ON LINKED PROJECTIVE SPACES

Renan Santos

December 2021

Contents

Abstract

A limit linear series over a nodal curve naturally gives rise to a structure called linked net of vector spaces. It is a quiver representation over a $\mathbb{Z}^n$-quiver, with properties that mimic the properties coming from the geometry of limit linear series. To any exact linked net of vector spaces $\mathfrak{g}$ with finite support, we associate a scheme $\mathbb{LP}(\mathfrak{g})$, called the linked projective space, which is the space that parametrizes sub-representations of dimension 1 of the linked net. The main result is the following: if $\mathfrak{g}$ is an exact linked net of vector spaces with finite support over a $\mathbb{Z}^2$-quiver, then the scheme $\mathbb{LP}(\mathfrak{g})$ is pure dimensional and all its components are rational. We also give an explicit description of the components using an equivalence relation in the vertices of the base quiver and a structural theorem that classifies all linked nets of vector spaces of dimension 1.

We use the main result to show that, for g with dimension 2, the scheme LP(g) is a deformation of the diagonal inside a product of projective spaces, thus they have the same Hilbert polynomial.

Keywords: limit linear series, linked nets of vector spaces, linked projective space.

Resumo

Uma série linear limite sobre uma curva nodal naturalmente dá origem a uma estrutura chamada rede ligada de espaços vetoriais. É uma representação de quiver sobre um $\mathbb{Z}^n$- quiver, com propriedades que imitam as propriedades vindas da geometria das séries lineares limite.

A toda rede ligada de espaços vetoriais $\eth$ com suporte finito, associamos um esquema $\mathbb{LP}(\mathfrak{g})$, chamado espaço projetivo ligado, que é o espaço que parametriza sub- representações de dimensão 1 da rede ligada.

O principal resultado é o seguinte: se $\eth$ é uma rede ligada de espaços vetoriais exata com suporte finito sobre um $\mathbb{Z}^2$-quiver, então o esquema $\mathbb{LP}(\mathfrak{g})$ é de dimensão pura e todas as suas componentes são racionais. Também damos uma descrição explícita das componentes usando uma relação de equivalência entre os vértices do quiver de base e um teorema de estrutura que classifica todos as redes ligadas de espaços vetoriais de dimensão 1.

Usamos o resultado principal para provar que, para $\eth$ de dimensão 2, o esquema $\mathbb{LP}(\mathfrak{g})$ é uma deformação da diagonal dentro de um produto de espaços projetivos e, portanto, eles tem o mesmo polinômio de Hilbert.

Palavras-chave: séries lineares limite, redes ligadas de espaços vetoriais, espaços projetivos ligados.

Agradecimentos

Ao meu orientador, Eduardo Esteves, pela paciência e por todo o conhecimento compartilhado.

Aos meus colegas de matemática, pelos momentos e que compartilhamos e conversas que tivemos, tanto as relacionadas à matemática, mas especialmente as não relacionadas à matemática.

Um agradecimento especial ao Eduardo Garcez, pelas lições de vida; ao Rafael Alves, por nos alegrar com suas risadas; ao Matheus Sousa, pelos seus conselhos; ao Hugo Araújo, por sempre trazer as discussões mais interessantes.

Um agradecimento também à galera do Portão Vermelho, a galera do Catete e a galera do Edifício Fiorio. A vida no Rio de Janeiro seria muito mais cinza sem vocês.

Ao pessoal do vôlei do Clube dos Macacos, por tornar as segundas-feiras mais animadas.

À APG do IMPA, por ter me proporcionado a oportunidade de ser um dos anfitriões do Cine Möbius. Foi um enorme prazer compartilhar com todos um pouco da minha paixão pela sétima arte.

Às meninas do Clube de Leitura do IMPA, Cynthia e Tainara, e do Clube do Livro Carioca, Carla e Juliana, por organizarem esses espaços para literatura. Afinal, nem só de números vive o matemático.

Ao IMPA por todo o suporte e acolhimento que recebi durante o tempo que aqui estive.

À CAPES pelo apoio financeiro.

Introduction

The main goal of this thesis is to generalize results by Esteves and Osserman in [1] and Santana in [2] to linked nets of vector spaces. As a consequence, we will obtain results also for limit linear series over compact type curves with three components. Linked nets of vector spaces are quiver representations that capture the linear nature of limit linear series. We then study the linked projective schemes associated to them, which are generalizations of the schemes $\mathbb{P}(\mathfrak{g})$ and $\mathbb{LP}(\mathfrak{g})$ studied in [1] and [2], respectively.

Let's recall in more details what has been done in the theory so far.

Let X be a nodal curve with two smooth components Y and Z meeting at the node P. For smooth curves we have the well known concept of a linear series (or linear system in some books). A limit linear series is a way to generalize it to non-smooth curves, like our curve X. Roughly speaking, a limit linear series $\mathfrak{g}$ over X is the data of two linear series, one over each of the components Y and Z, that satisfy very natural compatibility conditions.

Actually, there have been at least two definitions of limit linear series. Eisenbud and Harris can be considered the fathers of the theory. Their paper [3] was the first to introduce the concept in a formal way, relating limit linear series with degeneration of curves. Meaning, we can consider X as the limit of a flat family $\mathcal{X} \longrightarrow B$ and a limit linear series $\mathfrak{g}$ over X is somehow the "limit" of a linear series over the generic fiber $\mathcal{X}_\eta$ of $\mathcal{X}/B$. As mentioned above, this $\mathfrak{g}$ is built of two linear series $\mathfrak{g}_Y$ and $\mathfrak{g}_Z$ over Y and Z respectively, and satisfies certain inequalities involving their vanishing orders at P. If one of the inequalities is strict we say $\mathfrak{g}$ is *crude*, otherwise it is *refined*. Eisenbud and Harris also constructed a moduli space $G_d^{r,\mathrm{EH}}(X)$ for limit linear series of degree d and dimension r over X. The set of refined ones is open.

The definition of limit linear series by Osserman is slightly different from that by Eisenbud and Harris, but is a natural generalization. When passing to the limit, Eisenbud and Harris considered only the two linear series on X of maximum degree over each component, Y and Z, whereas Osserman accounts for all possible non-negative degrees. In [4], Osserman also constructed a moduli space for his limit linear series, $G_d^{r,\mathrm{Oss}}(X)$, which behaves functorially better than $G_d^{r,\mathrm{EH}}(X)$ and also contains the Eisenbud–Harris refined limit linear series as an open set. The refined limit linear series are actually contained in another open set in Osserman space, that of the *exact* ones.

In what follows, we consider Osserman's concept of limit linear series.

Esteves and Osserman [1] considered the Abel map

$$A_d : S^d(X) \longrightarrow \mathrm{Pic}^{(d,0)}(X).$$

Given a line bundle of degree $(d, 0)$ over X, say $\mathcal{L} \in \mathrm{Pic}^{(d,0)}(X)$, they constructed a scheme $\mathbb{P}(\mathfrak{g}) \subset A_d^{-1}(\mathcal{L})$, for each exact limit linear series $\mathfrak{g}$ which has $\mathcal{L}$ as its underlying line bundle. It parameterizes limits of divisors if $\mathfrak{g}$ arises from a degeneration.

The main result in [1] is that $\mathbb{P}(\mathfrak{g})$ is reduced, connected, Cohen–Macaulay of dimension r, where r is the rank of $\mathfrak{g}$. Also, $\mathbb{P}(\mathfrak{g})$ can be embedded into $\mathbb{P}^r \times \mathbb{P}^r$ and, inside this product, its Hilbert polynomial is the same as that of the diagonal. Esteves and Osserman also proved that their construction is compatible with one-parameter smooth deformations, that is, if $\mathfrak{g}$ arises from a degeneration, then $\mathbb{P}(\mathfrak{g})$ is the degeneration of the projective space associated to the generic linear series.

We point out that $\mathbb{P}(\mathfrak{g})$ is defined as the Zariski closure of a particular subset $\mathbb{P}(\mathfrak{g})^*$ of the symmetric product $S^d(X)$. Passing to the closure is not a very practical way of defining an object, because one can lose control of what appears on the boundary.

Following Esteves' and Osserman's steps, Santana [2] generalizes the ideas in [1] to a more functorial approach. Using works by Osserman on linked Grassmannians in [4] and [5] as theoretical ground base, he defined another scheme associated to a limit linear series $\mathfrak{g}$ over a nodal curve X, the *linked projective space* $\mathbb{LP}(\mathfrak{g})$. This scheme has some advantages. First, the definition is functorial, since it is based on the theory of linked Grassmannians. Second, $\mathbb{LP}(\mathfrak{g})$ is the Zariski closure of $\mathbb{P}(\mathfrak{g})^*$ inside the Hilbert scheme $\mathrm{Hilb}^d(X)$. In other words, we have a nicer compactification for $\mathbb{P}(\mathfrak{g})^*$. There is also an embedding $\mathbb{LP}(\mathfrak{g}) \longrightarrow \prod \mathbb{P}^r$ of the linked projective space into a product of projective spaces.

Santana proved the analogous results in Esteves–Osserman for linked projective spaces. For a limit linear series $\mathfrak{g}$ of rank r, the scheme $\mathbb{LP}(\mathfrak{g})$ is reduced, connected, Cohen–Macaulay of dimension r and the Hilbert polynomial is the same as that of the diagonal. It is also compatible with one-parameter smoothing deformations.

All the results commented above were done for limit linear series over compact-type curves X with two components. The work in this thesis started with the idea to deal with limit linear series over compact type curves with three components. Although Eisenbud and Harris had defined the notion of limit linear series over compact-type curves with several components in [3], most of the theory done so far is for nodal curves with two components. One reason for that, as cited by Rizzo in [6], is to avoid "combinatorics". But there are works on that subject. For instance, Muñoz in [7] has proved that for compact type curves with three components, the Hilbert polynomial of $\mathbb{P}(\mathfrak{g})$ is the same as that of the diagonal, if $\mathfrak{g}$ is an exact limit linear series arising as the unique extension of a refined one. In section 8.2 however we give an example of an exact limit linear series whose scheme $\mathbb{P}(\mathfrak{g})$ is not a degeneration of the diagonal.

Limit linear series over curves with two components give rise to quiver representations (with certain properties, coming from the geometry of the subject), where the quiver is very simple, as in Figure 1.

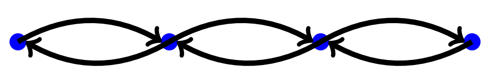

Figure 1: The base quiver for $\mathfrak{g}$ of degree 3 over a two component curve

We can do the study of such $\mathfrak{g}$ without considering the combinatorics of the base quiver. When we pass to three components curves, the situation becomes more complex. For a compact type curve with three components, the base quiver of a limit linear series is now a grid in $\mathbb{Z}^2$ as shown in Figure 2.

In this situation, we can not avoid a little bit of combinatorics.

Also in [2], Santana introduced another concept associated to a limit linear series: the concept of a *linked chain of vector spaces*. Certain chains arise from a limit linear series: just keep the vector spaces, forgetting the bundles. Basically, a linked chain is a quiver representation for the type of quivers in Figure 1 that satisfies the same linear algebraic properties of those arising from limit linear series. In fact, Santana's $\mathbb{LP}(\mathfrak{g})$ is defined for $\mathfrak{g}$ being a linked chain of vector spaces and all the results mentioned above are still valid.

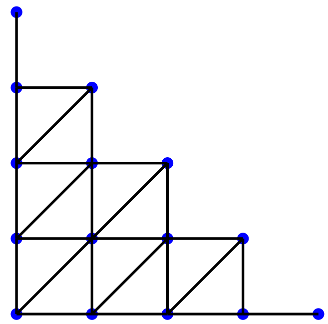

Figure 2: The base quiver for $\mathfrak{g}$ of degree 4 over a compact type curve with three components (all edges are double opposing arrows)

Given a special type of quiver Q we can define a *linked net of vector spaces* as a representation of G satisfying certain conditions. The special quiver is what we call a $\mathbb{Z}^n$*-quiver*. The quivers from Figures 1 and 2 are $\mathbb{Z}$ and $\mathbb{Z}^2$ quivers, respectively. If $\mathfrak{g}$ is a linked net of vector spaces over a $\mathbb{Z}^n$-quiver G, we can define $\mathbb{LP}(\mathfrak{g})$ simply as the scheme parameterizing subrepresentations of pure dimension 1.

We focus on linked nets over $\mathbb{Z}^2$-quivers Q, because they are the objects that arise in the study of limit linear series over compact type curves with three components. In Theorem 7.1, we prove that the scheme $\mathbb{LP}(\mathfrak{g})$ is pure dimensional and all the components are birational to $\mathbb{P}^r$. We also explicitly describe its components. For every vertex $\underline{d}$ of Q, we define $\mathbb{LP}(\mathfrak{g})_{\underline{d}}$ as the closure of the open set $\mathbb{LP}(\mathfrak{g})^*_{\underline{d}}$ of the subrepresentations $(I_{\underline{v}}) \in \mathbb{LP}(\mathfrak{g})$ generated by $I_{\underline{d}}$. Theorem 3.1 guarantees the union of all these open sets $\mathbb{LP}(\mathfrak{g})^*_{\underline{d}}$ is the set of exact points. By Proposition 7.6, the non-exact points of $\mathbb{LP}(\mathfrak{g})$ are in the intersection of the $\mathbb{LP}(\mathfrak{g})_{\underline{d}}$. Therefore, the set of exact points of $\mathbb{LP}(\mathfrak{g})$ is the non-singular locus.

If the dimension of $\mathfrak{g}$ is 2, we prove that $\mathbb{LP}(\mathfrak{g})$ is a flat deformation of the diagonal, hence we know its Hilbert polynomial. This result is a consequence of the characterization of subschemes of $(\mathbb{P}^1)^n$ with the same Hilbert polynomial as the diagonal by Cartwright and Sturmfels [8].

We also deal with the concept of a *simple basis* of a linked net. Roughly speaking, it is a set of vectors that forms a basis for the vector space associated to each vertex of the quiver, and diagonalize all maps of the representation at the same time. In general, it is a stronger condition than exactness (see Proposition 5.1), though for linked chains as in Figure 1, they are equivalent, as we prove in

Proposition 5.3. For general linked nets, there are exact ones that do not admit simple bases (see the example in Section 8.2).

We also study the scheme $\mathbb{P}(\mathfrak{g})$ for linked nets of vector spaces with finite support in $\mathbb{N}^3(\leq d)$. Here we also see the importance of working under the hypothesis of there being a simple basis. We have an example in Section 8.2 of a $\mathfrak{g}$ which is exact, but does not admit a simple basis and his associated scheme $\mathbb{P}(\mathfrak{g})$ does not have the Hilbert polynomial of the diagonal. In fact, it does not even have the same Chow class of the diagonal. We developed a practical algorithm to calculate the Chow class of $\mathbb{P}(\mathfrak{g})$ in terms of certain numerical data associated to $\mathfrak{g}$ (see Theorem 6.1).

Here is the structure of this thesis.

In Chapter 1 we review the classical theory of limit linear series introduced by Eisenbud and Harris and modified by Osserman; we also comment the results by Esteves and Osserman on $\mathbb{P}(\mathfrak{g})$ and of Santana on $\mathbb{LP}(\mathfrak{g})$.

Chapter 2 is where we generalize the concepts and ideas presented in Chapter 1 to the context of compact type curves with several components. In summary, let C denote a nodal and connected curve with $n+1$ components $X_0, X_1, \ldots, X_n$. Let $\mathcal{X} \longrightarrow B$ be a regular smoothing for C, $\mathcal{L}_\eta$ be a line bundle on the generic fiber $\mathcal{X}_\eta$ and d its degree. We can consider the twisted line bundle

$$\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(a_0X_0 + \cdots + a_nX_n).$$

Denote by $\mathcal{L}_{\underline{d}}$ the extension of $\mathcal{L}_\eta$ whose restriction to C has multi-degree $\underline{d}$. If we twist $\mathcal{L}_{\underline{d}}$ by

$$D = \sum_{i=0}^{n} a_i X_i$$

we will get another extension $\mathcal{L}_{\underline{e}}$. If we let v_i denote the multi-degree of $\mathcal{O}_{\mathcal{X}}(X_i)$ over C for $i = 0, \ldots, n$, then

$$\underline{e} = \underline{d} + \sum_{i=0}^{n} a_i v_i.$$

Twisting by $\mathcal{O}_{\mathcal{X}}(D)$ gives us a natural map

$$\varphi_{\underline{e}}^{\underline{d}} = \varphi_D^{\underline{d}} \colon \mathcal{L}_{\underline{d}} \longrightarrow \mathcal{L}_{\underline{e}}$$

Consider now the sheaves $L_{\underline{d}} := \mathcal{L}_{\underline{d}|_C}$. The map $\mathcal{L}_{\underline{d}} \to \mathcal{L}_{\underline{e}}$ induces the map

$$\varphi_{\underline{e}}^{\underline{d}} = \varphi_D^{\underline{d}} \colon L_{\underline{d}} \longrightarrow L_{\underline{e}}.$$

Let $\mathcal{V}_\eta \subset H^0(\mathcal{X}_\eta, \mathcal{L}_\eta)$. For each extension $\mathcal{L}_{\underline{d}}$ of $\mathcal{L}_\eta$ there exists an extension $\mathcal{V}_{\underline{d}} \subseteq H^0(\mathcal{X}, \mathcal{L})$ of $\mathcal{V}_\eta$, given by

$$\mathcal{V}_{\underline{d}} := \mathcal{V}_\eta \cap H^0(\mathcal{X}, \mathcal{L}_{\underline{d}}).$$

For each $D = \sum a_i X_i$ with $\min(a_i) = 0$, the map $\varphi_D^{\underline{d}}$ induces the map

$$\varphi_{\underline{e}}^{\underline{d}} = \varphi_D^{\underline{d}} \colon \mathcal{V}_{\underline{d}} \longrightarrow \mathcal{V}_{\underline{e}},$$

where $\underline{e} := D \cdot \underline{d}$. Letting $V_{\underline{d}}$ be the image of $\mathcal{V}_{\underline{d}}$ by the restriction map $H^0(\mathcal{X}, \mathcal{L}_{\underline{d}}) \longrightarrow H^0(C, L_{\underline{d}})$ for each $\underline{d}$, the above map induces the map

$$\varphi_{\underline{e}}^{\underline{d}} = \varphi_D^{\underline{d}} \colon V_{\underline{d}} \longrightarrow V_{\underline{e}}.$$

In the end of the day, we have the data consisting of line bundles $L_{\underline{d}}$ on C and vector spaces $V_{\underline{d}} \subset H^0(C, L_{\underline{d}})$ "linked" by maps $L_{\underline{d}} \longrightarrow L_{\underline{e}}$, which induce maps $V_{\underline{d}} \longrightarrow V_{\underline{e}}$. That's what we call a limit linear series over C.

In Chapter 3 we introduce the $\mathbb{Z}^n$-quivers and the linked nets of vector spaces. Let Q be a quiver, G its set of vertices and A its set of arrows. Let $n \in \mathbb{N}$. A *$\mathbb{Z}^n$-structure* on Q is a decomposition of A in subsets $A_0, \dots, A_n$ satisfying the following three properties:

1. For each vertex of Q and each $i = 0, \dots, n$ there is a unique arrow in A_i leaving the vertex.

For each path γ in Q let $\gamma(i)$ be the number of arrows of A_i it contains. If $\gamma(i) = 0$ for some i then γ is called *admissible*.

2. For each two distinct vertices $v_1, v_2 \in G$ there is an admissible path γ in Q connecting v_1 to v_2.

3. Two paths γ_1 and γ_2 in Q have the same initial and final vertices if and only if $\gamma_1(i) - \gamma_2(i)$ is constant for $i \in \{0, \dots, n\}$.

A quiver with a $\mathbb{Z}^n$-structure is called a *$\mathbb{Z}^n$-quiver*.

Let Q be a $\mathbb{Z}^n$-quiver, G its vertex set and A its arrow set. Let $A_0, \dots, A_n$ be a decomposition of A giving Q a $\mathbb{Z}^n$-structure. A *linked net of vector spaces* is a quiver representation of Q of pure dimension satisfying the following three additional properties:

1. If γ_1 and γ_2 are two admissible paths connecting the same two vertices then $\varphi_{\gamma_1} = \varphi_{\gamma_2}$.

2. If γ is a non-admissible path then $\varphi_\gamma = 0$.

3. If γ_1 and γ_2 are two simple admissible paths leaving the same vertex such that $\gamma_1(i) = 0$ or $\gamma_2(i) = 0$ for every i then $\mathrm{Ker}(\varphi_{\gamma_1}) \cap \mathrm{Ker}(\varphi_{\gamma_2}) = 0$.

In Chapter 4 we prove the main results concerning the structure of linked nets of dimension 1 over $\mathbb{Z}^2$-quivers:

Theorem 4.1. *Let $\mathfrak{g}$ be a linked net of vector spaces of dimension* 1 *with finite support over a $\mathbb{Z}^2$-quiver. Then $\mathfrak{g}$ is generated (perhaps not minimally) by a triangle, i.e., three vertices, pairwise adjacent.*

We describe in details all possible configurations for exact and non-exact linked nets of vector spaces of dimension 1. The exact ones are generated by a single vertex. The non-exact ones can be generated by a pair of adjacent vertices (there are three configurations on that case) or a triple of pairwise adjacent vertices (there are two configurations on that case). If one drop the assumption that the linked net has finite support, a “infinite corridor” configuration is also possible, where we can say that the generator is “at infinity”.

Chapter 5 is dedicated to the study of simple bases.

Let $\mathfrak{g}$ be a linked net of vector spaces of dimension r over a $\mathbb{Z}^n$-quiver with vertex set G. A simple basis for $\mathfrak{g}$ is a collection of r vertices $w_1, \dots, w_r$ and r vectors $s_i \in V_{w_i}$ for $i = 1, \dots, r$, such that:

$$\{s_1|_{V_w}, \dots, s_r|_{V_w}\} \text{ is a basis for } V_w \ \ \forall w \in G.$$

We have two important results about simple basis:

Proposition 5.1. *Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver. If $\mathfrak{g}$ has a simple basis, then $\mathfrak{g}$ is exact and has finite support.*

And a partial converse for $n = 1$:

Proposition 5.2. *Let $\mathfrak{g}$ be a linked chain of vector spaces of dimension r. Then the following statements are equivalent:*

a) $\mathfrak{g}$ admits a simple basis.

b) $\mathfrak{g}$ is exact of finite support.

In Chapter 6 we focus on the scheme $\mathbb{P}(\mathfrak{g})$ and prove the validity of the algorithm that allows us to calculate its Chow class. Our result is:

Theorem 6.1. *Let $\mathfrak{g}$ be an exact linked net of vector spaces of dimension $r+1$ over the standard $\mathbb{Z}^2$-quiver with finite support on $\mathbb{N}^2(\leq d)$. Let $\underline{d} \in \mathbb{N}^2(\leq d)$. Let i, j, k be integers satisfying $i+j+k = 2r$ and $0 \leq i, j, k \leq r$. Then the term $h_0^i h_1^j h_2^k$ appears in the expression of the class $\left[\mathbb{P}(\mathfrak{g})_{\underline{d}}\right]$ in the Chow ring of $\mathbb{P}^r \times \mathbb{P}^r \times \mathbb{P}^r$ (and then with coefficient 1) if and only if the following relations are true:*

$$\begin{aligned} p^{\underline{d}}_{Y+Z} &\leq i, & p^{\underline{d}}_{X} &\leq r-i,\\ p^{\underline{d}}_{X+Z} &\leq j, & p^{\underline{d}}_{Y} &\leq r-j,\\ p^{\underline{d}}_{X+Y} &\leq k, & p^{\underline{d}}_{Z} &\leq r-k. \end{aligned}$$

The study of $\mathbb{LP}(\mathfrak{g})$ is done in Chapter 7, where we prove the main structural theorem and the results on the Hilbert polynomial. The main theorem of this thesis is:

Theorem 7.1. *Let $\mathfrak{g}$ be an exact linked net of dimension $r+1$ with finite support over a $\mathbb{Z}^2$-quiver. Then the scheme $\mathbb{LP}(\mathfrak{g})$ is of pure dimension r, and all of its irreducible components are rational. More precisely, the components are the non-empty schemes $\mathbb{LP}(\mathfrak{g})_v$ and there is a one to one correspondence between the components of $\mathbb{LP}(\mathfrak{g})$ and the equivalence classes of $\mathfrak{g}$. Furthermore, the set of exact points of $\mathbb{LP}(\mathfrak{g})$ is its nonsingular locus.*

For dimension 2, we were able to calculate the Hilbert polynomial:

Theorem 7.2. *Let $\mathfrak{g}$ be a linked net of vector spaces of dimension 2 over a $\mathbb{Z}^2$-quiver. Let H be a finite set of vertices supporting $\mathfrak{g}$ such that $P(H) = H$. Then*

$$\mathbb{LP}(\mathfrak{g}) \subset \prod_{v \in H} \mathbb{P}(V_v)$$

is a connected union of projective lines, $Z_1 \cup \cdots \cup Z_m$, with the following property: for each $v \in H$ there is a unique $j \in \{1, \ldots, m\}$ such that the projection $Z_j \to \mathbb{P}(V_v)$ is an isomorphism. Furthermore, $\mathbb{LP}(\mathfrak{g})$ is a degeneration of the diagonal and the multivariate Hilbert polynomial of $\mathbb{LP}(\mathfrak{g})$ is

$$P(t_1, \ldots, t_N) = 1 + t_1 + \cdots + t_N,$$

where $N := \#H$.

