Degenerations of linear series to curves with three components, using quiver representations

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I dedicate this thesis to my beloved mother Felícia Vital.

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Abstract

General degenerations of linear series to nodal curves with $n+1$ components yield exact linked nets of vector spaces with finite support over $\mathbb{Z}^n$-quivers, which are special quiver representations of pure dimension. The linked projective space associated to a net is the associated quiver Grassmannian of subrepresentations of pure dimension 1, viewed in the product of the projectivized spaces associated to the representation.

Of fundamental importance is whether an exact linked net admits a simple basis, or equivalently, a complete decomposition of the quiver representation in subrepresentations that are exact linked nets of dimension 1. Indeed, if it does, we can prove, at least for $n \leq 2$, that its associated linked projective space has multivariate Hilbert polynomial equal to that of the diagonal. This is the case for exact linked nets of finite support if $n = 1$, but not always for $n > 1$, as we exemplify in the thesis. At any rate, we give a local characterization for when an exact linked net of vector spaces with finite support over a $\mathbb{Z}^n$-quiver admits a simple basis.

Also, we consider exact linked nets of vector spaces with finite support over a $\mathbb{Z}^2$-quiver and prove that the associated linked projective spaces are Cohen–Macaulay, reduced and of pure dimension. As a consequence, we show that for those nets that can be properly deformed, for instance, those with simple bases but also all of those arising from degenerations, the multivariate Hilbert polynomials of the associated linked projective spaces are equal to that of the diagonal.

We finish by describing the correspondence between exact linked nets of vector spaces with finite support over $\mathbb{Z}^1$-quivers and complete collineations.

Keywords: Linear series, degenerations, quiver representations, simple bases, linked projective spaces, complete collineations.

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Chapter 1

Introduction

The goal of this thesis is to advance with the theory of limit linear series for nodal curves with several components, including those not of compact type. The term "limit linear series" was coined by Eisenbud and Harris [3] in the 1980's to describe certain data arising from the study of degenerations of linear series on families of smooth curves degenerating to curves of compact type. Using these data, Eisenbud and Harris were able to obtain remarkable results as: a shorter proof of the Brill–Noether Theorem [1], a shorter proof of the Gieseker–Petri Theorem [2], a proof that the moduli of curves is of general type in genus at least 24 [5] and a partial solution to Hurwitz question on Weierstrass semigroups [4].

The work by Eisenbud and Harris was done in characteristic zero; it was clear that the definitions were not correct in positive characteristic. In the 2000's, Osserman [14] proposed a slightly different approach that would not only work in any characteristic, but would render the whole study more natural, functorial and complete. Whereas, considering degenerations of linear series on families of smooth curves degenerating to curves of compact type, Eisenbud and Harris picked certain linear series that occurred as limits, those they called focused on the components of the limit curve, Osserman considered many more, in fact, all of the linear series that could in principle yield limits of the divisors along the family. By considering more data, Osserman got rid of certain conditions on order (of vanishing) sequences used by Eisenbud and Harris, sequences which are notorially "pathological" in positive characteristic.

Most of the work Osserman did was concentrated on curves of compact type with two components, the simplest reducible nodal curves there are. It was clear what data to consider for curves of compact type with more components, even curves of noncompact type, and he started work on those in [15].

But as far as limits of divisors were concerned, the most important works focused on two-component curves of compact type. In [14] Osserman described what he called *linked Grassmannians*, which would serve to parametrize a moduli scheme of limit linear series. It was later observed by Daniel Santana Rocha in his thesis [16], extending work by Esteves and Osserman [6], that a certain linked Grassmannian of subspaces of dimension 1 was a parameter space for limits of divisors along the family. It was fundamental to Santana's approach the work by Helm and Osserman [11] showing the flatness of the linked Grasmannian. All of the works cited in this paragraph are restricted to the two-component case.

As mentioned above, in [15] Osserman considered curves of noncompact type, and it became clear that the objects called linked Grassmannians were in fact quiver Grassmannians associated to certain quiver representations. It was in fact a simple but remarkable idea. The crux of the problem in dealing with a degeneration of linear series on a family of smooth curves degenerating to a nodal curve C is that, if C is reducible, there are several limits of the line bundles associated to the linear series. Considering a degeneration of curves along a general direction, thus a *general degeneration*, the limits are line bundles L_u themselves. And the associated spaces of sections degenerate to a space of sections V_u of the limit line bundle L_u for each such limit, thus obtaining a linear series on C as limit of the family for each limit line bundle.

How to deal with so much data? The limit line bundles L_u are determined by their *multidegrees*, that is, the set of their degrees on each component of C. Not all multidegrees are achievable, so that is the first important information, the set Q_0 of achieved multidegrees. In fact, knowing the family of curves, just one multidegree is enough. The class of that multidegree in the quotient of the multidegree group by subgroup generated by the rows (or columns) of the intersection table of the components of the limit curve in the total space of the family is the set Q_0. That gives us, not only a set, but a quiver $Q = (Q_0, Q_1, s, t)$! The arrow set Q_1 is the subset of $Q_0 \times Q_0$ of pairs of multidegrees (u, v) whose difference $v - u$ is a row (or column) of the intersection table. We obtain a special quiver Q, one whose arrow set Q_1 decomposes naturally according to which row of the intersection table is used. This is a $\mathbb{Z}^n$*-quiver*, where $n+1$ is the number of components of C, as first defined by Santos [17]; see our Definition 9.

Furthermore, it can be shown that the limit line bundles L_u form in fact a representation of the quiver Q in the category of line bundles over C. In other words, not only is there a limit line bundle L_u for each $u \in Q_0$, but a map of line bundles $\varphi_u^v \colon L_u \to L_v$ for each $(u, v) \in Q_1$ satisfying certain properties. The maps induce linear maps of vector spaces $\varphi_u^v \colon V_u \to V_v$ for each $(u, v) \in Q_1$. The vector spaces V_u and the maps φ_u^v form a representation $\mathfrak{g}$ (in the category of vector spaces) of Q of pure dimension satisfying certain properties. These properties were identified by Santos [17], who coined the term *linked net of vector spaces* for the quiver representations satisfying these properties; see our Definition 11.

How about the limits on C of the divisors associated to the linear series of the family? These are not considered at all in [15]. They form a subset of the Hilbert scheme Hilb_C, parameterizing subschemes of C. It was shown by Santana [16] that the subset is a reduced closed subscheme of Hilb_C isomorphic to $\mathbb{LP}(\mathfrak{g})$, the quiver Grassmannian of subrepresentations of $\mathfrak{g}$ of pure dimension 1 in the case C is of compact type with two components. Though the embedding of $\mathbb{LP}(\mathfrak{g})$ in Hilb_C depends on the limit line bundles L_u, abstractly $\mathbb{LP}(\mathfrak{g})$ depends only on the representation $\mathfrak{g}$. The interest thus arose on the study of linked nets $\mathfrak{g}$ of vector spaces over $\mathbb{Z}^n$-quivers and the associated $\mathbb{LP}(\mathfrak{g})$, initiated by Santos [17]. Would it be possible to extend Santana's work for $n > 1$, or at least $n = 2$?

Given a $\mathbb{Z}^n$-quiver $Q = (Q_0, Q_1, s, t)$ and a linked net of vector spaces $\mathfrak{g}$ over Q, in general, $\mathbb{LP}(\mathfrak{g})$ would not be a scheme. But the linked nets that arise are of *finite support*, meaning that there is a finite subset $H \subset Q_0$ such that for each $v \in Q_0$ there are $u \in H$ and a path γ connecting u to v such that the composition φ_γ of the maps of $\mathfrak{g}$ along γ is an isomorphism; see our Definition 13. In this case, $\mathbb{LP}(\mathfrak{g})$ is a scheme. Indeed, Santos [17] showed that if, by possibly enlarging the finite set H, we have that $P(H) = H$, that is, if H is equal to its hull (see our Definition 14), then $\mathbb{LP}(\mathfrak{g})$ is equal to the quiver Grassmannian $\mathbb{LP}(\mathfrak{g})_H$ of subrepresentations of $\mathfrak{g}|_H$ of pure dimension 1, which is thus a subscheme of the product $\prod_{u \in H} \mathbb{P}(V_u)$.

Another important property of the linked nets $\mathfrak{g}$ that arise from degenerations is exactness; see our Definition 12. So suppose $\mathfrak{g}$ is an exact linked net of vector spaces of finite support over a $\mathbb{Z}^n$-quiver Q. For $n = 1$, Santana observed that $\mathbb{LP}(\mathfrak{g})$ is reduced, Cohen–Macaulay, of pure dimension and with multivariate Hilbert polynomial equal to that of the diagonal in $\prod_{u \in H} \mathbb{P}(V_u)$. The work supporting that observation had already been done by Helm and Osserman in [14] and [11]. What can we say for $n > 1$?

For $n = 2$, Santos [17] showed that $\mathbb{LP}(\mathfrak{g})$ is generically reduced and of pure dimension. He described as well the points of $\mathbb{LP}(\mathfrak{g})$ is terms of properties of the subrepresentations. Also, he concluded that the multivariate Hilbert polynomial of $\mathbb{LP}(\mathfrak{g})$ is equal to that of the diagonal in certain very special cases (for instance, if the dimension of $\mathfrak{g}$ is at most 2.)

Here we go further (for $n = 2$): We prove that $\mathbb{LP}(\mathfrak{g})$ is Cohen–Macaulay, and thus reduced; see our Theorem 42. And we proved that if $\mathfrak{g}$ admits a simple basis or alternatively arises from a degeneration of linear series, then the multivariate Hilbert polynomial of $\mathbb{LP}(\mathfrak{g})$ is equal to that of the diagonal; see our Theorem 44. More explicitly:

Theorem A: Let $\mathfrak{g}$ be an exact linked net of vector spaces of dimension $r+1$ and finite support over a $\mathbb{Z}^2$-quiver. Then $\mathbb{LP}(\mathfrak{g})$ is Cohen–Macaulay and reduced with pure dimension r. Furthermore, if $\mathfrak{g}$ admits a simple basis or arises from a degeneration of linear series, then the multivariate Hilbert polynomial of $\mathbb{LP}(\mathfrak{g})$ is equal to that of the diagonal.

We say that a linked net of vector spaces $\mathfrak{g}$ over a $\mathbb{Z}^n$-quiver *admits a simple basis* if it admits a complete decomposition as a direct sum of subrepresentations that are themselves exact linked nets of dimension 1. These subrepresentations are irreducible, so a consequence is that $\mathfrak{g}$ is completely reducible.

In their seminal work [3] Eisenbud and Harris showed the existence of what they called an *adapted basis*; see [3, lemma 2.3]. It did play a fundamental role in their work. A related structure appeared as well in Osserman's seminal work [14], in the proof of Lemma A.12. Esteves and Osserman identified it in their work and started calling it a *simple basis* in [6, lemma 2.3]. Theirs was an important lemma. It appeared as well in Santana's thesis [16]. All of these structures translate to the definition we gave above for linked nets over $\mathbb{Z}^1$-quivers.

Santos [17] showed that linked nets of vector spaces admitting a simple basis are exact and of finite support. The converse was known over $\mathbb{Z}^1$-quivers, as explained above. It is proved here for linked nets of dimension 1, our Theorem 30. The proof appears as well in Santos [17]. One of the first questions that arose in the theory was whether the converse would hold in general, or at least for linked nets arising from degenerations. That turned out to be false, and the first counter-example found is our Example 39 of a linked net $\mathfrak{g}$ of dimension 2 arising from a degeneration of linear series along a pencil of cubics degenerating to the triangle.

But when does an exact linked net $\mathfrak{g}$ of finite support over a $\mathbb{Z}^n$-quiver Q admits a simple basis? Admitting a simple basis is clearly a global property, but we could find a local characterization of it! More precisely, there is a property that is checked at each vertex of a given finite support of $\mathfrak{g}$ that, if satisfied at each such vertex, is equivalent to admitting a simple basis. We call it the *intersection property*; see Definition 31. Roughly, $\mathfrak{g}$ satisfies the intersection property at a vertex u of Q if the intersection of kernels of the maps of $\mathfrak{g}$ associated to certain paths leaving u distributes with respect to sums. And here is our main result concerning simple bases, a combination of Proposition 32 with Theorem 37:

Theorem B: An exact linked net $\mathfrak{g}$ of vector spaces with finite support over a $\mathbb{Z}^n$-quiver admits a simple basis if and only if it satisfies the intersection property at each vertex of its support.

To apply Theorem B, it is thus important to identify the support of $\mathfrak{g}$. The smallest the support is, the fastest it is to check whether $\mathfrak{g}$ has a simple basis. But in small examples, just knowing a support is enough. Luckily, for linked nets arising from degenerations of linear series, where the vertex set of the $\mathbb{Z}^n$-quiver is a set of multidegrees, a support is easy to identify: the subset of the vertex set consisting of *effective* multidegrees, those that are nonnegative on each component of the limit curve. We call this subset the *effective locus* of the quiver. It is easy to describe for a limit curve of compact type, but not as easy in general. In Chapter 3 we describe the effective loci for the limit curve equal to the triangle in the plane, and present lots of pictures!

Finally, it was pointed out to us by Daniel Santana Rocha, who learned about them from Evangelos Routis, the existence of a connection between exact linked nets of finite support over $\mathbb{Z}^1$-quivers and complete collineations. According to Thaddeus [19], "moduli spaces of complete collineations [were] introduced and explored by many of the leading lights of 19th-century algebraic geometry, such as Chasles, Schubert, Hirst, and Giambelli. They are roughly compactifications of the spaces of linear maps of a fixed rank between two fixed vector spaces, in which the boundary added is a divisor with normal crossings. This renders them useful in solving many enumerative problems on linear maps, and they are famous as much for the intricacy of the resulting formulas as for the elegance and symmetry of the underlying geometry."

It was Thaddeus himself that, when revisiting the theory, established the link to linked nets, though they do not appear in his work under this name. In fact, he found many ways to describe the moduli spaces of complete collineations, compared to the "old way," which is beautifully described with applications in [20]. Together with a visiting undergraduate student to IMPA, Olivier Bernard, we wrote the connection with details in Chapter 6, expanding on Thaddeus's exposition. It is a beautiful and remarkable connection and it is tantalizing to think what possible connections and applications linked nets over $\mathbb{Z}^n$-quivers for $n > 1$ could have, beyond those dealt with in this thesis!

In fact, several more concrete questions remain open. First, is it possible to view $\mathbb{LP}(\mathfrak{g})$ in Hilb_C when $\mathfrak{g}$ arises from a degeneration of linear series to a limit curve C with more than two components? Second, is the quiver Grassmannian of subrepresentations of pure dimension $r > 1$ of an exact linked net $\mathfrak{g}$ of finite support over a $\mathbb{Z}^n$-quiver with $n > 1$ Cohen–Macaulay, reduced, of pure dimension? How about its multivariate Hilbert polynomial in the product of Grassmannians, is it equal to that of the diagonal? Surely, these Grassmannians must play an important role in understanding limits of linear series, they are generalizations of Osserman's linked Grassmannians. All of these questions are open even for $n = 2$. And of course, there is the obvious question: Is it possible to extend our work for $n > 2$?

In short, in Chapter 2 we define most of our basic objects of study: quivers, their representations, $\mathbb{Z}^n$-quivers, linked nets, exactness, support, simple bases, hulls, and show how linked nets arise from degenerations of linear series. In Chapter 3, we describe effective loci of $\mathbb{Z}^n$-quivers arising from degenerations. In Chapter 4 we first prove directly that linked nets over $\mathbb{Z}^2$-quivers supported in a triangle admit simple bases, then show that exact linked nets of dimension 1 and finite support over $\mathbb{Z}^n$-quivers admit simple bases, state and prove Theorem B and then give our example of a linked net arising from a degeneration of linear series which does not admit a simple basis. In Chapter 5, we define the linked projective space and prove Theorem A. Finally, in Chapter 6 we describe the connection between complete collineations and linked nets.

Chapter 2

Quiver representations and degenerations of linear series

In this chapter we introduce certain basic definitions for the development of the thesis as, for example, quivers, quiver representations and degenerations of linear series.

2.1 Quivers and their representations

We start with the most basic definition of this chapter.

Definition 1. A **quiver** $Q = (Q_0, Q_1, s, t)$ consists of:

- Sets, Q_0, whose elements we call vertices, and Q_1, whose elements we call arrows;
- Maps $s : Q_1 \longrightarrow Q_0$ and $t : Q_1 \longrightarrow Q_0$ sending an arrow to vertices, which we call its initial and terminal vertices, respectively.

Given an element α in Q_1 we represent it by drawing an arrow from its initial vertex to its terminal vertex, $s(\alpha) \xrightarrow{\alpha} t(\alpha)$.

Definition 2. A **morphism** between two quivers

$$Q = (Q_0, Q_1, s, t) \quad \text{and} \quad Q' = (Q'_0, Q'_1, s', t')$$

is a pair $F = (F_0, F_1)$, where $F_0 : Q_0 \longrightarrow Q'_0$ and $F_1 : Q_1 \longrightarrow Q'_1$ are maps such that the following two diagrams are commutative:

$$\begin{array}{ccc} Q_1 & \xrightarrow{s} & Q_0 \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_0} \\ Q'_1 & \xrightarrow[s']{} & Q'_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} Q_1 & \xrightarrow{t} & Q_0 \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_0} \\ Q'_1 & \xrightarrow[t']{} & Q'_0 \end{array} \tag{2.1}$$

A morphism of quiver can identify vertices, but cannot invert arrows. To be more explicit, we give two examples.

Example 3. Consider the following two quivers

$$Q:\ 1 \xrightarrow{\alpha} 2 \quad \text{and} \quad Q':\ \begin{tikzcd} 3 \arrow[loop right, "\beta"] \end{tikzcd}$$

Let $F = (F_0, F_1) : Q \longrightarrow Q'$ be a morphism of quivers defined by $F_0(1) = F_0(2) = 3$ and $F_1(\alpha) = \beta$. As $F_0 s(\alpha) = s' F_1(\alpha) = 3$ and $F_0 t(\alpha) = t' F_1(\alpha) = 3$, the commutativity of the diagrams in (2.1) holds for F. Thus, it is a quiver morphism which identifies the vertices 1 and 2.

Example 4. In this example, we illustrate that a quiver morphism cannot invert arrows. To this end, consider two quivers Q and Q' defined as

$$Q : 1 \xrightarrow{\ \alpha\ } 2 \quad \text{and} \quad Q' : 3 \xrightarrow{\ \beta\ } 4.$$

When we try to define a morphism

$$F = (F_0, F_1) : Q \longrightarrow Q'$$

by $F_1(\alpha) = \beta$ and $F_0(1) = 4$, $F_0(2) = 3$ we notice that we cannot have commutative diagrams as in (2.1). More precisely,

$$s' F_1(\alpha) = 3 \neq 4 = F_0 s(\alpha) \quad \text{and} \quad t' F_1(\alpha) = 4 \neq 3 = F_0 t(\alpha).$$

As claimed, a morphism of quivers cannot invert arrows.

Given two morphisms of quivers, $(F_0, F_1) : (Q_0, Q_1, s, t) \longrightarrow (Q'_0, Q'_1, s', t')$ and $(F'_0, F'_1) : (Q'_0, Q'_1, s', t') \longrightarrow (Q''_0, Q''_1, s'', t'')$, their composition is defined as

$$(F'_0, F'_1) \circ (F_0, F_1) := (F'_0 \circ F_0, F'_1 \circ F_1) : (Q_0, Q_1, s, t) \longrightarrow (Q''_0, Q''_1, s'', t'').$$

The identity morphism of a quiver Q exists and is given by $\mathrm{id}_Q := (\mathrm{id}_{Q_0}, \mathrm{id}_{Q_1})$. We say that two quivers Q and Q' are **isomorphic** if there exist morphisms $F : Q \to Q'$ and $G : Q' \to Q$ such that $F \circ G = \mathrm{id}_{Q'}$ and $G \circ F = \mathrm{id}_Q$.

Having introduced the category of quivers, whose objects are quivers and whose morphisms are quiver morphisms, we now define quiver representations.

Definition 5. Let $Q = (Q_0, Q_1, s, t)$ be a quiver. A **representation**

$$M = (V_v, \varphi_\alpha)_{v \in Q_0, \alpha \in Q_1}$$

of Q is a collection of k-vector spaces V_v, one for each vertex $v \in Q_0$, and a collection of k-linear maps $\varphi_\alpha : V_{s(\alpha)} \longrightarrow V_{t(\alpha)}$, one for each arrow $\alpha \in Q_1$.

The **dimension vector** of M is defined as $\underline{\dim} M := (\dim V_v)_{v \in Q_0}$. We say that M is of pure (or has pure) dimension if $\dim V_v = r$ for each vertex $v \in Q_0$. In this case we write $\dim M = r$. A representation M is finite-dimensional if each V_v has finite dimension. For further information about the basic theory of quiver representations the reader can consult [18].

To illustrate these definitions, we give examples.

Example 6. Let Q be the quiver

$$
\begin{tikzcd} Q: \; 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\sigma"] \end{tikzcd}
$$

where $Q_0 = \{1,2\}$, $Q_1 = \{\alpha, \sigma\}$, and the maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ are given by $s(\alpha) = t(\sigma) = 1$ and $s(\sigma) = t(\alpha) = 2$. We have a representation of Q as follows:

$$
\begin{tikzcd} M: \; V_1 \arrow[r, bend left, "\varphi_\alpha"] & V_2 \arrow[l, bend left, "\varphi_\sigma"] \end{tikzcd}
$$

where the vector spaces V_1 and V_2 are equal to k^2, and the k-linear maps are given by

$$
\varphi_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \varphi_\sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
$$

Definition 7. Let $M = (M_v, \varphi_\alpha)$ and $M' = (M'_v, \varphi'_\alpha)$ be two representations of a quiver Q. A **morphism of representations** $f : M \longrightarrow M'$ is a collection $(f_v)_{v \in Q_0}$ of linear maps $f_v : M_v \longrightarrow M'_v$ where for each arrow $v_1 \xrightarrow{\alpha} v_2$ of Q_1 the diagram

$$
\begin{tikzcd}
M_{v_1} \arrow[r, "\varphi_\alpha"] \arrow[d, "f_{v_1}"'] & M_{v_2} \arrow[d, "f_{v_2}"] \\
M'_{v_1} \arrow[r, "\varphi'_\alpha"'] & M'_{v_2}
\end{tikzcd}
$$

is commutative.

We say that f is **injective** if all the f_v are injective maps. A **subrepresentation** M' of M is just a representation M' together with an injective morphism $M' \hookrightarrow M$.

Once we fix a quiver Q, the above definitions give us a category $\mathrm{Rep}(Q)$ whose objects are representations of Q and whose morphisms are morphisms of representations.

