty$ is the limit of such surfaces, $\breve{\Sigma}_\infty$ is itself simply connected. Proceeding as in Lemma 3.16, we obtain that $\breve{\Sigma}_\infty$ is a vertical graph defined on Ω assuming the values $+\infty$ on $\gamma_1\cup\gamma_2$ and $-\infty$ on $\eta_1\cup\eta_2$. By Theorem 3.1 of [14], we obtain the equality $a(\partial\Omega)=P(\partial\Omega)$, following the notation of the reference, and it means that $d_{\mathbb{M}}(\gamma_1,\gamma_2)\geq 2ln(\sqrt{2}+1)$, a contradiction. □

Now, we are going to prove Theorem 3.21, the main result of the chapter.

Proof. Taking the sequence $(\Sigma_n)_n$, Lemma 3.29 and Proposition 3.33 guarantee that the sequences $(p_n^+)_n$ and $(p_n^-)_n$ are bounded, so those points have subsequences converging to p_∞^+ and p_∞^-, those points being symmetric with respect to the slice $\mathbb{M}\times\{0\}$. As in Proposition 3.33, we obtain, using Appendix B of [7] and Lemma 3.15, the existence of U (resp. U'), which is a neighborhood of p_∞^+ (resp. p_∞^-) such that there is a surface Σ_∞ containing p_∞^+ and p_∞^- and $(\Sigma_n\cap U)_n$ (resp. $(\Sigma_n\cap U')_n$) converges to $\Sigma_\infty\cap U$ (resp. $\Sigma_\infty\cap U'$) with multiplicity one.

Again, by Lemma 3.31, for M large enough, each minimal surface $\Sigma_n \cap \{|z| > M\}$ is a normal graph over $\gamma_i^n \times \mathbb{R}$ lying below the graph of $u^* + sup_n h^+(\Sigma_n)$ and above the graph of $-u^* + inf_n h^-(\Sigma_n)$. Passing the limit, the surface $\Sigma_\infty \cap \{|z| > M\}$ is also a normal graph over $\gamma_i^n \times \mathbb{R}$ which lies below the graph of $u^* + sup_n h^+(\Sigma_n)$ and above the graph of $-u^* + inf_n h^-(\Sigma_n)$ (consequently, it is asymptotic to $(\gamma_1 \cup \gamma_2) \times \mathbb{R}$ as $z \to \infty$ and $z \to -\infty$), and it is a limit of multiplicity 1.

For each vertex q of Ω, by Lemma 3.32, there exists a region $\mathcal{M}_q$ bounded by a complete geodesic of $\mathbb{M}$ perpendicular to γ_i (we assume q is an endpoint of γ_i) and q is contained in the boundary at infinity of $\mathcal{M}_q$ such that $\Sigma_\infty \cap (\mathcal{M}_q \times \mathbb{R})$ is a normal graph on $(\mathcal{M}_q \cap \gamma_i) \times \mathbb{R}$. In particular, the boundary at infinity of Σ_∞ is given by the four vertical lines at the vertices of Ω. Moreover, outside the compact $(\Omega \setminus \cup_{q \in \partial^2 \Omega} \mathcal{M}_q) \times \mathbb{R}$, Σ_∞ is a normal graph on $\gamma_i \times \mathbb{R}$ and is asymptotic to $\gamma_i \times \mathbb{R}$, for $i = 1, 2$.

The proof that Σ_∞ is a topological annulus and that this is a bigraph is analogous to the one in Theorem 3.1. $\square$

CHAPTER 4

Minimal surfaces of finite total curvature in $\mathbb{M} \times \mathbb{R}$

4.1 Introduction

The goal of this chapter is to study minimal surfaces in $\mathbb{M} \times \mathbb{R}$ having finite total curvature, where $\mathbb{M}$ is a Hadamard manifold with pinched sectional ccurvature. The main result gives a formula to compute the total curvature in terms of topological, geometrical and conformal data of the minimal surface. In particular, we prove the total curvature is an integral multiple of 2π.

4.2 Preliminaries

Let $X : \Sigma \to \mathbb{M} \times \mathbb{R}$ be a minimal conformal immersion of the surface Σ in $\mathbb{M} \times \mathbb{R}$, where $\mathbb{M}$ is a Hadamard surface satisfying $-a^2 \leq K_{\mathbb{M}} \leq -b^2$, for positive constants a and b. We can decompose the immersion X as (h, f), where h and f are the projections of X in the first and second factors of $\mathbb{M} \times \mathbb{R}$, respectively. Since X is minimal, the maps h and f are harmonic.

We consider local conformal parameters for a simply-connected open domain $\Omega \subset \Sigma$, given by $w = u + iv$. In $\mathbb{M}$, we can take global conformal parameters $z = x + iy$ such that $\mathbb{M}$ is isometric to $(\mathbb{D}, \frac{4\alpha(z)^2}{(1-|z|^2)^2}|dz|^2)$, where α is a smooth function bounded between two positive constants (see [23]). With these notations, we can write the equation satisfied by the harmonic map h:

$$\sigma h_{w\bar{w}} + 2(\sigma_z \circ h)h_w h_{\bar{w}} = 0,$$

where $\sigma(z)^2 = \frac{4\alpha(z)^2}{(1-|z|^2)^2}$.

Associated to this map, we have the holomorphic *Hopf differential* of h, given by

$$\mathcal{Q}(h) = (\sigma \circ h)^2 h_w \bar{h}_w dw^2$$

(for short, we write ϕ for $(\sigma \circ h)^2 h_w \bar{h}_w$).

Since X is a conformal immersion, the following equalities hold:

$$\sigma^2 |h_u|^2 + f_u^2 = \sigma^2 |h_v|^2 + f_v^2;$$
$$\sigma^2 \langle h_u, h_v \rangle + f_u f_v = 0.$$

A trivial consequence of the above equations is that $\phi = -f_w^2$, hence the zeroes of ϕ have even order. Furthermore, we define η as the holomorphic 1-form in Ω given by $\eta = -2i\sqrt{\phi(w)}dw$, where the square root of ϕ is chosen in such a way that

$$f = Re \int_w \eta. \tag{4.1}$$

Considering N the unit normal vector field along Σ, denote by N_3 the function $\langle N, \partial_t \rangle$, where t is a global parameter for $\mathbb{R}$. Define ξ as the function given by $\xi := tanh^{-1}(N_3)$. We can conclude from [40] that the function ξ satisfies the sinh-Gordon equation:

$$\Delta_0 \xi = -2K_{\mathbb{M}} sinh(2\xi)|\phi|, \tag{4.2}$$

where Δ_0 stands for the Euclidean Laplacian.

Writing the metric of Σ in terms of ξ, we have:

$$ds^2 = cosh^2(\xi)|\eta|^2 = 4cosh^2(\xi)|\phi||dz|^2.$$

Finally, we denote by K_Σ the Gaussian curvature of Σ. The Gauss equation states that

$$K_\Sigma = K_{\mathbb{M}\times\mathbb{R}}(X_u, X_v) + K_{ext},$$

where K_{ext} is the extrinsic curvature of Σ. Since X is minimal and the sectional curvature of $\mathbb{M} \times \mathbb{R}$ is nonpositive, the curvature K_Σ is nonpositive. The total curvature of Σ is defined by

$$C(\Sigma) = \int_\Sigma K_\Sigma dA.$$

4.3 Minimal surfaces of finite total curvature

We are going to prove the following result:

Theorem 4.1. *Let X be a complete minimal immersion of Σ in $\mathbb{M}\times\mathbb{R}$ with finite total curvature. Then*

1. *$\mathcal{Q}$ is holomorphic on $S\backslash\{p_1,\cdots,p_n\}$ and extends meromorphically to each puncture. Moreover, parametrizing a neighborhood of each puncture p_j by the exterior of a disc and writing*

$$\mathcal{Q}(z)=\left(\sum_{k\geq 1}\frac{a_{-k}}{z^k}+P_j(z)\right)^2 dz^2$$

around p_j, where P_j is a polynomial function, then P_j is not identically zero. We denote the degree of P_j by m_j.

2. *The third coordinate of the unit normal vector N_3 converges to 0 uniformly at each puncture.*

3. *The total curvature is a multiple of 2π. More precisely, the following equality holds:*

$$\int_\Sigma K_\Sigma = 2\pi(2-2g-2n-\sum_{k=1}^n m_k).$$

Definition. We say that m_j is the *degree* of p_j.

Proof. It is well known that Σ is conformally equivalent to $S\backslash\{p_1,\cdots,p_n\}$, a compact Riemann surface S punctured in a finite number of points. This follows directly from Huber's theorem (see [21]).

1. For $j=1,\ldots,n$, let U_j be a neighborhood of p_j such that $U_j\cap U_k=\emptyset$ if $j\neq k$ and there exists a biholomorphism $\psi_j:D(0,1)\to U_j$ mapping 0 to p_j. If $0<r<1$, define $U_j(r)$ by $\psi_j(D(0,r))$, the set $S(r)$ by $S\backslash\cup_{k=1}^n U_k(r)$ and S^* by $S\backslash\{p_1,\cdots,p_n\}$. Around p_j, we can take $U_j(r)\backslash\{p_j\}$ as a neighborhood of this puncture in S^*, and the corresponding end representative of Σ can be parametrized by $A(1/r):=\mathbb{C}\backslash\overline{D(0,1/r)}$. From now on, if $R_j>1$, we denote by E_j the end representative of Σ corresponding to $U_j(R_j^{-1})\backslash\{p_j\}$. In these coordinates, the metric is given by

$$ds^2 := \lambda^2|dz|^2 = 4cosh^2(\xi)|\phi||dz|^2.$$

If $u := logcosh^2(\xi)$, we have that

$$\Delta_0 u = \tfrac{2||\nabla_0\xi||^2}{cosh^2(\xi)} + 2tanh(\xi)\Delta_0\xi = \tfrac{2||\nabla_0\xi||^2}{cosh^2(\xi)} - 8K_{\mathbb{M}}sinh^2(\xi)|\phi| \geq 0.$$

Clearly, u is a subharmonic function.

Claim. *The quadratic differential $\mathcal{Q}$ has a finite number of zeroes in S.*

Proof. Clearly, the number of zeroes in $S(r)$ is finite, since they are isolated and $S(r)$ is compact. Fix $j \in \{1, \ldots, n\}$. If $\mathcal{Z}_j$ is the set of zeroes of ϕ in E_j, we have that

$$\Delta_0 log|\phi| = \textstyle\sum_{z\in\mathcal{Z}_j} 2\pi m(z)\delta_z,$$

where $m(z)$ is the multiplicity of z as a zero of ϕ.