Finally, Chapter 8 is a collection of examples.

Chapter 1

Limit linear series

1.1 The notion of limit linear series

All schemes in this thesis are over a characteristic zero algebraically closed field.

From now on, C will be a smooth connected projective curve of genus g. A *linear series* on C of degree d and rank r is a pair $\mathfrak{g} = (L, V)$, where L is a line bundle over C of degree d and $V \subset H^0(C, L)$ is a $(r+1)$-dimensional space of global sections of L.

Linear series are well known objects. Each rank r linear series $\mathfrak{g}$ over C gives rise to a rational map $C \dashrightarrow \mathbb{P}^r$. The map is not defined at the so called base points; the set of base points is a finite subset of C. There is a projective moduli space $G^r_d(C)$ parametrizing linear series on C of rank r and degree d.

A profound result about linear series is the Brill–Noether Theorem.

Theorem 1.1. (Brill–Noether) A general smooth connected projective curve C admits a linear series of rank r and degree d if, and only if

$$\rho(g, d, r) := (r+1)(d-r) - gr \geq 0.$$

Moreover, if that's the case, the moduli space $G^r_d(C)$ is of pure dimension $\rho(g, d, r)$.

As Esteves explains in his notes [9] the proof given by Brill and Noether was incomplete. Decades later, the "if" part was independently proved by Kempf and by Kleiman and Laksov. For the "only if" part, Severi suggested a degeneration argument, based on ideas by Castelnouvo. The idea is to consider a family of smooth curves degenerating to a general rational nodal curve X_0. Severi thought that if linear series of a certain rank and degree existed on the smooth curves in the family, then linear series of the same kind would exist on X_0, by a passage to the limit. That was not necessarily true, though.

This idea was improved by Griffiths and Harris and, later on, Eisenbud and Harris observed that the argument could be simplified if one considered degenerations to rational cuspidal curves. They also noted that one could replace the irreducible cuspidal curve by a semi-stable model of it: a rational line with an elliptic tail attached to each cusp in the original curve. As Eisenbud and Harris pointed out in their article [3], in order to study smooth curves by using degenerations to reducible curves, one should understand what happens to linear series in the course of such degeneration. That was the driving motivation to start a theory of limit linear series.

Roughly speaking, a limit linear series somehow generalizes the concept of linear series to singular curves. Through the last decades, a few slightly different definitions came out. We will make a brief description of these approaches.

1.1.1 Eisenbud and Harris approach

The ideia of dealing with limit linear series goes back to work by Eisenbud and Harris in the eighties. Their techniques are so powerful they were able to prove results on existence of Weierstrass points of certain types and enumeration of limit linear series, to name a few. See their paper [3] for more details.

We will consider $X = Y \cup Z$ to be a singular curve with two smooth components Y and Z intersecting at the only singular point $P = Y \cap Z$, which we will ask to be a node. This theory also works with more general singular curves, but here we will focus on that type, for simplicity.

Definition 1.1. Let X be a nodal curve with smooth components Y and Z meeting transversally at P. A *regular smoothing* of X is a flat and projective map $\pi : \mathcal{X} \longrightarrow B$, where B is the spectrum of a discrete valuation ring with algebraically closed residue field, $\mathcal{X}$ is regular, the generic fiber of π is smooth and the special fiber is isomorphic to X.

Now we fix a regular smoothing $\mathcal{X} \longrightarrow B$ of X. Being $\mathcal{X}$ regular, we have that Y and Z are Cartier divisors of $\mathcal{X}$; actually, every Cartier divisor of $\mathcal{X}$ supported in X is a linear combination of Y and Z. But $\mathcal{O}_{\mathcal{X}}(Y+Z) \cong \mathcal{O}_{\mathcal{X}}$, so it is enough to look at the divisors that are multiples of one component. Also, given an invertible sheaf $\mathcal{L}_\eta$ on the fiber $\mathcal{X}_\eta$, there is an invertible extension $\mathcal{L}$ over $\mathcal{X}$. The extension is not unique. Indeed, for each $i \in \mathbb{Z}$, $\mathcal{L}(iY) := \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(iY)$ is also an extension of $\mathcal{L}_\eta$ and these are all the invertible extensions.

When we specify the degree of $\mathcal{L}$ on Y and Z, we have uniqueness. For each linear series $(\mathcal{L}_\eta, \mathcal{V}_\eta)$ over $\mathcal{X}_\eta$ of degree d and rank r, there is a collection of extensions $(\mathcal{L}_i, \mathcal{V}_i)$ on $\mathcal{X}$, where $\mathcal{L}_i$ is characterized by having bi-degree $(d-i, i)$ on X, meaning that the restriction of $\mathcal{L}_i$ to Y has degree $d-i$ and to Z has degree i.

What Eisenbud and Harris did was to consider just the extremal degree linear series $(\mathcal{L}_0, \mathcal{V}_0)$ and $(\mathcal{L}_d, \mathcal{V}_d)$. More precisely, they considered the restricted sheaves $L^Y = \mathcal{L}_{0|_Y}$ and $L^Z = \mathcal{L}_{d|_Z}$, as well as $V^Y = V_{0|_Y}$ and $V^Z = V_{d|_Z}$. In this way we have two linear series of degree d over the components of X: (L^Y, V^Y) over Y and (L^Z, V^Z) over Z. This concept first appeared in their article [3].

In summary, we started with a linear series $(\mathcal{L}_\eta, \mathcal{V}_\eta)$ on the generic fiber of the family $\mathcal{X} \longrightarrow B$ and we obtained a "limit linear series": $\{(L^Y, V^Y), (L^Z, V^Z)\}$ on the limit of the family, i.e., on the special fiber X.

The objects that arise this way satisfy a crucial property, as the following proposition asserts:

Proposition 1.1. With the terminology and notation above, if $\epsilon_i^Y, \epsilon_i^Z$ are the orders of vanishing at $P = Y \cap Z$ of (L^Y, V^Y) and (L^Z, V^Z), in increasing order, then, for each $i = 0, \ldots, r$:

$$\epsilon_i^Y + \epsilon_{r-i}^Z \geq d$$

Considering all that was explained above, we take that property to formally define a limit linear series a la Eisenbud and Harris:

Definition 1.2. Let X be a nodal curve with two smooth components Y and Z meeting at the only node P. A *limit linear series* on X of degree d and rank r is a pair $\mathfrak{g} = \{(L^Y, V^Y), (L^Z, V^Z)\}$, where (L^Y, V^Y) and (L^Z, V^Z) are linear series of degree d and rank r on Y and Z, respectively, satisfying the condition:

$$\epsilon_i^Y + \epsilon_{r-i}^Z \geq d \; \forall \, i = 0, \ldots, r,$$

where ϵ_i^Y (resp. ϵ_i^Z) are the orders of vanishing of (L^Y, V^Y) (resp. (L^Z, V^Z)) at P.

If all inequalities are equalities, then we say $\mathfrak{g}$ is *refined*. Otherwise, it is called *crude*.

There is a moduli space $G_d^{r,\mathrm{EH}}(X)$ which parametrizes such objects, the Eisenbud–Harris limit linear series scheme of X. It is a projective subscheme of $G_d^r(Y) \times G_d^r(Z)$. The refined limit linear series form an open set.

1.1.2 Osserman's approach

Years later, Osserman came up with a different approach to the subject. Instead of looking only at the "extremal degrees" $(d, 0)$ and $(0, d)$, Osserman's concept of limit linear series considers all possible bidegrees $(d - i, i)$ for $i = 0, \ldots, d$.

We start, as before, with a regular smoothing $\pi : \mathcal{X} \longrightarrow B$ of X and a linear series $(\mathcal{L}_\eta, \mathcal{V}_\eta)$ of degree d and dimension $r + 1$ over the generic fiber X_η. It gives rise to a sequence of extensions $(\mathcal{L}_i, \mathcal{V}_i)$ over $\mathcal{X}$ such that the restrictions of $\mathcal{L}_i$ to Y and to Z have degrees $d - i$ and i, respectively, for each $i = 0, \ldots, d$. Since $\mathcal{L}_{i+1} = \mathcal{L}_i(Y)$, we have a natural morphism

$$\mathcal{L}_i \longrightarrow \mathcal{L}_{i+1}$$

In the other direction, we have:

$$\mathcal{L}_{i+1} = \mathcal{L}_i(Y) \cong \mathcal{L}_i(-Z) \longrightarrow \mathcal{L}_i.$$

These maps of sheaves induce maps on the global sections, which induce maps on the modules:

$$\varphi^i : \mathcal{V}_i \longrightarrow \mathcal{V}_{i+1} \text{ and } \varphi_i : \mathcal{V}_{i+1} \longrightarrow \mathcal{V}_i$$

The sheaf $L_i = \mathcal{L}_{i|_X}$ is determined by its restrictions to Y and Z. The following identifications hold:

$$L_{i|_Z} = \mathcal{L}_0(iY)_{|_Z} \cong L_{0|_Z}(iP) \tag{1.1}$$

$$L_{i|_Y} = \mathcal{L}_0(-iZ)_{|_Y} \cong L_{0|_Y}(-iP) \tag{1.2}$$

Considering that $L_i \subset L_{0|_Y} \oplus L_{d|_Z}$, we can view the map $L_i \longrightarrow L_{i+1}$ as induced by the zero map on $L_{0|_Y}$ and the projection of the canonical inclusion on $L_{d|_Z}$. Analogously for $L_{i+1} \longrightarrow L_i$. Let V_i be the image of $\mathcal{V}_i$ by the restriction map $H^0(\mathcal{X}, \mathcal{L}_i) \longrightarrow H^0(X, L_i)$. We have that V_i is mapped into V_{i+1} and vice-versa.

To present our new definition, we focus on the curve X and we pick up an invertible sheaf L of bidegree $(d,0)$ on X. For each i, we define the sheaf L_i, whose restrictions to Y and Z are $L_{|_Y}(-iP)$ and $L_{|_Z}(iP)$, respectively. There are morphisms:

$$\varphi^i : L_i \longrightarrow L_{i|_Z} = L_{i+1|_Z}(-P) \longrightarrow L_{i+1}$$
$$\varphi_{i+1} : L_{i+1} \longrightarrow L_{i+1|_Y} = L_{i|_Y}(-P) \longrightarrow L_i$$

And the relations hold: $\varphi^i \circ \varphi_{i+1} = 0$ and $\varphi_{i+1} \circ \varphi^i = 0$.

Now we can define a limit linear series a la Osserman.

Definition 1.3. A *limit linear series* $\mathfrak{g} = (L, V_0, \dots, V_d)$ of degree d and rank r over X is the data consisting of an invertible sheaf L on X of bidegree $(d,0)$ and a collection of $(r+1)$-dimensional vector spaces $V_i \subset H^0(X, L_i)$ such that $\varphi^i(V_i) \subset V_{i+1}$ and $\varphi_{i+1}(V_{i+1}) \subset V_i$.

We denote by V_i^Y (respectively, V_i^Z) the subspace of V_i of sections that vanish on Y (respectively, on Z). They are the kernels of the maps $V_i \longrightarrow V_{i-1}$ and $V_i \longrightarrow V_{i+1}$, respectively.

We say that $\mathfrak{g}$ is *exact* when for every i:

$$\operatorname{Im}(V_i \longrightarrow V_{i+1}) = V_{i+1}^Y = \operatorname{Ker}(V_{i+1} \longrightarrow V_i)$$
$$\operatorname{Im}(V_{i+1} \longrightarrow V_i) = V_i^Z = \operatorname{Ker}(V_i \longrightarrow V_{i+1}).$$

Remark 1.1. It is important to point out that every limit linear series $\mathfrak{g}$ on X arising as a limit of a linear series $(\mathcal{L}_\eta, \mathcal{V}_\eta)$ on the generic fiber $\mathcal{X}_\eta$ of a regular smoothing $\mathcal{X} \longrightarrow B$ of X is exact. See Esteves–Osserman [1].

The definition above can be reformulated in the language of quiver representations. We briefly recall this concept.

Definition 1.4. A *quiver* $Q = (V, E, s, t)$ is the data consisting of:

- a set V, called the set of vertices;
- a set E, called the set of arrows;
- a map $s : E \longrightarrow V$ thought as sending an arrow to its starting (or initial) vertex;
- a map $t : E \longrightarrow V$ thought as sending an arrow to its terminal (or final) vertex.

A quiver is *finite* if both V and E are finite.

There is thus a natural map $(s,t)\colon E \longrightarrow V \times V$ associated to a quiver $Q = (V, E, s, t)$. The quivers we study are such that (s,t) is injective, and thus E can be presented as a subset of $V \times V$. When (s,t) is a bijection, we say that Q is the *complete* quiver with vertices in V.

Example 1.1. Consider the quiver $\mathbb{N}(d) = (V, E, s, t)$ where

- $V = \{0, 1, 2, \ldots, d\}$,
- $E = \{(0,1), (1,2), \ldots, (d-1, d)\} \cup \{(d, d-1), \ldots, (2,1), (1,0)\} \subset V \times V$.

A pictorial representation of this quiver for $d = 3$ is shown in the figure below:

Definition 1.5. Fix a field k and let $Q = (V, E, s, t)$ be a quiver. A *representation* $\mathfrak{q} = (V_i, \varphi_\alpha)_{i \in V, \alpha \in E}$ of Q is the assignment of a k-vector space V_i to each vertex $i \in V$ and a k-linear map $\varphi_\alpha\colon V_{s(\alpha)} \longrightarrow V_{t(\alpha)}$ to each arrow $\alpha \in E$.

A representation $\mathfrak{q}$ is said to be finite-dimensional if each V_i is finite-dimensional.

Remark 1.2. With this terminology in mind, we see that a limit linear series $\mathfrak{g}$ on X of degree d is a representation of the quiver in Example 1.1.

There is also a moduli space $G_d^{r,\mathrm{Oss}}(X)$ which parametrizes limit linear series (a la Osserman) of rank r and degree d over X. There is an obvious forgetful map

$$G_d^{r,\mathrm{Oss}}(X) \longrightarrow G_d^r(Y) \times G_d^r(Z)$$

that takes $\mathfrak{g} = (L, V_0, \ldots, V_d)$ to $((L_{0|_Y}, V_{0|_Y}), (L_{d|_Z}, V_{d|_Z}))$. This map induces a surjective morphism

$$G_d^{r,\mathrm{Oss}}(X) \longrightarrow G_d^{r,\mathrm{EH}}(X).$$

This map is actually an isomorphism over the open subscheme of refined limit linear series. It is also true that if X is general, then the exact limit linear series $\mathfrak{g}$ over X form a dense open subset of $G_d^{r,\mathrm{Oss}}(X)$. All these results are proved in [4].

1.2 Esteves–Osserman result

In their paper [1], Esteves and Osserman have found a relation between limit linear series $\mathfrak{g}$ over nodal cuves with two components and the fibers of a certain Abel map. They described a closed subscheme $\mathbb{P}(\mathfrak{g})$ of the fiber and proved it is, in a sense which will be made precise below, a degeneration of the diagonal. This allows them to calculate the multivariate Hilbert polynomial of $\mathbb{P}(\mathfrak{g})$.

Recall our curve $X = Y \cup Z$ with $P = Y \cap Z$. For each $d > 0$ there is the Abel map:

$$A_d \colon S^d(X) \longrightarrow \mathrm{Pic}^d(X)$$

where $S^d(X)$ is the d-th symmetric product of X and $\mathrm{Pic}^d(X)$ parametrizes *equivalence classes* of line bundles of total degree d. Two line bundles L_1 and L_2 are *equivalent* if there exists an integer j such that $L_{1|_Y} \cong L_{2|_Y}(-jP)$ and $L_{1|_Z} \cong L_{2|_Z}(jP)$.

This A_d is constructed in the following way: given a 0-cycle D on X of degree d, write $D = D_Y + D_Z$, where D_Y and D_Z are 0-cycles supported in Y and Z respectively. The image of D by A_d is the class of the line bundle on X whose restrictions to Y and Z are $\mathcal{O}_Y(D_Y)$ and $\mathcal{O}_Z(D_Z)$, respectively. This definition does not depend on the way we write D as $D_Y + D_Z$.

A point of $\mathrm{Pic}^d(X)$ has a unique representative L of degree d on Y and degree 0 on Z. Let L_i be the twists defined as in Equations (1.1) and also set $\Gamma_Y^i := \Gamma(Y, L_{d-i|_Y})$ and $\Gamma_Z^i := \Gamma(Z, L_{i|_Z})$, for each $i = 0, \dots, d$. By sending the class of a non-zero section s to the 0-cycle $\mathrm{div}(s)$ we get closed embeddings

$$\mathbb{P}(\Gamma_Y^i) \longrightarrow S^i(Y)$$
$$\mathbb{P}(\Gamma_Z^i) \longrightarrow S^i(Z).$$

There is also a natural embedding

$$S^{d-i}(Y) \times S^i(Z) \longrightarrow S^d(X)$$

which sends a pair of 0-cycles to their sum. Therefore we can consider the composition

$$\mathbb{P}(\Gamma_Y^{d-i}) \times \mathbb{P}(\Gamma_Z^i) \longrightarrow S^{d-i}(Y) \times S^i(Z) \longrightarrow S^d(X).$$

The union of the images of all these maps for $i \in \{0, \dots, d\}$ is precisely the fiber over the class of L of the Abel map A_d. So, if we forget the embeddings and abuse notation, we can write

$$A_d^{-1}(L) = \bigcup_{i=0}^{d} \mathbb{P}(\Gamma_Y^{d-i}) \times \mathbb{P}(\Gamma_Z^{i}).$$

Sending a 0-cycle D to $D+P$ will give us inclusions

$$\begin{aligned} S^i(Y) &\longrightarrow S^{i+1}(Y) \\ S^i(Z) &\longrightarrow S^{i+1}(Z). \end{aligned}$$

These inclusions take $\mathbb{P}(\Gamma_Y^i)$ inside $\mathbb{P}(\Gamma_Y^{i+1})$ and $\mathbb{P}(\Gamma_Z^i)$ inside $\mathbb{P}(\Gamma_Z^{i+1})$. So, again abusing notation, we have chains of subschemes:

$$\begin{aligned} \mathbb{P}(\Gamma_Y^0) \subset \mathbb{P}(\Gamma_Y^1) \subset \cdots \subset \mathbb{P}(\Gamma_Y^d) \subset S^d(Y) \\ \mathbb{P}(\Gamma_Z^0) \subset \mathbb{P}(\Gamma_Z^1) \subset \cdots \subset \mathbb{P}(\Gamma_Z^d) \subset S^d(Z). \end{aligned}$$

So, we can consider the fiber $A_d^{-1}(L)$ inside $\mathbb{P}(\Gamma_Y^d) \times \mathbb{P}(\Gamma_Z^d)$ instead of $S^d(X)$.

To each limit linear series $\mathfrak{g} = (L, V_0, \ldots, V_d)$ of degree d on X, Esteves and Osserman constructed a subscheme $\mathbb{P}(\mathfrak{g}) \subset S^d(X)$ which actually lies on the fiber over the class of $L \in \mathrm{Pic}^d(X)$, in the following way: Consider

$$V_i^* = \{s \in V_i \mid s_{|_Y} \neq 0 \text{ and } s_{|_Z} \neq 0\}.$$

Then $\mathbb{P}(\mathfrak{g})$ is defined as the closure of

$$\mathbb{P}(\mathfrak{g})^* = \bigcup_{i=0}^{d} \{Z(s) \mid s \in V_i^*\} \subset S^d(X)$$

with the reduced scheme structure. Here, $Z(s)$ denotes the zero-divisor of s.

We set $\mathbb{P}(\mathfrak{g})_i$ to be the closure of $\{Z(s) \mid s \in V_i^*\}$. We can think of $\mathbb{P}(\mathfrak{g})_i$ as the closure of the image of the rational map

$$\mathbb{P}(V_i) \dashrightarrow \mathbb{P}(V_0) \times \mathbb{P}(V_d).$$

This map is not defined precisely on $\mathbb{P}(V_i^Y) \cup \mathbb{P}(V_i^Z)$.

In other words, we look at the image of each V_i into the extremal spaces V_0 and V_d. We will use this interpretation later to generalize $\mathbb{P}(\mathfrak{g})$ to curves with three components.

Esteves and Osserman obtained a good description of $\mathbb{P}(\mathfrak{g})$:

Theorem 1.2. If $\mathfrak{g} = (L, V_0, \ldots, V_d)$ is an exact limit linear series on X of degree d and rank r, then $\mathbb{P}(\mathfrak{g})$ is reduced, connected, Cohen–Macaulay of pure dimension r, and has Hilbert polynomial

$$P(s,t) = \binom{s+t+r}{r}$$

in $\mathbb{P}(V_0)\times\mathbb{P}(V_d)$. It is also a flat degeneration of the diagonal $\Delta \subset \mathbb{P}(V_0)\times\mathbb{P}(V_d)$. In particular, $[\mathbb{P}(\mathfrak{g})] = [\Delta]$ in the Chow ring $\mathcal{A}(\mathbb{P}(V_0)\times\mathbb{P}(V_d))$.

The full proof can be read in [1]. It depends on an important result, which appears in the following lemma:

Lemma 1.1. If $\mathfrak{g} = (L, V_0, \ldots, V_d)$ is an exact limit linear series of rank r, then there exist indices $i_0 \leq i_1 \cdots \leq i_r$ and sections $s_j \in V_{i_j}$ that satisfy the following properties:

1. for each $i \in \{0, \ldots, d\}$, the set $\{s_j \mid i_j = i\}$ forms a basis to $V_i/(V_i^Y \oplus V_i^Z)$;

2. the iterated images of all the s_j in each V_i form a basis of V_i.

The set $\{s_0, \ldots, s_r\}$ is what we call a *simple basis* for $\mathfrak{g}$. The converse holds, in fact: if $\mathfrak{g}$ has a simple basis, then $\mathfrak{g}$ must be exact. See Lemma 5.1 on Page 78 for a proof of this fact in a quite more general context.

Esteves and Osserman also studied the behavior of the scheme $\mathbb{P}(\mathfrak{g})$ in regular smoothing families $\mathcal{X}/B$. We start with a linear series $(\mathcal{L}_\eta, V_\eta)$ of degree d on the generic fiber $\mathcal{X}_\eta$. As observed before, it induces an exact limit linear series $\mathfrak{g} = (L, V_0, \ldots, V_d)$ on X, which we think of the limit of the linear series $(\mathcal{L}_\eta, V_\eta)$. The result in [1] is:

Theorem 1.3. Let $\mathcal{X} \longrightarrow B$ be a regular smoothing of X and $(\mathcal{L}_\eta, V_\eta)$ a linear series over the generic fiber $\mathcal{X}_\eta$ of degree d and rank r. Let $\mathfrak{g}$ be the limit of $(\mathcal{L}_\eta, V_\eta)$. Let also $\overline{\mathbb{P}(V_\eta)}$ be the closure of $\mathbb{P}(V_\eta)$ inside the relative symmetric product $S^d(\mathcal{X}/B)$. Then $\mathbb{P}(\mathfrak{g}) = S^d(X) \cap \overline{\mathbb{P}(V_\eta)}$.