Definition 8. Let $M = (M_v, \varphi_\alpha)$ and $N = (N_v, \tau_\alpha)$ be two representations of Q. Then

$$M \oplus N = \left(M_v \oplus N_v, \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \tau_\alpha \end{bmatrix} \right)_{v \in Q_0,\, \alpha \in Q_1}$$

is a representation of Q called the **direct sum** of M and N.

A representation $M \in \operatorname{Rep} Q$ is **indecomposable** if whenever $M \cong M_1 \oplus M_2$ either $M_1 = 0$ or $M_2 = 0$.

Although we define representations using k-vector spaces, that is, we define the category $\operatorname{Rep}_k(Q)$ of representations of Q over the field k, we can replace "k-vector space" and "linear maps" by

1. "R modules" and " "morphisms of R-modules", or
2. "$\mathscr{O}_X$-modules" and "morphisms of $\mathscr{O}_X$-modules",

to obtain $\operatorname{Rep}_R(Q)$, the category of representations of Q on R-modules and $\operatorname{Rep}_{\mathscr{O}_X}(Q)$, the category of representations of Q on $\mathscr{O}_X$-modules.

We will now introduce the quivers we will study in the thesis.

For each non-negative integers d and n define

$$\mathbb{Z}^{n+1}(d) := \{(d_0, \cdots, d_n) \in \mathbb{Z}^{n+1} \mid \sum d_i = d\} \text{ and}$$
$$\mathbb{N}^{n+1}(d) := \{(d_0, \cdots, d_n) \in \mathbb{Z}^{n+1}(d) \mid d_i \geq 0 \text{ for each } i\}.$$

Let $u_0, \cdots, u_n$ be elements in $\mathbb{Z}^{n+1}(0)$ such that their sum is null and any proper subset of them is $\mathbb{Q}$-linearly independent. To an element u in $\mathbb{Z}^{n+1}(d)$ we associate a quiver $Q(u; u_0, \cdots, u_n)$ whose set of vertices is

$$Q_0 := u + \mathbb{Z}u_0 + \cdots + \mathbb{Z}u_n \subseteq \mathbb{Z}^{d+1}(d)$$

and whose set of arrows $A \subseteq Q_0 \times Q_0$ is such that (u, u') is an arrow from u to u' if

$$u' = u + u_i$$

for some i in $\{0, \cdots, n\}$. It is a $\mathbb{Z}^n$-quiver, according to the following definition.

Definition 9. Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $n \in \mathbb{N}$. A $\mathbb{Z}^n$-**structure** on Q is a decomposition of the set of arrows Q_1 into subsets $A_0, \cdots, A_n$ satisfies the following three conditions:

1. For each vertex v in Q_0 and each $i = 0, \cdots, n$ there is a unique arrow in A_i leaving it.

For a path γ in Q, let $\gamma(i)$ be the number of arrows of A_i that is contained in γ.

2. For each two distinct vertices v_1 and v_2 in Q_0 there is a path γ in Q connecting v_1 to v_2 such that $\gamma(i) = 0$ for some i.

3. Every two paths γ_1 and γ_2 in Q leaving the same vertex arrive at the same vertex if and only if $\gamma_1(i) - \gamma_2(i)$ is constant for $i \in \{0, \cdots, n\}$.

A quiver with a $\mathbb{Z}^n$-structure is called a $\mathbb{Z}^n$**-quiver**.

For a fixed n, by [17, prop. 3.1] all $\mathbb{Z}^n$-quivers are isomorphic. Given a $\mathbb{Z}^n$-quiver Q, a path γ for which $\gamma(i) = 0$ for some i is called **admissible**. Also, a path γ for which $\gamma(i) \leq 1$ for all i is called **simple**.

The quiver $Q(u; u_0, \cdots, u_n)$, is a $\mathbb{Z}^n$-quiver with A_i being the set of arrows $v_1 \longrightarrow v_2$ suth that

$$v_2 = v_1 + u_i.$$

2.2 Degenerations of linear series

Let B be the spectrum of $\mathbb{C}[[T]]$ with η and θ being its generic and special point, respectively. Let C be a nodal curve and $X_0, \ldots, X_n$ its irreducible components.

Definition 10. A **smoothing** of C is a flat, projective morphism $\pi : \mathfrak{X} \to B$, with smooth generic fibre $\mathfrak{X}_\eta$, and an isomorphism $\mathfrak{X}_\theta \cong C$. We say that the smoothing is **regular** when $\mathfrak{X}$ is regular.

Let $\pi : \mathfrak{X} \to B$ be a regular smoothing of C. We identify the special fibre with C. Let L_η be a line bundle on the generic fibre $\mathfrak{X}_\eta$ with degree d. As $\mathfrak{X}$ is regular, each X_i is a Cartier divisor and there exists a line bundle extension $\mathscr{L}$ of L_η to $\mathfrak{X}$. For each $D = \sum n_i X_i$, for integers n_i, the sheaf $\mathscr{L}(D) := \mathscr{L} \otimes \mathscr{O}_{\mathfrak{X}}(D)$ is also a line bundle extension of L_η. Denote

$$\mathscr{L}_v := \mathscr{L}(D)$$

where v is the multidegree of $\mathscr{L}(D)_{|C}$. Letting $u_i := (\cdots, \deg \mathscr{O}_{\mathfrak{X}}(X_i)_{|X_j}, \cdots)$ and $u = (\cdots, \deg \mathscr{L}_{|X_j}, \cdots)$ be the multidegrees of $\mathscr{O}_{\mathfrak{X}}(X_i)_{|C}$ for $i = 0, \ldots, n$ and $\mathscr{L}_{|C}$ respectively, we have $v = u + \sum n_i u_i$. We also write $v = D \cdot u$. Notice that $v \in \mathbb{Z}^{n+1}(d)$. Put $L_v := \mathscr{L}_{v|C}$.

For each $D = \sum n_i X_i$ with $\min\{n_i\} = 0$, twisting by $\mathscr{O}_{\mathfrak{X}}(D)$ gives us natural morphisms

$$\varphi^{v_1}_{v_2} : \mathscr{L}_{v_1} \longrightarrow \mathscr{L}_{v_2}, \tag{2.2}$$

where $v_2 = D \cdot v_1$, which are nonzero when restricted to C. The composition $\varphi_{v_1}^{v_2}\varphi_{v_2}^{v_1}$ is the multiplication by T^m, where T is the uniformizer parameter of B and $m = \max\{n_i\}$. The map above induces (with abuse of notation)

$$\varphi_{v_2}^{v_1} : L_{v_1} \longrightarrow L_{v_2}.$$

If $D = D_1 + D_2$ for effective divisors D_1 and D_2, then $\varphi_{v_3}^{v_1} = \varphi_{v_3}^{v_2}\varphi_{v_2}^{v_1}$ where $v_2 = D_1 \cdot v_1$ and $v_3 = D_2 \cdot v_2$.

Let V_η be a subspace of $H^0(\mathfrak{X}_\eta, \mathscr{L}_\eta)$. For each extension $\mathscr{L}_v$ of L_η there is an extension $\mathscr{V}_v$ of V_η given by

$$\mathscr{V}_v := \{s \in H^0(\mathfrak{X}, \mathscr{L}_{v'}) \mid s_{|\mathfrak{X}_\eta} \in V_\eta\}.$$

Set $V_v := \mathscr{V}_{v|C} \subseteq H^0(C, L_v)$. For each $D = \sum n_i X_i$ with $\min\{n_i\} = 0$ the map $\varphi_{v_1}^{v_2}$ induces a map

$$\varphi_{v_2}^{v_1} : \mathscr{V}_{v_1} \longrightarrow \mathscr{V}_{v_2},$$

which induces upon restriction:

$$\varphi_{v_2}^{v_1} : V_{v_1} \longrightarrow V_{v_2}.$$

Let $Q_0 \subseteq \mathbb{Z}^{n+1}(d)$ be the subset consisting of the multidegrees on C of the extensions of L_η. Then $Q_0 = u + \mathbb{Z}u_0 + \cdots + \mathbb{Z}u_n$. Let $Q_1 \subset Q_0 \times Q_0$ be the subset of pairs (v_1, v_2) such that $v_2 = X_i \cdot v_1$ for some $i = 0, \cdots, n$. With this data, we have a $\mathbb{Z}^n$-quiver:

$$Q = (Q_0, Q_1, s, t) = Q(u; u_0, \cdots, u_n). \tag{2.3}$$

Furthermore, the V_v for $v \in Q_0$ and the $\varphi_{v_2}^{v_1} \colon V_{v_1} \to V_{v_2}$ for $(v_1, v_2) \in Q_1$ form a representation $\mathfrak{g}$ of Q. It is a special representation: It can be shown [17, § 2] to be an exact linked net of vector spaces over Q with (finite) support on the **effective locus** of Q, the set $Q_0 \cap \mathbb{N}^{n+1}(d)$, according to Definitions 11, 12 and 13 below.

Definition 11. Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $A_0, \cdots, A_n$ be a decomposition of Q_1 giving Q a $\mathbb{Z}^n$-structure. A **linked net of vector spaces** $\mathfrak{g} = (V_v, \varphi_\alpha)$ is a quiver representation of Q of pure dimension satisfying the following three properties.

1. If γ_1 and γ_2 are admissible paths connecting the same two vertices, then $\varphi_{\gamma_1} = \varphi_{\gamma_2}$.

2. If γ is a non-admissible path, then $\varphi_\gamma = 0$.

3. If γ_1 and γ_2 are simple admissible paths leaving the same vertex such that $\gamma_1(i) = 0$ or $\gamma_2(i) = 0$ for each i, then $\mathrm{Ker}(\varphi_{\gamma_1}) \cap \mathrm{Ker}(\varphi_{\gamma_2}) = 0$.

Given an admissible path γ connecting v to v', as φ_γ does not depend on the choice of such a path, we write $\varphi^v_{v'} := \varphi_\gamma$. In fact, φ_γ depends only on $I_\gamma := \{i \mid \gamma(i) > 0\}$ and the initial vertex v of γ; thus we also write $\varphi^v_{I_\gamma} := \varphi_\gamma$, and we drop the superscript v if v is clear from the context. Again because of the independence from the path, given $s \in V_v$ we write $s|_{V_{v'}} := \varphi^v_{v'}(s)$.

Notice that $\varphi^{v_2}_{v_1}\varphi^{v_1}_{v_2} = 0$ for each pair (v_1, v_2) of distinct vertices of the quiver.

Definition 12. Let $Q = (Q_0, Q_1, s, t)$ be a $\mathbb{Z}^n$-quiver and $\mathfrak{g}$ be a linked net of vector spaces over it. Two vertices $v_1, v_2 \in Q_0$ are called **neighbors** if there is a simple path connecting v_1 to v_2. The linked net $\mathfrak{g}$ is called **exact** if $\operatorname{Im}\varphi^{v_1}_{v_2} = \operatorname{Ker}(\varphi^{v_2}_{v_1})$ for each pair (v_1, v_2) of distinct neighboring vertices of the quiver.

Definition 13. Let $Q = (Q_0, Q_1, s, t)$ be a $\mathbb{Z}^n$-quiver and $\mathfrak{g}$ be a linked net of vector spaces over it. We say that $\mathfrak{g}$ has **support** on a subset $H \subseteq Q_0$ if for each $v \in Q_0$ there exists $v' \in H$ such that $\varphi^{v'}_{v}$ is an isomorphim. If H is finite, we say that $\mathfrak{g}$ has **finite support** (on H).

The linked net of vector spaces $\mathfrak{g}$ over the quiver Q in (2.3) arising from a degeneration of linear series is indeed a subrepresentation of the representation of global sections induced by the representation of the quiver, $v \mapsto L_v$ and $(v_1, v_2) \mapsto \varphi^{v_1}_{v_2} : L_{v_1} \to L_{v_2}$, in the category of line bundles. You can see this in [15, def. 2.21]. Here we focus on the linear aspects of the degeneration though.

We need a few more definitions and results.

Definition 14. Let $Q = (Q_0, Q_1, s, t)$ be a $\mathbb{Z}^n$-quiver and H a non-empty subset of Q_0. Let $P(H)$ be the set of all $v \in Q_0$ such that for each $i = 0, \cdots, n$ there are $v' \in H$ and a path γ from v' to v with $\gamma(i) = 0$. We call $P(H)$ the **hull** of H.

Proposition 15 (Esteves et al, [8, Prop. 3.4])**.** Let $Q = (Q_0, Q_1, s, t)$ be a $\mathbb{Z}^n$-quiver and $H \subseteq Q_0$ be non-empty. Then the following three statements hold:

1. $H \subseteq P(H)$.
2. If H is finite, so is $P(H)$.
3. $P(P(H)) = P(H)$.

Lemma 16. Let $\mathfrak{g}$ be a representation over a $\mathbb{Z}^n$-quiver Q satisfying items 1 and 2 in Definition 11. Then each of the following statements imply the next:

1. $\mathfrak{g}$ is a linked net

2. For each vertex v of Q and each $i = 0, \dots, n$, we have

$$\operatorname{Ker}(\varphi_a) \cap \operatorname{Im}(\varphi_b) = 0,$$

where a is the i-arrow leaving v and b is the i-arrow arriving at v.

3. For each admissible path γ in Q,

$$\operatorname{Ker}(\varphi_\gamma) = \operatorname{Ker}(\varphi_\mu), \tag{2.4}$$

where μ is any simple admissible path leaving the same vertex of Q as γ such that $\mu(i) > 0$ if and only if $\gamma(i) > 0$ for each $i = 0, \dots, n$.

If $\mathfrak{g}$ is exact then all three statements are equivalent.

Proof. Statement 1 clearly implies Statement 2, as $\varphi_\mu \varphi_b = 0$ for each simple admissible path μ leaving v satisfying $\mu(j) = 1$ for $j \neq i$.

Assume Statement 2. We make a claim. Let μ be a nontrivial admissible path and $i \in \{0, \dots, n\}$ such that $\mu(i) > 0$. Let w be the final vertex of μ and a an arrow in A_i leaving w. We claim that

$$\operatorname{Ker}(\varphi_a) \cap \operatorname{Im}(\varphi_\mu) = 0.$$

Indeed, let b the arrow in A_i arriving in w. Let v be its initial vertex. Let β be a path arriving in v satisfying $\beta(j) = \mu(j)$ for each $j \neq i$ and $\beta(i) = \mu(i) - 1$. The concatenation of β with b is an admissible path that leaves and arrives at the same vertex as μ, whence $\varphi_\mu = \varphi_b \varphi_\beta$, and so $\operatorname{Im}(\varphi_\mu) \subseteq \operatorname{Im}(\varphi_b)$. We may thus assume $\mu = b$ and apply Statement 2.

We prove Statement 3. Let v be a vertex of Q and γ an admissible path in Q leaving v. Let μ be a path as in Statement 3. We proceed by induction on the length of γ. If γ has length 0 or 1, Equation (2.4) holds trivially. It holds as well if γ is simple, as then $\varphi_\gamma = \varphi_\mu$. So we may assume there is $i \in \{0, 1, \dots, n\}$ such that $\gamma(i) > 2$. There is an admissible path β leaving v whose last arrow b is of type i such that $\gamma(j) = \beta(j)$ for each $j \neq i$ and $\gamma(i) = \beta(i) + 1$. Let a be an arrow of type i leaving the final vertex of b. Then $\varphi_\gamma = \varphi_a \varphi_\beta$. By Statement 3, ora rather the claim, $\operatorname{Ker}(\varphi_a \varphi_\beta) = \operatorname{Ker}(\varphi_\beta)$. By induction, $\operatorname{Ker}(\varphi_\beta) = \operatorname{Ker}(\varphi_\alpha)$ for any simple admissible path α leaving v such that $\alpha(j) > 0$ if and only if $\beta(j) > 0$, or equivalently, if and only if $\gamma(j) > 0$ for each $j = 0, \dots, n$. We may thus put $\alpha := \mu$. Statement 3 is proved.

Assume now that $\mathfrak{g}$ is exact and Statement 3 holds. Let γ_1 and γ_2 be two simple admissible paths in Q leaving the same vertex such that $\gamma_1(i) = 0$ or $\gamma_2(i) = 0$ for each i. Let v be their initial vertex. To show $\mathfrak{g}$ is a linked net we may assume γ_1 and γ_2 are both nontrivial. Let μ_2 be a simple admissible path arriving at v such that $\gamma_2(i) + \mu_2(i) = 1$ for every i. Since $\mathfrak{g}$ is exact, $\mathrm{Ker}(\varphi_{\gamma_2}) = \mathrm{Im}(\varphi_{\mu_2})$. Thus

$$\mathrm{Ker}(\varphi_{\gamma_1}) \cap \mathrm{Ker}(\varphi_{\gamma_2}) = \varphi_{\mu_2}\big(\mathrm{Ker}(\varphi_{\gamma_1}\varphi_{\mu_2})\big).$$

Now, for each i such that $\gamma_1(i) = 1$ we must have $\gamma_2(i) = 0$, and hence $\mu_2(i) = 1$. Then Statement 3 implies that

$$\mathrm{Ker}(\varphi_{\gamma_1}\varphi_{\mu_2}) = \mathrm{Ker}(\varphi_{\mu_2}),$$

and hence $\mathrm{Ker}(\varphi_{\gamma_1}) \cap \mathrm{Ker}(\varphi_{\gamma_2}) = 0$. Statement 1 is proved. □

Let (v_1, v_2) and (v_2, v_3) be two arrows in the same A_i. If $\mathfrak{g}$ is exact, then

$$\mathrm{Im}\,\varphi_{v_2}^{v_1} \cap \mathrm{Ker}\varphi_{v_3}^{v_2} = 0.$$

Indeed, the equality is just a particular case of Property (3) in Definition 11. This implies that if $\varphi_{v_2}^{v_1}$ is an isomorphism then so is $\varphi_{v_3}^{v_2}$ is, and if $\varphi_{v_3}^{v_2}$ is zero then so is $\varphi_{v_1}^{v_2}$.

Lemma 17. Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver. Let γ_1 and γ_2 be two simple admissible paths leaving a vertex v. Let $I_\ell := \{i \,|\, \gamma_\ell(i) > 0\}$ for $\ell = 1, 2$ and put $I := I_1 \cap I_2$. Then

$$\mathrm{Ker}\varphi_{I_1}^{v} \cap \mathrm{Ker}\varphi_{I_2}^{v} = \mathrm{Ker}\varphi_{I}^{v}.$$

Proof. Of course, $\varphi_{I_\ell}^{v} = \varphi_{\gamma_\ell}$ for $\ell = 1, 2$. We can write $\varphi_{\gamma_1} = \varphi_{\gamma_1'}\varphi_I^v$ and $\varphi_{\gamma_2} = \varphi_{\gamma_2'}\varphi_I^v$ with γ_1' and γ_2' satisfying $\gamma_1'(i) > 0$ only if $\gamma_2'(i) = 0$. Clearly, $\mathrm{Ker}\varphi_{\gamma_1} \cap \mathrm{Ker}\varphi_{\gamma_2} \supseteq \mathrm{Ker}\varphi_I^v$. But also, since $\mathrm{Ker}\varphi_{\gamma_1'} \cap \mathrm{Ker}\varphi_{\gamma_2'}$ is trivial, the reverse inclusion holds. □

Definition 18. Let $\mathfrak{g}$ be a linked net of vector spaces of dimension r over a $\mathbb{Z}^n$-quiver Q. A collection of vertices $v_1, \cdots, v_m$ of Q and vectors $s_i \in V_{v_i}$ for $i = 1, \cdots, m$ such that

$$\{s_1|_{V_v}, \cdots, s_m|_{V_v}\} \text{ generates } V_v \text{ for each vertex } v \text{ of } Q$$

is called a **set of generators** of $\mathfrak{g}$. If $m = r$, it is called a **simple basis**.

Chapter 3

Classification of effective loci

The $\mathbb{Z}^n$-quivers arising from degenerations of linear series are the $Q := Q(u; u_0, \dots, u_n)$ in (2.3). Here, $u \in \mathbb{Z}^{n+1}(d)$ for a non-negative integer d. And $u_0, \dots, u_n \in \mathbb{Z}^{n+1}(0)$ are elements whose sum is zero and such that any proper subset of them is $\mathbb{Q}$-linearly independent. The vertex set of Q is

$$Q_0 := u + \mathbb{Z}u_0 + \cdots + \mathbb{Z}u_n \subseteq \mathbb{Z}^{n+1}(d),$$

and the arrow set Q_1 can be viewed as the subset of $Q_0 \times Q_0$ of pairs (u, u') such that $u' = u + u_i$ for some i. Its **effective locus** is the set $H := Q_0 \cap \mathbb{N}^{n+1}(d)$. This is a finite set. Furthermore, every linked net of vector spaces over Q that arises from a degeneration of linear series has support in H. It is thus interesting to describe the effective locus of Q.

Furthermore, we would like to describe the **effective quiver** associated to Q. This is the quiver obtained from Q by adding to the full subquiver of Q with support on H an arrow between each two vertices of H that are neighbors but cannot be connected by a simple admissible path passing only through vertices in H.

The quivers arising from degenerations to a curve of compact type with two components are the $Q(u; u_0, u_1)$ for $u := (d, 0)$ and $u_0 = -u_1 = (-1, 1)$. The effective locus is $\mathbb{N}^2(d)$ and the associated effective quiver is as described below:

$$\mathbb{N}^2(d): \quad (d,0) \longleftrightarrow (d-1,1) \qquad \cdots \qquad (1,d-1) \longleftrightarrow (0,d)$$

Replacing each edge of the Dynkin diagram

$$A_{d+1}: 0 \;\text{——}\; 1 \;\text{——}\; 2 \;\text{——}\; \cdots \;\text{——}\; d-1 \;\text{——}\; d$$

by two arrows in opposite directions we obtain that quiver. You can consult [18] for more details concerning the Dynking diagrams.

The quivers arising from degenerations to a curve of compact type with three components are the $Q(u;u_0,u_1,u_2)$ for $u := (d,0,0)$ and $u_0 = (-1,1,0)$, $u_1 = (1,-2,1)$ and $u_3 = (0,1,-1)$. The effective locus is $\mathbb{N}^3(d)$ and the associated effective quiver is as described below:

$$\mathbb{N}^3(d):\quad \begin{tikzcd} (0,0,d) \arrow[d, leftrightarrow] \\ (0,1,d-1) \arrow[r, leftrightarrow] & (1,0,d-1) \\ \vdots & \vdots \\ (0,d-1,1) \arrow[d] & (1,d-2,1) \arrow[l] \arrow[d] & \cdots & (d-1,1) \arrow[d, leftrightarrow] \\ (0,d,0) \arrow[ur] & (1,d-1,0) \arrow[l] & \cdots & (d-1,1,0) \arrow[r, leftrightarrow] & (d,0,0) \end{tikzcd}$$

It is indeed true that the effective loci of quivers arising from degenerations to curves of compact type are the $\mathbb{N}^{n+1}(d)$, where $n+1$ is the number of components of the curve. The situation is more interesting for curves not of compact type.