It is well-known that $-K_\Sigma\lambda^2 = \Delta_0 log\lambda$, hence the following equality holds:

$$-2K_\Sigma\lambda^2 = \Delta_0 u + \Delta_0 log|\phi|. \tag{4.3}$$

Denote by $A(R_j, R)$ the annulus $\{z \in \mathbb{C}; R_j \leq |z| \leq R\}$ and by D_ϵ the union of discs of radius ϵ around the points of $\mathcal{Z}_j$. Integrating the identity (4.3) over $A(R_j, R)\backslash D_\epsilon$, we have that

$$-2\int_{A(R_j,R)\backslash D_\epsilon} K_\Sigma\lambda^2 = \int_{A(R_j,R)\backslash D_\epsilon} \Delta_0 u = \int_{\partial A(R_j,R)} \partial_\nu u + \int_{\partial D_\epsilon} \partial_\nu u. \tag{4.4}$$

In a neighborhood of $w \in \mathcal{Z}_j$, the function $u + m(w)log|z-w|$ is regular and smooth. Since ν points inside D_ϵ, we have that

$$lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \partial_\nu u = 2\pi m(w).$$

Substituting into (4.4),

$$-2\int_{A(R_j,R)} K_\Sigma\lambda^2 = \sum_{w\in\mathcal{Z}_j} 2\pi m(w) + \int_{\partial A(R_j,R)} \partial_\nu u,$$

therefore

$$\int_{\mathbb{S}^1} \partial_r u(R,\theta)Rd\theta = \int_{\mathbb{S}^1} \partial_r u(R_j,\theta)R_j d\theta - 2\int_{A(R_j,R)} K_\Sigma \lambda^2 - \sum_{w\in\mathcal{Z}_j} 2\pi m(w). \tag{4.5}$$

Let $U(r) = \int_{\mathbb{S}^1} u(r,\theta)d\theta$. Clearly, the function U is continuous and the derivative of U is well defined when $\{|z| = r\}$ has no zeroes of ϕ (we can suppose this is the case for $r = R_j$). In principle, U would take values on $[0, +\infty]$, but it only takes real values. In fact, if $r^\infty \in U^{-1}(+\infty)$, we have that, when $r^\infty - r'$ is a small enough positive number, then $[r', r^\infty) \cap U^{-1}(+\infty) = \emptyset$ and $|U'|$ is uniformly bounded in (r', r^∞) by a constant D, by the identity (4.5). If $r \in (r', r^\infty)$, we have

$$U(r) = U(r') + \int_{r'}^{r} U'(x)dx \le U(r') + D(r - r'),$$

thus $U(r^\infty) \le U(r') + D(r^\infty - r')$ and $U(r^\infty)$ is finite, a contradiction. If the number of zeroes of ϕ is infinite, for large R, we have that

$$R\partial_r U(R) \le -1.$$

Hence, when R is large, the function U is decreasing and $U(R) \le C - logR$, thus $U(R) < 0$ for some R, a contradiction, because $U \ge 0$. □

A trivial corollary of last claim is that $\int_{A(R_j,R)} \Delta_0 u$ is nonnegative and bounded from above by $-2C(\Sigma)$, consequently the integral $\int_{A(R_j,R)} \Delta_0 u$ is uniformly bounded on (R_j, ∞).

Claim. *The inequality $cosh^2(\xi)|\phi| \le B|z|^B|\phi|$ holds in $A(R_j)$, for a positive constant $B > 0$ and for sufficiently large $R_j > 0$.*

Proof. This follows the same ideas of the analogous result in [19]. □

Claim. *The differential $\mathcal{Q}$ is holomorphic on $S\backslash\{p_1, \cdots, p_n\}$ and extends meromorphically to each puncture.*

Proof. Considering R_j to be large enough, we can take B as an even integer and ϕ as a function without zeroes in $A(R_j)$. If $\pi : \widetilde{A(R_j)} \to A(R_j)$ is the double cover of $A(R_j)$, we have that $(Bz^B\phi) \circ \pi$ is the square of a holomorphic function ρ. We obtain that $(cosh(\xi)|\phi|^{\frac{1}{2}}) \circ \pi \le |\rho|$, and by Lemma 9.6 of [32], since X is a complete immersion, the function ρ extends meromorphically to infinity, hence we can extend ϕ meromorphically to the punctures. □

Claim. *If the differential $\mathcal{Q}$ is written as*

$$\mathcal{Q}(z) = \left(\sum_{k \geq 1} \frac{a_{-k}}{z^k} + P_j(z) \right)^2 dz^2$$

around p_j, where P_j is a polynomial function, then P_j is not identically zero.

Proof. First, we are going to prove the claim when $a_{-1} = 0$. In fact, if the claim is false in this case, then, up to a conformal change of coordinates, we can suppose that $\mathcal{Q}(z) = z^{2k_j} dz^2$, for some integer k_j satisfying $k_j \leq -2$. In this situation, the integral $\int_{A(R_j)} |\phi(z)| dz$ is finite. Therefore, we obtain that

$$\begin{aligned} &\int_{E_j} -K_\Sigma dA = \int_{A(R_j)} \Delta_0 log\lambda dz = \int_{A(R_j)} \Delta_0 u dz \\ &\geq \int_{A(R_j)} \frac{2||\nabla_0 \xi||^2}{cosh^2(\xi)} dz - \int_{A(R_j)} 8K_{\mathbb{M}} sinh^2(\xi)|\phi| dz \\ &\geq \int_{A(R_j)} 8b^2 sinh^2(\xi)|\phi| dz, \end{aligned}$$

consequently the following inequality holds:

$$\int_{A(R_j)} 8b^2|\phi|dz - \int_{E_j} K_\Sigma dA \geq \int_{A(R_j)} 8b^2 cosh^2(\xi)|\phi|dz.$$

We conclude that $Area(E_j) = \int_{A(R_j)} 4cosh^2(\xi)|\phi|dz$ is finite, which contradicts the fact that a complete end of Σ must have infinite area (see Remark 4 in the Appendix of [12]).

Now we prove the claim when a_{-1} is nonzero. Indeed, suppose the end associated to p_j satisfies $a_{-1} \neq 0$ and $P_j \equiv 0$. For a conformal parameter z in E_j satisfying $\mathcal{Q}(z) = -c_j^2 z^{-2} dz^2$, for some $c_j > 0$, we obtain the equality

$$f(z) = 2c_j Re(\int_z u^{-1} du) = 2c_j log(|z|/R).$$

We conclude that the intersection of E_j with $\mathbb{M} \times \{t\}$ is a compact curve, for $t \geq 0$, and that E_j is properly immersed.

Since $K_{\mathbb{M}} \leq -b^2$, we can take a vertical rotational catenoid $\mathcal{C}$ in $\mathbb{M} \times \mathbb{R}$ whose mean curvature vector field points inwards, whose height is bounded and such that ∂E_j is disjoint from all vertical translations of $\mathcal{C}$ (see the Appendix for the meaning of "inwards" and the existence of such catenoid). Then, if $T_x(\mathcal{C})$ is a vertical translation of $\mathcal{C}$ by $x \in \mathbb{R}$, we have $T_{-n}(\mathcal{C}) \cap E_j$ is empty for large enough $n \in \mathbb{N}$. Moving the catenoid vertically in the positive direction, since the catenoid can not have a first point of contact with E_j, by the maximum principle, we have that E_j is cylindrically bounded, and it has unbounded height. But this contradicts Proposition 5.2, then P_j must not be identically zero. □

Remark. Since the polynomials P_j are not identically zero, we can conformally parametrize E_j such that, by Theorem 6.4 of [41], the Hopf differential of h near p_j satisfies

$$\mathcal{Q}(z) = \left((m_j + 1)z^{m_j} + \frac{c_j i}{z}\right)^2 dz^2,$$

for some $c_j \in \mathbb{R}$ (the coefficient c_j is real because the function f is well defined by (4.1)). We are going to assume this expression, unless otherwise stated.

Remark. There are several manners to prove that P_j is not the zero polynomial when $a_{-1} \neq 0$. Consider in $\mathbb{M}$ the Fermi coordinates given by $\phi(s,t) = exp_{\alpha(t)}(sJ\alpha'(t))$, for $(s,t) \in \mathbb{R}^2$ and some geodesic α which does not intersect $h(\partial E_j)$. In order to prove that E_j is cylindrically bounded, we could use the barriers defined by the graph of the function

$$f(s) = \frac{1}{k}log(tanh(\frac{ks}{2})), s > 0,$$

where $k \in (0, b)$. Supposing that $h(\partial E_j)$ is contained in the region $\{\phi(s,t) \in \mathbb{M}; s < 0\}$, we have that the mean curvature vector field of the graph of f points upwards (see [14] for the proof), and proceeding as before, we conclude that $h(E_j)$ is contained in the convex hull of $h(\partial E_j)$, therefore $E_j \subset D(p,R) \times \mathbb{R}$, for some $p \in \mathbb{M}$, $R > 0$. In addition, we can prove that E_j can not be cylindrically bounded considering a family of rotational catenoids with mean curvature vector field pointing inwards. We suppose this family varies from a surface containing $D(p,R) \times \mathbb{R}$ in its complement to a double-sheeted covering of a horizontal slice $\mathbb{H}^2 \times \{t\}$, for a sufficiently large $t > 0$ (the existence of this family of catenoids is guaranteed in the Appendix). Then, when we vary the catenoids, we obtain a first point of contact of E_j and one of the annuli, a contradiction to the maximum principle (see [37]).

2. We prove here that N_3 goes to 0 in each puncture. We choose R_j large enough to guarantee that ϕ has no zeroes in E_j and, in this situation, it is clear that the metric $g_\phi = |\phi(z)||dz|^2$ is flat. Denoting by $D_{|\phi|}(z,r)$ a disc in E_j with respect to the metric g_ϕ centered in z of radius r, by Proposition 2.1 and Lemma 2.4 of [18] (which also can be applied to this context), there exist positive constants R' and c' such that, if $|z| > R'$, then $F := \int \sqrt{\phi}dz$ is well defined in $D_{|\phi|}(z, c'|z|)$ and it is a conformal diffeomorphism onto its image. If w are the coordinates in $D_{|\phi|}(z, c'|z|)$ induced by F such that $w(z) := F(z) = 0$, we have that $g_\phi = |dw|^2$. Therefore, if $|z| > max\{1/c', R'\}$, define in $D_{|\phi|}(z,1)$ the metric

$$d\mu^2 = \sigma^2|dw|^2 := \frac{4\alpha(w)^2}{(1 - d_{|\phi|}(w,0)^2)^2}|dw|^2,$$

where d_ϕ is the distance function in the metric $|dw|^2$. Notice that this metric is precisely the metric of $\mathbb{M}$ in the disc $D_{|\phi|}(z,1)$. Then its curvature function, denoted by $\tilde{K}$, satisfies the inequalities $-a^2 \le \tilde{K} \le -b^2$.

The functions ξ and $\tilde{\xi} := log\sigma$ satisfy

$$\begin{aligned} \Delta_{|\phi|}\xi &= -2K_{\mathbb{M}} sinh(2\xi); \\ \Delta_{|\phi|}\tilde{\xi} &= -\tilde{K}e^{2\tilde{\xi}}. \end{aligned}$$

If $\eta := \xi - \tilde{\xi}$, we have

$$\begin{aligned} \Delta_{|\phi|}\eta &= -K_{\mathbb{M}}(e^{2\xi} - e^{-2\xi} - \frac{\tilde{K}}{K_{\mathbb{M}}}e^{2\tilde{\xi}}) \\ &\ge b^2e^{2\xi} - a^2e^{-2\xi} - a^2e^{2\tilde{\xi}} \\ &\ge e^{2\tilde{\xi}}(b^2e^{2\eta} - a^2Ce^{-2\eta} - a^2), \end{aligned}$$

where $C := max_{w \in D_{|\phi|}(z,1)} e^{-4\tilde{\xi}(w)}$. Since η goes to $-\infty$ as w goes to $\partial D_{|\phi|}(z,1)$, we have that η is bounded from above and it has a maximum at a point $p_0 \in D_{|\phi|}(z,1)$. Obviously, $\Delta_{|\phi|}\eta(p_0) \le 0$, then, at this point,

$$
\begin{aligned}
-e^{2\tilde{\xi}}K_{\mathbb{M}}(e^{2\eta} - e^{-4\tilde{\xi}}e^{-2\eta} - \frac{\tilde{K}}{K_{\mathbb{M}}}) \leq 0 \leftrightarrow \\
e^{2\eta} - e^{-4\tilde{\xi}}e^{-2\eta} \leq \frac{\tilde{K}}{K_{\mathbb{M}}} \leq \frac{a^2}{b^2} \leftrightarrow \\
e^{4\eta} - \frac{a^2}{b^2}e^{2\eta} - e^{-4\tilde{\xi}} \leq 0 \leftrightarrow \\
2e^{2\eta(p_0)} \leq \frac{a^2}{b^2} + \sqrt{\frac{a^4}{b^4} + 4C} =: 2C_1.
\end{aligned}
$$

Since η maximizes at p_0, we obtain that $\eta \leq \eta(p_0) \leq log\sqrt{C_1}$, hence we conclude the inequality $\xi \leq \tilde{\xi} + log\sqrt{C_1}$. We can apply the same reasoning to $-\xi$ instead of ξ, then we have that, at $w = 0$,

$$
|\xi(0)| \leq \tilde{\xi}(0) + log\sqrt{C_1} \leq \sup_{\mathbb{D}} log(2\alpha) + log\sqrt{C_1} =: C_2,
$$

and this implies that $|\xi(z)| \leq C_2$ if $|z| > max\{1/c', R'\}$.