1.3 Linked Grassmannians and linked projective spaces

We first introduce the concept of a linked chain of vector spaces. These objects arise naturally in the study of limit linear series over curves with two components. Later on, when we focus on more general curves, a new concept will appear, which will generalize linked chains.

Definition 1.6. A *linked chain of vector spaces (l.c.v.)* of degree d is a collection $\mathfrak{g}$ of vector spaces V_i for $i = 0, \dots, d$ and linear maps $\varphi^i \colon V_i \longrightarrow V_{i+1}$ and $\varphi_i \colon V_i \longrightarrow V_{i-1}$ between them such that for each i:

a) $\varphi_{i+1} \circ \varphi^i = 0$ and $\varphi^i \circ \varphi_{i+1} = 0$,

b) $\mathrm{Ker}(\varphi^{i+1} \circ \varphi^i) = \mathrm{Ker}(\varphi^i)$ and $\mathrm{Ker}(\varphi_i \circ \varphi_{i+1}) = \mathrm{Ker}(\varphi_{i+1})$,

c) $\mathrm{Ker}(\varphi^i) \cap \mathrm{Ker}(\varphi_i) = 0$.

If dim $V_i = r + 1$ for each $i = 0, \dots, d$, we say that $\mathfrak{g}$ is *pure dimensional* of dimension $r + 1$.

We say that $\mathfrak{g}$ is *exact* if it is pure dimensional and for each $i = 0, \dots, d - 1$

$$\mathrm{Im}(\varphi^i) = \mathrm{Ker}(\varphi_{i+1}),$$
$$\mathrm{Im}(\varphi_{i+1}) = \mathrm{Ker}(\varphi^i).$$

Before moving further, we will establish some notation. If $j > i$, denote by $\varphi^{i\longrightarrow j}$ the composition $\varphi^{j-1} \circ \varphi^{j-2} \circ \cdots \circ \varphi^{i+1} \circ \varphi^i$. If $j < i$, denote by $\varphi_{i\longrightarrow j}$ the composition $\varphi_{j+1} \circ \varphi_{j+2} \circ \cdots \circ \varphi_{i-1} \circ \varphi_i$. In any case, using Condition b) and induction, we see that $\mathrm{Ker}(\varphi_{i\longrightarrow j}) = \mathrm{Ker}(\varphi_i)$ for $j < i$ and $\mathrm{Ker}(\varphi^{i\longrightarrow j}) = \mathrm{Ker}(\varphi^i)$ for $j > i$.

Example 1.2. Of course, a limit linear series of degree d (on X) gives rise to a linked chain of vector spaces of degree d. The definition of a l.c.v. captures the linear properties of a l.l.s.

Santana characterized in [2] all l.c.v. of pure dimension 1:

Proposition 1.2. (Santana) Let $\mathfrak{g} = (I_i, \varphi^i, \varphi_i)$ be a linked chain of vector spaces of pure dimension 1 and degree d.

1. If $\mathfrak{g}$ is exact, then there exists an index i_0 such that $I_j = \varphi_{i_0\longrightarrow j}(I_{i_0})$ for every $j = 0, \dots, d$.

2. If $\mathfrak{g}$ is not exact, then there exists $i_0 < d$ such that $I_j = \varphi_{i_0 \longrightarrow j}(I_{i_0})$ for $j = 0, \dots, i_0 - 1$ and $I_j = \varphi_{i_0+1 \longrightarrow j}(I_{i_0+1})$ for $j = i_0 + 2, \dots, d$.

Putting in simple words, if $\mathfrak{g}$ is exact, then it is generated by only one index. If $\mathfrak{g}$ is not exact, it is generated by two consecutive indexes. In Theorem 4.2 we will see the analogous result for linked nets in $\mathbb{Z}^2$.

We will make this study a little more functorial. Let $\mathcal{E}$ be a vector bundle over a scheme S. Denote by $\mathbb{G}(s, \mathcal{E}/S)$ the relative Grassmannian of subspaces of the fibers of $\mathcal{E}/S$ of dimension s and $\mathbb{P}(\mathcal{E}/S) = \mathbb{G}(1, \mathcal{E}/S)$.

We will define the *linked Grassmannian functor* $\mathcal{LG}$. For this, consider locally free sheaves $\mathcal{E}_1, \dots, \mathcal{E}_n$ of rank r over an integral and Cohen–Macaulay scheme S. Fix bundle maps $f_i : \mathcal{E}_i \longrightarrow \mathcal{E}_{i+1}$ and $g_i : \mathcal{E}_{i+1} \longrightarrow \mathcal{E}_i$. For each positive integer $s \leq r$, define the functor

$$\mathcal{LG} = \mathcal{LG}(s, \{\mathcal{E}_i\}, \{f_i, g_i\}) : \mathfrak{Sch}_S \longrightarrow \mathfrak{Set}$$

which associates to each S-scheme T the set of sub-bundles $V_i \subset \mathcal{E}_{i,T}$ of rank s satisfying $f_{i,T}(V_i) \subset V_{i+1}$ and $g_{i,T}(V_{i+1}) \subset V_i$, for all $i = 1, \dots, n$.

We say that the data $(\mathcal{E}_i, f_i, g_i)$ is exact when:

1. $\forall i$, $\exists t \in \mathcal{O}_S$ such that $f_i \circ g_i$ and $g_i \circ f_i$ are multiplication by t;

2. if $t(x) = 0$, then $\mathrm{Ker}(f_i(x)) = \mathrm{Im}(g_i(x))$ and $\mathrm{Ker}(g_i(x)) = \mathrm{Im}(f_i(x))$;

3. $\forall x \in S$, $\mathrm{Im}(f_i(x)) \cap \mathrm{Ker}(f_{i+1}(x)) = 0$ and $\mathrm{Im}(g_{i+1}(x)) \cap \mathrm{Ker}(g_i(x)) = 0$.

Proposition 1.3. The functor $\mathcal{LG}(r, \{\mathcal{E}_i\}, \{f_i, g_i\})$ is represented by a projective scheme $\mathbb{LG}$ over S of relative dimension $r(d-r)$, which is naturally a closed subscheme of a product of relative Grassmannians over S. Moreover, if the data $(\mathcal{E}_i, f_i, g_i)$ is exact, then $\mathbb{LG}$ is Cohen–Macaulay and flat over S.

Proof. See Lemma A.3 in [4] and Theorem 4.1 in [5] $\square$

Let $\mathfrak{g} = (V_0, \dots, V_d, \varphi^i, \varphi_i)$ be a l.c.v. and S a k-scheme. A family of sub-l.c.v. of pure dimension s over S inside $\mathfrak{g}$ is a collection

$$\mathcal{W}_i \subset V_i \otimes \mathcal{O}_S$$

of locally free sheaves of rank s over S with locally free quotients and compatible with the maps: $\varphi^i(\mathcal{W}_i) \subset \mathcal{W}_{i+1}$ and $\varphi_{i+1}(\mathcal{W}_{i+1}) \subset \mathcal{W}_i$.

If $S = \mathrm{Spec}(k)$, then we define the *linked projective space* associated to $\mathfrak{g}$:

$$\mathbb{LP}(\mathfrak{g}) := \mathbb{LG}(1, \{V_i\}, \{\varphi^i, \varphi_i\}).$$

If $\mathfrak{g}$ is exact of dimension r, $\mathbb{LP}(\mathfrak{g})$ is reduced, Cohen–Macaulay of pure dimension r.

In addition, there is a natural inclusion

$$\mathbb{LP}(\mathfrak{g}) \longrightarrow \prod_{i=0}^{d} \mathbb{P}(V_i)$$

The main result concerning these objects in [2] is:

Proposition 1.4. Let $\mathfrak{g} = (V_0, \ldots, V_d, \varphi^i, \varphi_i)$ be an exact linked chain of vector spaces of pure dimension r and degree d. Then the multivariate Hilbert polynomial of $\mathbb{LP}(\mathfrak{g}) \subset \prod \mathbb{P}(V_i)$ is

$$P(t_0, \ldots, t_d) = \binom{t_0 + \cdots + t_d + r}{r}$$

Santana, in fact, did more. He showed that the set of exact points

$$\mathbb{LP}(\mathfrak{g})^* = \bigcup_{i=0}^{d} \mathbb{LP}(\mathfrak{g})_i^*$$

is an open dense set, and

$$\mathbb{LP}(\mathfrak{g})_i^* = \{(I_j, \varphi^j, \varphi_j) \in \mathbb{LP}(\mathfrak{g}) \mid I_j \text{ is the image of } I_i \, \forall j\}$$

is nonsingular for every i. These open sets are pairwise disjoint. Each $\mathbb{LP}(\mathfrak{g})_i$, the closure of $\mathbb{LP}(\mathfrak{g})_i^*$, is reduced, nonsingular and one of the irreducible components of $\mathbb{LP}(\mathfrak{g})$. The intersection $\mathbb{LP}(\mathfrak{g})_i \cap \mathbb{LP}(\mathfrak{g})_j$ is non-empty if and only if $|i - j| = 1$ and, in this case, $\mathbb{LP}(\mathfrak{g})_i \cap \mathbb{LP}(\mathfrak{g})_{i+1}$ is the set of $(I_j, \varphi^j, \varphi_j) \in \mathbb{LP}(\mathfrak{g})$ such that $\varphi_{i \longrightarrow j}(I_i) = I_j$ if $j < i$ and $\varphi^{i+1 \longrightarrow j}(I_{i+1}) = I_j$ if $j > i + 1$. By the characterization in Proposition 1.2, these are all the non-exact points. Hence, the set of exact points $\mathbb{LP}(\mathfrak{g})^*$ is the nonsingular locus of $\mathbb{LP}(\mathfrak{g})$.

Chapter 2

Limit linear series: generalization

The aim of this chapter is to generalize Osserman's limit linear series to nodal curves. As we have mentioned in Chapter 1, Eisenbud and Harris have defined limit linear series over curves of compact type, but they only considered the extremal degrees. Here we describe an approach closer to Osserman's.

2.1 Nodal curves

First, we introduce certain quivers. Let d and n be positive integers. Set

$$\mathbb{Z}^{n+1}(d) := \{(d_0, d_1, \ldots, d_n) \in \mathbb{Z}^{n+1} \mid d_0 + \cdots + d_n = d\},$$
$$\mathbb{N}^{n+1}(d) := \{(d_0, d_1, \ldots, d_n) \in \mathbb{Z}^{n+1}(d) \mid d_i \geq 0, \forall i\}.$$

Let $v_0, \ldots, v_n \in \mathbb{Z}^{n+1}(0)$ such that their sum is zero and any proper subset of them is linearly independent over $\mathbb{Q}$. Let $\underline{d} \in \mathbb{Z}^{n+1}(d)$. We associate to these data a quiver $Q(\underline{d}, v_0, \ldots, v_n)$: its set of vertices is

$$G := \underline{d} + \mathbb{Z}v_0 + \cdots + \mathbb{Z}v_n \subseteq \mathbb{Z}^{n+1}(d),$$

and its set of arrows is the subset $A \subseteq G \times G$, where $(\underline{d}, \underline{e}) \in A$ if and only if there is a proper subset $I \subset \{0, 1, \ldots, n\}$ such that

$$\underline{e} = \underline{d} + \sum_{i \in I} v_i.$$

(Note that if I exists then I is unique.) In this case, we denote $I(\underline{d}) := \underline{e}$. Note that if there an arrow connecting $\underline{d}$ to $\underline{e}$ if and only if there is an arrow connecting $\underline{e}$ to $\underline{d}$, because if $\underline{e} = I(\underline{d})$, then $\underline{d} = J(\underline{e})$, where J is the complement of I in $\{0, 1, \ldots, n\}$.

The quivers $Q(\underline{d}, v_0, \ldots, v_n)$ arise in the study of degenerations of linear series. Indeed, let C denote a nodal and connected curve with $n+1$ components $X_0, X_1, \ldots, X_n$. Let $\mathcal{X} \longrightarrow B$ be a regular smoothing for C.

As in the particular case of two-component curves, we want to consider a family of lines bundles on $\mathcal{X}$ and see how it behaves on the special fiber. Let thus $\mathcal{L}_\eta$ be a line bundle on the generic fiber $\mathcal{X}_\eta$. Let d denote its degree. It has an extension $\mathcal{L}$ over $\mathcal{X}$. The extension is not unique. The components $X_0, X_1, \ldots, X_n$ are Cartier divisors on $\mathcal{X}$, so for each $(a_0, a_1, \ldots, a_n) \in \mathbb{Z}^{n+1}$ we can consider the twisted line bundle

$$\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(a_0 X_0 + \cdots + a_n X_n).$$

These are all possible extensions of $\mathcal{L}_\eta$. They are distinguished by their multidegrees $(\deg \mathcal{L}|_{X_0}, \ldots, \deg \mathcal{L}|_{X_n})$. We will thus denote by $\mathcal{L}_{\underline{d}}$ the extension of $\mathcal{L}_\eta$ whose restriction to C has multi-degree $\underline{d}$. Notice that $\underline{d} \in \mathbb{Z}^{n+1}(d)$.

If we twist $\mathcal{L}_{\underline{d}}$ by

$$D = \sum_{i=0}^{n} a_i X_i$$

we will get another extension $\mathcal{L}_{\underline{e}}$. If we let v_i denote the multi-degree of $\mathcal{O}_{\mathcal{X}}(X_i)$ over C for $i = 0, \ldots, n$, then

$$\underline{e} = \underline{d} + \sum_{i=0}^{n} a_i v_i.$$

More explicitly, expressing $\underline{d} = (d_0, \ldots, d_n)$ and $\underline{e} = (e_0, \ldots, e_n)$ we have:

$$e_j = d_j + X_j \cdot \left(\sum a_i X_i \right) = d_j + X_j \cdot D \tag{2.1}$$

where the dot $\cdot$ denotes the intersection number. We write $\underline{e} := D \cdot \underline{d}$.

Notice that $\underline{e} = D' \cdot \underline{d}$ for $D' = \sum a'_i X_i$ if and only if $a'_i - a_i$ does not depend on i. It follows that there is a unique $D' = \sum a'_i X_i$ with $\min(a'_i) = 0$ such that $\underline{e} = D' \cdot \underline{d}$. If $\max(a'_i) = 1$ we identify D' with $I := \{i \mid a'_i > 0\}$ and put $I \cdot \underline{d} := \underline{e}$. In this case we say that $\underline{d}$ and $\underline{e}$ are *neighbors*. If in addition I has one element only, we say $\underline{d}$ and $\underline{e}$ are *adjacent*.

Twisting by $\mathcal{O}_{\mathcal{X}}(D)$ gives us a natural map

$$\varphi_{\underline{e}}^{\underline{d}} = \varphi_{D}^{\underline{d}} \colon \mathcal{L}_{\underline{d}} \longrightarrow \mathcal{L}_{\underline{e}}$$

for each $D = \sum a_i X_i$ with $\min(a_i) = 0$, where $\underline{e} := D \cdot \underline{d}$. Notice that

$$\varphi_{\underline{d}}^{\underline{e}} \circ \varphi_{\underline{e}}^{\underline{d}}$$

is the multiplication by t^a, where $a := \max(a_i)$ and t is the uniformizer parameter for B. When $a = 1$ we will also denote $\varphi_{D}^{\underline{d}}$ by $\varphi_{I}^{\underline{d}}$, where $I := \{i \mid a_i > 0\}$.

Consider now the sheaves $L_{\underline{d}} := \mathcal{L}_{\underline{d}|_C}$. The map $\mathcal{L}_{\underline{d}} \to \mathcal{L}_{\underline{e}}$ induces the map

$$\varphi_{\underline{e}}^{\underline{d}} = \varphi_{D}^{\underline{d}} \colon L_{\underline{d}} \longrightarrow L_{\underline{e}}.$$

(Notice the abuse of notation.) It restricts to an injection on each component X_j with $a_j = 0$, and to the zero map otherwise.

If we break up D as $D = E + F$, where also E and F are effective, then

$$\varphi_{\underline{e}}^{\underline{d}} = \varphi_{\underline{e}}^{\underline{f}} \circ \varphi_{\underline{f}}^{\underline{d}},$$

where $\underline{f} := F \cdot \underline{d}$, and thus $\underline{e} = E \cdot \underline{f}$. It follows that all the $\varphi_{\underline{e}}^{\underline{d}}$ are determined by the $\varphi_{\underline{e}}^{\underline{d}}$ for neighboring or even adjacent $\underline{d}$ and $\underline{e}$.

Let $\mathcal{V}_\eta \subset H^0(\mathcal{X}_\eta, \mathcal{L}_\eta)$. For each extension $\mathcal{L}_{\underline{d}}$ of $\mathcal{L}_\eta$ there exists an extension $\mathcal{V}_{\underline{d}} \subseteq H^0(\mathcal{X}, \mathcal{L})$ of $\mathcal{V}_\eta$, given by

$$\mathcal{V}_{\underline{d}} := \mathcal{V}_\eta \cap H^0(\mathcal{X}, \mathcal{L}_{\underline{d}}).$$

For each $D = \sum a_i X_i$ with $\min(a_i) = 0$, the map $\varphi_{D}^{\underline{d}}$ induces the map

$$\varphi_{\underline{e}}^{\underline{d}} = \varphi_{D}^{\underline{d}} \colon \mathcal{V}_{\underline{d}} \longrightarrow \mathcal{V}_{\underline{e}},$$

where $\underline{e} := D \cdot \underline{d}$. Letting $V_{\underline{d}}$ be the image of $\mathcal{V}_{\underline{d}}$ by the restriction map

$H^0(\mathcal{X}, \mathcal{L}_{\underline{d}}) \longrightarrow H^0(C, L_{\underline{d}})$ for each $\underline{d}$, the above map induces the map

$$\varphi^{\underline{d}}_{\underline{e}} = \varphi^{\underline{d}}_{D} \colon V_{\underline{d}} \longrightarrow V_{\underline{e}}.$$

(Notice the abuse of notation.)

In short, the regular smoothing $\mathcal{X} \longrightarrow B$ of the curve C and a linear series $(\mathcal{L}_\eta, \mathcal{V}_\eta)$ on the generic fiber $\mathcal{X}_\eta$ give rise to the data consisting of line bundles $L_{\underline{d}}$ on C and vector spaces $V_{\underline{d}} \subset H^0(C, L_{\underline{d}})$ "linked" by maps $L_{\underline{d}} \longrightarrow L_{\underline{e}}$, which induce maps $V_{\underline{d}} \longrightarrow V_{\underline{e}}$.

Another way of expressing this is as follows: let $G \subseteq \mathbb{Z}^{n+1}(d)$ be the subset of the multidegrees over C of the extensions of $\mathcal{L}_\eta$. If $\underline{d} \in G$ then

$$G = \underline{d} + \mathbb{Z}v_0 + \cdots + \mathbb{Z}v_n.$$

Let $A \subseteq G \times G$ be the subset of adjacent pairs $(\underline{d}, \underline{e})$. Let $Q = Q(\underline{d}, v_0, \ldots, v_n)$ be the corresponding quiver. Then we may view the $L_{\underline{d}}$ and the maps $L_{\underline{d}} \longrightarrow L_{\underline{e}}$ as a representation of Q in the category of line bundles over C. This representation induces a representation of Q in the category of vector spaces, under the functor of global sections. The $V_{\underline{d}}$ and the maps $V_{\underline{d}} \longrightarrow V_{\underline{e}}$ are thus a subrepresentation of the latter.

The quiver Q is a $\mathbb{Z}^n$-*quiver* (see Definition 3.1). Indeed, A decomposes in subsets $A_0, \ldots, A_n$, where $(\underline{d}, \underline{e}) \in A_i$ if $\underline{e} = \{i\} \cdot \underline{d}$. This decomposition satisfies the properties listed in Definition 3.1. Furthermore, the $V_{\underline{d}}$ and the maps $V_{\underline{d}} \longrightarrow V_{\underline{e}}$ give us a special representation $\mathfrak{g}$ of Q, namely, a *linked net of vector spaces*, as it satisfies the properties listed in Definition 3.6. We invite the reader to browse ahead and check that the properties just mentioned follow from the discussion above.

Furthermore, $\mathfrak{g}$ is exact; see Definition 3.7. Indeed, it is clear that

$$\mathrm{Im}(\varphi^{\underline{d}}_{\underline{e}}) \subseteq \mathrm{Ker}(\varphi^{\underline{e}}_{\underline{d}})$$

for neighboring $\underline{d}$ and $\underline{e}$. But let $s \in \mathrm{Ker}(\varphi^{\underline{e}}_{\underline{d}})$. Then $s = \widetilde{s}_{|C}$ for some $\widetilde{s} \in \mathcal{V}_{\underline{e}}$. Say $\underline{e} = I \cdot \underline{d}$. Then $\underline{d} = J \cdot \underline{e}$, where $J := \{0, \ldots, n\} - I$. Since $\varphi^{\underline{e}}_{\underline{d}} : \mathcal{L}_{\underline{e}} \longrightarrow \mathcal{L}_{\underline{d}}$ is injective on each component X_j with $j \notin J$, we have that $\widetilde{s}$ vanishes on each component X_j with $j \in I$. Since $\varphi^{\underline{d}}_{\underline{e}} : \mathcal{L}_{\underline{d}} \to \mathcal{L}_{\underline{e}}$ is multiplication by $\sum_{j \in I} X_j$, it follows that $\widetilde{s} = \varphi^{\underline{d}}_{\underline{e}}(\widetilde{u})$ for a certain section $\widetilde{u}$ of $\mathcal{L}_{\underline{d}}$. Since $\widetilde{s}$ and $\widetilde{u}$ agree on the generic fiber, $\widetilde{u} \in \mathcal{V}_{\underline{d}}$. It follows that $s = \varphi^{\underline{d}}_{\underline{e}}(u)$, where $u := \widetilde{u}|_C$.

Finally, $\mathfrak{g}$ has finite support on $G \cap \mathbb{N}(d)$; see Definition 3.9. Indeed, let

$\underline{d} = (d_0, \dots, d_n) \in G$. Suppose $d_j < 0$ for some j. Then every section in $\mathcal{V}_{\underline{d}}$ vanishes on X_j, whence $\varphi^{e}_{\underline{d}}\colon \mathcal{V}_{\underline{e}} \longrightarrow \mathcal{V}_{\underline{d}}$ is surjective, thus an isomosphism, for $\underline{e}$ satisfying $\underline{d} = \{j\} \cdot \underline{e}$. If $\underline{e} \notin \mathbb{N}(d)$ repeat the process. The process must end because a nonzero global section of $\mathcal{L}_{\underline{d}}$ vanishes to finite order on the components X_i.

In the next chapter we will start the study of exact linked nets of vector spaces with finite support over $\mathbb{Z}^m$-quivers, in fuller generality.

Example 2.1. Assume C is of *compact type*. The name comes from the fact that each component of the Picard scheme of C, parametrizing invertible sheaves with a fixed multidegree $(d_0, \dots, d_n)$, is projective. In fact, each such component is isomorphic to the product of the components of the Picard schemes of the X_i parametrizing invertible sheaves of degrees d_i, the isomorphism being given by restriction.

Then $G = \mathbb{Z}^{n+1}(d)$. Indeed, fix $\underline{d} = (d_0, \dots, d_n) \in G$. For each $\underline{e} = (e_0, \dots, e_n) \in \mathbb{Z}^{n+1}(d)$. Consider the intersection matrix:

$$M = \begin{bmatrix} X_1 \cdot X_1 & X_1 \cdot X_2 & \dots & X_1 \cdot X_n \\ X_2 \cdot X_1 & X_2 \cdot X_2 & \dots & X_2 \cdot X_n \\ \dots & \dots & \dots & \\ X_n \cdot X_1 & X_n \cdot X_2 & \dots & X_n \cdot X_n \end{bmatrix}. \tag{2.2}$$

To show that $\underline{e} \in G$ we need only find an integer solution to the system $M(x_1, \dots, x_n)^T = (e_1 - d_1, \dots, e_n - d_n)$. This is only possible for every $\underline{e}$ if, and only if, the determinant of M is ± 1. Since $X_1 + \dots + X_n \equiv -X_0$, we see that $\det(M)$ is equal to the determinant of

$$\begin{bmatrix} -X_1 \cdot X_0 & X_1 \cdot X_2 & \dots & X_1 \cdot X_n \\ -X_2 \cdot X_0 & X_2 \cdot X_2 & \dots & X_2 \cdot X_n \\ \dots & \dots & \dots & \\ -X_n \cdot X_0 & X_n \cdot X_2 & \dots & X_n \cdot X_n \end{bmatrix}$$

The curve C is of compact type, so its dual graph is a tree. Since every tree has a leaf, after possibly relabeling the indexes, we can suppose that $X_0 \cdot X_1 = 1$ and $X_0 \cdot X_j = 0$ for $j > 1$ and then we conclude by induction on the number of components that M has determinant 1 or -1.