For instance, consider degenerations to a "triangular" curve, the nodal curve with three components and a single intersection between each two components. The associated quivers are the $Q := Q(v;v_0,v_1,v_2)$ for $v := (a,b,c) \in \mathbb{N}^3(d)$ and

$$v_0 = (-2,1,1),\ v_1 = (1,-2,1) \text{ and } v_2 = (1,1,-2) \tag{3.1}$$

Here is the picture of the arrows arriving and leaving (a,b,c), indicating their types:

$$\begin{tikzcd} & (a-2,b+1,c+1) & \\ (a-1,b-1,c+2) \arrow[dr, "\{2\}"'] & & (a-1,b+2,c-1) \arrow[dl, "\{1\}"] \\ & (a,b,c) \arrow[uu, "\{0\}"] \arrow[dl, "\{1\}"'] \arrow[dr, "\{2\}"] & \\ (a+1,b-2,c+1) & & (a+1,b+1,c-2) \\ & (a+2,b-1,c-1), \arrow[uu, "\{0\}"] & \end{tikzcd} \tag{3.2}$$

We will classify, in terms of the v, the effective quivers associated to the Q. Rather than on the full knowledge of v, the classification depends on certain numerical invariants derived from v, first of all the degree of the quiver.

Definition 19. The **degree** of Q, and of any of its full subquivers, is $d = a+b+c$.

Lemma 20. For each integer $d \geq 1$ there are exactly three effective loci of quivers Q of degree d. They are disjoint. If $d \geq 2$, one and only one of the vectors $(0,0,d)$, $(0,1,d-1)$ and $(0,2,d-2)$ belongs to each of them.

Proof: For $d = 1$, the effective loci are $\{(1,0,0)\}$, $\{(0,1,0)\}$ and $\{(0,0,1)\}$. If $d \geq 2$, in the effective locus of each Q there is always a vertex of the form $(0,a,d-a)$ for $0 \leq a \leq 2$. Since the arrows are as described in (3.2), a is uniquely determined. So, if $d \geq 2$ there are exactly three effective loci, those obtained "starting from" the vertices $(0,0,d)$, $(0,1,d-1)$ and $(0,2,d-2)$. $\square$

Due to the lemma, for each integer $d \geq 1$ and each $p = 0,1,2$, we will use the notation Q_p^d for the effective quiver associated to $Q := Q(v;v_0,v_1,v_2)$ for $v := (0,p,d-p)$ and v_0,v_1,v_2 as in (3.1).

Proposition 21. Let d be a positive integer. If $d \not\equiv 0 \pmod 3$, then the quivers Q_0^d, Q_1^d, Q_2^d are isomorphic. If $d \equiv 0 \pmod 3$, then there are just two of them up to isomorphism.

Proof: The three quivers are "generated" by the vertices $(0,0,d)$, $(0,1,d-1)$ and $(0,2,d-2)$, respectively.

The key observation in the proof is this: Given $(a,b,c) \in Q_p^d$, any effective vertex of the form $(a,b+3m,c-3m)$ or of the form $(a+3m,b,c-3m)$ is in Q_p^d as well. We call the passage from (a,b,c) to $(a,b+3,c-3)$ or $(a+3,b,c-3)$ a "two step" procedure. In fact, the vertex $(a-1,b+2,c-1)$ is connected to each of (a,b,c) and $(a,b+3,c-3)$ by an arrow, while $(a+2,b-1,c-1)$ is connected to each of (a,b,c) and $(a+3,b,c-3)$ by an arrow.

If $d = 3m+1$, consider the table of vertices belonging to each quiver:

$$\begin{array}{ccc}
Q_0^d & Q_1^d & Q_2^d \\[1ex]
(0,0,3m+1) & (0,1,3m) & (0,2,3m-1) \\
 & \Big\downarrow{\scriptstyle 2m \text{ steps}} & \Big\downarrow{\scriptstyle \{1\}} \\
 & (0,3m+1,0) & (1,0,3m) \\
 & & \Big\downarrow{\scriptstyle 2m \text{ steps}} \\
 & & (3m+1,0,0)
\end{array}$$

Due to the total symmetry of the problem, we see that $Q_1^d \cong Q_2^d \cong Q_3^d$.

If $d = 3m+2$, a similar argument works as well:

$$\begin{tikzcd}
Q_0^d & Q_1^d & Q_2^d \\
(0,0,3m+2) & (0,1,3m+1) & (0,2,3m) \arrow[d, "2m \text{ steps}"] \\
 & (2,0,3m) \arrow[u, "\{0\}"'] \arrow[d, "2m \text{ steps}"] & (0,3m+2,0) \\
 & (3m+2,0,0) &
\end{tikzcd}$$

As before, we conclude that $Q_1^d \cong Q_2^d \cong Q_3^d$.

For the last case we need a little more effort. Let $d = 3m$. Consider the table:

$$\begin{tikzcd}
Q_1^d & Q_2^d \\
(0,1,3m-1) & (0,2,3m-2) \arrow[d, "\{1\}"] \\
 & (1,0,3m-1)
\end{tikzcd}$$

Due to the symmetry, $Q_1^d \cong Q_2^d$. Now, denote by $V(Q_p^d)$ the set of vertices of Q_p^d. Fixing the first coordinate of a vertex and counting how many vertices there are in $V(Q_0^d)$ with that fixed first coordinate we have:

$$
\begin{array}{ccc}
(a,b,c) & & \begin{array}{c}\text{number of vertices}\\ \text{with first entry } a\end{array} \\
(0,0,3m) & \text{—————} & \left\lfloor \frac{3m}{3} \right\rfloor + 1 \\
\Big\downarrow{\scriptstyle \{2\}} & & \\
(1,1,3m-2) & \text{—————} & \left\lfloor \frac{3m-2}{3} \right\rfloor + 1 \\
\Big\downarrow{\scriptstyle \{2\}} & & \\
(2,2,3m-4) & \text{—————} & \left\lfloor \frac{3m-4}{3} \right\rfloor + 1 \\
\Big\downarrow{\scriptstyle \{1\}} & & \\
(3,0,3m-3) & \text{—————} & \left\lfloor \frac{3m-3}{3} \right\rfloor + 1 \\
\Big\downarrow{\scriptstyle \{2\}} & & \\
\vdots & & \vdots \\
\Big\downarrow{\scriptstyle \{2\}} & & \\
(3m-3,0,3) & \text{—————} & \left\lfloor \frac{3}{3} \right\rfloor + 1 \\
\Big\downarrow{\scriptstyle \{2\}} & & \\
(3m-2,1,1) & \text{—————} & \left\lfloor \frac{1}{3} \right\rfloor + 1 \\
\Big\downarrow{\scriptstyle \{0\}} & & \\
(3m-4,2,2) & \text{—————} & \left\lfloor \frac{2}{3} \right\rfloor + 1 \\
{\scriptstyle \{0\}\{0\}}\Big\uparrow & & \\
(3m,0,0) & \text{—————} & \left\lfloor \frac{0}{3} \right\rfloor + 1.
\end{array}
$$

Summing all these contributions we obtain

$$
\begin{aligned}
\#V(Q_0^d) &= \left\lfloor \frac{3m}{3} \right\rfloor + 1 + \sum_{i=0}^{3m-2} \left(\left\lfloor \frac{i}{3} \right\rfloor + 1 \right) = \left\lfloor \frac{3m}{3} \right\rfloor + 3m + \sum_{i=0}^{3m-2} \left\lfloor \frac{i}{3} \right\rfloor \\
&= 3m + 1 + \left\lfloor \frac{3m-1}{3} \right\rfloor + \sum_{i=0}^{3m-2} \left\lfloor \frac{i}{3} \right\rfloor = d + 1 + \sum_{i=0}^{d-1} \left\lfloor \frac{i}{3} \right\rfloor .
\end{aligned}
$$

Proceeding the same way for Q_1^d and Q_2^d we find that

$$\#V(Q_1^d) = \#V(Q_2^d) = \#V(Q_0^d) - 1.$$

This shows that $Q_0^d \ncong Q_1^d \cong Q_2^d$. $\square$

From the proposition we obtain, as a corollary, the number of vertices of each Q_p^d.

Corollary 22. Let d be a positive integer. If $d \not\equiv 0 \pmod 3$ then

$$\#V(Q_p^d) = \frac{(d+2)(d+1)}{6}$$

for every p. If $d \equiv 0 \pmod 3$ then

$$\#V(Q_0^d) = \left\lceil \frac{(d+2)(d+1)}{6} \right\rceil \quad \text{and} \quad \#V(Q_1^d) = \#V(Q_2^d) = \left\lfloor \frac{(d+2)(d+1)}{6} \right\rfloor.$$

In particular, the growth of the number of vertices of Q_p^d is quadratic on d.

Proof: Observe first that for each d the Q_p^d are disjoint and cover $\mathbb{N}^3(d)$. Thus

$$\sum_{p=0}^{2} \#V(Q_p^d) = \#\mathbb{N}^3(d) = \binom{d+2}{2}. \tag{3.3}$$

By Proposition 21, $Q_0^d \cong Q_1^d \cong Q_2^d$, whenever $d \not\equiv 0 \pmod 3$. In particular, $\#V(Q_p^d)$ does not depend on p. The first statement of the corollary follows now from (3.3).

Suppose $d = 3m$. By Proposition 21,

$$\#V(Q_0^d) - 1 = \#V(Q_1^d) = \#V(Q_2^d)$$

Thus, it follows from (3.3) that

$$3\#V(Q_1^d) = \frac{(d+2)(d+1)}{2} - 1,$$

that is,

$$\#V(Q_1^d) = \#V(Q_2^d) = \left\lfloor \frac{(d+2)(d+1)}{6} \right\rfloor.$$

As $d \equiv 0 \pmod 3$, it follows that

$$\frac{(d+2)(d+1)}{2} \equiv 1 \pmod 3,$$

which allows us to write

$$\#V(Q_0^d) = \left\lceil \frac{(d+2)(d+1)}{6} \right\rceil.$$

$\square$

3.1 Pictures of the quivers Q_p^d

Let d be a positive integer. If d is not divisible by 3, the quivers Q_p^d are isomorphic; we use Q^d to indicate one of them. If $d = 3m$ we use Q_1^{3m} to indicate Q_2^{3m} as well, as they are isomorphic.

Here we list some of these quivers.

Example 23. Here you see the quivers Q_0^{3m} for $m = 1, 2, 3$.

$$
Q_0^3 : \quad
\begin{tikzcd}
(0,3,0) \arrow[dr, bend left] & & (0,0,3) \arrow[dl, bend left] \\
& (1,1,1) \arrow[ul, bend left] \arrow[ur, bend left] \arrow[d, bend left] & \\
& (3,0,0) \arrow[u, bend left] &
\end{tikzcd}
$$

$$
Q_0^6 : \quad
\begin{tikzcd}
(0,6,0) \arrow[dr, bend left] & & (0,3,3) \arrow[dl] \arrow[dr] & & (0,0,6) \arrow[dl, bend left] \\
& (1,4,1) \arrow[ul, bend left] \arrow[dr] & & (1,1,4) \arrow[ur, bend left] \arrow[dl] & \\
& & (2,2,2) \arrow[uu] \arrow[dl] \arrow[dr] & & \\
& (3,3,0) \arrow[uu] \arrow[dr] & & (3,0,3) \arrow[uu] \arrow[dl] & \\
& & (4,1,1) \arrow[uu] \arrow[d, bend left] & & \\
& & (6,0,0) \arrow[u, bend left] & &
\end{tikzcd}
$$

$$
Q_0^9 : \quad
\begin{tikzcd}
(0,9,0) \arrow[dr, bend left] & & (0,6,3) \arrow[dl] \arrow[dr] & & (0,3,6) \arrow[dl] \arrow[dr] & & (0,0,9) \arrow[dl, bend left] \\
& (1,7,1) \arrow[ul, bend left] \arrow[dr] & & (1,4,4) \arrow[dl] \arrow[dr] & & (1,1,7) \arrow[ur, bend left] \arrow[dl] & \\
& & (2,5,2) \arrow[uu] \arrow[dl] \arrow[dr] & & (2,2,5) \arrow[uu] \arrow[dl] \arrow[dr] & & \\
& (3,6,0) \arrow[uu] \arrow[dr] & & (3,3,3) \arrow[uu] \arrow[dl] \arrow[dr] & & (3,0,6) \arrow[uu] \arrow[dl] & \\
& & (4,4,1) \arrow[uu] \arrow[dr] & & (4,1,4) \arrow[uu] \arrow[dl] & & \\
& & & (5,2,2) \arrow[uu] \arrow[dl] \arrow[dr] & & & \\
& & (6,3,0) \arrow[uu] \arrow[dr] & & (6,0,3) \arrow[uu] \arrow[dl] & & \\
& & & (7,1,1) \arrow[uu] \arrow[d, bend left] & & & \\
& & & (9,0,0) \arrow[u, bend left] & & &
\end{tikzcd}
$$

Example 24. Here you see the quivers Q_1^{3m} for $m = 1, 2, 3$.

$$Q_1^3 : \begin{tikzcd}
(0,2,1) \arrow[dr] & \\
& (1,0,2) \arrow[dl] \\
(2,1,0) \arrow[uu] &
\end{tikzcd}$$

$$Q_1^6 : \begin{tikzcd}
(0,5,1) \arrow[dr] & & (0,2,4) \arrow[dl] \arrow[dr] & \\
& (1,3,2) \arrow[dl] \arrow[dr] & & (1,0,5) \arrow[dl] \\
(2,4,0) \arrow[uu] \arrow[dr] & & (2,1,3) \arrow[uu] \arrow[dl] & \\
& (3,2,1) \arrow[uu] \arrow[dr] & & \\
& & (4,0,2) \arrow[uu] \arrow[dl] & \\
& (5,1,0) \arrow[uu] & &
\end{tikzcd}$$

$$Q_1^9 : \begin{tikzcd}
(0,8,1) \arrow[dr] & & (0,5,4) \arrow[dl] \arrow[dr] & & (0,2,7) \arrow[dl] \arrow[dr] & \\
& (1,6,2) \arrow[dl] \arrow[dr] & & (1,3,5) \arrow[dl] \arrow[dr] & & (1,0,8) \arrow[dl] \\
(2,7,0) \arrow[uu] \arrow[dr] & & (2,4,3) \arrow[uu] \arrow[dl] \arrow[dr] & & (2,1,6) \arrow[uu] \arrow[dl] & \\
& (3,5,1) \arrow[uu] \arrow[dr] & & (3,2,4) \arrow[uu] \arrow[dl] \arrow[dr] & & \\
& & (4,3,2) \arrow[uu] \arrow[dl] \arrow[dr] & & (4,0,5) \arrow[uu] \arrow[dl] & \\
& (5,4,0) \arrow[uu] \arrow[dr] & & (5,1,3) \arrow[uu] \arrow[dl] & & \\
& & (6,2,1) \arrow[uu] \arrow[dr] & & & \\
& & & (7,0,2) \arrow[uu] \arrow[dl] & & \\
& & (8,1,0) \arrow[uu] & & &
\end{tikzcd}$$

Example 25. Here you see Q^{3m+1} for $m = 0, 1, 2$.

$$Q^1 : \ (1,0,0)$$

$$Q^4 : \begin{tikzcd}
& (0,2,2) \arrow[dl] \arrow[dr] & \\
(1,3,0) \arrow[dr] & & (1,0,3) \arrow[dl] \\
& (2,1,1) \arrow[uu] \arrow[d, bend left] & \\
& (4,0,0) \arrow[u, bend left] &
\end{tikzcd}$$

$$Q^7 : \begin{tikzcd}
& (0,5,2) \arrow[dl] \arrow[dr] & & (0,2,5) \arrow[dl] \arrow[dr] & \\
(1,6,0) \arrow[dr] & & (1,3,3) \arrow[dl] \arrow[dr] & & (1,0,6) \arrow[dl] \\
& (2,4,1) \arrow[uu] \arrow[dr] & & (2,1,4) \arrow[uu] \arrow[dl] & \\
& & (3,2,2) \arrow[uu] \arrow[dl] \arrow[dr] & & \\
& (4,3,0) \arrow[uu] \arrow[dr] & & (4,0,3) \arrow[uu] \arrow[dl] & \\
& & (5,1,1) \arrow[uu] \arrow[d, bend left] & & \\
& & (7,0,0) \arrow[u, bend left] & &
\end{tikzcd}$$

Example 26. Here you see Q^{3m+2} for $m = 0, 1, 2$.

$$Q^2 : \begin{tikzcd}
(0,1,1) \arrow[d, bend left] \\
(2,0,0) \arrow[u, bend left]
\end{tikzcd}$$

$$Q^5 : \begin{tikzcd}
(0,4,1) \arrow[dr] & & (0,1,4) \arrow[dl] \\
& (1,2,2) \arrow[dl] \arrow[dr] & \\
(2,3,0) \arrow[uu] \arrow[dr] & & (2,0,3) \arrow[uu] \arrow[dl] \\
& (3,1,1) \arrow[uu] \arrow[d, bend left] & \\
& (5,0,0) \arrow[u, bend left] &
\end{tikzcd}$$

$$Q^8 : \begin{tikzcd}
(0,7,1) \arrow[dr] & & (0,4,4) \arrow[dl] \arrow[dr] & & (0,1,7) \arrow[dl] \\
& (1,5,2) \arrow[dl] \arrow[dr] & & (1,2,5) \arrow[dl] \arrow[dr] & \\
(2,6,0) \arrow[uu] \arrow[dr] & & (2,3,3) \arrow[uu] \arrow[dl] \arrow[dr] & & (2,0,6) \arrow[uu] \arrow[dl] \\
& (3,4,1) \arrow[uu] \arrow[dr] & & (3,1,4) \arrow[uu] \arrow[dl] & \\
& & (4,2,2) \arrow[uu] \arrow[dl] \arrow[dr] & & \\
& (5,3,0) \arrow[uu] \arrow[dr] & & (5,0,3) \arrow[uu] \arrow[dl] & \\
& & (6,1,1) \arrow[uu] \arrow[d, bend left] & & \\
& & (8,0,0) \arrow[u, bend left] & &
\end{tikzcd}$$

Chapter 4

Simple Bases

In this chapter we drive our attention to simple bases for exact linked nets of vector spaces over $\mathbb{Z}^2$-quivers. Since the work by Eisenbud and Harris [3] — where they called adapted basis what we call simple basis — it is known the existence of a simple basis for a linked net of vector spaces over a $\mathbb{Z}^1$-quiver with finite support.

Simple bases were used by Esteves and Osserman in [6] and by Santana in [16] to assist and simplify the computation of the multivariate Hilbert polynomial of the *linked projective space* $\mathbb{LP}(\mathfrak{g})$. Given the existence of a simple basis, we can ensure that the multivariate Hilbert polynomial of $\mathbb{LP}(\mathfrak{g})$ is equal to that of the diagonal,

$$\mathrm{Hilb}_{\mathbb{LP}(\mathfrak{g})}(x_0,\cdots,x_n)=\binom{r+\sum x_i}{r}.$$

You can see more details about this in Chapter 5.

In contrast to the case of linked nets of vector spaces over $\mathbb{Z}^1$-quivers with finite support, linked nets of vector spaces over $\mathbb{Z}^n$-quivers, for $n>1$, with finite support may not admit a simple basis. In this chapter we characterise under what circumstances a given exact linked net of vector spaces over a $\mathbb{Z}^n$-quiver with finite support admits a simple basis.

4.1 Simple basis for exact linked nets with support on Q_1^3

Given a $\mathbb{Z}^2$-quiver, a **triangle** is the full subquiver supported on a set of three pairwise neighboring vertices. It is denoted Q_1^3; see (4.1).

Theorem 27. An exact linked net of vector spaces over a $\mathbb{Z}^2$-quiver with finite support contained in a triangle admits a simple basis.

Proof: Denote by Q_1^3 the triangle. Let v_0, v_1, v_2 denote its vertices and $\alpha_0, \alpha_1, \alpha_2$ its arrows, disposed as described below.

$$
\begin{tikzcd}
 & v_0 \arrow[dr, "\alpha_2"] & \\
Q_1^3 & & v_1 \arrow[dl, "\alpha_0"] \\
 & v_2 \arrow[uu, "\alpha_1"] &
\end{tikzcd}
\tag{4.1}
$$

Let $\mathfrak{g}$ be an exact linked net of vector spaces supported on $H \subseteq \{v_0, v_1, v_2\}$. For each $v' \notin H$ there exists $v \in H$ such that $\varphi_{v'}^{v}$ is an isomorphism. Thus, to show that $\mathfrak{g}$ admits a simple basis it is enough to show that there are vectors in the V_{v_j} whose images in V_{v_i} form a basis for each $i = 0, 1, 2$.