Take $z \in \mathbb{C}$ such that $|z| \geq max\{r/c', R'\}$. Using Euclidean coordinates $x + iy$ in $D_{|\phi|}(z, r)$, define the function $\Psi : D_{|\phi|}(z, r) \to \mathbb{R}$ as

$$
\Psi(x, y) = \frac{C_2}{cosh(br)} cosh(\sqrt{2}bx)cosh(\sqrt{2}by),
$$

we have $\Delta_0\Psi = 4b^2\Psi$, and $\Psi \geq C_2 \geq \xi$ in $\partial D_{|\phi|}(z, r)$. Moreover, $\Psi \geq \xi$ in $D_{|\phi|}(z, r)$. In fact, if $\Psi - \xi$ admits a negative minimum at p_0, it would be in the interior of the disc, therefore $\xi(p_0) > \Psi(p_0) \geq 0$ and $\Delta_0(\Psi - \xi)(p_0) \geq 0$. On the other hand, we have at p_0 that

$$
\Delta_0(\Psi - \xi) = 4b^2\Psi + 2K_{\mathbb{M}}sinh(2\xi) \leq 4(b^2\Psi + K_{\mathbb{M}}\xi) \leq 4b^2(\Psi - \xi) < 0,
$$

a contradiction. Analogously, $\Psi \geq -\xi$, and then $\Psi \geq |\xi|$. Therefore, evaluating at z, $|\xi(z)| \leq C_2/cosh(br)$. Consequently, we conclude that

$$
|\xi(z)| \leq 2C_2e^{-c'|z|}. \tag{4.6}
$$

This estimate implies that $|\xi| \to 0$ at the punctures. Consequently, the tangent planes become vertical at infinity.

Remark. It is easy to verify that, for any $\epsilon \in (0, 1)$, there exists $\delta = \delta(\epsilon)$ and $R = R(\epsilon)$ such that the disc $D_{|\phi|}(z, \delta|z|^{m_j+1})$ is contained in $D(z, \epsilon|z|)$, for all $z \in \mathbb{C}$ satisfying $|z| > R$.

3. We finally prove the last statement. Recall that we can parametrize E_j by $A(R_j)$ such that the Hopf differential of h has the expression

$$\mathcal{Q}(z) = \left((m_j+1)z^{m_j} + \frac{c_j i}{z}\right)^2 dz^2,$$

for some $c_j \in \mathbb{R}$. Without loss of generality, we can assume that

$$R_j^{m_j+1} > 1 + (4\pi|c_j|/cos(\pi/10)). \tag{4.7}$$

Then, we can locally define the map

$$F(z) := \int \sqrt{\phi(z)}dz = \int (m_j+1)z^{m_j} + \frac{c_j i}{z}dz.$$

It is clear that ImF is globally well defined, and if θ is a locally defined argument function, we have

$$ImF(z) = c_j log|z| + |z|^{m_j+1} sin((m_j+1)\theta)$$

and, locally,

$$ReF(z) = -c_j\theta + |z|^{m_j+1} cos((m_j+1)\theta).$$

From now on, given a simply connected domain $\Omega \subset A(R_j)$, we denote by F_Ω a branch of F defined on Ω.

Consider now the following concept:

Definition. Given a piecewise smooth continuous curve $\gamma : [0,l] \to \mathbb{C}$, a *generalized lift* of γ is a piecewise smooth continuous curve $\beta : [0,l] \to A(R_j)$ such that there exists a partition $0 = t_0 < t_1 < \cdots < t_{n+1} = l$, for some $n \in \mathbb{N}$ and domains $D_i \subset A(R_j)$, $i = 0, \cdots, n$, where we can define a branch of the logarithm, such that

- $\beta([t_i, t_{i+1}]) \subset D_i$, $i = 0, \cdots, n$;
- γ is the juxtaposition of the paths $F_{D_0}(\beta|_{[t_0,t_1]}), \cdots, F_{D_n}(\beta|_{[t_n,t_{n+1}]})$, in this order.

This result is crucial for the proof:

Lemma 4.2. *Fix $C > 0$. Let $\gamma_1^C : [0, 8C] \to \mathbb{C}$ be the curve given by*

$$\gamma_1^C(t) = \begin{cases} C + it, & t \in [0, C]; \\ 2C - t + iC, & t \in [C, 3C]; \\ -C + i(4C - t) & t \in [3C, 5C]; \\ t - 6C - iC & t \in [5C, 7C]; \\ C + i(t - 8C) & t \in [7C, 8C]. \end{cases}$$

Let also $\gamma^C : [0, 8(m_j + 1)C] \to \mathbb{C}$ be the curve γ_1^C traversed $m_j + 1$ times. Then, for C sufficiently large, the curve γ^C admits a generalized lift $\tilde{\gamma}^C$ which starts and finishes at the same connected component of $(ImF)^{-1}(0)$.

Proof. Suppose first that $c_j = 0$. Hence $F : A(R_j) \to A(R_j^{m_j+1})$ is a well-defined covering map, and if $C > R_j^{m_j+1}$, it is enough to take the usual lift of γ^C.

Now, suppose c_j is nonzero. It is known (see [18]) that, if R_j is large enough, the set $(ImF)^{-1}(0)$ consists of $2(m_j + 1)$ connected components, denoted by $l_0, \cdots, l_{2m_j+1}$, and each of them is a smooth curve whose boundary is a point in $\{z; |z| = R_j\}$ and, for each $k \in \{0, \cdots, 2m_j + 1\}$, the curve l_k is contained in the region

$$\left\{ z \in A(R_j); \frac{k\pi}{m_j + 1} - \frac{\pi}{10(m_j + 1)} < arg(z) < \frac{k\pi}{m_j + 1} + \frac{\pi}{10(m_j + 1)} \right\}.$$

In addition, let Δ_k be the domain

$$\left\{ z \in A(R_j); \frac{k\pi}{m_j + 1} - \frac{\pi}{10(m_j + 1)} < arg(z) < \frac{(k+1)\pi}{m_j + 1} + \frac{\pi}{10(m_j + 1)} \right\},$$

and let Ω_k be the (open) subdomain of Δ_k bounded by l_k, l_{k+1} and $\{z; |z| = R_j\}$ (here, $l_{2m_j+2} = l_0$). We can consider an argument function in Δ_k taking values in the interval

$$\left(\frac{k\pi}{m_j + 1} - \frac{\pi}{10(m_j + 1)}, \frac{(k+1)\pi}{m_j + 1} + \frac{\pi}{10(m_j + 1)} \right),$$

then we can define F_k as F_{Δ_k}. The assumption (4.7) implies that $ReF_k(z)$ is positive if $z \in l_{2k}$. In fact, when $z \in l_{2k}$, we have

$$ReF_k(z) = |z|^{m_j+1} cos[(m_j + 1) arg z] - c_j arg z \geq \\ R_j^{m_j+1} cos(\pi/10) - 4\pi |c_j| > 1 - cos(\pi/10) > 0.$$

The same argument proves that $ReF_k(z)$ is negative if $z \in l_{2k+1}$. Since ϕ never vanishes in $A(R_j)$ (we can choose R_j to be large enough), the derivative of ReF_k is never zero along l_k. Hence, since $ReF_k(z)$ tends to $+\infty$ along l_0 as z diverges along l_0, we have that, for some sufficiently large $C > 0$, there is a unique point $p \in l_0$ such that $F_0(p) = C$. In particular, we can choose $C > max\{M_0, M_1\}$, where $M_0 := R_j^{m_j+1} + 4\pi|c_j|$ and $M_1 := max\{|ImF(z)|; |z| = R_j\}$.

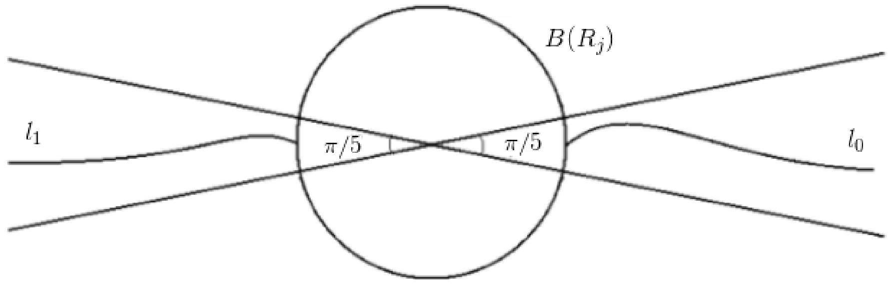

Figure 4.1: Curves l_k when $m_j = 0$

In order to construct $\tilde{\gamma}^C$, the first step is to obtain a (usual) lift of $\gamma^C|_{[0,4C]}$ with respect to $F_0 : \Delta_0 \to \mathbb{C}$. Consider the number

$$t^* := sup\{t \in [0, 4C]; \exists \beta_t : [0, t] \to \overline{\Omega}_0, \beta_t(0) = p \,\text{and}\, F_0 \circ \beta_t = \gamma^C|_{[0,t]}\}.$$

Since ϕ does not have zeroes in $A(R_j)$, by the Inverse Function Theorem and the fact that F_0 preserves orientation, there exists a path $\beta_\delta : [0, \delta] \to \overline{\Omega}_0$ satisfying $\beta_\delta(0) = p$ and $F_0 \circ \beta_\delta = \gamma^C|_{[0,\delta]}$, for some $\delta \in (0, 4C)$. Hence $t^* > 0$. Moreover, we can define a lift $\hat{\beta} : [0, t^*) \to \overline{\Omega}_0$ of $\gamma^C|_{[0,t^*)}$ starting at p.

Now, we prove that we can extend $\hat{\beta}$ to $[0, t^*]$, taking values in $\overline{\Omega}_0$. In order to do this, take a sequence $(t_n)_{n\in\mathbb{N}}$ in $[0, t^*)$ converging to t^*. We know that either $|ReF_0(\hat{\beta}(t_n))| = C$, for all n, or $ImF_0(\hat{\beta}(t_n)) = C$, for all n, up to taking a subsequence.