Example 2.2. There is only one tree with three vertices, so there is only one possible configuration for a compact type curve with three components. We rename the components X_0, X_1, X_2 to X, Y, Z respectively, and order them in such a way that we have the following table of multiplication:

$$\begin{aligned} X \cdot X &= -1 & Y \cdot Y &= -2 \\ X \cdot Y &= 1 & Y \cdot Z &= 1 \\ X \cdot Z &= 0 & Z \cdot Z &= -1 \end{aligned}$$

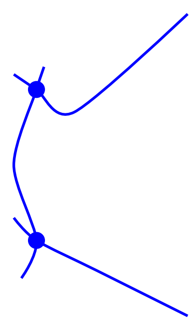

Figure 2.1: Compact type curve with three components

For each integer d there is an associated quiver with vertices in $\mathbb{Z}^3(d)$, and for each regular smoothing of C and linear series on the generic fiber of the smoothing we obtain a linked net of vector spaces with support in $\mathbb{N}^3(d)$. We may drop the degree corresponding to the component Y, thus projecting to $\mathbb{Z}^2$. The projection induces a bijection $\mathbb{Z}^3(d) \to \mathbb{Z}^2$ that takes $\mathbb{N}^3(d)$ to

$$\mathbb{N}^2(\leq d) := \{(i, j) \in \mathbb{Z}^2 \mid 0 \leq i, j \leq i + j \leq d\}.$$

The vectors v_0, v_1 and v_2 are thus projected to $v_X := (-1, 0)$, $v_Y := (1, 1)$ and $v_Z := (0, -1)$, respectively. We use the vertices of $\mathbb{Z}^2$ as indices for the linked net of vector spaces and rename the corresponding maps, substituting

$$\varphi_X,\ \varphi_Y,\ \varphi_Z,\ \varphi_{X+Y},\ \varphi_{X+Z} \text{ and } \varphi_{Y+Z}$$

for

$$\varphi_{\{0\}},\ \varphi_{\{1\}},\ \varphi_{\{2\}},\ \varphi_{\{0,1\}},\ \varphi_{\{0,2\}} \text{ and } \varphi_{\{1,2\}},$$

respectively. There are thus six maps with domain $V_{(i,j)}$;

$$\begin{aligned}
\varphi_X : V_{(i,j)} &\longrightarrow V_{(i-1,j)},\\
\varphi_Y : V_{(i,j)} &\longrightarrow V_{(i+1,j+1)},\\
\varphi_Z : V_{(i,j)} &\longrightarrow V_{(i,j-1)},\\
\varphi_{X+Y} : V_{(i,j)} &\longrightarrow V_{(i,j+1)},\\
\varphi_{X+Z} : V_{(i,j)} &\longrightarrow V_{(i-1,j-1)},\\
\varphi_{Y+Z} : V_{(i,j)} &\longrightarrow V_{(i+1,j)}.
\end{aligned}$$

In Figure 2.2 below we have a pictorial representation of the complete subquiver supported on $\mathbb{N}^2(\leq d)$ for $d = 4$.

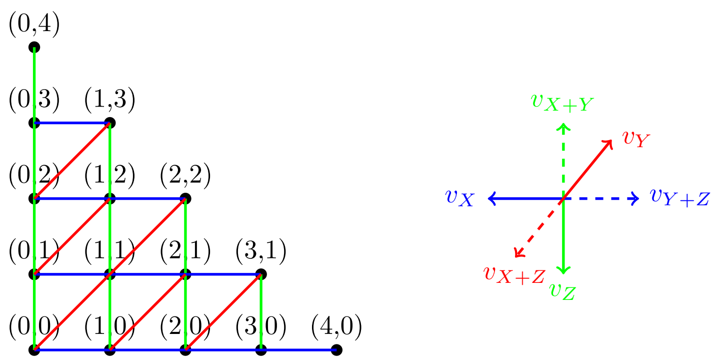

Figure 2.2: The quiver on $\mathbb{N}^2(\leq 4)$ for the compact type curve

How about non-compact type curves? Then $G = \underline{d} + \mathbb{Z}v_0 + \cdots + \mathbb{Z}v_n$ is different from $\mathbb{Z}(d)$. In fact, the number of possibilities for G is the absolute value of the determinant of (2.2). The next example illustrate this.

Example 2.3. Consider C to be the "triangle curve": $C = X \cup Y \cup Z$, where each component intersects each other at one point; see Figure 2.3. Its dual graph is a cycle with three vertices.

The multiplication table is: $X \cdot X = Y \cdot Y = Z \cdot Z = -2$ and $X \cdot Y = X \cdot Z = Y \cdot Z = 1$. As before, we project to $\mathbb{Z}^2$, now forgetting the degree on X. The vectors v_0, v_1 and v_2 are thus sent to $v_X = (1, 1)$, $v_Y = (-2, 1)$ and $v_Z = (1, -2)$. In this case, forgetting v_X, the matrix (2.2) is

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

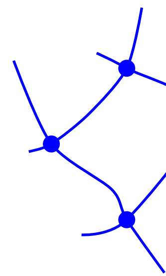

Figure 2.3: Non-compact type curve: the "triangle" curve

Its determinant is equal to 3. And there are three possibilities for the intersection of the image of G in $\mathbb{Z}^2$ with $\mathbb{N}^2(\leq d)$, marked in red, blue and green in Figure 2.4 below, for $d = 4$:

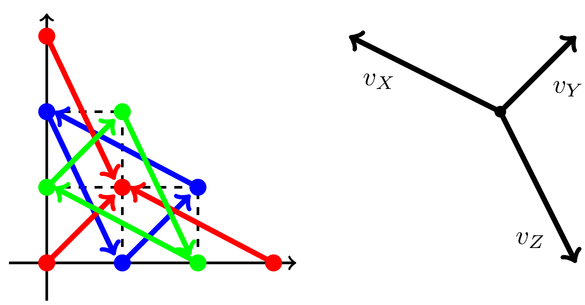

Figure 2.4: The quivers on $\mathbb{N}^2(\leq 4)$ for the "triangle" curve

Chapter 3

Linked nets over $\mathbb{Z}^n$-quivers

In this chapter we introduce $\mathbb{Z}^n$-quivers and linked nets of vector spaces over them. As we have seen in the last chapter, these objects arise naturally in the study of limit linear series. But here we study them independently.

In the first section we define and study $\mathbb{Z}^n$-quivers. And in the second section we define and study linked nets. In the second section we prove in Theorem 3.1 that an exact one-dimensional linked net with finite support has simple basis, or equivalently, has support on a vertex.

3.1 $\mathbb{Z}^n$-quivers

Definition 3.1. Let Q be a quiver, G its set of vertices and A its set of arrows. Let $n \in \mathbb{N}$. A $\mathbb{Z}^n$-*structure* on Q is a decomposition of A in subsets $A_0, \dots, A_n$ satisfying the following three properties:

1. For each vertex of Q and each $i = 0, \dots, n$ there is a unique arrow in A_i leaving the vertex.

For each path γ in Q let $\gamma(i)$ be the number of arrows of A_i it contains. If $\gamma(i) = 0$ for some i then γ is called *admissible*.

2. For each two distinct vertices $v_1, v_2 \in G$ there is an admissible path γ in Q connecting v_1 to v_2.

3. Two paths γ_1 and γ_2 in Q have the same initial and final vertices if and only if $\gamma_1(i) - \gamma_2(i)$ is constant for $i \in \{0, \dots, n\}$.

A quiver with a $\mathbb{Z}^n$-structure is called a *$\mathbb{Z}^n$-quiver*.

Given a *$\mathbb{Z}^n$-structure* on a quiver Q, any path γ for which for which $\gamma(i) \leq 1$ for every i is called *simple*.

Considering the trivial path, it follows from Property 3 above that a simple non-admissible path is a cycle. As a consequence, if v_1 is connected to v_2 by a simple admissible path, so is v_2 to v_1, in which case we say that v_1 and v_2 are *neighbors*. It follows as well from Property 3 that every two admissible paths γ_1 and γ_2 connecting the same two vertices satisfy $\gamma_1(i) = \gamma_2(i)$ for $i \in \{0, \dots, n\}$.

For each vertex v of Q and each $i = 0, \dots, n$ there is a unique arrow in A_i arriving at v. Indeed, consider the simple circular path γ leaving v whose last arrow a is in A_i; this arrow arrives at v. Write $\gamma = a + \gamma'$. If b is another arrow of A_i arriving at v, we must have that $\gamma' + b$ is circular, and thus γ' arrives at the initial point of b. But also γ' arrives at the initial point of a. Both points are the same, whence $a = b$ by Property 1.

Also, two paths γ_1 and γ_2 arriving at the same vertex w leave from the same vertex if and only if $\gamma_1(i) - \gamma_2(i)$ is constant for $i \in \{0, \dots, n\}$. Indeed, the "only if" statement follows from Property 3 directly. Conversely, suppose $\gamma_1(i) - \gamma_2(i)$ is constant for $i \in \{0, \dots, n\}$. By Property 1, there are paths γ_1' and γ_2' leaving w such that $\gamma_1'(i) + \gamma_1(i)$ and $\gamma_2'(i) + \gamma_2(i)$ are constant for $i \in \{0, \dots, n\}$. Then $\gamma_1'(i) - \gamma_2'(i)$ is constant for $i \in \{0, \dots, n\}$, and hence it follows from Property 3 that γ_1' and γ_2' arrive at the same vertex v. Also from Property 3, the concatenations of γ_1 with γ_1' and of γ_2 with γ_2' arrive at the initial vertices of γ_1 and γ_2, respectively, which is thus v.

For each vertex $v \in G$ and each proper subset $I \subsetneq \{0, \dots, n\}$ we denote by $I(v)$ or $I \cdot v$ the terminal vertex of a simple admissible path γ leaving v with $\gamma(i) > 0$ if and only if $i \in I$. It follows from Property 3 that $I(v)$ does not depend on the choice of γ.

The terminology is justified by the following example.

Example 3.1. Let $n \in \mathbb{N}$. Let $v_0, v_1, \dots, v_n \in \mathbb{Z}^n$ satisfying the following two properties:

1. $v_0 + \cdots + v_n = 0$.

2. Every proper subset of $\{v_0, \dots, v_n\}$ is linearly independent (considering $\mathbb{Z}^n$ inside the $\mathbb{Q}$-vector space $\mathbb{Q}^n$).

Let $\underline{d} \in \mathbb{Z}^n$. Put $G := \underline{d} + \mathbb{Z}v_0 + \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$. Consider the (complete) quiver with vertices in G and one arrow connecting any two vertices. For each

$i = 0, \dots, n$ let A_i be the set of arrows connecting a vertex $\underline{e}$ to a vertex $\underline{f}$ such that $\underline{f} - \underline{e} = v_i$. Let Q be the spanning subquiver whose arrow set is $A := \bigcup_i A_i$. Then the A_i give a $\mathbb{Z}^n$-structure to Q.

We call $v_0, v_1, \dots, v_n$ the *generating vectors* of Q.

In fact, the above example is the only one.

Proposition 3.1. All $\mathbb{Z}^n$-quivers are equivalent.

Proof. Let Q be the $\mathbb{Z}^n$-quiver of the example above. Let Q' be another $\mathbb{Z}^n$-quiver, with vertex set G', arrow set A' and $\mathbb{Z}^n$-structure given by the decomposition $A' = \bigcup A'_i$. It is enough to show that these two $\mathbf{Z}^n$-quivers are equivalent.

We define a bijection $f_0 \colon G \longrightarrow G'$ as follows: Pick any vertex $v \in G'$ and put $f_0(\underline{d}) := v$. Then, for any choice of integers $\ell_0, \dots, \ell_n$ with $\min(\ell_i) = 0$, let $f_0(\underline{d} + \sum_i \ell_i v_i)$ be the vertex of Q' obtained as the final point of a path γ in Q' with initial point v satisfying $\gamma(i) = \ell_i$ for each i. That γ exists follows from Property 1 of Q', and that its final point does not depend on the chosen γ follows from Property 3. Now, any point of G is expressed as $\underline{d} + \sum_i \ell_i v_i$ for unique integers ℓ_i with $\min(\ell_i) = 0$. Thus f_0 is well-defined. It is surjective by Property 2 of Q'. It is injective by Property 3.

We define now a bijection $f_1 \colon A \longrightarrow A'$ as follows: For each $i = 0, \dots, n$ and each arrow $a \in A_i$, let $f_1(a)$ be the arrow in A'_i leaving the vertex $f_0(\underline{e})$, where $\underline{e}$ is the initial point of a; it is unique by Property 1 of Q'. Clearly, $f_1(A_i) \subseteq A'_i$ for each i. Also, f_1 has an inverse, mapping for each $i = 0, \dots, n$ an arrow $a' \in A'_i$ to the arrow $a \in A$ connecting $\underline{e}$ to $\underline{e} + v_i$, where $f_0(\underline{e})$ is the initial vertex of a'.

Finally, f_1 is compatible with f_0. Indeed, let $a \in A$, with initial point $\underline{e}$ and terminal point $\underline{h}$. Then $a \in A_j$ with $\underline{h} = \underline{e} + v_j$. Set $a' := f_1(a)$. By definition, $a' \in A_j$ and $w := f_0(\underline{e})$ is its initial point. Let z be its terminal point. It remains to show that $z = f_0(\underline{h})$. Now,

$$\underline{e} = \underline{d} + \sum_i \ell_i v_i \quad \text{and} \quad \underline{h} = \underline{d} + \sum_i m_i v_i$$

for unique integers ℓ_i and m_i with $\min(\ell_i) = \min(m_i) = 0$. Also, since $w = f_0(\underline{e})$, there is a path γ in Q' connecting v to w satisfying $\gamma(b) = \ell_b$ for each b. Then $\epsilon := a' + \gamma$ is a path in Q' connecting v to z. We analyze two cases:

First, assume that there is i different from j with $\ell_i = 0$. Then $m_b = \ell_b$ for each $b \neq j$ and $m_j = \ell_j + 1$. Then $\epsilon(b) = m_b$ for each b. By definition, $z = f_0(\underline{h})$.

Finally, assume that $\ell_i > 0$ for each i different from j, whence $\ell_j = 0$. Then $m_b = \ell_b - 1$ for each $b \neq j$ and $m_j = \ell_j$. It follows that $\epsilon(b) = m_b + 1$ for each b. By Property 3 of Q', any path ρ leaving v with $\rho(b) = m_b$ for each b arrives at the same vertex as ϵ, which is z. Thus, by definition again, $z = f_0(\underline{h})$. $\square$

Remark 3.1. In view of Proposition 3.1, we may and will often assume that our $\mathbb{Z}^n$-quiver is given as in Example 3.1, and assume the notation therein. We may also assume $\underline{d} = 0$. In the special case $n = 2$, a case we will mostly concentrate on, we will assume $v_0 = (-1, 0)$, $v_1 = (1, 1)$ and $v_2 = (0, -1)$. We call this quiver the *standard* $\mathbb{Z}^2$*-quiver*.

Definition 3.2. Let Q be a $\mathbb{Z}^n$-quiver. Its *expansion* is the quiver $\widetilde{Q}$ with same vertex set as Q, but with arrow set equal to the set of all simple admissible paths of Q.

In other words, for each two neighboring vertices of Q, there are two arrows in $\widetilde{Q}$ connecting one to the other in reverse directions. Since arrows of $\widetilde{Q}$ are simple admissible paths, we may view Q as a spanning subquiver of $\widetilde{Q}$.

Proposition 3.2. Let Q be a $\mathbb{Z}^n$-quiver. Let H be a finite collection of vertices of Q. Then, for each $i = 0, \dots, n$ there is a unique vertex w of Q satisfying the following two properties:

1. For each $v \in H$ there is a path γ in Q connecting v to w with $\gamma(i) = 0$.

2. For each other vertex w' with the same property there is a path γ in Q connecting w to w' with $\gamma(i) = 0$.

Proof. Fix a vertex u of H. For simplicity, assume $i = 0$. For each $v \in H$, let γ_v be an admissible path connecting u to v. For each $j = 0, \dots, n$ let

$$m_j := \max\Big(\gamma_v(j) - \gamma_v(0) \,|\, v \in H\Big).$$

Since γ_u is the trivial path, $m_j \geq 0$ for every j. Let γ be any path leaving u with $\gamma(j) = m_j$ for each $j = 0, \dots, n$. Let w_0 be its terminal point. We claim w_0 satisfies Properties 1 and 2. For simplicity, set $w := w_0$.

Indeed, observe first that $\gamma(0) = 0$. Let $v \in H$. Since $\gamma(j) \geq \gamma_v(j) - \gamma_v(0)$ for every j, there is a path ϵ_v leaving v with $\epsilon_v(j) = \gamma(j) - \gamma_v(j) + \gamma_v(0)$ for every j. Notice that $\epsilon_v(0) = 0$. Now, the concatenation $\gamma_v\epsilon_v$ is a path leaving u such that $\epsilon_v(j) + \gamma_v(j) - \gamma(j)$ is constant, equal to $\gamma_v(0)$ for $j \in \{0, \dots, n\}$. Thus

$\gamma_v\epsilon_v$ and γ arrive at the same point, namely w, which is thus the end point of ϵ_v. Property 1 is verified for w.

Let now w' be another vertex satisfying the same property, that is, for each $v \in H$, there is a path μ_v connecting v to w' with $\mu_v(0) = 0$. Then $\gamma_v\mu_v$ connects u to w' for each $v \in H$. It follows that, for each $v, v' \in H$,

$$\mu_v(j) + \gamma_v(j) - \mu_{v'}(j) - \gamma_{v'}(j) \text{ is constant for } j \in \{0, \dots, n\}.$$

Fix $v \in H$ and put $\ell_j := \mu_v(j) - \epsilon_v(j)$. Then $\ell_0 = 0$. We claim that $\ell_j \geq 0$ for every j. Indeed, for each $j = 0, \dots, n$ there is v_j such that $\gamma(j) = \gamma_{v_j}(j) - \gamma_{v_j}(0)$. Then

$$\mu_v(j)+\gamma_v(j)-\mu_{v_j}(j)-\gamma_{v_j}(j) = \mu_v(0)+\gamma_v(0)-\mu_{v_j}(0)-\gamma_{v_j}(0) = \gamma_v(0)-\gamma_{v_j}(0),$$

whence

$$\begin{aligned}
\ell_j =& \mu_v(j) - \epsilon_v(j) \\
=& \mu_v(j) - \gamma(j) + \gamma_v(j) - \gamma_v(0) \\
=& \mu_v(j) - \gamma_{v_j}(j) + \gamma_{v_j}(0) + \gamma_v(j) - \gamma_v(0) = \mu_{v_j}(j) \geq 0.
\end{aligned}$$

Let μ be a path leaving w with $\mu(j) = \ell_j$ for every j. Then $\mu(0) = 0$. Also, $\mu(j) + \epsilon_v(j) = \mu_v(j)$ for every j, whence the path $\mu + \epsilon_v$ arrives at the same vertex as μ_v, namely w'. Thus μ connects w to w'.

Finally, w is unique. Indeed, if there were another w' satisfying Properties 1 and 2, there would be two paths, γ connecting w to w' and γ' connecting w' to w, with $\gamma(0) = \gamma'(0) = 0$. Composing, we would get an nontrivial circular path μ with $\mu(0) = 0$. That is not possible. □

Definition 3.3. Let Q be a $\mathbb{Z}^n$ -quiver. For each vertex v of Q and each proper subset $I \subseteq \{0, \dots, n\}$, the *$I$-cone of v*, denoted $C_I(v)$, is the set of end points of all the paths γ leaving v for which $\gamma(i) = 0$ if $i \notin I$.

It follows from Proposition 3.2 that, for any finite subset H or vertices of Q and for each $i = 0, \dots, n$, the vertices u of Q for which there is a path γ with $\gamma(i) = 0$ connecting z to u for each $z \in H$ form a cone $C_{J_i}(w_i)$, for a unique vertex w_i, where $J_i := \{0, \dots, n\} - \{i\}$.

Definition 3.4. We call w_i the *i-th bound* of H.

Proposition 3.3. Let Q be a $\mathbb{Z}^n$-quiver, and let $z_0, \ldots, z_n$ be vertices of Q. For each $j = 0, \ldots, n$, let $I_j := \{0, \ldots, n\} - \{j\}$. Then

$$\bigcap_{j=0}^{n} C_{I_j}(z_j)$$

is finite

Proof. As all $\mathbb{Z}^n$-quivers are equivalent by Proposition 3.1, assume that Q is as in Example 3.1 with $\underline{d} = 0$, with $v_1, \ldots, v_n$ forming the canonical basis of $\mathbb{Z}^n$ and with $v_0 = (-1, \ldots, -1)$. Then the jth coordinate of a vertex in $C_{I_j}(z_j)$ is bounded above by the jth coordinate of z_j for each $j = 1, \ldots, n$. And the jth coordinate of a vertex in $C_{I_0}(z_0)$ is bounded below by the jth coordinate of z_0 for each $j = 1, \ldots, n$. Thus the vertices of the intersection have bounded coordinates, and hence there are finitely many of them. □

Definition 3.5. Let Q be a $\mathbb{Z}^n$ -quiver with vertex set G. Let H be a non-empty finite subset of G. Let $P(H)$ be the set of all $v \in G$ such that for each $i = 0, \ldots, n$ there are $z \in H$ and a path γ connecting z to v with $\gamma(i) = 0$. Call $P(H)$ the *hull* of H.

Proposition 3.4. Let Q be a $\mathbb{Z}^n$ -quiver. Let H be a non-empty finite set of vertices of Q. Then the following two statements hold:

1. $H \subseteq P(H)$.

2. If H is finite, so is $P(H)$.

3. $P(P(H)) = P(H)$.

Proof. Statement 1 is clear: If $v \in H$ then the trivial path γ connects a vertex of H to v and satisfies $\gamma(i) = 0$ for every i.

As for Statement 2, observe that

$$P(H) = \bigcup_{f \in H^{\{0,\ldots,n\}}} \bigcap_{j=0}^{n} C_{I_j}(f(j)).$$

If H is finite, so is H^{n+1}, and thus $P(H)$ is the finite union of finite sets by Proposition 3.3.

As for Statement 3, the inclusion $P(H) \subseteq P(P(H))$ follows from Statement 1. In addition, for each $v \in P(P(H))$ and $i \in \{0, \ldots, n\}$ there are $w_i \in P(H)$

and a path γ_i connecting w_i to v with $\gamma_i(i) = 0$. Since $w_i \in P(H)$, there are $z_i \in H$ and a path μ_i connecting z_i to w_i with $\mu_i(i) = 0$. The concatenation of μ_i with γ_i is a path ν_i connecting z_i to v with $\nu_i(i) = 0$. As this holds for each $i = 0, \dots, n$, it follows that $v \in P(H)$. As this holds for each $v \in P(P(H))$, we have $P(P(H)) \subseteq P(H)$. $\square$

There are other ways of characterizing $P(H)$:

Proposition 3.5. Let Q be a $\mathbb{Z}^n$ -quiver. Let H be a non-empty finite set of vertices of Q. Let v be a vertex of Q. Then the following statements are equivalent:

1. $v \notin P(H)$.
2. There is a vertex w of Q such that there is an admissible path from z to v for each $z \in H$ passing through w.
3. There is a vertex w of $P(H)$ such that there is an admissible path from z to v for each $z \in H$ passing through w. If $P(H) = H$ then w is unique.
4. There are a vertex w of Q and a nonempty proper subset I of $\{0, \dots, n\}$ such that $v \in C_I(w) - \{w\}$ and for each admissible path γ connecting a vertex of H to w there is $i \notin I$ with $\gamma(i) = 0$.