To abbreviate the notation, subindices in $\mathbb{Z}$ will mean their rests modulo 3. For each i we must choose vectors in V_{v_i} spanning a subspace U_i such that $V_{v_i} = U_i + \operatorname{Im} \varphi_{v_i}^{v_{i-1}}$ and $\varphi_{v_{i+2}}^{v_i}|_{U_i}$ is injective. Due to exactness, this is equivalent to requiring that the composition

$$
U_i \longrightarrow V_{v_i} \longrightarrow \frac{V_{v_i}}{\operatorname{Im} \varphi_{v_i}^{v_{i-1}}}
\tag{4.2}
$$

be an isomorphism. Moreover, as we want that bases of the U_i form a simple basis for $\mathfrak{g}$, the dimension of the U_i must sum up to r, thus

$$
\sum_{i=0}^{2} \dim \frac{V_{v_i}}{\operatorname{Im}\left(\varphi_{v_i}^{v_{i-1}}\right)} = r.
$$

If we denote $r_i := \operatorname{rank}\left(\varphi_{v_{i+1}}^{v_i}\right)$ for each i, we can rewrite the above equality as

$$
r_0 + r_1 + r_2 = 2r.
\tag{4.3}
$$

We claim that (4.3) holds. Indeed, for each i consider the following exact sequence:

$$
0 \longrightarrow \operatorname{Ker}\left(\varphi_{v_i}^{v_{i+2}}|_{\operatorname{Im}\left(\varphi_{v_{i+2}}^{v_{i+1}}\right)}\right) \longrightarrow \operatorname{Im}\left(\varphi_{v_{i+2}}^{v_{i+1}}\right) \xrightarrow{\varphi_{v_i}^{v_{i+2}}} \operatorname{Im}\left(\varphi_{v_i}^{v_{i+1}}\right) \longrightarrow 0.
$$

Since $\mathfrak{g}$ is exact we have

$$
\operatorname{Ker}\left(\varphi_{v_i}^{v_{i+2}}|_{\operatorname{Im}\left(\varphi_{v_{i+2}}^{v_{i+1}}\right)}\right) = \operatorname{Im}\left(\varphi_{v_{i+2}}^{v_{i+1}}\right) \cap \operatorname{Ker}\left(\varphi_{v_i}^{v_{i+2}}\right) = \operatorname{Im}\left(\varphi_{v_{i+2}}^{v_{i+1}}\right) \cap \operatorname{Im}\left(\varphi_{v_{i+2}}^{v_i}\right) = \operatorname{Im}\left(\varphi_{v_{i+2}}^{v_i}\right).
$$

Also, $r - r_{i+2} = \operatorname{rank}(\varphi^{v_i}_{v_{i+2}})$ and $r - r_i = \operatorname{rank}(\varphi^{v_{i+1}}_{v_i})$. It follows that

$$r - r_{i+2} = \operatorname{rank}(\varphi^{v_i}_{v_{i+2}}) = r_{i+1} - \operatorname{rank}(\varphi^{v_{i+1}}_{v_i}) = r_{i+1} - (r - r_i).$$

Now, for each $i = 0, 1, 2$, let $U_i \subseteq V_{v_i}$ be a subspace such that (4.2) is an isomorphism, and let $\{w^i_1, \cdots, w^i_{r-r_{i+2}}\}$ be a basis of U_i. By Equation (4.3), the number of these elements sum up to r. We claim that $\varphi^{v_i}_{v_{i+1}}(w^i_1), \cdots, \varphi^{v_i}_{v_{i+1}}(w^i_{r-r_{i+2}}), w^{i+1}_1, \cdots, w^{i+1}_{r-r_i}$ are linearly independent in $V_{v_{i+1}}$. Indeed, suppose that

$$\sum_{j=1}^{r-r_{i+2}} a_j \varphi^{v_i}_{v_{i+1}}(w^i_j) + \sum_{j=1}^{r-r_i} b_j w^{i+1}_j = 0.$$

Considering the above equality in $V_{v_{i+1}}/\operatorname{Im}\left(\varphi^{v_i}_{v_{i+1}}\right)$ we see that each b_j is zero. In addition, observe that $\varphi^{v_i}_{v_{i+1}}|_{U_i}$ is injective, because $\operatorname{Ker}\left(\varphi^{v_i}_{v_{i+1}}\right) \subseteq \operatorname{Ker}\left(\varphi^{v_i}_{v_{i+2}}\right)$. Thus, every a_j is zero as well. It follows that

$$\varphi^{v_i}_{v_{i+1}}(U_i) \oplus U_{i+1} \subseteq V_{v_{i+1}}.$$

Now suppose that

$$\varphi^{v_{i+1}}_{v_{i+2}} \left(\sum_{j=1}^{r-r_{i+2}} a_j \varphi^{v_i}_{v_{i+1}}(w^i_j) + \sum_{j=1}^{r-r_i} b_j w^{i+1}_j \right) = 0.$$

As $\operatorname{Ker}(\varphi^{v_{i+1}}_{v_{i+2}}) \subseteq \operatorname{Im}(\varphi^{v_i}_{v_{i+1}})$ we conclude that all the b_j are zero. But $\operatorname{Ker}(\varphi^{v_i}_{v_{i+2}}) = \operatorname{Im}(\varphi^{v_{i+2}}_{v_i})$, and because the w^i_j form a lifting of a basis of $V_{v_i}/\operatorname{Im}(\varphi^{v_{i+2}}_{v_i})$ we see that all the a_j are zero as well. This shows that $\varphi^{v_{i+1}}_{v_{i+2}}|_{\varphi^{v_i}_{v_{i+1}}(U_i)\oplus U_{i+1}}$ is injective.

Finally, suppose

$$\sum_{j=1}^{r-r_{i+2}} a_j \varphi^{v_i}_{v_{i+2}}(w^i_j) + \sum_{j=1}^{r-r_i} b_j \varphi^{v_{i+1}}_{v_{i+2}}(w^{i+1}_j) + \sum_{j=1}^{r-r_{i+1}} c_j w^{i+2}_j = 0.$$

Considering the above equality modulo $\operatorname{Im}(\varphi^{v_{i+1}}_{v_{i+2}})$ we conclude that each c_j is zero. As $\varphi^{v_{i+1}}_{v_{i+2}}|_{\varphi^{v_i}_{v_{i+1}}(U_i)\oplus U_{i+1}}$ is injective, $\sum_{j=1}^{r-r_{i+2}} a_j \varphi^{v_i}_{v_{i+1}}(w^i_j) + \sum_{j=1}^{r-r_i} b_j w^{i+1}_j = 0$, and thus all the a_j and the b_j are zero as well. This shows that

$$\{w^i_1|_{V_{i+2}}, \cdots, w^i_{r-r_{i+2}}|_{V_{i+2}}, w^{i+1}_1|_{V_{i+2}}, \cdots, w^{i+1}_{r-r_i}|_{V_{i+1}}, w^{i+2}_1, \cdots, w^{i+2}_{r-r_{i+1}}\}$$

form a linearly independent set and thus a basis of V_{i+2} for each i. It follows that the w^i_j form a simple basis for $\mathfrak{g}$. $\square$

Remark 28. For an exact linked net of vector spaces $\mathfrak{g}$ over a $\mathbb{Z}^2$-quiver and any triangle $Q_1^3 = \{v_0, v_1, v_2\}$, the proof shows that there are elements $w_j^i \in V_{v_i}$ whose images in each V_{v_ℓ} form a basis.

Example 29. Let $\mathfrak{g}$ be the exact linked net over a $\mathbb{Z}^2$-quiver with support on a triangle Q_1^3 whose restriction to the triangle is of the form:

$$
\begin{tikzcd}
 & V_0 \arrow[dr, "E_{22}+E_{33}"] & \\
\mathfrak{g} & & V_1 \arrow[dl, "E_{11}+E_{22}"] \\
 & V_2 \arrow[uu, "E_{11}+E_{33}"] &
\end{tikzcd}
$$

where $V_i = k^3$ and $E_{ii} = [e_{ij}]$ is the 3×3 matrix zero everywhere but $e_{ii} = 1$. A simple basis for $\mathfrak{g}$ is $\{e_2^0, e_1^1, e_3^2\}$, where e_j^i is the j-th vector in the canonical basis of V_i for each i and j.

To illustrate the notion of a simple basis, we give an example of an exact linked net of vector spaces over a $\mathbb{Z}^2$-quiver with dimension 1 and support contained in the quiver Q_0^6.

$$
\begin{tikzcd}
\mathbb{C}_{(0,6,0)} \arrow[dr, shift left, "0"] & & \mathbb{C}_{(0,3,3)} \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & & \mathbb{C}_{(0,0,6)} \arrow[dl, shift left, "0"] \\
 & \mathbb{C}_{(1,4,1)} \arrow[ul, shift left, "\cong"] \arrow[dr, "0"] & & \mathbb{C}_{(1,1,4)} \arrow[ur, shift left, "\cong"] \arrow[dl, "0"'] & \\
\mathfrak{g}: & & \mathbb{C}_{(2,2,2)} \arrow[uu, "\cong"] \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & & \\
 & \mathbb{C}_{(3,3,0)} \arrow[uu, "\cong"] \arrow[dr, "0"] & & \mathbb{C}_{(3,0,3)} \arrow[uu, "\cong"] \arrow[dl, "0"'] & \\
 & & \mathbb{C}_{(4,1,1)} \arrow[uu, "\cong"] \arrow[d, shift left, "0"] & & \\
 & & \mathbb{C}_{(6,0,0)} \arrow[u, shift left, "\cong"] & &
\end{tikzcd}
$$

A simple basis of $\mathfrak{g}$ is $\{e_1\}$ in $\mathbb{C}_{(6,0,0)}$.

4.2 Simple bases in the general case

The strategy to prove the main result of this chapter is use induction on the dimension of the exact linked net of vector spaces. Thus, the first step on this journey is to prove that the result is true for exact linked nets of vector spaces of dimension one.

Theorem 30. An exact linked net $\mathfrak{g}$ of vector spaces of dimension one over a $\mathbb{Z}^n$-quiver with finite support admits a simple basis.

Proof. First we show uniqueness. Suppose that there exist $w \in V_v$ and $s' \in V_{v'}$, each being a simple basis of $\mathfrak{g}$. In this case, there are $a, b \in k^*$ such that $\varphi^v_{v'}(s) = as'$ and $\varphi^{v'}_v(s') = bs$, and thus $\varphi^{v'}_v \varphi^v_{v'}(s) = abs$, which implies that $v = v'$.

As $\mathfrak{g}$ has finite support, say H, it is enough to show that there exists $v \in H$ such that $\varphi^v_{v'}(V_v) = V_{v'}$ for each $v' \in H$.

We prove first that there exists a vertex $v \in H$ such that $\operatorname{Im} \varphi_\gamma = 0$ for each nontrivial admissible path γ arriving at v. To this end, suppose not. Write $Q = (Q_0, Q_1, s, t)$. Pick any $v_0 \in H$ and any nonzero $s_0 \in V_{v_0}$. By contradiction hypothesis, there exist a nontrivial admissible path γ_1 with $t(\gamma_1) = v_0$ and $s_1 \in V_{s(\gamma_1)}$ with $\varphi_{\gamma_1}(s_1) = s_0$. Since $\mathfrak{g}$ is supported in H, we may assume $s(\gamma_1) \in H$. We keep repeating this procedure to obtain a sequence of paths γ_i with $s(\gamma_i) \in H$ and elements s_i such that $\varphi_{\gamma_i}(s_i) = s_{i-1}$ for each $i \geq 1$. As H is finite, there will be i, j with $i < j$ such that $\mu := \gamma_j \cdots \gamma_{i+2}\gamma_{i+1}$ is a nontrivial circuit, and thus a non-admissible path. But then $\varphi_\mu = 0$ and $\varphi_\mu(s_j) = s_i$, a contradiction.

Let $v \in H$ be such that $\operatorname{Im} \varphi_\gamma = 0$ for each nontrivial admissible path γ arriving at v. By exactness, φ_γ is injective for each simple admissible path γ leaving v. Thus, by Lemma 16, this v generates $\mathfrak{g}$. □

Definition 31. An exact linked net of vector spaces over a $\mathbb{Z}^n$-quiver **satisfies the intersection property at a vertex** v if for every collection $I_0, I_1, \ldots, I_m$ of subsets of $\{0, 1, \ldots, n\}$ we have

$$\left(\sum_{\ell=1}^{m} \operatorname{Ker}(\varphi^v_{I_\ell}) \right) \cap \operatorname{Ker}(\varphi^v_{I_0}) = \sum_{\ell=1}^{m} \operatorname{Ker}(\varphi^v_{I_\ell \cap I_0}).$$

If a I_i is empty we consider that $\varphi^v_{I_i}$ is an automorphism of V_v; in this case the equality is either trivial, for $i = 0$, or reduces to one involving fewer I_ℓ. On the other hand, if a I_i is equal to $\{0, 1, \ldots, n\}$ then $\varphi^v_{I_i}$ is zero and its kernel is the whole V_v; in this case the equality holds trivially. Hence, to check the intersection property we need only consider I_i which are non-empty proper subsets of $\{0, 1, \ldots, n\}$. Finally, we say the linked net **satisfies the intersection property** if it satisfies it at every vertex.

Proposition 32. If a linked net of vector spaces over a $\mathbb{Z}^n$-quiver has support in a collection of vertices H, then it satisfies the intersection property at every vertex if it satisfies it at each vertex of H.

Proof. Indeed, let v be a vertex of the quiver not in H. Then there is $w \in H$ such that φ^w_v is an isomorphism. Let γ be an admissible path connecting w to v, and put $J := \{i \,|\, \gamma(i) > 0\}$. Let $I_0, \ldots, I_m$ be a collection of proper subsets of $\{0, \ldots, n\}$. For each $i = 1, \ldots, m$, let $s_i \in \operatorname{Ker}(\varphi^v_{I_i})$. Suppose $\varphi^v_{I_0}(s_1 + \cdots + s_m) = 0$. Then $s_i = \varphi^w_v(t_i)$ for a certain t_i for each $i = 1, \ldots, m$. Set $J_i := J \cup I_i$ for each $i = 0, \ldots, m$. It follows from Lemma 16 that $\varphi^w_{J_i}(t_i) = 0$ for each $i = 1, \ldots, m$ and $\varphi^w_{J_0}(t_1 + \cdots + t_m) = 0$. If the linked net satisfies the intersection property at w, we may write $t_1 + \cdots + t_m = t'_1 + \cdots + t'_m$, where $t'_i \in \operatorname{Ker}(\varphi^w_{J_i \cap J_0})$ for $i = 1, \ldots, m$. Set $s'_i := \varphi^w_v(t'_i)$ for $i = 1, \ldots, m$. Then $s'_i \in \operatorname{Ker}(\varphi^w_{I_i \cap I_0})$ for each i, and clearly $s_1 + \cdots + s_m = s'_1 + \cdots + s'_m$. □

Proposition 33. If an exact linked net of vector spaces over a $\mathbb{Z}^n$-quiver admits a simple basis, then it satisfies the intersection property.

Proof. Let v be a vertex of the quiver, and $I_0, I_1, \ldots, I_m$ subsets of $\{0, 1, \ldots, n\}$. Lemma 17 implies that $\operatorname{Ker}(\varphi^v_{I_i}) \cap \operatorname{Ker}(\varphi^v_{I_0}) = \operatorname{Ker}(\varphi^v_{I_i \cap I_0})$ for each i, and thus it follows that

$$\left(\sum_{\ell=1}^{m} \operatorname{Ker}(\varphi^v_{I_\ell}) \right) \cap \operatorname{Ker}(\varphi^v_{I_0}) \supseteq \sum_{\ell=1}^{m} \operatorname{Ker}(\varphi^v_{I_\ell \cap I_0}).$$

For the opposite inclusion, assume the linked net has a simple basis, say $\{s_0, \ldots, s_r\}$ with $s_j \in V_{v_j}$. For each j, let μ_j be an admissible path connecting v_j to v. For each $i = 0, 1, \ldots, m$, let γ_i be a path leaving v with $I_{\gamma_i} = I_i$. Then $\operatorname{Ker}(\varphi^v_{I_i})$ is generated by certain $\varphi^{v_j}_v(s_j)$, namely

$$\left\langle \varphi^{v_j}_v(s_j) \;\middle|\; \text{the path } \mu_j\gamma_i \text{ is not admissible} \right\rangle.$$

Analogously for $\left(\sum_{\ell=1}^{m} \operatorname{Ker}(\varphi^v_{I_\ell}) \right) \cap \operatorname{Ker}(\varphi^v_{I_0})$, we have

$$\left\langle \varphi^{v_j}_v(s_j) \;\middle|\; \begin{array}{l} \text{the path } \mu_j\gamma_0 \text{ is not admissible and} \\ \mu_j\gamma_\ell \text{ is not admissible for some } \ell > 0 \end{array} \right\rangle.$$

These conditions on the $\mu_j\gamma_\ell$ are equivalent to

$$\begin{cases} \mu_j(i) + \gamma_0(i) > 0 \text{ for all } i, \\ \mu_j(i) + \gamma_\ell(i) > 0 \text{ for some } \ell > 0 \text{ and for all } i, \end{cases}$$

which implies that there is $\ell > 0$ such that

$$\mu_j(i) + \min\{\gamma_\ell(i), \gamma_0(i)\} > 0 \text{ for all } i.$$

Now, let $\gamma_{l,0}$ be any path leaving v with $I_{\gamma_{l,0}} = I_l \cap I_0$ for each $l = 1, \ldots, m$. Then we have $\min\{\gamma_l(i), \gamma_0(i)\} = \gamma_{l,0}(i)$ for each i. Furthermore, since $\sum_{\ell=1}^{m} \operatorname{Ker}(\varphi^v_{I_\ell \cap I_0})$ can be described as

$$\left\langle \varphi_v^{v_j}(s_j) \;\middle|\; \mu_j \gamma_{\ell,0} \text{ is not admissible for some } \ell > 0 \right\rangle,$$

the opposite inclusion follows. $\square$

Proposition 34. Let $\mathfrak{g}$ be an exact linked net of vector spaces with finite support over a $\mathbb{Z}^n$-quiver Q. Then there exists a vertex v in Q such that

$$U_v := \frac{V_v}{\sum\limits_{t(\gamma)=v} \mathrm{Im}(\varphi_\gamma)} \neq 0,$$

where γ runs through all nontrivial simple admissible paths arriving at v. Furthermore, let $s \in V_v$ be such that $\bar{s} \neq 0$ in U_v. Then the $\varphi_u^v(s)$ generate a one-dimensional exact linked subnet of $\mathfrak{g}$.

Proof. The existence of v follows from an argument similar to that used in the proof of Theorem 30; see [8, lemma 6.3].

In addition, it follows from the exactness of $\mathfrak{g}$ and Lemma 16 that the $\varphi_u^v(s)$ are nonzero. Put $W_u := \langle \varphi_u^v(s) \rangle$ for each vertex u. Given two vertices u_1, u_2 we have that either $\varphi_{u_2}^{u_1}(\varphi_{u_1}^v(s)) = \varphi_{u_2}^v(s)$ or $\varphi_{u_2}^{u_1}(\varphi_{u_1}^v(s)) = 0$; in any case $\varphi_{u_2}^{u_1}(W_{u_1}) \subseteq W_{u_2}$. Thus the W_u form an one-dimensional subrepresentation $\mathfrak{g}_s$ of $\mathfrak{g}$, which then clearly satisfies the conditions of a linked net in Definition 11.

Furthermore, if u_1 and u_2 are neighbors then either $\varphi_{u_2}^{u_1}\varphi_{u_1}^v = \varphi_{u_2}^v$ or $\varphi_{u_1}^{u_2}\varphi_{u_2}^v = \varphi_{u_1}^v$. Thus, either $\varphi_{u_2}^{u_1}|_{W_{u_1}}$ is an isomorphism or $\varphi_{u_1}^{u_2}|_{W_{u_2}}$ is. In the first case, it occur that $\mathrm{Im}\left(\varphi_{u_2}^{u_1}|_{W_{u_1}}\right) = W_{u_2}$, whereas in the second case, $\mathrm{Ker}\left(\varphi_{u_1}^{u_2}|_{W_{u_2}}\right) = 0$. As in any case $\mathrm{Im}\left(\varphi_{u_2}^{u_1}|_{W_{u_1}}\right) \subseteq \mathrm{Ker}\left(\varphi_{u_1}^{u_2}|_{W_{u_2}}\right)$, equality holds. Thus $\mathfrak{g}_s$ is exact. $\square$

Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver. Let v be a vertex of the quiver. An element $s \in V_v$ will be called a **section of $\mathfrak{g}$ at v**. The section s is called **primitive** if

$$s \in V_v - \sum_{\alpha} \mathrm{Im}(\varphi_\alpha),$$

where α runs through all arrows arriving at v. If s is primitive then Lemma 34 shows that the $\varphi_u^v(s)$ generate a one-dimensional exact linked subnet of $\mathfrak{g}$ that we denote $\mathfrak{g}_s$.

Lemma 35. Let $\mathfrak{g}$ be an exact linked net of vector spaces over a $\mathbb{Z}^n$-quiver satisfying the intersection property at a vertex v. Let $I_0, I_1, \ldots, I_m$ be subsets of $\{0, 1, \ldots, n\}$. Let

$I'_0 := \{0,1,\dots,n\} - I_0$. For each $s \in V_v$:

$$\text{If} \quad \varphi_{I'_0}(s) \in \sum_{i=1}^{m} \operatorname{Im}(\varphi_{I_i}) \quad \text{then} \quad s \in \sum_{i=0}^{m} \operatorname{Im}(\varphi_{I_i \cap I_0}).$$

Proof. Let I'_i be the complement of I_i in $\{0,1,\dots,n\}$ for $i = 1,2,\dots,m$. Since $\mathfrak{g}$ is exact,

$$\varphi_{I'_0}(s) \in \Big(\sum_{i=1}^{m} \operatorname{Ker}(\varphi_{I'_i})\Big) \cap \operatorname{Ker}(\varphi_{I_0})$$

and thus, by the intersection property,

$$\varphi_{I'_0}(s) \in \sum_{i=1}^{m} \operatorname{Ker}(\varphi_{I'_i \cap I_0}).$$

Using that $\mathfrak{g}$ is exact again, we obtain

$$\varphi_{I'_0}(s) \in \sum_{i=1}^{m} \operatorname{Im}(\varphi_{I_i \cup I'_0}).$$

In other words, there are $s_1,\dots,s_m$ such that

$$\varphi_{I'_0}\Big(s - \sum_{i=1}^{m} \varphi_{I_i \cap I_0}(s_i)\Big) = 0,$$

whence the statement of the lemma, again by the exactness of $\mathfrak{g}$. □

The next Proposition is fundamental to prove Theorem 37.

Proposition 36. Let $\mathfrak{g}$ be an exact linked net of vector spaces with finite support over a $\mathbb{Z}^n$-quiver Q. Let s be a primitive section of $\mathfrak{g}$ at some vertex. If $\mathfrak{g}$ satisfies the intersection property then the quotient $\mathfrak{g}/\mathfrak{g}_s$ is an exact linked net of dimension $\dim \mathfrak{g} - 1$ satisfying the intersection property.

Proof. Let v be the vertex of Q such that $s \in V_v$. The quotient representation $\mathfrak{g}/\mathfrak{g}_s$ clearly satisfies Conditions (1) and (2) in Definition 11. Once we establish Condition (3) as well, and thus prove $\mathfrak{g}/\mathfrak{g}_s$ is a linked net, exactness is not difficult to show.

Indeed, let u_1 and u_2 be distinct neighboring vertices and let $x \in V_{u_2}$ such that $\varphi_{u_1}^{u_2}(x) = c\varphi_{u_1}^{v}(s)$ for some $c \in k$. Since $\varphi_{u_2}^{u_1}\varphi_{u_1}^{u_2} = 0$, it follows that $c\varphi_{u_2}^{u_1}\varphi_{u_1}^{v}(s) = 0$. If $c = 0$ then $\varphi_{u_1}^{u_2}(x) = 0$ and thus $x \in \operatorname{Im}(\varphi_{u_2}^{u_1})$ by the exactness of $\mathfrak{g}$. If $c \neq 0$ then $\varphi_{u_2}^{u_1}\varphi_{u_1}^{v}(s) = 0$ and thus $\varphi_{u_1}^{v}(s) = c'\varphi_{u_1}^{u_2}\varphi_{u_2}^{v}(s)$ for some c' in k, by the exactness of $\mathfrak{g}_s$. But then $\varphi_{u_1}^{u_2}(x - cc'\varphi_{u_2}^{v}(s)) = 0$ and thus $x = \varphi_{u_2}^{u_1}(y) + cc'\varphi_{u_2}^{v}(s)$ for some $y \in V_{u_1}$ by the exactness of $\mathfrak{g}$. At any rate, the kernel of the map induced by $\varphi_{u_1}^{u_2}$ on $\mathfrak{g}/\mathfrak{g}_s$ is the image of the map induced by $\varphi_{u_2}^{u_1}$. It follows that $\mathfrak{g}/\mathfrak{g}_s$ is exact.