Using the expression of F_0, we can conclude that $(\hat{\beta}(t_n))_n$ is bounded. Hence, for any sequence $(t_n)_{n\in\mathbb{N}}$ in $[0, t^*)$ converging to t^*, the sequence $(\hat{\beta}(t_n))_n$ has an accumulation point in $\overline{\Omega}_0$ (up to taking a subsequence, we can suppose that $(\hat{\beta}(t_n))_n$ converges). Suppose $(\hat{\beta}(t_n))_n$ converges to a point in $\{z; |z| = R_j\}$. If $|ReF_0(\hat{\beta}(t_n))| = C$, for all n, we have that

$$C \leq |arg(\hat{\beta}(t_n))c_j| + |\hat{\beta}(t_n)|^{m_j+1} \leq 2\pi|c_j| + |\hat{\beta}(t_n)|^{m_j+1},$$

and taking limits, we conclude that $C \leq 2\pi|c_j| + R_j^{m_j+1} < M_0$, a contradiction. Since $C > M_1$, it is not possible that $ImF_0(\hat{\beta}(t_n)) = C$, for all n, therefore $(\hat{\beta}(t_n))_n$ does not converge to a point in $\{z; |z| = R_j\}$.

If the sequence $(\hat{\beta}(t_n))_n$ converges to a point $q \in \Omega_0$, by continuity, we have that $F_0(q) \in \gamma^C([0, 4C])$ and that $\gamma^C(t^*) = F_0(q)$. Taking a neighborhood $U \subset \Omega_0$ of q such that $F_0|_U$ is a diffeomorphism onto its image, there exists $\delta > 0$ such that $\gamma([t^* - \delta, t^* + \delta]) \subset U$. Therefore, we can define $\beta_{t^*+\delta} : [0, t^* + \delta] \to \Omega_0$ as

$$\beta_{t^*+\delta}(t) = \begin{cases} \hat{\beta}(t), & t \in [0, t^*); \\ F_0^{-1}(\gamma^C(t)), & t \in (t^* - \delta, t^* + \delta], \end{cases}$$

and we deduce that $t^* + \delta \leq t^*$, a contradiction.

It remains to analyze the case when $(\hat{\beta}(t_n))_n$ converges to a point q in $l_0 \cup l_1$. In particular, F_0 is defined at q and $F_0(q)$ is a real number. Proceeding as before, we can smoothly extend $\hat{\beta}$ to $\beta_{t^*} : [0, t^*] \to \overline{\Omega}_0$ satisfying $\beta_{t^*}(t^*) = q$. If $q \in l_0$, since $|ReF_0(\hat{\beta}(t_n))| = C$ and $ReF_0 > 0$ along l_0, we conclude that $ReF_0(\beta_{t^*}(t^*)) = ReF_0(q) = C$, then $p = q$. Since β_{t^*} is a lift of $\gamma^C|_{[0,t^*]}$ and $t^* \leq 4C$, we obtain that $t^* \in [0, C]$. Furthermore, $ImF_0(\beta_{t^*})$ must be strictly increasing along $[0, t^*]$, but $ImF_0(\beta_{t^*}(t^*)) = ImF_0(\beta_{t^*}(0))$, a contradiction. Therefore, $q \in l_1$, $t^* = 4C$, and $F_0(\beta_{4C}(4C)) = -C$.

Inductively, for $k = 1, \cdots, 2m_j + 1$, we have a curve $\beta_{4kC} : [4kC, 4(k+1)C] \to \overline{\Omega}_k$ starting at $\beta_{4(k-1)C}(4kC)$, lifting $\gamma^C|_{[4kC,4(k+1)C]}$ with respect to $F_k : \Delta_k \to \mathbb{C}$, and $\beta_{4kC}(4(k+1)C) \in l_{k+1}$. Finally, we define $\tilde{\gamma}^C$ as the juxtaposition of $\beta_{4C}, \cdots, \beta_{8(m_j+1)C}$, in this order. Evidently, the point $\tilde{\gamma}^C(8(m_j + 1)C)$ is in $l_{2m_j+2} = l_0$, as well as $\tilde{\gamma}^C(0)$. □

A consequence of the arguments of the preceding proof is that we can cover $A(R_j)$ by domains Δ_k, $k = 0, \cdots, 2m_j+1$, where we can define an integral of $\sqrt{\phi}$, denoted by $F_k : \Delta_k \to \mathbb{C}$ (the domains Δ_k from Lemma 4.2 can also be considered in $A(R_j)$ when $c_j = 0$). Since the argument functions used to define the maps F_k are bounded from above by 4π, in absolute value, we have that there exist $R^*, C_* > 0$ independent on

k such that, when $|z| > R^*$, the following inequality holds:

$$C_*|z|^{m_j+1} > |F_k(z)| > C_*^{-1}|z|^{m_j+1}.$$

Let $P(C, p_j)$ be the curve obtained from $\tilde{\gamma}^C$ when we connect $\tilde{\gamma}^C(0)$ and $\tilde{\gamma}^C(8(m_j+1)C)$ by the shortest curve segment in l_0.

We state two properties of $P(C, p_j)$, whose proofs can be deduced by the arguments in the demonstration of Lemma 4.2.

(a) $P(C, p_j)$ is a simple, piecewise smooth closed curve. If $c_j = 0$, it has $4(m_j+1)$ vertices, all of them having internal angle $\frac{\pi}{2}$; if $c_j \neq 0$, it has $4(m_j+1)+2$ vertices, one of them having internal angle $\frac{3\pi}{2}$, and the other ones having internal angle $\frac{\pi}{2}$.

(b) If $R \geq R_j$, there exists $\widetilde{C} = \widetilde{C}(R)$ such that, if $C > \widetilde{C}$, the bounded region determined by $P(C, p_j)$ contains $D(0, R)$.

For $k = 0, \cdots, m_j$ and $l = 0, 1$, let $A_k^l(C)$ be the arc

$$\tilde{\gamma}^C([(8k+4l+1)C, (8k+4l+3)C]).$$

By construction, $A_k^l(C)$ is bijectively mapped onto a subset of $\{w \in \mathbb{C}; |Imw| = C\}$ by the map F_{2k+l}. Let also $B_k^l(C)$ be the arc

$$\tilde{\gamma}^C([(8k+4l)C, (8k+4l+1)C] \cup [(8k+4l+3)C, (8k+4l+4)C]),$$

for $k = 0, \cdots, m_j$ and $l = 0, 1$. Each of these curves are one-to-one mapped onto a subset of $\{w \in \mathbb{C}; |Rew| = C\}$ by the map F_{2k+l}. Denote by $B^*(C)$ the (possibly degenerate) compact arc of $P(C, p_j)$ lying in l_0 which connects $\tilde{\gamma}^C(0)$ and $\tilde{\gamma}^C(8(m_j+1)C)$. We are going to denote by $\mathcal{I}(C)$ and $\mathcal{R}(C)$ the union of the curves $A_k^l(C)$ and $B_k^l(C)$, respectively. It is true that there is a small neighborhood V of $A_k^l(C)$ contained in Ω_{2k+l} such that $F_{2k+l} : V \to F_{2k+l}(V)$ is a conformal diffeomorphism. A similar property holds for the curves $B_k^l(C)$ and for $B^*(C)$.

We now proceed to the proof. Choose r small enough such that $R_j < r^{-1}$, for all j. Applying Gauss-Bonnet on $S(r)$, we obtain that

$$\int_{S(r)} K_\Sigma dA + \int_{\partial S(r)} \kappa_g = 2\pi(2 - 2g - n). \tag{4.8}$$

Consider, in the z-plane, the annulus $\Omega(C,r,p_j)$ in $\mathbb{C}$ bounded by the union of two curves: the circle $\{|z| = r^{-1}\}$ and the curve $P(C,p_j)$. Again, by Gauss-Bonnet, we have

$$\int_{\Omega(C,r,p_j)} K_\Sigma dA + \int_{P(C,p_j)} \kappa_g - \int_{\{|z|=r^{-1}\}} \kappa_g = -2\pi(m_j+1). \tag{4.9}$$

Summing Equation (4.8) with the equations in (4.9) for all j, we obtain

$$\int_{\tilde{S}(C)} K_\Sigma dA + \sum_{j=1}^{n} \int_{P(C,p_j)} \kappa_g = 2\pi(2-2g-2n-\sum_{j=1}^{n} m_j),$$

where $\tilde{S}(C) = S(r) \cup [\bigcup_{j=1}^{n} \Omega(C,r,p_j)]$. As C goes to infinity, $\tilde{S}(C)$ goes to $S^* \cong \Sigma$. It is enough to prove that $\int_{P(C,p_j)} \kappa_g$ goes to zero as C goes to $+\infty$.

For each $k \in \{0,\cdots,2m_j+1\}$, we know that $ImF^{-1}(0) \cap \Omega_k$ is at a positive distance from the lines that bound Δ_k. Then, there exist positive numbers δ_k, ϵ_k and R_k^* such that $D_{|\phi|}(z,\delta_k|z|^{m_j+1}) \subset D(z,\epsilon_k|z|)$, for all $z \in A(R_k)$ satisfying $|z| > R_k^*$. Moreover, choosing ϵ_k to be small enough, we can assure that $D(z,\epsilon_k|z|) \subset \Delta_k$ when $z \in \Omega_k$ and $|z| > R_k^*$.

If $\epsilon^{(0)} := min\{\epsilon_0,\cdots,\epsilon_{2m_j+1}\}$, we take $R^* > 0$ such that, if $|z| > R^*$, there exists $k \in \{0,\cdots,2m_j+1\}$ depending on z such that the following properties hold:

- $D_{|\phi|}(z,1) \subset D(z,\epsilon^{(0)}|z|) \subset \Delta_k$;
- $F_k : D_{|\phi|}(z,1) \to F_k(D_{|\phi|}(z,1))$ is a conformal diffeomorphism;
- $C_*|z|^{m_j+1} > |F_k(z)| > C_*^{-1}|z|^{m_j+1}$;
- There exist positive constants $\widehat{C}$ and $\hat{c}$, not depending on k, such that
$$\sup_{D_{|\phi|}(z,1)} |\xi| \leq \widehat{C} e^{-\hat{c}|z|};$$
- $\sup_{D_{|\phi|}(z,1)} cosh(2\xi) \leq 2.$

We can consider w-coordinates in $D_{|\phi|}(z,1)$ induced by F_k, k depending on z (notation: $w := F_k(z)$); in these parameters, the function ξ satisfies the equation

$$\Delta_{|\phi|}\xi = -2K_{\mathbb{M}} sinh(2\xi). \tag{4.10}$$

If z satisfies $|z| > R^*$, define $B_1(z)$ as $D_{|\phi|}(z,1)$. By Theorem 3.9 of [15], we can conclude the following interior gradient estimate for the Poisson equation:

$$\sup_{B_{1/2}(z)} ||\nabla\xi|| \leq \widetilde{C}(\sup_{B_1(z)} |\xi| + \sup_{B_1(z)} |2K_{\mathbb{M}} sinh(2\xi)|),$$

for a universal constant $\widetilde{C}$. Since $\sup\limits_{B_1(z)} cosh(2\xi) \leq 2$, we obtain that

$$\sup_{B_1(z)} |sinh(2\xi)| \leq 4 \sup_{B_1(z)} |\xi|.$$

Therefore, we have the estimate

$$\sup_{B_{1/2}(z)} ||\nabla\xi|| \leq 9\widetilde{C} max\{1, a^2\} \sup_{B_1(z)} |\xi|.$$

By the properties stated above, we rewrite the estimate as

$$\sup_{B_{1/2}(z)} ||\nabla\xi|| \leq \widetilde{C} e^{-\tilde{c}|w|^{(m_j+1)^{-1}}},$$

renaming $9\widetilde{C} max\{1, a^2\}\widehat{C}$ by $\widetilde{C}$, for simplicity. Clearly, $\widetilde{C}$ does not depend on k. In particular, we conclude that

$$||\nabla\xi(w)|| \leq \widetilde{C} e^{-\tilde{c}|w|^{m'_j}}, \tag{4.11}$$

for $m'_j := (m_j + 1)^{-1}$.