Proof. Assume Statement 1. By definition, there is $i \in \{0, \dots, n\}$ such that for each $z \in H$ we have $\gamma_z(i) > 0$ for each admissible path γ_z connecting z to v. Let e be the i-arrow arriving at v, and w its initial vertex. For each $z \in H$ let μ_z be a path arriving at w with $\mu_z(j) = \gamma_z(j)$ for each $j \neq i$ and $\mu_z(i) = \gamma_z(i) - 1$. The concatenation of μ_z with e is an admissible path ρ_z satisfying $\rho_z(i) = \gamma(i)$ for every i. Since ρ_z and γ arrive at the same vertex, v, we have that they leave from the same vertex, z. So ρ_z is an admissible path from z to v passing through w, such that there is an admissible path from z to v for each $z \in H$ passing through w.

Assume Statement 2. If $w \notin P(H)$ then we apply the above argument again. As H is non-empty, and as an admissible path from $z \in H$ to v has finite length, the argument cannot be repeated indefinitely. Thus there is a vertex $w \in P(H)$ such that there is an admissible path from z to v for each $z \in H$ passing through w. If w' is another vertex of $P(H)$ with the same property, and $P(H) = H$, then there admissible paths from w to v passing through w' and from w' to v passing through w. Then an admissible path from w' to v has length at most that of an admissible path from w to v, and length at least that of an admissible path from w to v, with equality only if an admissible path from w to w' has length zero, that is, $w' = w$.

Assume Statement 3. Let μ be an admissible path from w to v and put $I := \{i \mid \mu(i) > 0\}$. Then I is a proper non-empty subset of $\{0, \dots, n\}$ and $v \in C_I(w) = \{w\}$. Moreover, since there is an admissible path γ_z connecting z to v through w, for each $z \in H$, then $\gamma_z(i) > 0$ for every $i \in I$, and thus $\gamma_z(i) = 0$ for some $i \notin I$. Of course, also $\gamma(i) = 0$ for any admissible path connecting z to v.

Assume Statement 4. Let Let μ be an admissible path from w to v. Put $J := \{i \mid \mu(i) > 0\}$. Then $J \subseteq I$. For each $z \in H$ let γ_z be an admissible path from z to w. Since $\gamma_z(i) = 0$ for some $i \notin I$, it follows that the concatenation ρ_z of γ_z with μ is an admissible path connecting z to v such that $\rho_z(i) > 0$ for each $i \in J$. Thus $v \notin P(H)$. □

Proposition 3.6. Let Q be a $\mathbb{Z}^n$-quiver and H a finite collection of vertices of Q. For each $i = 0, \dots, n$, let w_i be the ith bound of H and put $J_i := \{0, \dots, n\} - \{i\}$. Then the following statements hold:

1. For each $z \in H$ and $u \in C_{J_i}(w_i)$ there is an admissible path from z to u through w_i.

2. $C_{J_i}(w_i) \cap P(H) \subseteq \{w_i\}$.

Proof. By definition, for each $z \in H$ there is an admissible path γ from z to w_i with $\gamma(i) = 0$. Also, for each $u \in C_{J_i}(w_i)$ there is an admissible path μ from w_i to u. The concatenation of γ with μ is an admissible path from z to u through w_i. Statement 1 is proved. As for Statement 2, observe that any $v \in C_{J_i}(w_i) - \{w_i\}$ satisfies the conditions in Statement 3 of Proposition 3.5, for $w = w_i$ and $I = J_i$, and thus $v \notin P(H)$. □

Example 3.2. Let Q be the $\mathbb{Z}^n$-quiver of Example 3.1. Suppose $n = 2$ and $\underline{d} = 0$. Suppose as well that $v_0 = (-1, 0)$, $v_1 = (1, 1)$ and $v_2 = (0, -1)$. Then $G = \mathbb{Z}^2$. Let d be a positive integer and let $H := \mathbb{N}^2(\leq d)$, where

$$\mathbb{N}^2(\leq d) := \{(i, j) \in \mathbb{Z}^2_{\geq 0} \mid i + j \leq d\}.$$

Then $(d, 0)$, $(0, 0)$ and $(0, d)$ are the bounds of H, from the 0-th to the second. Furthermore, $P(H) = H$.

Indeed, consider $w := (0,0)$. For each $z := (x,y) \in H$, the path γ connecting z to w going through $(x-1,y), \dots, (0,y), (0,y-1), \dots, (0,1)$ satisfies $\gamma(1) = 0$. Let $w' = (a,b) \in \mathbb{Z}^2$ such that for each $z := (x,y) \in H$ there is a path μ connecting z to w' with $\mu(1) = 0$. Then $a \le x$ and $b \le y$ for each pair (x,y) in H, thus $a \le 0$ and $b \le 0$. But then there is clearly a path ν connecting w to w' passing through $(-1,0), \dots, (a,0), (a,-1), \dots, (a,b+1)$, thus satisfying $\nu(1) = 0$. Hence w is the first bound of H. Similarly, we show that $(d,0)$ is the 0-th bound and $(0,d)$ is the second bound of H.

Now, $H \subseteq P(H)$ by Proposition 3.4. To show the reverse inclusion, let $v \in \mathbb{Z}^2 - H$. Then $v \in C_{I_w}(w) - \{w\}$ for a certain $w = (i,j) \in H$ with $i = 0$, $j = 0$ or $i + j = d$, where I_w is chosen according to w as follows:

$$I_w = \begin{cases} \{0\} & \text{if } i = 0 \text{ and } 0 < j < d, \\ \{1\} & \text{if } i > 0,\ j > 0 \text{ and } i + j = d, \\ \{2\} & \text{if } 0 < i < d \text{ and } j = 0, \\ \{0,2\} & \text{if } i = 0 \text{ and } j = 0, \\ \{0,1\} & \text{if } i = 0 \text{ and } j = d, \\ \{1,2\} & \text{if } i = d \text{ and } j = 0. \end{cases}$$

In each case, one verifies that for each admissible path γ connecting a vertex of H to w there is $i \in I_w$ with $\gamma(i) = 0$. Thus, it follows from Proposition 3.5 that $v \notin P(H)$. We conclude that $P(H) = H$.

The following lemma will be very useful later.

Lemma 3.1. Let Q be a $\mathbb{Z}^n$-quiver with vertex set G. Let $u, v, w \in G$. Then the following first two statements are equivalent and imply the last two:

1. There is an admissible path from u to w through v.
2. The concatenation of every admissible path from u to v and every admissible path from v to w is admissible.
3. If $v \neq w$ then all paths from u to v through w are not admissible.
4. If $u \neq v$ then all paths from v to w through u are not admissible.

Furthermore, if v and w are neighbors then the third statement is equivalent to the first two, and if u and v are neighbors then the fourth statement is equivalent to the first two.

Proof. Assume Statement 1. Let γ be an admissible path from u to w through v. It is the concatenation of a path γ_1 from u to v with a path γ_2 from v to w, both admissible. If β is an admissible path from u to v and ν is one from v to w, then $\beta(i) = \gamma_1(i)$ and $\nu(i) = \gamma_2(i)$, and thus $\beta(i) + \nu(i) = \gamma(i)$ for every i. Since γ is admissible, it follows that the concatenation of β with ν is admissible.

Assume Statement 2 now. Since there are admissible paths from u to v and from v to w, Statement 1 holds. Furthermore, assume $v \neq w$. Let γ be a path from u to w and μ one from w to v. Assume by contradiction to Statement 3 that the concatenation of γ with μ is admissible, call it β. Then γ and μ are admissible. Let ν be any path leaving v such that $\nu(i) = m - \mu(i)$ for each $i = 0, 1, \ldots, n$, where $m := \max(\mu(i))$. Since $v \neq w$ we have $m > 0$. Also, ν is admissible. Since $\nu(i) + \mu(i)$ is constant for $i = 0, \ldots, n$, the path ν arrives back to w. Now, if $\beta(i) = 0$ then $\gamma(i) = \mu(i) = 0$ and thus $\nu(i) = m > 0$. Then the concatenation of β with μ is not admissible, contradicting Statement 2.

Assume Statement 2 again. Assume $u \neq v$. Let γ be a path from v to u and μ one from u to w. Assume by contradiction to Statement 4 that the concatenation of γ with μ is admissible, call it β. Then γ and μ are admissible. Let ν be any path leaving u such that $\nu(i) = m - \gamma(i)$ for each $i = 0, 1, \ldots, n$, where $m := \max(\gamma(i))$. Since $u \neq v$ we have $m > 0$. Also, ν is admissible. Since $\nu(i) + \gamma(i)$ is constant for $i = 0, \ldots, n$, the path ν arrives back to v. Now, if $\beta(i) = 0$ then $\gamma(i) = \mu(i) = 0$ and thus $\nu(i) = m > 0$. Then the concatenation of nu with β is not admissible, contradicting Statement 2.

Let now β be an admissible path from u to v and ν one from v to w. Assume that their concatenation ρ is not admissible. Let γ be an admissible path connecting u to w. Then both γ and ρ connect u to w. If β or ν is simple, it follows that

$$\gamma(i) = \beta(i) + \nu(i) - 1 \quad \text{for every } i.$$

If v and w are neighbors (thus $v \neq w$), then ν is simple. So is μ, any chosen admissible path from w to v satisfying $\mu(i) + \nu(i) = 1$ for every i. But then

$$\beta(i) = \gamma(i) + \mu(i) \quad \text{for every } i.$$

Since β is admissible, so is the concatenation of γ with μ, contradicting Statement 3.

If u and v are neighbors (thus $u \neq v$), then β is simple. So is α, any chosen admissible path connecting v to u satisfying $\alpha(i) + \beta(i) = 1$ for every i. But then

$$\nu(i) = \gamma(i) + \alpha(i) \quad \text{for every } i.$$

Since ν is admissible, so is the concatenation of α with γ, contradicting Statement 4. $\square$

3.2 Linked nets of vector spaces

Given a representation $\mathfrak{g}$ of a quiver Q, we will let $V_v^{\mathfrak{g}}$ be the vector space associated to each vertex v of Q and $\varphi_\gamma^{\mathfrak{g}}$ be the linear map associated to each path γ in Q. When the representation is clear from the context we drop the superscript.

Definition 3.6. Let Q be a $\mathbb{Z}^n$-quiver, G its vertex set and A its arrow set. Let $A_0, \dots, A_n$ be a decomposition of A giving Q a $\mathbb{Z}^n$-structure. A *linked net of vector spaces* is a quiver representation of Q of pure dimension (i.e. all vector space associated to each vertex have the same dimension) satisfying the following three additional properties:

1. If γ_1 and γ_2 are two admissible paths connecting the same two vertices then $\varphi_{\gamma_1} = \varphi_{\gamma_2}$.

2. If γ is a non-admissible path then $\varphi_\gamma = 0$.

3. If γ_1 and γ_2 are two simple admissible paths leaving the same vertex such that $\gamma_1(i) = 0$ or $\gamma_2(i) = 0$ for every i then $\mathrm{Ker}(\varphi_{\gamma_1}) \cap \mathrm{Ker}(\varphi_{\gamma_2}) = 0$.

It follows from Property 1 above that the linear map φ_γ associated to a simple admissible path γ is determined by the initial vertex v of γ and the set $I := \{i \,|\, \gamma(i) = 1\}$. We will thus write $\varphi_I^v := \varphi_\gamma$ in this case. When v is clear from the context the superscript is omitted.

Also, given two vertices v_1 and v_2 of Q, there is an admissible path γ connecting v_1 to v_2, and since φ_γ does not depend on the choice of γ, we write $\varphi_{v_2}^{v_1} := \varphi_\gamma$. Because of the independence, given $s \in V_{v_1}$ we put

$$s_{|V_{v_2}} := \varphi_{v_2}^{v_1}(s).$$

Lemma 3.2. Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver Q. For each vertex v and each admissible path γ leaving v,

$$\mathrm{Ker}(\varphi_\gamma) = \mathrm{Ker}(\varphi_I^v),$$

where $I := \{i \,|\, \gamma(i) > 0\}$.

Proof. Argue by induction on the length of γ. If the length of γ is at most 1, then γ is simple. And if γ is simple the result follows.

Assume now that γ is not simple. Let μ be any simple path leaving v satisfying $\mu(i) = 1$ if and only if $i \in I$. Then $\varphi_\mu = \varphi_I^v$. Let w be the terminal point of μ. Let ϵ be any path leaving w such that $\gamma(i) = \epsilon(i) + \mu(i)$ for every i. Then γ and $\epsilon + \mu$ connect the same two vertices.

Let ρ be the simple admissible path leaving w such that $J := \{i \mid \rho(i) = 1\} = \{i \mid \epsilon(i) > 0\}$. By the induction hypothesis, $\ker \varphi_\epsilon = \varphi_\rho$. Notice that $\rho(i) = 0$ if $\mu(i) = 0$.

Since $\varphi_\gamma = \varphi_{\epsilon+\mu} = \varphi_\epsilon \varphi_\mu$, we may assume that $\gamma = \epsilon + \mu$ with ϵ simple and satisfying $\epsilon(i) = 0$ if $\mu(i) = 0$.

Let now $s \in V_v$ such that $\varphi_\gamma(s) = 0$. Set $t := \varphi_\mu(s)$. Then $\varphi_\epsilon(t) = 0$ and $\varphi_\nu(t) = 0$ where ν is any simple admissible path leaving w such that $\nu + \mu$ is simple and circular. But if $\epsilon(i) > 0$ then $\mu(i) > 0$, and hence $\nu(i) = 0$. In other words, either $\epsilon(i) = 0$ or $\nu(i) = 0$ for every i, whence $\operatorname{Ker}(\varphi_\epsilon) \cap \operatorname{Ker}(\varphi_\nu) = 0$. Thus $t = 0$.

We have just proved that $\operatorname{Ker}(\varphi_\gamma) \subseteq \operatorname{Ker}(\varphi_\mu)$. The reverse inclusion is obvious. Thus

$$\operatorname{Ker}(\varphi_\gamma) = \operatorname{Ker}(\varphi_\mu) = \operatorname{Ker}(\varphi_I^v).$$

□

Given a linked net of vector spaces $\mathfrak{g}$ over a $\mathbb{Z}^n$-quiver with vertex set G, and given $v, w \in G$ we set

$$K_w^v := \operatorname{Ker}\big(\varphi_w^v \colon V_v \longrightarrow V_w\big), \quad \text{and} \quad p_w^v := \dim K_w^v.$$

As before, if w is the terminal point of a simple admissible path γ leaving v, we set

$$K_I^v := K_w^v \quad \text{and} \quad p_I^v = p_w^v, \quad \text{where} \quad I := \{i \,|\, \gamma(i) > 0\}.$$

When it is clear from the context, we will omit the index v.

The numbers p_I^v play an important role in the subject. See the constructions in Section 8.1 for a taste of it. The next result asserts certain relations between these numbers.

Lemma 3.3. Let $\mathfrak{g}$ be a linked net of vector spaces of pure dimension r over a $\mathbb{Z}^n$-quiver with vertex set G. Let I and J be proper subsets of $\{0, \dots, n\}$ and

$v, w \in G$. Then:

1. If $I \subset J$ then $K_I^v \subseteq K_J^v$; in particular, $p_I^v \leq p_J^v$.

2. If $I \cap J = \emptyset$ then $K_I^v \cap K_J^v = 0$; in particular, $p_I^v + p_J^v \leq r$.

3. $K_I^v \cap K_J^v = K_{I\cap J}^v$; in particular,$p_I^v + p_J^v \leq r + p_{I\cap J}^v$.

4. The sum $\sum_i K_{\{i\}}^v$ is direct; in particular, $\sum_{i=0}^n p_{\{i\}}^v \leq r$.

5. If $w = I(v)$ and $J = \{0, 1, \ldots, n\} - I$ then $p_I^v + p_J^w \geq r$.

6. If $I \cap J = \emptyset$ and $w = J(v)$ then $p_I^v \leq p_L^w$, where $L := \{0, 1, \ldots, n\} - J$. Likewise, if $I \subseteq J$ and $w = L(v)$ then $p_I^v \leq p_J^w$.

Proof. Considering Statement 3, observe that $\varphi_I = \varphi_{I'}\varphi_{I\cap J}$ and $\varphi_J = \varphi_{J'}\varphi_{I\cap J}$, where $I' := I - I \cap J$ and $J' := J - I \cap J$. Thus $K_{I\cap J}^v \subseteq K_I^v \cap K_J^v$. Moreover, equality holds, as letting $w := (I \cap J)(v)$, we have that $K_{I'}^w \cap K_{J'}^w = 0$. Then $K_I^v + K_J^v$ has dimension $p_I^v + p_J^v - p_{I\cap J}^v$, which is thus smaller than r.

Statements 1 and 2 follow from Statement 3.

As for Statement 4, observe that $\sum_{j\neq i} K_{\{j\}}^v \subseteq K_{L_i}^v$, where $L_i = \{0, \ldots, n\} - \{i\}$, and $K_{\{i\}}^v \cap K_{L_i}^v = 0$ for each $i = 0, \ldots, n$.

Statement 5 follows from the fact that $\mathrm{Im}(\varphi_w^v) \subseteq \mathrm{Ker}(\varphi_v^w)$ and from the Rank Nullity Theorem applied to φ_w^v.

As for Statement 6 observe first that if $I \cap J = \emptyset$ then $K_I^v \cap K_J^v = 0$, whence $\varphi_J^v|_{K_I^v}$ is injective. Also, $\varphi_L^w \circ \varphi_J^v = 0$. So

$$p_I^v \leq \dim \mathrm{Im}\big(\varphi_J^v\big) \leq \dim K_L^w = p_L^w..$$

The second statement in 6 follows from the first. □

If v_1 and v_2 are neighboring vertices, then Property 2 of a linked net of vector spaces implies that $\varphi_{v_1}^{v_2}\varphi_{v_2}^{v_1} = 0$. The reverse inclusion is the subject of the following definition.

Definition 3.7. A linked net of vector spaces of a quiver Q is called *exact* if for each two neighboring vertices v_1 and v_2 of Q we have

$$\mathrm{Im}(\varphi_{v_2}^{v_1}) = \mathrm{Ker}(\varphi_{v_1}^{v_2}).$$

Definition 3.8. Let $\mathfrak{g}$ be a linked net of vector spaces of dimension r over a $\mathbb{Z}^n$-quiver Q. A collection of vertices $v_1, \ldots, v_m$ of Q and vectors $s_i \in V_{v_i}$ for $i = 1, \ldots, m$ such that

$$\{s_1|_{V_w}, \ldots, s_r|_{V_w}\} \text{ generates } V_w \text{ for each vertex } w \text{ of } Q$$

is called a *set of generators* of $\mathfrak{g}$; if $m = r$, it is called a *simple basis*. We will also say that $\{v_1, \ldots, v_m\}$ *generate* $\mathfrak{g}$ and that $\mathfrak{g}$ is finitely generated.

We will see in Chapter 5 that if $\mathfrak{g}$ admits a simple basis, then it is exact (see Proposition 5.1).

Definition 3.9. Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver Q with vertex set G. We say that $\mathfrak{g}$ has *finite support* if there exists a finite subset $H \subset G$ such that for each $w \in G - H$ there exists $v \in H$ such that φ^v_w is an isomorphism. We say that $\mathfrak{g}$ has *finite support on* H.

Once $\mathfrak{g}$ has finite support on a certain H, we may remove vertices from H, if necessary, to assume that H is minimal. Then H is unique by the following proposition.

Proposition 3.7. Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver Q with vertex set G. Assume $\mathfrak{g}$ has finite support on $H \subset G$ and on $H' \subset G$. Suppose H' is minimal for this property. Then $H' \subseteq H$. Furthermore, for each $v, w \in H'$ distinct, φ^v_w is not an isomorphism.

Proof. Suppose by contradiction that there exist distinct $v, w \in H'$ such that the map $\varphi^v_w \colon V_v \longrightarrow V_w$ is an isomorphism. We claim that $\mathfrak{g}$ has support on $H' - \{w\}$, a contradiction. Indeed, take $u \in G$. By hypothesis, there exists $z \in H'$ such that the map $\varphi^z_u \colon V_z \longrightarrow V_u$ is an isomorphism. If $z \neq w$, we are done. If $z = w$ then the composition $\varphi^w_u \varphi^v_w \colon V_v \longrightarrow V_w \longrightarrow V_u$ is an isomorphism too. It follows that there is an admissible path connecting v to u via w, and thus $\varphi^v_u = \varphi^w_u \varphi^v_w$, also an isomorphism. Hence, we can remove w from H'.

Now, suppose that $v \notin H$, for some $v \in H'$. Since $\mathfrak{g}$ is supported on H, there exists $w \in H$ such that $\varphi^w_v \colon V_w \longrightarrow V_v$ is an isomorphism. Hence, $\varphi^v_w \colon V_v \longrightarrow V_w$ is zero. Since $\mathfrak{g}$ is supported on H' as well, there is $u \in H'$ such that $\varphi^u_w \colon V_u \longrightarrow V_w$ is an isomorphism. Clearly, $u \neq v$. But then the composition $\varphi^w_v \varphi^u_w$ is an isomorphism, and thus agrees with φ^u_v. But this contradicts what we have just proved above. Therefore $H' \subseteq H$. $\square$

Proposition 3.8. Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver Q. Let H be a set of vertices of $\mathfrak{g}$. If H generates $\mathfrak{g}$ then $\mathfrak{g}$ has support on $P(H)$. In addition, $\mathfrak{g}$ is finitely generated if and only if it has finite support.

Proof. We claim that if $\mathfrak{g}$ has support in a finite set of vertices S then S generates $\mathfrak{g}$. In fact, for each $v \in S$, let $s_1^v, \dots, s_r^v$ be a basis of V_v; then, for each vertex w of the quiver, there is $v \in S$ such that $s_1^v|_{V_w}, \dots, s_r^v|_{V_w}$ generate V_w.

Because of Proposition 3.4, it remains to prove the first statement. Suppose then that H generates $\mathfrak{g}$. Let $v \notin P(H)$. Then, by Proposition 3.5, there are a vertex w different from v such that there is an admissible path from each $z \in H$ to v passing through w. Thus $s|_{V_v} = \varphi_v^w(s|_{V_w})$ for each $s \in V_z$ for each $z \in H$. Since H generates $\mathfrak{g}$, it follows that φ_v^w is surjective, whence an isomorphism. If $w \notin P(H)$ we may repeat the argument. The argument cannot be repeated indefinitely as an admissible path from any $z \in H$ to a given vertex of Q cannot factor through infinitely many vertices. $\square$

Example 3.3. For each linked net of vector spaces over the quiver Q of Example 3.2 with finite support on $\mathbb{N}^2(\leq d)$ the linear maps associated to the blue arrows, as in Figure Figure 3.1 is shown for the special case of $d \leq 4$, are isomorphisms.

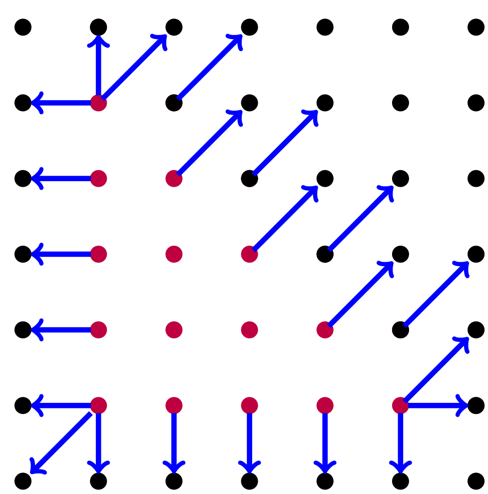

Figure 3.1: A linked net of vector spaces with finite support on $\mathbb{N}^2(\leq 4)$.

Theorem 3.1. Let $\mathfrak{g}$ be a linked chain of vector spaces of dimension one over a $\mathbb{Z}^n$-quiver Q. If $\mathfrak{g}$ has finite support and is exact then $\mathfrak{g}$ admits a simple basis.

More precisely, there are a vertex v in the support of $\mathfrak{g}$ and a vector $s \in V_v$ such that $s|_{V_w}$ generates V_w for every vertex w of Q.

Proof. We prove first that there are a vertex v of Q and $s \in V_v$ such that s is not in the image of φ_γ for any simple admissible path γ arriving at v. Indeed, suppose not. Let H be the (finite) support of $\mathfrak{g}$.

Let v_0 be any vertex of Q and $t_0 \in V_{v_0}$ any nonzero vector. If $v_0 \notin H$ then there is an admissible path γ_0 connecting a vertex $w_0 \in H$ with v_0 such that φ_{γ_0} is an isomorphism. Then there is $s_0 \in V_{w_0}$ such that $\varphi_{\gamma_0}(s_0) = t_0$. By contradiction hypothesis, there are a simple admissible path μ_0 connecting a vertex v_1 of Q to w_0 and $t_1 \in V_{v_1}$ such that $\varphi_{\mu_1}(t_1) = s_0$.