Next, we verify item (3) of Definition 11 for $\mathfrak{g}/\mathfrak{g}_s$. By Lemma 16, that item is equivalent to the following statement:

Claim. Let u be a vertex of Q and $i \in \{0,1,\cdots,n\}$. Let α be an i-arrow leaving u and β an i-arrow arriving at u. Let z be the initial vertex of β and w the final vertex of α, as below.

$$z \xrightarrow{\beta} u \xrightarrow{\alpha} w$$

Let x be a section of $\mathfrak{g}$ at z such that $\varphi_\alpha\varphi_\beta(x)$ is a section of $\mathfrak{g}_s$ at w. Then $\varphi_\beta(x)$ is a section of $\mathfrak{g}_s$ at u.

We will now verify the claim. If $\varphi_\alpha\varphi_\beta(x) = 0$ then $\varphi_\beta(x) = 0$, as $\mathfrak{g}$ itself satisfies item (3) of Definition 11 and hence the claim by Lemma 16. We may thus assume $\varphi_\alpha\varphi_\beta(x) \neq 0$.

Write $\varphi_\alpha\varphi_\beta(x) = c\varphi_w^v(s)$ for some nonzero scalar c. We may assume $c = 1$ and for simplicity we do assume it. Let ρ be a simple admissible path connecting w to u. Since $\varphi_\rho\varphi_\alpha = 0$, we have $\varphi_\rho\varphi_w^v(s) = 0$. It follows that there is an admissible path from v to w passing through u, and hence

$$\varphi_\alpha(\varphi_\beta(x) - \varphi_u^v(s)) = 0. \tag{4.4}$$

If there were an admissible path from v to u passing through z, then we would have $\varphi_\alpha\varphi_\beta(x - \varphi_z^v(s)) = 0$. It would follow that $\varphi_\beta(x - \varphi_z^v(s)) = 0$, as observed before, so $\varphi_\beta(x) = \varphi_u^v(s)$ as wished.

Otherwise, $\gamma(i) = 0$, where γ is an admissible path connecting v to u. Let

$$J := \{j \mid \gamma(j) = 0\} \quad \text{and} \quad K := \{0,\dots,n\} - \{i\}.$$

Then Equation (4.4) implies that

$$\varphi_u^v(s) \in \Big(\operatorname{Ker}(\varphi_K^u) + \operatorname{Ker}(\varphi_{\{i\}}^u)\Big) \cap \operatorname{Ker}(\varphi_J^u).$$

Since $\mathfrak{g}$ satisfies the intersection property, it follows that

$$\varphi_u^v(s) \in \operatorname{Ker}(\varphi_{K\cap J}^u) + \operatorname{Ker}(\varphi_{\{i\}}^u).$$

Now, we can write $\varphi_u^v = \varphi_{K_p}\cdots\varphi_{K_1}$ for certain proper subsets $K_1,\dots,K_p$ of $\{0,\cdots,n\}$ satisfying $K_1 \subseteq \cdots \subseteq K_p$ and $K_p = \{j \mid \gamma(j) > 0\}$. In particular, $i \notin K_p$. Applying Lemma 35 repeatedly, we can conclude that

$$s \in \operatorname{Im}(\varphi_{J-\{i\}}) + \operatorname{Im}(\varphi_{\{i\}}),$$

which implies that $J = \{i\}$. But in this case $\varphi_\alpha \varphi_u^v(s) = 0$, and hence $\varphi_\alpha \varphi_\beta(x) = 0$ from Equation (4.4), contradicting our assumption.

To finish the proof we need now show that $\mathfrak{g}/\mathfrak{g}_s$ satisfies the intersection property. To this end, let u be a vertex of Q and $I_0, I_1, \dots, I_m$ be a collection of subsets of $\{0,1,\dots,n\}$. For each $i = 1,2,\dots,m$ let s_i be a section of $\mathfrak{g}$ at u such that $t_i := \varphi_{I_i}^u(s_i)$ is a section of $\mathfrak{g}_s$. Furthermore, assume that $t := \varphi_{I_0}^u(s_1 + \cdots + s_m)$ is a section of $\mathfrak{g}_s$. We need to show that

$$s_1 + \cdots + s_m = s_1' + \cdots + s_m'$$

for sections s_i' of $\mathfrak{g}$ at u such that $\varphi_{I_0 \cap I_i}^u(s_i')$ is a section of $\mathfrak{g}_s$ for $i = 1,2,\dots,m$.

Since $\mathfrak{g}_s$ is exact, for each $i = 1,\dots,m$ there is a section y_i of $\mathfrak{g}_s$ at u such that $t_i := \varphi_{I_i}^u(y_i)$. We may thus suppose that $\varphi_{I_i}^u(s_i) = 0$ for each i. Also, there is a section y of $\mathfrak{g}_s$ at u such that $\varphi_{I_0}^u(y) = t$. Then $\varphi_{I_0}^u(s_1 + \cdots + s_m - y) = 0$. If $y = 0$ we may use that $\mathfrak{g}$ satisfies the intersection property at u to conclude.

Assume now that $y \neq 0$. Since $\mathfrak{g}$ is exact, we have

$$y \in \sum_{i=0}^{m} \mathrm{Im}(\varphi_{J_i}),$$

where $J_i := \{0,\dots,n\} - I_i$ for each $i = 0,1,\dots,m$. Write $y = c\varphi_u^v(s)$ for a nonzero scalar c. Again, we can write $\varphi_u^v = \varphi_{K_p} \cdots \varphi_{K_1}$ for certain proper subsets $K_1,\dots,K_p$ of $\{0,\dots,n\}$ satisfying $K_1 \subseteq \cdots \subseteq K_p$. Let $K_i' := \{0,1,\dots,n\} - K_i$ for each $i = 1,\dots,p$. And again, applying Lemma 35 repeatedly, we can conclude that

$$s \in \sum_{i=0}^{m} \mathrm{Im}(\varphi_{J_i \cap K_p'}).$$

Since s is primitive, $J_j \cap K_p' = \emptyset$ for some j, or equivalently $I_j \cup K_p = \{0,1,\dots,n\}$. It follows that $\varphi_{I_j}^u(y) = 0$. We may thus replace s_j by $s_j - y$ and thus assume that $y = 0$, the case we have already analyzed. □

Theorem 37. An exact linked net $\mathfrak{g}$ of vector spaces with finite support over a $\mathbb{Z}^n$-quiver admits a simple basis if and only if it satisfies the intersection property.

Proof. If $\mathfrak{g}$ has a simple basis, then by Proposition 33 the linked net $\mathfrak{g}$ satisfies the intersection property. To show the converse, we proceed by induction on $r = \dim \mathfrak{g}$. If $r = 1$, Theorem 30 says that $\mathfrak{g}$ has a simple basis.

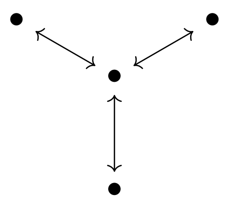
Now suppose that the converse is true for any exact linked net of dimension $l < r$. Using Proposition 34, we obtain an one-dimensional exact linked subnet $\mathfrak{g}_s$ of $\mathfrak{g}$. By Proposition 36, the quotient $\mathfrak{g}/\mathfrak{g}_s$ is an exact linked net satisfying the intersection property as well. Thus, by the induction hypothesis, $\mathfrak{g}/\mathfrak{g}_s$ admits a simple basis. Also, by Theorem 30, the representation $\mathfrak{g}_s$ admits a simple basis as well.

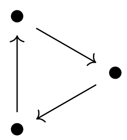
Let $\mathscr{B}_1 := \{w_{i_1}\}$ be a simple basis of $\mathfrak{g}_s$, and $\mathscr{B}_2 := \{\overline{w}_{i_2}, \cdots, \overline{w}_{i_r}\}$ be a simple basis of $\mathfrak{g}/\mathfrak{g}_s$. With an easy and standard verification we conclude that $\{w_{i_1}, \ldots, w_{i_r}\}$ is a simple basis of $\mathfrak{g}$. □

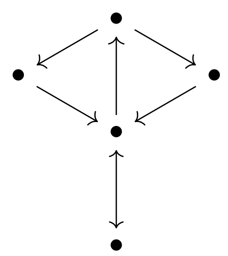
It was recently made known to the author that G. Munõz in [12] and [13] uses a condition similar to the intersection property to show the existence of extensions of refined limit linear series.

Corollary 38. Any exact linked net of vector spaces with finite support contained in Q_0^3, Q_1^3 or Q^4 has a simple basis.

Proof: The quivers Q_0^3, Q_1^3 and Q^4 appeared in Section 3:

Q_0^3 : 

Q_1^3 : 

Q^4 : 

They have few vertices. As it is enough to check the intersection property at the vertices in the support of a linked net, one can quickly verify that in any of the cases, the linked net satisfies the intersection property. Thus Theorem 37 yields the existence of a simple basis. □

This corollary gives another proof to the fact that an exact linked net of vector spaces with support on Q_1^3 admits a simple basis, given in Section 4.1. On the other hand, for $d \geq 5$ and any p, there may be exact linked nets with support in Q_p^d which do not satisfy the intersection property.

4.3 Examples

Here we give examples of exact linked nets which do not admit simple bases.

Example 39. Here we present an exact linked net $\mathfrak{g}$ of dimension 2 with support in Q_0^6 which does not satisfy the intersection property. Right after, we show that it occurs as a limit of linear series. Here's it:

$$
\mathfrak{g}:\quad
\begin{tikzcd}[column sep=small]
\mathbb{C}^2_{(0,6,0)} \arrow[rr, bend left, "0"] & & & & \mathbb{C}^2_{(0,3,3)} \arrow[dll, "{\begin{bmatrix}1&-1\\0&0\end{bmatrix}}"'] \arrow[drr, "{\begin{bmatrix}1&0\\0&0\end{bmatrix}}"] & & & & \mathbb{C}^2_{(0,0,6)} \arrow[dll, bend left, "0"] \\
& & \mathbb{C}^2_{(1,4,1)} \arrow[ull, bend left, "\cong"] \arrow[drr, "{\begin{bmatrix}0&1\\0&1\end{bmatrix}}"] & & & & \mathbb{C}^2_{(1,1,4)} \arrow[urr, bend left, "\cong"] \arrow[dll, "{\begin{bmatrix}0&0\\0&1\end{bmatrix}}"'] & & \\
& & & & \mathbb{C}^2_{(2,2,2)} \arrow[uu, "{\begin{bmatrix}1&0\\0&1\end{bmatrix}}"] \arrow[dll, "{\begin{bmatrix}1&0\\0&1\end{bmatrix}}"'] \arrow[drr, "{\begin{bmatrix}1&0\\0&1\end{bmatrix}}"] & & & & \\
& & \mathbb{C}^2_{(3,3,0)} \arrow[uu, "{\begin{bmatrix}1&-1\\0&0\end{bmatrix}}"] \arrow[drr, "{\begin{bmatrix}0&0\\0&1\end{bmatrix}}"] & & & & \mathbb{C}^2_{(3,0,3)} \arrow[uu, "{\begin{bmatrix}1&0\\0&0\end{bmatrix}}"'] \arrow[dll, "{\begin{bmatrix}0&0\\0&1\end{bmatrix}}"'] & & \\
& & & & \mathbb{C}^2_{(4,4,4)} \arrow[uu, "{\begin{bmatrix}1&0\\0&0\end{bmatrix}}"] \arrow[d, bend left, "\cong"] & & & & \\
& & & & \mathbb{C}^2_{(6,0,0)} \arrow[u, bend left, "0"] & & & &
\end{tikzcd}
$$

At $\mathbb{C}^2_{(2,2,2)}$ we can verify that

$$\operatorname{Ker}\big(\varphi^{(2,2,2)}_{(1,4,1)}\big) = \langle e_1+e_2\rangle,\ \operatorname{Ker}\big(\varphi^{(2,2,2)}_{(4,1,1)}\big) = \langle e_1\rangle \ \text{ and } \ \operatorname{Ker}\big(\varphi^{(2,2,2)}_{(1,1,4)}\big) = \langle e_2\rangle.$$

On the other hand, $\operatorname{Ker}\big(\varphi^{(2,2,2)}_{(3,3,0)}\big) = 0$ and $\operatorname{Ker}\big(\varphi^{(2,2,2)}_{(3,0,3)}\big) = 0$. Thus

$$\Big(\operatorname{Ker}\big(\varphi^{(2,2,2)}_{(1,4,1)}\big) + \operatorname{Ker}\big(\varphi^{(2,2,2)}_{(1,1,4)}\big)\Big) \cap \operatorname{Ker}\big(\varphi^{(2,2,2)}_{(4,1,1)}\big) = \langle e_1\rangle \neq 0 = \operatorname{Ker}\big(\varphi^{(2,2,2)}_{(3,3,0)}\big) \oplus \operatorname{Ker}\big(\varphi^{(2,2,2)}_{(3,0,3)}\big).$$

Let us show that $\mathfrak{g}$ occurs as a limit of linear series. To this end, let $C_\Delta := V(XYZ) \subset \mathbb{P}^2$, which has components $X_0 = V(X)$, $X_1 = V(Y)$ and $X_2 = V(Z)$. Define the surface $\mathfrak{X} := V(XYZ - TF) \subset \mathbb{P}^2 \times B$ for a general cubic F. As F is general, the surface $\mathfrak{X}$ is regular. Let $\pi_1 : \mathbb{P}^2 \times B \longrightarrow \mathbb{P}^2$ be the projection and $\pi : \mathfrak{X} \longrightarrow B$ be the restriction to $\mathfrak{X}$ of the projection $\pi_2 : \mathbb{P}^2 \times B \longrightarrow B$. Then π is a regular smoothing of C_Δ.

Consider the invertible sheaf $\mathscr{L} = \mathscr{O}_{\mathfrak{X}}(2) := \pi_1^*(\mathscr{O}_{\mathbb{P}^2}(2))|_{\mathfrak{X}}$, which has multidegree $(2,2,2)$ on C_Δ The effective locus of the multidegree quiver associated to $\mathscr{L}$ is Q_0^6.

The coordinates X, Y, and Z of $\mathbb{P}^2$ can be thought of as sections of $\mathscr{O}_{\mathbb{P}^2}(1)$ and restrict to sections of $\mathscr{O}_{\mathfrak{X}}(1)$ which we denote by x, y and z, respectively. Consider the linear system V_η of section of $L_\eta := \mathscr{L}|_{\mathfrak{X}_\eta}$ generated by $x(y+z)$ and $z(y-x)$.

For each divisor $D = \sum n_i X_i$ with $\min\{n_i\} = 0$ the sheaf $\mathscr{L}(D)$ may be viewed as a subsheaf of $\mathscr{O}_{\mathfrak{X}}(2+\sum n_i)$ with B-flat quotient. We use the embedding to view limit sections as sections of the larger sheaf $\mathscr{O}_{\mathfrak{X}}(2+\sum n_i)$, which may be understood as polynomials of degree $2+\sum n_i$ on x,y,z.

In the figure below we describe bases for the spaces of limit sections in each effective multidegree as polynomials in x,y,z. As for the maps between these spaces, the maps labeled X, Y and Z are multiplication by x, y and z, respectively, whenever their targets are spaces of polynomials of higher degree than those at the source. Otherwise, they are that multiplication followed by division by F, keeping in mind that $xyz = 0$ on the curve C_Δ.

$$
\begin{tikzcd}
\{x^3z^2(y+z), Fxz(x+z)\} \arrow[dr, "Y"] & & \{x^2(x+z), xz(y-x)\} \arrow[dl, "Z"'] \arrow[dr, "Y"] & & \{x^3y^2(y+z), Fxy(y-x)\} \arrow[dl, "Z"'] \\
& \{x^2z(y+z), F(x+z)\} \arrow[dr, "Y"] & & \{x^2y(y+z), F(y-x)\} \arrow[dl, "Z"'] & \\
\mathfrak{g}: & & \{x(y+z), z(y-x)\} \arrow[uu, "X"] \arrow[dl, "Z"'] \arrow[dr, "Y"] & & \\
& \{xz(y+z), z^2(y-x)\} \arrow[uu, "X"] \arrow[dr, "Y"] & & \{xy(y+z), yz(y-x)\} \arrow[uu, "X"] \arrow[dl, "Z"'] & \\
& & \{F(y+z), yz^2(y-x)\} \arrow[uu, "X"] & & \\
& & \{Fyz(y+z), y^2z^3(y-x)\} \arrow[u, "X"] & &
\end{tikzcd}
$$

Using the bases, the maps are represented by the matrices in the first figure of the example. This shows that $\mathfrak{g}$ occurs as the limit along π of the linear series

$$(\mathscr{L}_{|\mathfrak{X}_\eta}, \langle x(y+z), z(y-x)\rangle_{|\mathfrak{X}_\eta}).$$

Example 40. Although the next example shows an exact linked net $\mathfrak{g}$ with no simple basis, it can be deformed over $\mathrm{Spec}(\mathbb{C}[[T]])$, and we can verify that the multivariate Hilbert polynomial of $\mathbb{LP}(\mathfrak{g})$ is equal to that of the diagonal in $\prod_{v\in Q^5}\mathbb{P}(V_v)$; more on this in Chapter 5.

The exact linked net $\mathfrak{g}$ of vector spaces in this example has support on the quiver Q^5, as classified in Chapter 3.

$$\mathfrak{g}:\quad \begin{tikzcd}
\mathbb{C}^3_6 \arrow[drr, "\begin{bmatrix}0&1&0\\0&1&0\\0&0&0\end{bmatrix}"] & & & & \mathbb{C}^3_7 \arrow[dll, "\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}"'] \\
& & \mathbb{C}^3_5 \arrow[dll, "\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}"'] \arrow[drr, "\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}"] & & \\
\mathbb{C}^3_3 \arrow[uu, "\begin{bmatrix}1&-1&0\\0&0&0\\0&0&1\end{bmatrix}"] \arrow[drr, "\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}"] & & & & \mathbb{C}^3_4 \arrow[uu, "\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}"'] \arrow[dll, "\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}"'] \\
& & \mathbb{C}^3_2 \arrow[uu, "\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}"] & & \\
& & \mathbb{C}^3_1 \arrow[u, "\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}"] & &
\end{tikzcd}$$

In the attempt to find a simple basis for $\mathfrak{g}$ we must pick — in terms of canonical bases — $e_1 \in \mathbb{C}^3_2$, $e_3 \in \mathbb{C}^3_5$, $e_2 \in \mathbb{C}^3_6$ and $e_2 \in \mathbb{C}^3_7$. As a necessary condition to obtain a simple basis is that we must pick exactly three vectors, we see that $\mathfrak{g}$ does not admit a simple basis.

As a less heuristic argument, we may show the intersection property fails at some vertex. Indeed, remember from Diagram (3.2) the convention we use in our pictures for the types of arrows:

$$\begin{tikzcd}
v_1 \arrow[dr, "\{2\}"] & \\
& v_2 \arrow[dl, "\{1\}"] \\
v_0 \arrow[uu, "\{0\}"] &
\end{tikzcd}$$

It is not difficult to verify that

$$\operatorname{Ker}\big(\varphi^5_{\{0,1\}}\big) = \langle e_1 + e_2\rangle,\ \operatorname{Ker}\big(\varphi^5_{\{0,2\}}\big) = \langle e_2\rangle \text{ and } \operatorname{Ker}\big(\varphi^5_{\{1,2\}}\big) = \langle e_1\rangle.$$

On the other hand,

$$\operatorname{Ker}\big(\varphi^5_{\{1\}}\big) = 0 \text{ and } \operatorname{Ker}\big(\varphi^5_{\{2\}}\big) = 0.$$

Thus

$$\Big(\mathrm{Ker}\big(\varphi^5_{\{0,1\}}\big)+\mathrm{Ker}\big(\varphi^5_{\{0,2\}}\big)\Big)\bigcap \mathrm{Ker}\big(\varphi^5_{\{1,2\}}\big)=\langle e_1\rangle\neq 0=\mathrm{Ker}\big(\varphi^5_{\{1\}}\big)+\mathrm{Ker}\big(\varphi^5_{\{2\}}\big).$$

Thus $\mathfrak{g}$ does not satisfy the intersection property at the vertex 5, and hence admits no simple basis by Theorem 37.

Chapter 5

Linked projective space

The goal of this Chapter is to prove that for an exact linked net of vector spaces $\mathfrak{g}$ over a $\mathbb{Z}^2$-quiver Q, its linked projective space $\mathbb{LP}(\mathfrak{g})$ is Cohen–Macaulay and reduced with pure dimension equal to $\dim \mathfrak{g} - 1$.

The first step is to prove Proposition 41, which states the result for exact linked nets of vector spaces with support contained in a Q_1^3 subquiver. After that, we use this proposition to prove Theorem 42.

5.1 The linked projective space

Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver Q. Assume also that $\mathfrak{g}$ has finite support H. We define the **linked projective space** of $\mathfrak{g}$, denoted $\mathbb{LP}(\mathfrak{g})_H$, as the quiver Grassmannian of pure one-dimensional subrepresentations of the restriction of $\mathfrak{g}$ to H, that is,

$$\mathbb{LP}(\mathfrak{g})_H := \{\big([w_v] \mid v \in H\big) \in \prod_{v \in H} \mathbb{P}(V_v) \mid \varphi_{v_2}^{v_1}(w_{v_1}) \wedge w_{v_2} = 0 \text{ for all } v_1, v_2 \in H\}.$$

Note that there exists an injection from the Grassmannian of pure one-dimensional subrepresentations of $\mathfrak{g}$ over the whole quiver Q to $\mathbb{LP}(\mathfrak{g})_H$, induced by restriction.

This restriction is also surjective when $P(H) = H$, you can see the detailed proof in [17, Prop. 7.1]. In this case, for each $v \in Q$ there is a unique vertex $b_v \in H$ such that for each $v_1 \in H$ there is an admissible path from v_1 to v passing through b_v. (Clearly, $b_v = v$ if $v \in H$.) Then, given a larger subset of vertices $H \subseteq H'$ with $P(H') = H'$, the map

$$\Psi : \prod_{v \in H} \mathbb{P}(V_v) \longrightarrow \prod_{v \in H'} \mathbb{P}(V_v)$$

carrying $([w_v] \mid v \in H)$ to $([\varphi_v^u(w_{b_u}) \mid u \in H'])$ is a graph. Furthermore, Esteves *et al.* [7, § 3] show that, not only does Ψ restrict to an isomorphism from $\mathbb{LP}(\mathfrak{g})_H$ to $\mathbb{LP}(\mathfrak{g})_{H'}$, but also that the multivariate Hilbert polynomial

$$\mathrm{Hilb}_{\mathbb{LP}(\mathfrak{g})_{H'}}(x_v \mid u \in H')$$

is obtained from $\mathrm{Hilb}_{\mathbb{LP}(\mathfrak{g})_H}(x_v \mid v \in H)$ by replacing each x_v for $v \in H$ by the sum of the x_u for all $u \in H'$ such that $b_u = v$, for the complete argument the reader can consult [17, § 7.1]. Thus, the multivariate Hilbert polynomial of $\mathbb{LP}(\mathfrak{g})_H$ is equal to that of the diagonal scheme of $\prod_{v \in H} \mathbb{P}(V_v)$ if and only if so is $\mathrm{Hilb}_{\mathbb{LP}(\mathfrak{g})_{H'}}$. When H is clear from context we will write just $\mathbb{LP}(\mathfrak{g})$, instead of $\mathbb{LP}(\mathfrak{g})_H$.