First, let us prove that $\int_{\mathcal{I}(C)} \kappa_g ds$ goes to 0 as C goes to $+\infty$. Fixing a curve $A_k^0(C)$ in $\mathcal{I}(C)$, we know that this curve can be parametrized as $\tau_C(x) = x + iC$, for $x \in [-C, C]$. An elementary computation shows that

$$\kappa_g = -\frac{sinh(\xi)\xi_y}{2cosh^2(\xi)}.$$

Along the curve τ_C, we have that, when $|w|$ is sufficiently large, by the estimate in (4.11),

$$|\xi_y(w)| \leq ||\nabla\xi(w)|| \leq \widetilde{C} e^{-\tilde{c}(|x|^{m'_j} + |C|^{m'_j})},$$

for positive constants $\widetilde{C}$ and $\tilde{c}$. Therefore, we have

$$
\begin{aligned}
\int_{\tau_C} |\kappa_g| ds &\leq \int_{-C}^{C} |\xi_y| dx \\
&\leq \widetilde{C} \int_{-\infty}^{+\infty} e^{-\tilde{c}(|x|^{m_j'} + |C|^{m_j'})} dx \\
&\leq \widetilde{C} e^{-\tilde{c}|C|^{m_j'}} \int_{-\infty}^{+\infty} e^{-\tilde{c}|x|^{m_j'}} dx,
\end{aligned}
$$

and the last term certainly goes to zero as C goes to $+\infty$. The same argument can be applied to $A_k^1(C)$, and then we conclude that $\int_{\mathcal{I}(C)} \kappa_g ds$ converges to zero as C goes to $+\infty$.

Now, we are going to prove that

$$\int_{\mathcal{R}(C)} \kappa_g \to 0 \text{ as } C \to +\infty.$$

Here, we compute the curvature of $\chi_C(y) = C + iy$ as a curve in Σ. Similar to the previous case, the geodesic curvature is given by

$$\kappa_g = -\frac{sinh(\xi)\xi_x}{2cosh^2(\xi)},$$

and the conclusion follows as in the first case.

We finally prove that $\int_{B^*(C)} \kappa_g \to 0$ as $C \to +\infty$. Using the w-coordinates induced by F_0, we have that $B^*(C)$ is contained in the real interval $[C - 2\pi|c_j|, C + 2\pi|c_j|]$ of the w-plane. Proceeding exactly as in the first case, we obtain the estimate

$$\int_{B^*(C)} |\kappa_g| ds \leq \int_{C-2\pi|c_j|}^{C+2\pi|c_j|} |\xi_y| dx \leq \widetilde{C} \int_{C-2\pi|c_j|}^{+\infty} e^{-\tilde{c}|x|^{m_j'}} dx,$$

which goes to 0 as C goes to $+\infty$.

$\square$

We emphasize that the Section 2 of [18] can be fully applied to minimal surfaces of finite total curvature in $\mathbb{M} \times \mathbb{R}$. In particular, following the same ideas presented in the section, we can prove the results below:

Proposition 4.3. *Let $X : \Sigma \to \mathbb{M} \times \mathbb{R}$ be a complete minimal immersion of finite total curvature.*

1. *Let p be an end of Σ. If $m_p \geq 0$ is the degree of p, then this end corresponds to $m_p + 1$ geodesics $\gamma_1, ..., \gamma_{m_p+1} \subset \mathbb{M}^2 \times \{+\infty\}$, $m_p + 1$ geodesics $\Gamma_1, ..., \Gamma_{m_p+1} \subset \mathbb{M}^2 \times \{-\infty\}$, and $2(m_p + 1)$ vertical straight lines (possibly some of them coincide) in $\partial_\infty \mathbb{M}^2 \times \mathbb{R}$, each one joining an endpoint of some γ_j to an endpoint of some Γ_j. Moreover, any end representative of p is asymptotic at infinity (in the sense presented in [17]) to the ideal polygon formed by the mentioned curves.*

2. *X is a proper immersion.*

3. *Given $p_0 \in \Sigma$, there exists a positive constant $\Lambda = \Lambda(p_0, \Sigma)$ such that*

$$|K_\Sigma(p)| \leq \Lambda e^{-d(p,p_0)},$$

for all $p \in \Sigma$, where d is the distance function in Σ.

4.4 Examples

In this section, we give some examples of minimal surfaces with finite total curvature in $\mathbb{M} \times \mathbb{R}$.

1. *Vertical planes.* The simplest examples are the vertical totally geodesic planes $\alpha \times \mathbb{R}$, where α is a horizontal geodesic. Their total curvature is zero, and these are the only surfaces satisfying this condition. In fact, let Σ be a minimal surface with vanishing total curvature. The Gauss equation states that

$$K_\Sigma = K_{\mathbb{M}\times\mathbb{R}}|_{G(\Sigma)} + K_{ext},$$

where K_Σ and K_{ext} are the intrinsic and extrinsic curvatures of Σ, respectively, and $K_{\mathbb{M}\times\mathbb{R}}|_{G(\Sigma)}$ is the sectional curvature of the ambient restricted to the Grassmanian of tangent planes of Σ. The curvature K_Σ is nonpositive, by the minimality of Σ, and thus K_Σ is identically zero, since the total curvature vanishes. It implies that $K_{\mathbb{M}\times\mathbb{R}}|_{G(\Sigma)} \equiv K_{ext} \equiv 0$, therefore Σ is a totally geodesic surface whose tangent planes are always vertical. Finally, given a vertical plane $P \in T_{(p,r)}(\mathbb{M} \times \mathbb{R})$, there is exactly one totally geodesic surface in $\mathbb{M} \times \mathbb{R}$ that is tangent to P, which is $\gamma_v \times \mathbb{R}$, where γ_v is the geodesic of $\mathbb{M}$ satisfying $\gamma_v'(0) = v \in (T_p\mathbb{M} \times \{0\}) \cap P$, $v \neq 0$, and the assertion is proved.

2. *Scherk graphs.* Let P an ideal geodesic polygon in $\mathbb{M}$ whose vertices are the points of infinity $p_1, \cdots, p_{2n} \in \partial_\infty \mathbb{M}$. Denote by A_i the complete geodesic connecting p_{2i-1} to p_{2i}, $i = 1, \cdots, n$, and by B_i the complete geodesic connecting p_{2i} to p_{2i+1}, $i = 1, \cdots, n$, where $p_{2n+1} := p_1$.

   Consider the family $\mathcal{H} = \{H_i\}_{i=1}^{2n}$, where for each $i = 1, \cdots, 2n$, H_i is a horocycle at p_i bounding an open horodisc F_i such that $H_i \cap H_j = \emptyset$ if $i \neq j$. Denote by $\tilde{A}_i$ the geodesic segment given by $A_i \backslash (\cup_{j=1}^{2n} F_j)$, and define $\tilde{B}_i$ in a similar way. Let $\gamma(i)$ be the geodesic segment connecting the two interior points of $H_i \cap P$ and denote by $P(\mathcal{H})$ the polygon

$$\bigcup_{i=1}^{n} (\tilde{A}_i \cup \tilde{B}_i) \cup \bigcup_{j=1}^{2n} \gamma(j)$$

   and $D(\mathcal{H})$ the domain bounded by $P(\mathcal{H})$.

   For a positive $r > 0$, let $G(r, \mathcal{H})$ be the graph of the minimal surface equation over $D(\mathcal{H})$ whose boundary data are given by r on $\cup_{i=1}^{n} \tilde{A}_i$ and zero elsewhere on $P(\mathcal{H})$.

   Define the following quantities:

$$a(P) = \sum_{i=1}^{n} |\tilde{A}_i|;$$
$$b(P) = \sum_{i=1}^{n} |\tilde{B}_i|.$$

   Let D the domain bounded by P. We say that a geodesic convex polygon Q is *inscribed* in D if the set of vertices of Q is contained in the set of vertices of P. Using the horocycles H_i, define

$$a(Q) = \sum_{\tilde{A}_i \subset Q} |\tilde{A}_i|;$$
$$b(Q) = \sum_{\tilde{B}_i \subset Q} |\tilde{B}_i|.$$

   We also define $|Q|$ as the sum of the lengths of the geodesic segments contained in the sides of Q and determined by the horocycles H_i. In [14], the authors proved the following theorem:

   Theorem 4.4. *There is a solution to the Dirichlet problem for the minimal surface equation in the domain D bounded by P with prescribed data $+\infty$ at A_i and $-\infty$ at B_i if and only if the following two conditions are satisfied:*

(a) $a(P) - b(P) = 0$,

(b) *For all inscribed polygons Q in D different from P there exist horocycles at the vertices such that*

$$2a(Q) < |Q| \text{ and } 2b(Q) < |Q|.$$

Moreover, the solution is unique up to additive constants.

The graph of the function described in the theorem are called the *Scherk graph* over D.

By the proof of Theorem 4.4 (see [8] and [14]), the Scherk graph Σ_n over D is a limit of the sequence of surfaces $(G_k := G(r_k, \mathcal{H}^k))_k$, where $(r_k)_k$ is a sequence going to $+\infty$ as k goes to $+\infty$ and, for each k, $\mathcal{H}^k = \{H_i^k\}_{i=1}^{2n}$ a family of horocycles of $\mathbb{M}$ such that $(H_i^k)_{i=1}^{\infty}$ is a sequence of nested horocycles at p_i converging to this point. Using Gauss-Bonnet on each G_k, we conclude that the total curvature of those surfaces is uniformly bounded from below by $2\pi(1-n)$, therefore Σ_n has finite total curvature.

In order to compute explicitly the total curvature of Σ_n, we notice that, since this surface is a graph, the coincidences mentioned in Proposition 4.3 do not happen. Therefore, we have that $m_p = n - 1$, following the notation of the same corollary. Consequently, applying the formula of Theorem 4.1, we conclude that the total curvature of Σ_n is precisely $2\pi(1 - n)$.

Following the same ideas in Theorem 6 in [37], we have the result below:

Proposition 4.5. *If Σ is a complete minimal surface of total curvature -2π in $\mathbb{M} \times \mathbb{R}$, then Σ is the Scherk minimal graph over an ideal quadrilateral in $\mathbb{M}$.*

3. *Horizontal catenoids.* In [35], the author constructs a class of minimal annuli with horizontal slices of symmetry. These catenoids C are similar to the ones constructed in [28] and [36]. They are limits of compact minimal annuli $(C_n)_{n\in\mathbb{N}}$ whose boundary components S_n^1 and S_n^2 are convex curves contained in the vertical planes P_n^1 and P_n^2, respectively. Denote by κ_n^i, $\hat{\kappa}_n^i$ and $\tilde{\kappa}_n^i$ the geodesic curvatures of S_n^i as a curve of C_n, P_n^i and $\mathbb{M} \times \mathbb{R}$, respectively. Clearly, we have that $\kappa_n^i \leq \tilde{\kappa}_n^i$, and since P_n^i is a totally geodesic submanifold of $\mathbb{M} \times \mathbb{R}$, the curvatures $\hat{\kappa}_n^i$

and $\tilde{\kappa}_n^i$ are equal, up to a sign. Moreover, for each i, the induced metric on P_n^i is Euclidean, thus the total curvature of S_n^i is 2π. Consequently, using Gauss-Bonnet,

$$\begin{aligned}
&\int_{C_n} K_{C_n} + \int_{\partial C_n} \kappa_{\partial C_n} = 0 \leftrightarrow \\
&\left|\int_{C_n} K_{C_n}\right| \leq \int_{\partial C_n} |\kappa_{\partial C_n}| \leftrightarrow \\
&\left|\int_{C_n} K_{C_n}\right| \leq \int_{S_n^1} |\kappa_n^1| + \int_{S_n^2} |\kappa_n^2| \leftrightarrow \\
&\left|\int_{C_n} K_{C_n}\right| \leq \int_{S_n^1} |\hat{\kappa}_n^1| + \int_{S_n^2} |\hat{\kappa}_n^2| = 4\pi.
\end{aligned}$$

Therefore, C has finite total curvature and its absolute value is at most 4π. On the other hand, by the formula of Theorem 4.1,

$$\left|\int_{C_n} K_{C_n}\right| \geq 4\pi,$$

thus $\int_C K_C = -4\pi$.