Proceeding as before, with v_1 and t_1 replacing v_0 and t_0, and so on, we end up with a sequence of vertices $v_0, v_1, \dots$ of Q, a sequence of vertices $w_0, w_1, \dots$ of H, admissible paths $\gamma_0, \gamma_1, \dots$ and $\mu_0, \mu_1, \dots$, and vectors $t_i \in V_{v_i}$ and $s_i \in V_{w_i}$ for each $i = 0, 1, \dots$ such that:

1. γ_i connects w_i to v_i and μ_i connects v_{i+1} to w_i for each $i = 0, 1, \dots,$

2. $\varphi_{\gamma_i}(s_i) = t_i$ and $\varphi_{\mu_i}(t_{i+1}) = s_i$ for each $i = 0, 1, \dots.$

Since H is finite, there are i and j with $i < j$ such that $w_i = w_j$. But, from our construction, the path

$$\rho := \gamma_{i+1}\mu_{i+1}\cdots\gamma_{j-1}\mu_{j-1}\gamma_j$$

is such that $\varphi_\rho(s_j) = s_i$, but ρ is cycle, thus $\varphi_\rho = 0$. We reached a contradiction.

Let now v be the vertex of Q and $s \in V_v$ such that s is not in the image of φ_γ for any simple admissible path γ arriving at v. Since $\mathfrak{g}$ is exact, it follows that $\varphi_\gamma(s) \neq 0$ for every simple admissible path γ leaving from v. But then $\varphi_\gamma(s) \neq 0$ for any admissible path γ, that is, $s|_{V_w} \neq 0$. Finally, since $\mathfrak{g}$ has dimension 1, it follows that $s|_{V_w}$ generates V_w for every vertex w of Q. ☐

Proposition 3.9. Let $\mathfrak{g}$ be a finitely generated linked net of vector spaces of dimension 1 over a $\mathbb{Z}^n$-quiver. Then there is a set of vertices H minimally generating $\mathfrak{g}$. Furthermore, the following two statements hold:

1. $\varphi_u^v = 0$ for each two neighboring $u, v \in H$.

2. If each two vertices of H are neighbors, then H is the unique set of vertices minimally generating $\mathfrak{g}$.

Proof. Let H be a finite set of vertices generating $\mathfrak{g}$. Up to replacing H by a smaller subset, we may choose H minimal.

We prove Statement 1 first. Assume there are neighboring $u, v \in H$ such that $\varphi_u^v \neq 0$. We claim that $H - \{u\}$ generates $\mathfrak{g}$. Indeed, since $\mathfrak{g}$ has dimension 1, a set of generators of $\mathfrak{g}$ is obtained by picking nonzero $s_w \in V_w$ for each $w \in H$. Also, φ_u^v is an isomorphism, and hence we may assume that $\varphi_u^v(s_v) = s_u$. Let w be any vertex of Q. Then V_w is generated by the $s_z|_{V_w}$ for $z \in H$. But, if there is an admissible path connecting v to w through u then $s_u|_{V_w} = s_v|_{V_w}$. And if not, since u and v are neighbors, it follows from Lemma 3.1 that there is an admissible path from u to w through v, and hence

$$s_u|_{V_w} = \varphi_w^v \varphi_v^u(s_u) = \varphi_w^v \varphi_v^u \varphi_u^v(s_v) = 0.$$

At any rate, V_w is generated by the $s_z|_{V_w}$ for $z \in H - \{u\}$.

As for Statement 2, assume each two vertices of H are neighbors. Let H' be another set of vertices generating $\mathfrak{g}$. Let $a \in H$. Then $\varphi_a^z(V_z) = V_a$ for some $z \in H'$. On the other hand, $\varphi_z^b(V_b) = V_z$ for some $b \in H$. Then $\varphi_a^z \varphi_z^b$ is nonzero, whence equal to φ_a^b. Now, a and b cannot be neighbors by Statement 1. Since each two vertices of H are neighbors, we must have $a = b$. But then, as $\varphi_a^z \varphi_z^a$ is nonzero, we must have $z = a$. It follows that $H \subseteq H'$. $\square$

Chapter 4

Linked nets of dimension 1 over $\mathbb{Z}^2$-quivers

Here we restrict our attention to linked nets of vector spaces of dimension 1 over $\mathbb{Z}^2$-quivers, not necessarily exact. Our Theorem 4.1 guarantees that those of finite support are minimally generated by a set of one, two or three neighboring vertices. The proof follows from Theorem 4.2, where we go further and describe those linked nets which do not have finite support as those that admit a so-called infinite corridor configuration, and Theorem 4.3, which rules out the infinite corridors in the case of finite support.

Definition 4.1. Given a $\mathbb{Z}^2$-quiver, a *point* is a vertex, a *segment* is a collection of two neighboring vertices, and a *triangle* is a collection of three vertices, each two neighbors.

Given a segment, one and only one of its vertices is connected to the other by an arrow. We can classify the segments in three groups according to the type of the arrow.

Each two vertices in a triangle form a segment. Given a triangle, let v, w, z denote its vertices. We may assume there is an arrow connecting v to w. If there were an arrow connecting z to w then z would not be a neighbor of v. Thus there is an arrow connecting w to z. By the same token there is an arrow connecting z to v. All three arrows must be distinct. Thus we can classify the triangles in two groups according to the types of the arrows.

We classify one-dimensional linked nets of vector spaces over $\mathbb{Z}^2$-quivers according to their minimal generating set, a point, a segment or a triangle, a consequence of our main theorem below.

Theorem 4.1. Let $\mathfrak{g}$ be a linked net of vector spaces of dimension 1 with finite support over a $\mathbb{Z}^2$-quiver. Then $\mathfrak{g}$ is generated by a triangle.

The proof of the theorem will be achieved by the end of the section, as a consequence of the very descriptive Theorem 4.2, where we describe as well the case where $\mathfrak{g}$ does not have finite support, and of Theorem 4.3.

Observe that the triangle may not minimally generate $\mathfrak{g}$: a minimal generating set could be a segment or a point of $\mathfrak{g}$. At any rate, the minimal generating set is unique by Proposition 3.9

In order to be able to draw figures explaining the following results, we may and will assume without loss of generality that we are given the standard $\mathbb{Z}^2$-quiver Q, the one with vertex set $\mathbb{Z}^2$ generated by the vectors $v_0 = (-1, 0)$, $v_1 = (1, 1)$ and $v_2 = (0, -1)$.

We will throughout replace 0, 1 and 2 by X, Y and Z, which is suggestive of how the quiver arises as a projection of the quiver associated to any degeneration to the simplest curve of three components there is, combinatorially speaking, that of compact type. For example, $v_X := v_0$, $v_Y := v_1$ and $v_Z := v_2$.

Let $\mathfrak{g}$ be a linked net of vector spaces $\mathfrak{g}$ over Q. We use

$$\varphi^{\underline{d}}_X,\ \varphi^{\underline{d}}_Y,\ \varphi^{\underline{d}}_Z,\ \varphi^{\underline{d}}_{X+Y},\ \varphi^{\underline{d}}_{X+Z} \text{ and } \varphi^{\underline{d}}_{Y+Z}$$

for

$$\varphi^{\underline{d}}_{\{0\}},\ \varphi^{\underline{d}}_{\{1\}},\ \varphi^{\underline{d}}_{\{2\}},\ \varphi^{\underline{d}}_{\{0,1\}},\ \varphi^{\underline{d}}_{\{0,2\}} \text{ and } \varphi^{\underline{d}}_{\{1,2\}},$$

respectively.

Also, we denote by $C^{X+Y}(\underline{d})$ the $\{0, 1\}$-cone of $\underline{d}$ for each $\underline{d} \in \mathbb{Z}^2$. The cones $C^{X+Z}(\underline{d})$ and $C^{Y+Z}(\underline{d})$ are defined analogously.

We let $K^{\underline{d}}_X$ denote the kernel of $\varphi^{\underline{d}}_X$. Analogously, we define $K^{\underline{d}}_Y$, $K^{\underline{d}}_Z$, $K^{\underline{d}}_{X+Y}$, $K^{\underline{d}}_{X+Z}$ and $K^{\underline{d}}_{Y+Z}$. Also, we let $p^{\underline{d}}_X$, $p^{\underline{d}}_Y$, $p^{\underline{d}}_Z$, $p^{\underline{d}}_{X+Y}$, $p^{\underline{d}}_{X+Z}$ and $p^{\underline{d}}_{Y+Z}$ denote their dimensions.

We let $X(\underline{d})$ or $X \cdot \underline{d}$ denote $\{0\} \cdot v$. And so forth.

We highlight Lemma 3.3 for the quiver Q, as it is the version we will use most in the thesis.

Lemma 4.1. Let $\mathfrak{g}$ be a linked net of vector spaces of dimension r over the standard $\mathbb{Z}^2$-quiver with vertex set G. For each $\underline{d} \in \mathbb{Z}^2$ and $\{R, S, T\} = \{X, Y, Z\}$ the following is true.

(a) $p^{\underline{d}}_R \leq \min\{p^{\underline{d}}_{R+S}, p^{\underline{d}}_{R+T}\}$

(b) $p^{\underline{d}}_{R+S} \leq r - p^{\underline{d}}_T$

(c) $p^{\underline{d}}_X + p^{\underline{d}}_Y + p^{\underline{d}}_Z \leq r$

(d) $p^{\underline{d}}_{R+S} + p^{\underline{d}}_{R+T} \leq r + p^{\underline{d}}_R$

(e) $p^{\underline{d}}_R + p^{\underline{d}}_S \leq p^{\underline{d}}_{R+S}$

(f) $p^{\underline{d}}_R \leq \min\{p^{S\cdot\underline{d}}_{R+T}, p^{T\cdot\underline{d}}_{R+S}, p^{(S+T)\cdot\underline{d}}_R\}$

(g) $p^{\underline{d}}_{R+S} \leq p^{T\cdot\underline{d}}_{R+S}$

(h) $p^{\underline{d}}_R + p^{R\cdot\underline{d}}_{S+T} \geq r$

Proof. Statement (a) is an application of Statement 1 in Lemma 3.3. Statement (b) follows from Statement 2 therein. Statement (c) follows from Statement 4. Statement (d) follows from Statement 3.

As for Statement (e), Statement 4 yields that the sum $K^{\underline{d}}_R + K^{\underline{d}}_S$ is direct, and since $K^{\underline{d}}_R$ and $K^{\underline{d}}_S$ are subspaces of $K^{\underline{d}}_{R+S}$, the inequality follows.

Statement (f) follows from Statement 6, its second part. Statement (g) follows from Statement 6, its first part. Statement (h) is an application of Statement 5. $\square$

If we assume that $\mathfrak{g}$ has finite support, we will assume without loss of generality that it has finite support on

$$\mathbb{N}^2(\leq d) := \{(i,j) \in \mathbb{Z}^2 \,|\, 0 \leq i, j \leq i+j \leq d\}$$

for some d.

If $\mathfrak{g}$ is one-dimensional, we replace V_v by I_v. Assume from now on that $\mathfrak{g}$ has dimension 1.

Remark 4.1. Note that $\varphi^{\underline{d}}_{R+S} \neq 0$ if and only if the restriction of $\mathfrak{g}$ to $C^{R+S}(\underline{d})$ is supported on $\{\underline{d}\}$, for all R, S distinct with $\{R, S\} \subset \{X, Y, Z\}$.

If we now suppose that $\mathfrak{g}$ is not exact, then there exists a pair of neighboring vertices $(\underline{d}, \underline{e})$ such that the maps $I_{\underline{d}} \longrightarrow I_{\underline{e}}$ and $I_{\underline{e}} \longrightarrow I_{\underline{d}}$ are both zero. By symmetry, we can suppose that $\underline{d} = (R+S)\cdot\underline{e}$ and $\underline{e} = T\cdot\underline{d}$, where $\{R, S, T\} = \{X, Y, Z\}$. We call $(I_{\underline{d}}, I_{\underline{e}})$ a *non-exact pair*.

Given a non-exact pair $(I_{\underline{d}}, I_{\underline{e}})$, since $\varphi_T^d = 0$, we must have that φ_{R+S}^d is an isomorphism; thus the restriction of $\mathfrak{g}$ to $C^{R+S}(\underline{d})$ is supported on $\{\underline{d}\}$. Likewise, since $\varphi_{R+S}^e = 0$, also φ_T^e is an isomorphism.

We say $(I_{\underline{d}}, I_{\underline{e}})$ is *perfect* or a *generating pair* if the maps φ_{R+T}^e and φ_{S+T}^e are both isomorphisms. This is equivalent to say that the restriction of $\mathfrak{g}$ to $C^{R+T}(\underline{e}) \cup C^{S+T}(\underline{e})$ is supported on $\{\underline{e}\}$. As the restriction of $\mathfrak{g}$ to $C^{R+S}(\underline{d})$ is supported on $\{\underline{d}\}$, if $(I_{\underline{d}}, I_{\underline{e}})$ is perfect, then $\mathfrak{g}$ has minimal support on the independent segment $\{\underline{d}, \underline{e}\}$.

Observe that since φ_T^e is an isomorphism, φ_{R+T}^e or φ_{S+T}^e is also an isomorphism. In the case that only one of them is an isomorphism, we say that the pair $(I_{\underline{d}}, I_{\underline{e}})$ is an *imperfect non-exact pair.* A non-exact pair gives rise to a sequence of several non-exact pairs (see Figures 4.4 to 4.8 for a visual reference). This is the content of the following lemma.

(We will employ many figures in the proofs. In all of them, the filled arrows imply the dashed arrows. Also, maps associated to red arrows are zero, those associated to blue arrows are not.)

Lemma 4.2. Let $\mathfrak{g}$ be a linked net of vector spaces of dimension 1 over the standard $\mathbb{Z}^2$-quiver. For $\{R, S, T\} = \{X, Y, Z\}$ and a non-exact pair $(I_{\underline{d}}, I_{\underline{e}})$ with $\underline{e} = R \cdot \underline{d}$, the following two statements are true:

i) Let $\underline{a} := (R + S) \cdot \underline{d}$ and $\underline{b} := S \cdot \underline{d}$. Suppose that $\varphi_S^e : I_{\underline{e}} \longrightarrow I_{\underline{a}}$ is an isomorphism. Then $(I_{\underline{a}}, I_{\underline{d}})$ and $(I_{\underline{b}}, I_{\underline{a}})$ are both non-exact pairs.

ii) Let $\underline{p} := (R + T) \cdot \underline{e}$ and $\underline{q} := T \cdot \underline{e}$. Suppose that $\varphi_S^p : I_{\underline{p}} \longrightarrow I_{\underline{e}}$ is an isomorphism. Then $(I_{\underline{q}}, I_{\underline{p}})$ and $(I_{\underline{e}}, I_{\underline{q}})$ are both non-exact pairs.

Proof. i) The map $\varphi_T : I_{\underline{a}} \longrightarrow I_{\underline{d}}$ cannot be an isomorphism, otherwise the map $\varphi_{S+T} : I_{\underline{e}} \longrightarrow I_{\underline{d}}$ would be an isomorphism too. The map $\varphi_{R+S} : I_{\underline{d}} \longrightarrow I_{\underline{a}}$ factors though $I_{\underline{e}}$, hence is zero, since $\varphi_R : I_{\underline{d}} \longrightarrow I_{\underline{e}}$ is zero. Thus $(I_{\underline{a}}, I_{\underline{d}})$ is a non-exact pair. See Figure 4.1 for a visual reference.

Now, $\varphi_S : I_{\underline{d}} \longrightarrow I_{\underline{b}}$ is an isomorphism, since $\varphi_R : I_{\underline{d}} \longrightarrow I_{\underline{e}}$ is zero. Therefore, $\varphi_R : I_{\underline{b}} \longrightarrow I_{\underline{a}}$ is zero, otherwise $\varphi_{R+S} : I_{\underline{d}} \longrightarrow I_{\underline{a}}$ would be an isomorphism. And the map $\varphi_{S+T} : I_{\underline{a}} \longrightarrow I_{\underline{b}}$ factors through $I_{\underline{d}}$, hence is zero, because $\varphi_T : I_{\underline{a}} \longrightarrow I_{\underline{d}}$ is zero. Thus $(I_{\underline{a}}, I_{\underline{b}})$ is a non-exact pair.

ii) We must have $\varphi_T : I_{\underline{e}} \longrightarrow I_{\underline{q}}$ be the zero map, otherwise $\varphi_{S+T} : I_{\underline{p}} \longrightarrow I_{\underline{q}}$ would be an isomorphism. If $\varphi_{R+S} : I_{\underline{q}} \longrightarrow I_{\underline{e}}$ is an isomorphism then $\varphi_{S+T} : I_{\underline{e}} \longrightarrow I_{\underline{d}}$ is also a isomorphism. But it is a contradiction, because $(I_{\underline{d}}, I_{\underline{e}})$ is

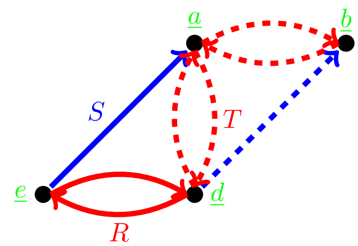

Figure 4.1: Lemma 4.2 part (i). Here, $R = X$ and $S = Y$.

a non-exact pair. Thus $I_{\underline{q}} \longrightarrow I_{\underline{e}}$ is zero and $(I_{\underline{e}}, I_{\underline{q}})$ is a non-exact pair. See Figure 4.2.

Since $\varphi_{R+S} : I_{\underline{q}} \longrightarrow I_{\underline{e}}$ factors through $I_{\underline{p}}$ and $\varphi_S : I_{\underline{p}} \longrightarrow I_{\underline{e}}$ is an isomorphism by hypothesis, we conclude that $I_{\underline{q}} \longrightarrow I_{\underline{a}}$ is zero. And finally, $\varphi_{S+T} : I_{\underline{p}} \longrightarrow I_{\underline{q}}$ factors though $I_{\underline{e}}$, so it is zero, because $\varphi_T : I_{\underline{e}} \longrightarrow I_{\underline{q}}$ is zero. Thus $(I_{\underline{q}}, I_{\underline{p}})$ is a non-exact pair.

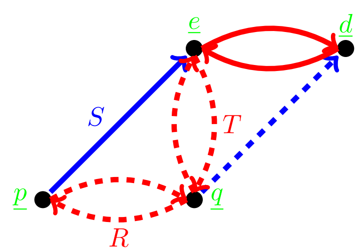

Figure 4.2: Lemma 4.2 part (ii). Here, $R = X$, $S = Y$ and $T = Z$.

$\square$

The above lemma is interesting because it shows us that a non-exact pair can be replicated one level down or one level up, given the appropriate hypothesis, i.e, the map φ_S is an isomorphism. The configuration is continuously replicated until the map φ_S ceases to be an isomorphism. It will be useful in inductive arguments very soon.

Lemma 4.3. Let $\mathfrak{g}$ be a linked net of dimension 1 over the standard $\mathbb{Z}^2$-quiver. For $\{R, S, T\} = \{X, Y, Z\}$ and non-exact pairs $(I_{\underline{b}}, I_{\underline{a}})$ and $(I_{\underline{c}}, I_{\underline{a}})$ where it holds that $\underline{a} = R \cdot \underline{b} = S \cdot \underline{c}$, if $\underline{d} = T \cdot \underline{a}$ then $(I_{\underline{a}}, I_{\underline{d}})$ is also a non-exact pair.

Proof. First, note that the map $\varphi_{R+S} : I_{\underline{d}} \longrightarrow I_{\underline{a}}$ must be zero, since it factors through $I_{\underline{b}}$ and $I_{\underline{c}}$. Suppose by contradiction that $\varphi_T : I_{\underline{a}} \longrightarrow I_{\underline{d}}$ is an isomorphism. Since $\ker(\varphi_R^d) \cap \ker(\varphi_S^d) = (0)$, one of the maps φ_R^d or φ_S^d is an isomorphism. Suppose the former is true. Then $\varphi_{R+T} : I_{\underline{a}} \longrightarrow I_{\underline{c}}$ would be an isomorphism, which is a contradiction.

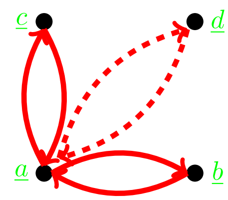

Figure 4.3: Lemma 4.3. Here, $R = X$, $S = Y$ and $T = Z$.

$\square$

We show in Figures 4.4 to 4.8 five types of configurations for a non-exact linked net $\mathfrak{g}$ of dimension 1 over the standard $\mathbb{Z}^2$-quiver. In each case, the vertices of the minimal support are marked in orange.

Types I, II and III admit perfect pairs $(I_{\underline{d}}, I_{\underline{e}})$, where $\underline{e} = T \cdot \underline{d}$ for T equal to X, Z and Y, respectively. The other two types of configurations contain only imperfect non-exact pairs, but they are special nevertheless, as an independent triangle $\{\underline{a}, \underline{b}, \underline{c}\}$ is the minimal support of $\mathfrak{g}$, where the vertices are ordered in such a way that the restriction of $\mathfrak{g}$ to $C^{X+Y}(\underline{a})$ is supported on $\{\underline{a}\}$, the restriction to $C^{X+Z}(\underline{b})$ is supported on $\{\underline{b}\}$ and that to $C^{Y+Z}(\underline{c})$ is supported on $\{\underline{c}\}$. Of course, types I, II and III are all the same configuration, after relabeling the generating vectors, and the same goes to types IV and V. But once we fix the order of the vectors v_X, v_Y, v_Z, having a distinction between those types is helpful to visualize the situation.

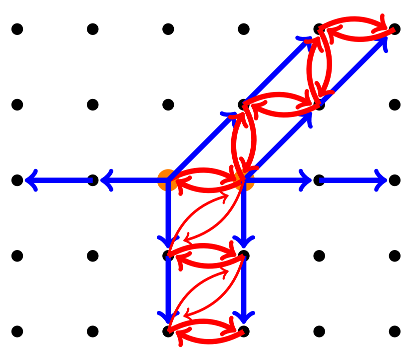

Figure 4.4: Non-exact configuration of type I

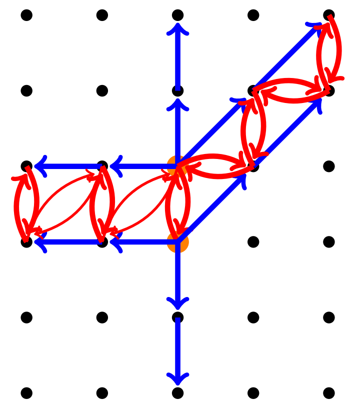

Figure 4.5: Non-exact configuration of type II

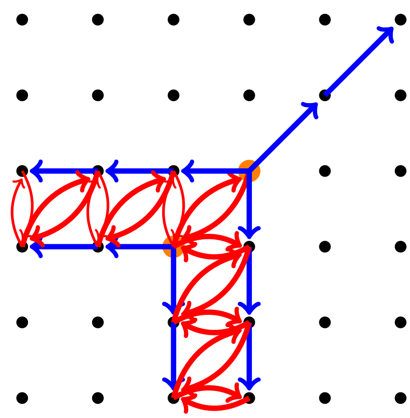

Figure 4.6: Non-exact configuration of type III

However, there exist linked nets of vector spaces $\mathfrak{g}$ of dimension 1 that are not supported on points, segments or triangles, in fact, that do not even have finite support. It is easy to come up with examples. In Figures 4.9 to 4.11 the red sequences of non-exact pairs never end. One can say that the support is "at infinity." If we take all spaces to be the field, associate the identity maps to the blue arrows and the zero maps to the red arrows, we have examples. Those three types are called *infinite corridor configurations*.

The next lemma says that we have a "horizontal" non-exact pair in almost every situation.