For each $v \in H$ define

$$\mathbb{LP}(\mathfrak{g})_v^* := \{ \big([\varphi_u^v(w)]\big) \in \mathbb{LP}(\mathfrak{g})_H \mid \varphi_u^v(w) \neq 0 \text{ for all } u \in H \}.$$

Observe the notation, that is, although we consider the linked projective space on H, in the definition above we omit H so that the notation does not become too heavy. Also, $\mathbb{LP}(\mathfrak{g})_v^*$ is the open set consisting of the one-dimensional linked subnets of $\mathfrak{g}$ which have simple bases at v. It is isomorphic to an open subscheme of $\mathbb{P}(V_v)$, and is thus irreducible of dimension $\dim \mathfrak{g} - 1$, if not empty.

Define

$$\mathbb{LP}(\mathfrak{g})_v := \overline{\mathbb{LP}(\mathfrak{g})_v^*} \quad \text{for each } v \in H,$$

the Zariski closure of $\mathbb{LP}(\mathfrak{g})_v^*$. Santos concludes in [17, § 7.1] that

$$\mathbb{LP}(\mathfrak{g}) = \bigcup_{v \in H} \mathbb{LP}(\mathfrak{g})_v.$$

In particular, $\mathbb{LP}(\mathfrak{g})$ is of pure dimension $\dim \mathfrak{g} - 1$.

5.2 Cohen–Macaulayness of $\mathbb{LP}(\mathfrak{g})$

Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver Q. Let $Q_\Delta = Q_1^3$ be a triangular subquiver of Q, the full subquiver supported on a triangle $H := \{v_0, v_1, v_2\}$. Suppose $\mathfrak{g}$ is exact and supported on H. Notice that $P(H) = H$. Let $r := \dim \mathfrak{g} - 1$.

$$\mathfrak{g}: \quad \begin{tikzcd} V_{v_1} \arrow[dr, "\varphi^{v_1}_{v_2}"] & \\ & V_{v_2} \arrow[dl, "\varphi^{v_2}_{v_0}"] \\ V_{v_0} \arrow[uu, "\varphi^{v_0}_{v_1}"] & \end{tikzcd} \tag{5.1}$$

As $\mathfrak{g}$ is exact, it has simple basis in Q_Δ by Theorem 37. Let us denote such a simple basis by

$$\mathscr{B} = \{\underbrace{e_1, \cdots, e_{r_0}}_{\text{in } V_{v_0}}, \underbrace{e_{r_0+1}, \cdots, e_{r_0+r_1}}_{\text{in } V_{v_1}}, \underbrace{e_{r_0+r_1+1}, \cdots, e_{r_0+r_1+r_2}}_{\text{in } V_{v_2}}\}.$$

Write $V_{v_i} = V_{i,0} \oplus V_{i,1} \oplus V_{i,2}$ for $i = 0, 1, 2$, where $V_{i,j}$ is the subspace generated by the images of the elements in the simple basis belonging to V_{v_j}. Then $r_j = \dim(V_{i,j})$ for each i and j. Also,

$$r_0 + r_1 + r_2 = r + 1.$$

The maps $\varphi^{v_0}_{v_1}$, $\varphi^{v_1}_{v_2}$ and $\varphi^{v_2}_{v_0}$ can be expressed by the diagonal matrices

$$M^0_1 = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 1 \end{bmatrix}, \quad M^1_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \quad \text{and} \quad M^2_0 = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix},$$

respectively.

Lemma 41. Let $\mathfrak{g}$ be an exact linked net of vector spaces over the $\mathbb{Z}^2$-quiver Q. If $\mathfrak{g}$ has support in a triangle then $\mathbb{LP}(\mathfrak{g})$ is Cohen–Macaulay.

Proof. Assume $\mathfrak{g}$ as before. We have

$$\mathbb{LP}(\mathfrak{g}) = \{([x_0], [x_1], [x_2]) \in \mathbb{P}^r \times \mathbb{P}^r \times \mathbb{P}^r \mid M^0_1 x_0 \wedge x_1 = M^1_2 x_1 \wedge x_2 = M^2_0 x_2 \wedge x_0 = 0\}.$$

Write $x_i = (x_{i,0}, x_{i,1}, x_{i,2})$ for $i = 0, 1, 2$. The defining equations of $\mathbb{LP}(\mathfrak{g})$ become

$$(x_{0,0}, x_{0,2}) \wedge (x_{1,0}, x_{1,2}) = 0 \tag{5.2}$$
$$(x_{0,0}, x_{0,2}) \otimes x_{1,1} = 0 \tag{5.3}$$
$$(x_{1,0}, x_{1,1}) \wedge (x_{2,0}, x_{2,1}) = 0 \tag{5.4}$$
$$(x_{1,0}, x_{1,1}) \otimes x_{2,2} = 0 \tag{5.5}$$
$$(x_{2,1}, x_{2,2}) \wedge (x_{0,1}, x_{0,2}) = 0 \tag{5.6}$$
$$(x_{2,1}, x_{2,2}) \otimes x_{0,0} = 0. \tag{5.7}$$

For each $i,j \in \{0,1,2\}$ let $D_{i,j}$ be the open subset of $\mathbb{P}^r \times \mathbb{P}^r \times \mathbb{P}^r$ where $x_{i,j} \neq 0$.

We claim that

$$\mathbb{LP}(\mathfrak{g}) \subseteq D_{0,1} \cup D_{1,2} \cup D_{2,0}.$$

Indeed, if

$$\mathfrak{h} = \big([x_0],[x_1],[x_2]\big) \in \mathbb{LP}(\mathfrak{g})$$

were such that $x_{0,1}$, $x_{1,2}$ and $x_{2,0}$ are zero, then $M_1^0 x_0$, $M_2^1 x_1$ and $M_0^2 x_2$ would be nonzero, and thus nonzero multiples of x_1, x_2 and x_0, respectively. But then $M_0^2 M_2^1 M_1^0 x_0$ would be nonzero, an absurd.

As Cohen–Macaulayness is a local property, and by symmetry, we need only prove $\mathbb{LP}(\mathfrak{g}) \cap D_{0,1}$ is Cohen–Macaulay. As $D_{0,1}$ is nonsingular of dimension $3r$ and $\mathbb{LP}(\mathfrak{g})$ has pure dimension r, we need only show that $\mathbb{LP}(\mathfrak{g}) \cap D_{0,1}$ is locally given by $2r$ equations in $D_{0,1}$.

Locally, the i-th entry of $x_{0,1}$ is nonzero for some i; we may suppose the value of the entry is 1. Let y be the i-th entry of $x_{2,1}$. Then Equation (5.6) is equivalent to $x_{2,1} = yx_{0,1}$ and $x_{2,2} = yx_{0,2}$, a total of $r_1 + r_2 - 1$ equations. This implies that $x_{2,0} \neq 0$ or $x_{2,1} \neq 0$. In other words,

$$\mathbb{LP}(\mathfrak{g}) \cap D_{0,1} = \Big(\mathbb{LP}(\mathfrak{g}) \cap D_{0,1} \cap D_{2,0}\Big) \cup \Big(\mathbb{LP}(\mathfrak{g}) \cap D_{0,1} \cap D_{2,1}\Big).$$

Let us first show that $\mathbb{LP}(\mathfrak{g}) \cap D_{0,1} \cap D_{2,0}$ is locally given by $2r$ equations in $D_{0,1} \cap D_{2,0}$. Locally, the j-th entry of $x_{2,0}$ is nonzero for some j; we may suppose the value of the entry is 1. Let z be the j-th entry of $x_{1,0}$. Then Equation (5.4) is equivalent to $x_{1,0} = zx_{2,0}$ and $x_{1,1} = zx_{2,1}$, a total of $r_0 + r_1 - 1$ equations. This implies that $x_{1,0} \neq 0$ or $x_{1,2} \neq 0$. In other words,

$$\mathbb{LP}(\mathfrak{g}) \cap D_{0,1} \cap D_{2,0} = \Big(\mathbb{LP}(\mathfrak{g}) \cap D_{0,1} \cap D_{2,0} \cap D_{1,0}\Big) \cup \Big(\mathbb{LP}(\mathfrak{g}) \cap D_{0,1} \cap D_{2,0} \cap D_{1,2}\Big).$$

Let us now look at the equations of $\mathbb{LP}(\mathfrak{g}) \cap D_{0,1} \cap D_{2,0} \cap D_{1,2}$ in $D_{0,1} \cap D_{2,0} \cap D_{1,2}$. Locally, the ℓ-th entry of $x_{1,2}$ is nonzero for some ℓ; we may suppose the value of the entry is 1. Let w be the ℓ-th entry of $x_{0,2}$. Then Equation (5.2) is equivalent to $x_{0,0} = wx_{1,0}$ and $x_{0,2} = wx_{1,2}$, at total of $r_0 + r_2 - 1$ equations. The remaining equations become

$$(wzx_{2,0}, wx_{1,2}) \otimes zyx_{0,1} = 0 \tag{5.8}$$

$$(zx_{2,0}, zyx_{0,1}) \otimes ywx_{1,2} = 0 \tag{5.9}$$

$$(yx_{0,1}, ywx_{1,2}) \otimes wzx_{2,0} = 0, \tag{5.10}$$

which are all equivalent to $ywz = 0$. We have thus

$$(r_1 + r_2 - 1) + (r_0 + r_1 - 1) + (r_0 + r_2 - 1) + 1 = 2r$$

equations.

Let us then look at the equations of $\mathbb{LP}(\mathfrak{g}) \cap D_{0,1} \cap D_{2,0} \cap D_{1,0}$ in $D_{0,1} \cap D_{2,0} \cap D_{1,0}$. As $x_{1,0} = zx_{2,0}$ and $x_{2,0} \neq 0$, we have that $x_{1,0} \neq 0$ is equivalent to $z \neq 0$; we may suppose $z = 1$. Let w now be the j-th entry of $x_{0,0}$. Then Equation (5.2) is equivalent to $x_{0,0} = wx_{1,0}$ and $x_{0,2} = wx_{1,2}$, again a total of $r_0 + r_2 - 1$ equations. And the remaining equations become

$$(wx_{2,0}, wx_{1,2}) \otimes yx_{0,1} = 0 \tag{5.11}$$
$$(x_{1,0}, yx_{0,1}) \otimes ywx_{1,2} = 0 \tag{5.12}$$
$$(yx_{0,1}, ywx_{1,2}) \otimes wx_{1,0} = 0, \tag{5.13}$$

which are equivalent to $yw = 0$. We have $2r$ equations, as before.

Similarly, let us show that $\mathbb{LP}(\mathfrak{g}) \cap D_{0,1} \cap D_{2,1}$ is locally given by $2r$ equations in $D_{0,1} \cap D_{2,1}$. As $x_{2,1} = yx_{0,1}$ and $x_{0,1} \neq 0$, we have that $x_{2,1} \neq 0$ is equivalent to $y \neq 0$; we may suppose $y = 1$. Let z be the i-th entry of $x_{1,1}$. Then Equation (5.4) is equivalent to $x_{1,1} = zx_{2,1}$ and $x_{1,0} = zx_{2,0}$, a total of $r_0 + r_1 - 1$ equations. This implies that $x_{1,1} \neq 0$ or $x_{1,2} \neq 0$. In other words,

$$\mathbb{LP}(\mathfrak{g}) \cap D_{0,1} \cap D_{2,1} = \Big(\mathbb{LP}(\mathfrak{g}) \cap D_{0,1} \cap D_{2,1} \cap D_{1,1}\Big) \cup \Big(\mathbb{LP}(\mathfrak{g}) \cap D_{0,1} \cap D_{2,1} \cap D_{1,2}\Big).$$

Let us then look at the equations of $\mathbb{LP}(\mathfrak{g}) \cap D_{0,1} \cap D_{2,1} \cap D_{1,2}$ in $D_{0,1} \cap D_{2,1} \cap D_{1,2}$. Locally, the j-th entry of $x_{1,2}$ is nonzero for some j; we may suppose the entry is 1. Let w be the j-th entry of $x_{0,2}$. Then Equation (5.2) is equivalent to $x_{0,0} = wx_{1,0}$ and $x_{0,2} = wx_{1,2}$, at total of $r_0 + r_2 - 1$ equations. And the remaining equations become

$$(wzx_{2,0}, wx_{1,2}) \otimes zx_{2,1} = 0 \tag{5.14}$$
$$(zx_{2,0}, zx_{2,1}) \otimes wx_{1,2} = 0 \tag{5.15}$$
$$(x_{2,1}, wx_{1,2}) \otimes wzx_{2,0} = 0, \tag{5.16}$$

which are equivalent to $wz = 0$. We have $2r$ equations.

Finally, let us look at the equations of $\mathbb{LP}(\mathfrak{g}) \cap D_{0,1} \cap D_{2,1} \cap D_{1,1}$ in $D_{0,1} \cap D_{2,1} \cap D_{1,1}$. As $x_{1,1} = zx_{2,1}$ and $x_{2,1} \neq 0$, we have that $x_{1,1} \neq 0$ is equivalent to $z \neq 0$; we may suppose $z = 1$. Equations (5.3), (5.5) and (5.7) are equivalent to $x_{0,0} = 0$ and $x_{0,2} = x_{2,2} = 0$, a total of $r_0 + 2r_2$ equations, whereas Equation (5.2) is a consequence of these. Under these equations, Equation (5.6) is equivalent to $x_{2,1} = x_{0,1}$, a total of $r_1 - 1$ equations. We have thus a total of

$$(r_1-1)+(r_0+r_1-1)+r_0+2r_2=2r$$

equations. $\square$

Now we can prove the following theorem:

Theorem 42. Let $\mathfrak{g}$ be an exact linked net of vector spaces over a $\mathbb{Z}^2$-quiver with finite support and dimension $r+1$. Then $\mathbb{LP}(\mathfrak{g})$ is Cohen–Macaulay and reduced with pure dimension r.

Proof. Since $\mathbb{LP}(\mathfrak{g})$ is generically nonsingular by [7, Thm. 8.2], if $\mathbb{LP}(\mathfrak{g})$ is Cohen–Macaulay then $\mathbb{LP}(\mathfrak{g})$ is reduced, by [9, Prop. 14.124]. Thus, it is enough to show that $\mathbb{LP}(\mathfrak{g})$ is Cohen–Macaulay.

Let H be a finite collection of vertices of Q supporting $\mathfrak{g}$ such that $P(H)=H$. Let $\mathfrak{h}=([s_v]\,|\,v\in H)\in\mathbb{LP}(\mathfrak{g})$.

By [7, Thm. 5.1], there is a triangle quiver Q_1^3 with vertices $\Delta=\{v_0,v_1,v_2\}$ generating $\mathfrak{h}$. We may assume that v_0 is connected to v_1 by an arrow $a_2\in A_2$, that v_1 is connected to v_2 by an arrow $a_1\in A_1$ and v_2 is connected to v_0 by an arrow $a_0\in A_0$. Let R_0 (resp. R_1, resp. R_2) be the collection of endpoints of paths γ leaving v_2 (resp. v_1, resp. v_0) with $\gamma(0)=0$ (resp. $\gamma(1)=0$, resp. $\gamma(2)=0$). The R_i are pairwise disjoint and their union is the whole vertex set of Q; see Figure 5.1.

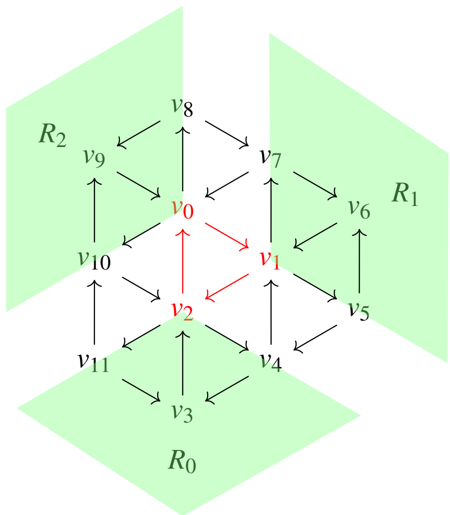


Fig. 5.1 $\mathfrak{h}$ generated by the vertices v_0, v_1 and v_2.

Let $\mathfrak{g}_\Delta$ be the representation of Q with the same restriction as $\mathfrak{g}$ to the full subquiver Q_1^3 of Q with vertices in Δ, but defined elsewhere by:

1. $V_u=V_{v_2}$ for each $u\in R_0$; likewise for R_1 and R_2.

2. $\varphi_b = \mathrm{id}_{V_{v_2}}$ for each arrow $b \in A_1 \cup A_2$ and $\varphi_b = 0$ for each arrow $b \in A_0$ connecting vertices of R_0; likewise for R_1 and R_2.

3. $\varphi_b = \varphi_{v_0}^{v_2}$ for each arrow b connecting a vertex of R_0 to one of R_2 and $\varphi_b = \varphi_{v_2}^{v_1}\varphi_{v_1}^{v_0}$ for each arrow b connecting a vertex of R_2 to one of R_0; likewise for the arrows connecting vertices of R_0 and R_1 and vertices of R_1 and R_2.

Then $\mathfrak{g}_\Delta$ is an exact linked net of vector spaces over Q with support in the triangle quiver Q_1^3, and hence $\mathbb{LP}(\mathfrak{g}_\Delta)$ is Cohen–Macaulay by Lemma 41.

Now, the $\mathfrak{x} = \big([x_v]\,|, v \in H\big) \in \mathbb{LP}(\mathfrak{g})$ generated by Δ form an open subscheme U_Δ given by $\varphi_v^{v_0}(x_{v_0}) \neq 0$ or $\varphi_v^{v_1}(x_{v_1}) \neq 0$ or $\varphi_v^{v_2}(x_{v_2}) \neq 0$ for each $v \in H$. For each such $\mathfrak{x}$ there is a corresponding $\mathfrak{y} = \big([y_v]\,|, v \in \Delta\big) \in \mathbb{LP}(\mathfrak{g}_\Delta)$ given by letting $y_v = x_{v_2}$ (resp. $y_v = x_{v_1}$, resp. $y_v = x_{v_0}$) for each $v \in R_0$ (resp. $v \in R_1$, resp. $v \in R_2$). Let

$$\Theta\colon U_\Delta \longrightarrow \mathbb{LP}(\mathfrak{g}_\Delta) \subseteq \mathbb{P}(V_{v_0}) \times \mathbb{P}(V_{v_1}) \times \mathbb{P}(V_{v_2})$$

be the map taking $\mathfrak{x}$ to $\mathfrak{y}$. Of course, $\mathfrak{y}$ determines $\mathfrak{x}$. The image of Θ is the open subset U of $\mathbb{LP}(\mathfrak{g}_\Delta)$ given by $\varphi_v^{v_0}(y_{v_0}) \neq 0$ or $\varphi_v^{v_1}(y_{v_1}) \neq 0$ or $\varphi_v^{v_2}(y_{v_2}) \neq 0$ for each $v \in H$. It follows that Θ is an isomorphism from U_Δ to U. Since $\mathbb{LP}(\mathfrak{g}_\Delta)$ is Cohen–Macaulay, so is U_Δ. As U_Δ is a neighborhood of $\mathfrak{h}$, we have that $\mathbb{LP}(\mathfrak{g})$ is Cohen–Macaulay around $\mathfrak{h}$. As $\mathfrak{h} \in \mathbb{LP}(\mathfrak{g})$ was arbitrary, $\mathbb{LP}(\mathfrak{g})$ is Cohen–Macaulay. □

Xiang He and Naizhen Zhang [10] proved the Cohen–Macaulayness of certain quiver Grassmannians but for other quivers and in the context of degenerations of Grassmannians constructed using convex lattice configurations in Bruhat–Tits buildings.

5.3 The multivariate Hilbert polynomial of $\mathbb{LP}(\mathfrak{g})$

Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^2$-quiver Q with finite support and dimension $r+1$. Let $B := \mathrm{Spec}(\mathbb{C}[[T]])$. We say that $\mathfrak{g}$ **extends** if there is a representation $\mathfrak{g}(T)$ of Q in the category of free $C[[T]]$-modules satisfying the following three properties:

1. $\mathfrak{g}$ is obtained from $\mathfrak{g}(T)$ by setting $T := 0$, that is, by tensoring the modules and maps giving $\mathfrak{g}(T)$ by the residue field of $\mathbb{C}[[T]]$.

2. All the maps in $\mathfrak{g}(T)$ extend to isomorphisms over the field of fractions of $\mathbb{C}[[T]]$.

3. If γ_1 and γ_2 are admissible paths connecting the same two vertices the corresponding maps of $\mathbb{C}[[T]]$-modules are equal.

If $\mathfrak{g}$ arises from a degeneration of linear series then $\mathfrak{g}$ extends over B. This is clear from the construction in Chapter 2, where the spaces V_v arise as quotients of $\mathbb{C}[[T]]$-modules $\mathscr{V}_v$ and the maps $\varphi_{v_2}^{v_1}: V_{v_1} \to V_{v_2}$ are induced from maps of $\mathbb{C}[[T]]$-modules $\varphi_{v_2}^{v_1}: \mathscr{V}_{v_1} \to \mathscr{V}_{v_2}$.

Proposition 43. Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^2$-quiver Q. If $\mathfrak{g}$ has a simple basis then $\mathfrak{g}$ extends.