4.5 Index of minimal surfaces in $\mathbb{M}^2 \times \mathbb{R}$

Here, we are going to add the extra assumption that $K_{\mathbb{M}}$, the sectional curvature of $\mathbb{M}$, satisfies $||\nabla_{\mathbb{M}} K_{\mathbb{M}}|| \in L^\infty(\mathbb{M})$. The main objective of this section is to prove the following result:

Theorem 4.6. *Let Σ be a complete oriented minimal surface with unit normal field N immersed in $\mathbb{M}^2 \times \mathbb{R}$. Let $\nu := g(N, \partial_t)$ be the vertical component of N, A the second fundamental form of Σ and K_Σ be the intrinsic curvature of Σ. Then:*

1. *If $\nu^2(1-\nu^2)^{1/2} \in L^1(\Sigma)$ and $|A| \in L^2(\Sigma)$, then the function $|A|$ tends to zero uniformly at infinity. In particular, if $\nu \in L^2(\Sigma)$ (or, equivalently, if Σ has finite total curvature), then ν and K_Σ converge to zero uniformly at infinity.*

2. *If $\nu^2(1-\nu^2)^{1/2} \in L^1(\Sigma)$ and $|A| \in L^2(\Sigma)$, then the Jacobi operator of Σ has finite index.*

This result generalizes a theorem of [5], proved in the context of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. We point out that the hypotheses in Theorem 4.6 are slightly more general that the finiteness of the finite total curvature. In fact, the theorem includes, for example, the horizontal slices of $\mathbb{M} \times \mathbb{R}$.

We start by the following proposition:

Proposition 4.7. *In the sense of distributions, the following formula holds:*

$$|A|\Delta|A| \leq -|A|^2 Ric(N,N) + 4|A|^2 \widetilde{K}_\Sigma - |A|^4 - \sqrt{2}|A|\nu^2 \langle \nabla_{\mathbb{M}} K_{\mathbb{M}}, N\rangle.$$

Proof. By the Simons' formula (see [29]), we have

$$\begin{aligned}\langle \Delta A, A\rangle = {} & (\sum_{k=1}^{2} |\nabla_{\tilde{e}_k} A|^2) - |A|^2 Ric(N,N) + 4|A|^2 \widetilde{K}_\Sigma - |A|^4 \\ & + \sum_{i,k,l=1}^{2} \langle (\tilde{\nabla}_{\tilde{e}_k} \tilde{R})(\tilde{e}_i, \tilde{e}_l)\tilde{e}_i, A(\tilde{e}_k, \tilde{e}_l)\rangle \\ & + \sum_{i,k,l=1}^{2} \langle (\tilde{\nabla}_{\tilde{e}_i} \tilde{R})(\tilde{e}_i, \tilde{e}_k)\tilde{e}_l, A(\tilde{e}_k, \tilde{e}_l)\rangle.\end{aligned}$$

Here, the basis $\{\tilde{e}_1, \tilde{e}_2\}$ is an orthonormal basis on Σ and $\widetilde{K}_\Sigma$ is the sectional curvature of $\mathbb{M} \times \mathbb{R}$ along Σ.

We can choose the basis $\{\tilde{e}_1, \tilde{e}_2\}$ conveniently, such that, for some point $p \in \Sigma$, the chosen frame is geodesic at p. Making the necessary computations, we obtain the following identity (at p):

$$\begin{aligned}& \sum_{i,k,l=1}^{2} \langle (\tilde{\nabla}_{\tilde{e}_k} \tilde{R})(\tilde{e}_i, \tilde{e}_l)\tilde{e}_i, A(\tilde{e}_k, \tilde{e}_l)\rangle + \sum_{i,k,l=1}^{2} \langle (\tilde{\nabla}_{\tilde{e}_i} \tilde{R})(\tilde{e}_i, \tilde{e}_k)\tilde{e}_l, A(\tilde{e}_k, \tilde{e}_l)\rangle \\ = & \sqrt{2}|A| \sum_{i\neq k} (-1)^k \langle (\tilde{\nabla}_{\tilde{e}_k} \tilde{R})(\tilde{e}_i, \tilde{e}_l)\tilde{e}_i, N\rangle \\ = & -\sqrt{2}|A|\nu^2 \langle \nabla_{\mathbb{M}} K_{\mathbb{M}}, N\rangle,\end{aligned}$$

and this can be extended to all Σ.

We now compare $|A|\Delta|A|$ with $\langle \Delta A, A\rangle$. For $r > 0$, define u_r as the function $\sqrt{|A|^2 + r^2}$. Clearly, u_r is a positive smooth function. Computing the Laplacian of u_r, we obtain the identity:

$$u_r \Delta u_r = \langle A, \Delta A\rangle + |\nabla u_r|^2 - |\nabla A|^2.$$

Moreover, we can make the following computations:

$$|\nabla u_r|^2 = \sum_{k=1}^{2} |\tilde{e}_k u_r|^2 = \sum_{k=1}^{2} u_r^{-1} |\langle \nabla_{\tilde{e}_k} A, A\rangle|^2 \leq \sum_{k=1}^{2} |\nabla_{\tilde{e}_k} A|^2 = |\nabla A|^2,$$

hence we conclude that $u_r \Delta u_r \leq \langle A, \Delta A\rangle$.

Furthermore, we calculate Δu, where $u := |A|$, in the sense of distributions. For a nonnegative function $\phi \in C_0^\infty(M)$, we have

$$\begin{aligned}
\int_M \phi \Delta u d\mu_M &= \int_M u \Delta \phi d\mu_M \\
&= lim_{r\to 0} \int_M u_r \Delta \phi d\mu_M \\
&= lim_{r\to 0} \int_M \phi \Delta u_r d\mu_M \\
&\leq lim_{r\to 0} \int_M \phi \langle A, \Delta A\rangle u_r^{-1} d\mu_M,
\end{aligned}$$

and when r goes to 0, the last term of the inequality chain goes to the integral $\int_M \phi \langle sgn(A), \Delta A\rangle d\mu_M$, where

$$sgn(A)(p) = \begin{cases} 0, & \text{if } A(p) = 0; \\ A(p)/|A(p)|, & \text{if } A(p) \neq 0. \end{cases}$$

We can conclude that $\Delta u - \langle sgn(A), \Delta A\rangle$ is a nonpositive distribution and, consequently, it is a nonpositive measure. Since $\langle sgn(A), \Delta A\rangle$ is a locally integrable function, it defines a signed measure, and obviously Δu is a signed measure satisfying

$$\Delta u \leq \langle sgn(A), \Delta A\rangle,$$

in the sense of measures. Therefore, we can multiply the inequality by u, and we obtain

$$u \Delta u \leq \langle A, \Delta A\rangle,$$

from where we get the inequality in the statement. □

To prove the first item of Proposition 4.6, we point out that, by Lemma 31 of [1], the function u is in $H^1_{loc}(\Sigma)$. Consequently, we can follow the same calculations of [4], obtaining the inequality

$$||\xi u^k||_4^2 \leq Ck(||\xi u^k||_2^2 + |||d\xi| u^k||_2^2 + ||\xi^2 u^{2k-1} \nu^2 \sqrt{1-\nu^2}||_1),$$

where $\xi \in C_0^\infty(\Sigma)$.

If u_1 is the restriction of u to the region of Σ where $|u| < 1$, we conclude that

$$||\xi u^k||_4^2 \leq 2Ck(||\xi u^k||_2^2 + |||d\xi|u^k||_2^2 + ||\xi^2 u_1^{2k-1}\nu^2\sqrt{1-\nu^2}||_1), \tag{4.12}$$

Consequently, we have that

$$||\xi u^k||_4^2 \leq C_1 k(||1_{supp\xi} u^k||_2^2 + ||\xi^2 u_1^{2k-1}\nu^2\sqrt{1-\nu^2}||_1), \tag{4.13}$$

if $|\xi| \leq 1$.

If the set where $u > 1$ is unbounded, when the area of $supp\xi$ is large enough, we conclude that $||\xi^2 u_1^{2k-1}\nu^2\sqrt{1-\nu^2}||_1 \leq C'||1_{supp\xi} u^k||_2^2$, and the inequality

$$||\xi u^k||_4^2 \leq C_1' k(||1_{supp\xi} u^k||_2^2) \tag{4.14}$$

would allow us to prove that $u(x) \to 0$ as $x \to \infty$, a contradiction. Hence $u \leq 1$ out of a compact set of Σ. From this information, if we multiply the metric of $\mathbb{M} \times \mathbb{R}$ by a constant $c > 0$, we have that the second fundamental form of Σ in this new ambient, denoted by $\tilde{A}$, is bounded in norm by 1 out of a compact set $K \subset \Sigma$, and then we have that, in $\Sigma \backslash K$, $u \leq c^{-1}$. Therefore, in fact, $u(x) \to 0$ as $x \to \infty$.

Furthermore, the function ν satisfies the equation

$$-\Delta\nu = -(K_{\mathbb{M}} \circ \pi)\nu^3 + (|A|^2 + K_{\mathbb{M}} \circ \pi)\nu,$$

where $\pi : \mathbb{M} \times \mathbb{R} \to \mathbb{M}$ is the projection in the first factor. Proceeding as before, we conclude that, if $\nu \in L^2(\Sigma)$, then $\nu \to 0$ uniformly at infinity. Since $2K_\Sigma = -|A|^2 + 2(K_{\mathbb{M}} \circ \pi)\nu^2$, the proof of the first item is finished.