Lemma 4.4. Let $\mathfrak{g}$ be a linked net of dimension 1 over the standard $\mathbb{Z}^2$-quiver. If $\mathfrak{g}$ is non-exact and does not admit a horizontal infinite corridor configuration,

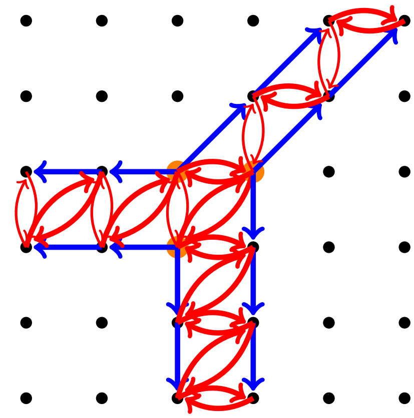

Figure 4.7: Non-exact configuration of type IV

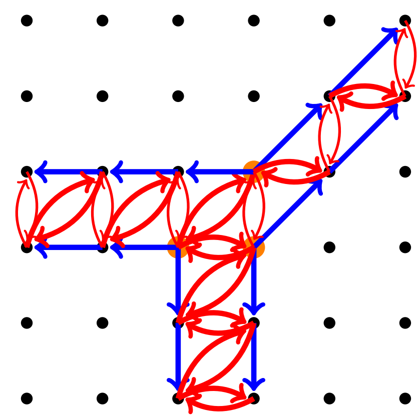

Figure 4.8: Non-exact configuration of type V

then there exists a non-exact pair $(I_{\underline{d}}, I_{\underline{e}})$ such that $\underline{e} = X \cdot \underline{d}$.

Proof. Since $\mathfrak{g}$ is non-exact, there exists a non-exact pair $(I_{\underline{a}}, I_{\underline{b}})$. If $\underline{b} = X \cdot \underline{a}$, we are done. Otherwise, we may suppose $\underline{b} = Z \cdot \underline{a}$ (the case $\underline{b} = Y \cdot \underline{a}$ is analogous). Define $\underline{c} = Y \cdot \underline{b}$ and $\underline{d} = Z \cdot \underline{c}$. Considering the map $\varphi_X : I_{\underline{d}} \longrightarrow I_{\underline{b}}$ we have two cases to analyze.

Case 1: The map $\varphi_X : I_{\underline{d}} \longrightarrow I_{\underline{b}}$ is zero.

In this case we have two possibilities. If $\varphi_{Y+Z} : I_{\underline{b}} \longrightarrow I_{\underline{d}}$ is also zero, then $(I_{\underline{d}}, I_{\underline{b}})$ is the non-exact pair we are looking for.

If $\varphi_{Y+Z} : I_{\underline{b}} \longrightarrow I_{\underline{d}}$ is an isomorphism, then $\varphi_Y : I_{\underline{b}} \longrightarrow I_{\underline{c}}$ is also an isomorphism. The map $\varphi_{Y+Z} : I_{\underline{a}} \longrightarrow I_{\underline{c}}$ is zero, because it factors through $I_{\underline{b}}$. And the map $\varphi_X : I_{\underline{c}} \longrightarrow I_{\underline{a}}$ can not be an isomorphism, because otherwise,

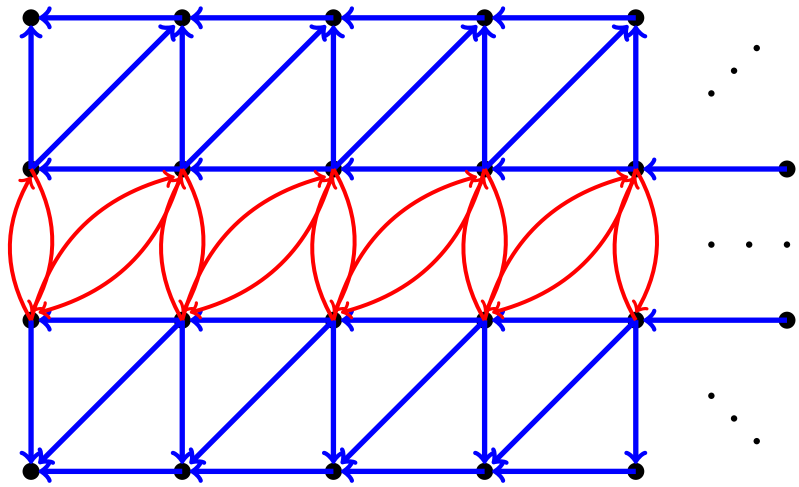

Figure 4.9: Horizontal infinite corridor configuration

$\varphi_{X+Y}: I_{\underline{b}} \longrightarrow I_{\underline{a}}$ would be an isomorphism too. Therefore, $(I_{\underline{c}}, I_{\underline{a}})$ is the non-exact pair we are looking for.

Case 2: The map $\varphi_X : I_{\underline{d}} \longrightarrow I_{\underline{b}}$ is an isomorphism.

In this case we are in the hypotheses of Lemma 4.2 (ii), taking $R = Z, S = X$ and $T = Y$. We conclude that $(I_{\underline{b}}, I_{\underline{c}})$ and $(I_{\underline{c}}, I_{\underline{d}})$ are both non-exact pairs and also that $\varphi_X : I_{\underline{c}} \longrightarrow I_{\underline{a}}$ is an isomorphism.

Note that we obtain a non-exact pair similar to the one we started with, i.e., $\underline{d} = Z \cdot \underline{c}$. We could have considered, in the first place, the "vertical" pair $(I_{\underline{c}}, I_{\underline{d}})$ instead of the "vertical" pair $(I_{\underline{a}}, I_{\underline{b}})$.

So we can proceed repeating the same arguments above. Consider the vertices $\underline{p} = Y \cdot \underline{d}$ and $\underline{q} = Z \cdot \underline{p}$. If $\varphi_X : I_{\underline{q}} \longrightarrow I_{\underline{d}}$ is zero we are in Case 1 again and it's over. If not, we come back to Case 2 and we will conclude that there exists another "vertical" non-exact pair to the right, namely, the pair $(I_{\underline{p}}, I_{\underline{q}})$.

So, the worst case scenario is when we continuously repeat Case 2 indefinitely. But that will produce an infinite horizontal corridor configuration, which is against the hypothesis. So, at some point of the argument, Case 2 can not occur again and then we will finish the proof.

□

The main theorem below says those five types along with the three infinite corridor configurations mentioned above cover all possibilities for configurations of non-exact linked nets of dimension 1.

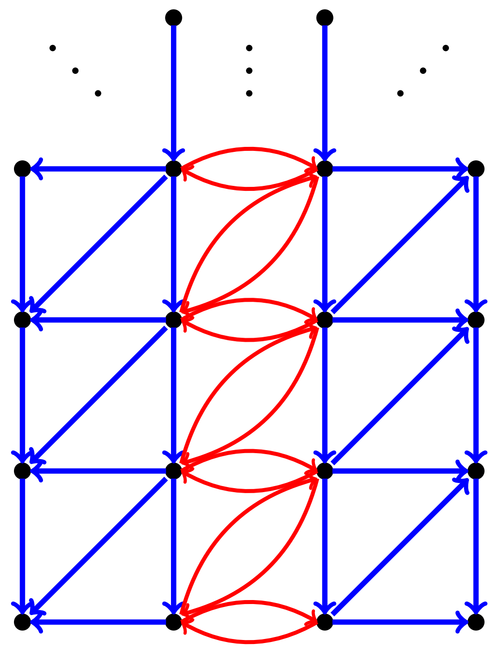

Figure 4.10: Vertical infinite corridor configuration.

Theorem 4.2. Let $\mathfrak{g}$ be a linked net of vector spaces of dimension 1 over the standard $\mathbb{Z}^2$-quiver. If $\mathfrak{g}$ is not exact, then it admits a configuration of type I, II, III, IV or V, or it admits an infinite corridor configuration.

Proof. If $\mathfrak{g}$ admits an infinite horizontal corridor configuration, then we are done. Otherwise, by Lemma 4.4, there exists a non-exact pair $(I_{\underline{d}}, I_{\underline{e}})$ such that $\underline{e} = X \cdot \underline{d}$. Without loss of generality, after a translation, we may suppose that $\underline{e} = (0,0)$ and $\underline{d} = (1,0)$. We do this to simplify the notation.

Denote $I'_i = I_{(i,i)}$ for each $i \in \mathbb{Z}$. Note that the map $\varphi_{X+Z} : I_{(1,1)} \longrightarrow I_{(0,0)}$ must be zero, because otherwise the map $\varphi_X : I_{(1,0)} \longrightarrow I_{(0,0)}$ would be an isomorphism. Thius implies that the maps $\varphi_Y : I'_i \longrightarrow I'_{i+1}$ are isomorphisms for all $i \geq 1$. We have a few cases to analyze.

CASE 1: The maps $I'_i \longrightarrow I'_{i+1}$ are isomorphisms for all $i \in \mathbb{Z}$. See Figure 4.14.

Since $\varphi_Y : I_{(-1,-1)} \longrightarrow I_{(0,0)}$ is an isomorphism, it follows from Lemma 4.2 (ii) that both pairs $(I_{(0,-1)}, I_{(-1,-1)})$ and $(I_{(0,0)}, I_{(0,-1)})$ are non-exact and also $\varphi_Y : I_{(0,-1)} \longrightarrow I_{(1,0)}$ is an isomorphism. Using induction and both parts of Lemma 4.2 we actually prove that $(I_{(i+1,i)}, I_{(i,i)})$ and $(I_{(i,i)}, I_{(i,i-1)})$ are non-

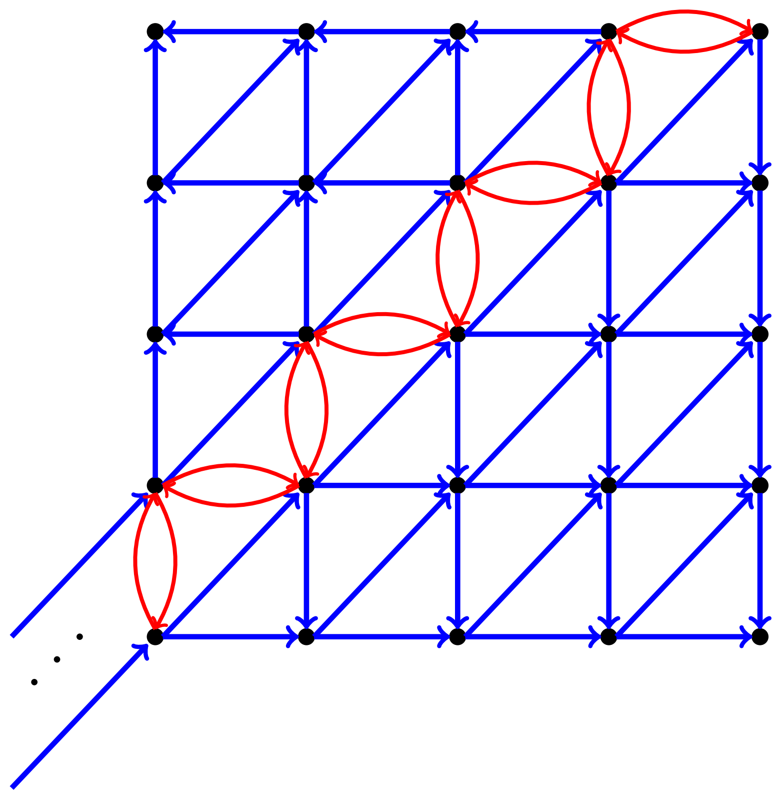

Figure 4.11: Diagonal infinite corridor configuration.

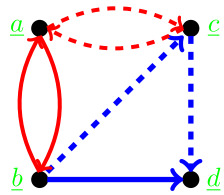

Figure 4.12: Lemma 4.4, Case 1, $I_{\underline{b}} \longrightarrow I_{\underline{d}}$ is an isomorphism.

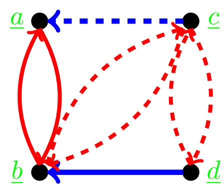

Figure 4.13: Lemma 4.4, Case 2.

exact pairs for all $i \in \mathbb{Z}$. Therefore, $\mathfrak{g}$ admits a diagonal infinite corridor configuration.

CASE 2: There exists $i_0 \in \mathbb{Z}$ such that $I'_i \longrightarrow I'_{i-1}$ is an isomorphism for all $i \leq i_0$ and $I'_i \longrightarrow I'_{i+1}$ is an isomorphism for all $i \geq i_0$. See Figure 4.15.

In this case we have that $i_0 \leq 0$, otherwise $I_{(1,1)} \longrightarrow I_{(0,0)}$ would be an isomorphism and we already saw this is impossible. As in Case 1, we can use Lemma 4.2 (ii) and induction to conclude that $(I_{(i+1,i)}, I_{(i,i)})$ and $(I_{(i,i)}, I_{(i,i-1)})$ are non-exact pairs for each $i = 0, -1, -2, \ldots, i_0 + 1$. We also have that the maps $\varphi_{X+Y} \colon I_{(i_0,i_0)} \longrightarrow I_{(i_0,i_0+1)}$ and $\varphi_{X+Z} \colon I_{(i_0,i_0)} \longrightarrow I_{(i_0-1,i_0-1)}$ are both isomorphisms. Therefore, $(I_{(i_0+1,i_0)}, I_{(i_0,i_0)})$ is a generating pair that induces a configuration of type I on $\mathfrak{g}$.

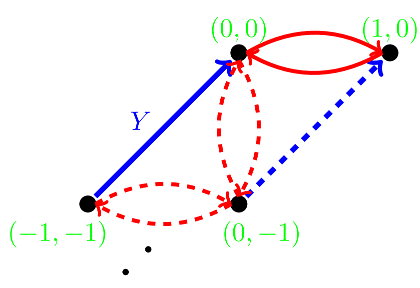

Figure 4.14: Proof of Theorem 4.2, Case 1.

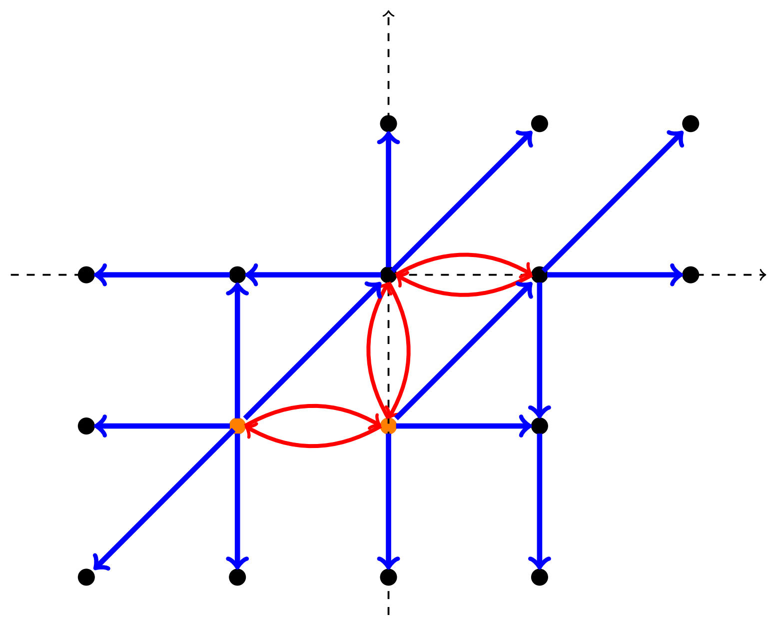

Figure 4.15: Proof of Theorem 4.2, Case 2: configuration of type I.

CASE 3: There exists $i_0 \in \mathbb{Z}$ such that $I'_i \longrightarrow I'_{i+1}$ is an isomorphism for all $i > i_0$, that $I'_i \longrightarrow I'_{i-1}$ is an isomorphism for all $i \leq i_0$ and $I'_{i_0} \longrightarrow I'_{i_0+1}$ and $I'_{i_0+1} \longrightarrow I'_{i_0}$ are both zero.

As we saw before, $i_0 > 0$ is impossible. If $i_0 < 0$ then we can apply Lemma 4.2 (ii) and induction again to conclude that $(I_{(i_0+2,i_0+1)}, I_{(i_0+1,i_0+1)})$ is a "horizontal" non-exact pair. So we could have started with that pair at the very beginning of the proof. Hence, without loss of generality, we will suppose that $i_0 = -1$. See Figure 4.16.

By Lemma 4.3, we see that $(I_{(0,0)}, I_{(0,-1)})$ is a non-exact pair. Hence, the maps $\varphi_{X+Y} \colon I_{(0,0)} \longrightarrow I_{(0,1)}$ and $\varphi_Z \colon I_{(0,-1)} \longrightarrow I_{(0,-2)}$ are both isomorphisms. The map $\varphi_{X+Z} \colon I_{(-1,-1)} \longrightarrow I_{(-2,-2)}$ is also an isomorphism, because $\varphi_Y \colon I_{(-1,-1)} \longrightarrow I_{(0,0)}$ is zero. The maps $\varphi_X \colon I_{(0,-1)} \longrightarrow I_{(-1,-1)}$ and $\varphi_Y \colon I_{(0,-1)} \longrightarrow I_{(1,0)}$ can not be both zero, because their kernels intersect trivially. We have then the following possibilities:

1. (See Figure 4.17) If $\varphi_X \colon I_{(0,-1)} \longrightarrow I_{(-1,-1)}$ and $\varphi_Y \colon I_{(0,-1)} \longrightarrow I_{(1,0)}$ are both isomorphisms, so are the maps $\varphi_{X+Z} \colon I_{(0-1)} \longrightarrow I_{(-1,-2)}$ and $\varphi_{Y+Z} \colon I_{(0,-1)} \longrightarrow I_{(1,-1)}$. Therefore, $(I_{(0,0)}, I_{(0,-1)})$ is a generating pair that induces a configuration of type II on $\mathfrak{g}$.

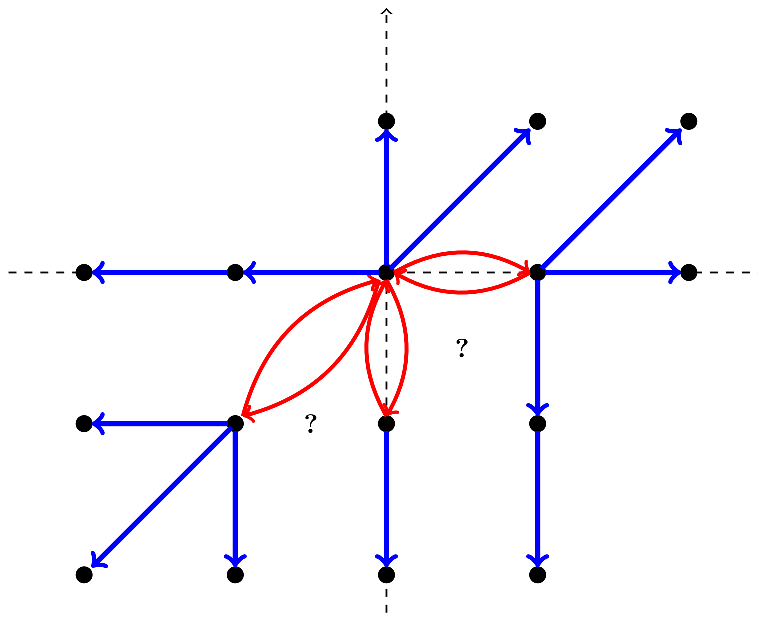

Figure 4.16: Proof of Theorem 4.2, Case 3, general setup when $i_0 = -1$.

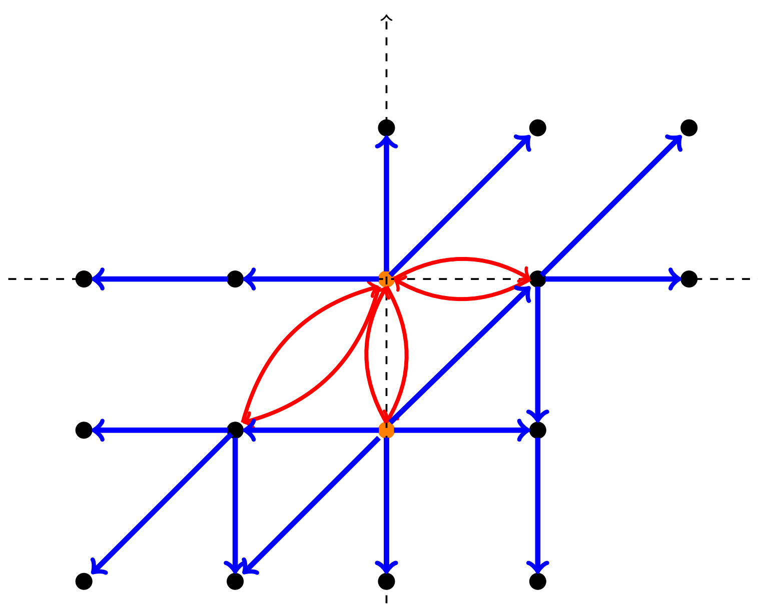

Figure 4.17: Proof of Theorem 4.2, Case 3, setup when $i_0 = -1$, first possibility: configuration of type II

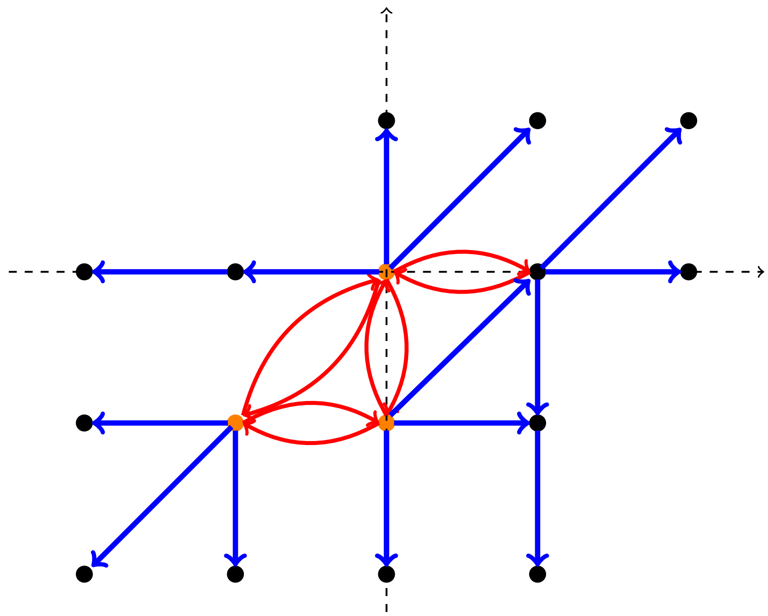

Figure 4.18: Proof of Theorem 4.2, Case 3, setup when $i_0 = -1$, second possibility: configuration of type V

2. (See Figure 4.18) If $I_{(0,-1)} \longrightarrow I_{(-1,-1)}$ is zero and $I_{(0,-1)} \longrightarrow I_{(1,0)}$ is an isomorphism, then $\varphi_{Y+Z} : I_{(0,-1)} \longrightarrow I_{(1,-1)}$ is an isomorphism. The map $\varphi_{Y+Z} : I_{(-1,-1)} \longrightarrow I_{(0,-1)}$ is zero because it factors through $I_{(0,0)}$. Therefore, $(I_{(0,0)}, I_{(-1,-1)}, I_{(0,-1)})$ is a generating triple that induces a configuration of type V on $\mathfrak{g}$.

3. (See Figure 4.19) If $\varphi_Y \colon I_{(0,-1)} \longrightarrow I_{(1,0)}$ is zero and $\varphi_X \colon I_{(0,-1)} \longrightarrow I_{(-1,-1)}$ is an isomorphism, then $\varphi_{X+Z} \colon I_{(0,-1)} \longrightarrow I_{(-1,-2)}$ is an isomorphism. Also, the map $\varphi_{X+Z} \colon I_{(1,0)} \longrightarrow I_{(0,-1)}$ is zero, because it factors through $I_{(0,0)}$. Therefore, $(I_{(0,0)}, I_{(0,-1)}, I_{(1,0)})$ is a generating triple that induces a configuration of type IV on $\mathfrak{g}$.

Now, suppose $i_0 = 0$. Then $(I_{(0,0)}, I_{(1,1)})$ is a non-exact pair. Put $I''_j := I_{(0,j)}$ for each $j \in \mathbb{Z}$. We have a few subcases.

Subcase 3.1: All maps $\varphi_Z \colon I''_{j+1} \longrightarrow I''_j$ are isomorphisms. See Figure 4.21.