Proof. If $\mathfrak{g}$ has a simple basis, we can express the arrow maps of $\mathfrak{g}$ using diagonal matrices with only 0 and 1 on the diagonal. Replacing the zeros on each such diagonal by T we obtain a representation $\mathfrak{g}(T)$. To show it extends $\mathfrak{g}$, let γ be an admissible path. Let v_1 be its initial point and v_2 its final point. Let $u_0, \dots, u_r$ be vertices of Q and s_i be a section of $\mathfrak{g}$ at u_i for each $i = 0, \dots, r$ forming a simple basis. For each $i = 0, \dots, r$, let μ_i be an admissible path connecting u_i to v_1 and let $a_i := \min(\mu_i(j) + \gamma(j))$. Then the composition of the arrow maps prescribed by γ yields a map of $\mathbb{C}[[T]]$-modules expressed by a diagonal matrix whose i-th diagonal entry is T^{a_i}. In particular, the same map is obtained replacing γ by any other admissible path connecting v_1 to v_2. It follows that $\mathfrak{g}(T)$ is an extension of $\mathfrak{g}$. □

Theorem 44. Let $\mathfrak{g}$ be an exact linked net of vector spaces over a $\mathbb{Z}^2$-quiver Q of dimension $r+1$. Assume $\mathfrak{g}$ has finite support H with $P(H) = H$. If $\mathfrak{g}$ extends then $\mathbb{LP}(\mathfrak{g}) \subseteq \prod_{v \in H} \mathbb{P}(V_v)$ has multivariate Hilbert polynomial

$$\mathrm{Hilb}_{\mathbb{LP}(\mathfrak{g})}(x_v \mid v \in H) = \binom{r + \sum x_v}{r}.$$

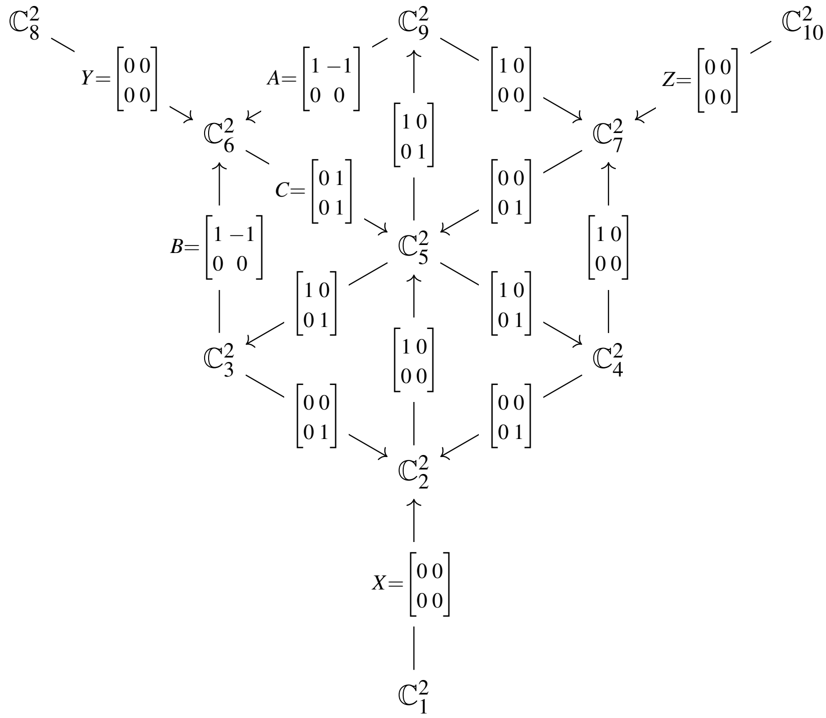
Proof: Let $\mathfrak{g}(T)$ be an extension of $\mathfrak{g}$. For each vertex v of Q, let $\mathscr{V}_v$ denote the corresponding $\mathbb{C}[[T]]$-module and for each two vertices v_1, v_2 of Q let $\varphi_{v_2}^{v_1}: \mathscr{V}_{v_1} \to \mathscr{V}_{v_2}$ be the corresponding map of $\mathbb{C}[[T]]$-modules. Let $\mathbb{LP}$ be the subscheme of $\prod_{v \in H} \mathbb{P}_B(\mathscr{V}_v)$ defined as $\mathbb{LP}(\mathfrak{g})$ was. Then the map $\pi: \mathbb{LP} \to B$ has special fibre equal to $\mathbb{LP}(\mathfrak{g})$ and generic fibre equal to the diagonal up to the action of (linear) automorphisms. As the multivariate Hilbert polynomial is constant on fibres of a flat map, it is enough to show that π is flat.

The geometric fibres of π are reduced, that over the special point of B so because of Theorem 42. The irreducible components of $\mathbb{LP}(\mathfrak{g})$ are the non-empty $\mathbb{LP}(\mathfrak{g})_v$, for vertices v of Q such that $\mathbb{LP}(\mathfrak{g})_v^* \neq \emptyset$. For each such v, let $s \in V_v$ such that $\varphi_u^v(s) \neq 0$ for each $u \in H$. Let $\sigma \in \mathscr{V}_v$ be a lift of s. Then the $\varphi_u^v(\sigma)$ yield a section of π passing through the point on $\mathbb{LP}(\mathfrak{g})_v^*$ corresponding to s. It follows that every irreducible component of $\mathbb{LP}$ dominates B. As a consequence, $\pi_{\mathrm{red}}: \mathbb{LP}_{\mathrm{red}} \to B$ is flat. But then $\mathbb{LP}$ is reduced and π is flat by Osserman, Lemma 6.13 [14]. □

5.4 Example

Example 45. We know that the exact linked net from Example 39 does not have a simple basis. In this example we show that, despite the absence of a simple basis, $\mathfrak{g}$ admits an extension $\mathfrak{g}(T)$ such that the composition of the maps of $\mathfrak{g}(T)$ along a triangle is T times the identity.

Recall $\mathfrak{g}$ on the effective locus H:



To define $\mathfrak{g}(T)$, we put T instead of zero on the diagonal of all diagonal matrices. As for the nondiagonal matrices A,B,C in the rhombus with vertices $3,5,6,9$, we do a base change to get diagonal matrices in the rhombus, replace the 0 on the diagonals by T, and change back to the original matrices.

To make it more precise, let us denote by $\mathscr{A}_i = (e^i_1, e^i_2)$ the canonical basis for each $i = 1,\dots,10$. The matrices in the above figure express the maps in these bases. Consider the images of $\mathscr{B} = (e^5_1, e^6_2)$ in the rhombus. They form bases:

$$\begin{aligned}
\mathscr{B}_3 &= (e^3_1, e^3_1 + e^3_2);\\
\mathscr{B}_5 &= (e^5_1, e^5_1 + e^5_2);\\
\mathscr{B}_6 &= (e^6_1, e^6_2);\\
\mathscr{B}_9 &= (e^9_1, e^9_1 + e^9_2).
\end{aligned}$$

Denote by R, S, U and V the basis change matrices of $\mathbb{C}^2_3, \mathbb{C}^2_5, \mathbb{C}^2_6$ and $\mathbb{C}^2_9$ respectively, taking $\mathscr{A}_i$ to $\mathscr{B}_i$ for $i = 3, 5, 6, 9$. Then

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

whereas

$$R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ R^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \ U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ V^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

They diagonalize the matrices A, B and C:

$$D(A) = U^{-1}AV = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ P(A) := I - D(A) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix};$$
$$D(B) = U^{-1}BR = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ P(B) := I - D(B) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix};$$
$$D(C) = S^{-1}CU = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \ P(C) := I - D(C) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then we define $A(T), B(T)$ and $C(T)$ by:

$$A(T) = B(T) = A + TUP(A)V^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & T \end{bmatrix} \text{ and } C(T) = C + TSP(C)U^{-1} = \begin{bmatrix} T & 1 \\ 0 & 1 \end{bmatrix}.$$

The representation $\mathfrak{g}(T)$ is

$$\begin{tikzcd}
\mathbb{C}^2_8 \arrow[dr, "{\begin{bmatrix}T&0\\0&T\end{bmatrix}}" description] & & \mathbb{C}^2_9 \arrow[dl, "{\begin{bmatrix}1&-1\\0&T\end{bmatrix}}" description] \arrow[dr, "{\begin{bmatrix}1&0\\0&T\end{bmatrix}}" description] & & \mathbb{C}^2_{10} \arrow[dl, "{\begin{bmatrix}T&0\\0&T\end{bmatrix}}" description] \\
& \mathbb{C}^2_6 \arrow[dr, "{\begin{bmatrix}T&1\\0&1\end{bmatrix}}" description] & & \mathbb{C}^2_7 \arrow[dl, "{\begin{bmatrix}T&0\\0&1\end{bmatrix}}" description] & \\
\mathfrak{g}(T) & & \mathbb{C}^2_5 \arrow[uu, "{\begin{bmatrix}1&0\\0&1\end{bmatrix}}" description] \arrow[dl, "{\begin{bmatrix}1&0\\0&1\end{bmatrix}}" description] \arrow[dr, "{\begin{bmatrix}1&0\\0&1\end{bmatrix}}" description] & & \\
& \mathbb{C}^2_3 \arrow[uu, "{\begin{bmatrix}1&-1\\0&T\end{bmatrix}}" description] \arrow[dr, "{\begin{bmatrix}T&0\\0&1\end{bmatrix}}" description] & & \mathbb{C}^2_4 \arrow[uu, "{\begin{bmatrix}1&0\\0&T\end{bmatrix}}" description] \arrow[dl, "{\begin{bmatrix}T&0\\0&1\end{bmatrix}}" description] & \\
& & \mathbb{C}^2_2 \arrow[uu, "{\begin{bmatrix}1&0\\0&T\end{bmatrix}}" description] & & \\
& & \mathbb{C}^2_1 \arrow[u, "{\begin{bmatrix}T&0\\0&T\end{bmatrix}}" description] & &
\end{tikzcd}$$

Then $\mathfrak{g}(T)$ is an extension of $\mathfrak{g}$. It follows that $\mathbb{LP}(\mathfrak{g})$ has multivariate Hilbert polynomial equal to that of the diagonal, that is:

$$\mathrm{Hilb}_{\mathbb{LP}(\mathfrak{g})}\big(x_v \mid v \in H\big) = \binom{1 + \sum x_v}{1} = \sum_{v \in H} x_v.$$

Chapter 6

Complete collineations and linked nets

6.1 Complete collineations and exact linked nets

In this chapter we show how to associate to a complete collineation an exact linked net of vector spaces with finite support over the standard $\mathbb{Z}^1$-quiver, the quiver whose vertex set is $\mathbb{Z}$ and whose arrow set is the set of pairs $(i,j) \in \mathbb{Z}^2$ with $|i-j| = 1$. We also show how to associate to an exact linked net of vectors spaces of finite support a complete collineation. And we show the two associations are mutually inverse, up isomorphism. More precisely, we show a bijective correspondence:

$$\left\{\begin{matrix}\text{exact linked nets of vector}\\ \text{spaces over the } \mathbb{Z}^1\text{-quiver}\end{matrix}\right\} / \sim \;\underset{\beta}{\overset{\alpha}{\rightleftarrows}}\; \left\{\text{complete collineations}\right\} / \sim$$

On both sides, "$\sim$" is a certain equivalence relation that will be clear later on.

To describe the process that associates a complete collineation to an exact linked net, recall that a linked net $\mathfrak{g}$ of finite support over the standard $\mathbb{Z}^1$-quiver is described by giving it on its support. After a translation, we may suppose the support of $\mathfrak{g}$ is in the set $\{0,\dots,d\}$ for a given integer d. We may suppose d is minimal. We call d the **length** of $\mathfrak{g}$. In other words, $\mathfrak{g}$ is the data of vector spaces $U_0,\cdots,U_d$ of the same dimension and linear maps φ^i and φ_i as below,

$$\mathfrak{g}:\; U_0 \underset{\varphi_0}{\overset{\varphi^0}{\rightleftarrows}} U_1 \underset{\varphi_1}{\overset{\varphi^1}{\rightleftarrows}} U_2 \qquad \cdots \qquad U_{d-1} \underset{\varphi_{d-1}}{\overset{\varphi^{d-1}}{\rightleftarrows}} U_d, \tag{6.1}$$

such that:

1. $\varphi^i \circ \varphi_i = 0$ and $\varphi_i \circ \varphi^i = 0$ for each i;

2. $\mathrm{Ker}(\varphi_{i-1}) \cap \mathrm{Ker}(\varphi^i) = 0$ for each $i = 1,\dots,d-1$;

3. φ_0 and φ^{d-1} are not isomorphisms.

It is exact when $\mathrm{Ker}(\varphi_i) = \mathrm{Im}(\varphi^i)$ and $\mathrm{Ker}(\varphi^i) = \mathrm{Im}(\varphi_i)$ for each $i = 0, \cdots, d-1$.

A **complete collineation** of **depth** d is a collection of $d+1$ linear maps $V_i \xrightarrow{\lambda_i} W_i$ for $i = 0, \cdots, d$, such that $V_{i+1} = \mathrm{Ker}(\lambda_i)$, $W_{i+1} = \mathrm{Coker}(\lambda_i)$ and λ_d is an isomorphism of nontrivial vector spaces. We represent a complete collineation by $\mathfrak{c} = (V_i, \lambda_i, W_i)_{i=0}^{d}$, or by a diagram

$$\mathfrak{c}: \quad \begin{tikzcd} V_0 \arrow[r, "\lambda_0"] \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\supseteq$}"] & W_0 \arrow[d, two heads, "r_1"] \\ V_1 \arrow[r, "\lambda_1"] \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\supseteq$}"] & W_1 \arrow[d, two heads, "r_2"] \\ \vdots \quad\quad \vdots \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\supseteq$}"] & \vdots \arrow[d, two heads, "r_d"] \\ V_d \arrow[r, "\lambda_d"] & W_d \end{tikzcd} \tag{6.2}$$

Often we omit the surjections r_i.

For more details concerning complete collineations you may consult [19, p. 474] or [20, p. 417].

We want to obtain an exact linked net from a complete collineation. This seems counter-intuitive because the conditions on an exact linked net are symmetric while those on a complete collineation are not. So, first we describe how to reverse a complete collineation, that is, a way to see that a complete collineation is symmetric too.

Proposition 46. To a complete collineation oriented from V_0 to W_0 we can associate a complete collineation oriented from W_0 to V_0, this is, from a given complete collineation

$$\begin{tikzcd} V_0 \arrow[r, "\lambda_0"] \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\supseteq$}"] & W_0 \arrow[d, two heads, "r_1"] & & V_0 \arrow[d, two heads] & W_0 \arrow[l, "\mu_0"'] \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\supseteq$}"] \\ V_1 \arrow[r, "\lambda_1"] \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\supseteq$}"] & W_1 \arrow[d, two heads, "r_2"] & \text{associate} & V_1' \arrow[d, two heads] & W_1' \arrow[l, "\mu_1"'] \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\supseteq$}"] \\ \vdots \quad\quad \vdots \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\supseteq$}"] & \vdots \arrow[d, two heads, "r_d"] & & \vdots \arrow[d, two heads] \quad\quad \vdots & \vdots \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\supseteq$}"] \\ V_d \arrow[r, "\lambda_d"] & W_d & & V_d' & W_d' \arrow[l, "\mu_d"'] \end{tikzcd}$$

Proof. Starting with a complete collineation from V_0 to W_0 we want to describe a complete collineation from W_0 to V_0. We have $2(d+1)$ vector spaces V_i and W_i and $d+1$ linear map

$$\lambda_i : V_i \xrightarrow{\lambda_i} W_i \quad \text{for } i = 0, \cdots, d$$

such that $V_{i+1} = \mathrm{Ker}(\lambda_i)$ and $W_{i+1} = \mathrm{Coker}(\lambda_i)$ for each i and λ_d is an isomorphism.

To describe our new complete collineation set

$$V_i' := \frac{V_0}{V_{d-i+1}} \text{ for } i = 0, \dots, d,$$

where we put $V_{d+1} := 0$. In this way we have a canonical surjective map from V_i' to V_{i+1}' for each $i = 0, \dots, d-1$. Define

$$W_i' := \mathrm{Ker}(s_{d-i+1}) \ \text{ with } \ s_i := r_i \circ \cdots \circ r_1 \circ r_0 \text{ for } i = 0, \dots, d,$$

where we let r_0 be the identity of W_0 and put $r_{d+1} := 0$. Thus notice that $W_0' = W_0$ and $0 \subsetneq W_d' \subset \cdots \subset W_{i+1}' \subset W_i' \subset \cdots \subset W_0'$.

Now we have to describe our new maps from the W_i' to the V_i'. Observe that λ_i induces an isomorphism between V_i/V_{i+1} and W_{d-i}'/W_{d+1-i}'. Indeed, we have a natural isomorphism $V_i/V_{i+1} \cong \mathrm{Im}(\lambda_i)$ induced by λ_i and

$$\mathrm{Ker}(s_{i+1}) = s_i^{-1}(\mathrm{Im}(\lambda_i)) \xrightarrow{s_i|_{s_i^{-1}(\mathrm{Im}(\lambda_i))}} \mathrm{Im}(\lambda_i),$$

but $\mathrm{Ker}(s_i|_{s_i^{-1}(\mathrm{Im}(\lambda_i))}) = \mathrm{Ker}(s_{i+1}) \cap \mathrm{Ker}(s_i) = \mathrm{Ker}(s_i)$, that is

$$\frac{W_{d-i}'}{W_{d+1-i}'} = \frac{\mathrm{Ker}(s_{i+1})}{\mathrm{Ker}(s_i)} \cong \mathrm{Im}(\lambda_i) \cong \frac{V_i}{V_{i+1}}.$$

We denote the inverse isomorphism from W_{d-i}'/W_{d+1-i}' to V_i/V_{i+1} by τ_i. We can now define the maps μ_i by means of the compositions

$$\begin{tikzcd} W_{d-i}' \arrow[r, two heads, "\pi_i"] \arrow[rrr, bend right, "\mu_{d-i}"'] & W_{d-i}'/W_{d+1-i}' \arrow[r, "\tau_i"] & V_i/V_{i+1} \arrow[r, hook] & V_0/V_{i+1} := V_{d-i}' \end{tikzcd}$$

Notice that $\mathrm{Ker}(\mu_i) = \mathrm{Ker}(\pi_{d-i}) = W_{i+1}'$. Also, $\mathrm{Coker}(\mu_i)$ can be described as

$$\mathrm{Coker}(\mu_i) = \frac{V_0/V_{d-i+1}}{V_{d-i}/V_{d-i+1}} \cong \frac{V_0}{V_{d-i}} = V_{i+1}'.$$

Therefore, the complete collineation is reversed from V_0 to W_0 in a complete collineation from W_0 to V_0. $\square$

We have a relation between λ_i and μ_{d-i} for each i that we can see in the diagram

$$\begin{tikzcd}
W_0/W'_{d+1-i} \cong W_i & V_i \arrow[l, "\lambda_i"'] \arrow[d] \\
W'_{d-i}/W'_{d+1-i} \arrow[u, hook] \arrow[r, shift right, "\tau_i"'] & V_i/V_{i+1} \arrow[l, shift right, "\tau_i^{-1}"'] \arrow[d, hook] \\
W'_{d-i} \arrow[u] \arrow[r, "\mu_{d-i}"'] & V_0/V_{i+1}
\end{tikzcd}$$

The above proposition shows that to give a complete collineation as in (6.2) is the same as giving the following data:

$$\mathfrak{c}:\quad \begin{tikzcd}
V_0/V_1 & & \operatorname{Ker}(s_1) \arrow[ll, "\mu_d"'] \\
\vdots \arrow[u, two heads] & \vdots & \vdots \arrow[u, phantom, "\cap" description] \\
V_0/V_d \arrow[u, two heads] & & \operatorname{Ker}(s_d) \arrow[ll, "\mu_1"'] \arrow[u, phantom, "\cap" description] \\
V_0 \arrow[u, two heads] \arrow[rr, shift right, "\lambda_0"'] & & W_0 \arrow[ll, shift right, "\mu_0"'] \arrow[u, phantom, "\cap" description] \arrow[d, two heads] \\
V_1 \arrow[u, phantom, "\cup" description] \arrow[rr, "\lambda_1"'] & & W_1 \arrow[d, two heads] \\
\vdots \arrow[u, phantom, "\cup" description] & \vdots & \vdots \arrow[d, two heads] \\
V_d \arrow[u, phantom, "\cup" description] \arrow[rr, "\lambda_d"'] & & W_d
\end{tikzcd}$$

To be more symmetrical, from now on we write a complete collineation as

$$
\begin{tikzcd}
 & V_0/V_1 & & W_d \arrow[ll, "\mu_d"'] \arrow[d, phantom, sloped, "\subseteq"] \\
 & \vdots \arrow[u, two heads] & \vdots & \vdots \arrow[d, phantom, sloped, "\subseteq"] \\
 & V_0/V_d \arrow[u, two heads] & & W_1 \arrow[ll, "\mu_1"'] \arrow[d, phantom, sloped, "\subseteq"] \\
\mathfrak{c}: & V_0 \arrow[u, two heads] \arrow[rr, shift right, "\lambda_0"'] \arrow[d, phantom, sloped, "\supseteq"] & & W_0 \arrow[ll, shift right, "\mu_0"'] \arrow[d, two heads] \\
 & V_1 \arrow[rr, "\lambda_1"'] \arrow[d, phantom, sloped, "\supseteq"] & & W_0/W_d \arrow[d, two heads] \\
 & \vdots \arrow[d, phantom, sloped, "\supseteq"] & \vdots & \vdots \arrow[d, two heads] \\
 & V_d \arrow[rr, "\lambda_d"'] & & W_0/W_1.
\end{tikzcd}
\tag{6.3}
$$

with $\mathrm{Ker}(\mu_i) = W_{i+1}$, $\mathrm{Ker}(\lambda_i) = V_{i+1}$, $\mathrm{Im}(\lambda_i) = W_{d-i}/W_{d+1-i}$ and $\mathrm{Im}(\mu_i) = V_{d-i}/V_{d+1-i}$.

6.2 From exact linked nets to complete collineations

To begin with, we will describe how to obtain a complete collineation from an exact linked net by first giving the data and then checking that it satisfies the properties of a complete collineation. In short, here we define the map α. Suppose we start with an exact linked net $\mathfrak{g}$ of length d as in (6.1). Then we simply put $V_0 := U_0$, $W_0 := U_d$ and $\lambda_0 := \varphi^{d-1} \circ \cdots \circ \varphi^0$. Also, for each $i = 1, ..., d$ we set $V_i := \mathrm{Im}(\varphi_0 \circ \cdots \circ \varphi_{i-1})$, $W_i = \mathrm{Im}(\varphi^{d-1} \circ \cdots \circ \varphi^{d-i})$ and

$$
\begin{aligned}
\lambda_i := \varphi^{d-1} \circ \cdots \circ \varphi^i \, : \, V_i &\longrightarrow W_0/W_{d+1-i} \\
\varphi_0 \circ \cdots \circ \varphi_{i-1}(x) &\mapsto [\varphi^{d-1} \circ \cdots \circ \varphi^i(x)].
\end{aligned}
$$

To check that λ_i is well defined, notice that

$$
\mathrm{Ker}(\varphi_0 \circ \cdots \circ \varphi_{i-1}) = \mathrm{Ker}(\varphi_{i-1}) = \mathrm{Im}(\varphi^{i-1}), \tag{6.4}
$$

and thus $\varphi_0 \circ \cdots \circ \varphi_{i-1}(x) = 0$ if and only if there is y such that $x = \varphi^{i-1}(y)$, and hence

$$
[\varphi^{d-1} \circ \cdots \circ \varphi^i(x)] = [\varphi^{d-1} \circ \cdots \circ \varphi^{i-1}(y)] = [0]
$$

in $\mathrm{Coker}(\varphi^{d-1}\circ\cdots\circ\varphi^{i-1})$. Thus λ_i is well defined. Also, we have verified that

$$\mathrm{Ker}(\lambda_i)=\mathrm{Im}\,(\varphi_0\circ\cdots\circ\varphi_i))=V_{i+1} \quad\text{and}\quad \mathrm{Coker}(\lambda_i)=W_0/\mathrm{Im}\,(\varphi^{d-1}\circ\cdots\circ\varphi^{i})=W_0/W_{d-i}.$$

Analogously we define the maps on the opposite direction:

$$\begin{aligned}\mu_i=\varphi_0\circ\cdots\circ\varphi_{d-1-i}:\ \mathrm{Im}\,(\varphi^{d-1}\circ\cdots\circ\varphi^{d-i}) &\longrightarrow V_0/\mathrm{Im}\,(\varphi_0\circ\cdots\circ\varphi_{d-i})\\ \varphi^{d-1}\circ\cdots\circ\varphi^{d-i}(x) &\mapsto [\varphi_0\circ\cdots\circ\varphi_{d-1-i}(x)].\end{aligned}$$

As before, we verify that $\mu_i\colon W_i\to V_0/V_{d+1-i}$ is well defined, and that $\mathrm{Ker}(\mu_i)=W_{i+1}$ and $\mathrm{Coker}(\mu_i)=V_0/V_{d-i}$.