To prove the second item of Proposition 4.6, it is enough to proceed as in [2]. However, we need to clarify some steps. If $B : \mathbb{M} \to \mathbb{R}$ is a Busemann function for $\mathbb{M}$ and $\hat{B} : \mathbb{M} \times \mathbb{R} \to \mathbb{R}$ the function given by $\hat{B}(p,t) = B(p)$. Denote by $g_{\mathbb{M}}, g$ and $\hat{g}$ the metrics of $\mathbb{M}$, Σ and the product metric of $\mathbb{M} \times \mathbb{R}$, respectively. We need to compute $\Delta_g B$. By Lemma 2.2 of [2], we have that

$$\Delta_g B = \Delta_{\hat{g}} \hat{B}|_\Sigma - Hess_{\hat{g}} \hat{B}(N, N). \tag{4.15}$$

Here, both Laplacians are defined as $div \circ grad$, in their respective metrics. Given a vector $w \in T(\mathbb{M} \times \mathbb{R})$, let its horizontal and vertical components given by w^h and w^v, respectively. With this notation, we have that

$$Hess_{\hat{g}} \hat{B}(w, w) = Hess_{\hat{g}} \hat{B}(w^h, w^h) = Hess_{g_{\mathbb{M}}} B(w^h, w^h).$$

We know that $\nabla_{\mathbb{M}}B$ is a unit vector field whose integral curves are the geodesics which pass through the center of the horocycles where B is constant (call this point $B(\infty)$). For a point $p \in \mathbb{M}$, let $\{e_1, e_2\}$ be an orthonormal local frame around p such that e_1 is tangent along the geodesic passing through p and $B(\infty)$ (call it γ_p) and e_2 is tangent along the horocycle centered in $B(\infty)$ passing through p (denoted by H_p). By the choice of e_1, we have that $e_1 = \pm\nabla f$ along γ_p, we have that $Hess_{g_{\mathbb{M}}}B(e_1, \cdot\,) \equiv 0$, therefore the equality holds:

$$Hess_{g_{\mathbb{M}}}B(w^h, w^h) = \langle w^h, e_2\rangle^2 Hess_{g_{\mathbb{M}}}B(e_2, e_2).$$

By the choice of e_2, it is clear that $Hess_{g_{\mathbb{M}}}B(e_2, e_2)(p)$ is the geodesic curvature of H_p with respect to the inward pointing normal vector field (we denote this function by $\kappa(p)$). In fact, $\nabla B(p)$ is the unit vector at p pointing in the direction which is opposite to $B(\infty)$ (see [14]). In an analogous reasoning, we can conclude that $\Delta_{\hat{g}}\hat{B}|_{\Sigma} = \kappa(p)$. Thus, by the identity (4.15), we have

$$\Delta_g B(p) = \kappa(p) - \langle N, e_2\rangle^2\kappa(p) \geq \nu^2\kappa(p).$$

Using comparison theorems, we have that $\kappa(p) \geq b^{-1}$, since $K_{\mathbb{M}} \leq -b^2$, therefore the following inequality holds:

$$\Delta_g B \geq b^{-1}\nu^2. \tag{4.16}$$

From now on, we consider the Jacobi operator of Σ, given by the expression

$$J_\Sigma = -\Delta - (1-\nu^2)(K_{\mathbb{M}} \circ \pi) - |A|^2.$$

Next, we consider the proposition below:

Proposition 4.8. *The spectrum of the operator $J_\Sigma + |A|^2$ is bounded from below by a positive constant C depending only on b.*

Proof. We start by the inequality

$$\int_\Sigma b^{-1}\nu^2 f^2 \leq \int_\Sigma \Delta_g B f^2 = \int_\Sigma \langle\nabla_g B, \nabla f^2\rangle \leq 2\int_\Sigma |f|.|\nabla f| \tag{4.17}$$

By the elementary inequality

$$\int_\Sigma 2|f|.|\nabla f| \leq 2b\int_\Sigma |\nabla f|^2 + \frac{1}{2b}\int_\Sigma f^2, \tag{4.18}$$

we conclude that

$$(2b)^{-1}\int_{\Sigma} f^2 \leq 2b\int_{\Sigma}|\nabla f|^2 + b^{-1}\int_{\Sigma}(1-\nu^2)f^2. \tag{4.19}$$

Since $K_{\mathbb{M}} \leq -b^2$, we obtain

$$(2b)^{-1}\int_{\Sigma} f^2 \leq 2b\int_{\Sigma}|\nabla f|^2 - b^{-3}\int_{\Sigma}(K_{\mathbb{M}}\circ\pi)(1-\nu^2)f^2. \tag{4.20}$$

The proposition is proved if we choose C to be $(2bmax(b^{-3}, 2b))^{-1}$. □

We finish the proof of the second item as in [2]. Sketching the arguments, we prove that the essential spectrum of J_Σ is bounded from below by a positive constant and, given that J_Σ is bounded from below, we conclude, by Proposition 1 of [3] that the index of J_Σ is finite.

CHAPTER 5

Appendix

In this chapter, we provide a detailed discussion about some basic results which are useful along this work.

5.1 Vertical annuli in $\mathbb{M} \times \mathbb{R}$

In this subsection, we study complete vertical rotational minimal catenoids in $\mathbb{H}^2 \times \mathbb{R}$. We prove that, when suitably placed in $\mathbb{M} \times \mathbb{R}$, their mean curvature vector fields do not vanish at any point. We also prove that, for a fixed point $p \in \mathbb{M}$ and positive number $R > 0$, there exists a positive number $h = h(p, R)$ such that there is no minimal annulus whose boundary is contained in the set $B_R(p) \times \{-h', h'\}$ for $h' > h$, where $B_R(p)$ is the open ball of radius R centered in p.

5.1.1 Comparing geometries

Around a point of $\mathbb{M}$, we consider polar coordinates (s, θ) on the surface, and the metric is given by $ds^2 + Gd\theta^2$, for some positive smooth function G of s and θ. In particular, when $\mathbb{M}$ is the hyperbolic space of curvature $-k^2$, $k > 0$ (notation: $\mathbb{H}^2(-k^2)$), we have that the function G is precisely $G^{(k)}(s, \theta) := sinh^2(ks)$.

Let us consider a rotational surface Σ in $\mathbb{M} \times \mathbb{R}$. We can parametrize it by $(s, \theta) \mapsto (s, \theta, h(s))$, and the associated coordinate frame is $\bar{\partial}_s = \partial_s + h'(s)\partial_z$

and $\bar{\partial}_\theta = \partial_\theta$ (here, we consider in $\mathbb{M}\times\mathbb{R}$ the coordinates (s,θ,z)). So, the vector field $N = (1+h'(s)^2)^{-\frac{1}{2}}(-h'(s)\partial_s + \partial_z)$ along Σ is normal and unitary, and the mean curvature with respect to it is given by

$$2H = \frac{1}{2G(1+h'(s)^2)^{\frac{3}{2}}}\left(2Gh''(s) + (1+h'(s)^2)h'(s)G_s\right).$$

Then the surface Σ is minimal if and only if

$$2Gh''(s) + (1+h'(s)^2)h'(s)G_s = 0.$$

In particular, when $\mathbb{M} = \mathbb{H}^2(-k^2)$, the equation becomes

$$sinh(ks)h''(s) + kcosh(ks)(1+h'(s)^2)h'(s) = 0. \tag{5.1}$$

Fix two constants $A, k > 0$ and let $R_{A,k} := \frac{arcsinh(A)}{k}$. Consider the function $h_{A,k} : [R_{A,k}, +\infty) \to \mathbb{R}$ defined by

$$h_{A,k}(s) = \int_{R_{A,k}}^{s} \frac{A}{\sqrt{sinh^2(kr) - A^2}}dr.$$

The following facts about $h_{A,k}$ are easy to verify:

- $h_{A,k} \in C^\infty((R_{A,k}, +\infty)) \cap C^0([R_{A,k}, +\infty))$;
- $h_{A,k}$ solves Equation 5.1 on the domain $(R_{A,k}, +\infty)$;
- $h'_{A,k} > 0$ and $lim_{s\to R_{A,k}} h'(s) = +\infty$.

In $\mathbb{H}^2(-k^2)\times\mathbb{R}$, define the subset

$$\mathcal{C}^{A,k} := \{(s,\theta,(-1)^j h_{A,k}(s)), s \geq R_{A,k}, j \in \{0,1\}\}.$$

Obviously, $\mathcal{C}^{A,k}$ is a complete vertical rotational minimal catenoid in the space $\mathbb{H}^2(-k^2)\times\mathbb{R}$.

We now define, in $\mathbb{M}\times\mathbb{R}$, the surface

$$\mathcal{C}^{A,k}_{\mathbb{M}} := \{(s,\theta,(-1)^j h_{A,k}(s)), s \geq R_{A,k}, j \in \{0,1\}\},$$

for some fixed polar coordinate system in $\mathbb{M}$. This surface is a complete vertical rotational annulus in $\mathbb{M}\times\mathbb{R}$. If the sectional curvature of $\mathbb{M}$ satisfies $-k_1^2 < K_{\mathbb{M}} < -k_2^2$, then, by a slight variation of Proposition 2 of [13], we have that

$$\frac{G_s^{(k_1)}}{G^{(k_1)}} > \frac{G_s}{G} > \frac{G_s^{(k_2)}}{G^{(k_2)}}. \tag{5.2}$$

By Equation 5.1, we obtain the inequalities

$$2Gh''_{A,k_1}(s) + (1 + h'_{A,k_1}(s)^2)h'_{A,k_1}(s)G_s < 0;$$
$$2Gh''_{A,k_2}(s) + (1 + h'_{A,k_2}(s)^2)h'_{A,k_2}(s)G_s > 0,$$

for any $A > 0, i = 1, 2$.

The catenoid $\mathcal{C}_{\mathbb{M}}^{A,k}$ separates $\mathbb{M} \times \mathbb{R}$ in two connected components. One of them contains $\mathbb{M} \times (T, +\infty)$, for some $T \in \mathbb{R}$, which we call the *inner region* of $\mathcal{C}_{\mathbb{M}}^{A,k}$. The other component is the *outer region* of the catenoid.

We say that the mean curvature vector field $\overrightarrow{H}_{A,k}$ of $\mathcal{C}_{\mathbb{M}}^{A,k}$ points *inwards* (resp. *outwards*) when it is nonzero everywhere and it points to the inner region (resp. to the outer region). With the above reasoning, we conclude the following result:

Proposition 5.1. *For a Hadamard surface $\mathbb{M}$, suppose that the inequalities $-k_1^2 < K_{\mathbb{M}} < -k_2^2$ hold. Then, for any positive A, the vector field $\overrightarrow{H}_{A,k_1}$ points outwards, while $\overrightarrow{H}_{A,k_2}$ points inwards.*

Remark. Concerning the variation of Proposition 2 of [13], we need to assure that the inequalities in (5.2) are strict, which is not done in the reference. Indeed, if $G^i(s,\theta) := sinh^2(k_i s)$, for $i = 1, 2$, it is true that the functions $f_\theta(s) = \frac{G_s^1(s,\theta)}{2G^1(s,\theta)}$ and $g_\theta(s) = \frac{G_s(s,\theta)}{2G(s,\theta)}$ satisfy the equations

$$f'_\theta + f_\theta^2 = k_1^2 > \frac{(-K_{\mathbb{M}}(\cdot\,,\theta) + k_1^2)}{2};\; g'_\theta + g_\theta^2 = -K_{\mathbb{M}}(\cdot\,,\theta) < \frac{(-K_{\mathbb{M}}(\cdot\,,\theta) + k_1^2)}{2}.$$

It is clear that f_θ and g_θ satisfy the conditions of Corollary 2.2 of [34] (see [13] for details about f_θ and g_θ). Then, defining

$$\phi_\theta(s) = s\int_0^s (f_\theta(t) - t^{-1})dt;$$
$$\psi_\theta(s) = s\int_0^s (g_\theta(t) - t^{-1})dt,$$

we can apply the ideas of Lemma 2.1 of [34]. Explicitly,

$$(\phi'_\theta\psi_\theta - \phi_\theta\psi'_\theta)'(s) \geq (k_1^2 + K_{\mathbb{M}}(s,\theta))\phi_\theta(s)\psi_\theta(s) \leftrightarrow$$
$$\frac{G_s^1(s,\theta)}{G^1(s,\theta)} - \frac{G_s(s,\theta)}{G(s,\theta)} \geq \frac{2\displaystyle\int_0^s (k_1^2 + K_{\mathbb{M}}(x,\theta))\phi_\theta(x)\psi_\theta(x)dx}{\phi_\theta\psi_\theta(s)},$$

then one of the strict inequalities in (5.2) was proved. The other one can be proved in a similar procedure.

5.1.2 Height bounds of minimal annuli

We prove here the following proposition.

Proposition 5.2. *If $\mathbb{M}$ is a Cartan-Hadamard manifold and if $B_R(p)$ is a compact subset of $\mathbb{M}$, there exists $h_0 > 0$ depending on p and R such that, for any two Jordan curves $\Lambda_1, \Lambda_2 \subset B_R(p)$ and $h' > h$, there is no minimal annulus in $\mathbb{M} \times \mathbb{R}$ whose boundary is given by $(\Lambda_1 \times \{0\}) \cup (\Lambda_2 \times \{h'\})$.*

Proof. Suppose, by contradiction, that there is a sequence $\{\Sigma_n\}_{n\in\mathbb{N}}$ of minimal annuli such that $\partial\Sigma_n \subset B_R(p) \times \{-h_n, h_n\}$, where $(h_n)_{n\in\mathbb{N}}$ is an increasing sequence of positive numbers which goes to $+\infty$. By [26], there is a minimal stable annuli S_n whose boundary is $\partial B_R(p) \times \{-h_n, h_n\}$ that minimizes area among the annuli contained in the unbounded component of $(\mathbb{M} \times [-h_n, h_n])\backslash\Sigma_n$. We then have area and curvature estimates for the sequence $(S_n)_n$ in compact sets, then, by a diagonal argument, we have that a subsequence of $(S_n)_n$ converges to a cylindrically bounded minimal annuli S. Since all the S_n are stable, the surface S also is. By Theorem 3 of [39], the second fundamental form of S is bounded.

Obviously, $S \subset B_R(p) \times \mathbb{R}$, and let R' the smallest number such that $S \subset B_{R'}(p) \times \mathbb{R}$ (by the maximum principle, this number exists). By the choice of R', we can choose a sequence $(s_n = (q_n, t_n))_{n\in\mathbb{N}}$ of points of S, $q_n \in \mathbb{M}, t_n \in \mathbb{R}$ such that $(q_n)_n$ converges to a point q in $\partial B_{R'}(p)$. We then consider, for each n, the surface S^n, a vertical translation of S such that $\bar{s}_n := (q_n, 0) \in S^n$. The points $\bar{s}_n$ have δ-neighborhoods on S^n that are graphs of functions F_n over the δ-disc in $T_{\bar{s}_n}S^n$ such that the set $||F_n||_{C^2}$ is uniformly bounded. Therefore, up to a subsequence, the sequence $(T_{\bar{s}_n}S^n)$ converges to a vertical plane P in $T_{(q,0)}(\mathbb{M} \times \mathbb{R})$, otherwise S would not be contained in $B_R(p) \times \mathbb{R}$, and the sequence of graphs of $(F_n)_n$ converges to a minimal graph over a δ-disc which intersects $\partial B_R(p) \times \mathbb{R}$ tangentially, which is impossible. $\square$

Bibliography

[1] P. Bérard. *Spectral Geometry - Direct and Inverse Problems*, volume 1207 of *Lecture Notes in Mathematics*. Springer-Verlag Berlin Heidelberg, 1986.

[2] P. Bérard, P. Castillon, and M. Cavalcante. Eigenvalue estimates for hypersurfaces in $\mathbb{H}^m \times \mathbb{R}$ and applications. *Pac. J. Math.*, 253:19–35, 2011. doi:10.2140/pjm.2011.253.19.

[3] P. Bérard, M. P. do Carmo, and W. Santos. The index of constant mean curvature surfaces in hyperbolic 3-space. *Math. Z.*, 224:313–326, 1997. doi:10.1007/PL00004288.

[4] P. Bérard, M. P. do Carmo, and W. Santos. Complete hypersurfaces with constant mean curvature and finite total curvature. *Ann. Glob. Anal. Geom.*, 16(3):273–290, 1998. doi:10.1023/A:1006542723958.

[5] P. Bérard and R. Sá Earp. Minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$, total curvature and index. *Bollettino dell'Unione Matematica Italiana*, 9(3):341–362, 2016. doi:10.1007/s40574-015-0050-0.

[6] M. Callahan, D. Hoffman, and W. Meeks III. The structure of singly-periodic minimal surfaces. *Invent Math.*, 99(1):455–481, 1990. doi:10.1007/BF01234428.

[7] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold iv; locally simply connected. *Ann. of Math.*, 160(2):573–615, 2004. doi:10.4007/annals.2004.160.573.

[8] P. Collin and H. Rosenberg. Construction of harmonic diffeomorphisms and minimal graphs. *Ann. Math.*, 172(3):1879–1906, 2010. doi:10.4007/annals.2010.172.1879.

[9] R. Sá Earp. Parabolic and hyperbolic screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$. *J. Aust. Math. Soc.*, 85.

[10] R. Sá Earp and E. Toubiana. Screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$. *Illinois J. Math.*, 49.

[11] Y. Fang. *Lectures on Minimal Surfaces in* $\mathbb{R}^3$, volume 35 of *Proceedings of the Centre for Mathematics and its Applications, Australian National University*. Australian National University, Centre for Mathematics and its Applications, 1996.

[12] K. Frensel. Stable complete surfaces with constant mean curvature. *Bol. Soc. Brasil. Mat.*, 27:129–144, 1996. doi:10.1007/BF01259356.

[13] J. A. Gálvez and V. Lozano. Existence of barriers for surfaces with prescribed curvatures in $\mathbb{M}^2 \times \mathbb{R}$. *J. Differ. Equ.*, 255(7):1828–1838, 2013. doi:10.1016/j.jde.2013.05.027.

[14] J. A. Gálvez and H. Rosenberg. Minimal surfaces and harmonic diffeomorphisms from the complex plane onto certain Hadamard surfaces. *Amer. J. Math.*, 132(5):1249–1273, 2010. URL: www.jstor.org/stable/40864609.

[15] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, second edition, 1983.

[16] L. Hauswirth. Generalized Riemann examples in three-dimensional manifolds. *Pacific Journal of Math.*, 224(1):91–117, 2006. URL: http://perso-math.univ-mlv.fr/users/hauswirth.laurent/art10.pdf.

[17] L. Hauswirth, A. Menezes, and M. Rodríguez. On the characterization of minimal surfaces with finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{PSL}_2(\mathbb{R}, \tau)$. *M. Calc. Var.*, 58(80), 2019. doi:10.1007/s00526-019-1505-4.

[18] L. Hauswirth, B. Nelli, R. Sá Earp, and E. Toubiana. Schoen's theorem and minimal annular surfaces in $\mathbb{H} \times \mathbb{R}$. *Advances in Mathematics*, 274:199–240, 2015. URL: https://arxiv.org/abs/1111.0851.

[19] L. Hauswirth and H. Rosenberg. Minimal surfaces of finite total curvature in $\mathbb{H} \times \mathbb{R}$. *Matemática Contemporânea*, 31:65–80, 2006. URL: http://w3.impa.br/~rosen/courbure.pdf.

[20] D. Hoffman and H. Karcher. Complete embedded minimal surfaces of finite total curvature. In R. Osserman, editor, *Geometry V*, pages 5–93. Springer, Berlin, 1997. doi:10.1007/978-3-662-03484-2_2.

[21] A. Huber. On subharmonic functions and differential geometry in the large. *Comm. Math. Helv.*, 32:13–72, 1957. URL: http://eudml.org/doc/139145.

[22] L. Jorge and W. Meeks III. The topology of complete minimal surfaces of finite total Gaussian curvature. *Topology*, 22:203–221, 1983. doi:10.1016/0040-9383(83)90032-0.

[23] P. Li, L. Tam, and J. Wang. Harmonic diffeomorphisms between Hadamard manifolds. *Tran. AMS*, 347:3645–3658, 1995. doi:10.2307/2155031.

[24] L. Mazet, M. Rodríguez, and H. Rosenberg. The Dirichlet problem for the minimal surface equation, with possible infinite boundary data, over domains in a Riemannian surface. *Proc. Lond. Math. Soc.*, 102(6):985–1023, 2011. doi:10.1112/plms/pdq032.

[25] W. H. Meeks III and S. T. Yau. The classical plateau problem and the topology of three-dimensional manifolds: The embedding of the solution given by Douglas-Morrey and an analytic proof of Dehn's lemma. *Topology*, 21(4):409–442, 1982. doi:10.1016/0040-9383(82)90021-0.

[26] W. H. Meeks III and S. T. Yau. The existence of embedded minimal surfaces and the problem of uniqueness. *Math. Z.*, 179(2):151–168, 1982. doi:10.1007/BF01214308.

[27] X. Mo and R. Osserman. On the Gauss map and total curvature of complete minimal surfaces and an extension of Fujimoto's theorem. *J. Differential Geom.*, 31(2):343–355, 1990. doi:10.4310/jdg/1214444316.

[28] F. Morabito and M. M. Rodríguez. Saddle towers and minimal k-noids in $\mathbb{H} \times \mathbb{R}$. *J. Inst. Math. Jussieu*, 11(2):333–349, 2012. doi:10.1017/S1474748011000107.

[29] A. Caminha Muniz Neto. *Tópicos de Geometria Diferencial*. Fronteiras da Matemática. SBM, 2014.

[30] B. Nelli and H. Rosenberg. Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. *Bull. Braz. Math. Soc. (N.S.)*, 33:263–292, 2002. doi:10.1007/s005740200013.

[31] M. H. Nguyen. *Minimal surfaces in three dimensional homogeneous manifolds*. PhD thesis, Mathématiques, Informatique et Télécommunications de Toulouse (MITT), 6 2016.

[32] R. Osserman. *A survey of minimal surfaces*. Dover Publications, second edition, 1986.

[33] J. Pérez and A. Ros. *Properly embedded minimal surfaces with finite total curvature*, pages 15–66. Springer Berlin Heidelberg, Berlin, Heidelberg, 2002. doi:10.1007/978-3-540-45609-4_2.

[34] S. Pigola, M. Rigoli, and A. G. Setti. *Vanishing and finiteness results in geometric analysis: a generalization of the Bochner technique*, volume 266 of *Progress in Mathematics*. Birkhäuser, 2008.

[35] R. Ponte. Minimal annuli in $\mathbb{M}^2 \times \mathbb{R}$. In preparation, 2019.

[36] J. Pyo. New complete embedded minimal surfaces in $\mathbb{M}^2 \times \mathbb{R}$. *Ann. Global Anal. Geom.*, 40:167–176, 2011. doi:10.1007/s10455-011-9251-7.

[37] J. Pyo and M. M. Rodríguez. Simply-connected minimal surfaces with finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$. *Int. Math. Res. Notices*, pages 2944–2954, 2014. doi:10.1093/imrn/rnt017.

[38] H. Rosenberg. Minimal surfaces in $m^2 \times \mathbb{R}$. *Illinois J. Math.*, 46.

[39] R. Schoen. Estimates for stable minimal surfaces in three-dimensional manifolds. In E. Bombieri, editor, *Seminar on minimal submanifolds*, Annals of Math. Studies. Princeton University Press, 1983.

[40] R. Schoen and S. T. Yau. *Lectures on harmonic maps*. Conf. Proc. Lecture Notes Geom. and Topology, II. International Press, 1997.

[41] K. Strebel. *Quadratic differentials*, volume 5 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, 1984.

[42] B. White. On the compactness theorem for embedded minimal surfaces in 3-manifolds with locally bounded area and genus. To appear in Comm. Anal. Geom. URL: https://arxiv.org/pdf/1503.02190.pdf.

[43] B. White. Complete surfaces of finite total curvature. *J. Diff. Geom.*, 26:315–326, 1987. doi:10.4310/jdg/1214441372.