Observe that $V_d\neq 0$, a consequence of (6.4) for $i=d$ and the fact that φ^{d-1} is not surjective. Likewise, $W_d\neq 0$.

We have thus obtained a complete collineation of depth d, which we denote $\alpha(\mathfrak{g})$:

$$\begin{tikzcd}
 & U_0/\mathrm{Im}\,(\varphi_0) & & \mathrm{Im}\,(\varphi^{d-1}\cdots\varphi^0) \arrow[ll, "\mu_d"'] \arrow[d, phantom, sloped, "\subseteq"] \\
 & \vdots \arrow[u, two heads] & \vdots & \vdots \arrow[d, phantom, sloped, "\subseteq"] \\
 & U_0/\mathrm{Im}\,(\varphi_0\cdots\varphi_{d-1}) \arrow[u, two heads] & & \mathrm{Im}\,(\varphi^{d-1}) \arrow[ll, "\mu_1"'] \arrow[d, phantom, sloped, "\subseteq"] \\
\alpha(\mathfrak{g}) & U_0 \arrow[u, two heads] \arrow[rr, shift right, "\lambda_0"'] \arrow[d, phantom, sloped, "\supseteq"] & & U_d \arrow[ll, shift right, "\mu_0"'] \arrow[d, two heads] \\
 & \mathrm{Im}\,(\varphi_0) \arrow[rr, "\lambda_1"'] \arrow[d, phantom, sloped, "\supseteq"] & & U_d/\mathrm{Im}\,(\varphi^{d-1}\cdots\varphi^0) \arrow[d, two heads] \\
 & \vdots \arrow[d, phantom, sloped, "\supseteq"] & \vdots & \vdots \arrow[d, two heads] \\
 & \mathrm{Im}\,(\varphi_0\cdots\varphi_{d-1}) \arrow[rr, "\lambda_d"'] & & U_d/\mathrm{Im}\,(\varphi^{d-1}).
\end{tikzcd}$$

6.3 From complete collineations to exact linked nets

Previously, we saw an exact linked net of length d as a representation

$$\mathfrak{g}:\ U_0 \underset{\varphi_0}{\overset{\varphi^0}{\rightleftarrows}} U_1 \underset{\varphi_1}{\overset{\varphi^1}{\rightleftarrows}} U_2 \qquad \cdots \qquad U_{d-1} \underset{\varphi_{d-1}}{\overset{\varphi^{d-1}}{\rightleftarrows}} U_d$$

where the U_i are abstract vector spaces of the same dimension n. We may see all the U_i embedded in $U_0 \oplus U_d$ using the given maps. For instance,

$$U_0 \cong \left\{ \left(x, \varphi^{d-1} \cdots \varphi^0(x)\right) \,\middle|\, \forall x \in U_0 \right\} \subseteq U_0 \oplus U_d.$$

In general,

$$U_i \cong \left\{ \left(\varphi_0 \circ \cdots \circ \varphi_{i-1}(x), \varphi^{d-1} \cdots \varphi^i(x)\right) \,\middle|\, \forall x \in U_i \right\} \subseteq U_0 \oplus U_d.$$

Indeed, the map

$$\begin{aligned} \sigma_i \,:\, U_i &\longrightarrow U_0 \oplus U_d \\ x \;&\mapsto\; (\varphi_0 \cdots \varphi_{i-1}(x), \varphi^{d-1} \cdots \varphi^i(x)) \end{aligned}$$

is an injection because

$$\mathrm{Ker}(\sigma_i) = \mathrm{Ker}(\varphi_0 \cdots \varphi_{i-1}) \cap \mathrm{Ker}(\varphi^{d-1} \cdots \varphi^i) = \mathrm{Ker}(\varphi_{i-1}) \cap \mathrm{Ker}(\varphi^i) = 0.$$

We let $\Gamma_i := \sigma_i(U_i)$ for each i and obtain from them an equivalent "simpler" linked net. Indeed, the following diagram is commutative:

$$\begin{tikzcd} U_i \arrow[r, hook, "\sigma_i"] \arrow[d, shift left, "\varphi^i"] & U_0 \oplus U_d \arrow[d, shift left, "\pi_d"] \\ U_{i+1} \arrow[r, hook, "\sigma_{i+1}"] \arrow[u, shift left, "\varphi_i"] & U_0 \oplus U_d \arrow[u, shift left, "\pi_0"] \end{tikzcd}$$

where $\pi_0(a,b) = a$ and $\pi_d(a,b) = b$. In symbols

$$\pi_d \circ \sigma_i = \sigma_{i+1} \circ \varphi^i \text{ and } \pi_0 \circ \sigma_{i+1} = \sigma_i \circ \varphi_i.$$

We have then an equivalent linked net embedded in $V_0 \oplus V_d$:

$$\mathfrak{g}' = (\Gamma_i, \pi_0|_{\Gamma_{i+1}}, \pi_d|_{\Gamma_i}). \tag{6.5}$$

We will associate to a complete collineation of depth d between spaces V_0 and W_0 an "embedded" exact linked net of length d, that is, we let $U_0 := V_0$ and $U_d := W_0$ and define subspaces $\Gamma_i \subseteq U_0 \oplus U_d$ for $i = 0, \ldots, d$ such that $(\Gamma_i, \pi_0|_{\Gamma_{i+1}}, \pi_d|_{\Gamma_i})$ is an exact linked net of length d.

Indeed, consider the complete collineation $\mathfrak{c}$ of depth d in (6.3). For each $i = 0, \dots, d$, the space Γ_i we will define will satisfy

$$V_{i+1} \oplus W_{d+1-i} \subseteq \Gamma_i \subseteq V_i \oplus W_{d-i} \subseteq V_0 \oplus W_0.$$

(We use $V_{d+1} = 0$ for $i = d$ and $W_{d+1} := 0$ for $i = 0$.) More precisely, we define Γ_i as follows. We saw that λ_i induces an isomorphism τ_i^{-1}, with inverse τ_i induced by μ_{d-i}:

$$\frac{W_{d-i}}{W_{d+1-i}} \underset{\tau_i^{-1}}{\overset{\tau_i}{\rightleftarrows}} \frac{V_i}{V_{i+1}}.$$

Then, letting q be the canonical projection and $\Gamma(\tau_i)$ the graph of τ_i, we have

$$\Gamma(\tau_i) = \Gamma(\tau_i^{-1}) \subset \frac{V_i}{V_{i+1}} \oplus \frac{W_{d-i}}{W_{d+1-i}} \cong \frac{V_i \oplus W_{d-i}}{V_{i+1} \oplus W_{d+1-i}} \xleftarrow{q} V_i \oplus W_{d-i}, \tag{6.6}$$

and we set

$$\Gamma_i := q^{-1}(\Gamma(\tau_i)) = q^{-1}(\Gamma(\tau_i^{-1}))$$

for each $i = 0, \dots, d$. First, we must verify that Γ_i has dimension $n := \dim V_0$. But

$$\dim \Gamma_i - \dim V_{i+1} - \dim W_{d+1-i} = \frac{1}{2}(\dim V_i - \dim V_{i+1} + \dim W_{d-i} - \dim W_{d+1-i}),$$

and thus

$$\dim \Gamma_i = \frac{1}{2}(\dim V_{i+1} + \dim W_{d+1-i} + \dim V_i + \dim W_{d-i}).$$

From the isomorphisms $V_i/V_{i+1} \cong \operatorname{Im} \lambda_i \cong W_{d-i}/W_{d+1-i}$ we obtain

$$\dim V_i + \dim W_{d+1-i} = \dim V_{i+1} + \dim W_{d-i} = \dim W_0 = n,$$

implying that $\dim \Gamma_i = n$.

Denote by p_1 and p_2 the projections of $V_0 \oplus W_0$ onto the first and second factors respectively. Also, let i_1 and i_2 be the inclusion of V_0 and W_0 in $V_0 \oplus W_0$ respectively.

Consider the following diagram:

$$\begin{tikzcd}
 & & W_{d-j} \arrow[dr, hook, "i_2"] & & \\
\Gamma_j \arrow[r, hook] \arrow[rrrr, bend left=40, "\varphi^j"] & V_j \oplus W_{d-j} \arrow[ur, two heads, "p_2"] & & V_{j+1} \oplus W_{d-j-1} \arrow[dl, two heads, "p_1"] & \Gamma_{j+1}. \arrow[l, hook'] \arrow[llll, bend left=40, "\varphi_j"] \\
 & & V_{j+1} \arrow[ul, hook', "i_1"] & &
\end{tikzcd} \qquad (6.7)$$

As $i_2 \circ p_2(\Gamma_j) = 0 \oplus W_{d-j} \subseteq \Gamma_{j+1}$ and $i_1 \circ p_1(\Gamma_j) = V_{j+1} \oplus 0 \subseteq \Gamma_j$, we can define, abusing notation,

$$\varphi^j := i_2 \circ p_2 \colon \Gamma_j \longrightarrow \Gamma_{j+1},$$
$$\varphi_j := i_1 \circ p_1 \colon \Gamma_{j+1} \longrightarrow \Gamma_j.$$

The desired linked net is thus

$$\beta(\mathfrak{c}) : \Gamma_0 \underset{\varphi_0}{\overset{\varphi^0}{\rightleftarrows}} \Gamma_1 \underset{\varphi_1}{\overset{\varphi^1}{\rightleftarrows}} \Gamma_2 \qquad \cdots \qquad \Gamma_{d-1} \underset{\varphi_{d-1}}{\overset{\varphi^{d-1}}{\rightleftarrows}} \Gamma_d. \qquad (6.8)$$

The conditions that a linked net must satisfy are easily verified for $\beta(\mathfrak{c})$:

Condition (1):

$$\varphi^j \circ \varphi_j = (i_2 \circ p_2) \circ (i_1 \circ p_1) = i_2 \circ \underbrace{p_2 \circ i_1}_{0} \circ p_1 = 0,$$
$$\varphi_j \circ \varphi^j = (i_1 \circ p_1) \circ (i_2 \circ p_2) = i_1 \circ \underbrace{p_1 \circ i_2}_{0} \circ p_2 = 0$$

Condition (2): As $\mathrm{Ker}(\varphi^{j+1}) \subseteq V_0 \oplus 0$ and $\mathrm{Ker}(\varphi_j) \subseteq 0 \oplus W_0$, it follows that

$$\mathrm{Ker}(\varphi^{j+1}) \cap \mathrm{Ker}(\varphi_j) = \{0\}.$$

Condition (3): Observe in diagram (6.7) that φ_0 factors through V_1 and this vector space has dimension smaller than n. Thus φ_0 cannot be an isomorphism. Analogously φ^{d-1} factors through W_1 which has dimension smaller than n. Thus φ^{d-1} cannot be an isomorphism as well.

Exactness follows easily as well:

$$\begin{aligned}\mathrm{Ker}(\varphi_j) &= (i_1 \circ p_1)^{-1}(0) \cap \Gamma_{j+1} = (0 \oplus W_{d-j-1}) \cap \Gamma_{j+1} \\ &= 0 \oplus W_{d-j} = i_2 \circ p_2(\Gamma_j) = \mathrm{Im}(\varphi^j).\end{aligned}$$

In a completely analogous way we can conclude that $\mathrm{Ker}(\varphi^j) = \mathrm{Im}(\varphi_j)$ as well.

6.4 Equivalence: linked nets and complete collineations

Having described how to pass from a complete collineation of depth d to an exact linked net of length d and vice versa, the correspondence will be proven once we show that the maps are mutually inverse.

$$\{\text{exact linked nets of length } d\}/\sim \;\underset{\beta}{\overset{\alpha}{\rightleftarrows}}\; \{\text{complete collineations of depth } d\}/\sim$$

Suppose we start with the complete collineation $\mathfrak{c}$ as in (6.3). Via β we get $\mathfrak{g} := \beta(\mathfrak{c})$ as in (6.8). So, we must show that $\alpha\beta(\mathfrak{c}) = \alpha(\mathfrak{g})$ is equivalent to $\mathfrak{c}$, that is, α is a left inverse of β. By definition

$$\Gamma_i = q^{-1}(\Gamma(\tau_i)) = q^{-1}(\Gamma(\tau_i^{-1})),$$

and the maps φ^i and φ_i are

$$\beta(\mathfrak{c}):\quad \begin{tikzcd} & W_0 \arrow[dr, hook, "i_2"] & \\ \Gamma_i \arrow[ur, "p_2"] \arrow[rr, shift left, "\varphi^i"] & & \Gamma_{i+1} \arrow[ll, shift left, "\varphi_i"] \arrow[dl, "p_1"] \\ & V_0 \arrow[ul, hook, "i_1"] & \end{tikzcd}$$

As we have seen, the exact linked net $\mathfrak{g}$ satisfies

$$\mathrm{Im}\,\varphi_i = V_{i+1} \oplus 0 \quad \text{and} \quad \mathrm{Im}(\varphi^i) = 0 \oplus W_{d-i} \quad \text{for } i = 0, \cdots, d-1. \tag{6.9}$$

For $\alpha(\beta(\mathfrak{c}))$ we put $V'_i := \mathrm{Im}(\varphi_0 \circ \cdots \circ \varphi_{i-1})$ and $W'_i = \mathrm{Im}(\varphi^{d-1} \circ \cdots \circ \varphi^{d-i})$ for $i = 1, \dots, d$. Also, we put $V'_0 = \Gamma_0$ and $W'_0 = \Gamma_d$. Then we have

$$\alpha\beta(\mathfrak{c}): \quad \begin{tikzcd}
\Gamma_0/\mathrm{Im}(\varphi_0) & & \mathrm{Im}(\varphi^{d-1}\circ\cdots\circ\varphi^0) \arrow[ll, "\mu'_d"'] \arrow[d, phantom, sloped, "\subseteq"] \\
\vdots \arrow[u, two heads] & \vdots & \vdots \arrow[d, phantom, sloped, "\subseteq"] \\
\Gamma_0/\mathrm{Im}(\varphi_0\circ\cdots\circ\varphi_{d-1}) \arrow[u, two heads] & & \mathrm{Im}(\varphi^{d-1}) \arrow[ll, "\mu'_1"'] \arrow[d, phantom, sloped, "\subseteq"] \\
\Gamma_0 \arrow[u, two heads] \arrow[rr, shift right, "\lambda'_0"'] \arrow[d, phantom, sloped, "\supseteq"] & & \Gamma_d \arrow[ll, shift right, "\mu'_0"'] \arrow[d, two heads] \\
\mathrm{Im}(\varphi_0) \arrow[rr, "\lambda'_1"'] \arrow[d, phantom, sloped, "\supseteq"] & & \Gamma_d/\mathrm{Im}(\varphi^{d-1}\circ\cdots\circ\varphi^0) \arrow[d, two heads] \\
\vdots \arrow[d, phantom, sloped, "\supseteq"] & \vdots & \vdots \arrow[d, two heads] \\
\mathrm{Im}(\varphi_0\circ\cdots\circ\varphi_{d-1}) \arrow[rr, "\lambda'_d"'] & & \Gamma_d/\mathrm{Im}(\varphi^{d-1})
\end{tikzcd}$$

Although $\alpha\beta(\mathfrak{c})$ is not $\mathfrak{c}$, they are isomorphic via the commutative diagram

$$\begin{tikzcd}
\mathrm{Im}(\varphi_0\circ\cdots\circ\varphi_{i-1}) \arrow[r, "\lambda'_i"] \arrow[d, shift right, "p_1"'] & \Gamma_d/\mathrm{Im}(\varphi^{d-1}\circ\cdots\circ\varphi^{i-1}) \arrow[d, shift left, "p_2"] \\
V_i \arrow[r, "\lambda_i"'] \arrow[u, shift right, "p_1^{-1}"'] & W_0/W_{d+1-i}. \arrow[u, shift left, "p_2^{-1}"]
\end{tikzcd} \tag{6.10}$$

The equalities in (6.9) show that p_1 and p_2 do induce the vertical isomorphisms in (6.10).

On the other hand, suppose we start with the exact linked net $\mathfrak{g}$ as in (6.3) and pass to $\mathfrak{c} = \alpha(\mathfrak{g})$. We want to show that $\beta\alpha(\mathfrak{g}) = \beta(\mathfrak{c})$ is equivalent to $\mathfrak{g}$. Recall that we have, setting $V_0 := U_0$ and $W_0 := U_d$:

$$
\alpha(\mathfrak{g}) : \quad
\begin{tikzcd}
\mathrm{Coker}(\varphi_0) & & \mathrm{Im}\,(\varphi^{d-1}\circ\cdots\circ\varphi^0) \arrow[ll, "\mu_d"'] \arrow[d, phantom, sloped, "\subseteq"] \\
\vdots \arrow[u, two heads] & \vdots & \vdots \arrow[d, phantom, sloped, "\subseteq"] \\
\mathrm{Coker}(\varphi_0\circ\cdots\circ\varphi_{d-1}) \arrow[u, two heads] & & \mathrm{Im}\,(\varphi^{d-1}) \arrow[ll, "\mu_1"'] \arrow[d, phantom, sloped, "\subseteq"] \\
V_0 \arrow[u, two heads] \arrow[rr, shift right, "\lambda_0"'] \arrow[d, phantom, sloped, "\supseteq"] & & V_d \arrow[ll, shift right, "\mu_0"'] \arrow[d, two heads] \\
\mathrm{Im}\,(\varphi_0) \arrow[rr, "\lambda_1"] \arrow[d, phantom, sloped, "\supseteq"] & & \mathrm{Coker}(\varphi^{d-1}\circ\cdots\circ\varphi^0) \arrow[d, two heads] \\
\vdots \arrow[d, phantom, sloped, "\supseteq"] & \vdots & \vdots \arrow[d, two heads] \\
\mathrm{Im}\,(\varphi_0\circ\cdots\circ\varphi_{d-1}) \arrow[rr, "\lambda_d"'] & & \mathrm{Coker}(\varphi^{d-1})
\end{tikzcd}
$$

with the maps

$$
\begin{aligned}
\lambda_i = \varphi^{d-1}\circ\cdots\circ\varphi^i : \mathrm{Im}\,(\varphi_0\circ\cdots\circ\varphi_{i-1}) &\longrightarrow \mathrm{Coker}(\varphi^{d-1}\circ\cdots\circ\varphi^{i-1}) \\
\varphi_0\circ\cdots\circ\varphi_{i-1}(x) &\mapsto [\varphi^{d-1}\circ\cdots\circ\varphi^i(x)], \\
\mu_i = \varphi_0\circ\cdots\circ\varphi_{d-1-i} : \mathrm{Im}\,(\varphi^{d-1}\circ\cdots\circ\varphi^{d-i}) &\longrightarrow \mathrm{Coker}(\varphi_0\circ\cdots\circ\varphi_{d-i}) \\
\varphi^{d-1}\circ\cdots\circ\varphi^{d-i}(x) &\mapsto [\varphi_0\circ\cdots\circ\varphi_{d-1-i}(x)],
\end{aligned}
$$

so we have isomorphisms

$$
\tau_i^{-1} : \frac{\mathrm{Im}\,(\varphi_0\circ\cdots\circ\varphi_{i-1})}{\mathrm{Im}\,(\varphi_0\circ\cdots\circ\varphi_i)} \longrightarrow \frac{\mathrm{Im}\,(\varphi^{d-1}\circ\cdots\circ\varphi^i)}{\mathrm{Im}\,(\varphi^{d-1}\circ\cdots\circ\varphi^{i-1})}.
$$

To construct $\beta\alpha(\mathfrak{g})$ we put $\Gamma_i := q^{-1}(\Gamma(\tau_i^{-1}))$ with q as in (6.6), which implies that

$$
\Gamma_i = \left\{ \left(\varphi_0\circ\cdots\varphi_{i-1}(x), \varphi^{d-1}\circ\cdots\varphi^i(x)\right) \middle| x\in U_i \right\}
$$

for $i = 0,\ldots,d$. We thus have that $\beta\alpha(\mathfrak{g})$ is the "embedded" linked net associated to $\mathfrak{g}$, namely $\mathfrak{g}'$ in (6.5), which is equivalent to $\mathfrak{g}$.

We have finally shown the correspondence, up to equivalence, between exact linked nets and complete collineations!

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Glossary of Notations

The next list describes several symbols that are used in the body of the text.

Abbreviations and Acronyms

lls	Acronym for Limit Linear Serie

Mathematical Symbols

$\mathbb{C}$	Complex numbers.
$\mathfrak{c}$	Complete collineation
$\mathfrak{g}$	linked net of vector spaces
Hilb_X	Hilbert polynomial of the scheme X
id_A	Identity morphism of the object A
$\mathbb{LP}(\mathfrak{g})$	linked projective space of $\mathfrak{g}$
$\mathbb{LP}(\mathfrak{g})_v^*$	subset of points of $\mathbb{LP}(\mathfrak{g})$ generated from v.
$\mathbb{LP}(g)_v$	Zariski closure of $\mathbb{LP}(\mathfrak{g})_v^*$
$\mathbb{N}$	Natural numbers, including zero.
$\mathbb{P}$	projective spcae
$\mathbb{P}(V)$	projective space of the vector space V
$\mathbb{Z}$	Integers numbers.
A_{d+1}^2	Quiver associated to a limit linear series of degree d on C_1
Q	Quiver
Q_p^d	Quiver associated a limit linear series of degree d on C_Δ
Q_Δ	triangle quiver

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