In this case, the map $\varphi_{Y+Z} \colon I_{(0,1)} \longrightarrow I_{(1,1)}$ is zero, since it factors through $I_{(0,0)}$. Also $\varphi_X \colon I_{(1,1)} \longrightarrow I_{(0,1)}$ cannot be an isomorphism, because the map $\varphi_{X+Y} \colon I_{(1,1)} \longrightarrow I_{(0,0)}$ would be an isomorphism too. Therefore, $(I_{(1,1)}, I_{(0,1)})$ is a non-exact pair. That also implies that $\varphi_{Y+Z} \colon I_{(1,1)} \longrightarrow I_{(2,1)}$ is an isomor-

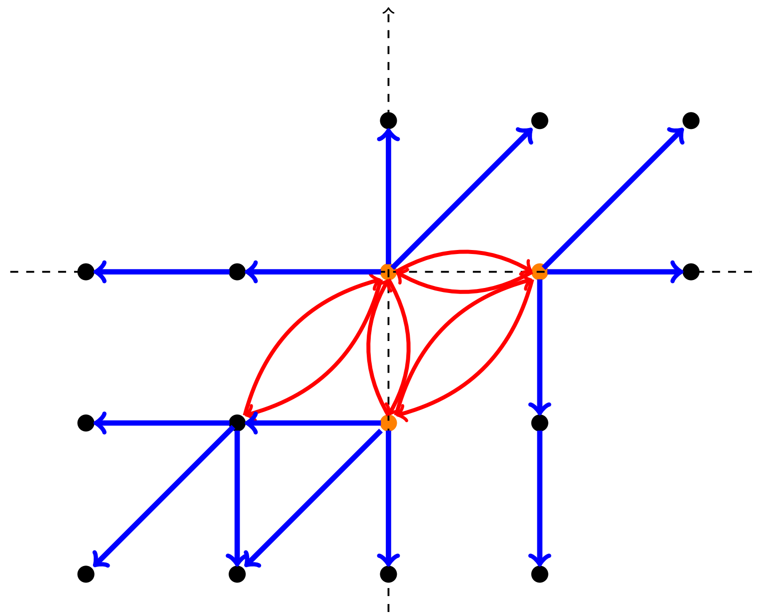

Figure 4.19: Proof of Theorem 4.2, Case 3, setup when $i_0 = -1$, third possibility: configuration of type IV

phism and hence $\varphi_Z \colon I_{(1,1)} \longrightarrow I_{(1,0)}$ is an isomorphism too. Using induction and repeating the above argument, we conclude that for every $j \in \mathbb{Z}$, the pairs $(I_{(1,j)}, I_{(0,j)})$ and $(I_{(0,j)}, I_{(1,j+1)})$ are non-exact and $\varphi_Z \colon I_{(1,j)} \longrightarrow I_{(1,j-1)}$ is an isomorphism. Therefore, $\mathfrak{g}$ admits an infinite vertical corridor configuration.

Subcase 3.2: There exists $j_0 \in \mathbb{Z}$ such that $\varphi_Z : I''_j \longrightarrow I''_{j-1}$ is an isomorphism for all $j \leq j_0$ and $\varphi_{X+Y} : I''_j \longrightarrow I''_{j+1}$ is an isomorphism for all $j \geq j_0$. See Figure 4.22.

In this case, as $\varphi_{X+Y} \colon I_{(0,0)} \longrightarrow I_{(0,1)}$ factors through $I_{(1,1)}$, hence is zero, we must have that $j_0 > 0$. Using the same arguments as in Subcase 3.1, we conclude that the pairs $(I_{(1,j)}, I_{(0,j)})$ and $(I_{(0,j-1)}, I_{(1,j)})$ are non-exact and the map $\varphi_Z : I_{(1,j)} \longrightarrow I_{(1,j-1)}$ is an isomorphism, for every $j \leq j_0$. Also, both $\varphi_{X+Y} : I_{(0,j_0)} \longrightarrow I_{(0,j_0+1)}$ and $\varphi_{X+Z} : I_{(0,j_0)} \longrightarrow I_{(-1,j_0-1)}$ are isomorphisms. Therefore, $(I_{(1,j_0)}, I_{(0,j_0)})$ is a generating pair that induces a configuration of type I on $\mathfrak{g}$.

Subcase 3.3: There exists $j_0 \in \mathbb{Z}$ such that $I''_j \longrightarrow I'_{j+1}$ is an isomorphism for all $j > j_0$, $I''_j \longrightarrow I''_{j-1}$ is an isomorphism for all $j \leq j_0$ and $I''_{j_0} \longrightarrow I''_{j_0+1}$ and $I''_{j_0+1} \longrightarrow I''_{j_0}$ are both zero. See Figure 4.23.

Again, since $\varphi_Z \colon I_{(0,0)} \longrightarrow I_{(0,-1)}$ is an isomorphism, we have $j_0 \geq 0$. Using

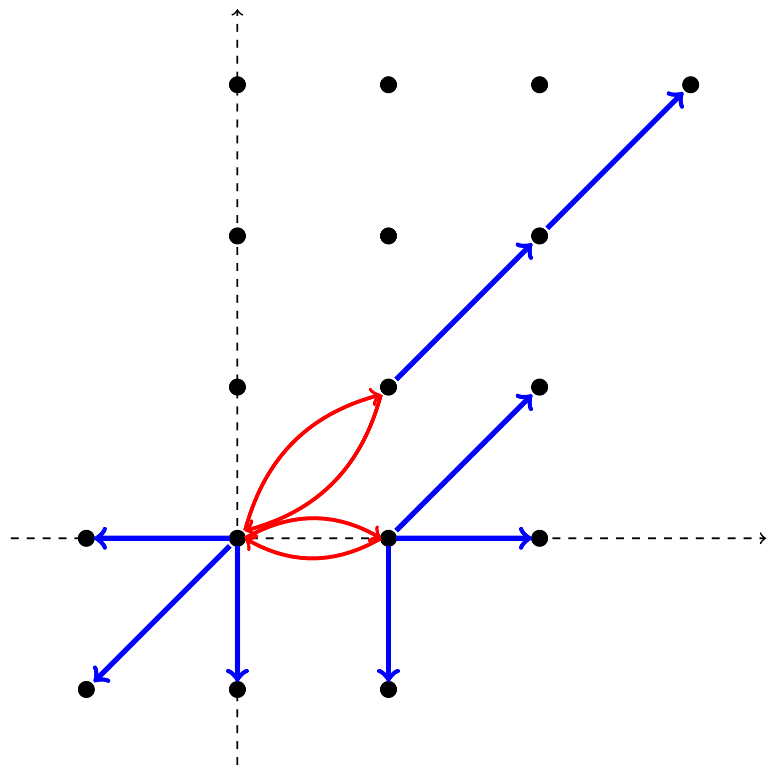

Figure 4.20: Proof of Theorem 4.2, Case 3, general setup when $i_0 = 0$.

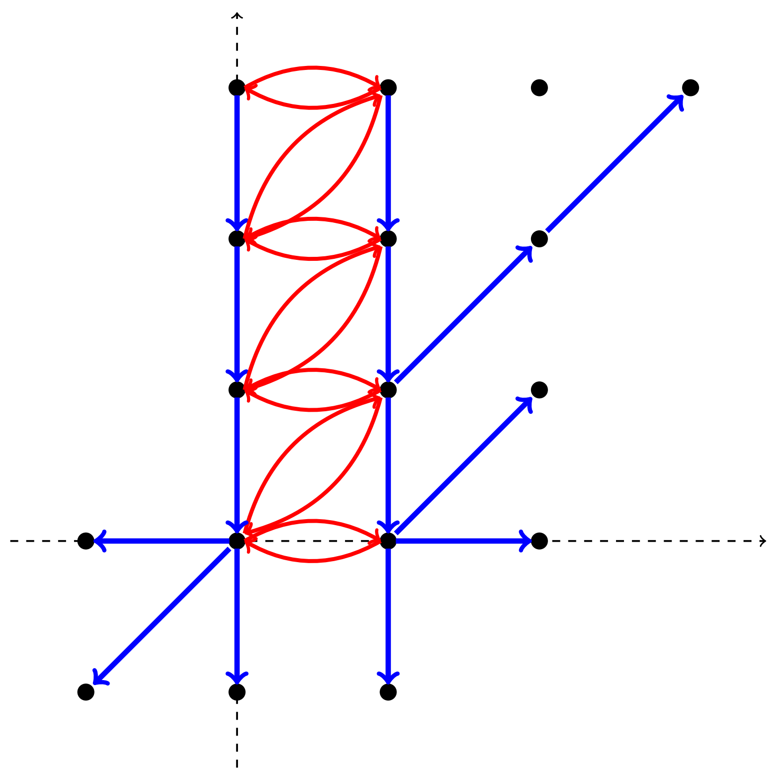

Figure 4.21: Proof of Theorem 4.2, Case 3, setup when $i_0 = 0$, Subcase 3.1: vertical infinite corridor configuration.

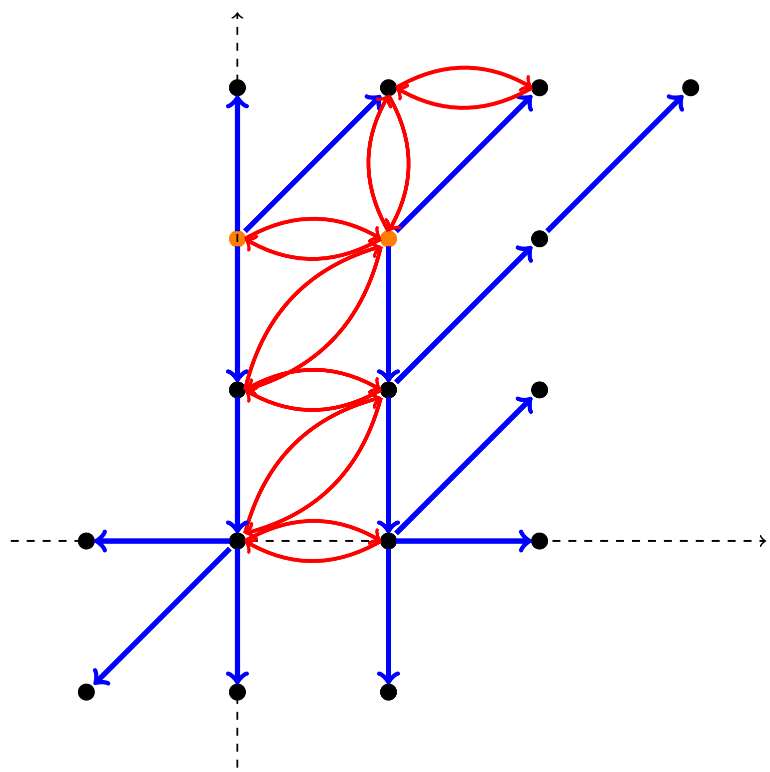

Figure 4.22: Proof of Theorem 4.2, Case 3, setup when $i_0 = 0$, Subcase 3.2: configuration of type I.

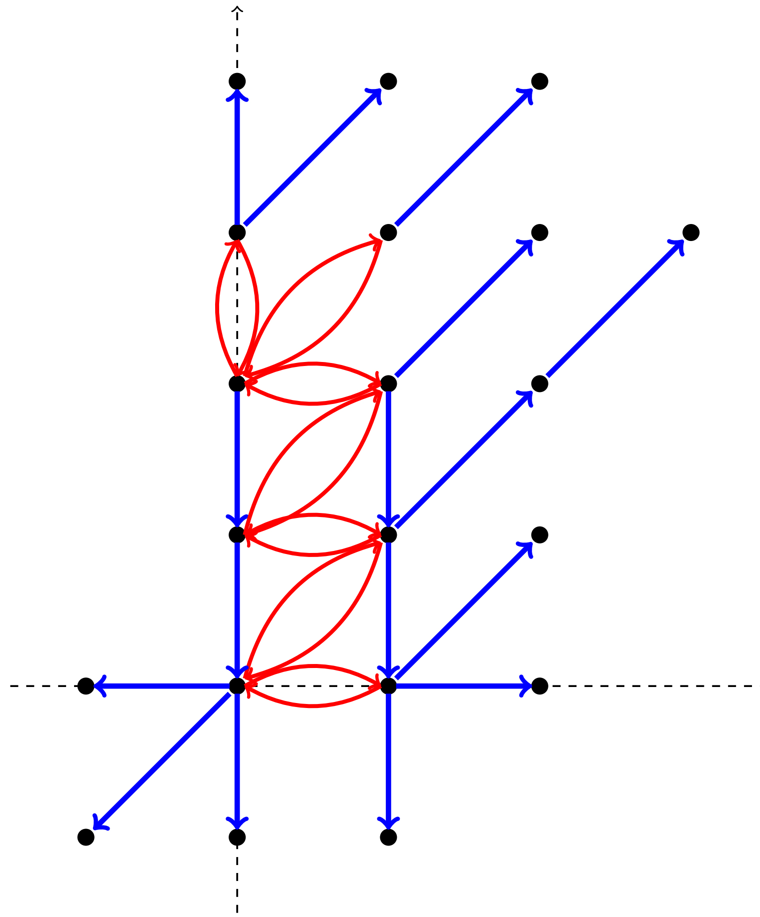

Figure 4.23: Proof of Theorem 4.2, Case 3, setup when $i_0 = 0$, Subcase 3.3: general configuration.

the same arguments as in Subcase 3.1, we conclude that the pairs $(I_{(1,j)}, I_{(0,j)})$ and $(I_{(0,j-1)}, I_{(1,j)})$ are non-exact and the map $\varphi_Z \colon I_{(1,j)} \longrightarrow I_{(1,j-1)}$ is an isomorphism for every $j \leq j_0$. Since $(I_{0,j_0+1}, I_{(0,j_0)})$ is an non-exact pair, Lemma 4.3 assure us that $(I_{(0,j_0)}, I_{(1,j_0+1)})$ is a non-exact pair too. Because of this, we may simplify the notation and suppose without loss of generality that $j_0 = 0$.

Now we have a situation similar to the one we discussed in Case 3, when $i_0 = -1$. Actually, it is the same situation, after relabeling the generating vectors, so we will make it brief. There are three possibilities:

1. If the maps $I_{(1,1)} \longrightarrow I_{(0,1)}$ and $I_{(1,1)} \longrightarrow I_{(1,0)}$ are both isomorphisms,

then $(I_{(0,0)}, I_{(1,1)})$ is a generating pair that induces a configuration of type III on $\mathfrak{g}$.

2. If $I_{(1,1)} \longrightarrow I_{(0,1)}$ is zero and $I_{(1,1)} \longrightarrow I_{(1,0)}$ is an isomorphism, then $(I_{(0,1)}, I_{(0,0)}, I_{(1,1)})$ is a generating triple that induces a configuration of type IV on $\mathfrak{g}$.

3. If $I_{(1,1)} \longrightarrow I_{(0,1)}$ is an isomorphism and $I_{(1,1)} \longrightarrow I_{(1,0)}$ is zero, then $(I_{(1,1)}, I_{(0,0)}, I_{(1,0)})$ is a generating triple that induces a configuration of type V on $\mathfrak{g}$.

$\square$

Now, assume that $\mathfrak{g}$ has finite support on $\mathbb{Z}^2(\leq d)$. We argue that the infinite corridor configurations do not occur.

Theorem 4.3. Let $\mathfrak{g}$ be a linked net of vector spaces of dimension 1 over the standard $\mathbb{N}^2$-quiver. Assume it has a finite support on $\mathbb{N}^2(\leq d)$ for some integer $d > 0$. Then $\mathfrak{g}$ does not admit an infinite corridor configuration. Furthermore, its minimal support is contained in $\mathbb{Z}^2(\leq d)$.

Proof. Suppose by contradiction, and without loss of generality, that $\mathfrak{g}$ admits a vertical infinite corridor configuration. Then the maps $\varphi_Z^{\underline{d}}$ are isomorphisms for all $\underline{d} \in \mathbb{Z}^2$. This is impossible when the representation has finite support. Finally, it follows from Proposition 3.7 that the minimal support is contained in $\mathbb{Z}^2(\leq d)$. $\square$

Chapter 5

Simple bases

In this chapter we explore in more details the concept of a simple basis introduced before. As far as this author knows, the idea of a simple basis appeared first hidden in the proof of Lemma A.12 of Osserman's article [4], on Page 25, but it was explicitly cited as an important property of exact limit linear series by Esteves and Osserman in their paper on Abel maps [1]. In fact, the property of admitting a simple basis is equivalent to being exact for limit linear series over compact type curves with two components. In Proposition 5.3 we prove this is also true for linked chains of vector spaces.

For several months we believed this result was also true for limit linear series over curves with three components. But we have found a counterexample to this fact (see Section 8.2). Nonetheless, it is still true that a limit linear series that admits a simple basis is exact, as we prove in Proposition 5.1.

Let's first recall our definition of a simple basis for linked nets of vector spaces:

Definition 5.1. Let $\mathfrak{g}$ be a linked net of vector spaces of dimension r over a $\mathbb{Z}^n$-quiver with vertex set G. A simple basis for $\mathfrak{g}$ is a collection of r vertices $w_1, \ldots, w_r$ and r vectors $s_i \in V_{w_i}$ for $i = 1, \ldots, r$, such that:

$$\{s_1|_{V_w}, \ldots, s_r|_{V_w}\} \text{ is a basis for } V_w \ \ \forall w \in G.$$

Observe that, if a collection of $s_i \in V_{w_i}$ form a simple basis, then

$$\varphi_u^w \left(s_{i|_{V_w}} \right) = \begin{cases} s_{i|_{V_u}} & \text{if } s_{i|_{V_w}} \notin \mathrm{Ker}(\varphi_u^w) \\ 0 & \text{if } s_{i|_{V_w}} \in \mathrm{Ker}(\varphi_u^w) \end{cases}.$$

Moreover, those $s_{i|_{V_w}}$ belonging to the kernel form a basis for $\mathrm{Ker}(\varphi_u^w)$, whereas the image of the others under φ_u^w form a basis for $\mathrm{Im}(\varphi_u^w)$. Indeed, if $\varphi_u^w(s_{i|_{V_w}})$ is nonzero then there is an admissible path connecting w_i to u passing through w, whence $\varphi_u^{w_i} = \varphi_u^w \varphi_w^{w_i}$ and thus $\varphi_u^w(s_{i|_{V_w}}) = s_{i|_{V_u}}$. As these $s_{i|_{V_u}}$ are linearly independent, they form a basis for the image of φ_u^w. Furthermore, if a linear combination of the $s_{i|_{V_w}}$ is mapped to zero under φ_u^w, the induced linear combination of the $s_{i|_{V_u}}$ is zero, and hence the kernel of φ_u^w is generated by those $s_{i|_{V_w}}$ mapped to zero.

In short, the maps φ_v^w are just projections with respect to the bases. In particular, a simple basis diagonalizes all the maps φ_v^w simultaneously.

Proposition 5.1. Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver. If $\mathfrak{g}$ has a simple basis, then $\mathfrak{g}$ is exact and has finite support.

Proof. First of all, $\mathfrak{g}$ has finite support by Proposition 3.8. Now, let $w_1, \ldots, w_r$ be vertices of the quiver and $s_i \in V_{w_i}$ for $i = 1, \ldots r$ forming a simple basis for $\mathfrak{g}$. Let u, w be neighboring vertices. As we have observed, those $s_{i|_{V_u}}$ belonging to $\mathrm{Ker}(V_u \longrightarrow V_w)$ form a basis for this kernel and the $s_{i|_{V_u}}$ for those $s_{i|_{V_w}}$ not belonging to $\mathrm{Ker}(V_w \longrightarrow V_u)$ form a basis of $\mathrm{Im}(V_w \longrightarrow V_u)$. So, proving that $\mathrm{Ker}(V_u \longrightarrow V_w) = \mathrm{Im}(V_w \longrightarrow V_u)$ is equivalent to proving that for each $i = 1, \ldots, r$:

$$s_{i|_{V_u}} \in \mathrm{Ker}(V_u \longrightarrow V_w) \iff s_{i|_{V_w}} \notin \mathrm{Ker}(V_w \longrightarrow V_u).$$

Suppose $s_{i|_{V_u}} \in \mathrm{Ker}(V_u \longrightarrow V_w)$. Then there is no admissible path connecting w_i to w through u. Since u and w are neighbors, it follows from Lemma 3.1 that there is an admissible path connecting w_i to u through w. But then

$$\varphi_u^w \left(s_{i|_{V_w}} \right) = s_{i|_{V_u}} \neq 0.$$

Conversely, suppose $s_{i|_{V_w}} \notin \mathrm{Ker}(V_w \longrightarrow V_u)$. Then there is an admissible path connecting w_i to u through w. By Lemma 3.1, the concatenation of every path from w_i to u with every path from u to w is not admissible, whence

$\varphi_w^u \varphi_u^{w_i} = 0$. It follows that
$$\varphi_w^u \left(s_{i|_{V_u}} \right) = 0.$$
□

The converse holds for $n = 1$, the case of linked chains of vector spaces, as Proposition 5.3 asserts. Before stating it, we need to somehow measure the non-exactness of a linked net of vector spaces and relate this to the property of having a simple basis.

Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver. For each vertex u of the quiver define the following subspaces of V_u:

$$\Upsilon_u := \sum_{w \in N(u)} \operatorname{Im}\left(\varphi_u^w \colon V_w \longrightarrow V_u\right) \quad \text{and} \quad \Sigma_u := \sum_{w \in N(u)} \operatorname{Ker}\left(\varphi_w^u \colon V_u \longrightarrow V_w\right).$$

where $N(u)$ is the set of vertices neighboring u. Set

$$\mathcal{V}(\mathfrak{g}) := \bigoplus_u \left(\frac{V_u}{\Upsilon_u} \right) \quad \text{and} \quad \mathcal{W}(\mathfrak{g}) := \bigoplus_u \left(\frac{V_u}{\Sigma_u} \right).$$

Clearly, $\mathcal{W}(\mathfrak{g})$ is a quotient of $\mathcal{V}(\mathfrak{g})$, and they are isomorphic if $\mathfrak{g}$ is exact.

Proposition 5.2. Let $\mathfrak{g}$ be a linked net of vector spaces of dimension r over a $\mathbb{Z}^n$-quiver. If $\mathfrak{g}$ has a simple basis, then $\dim \mathcal{V}(\mathfrak{g}) = r$.

Proof. Let $w_1, \dots, w_r$ be vertices of the quiver and $s_i \in V_{w_i}$ for $i = 1, \dots, r$ forming a simple basis for $\mathfrak{g}$. Let u be a vertex of the quiver. Then $s_i \notin \Upsilon_u$ for any i such that $w_i = u$. In fact, a stronger statement holds, finishing the proof: The classes of all the s_i with $w_i = u$ form a basis for the quotient space ${}^{V_u}/_{\Upsilon_u}$. Indeed, Υ_u is generated by $\varphi_u^w(s_{i|_{V_w}})$ for all $w \in N(u)$ and all $i = 1, \dots, r$. But if $w_i = u$ then $s_{i|_{V_w}} = \varphi_w^u(s_i)$ and hence $\varphi_u^w(s_{i|_{V_w}}) = 0$ for every $w \in N(u)$. On the other hand, if $w_i \neq u$, then each path from w_i to u goes through a neighbor w to u, whence $\varphi_u^w(s_{i|_{V_w}}) = s_{i|_{V_u}}$. It follows that Υ_u is generated by the $s_{i|_{V_u}}$ for $w_i \neq u$, and hence the classes of the s_i with $w_i = u$ form a basis for the quotient space ${}^{V_u}/_{\Upsilon_u}$. □

Also the converse to this statement holds if $n = 1$, as the next proposition asserts.

Proposition 5.3. Let $\mathfrak{g}$ be a linked chain of vector spaces of dimension r. Then the following statements hold:

1. $\dim \mathcal{W}(\mathfrak{g}) \leq r$.

2. If $\mathfrak{g}$ has finite support then $\mathcal{V}(\mathfrak{g})$ has finite dimension at least r.

Furthermore, the following statements are equivalent:

a) $\mathfrak{g}$ admits a simple basis.

b) $\dim \mathcal{W}(\mathfrak{g}) = r$.

c) $\dim \mathcal{V}(\mathfrak{g}) = r$.

d) $\mathfrak{g}$ is exact of finite support.

Proof. We may assume we are given the quiver Q whose vertex set is $\mathbb{Z}$ and whose arrow set $A \subseteq \mathbb{Z} \times \mathbb{Z}$ is the union of

$$A_0 := \{(i, i+1) \mid i \in \mathbb{Z}\} \quad \text{and} \quad A_1 := \{(i, i-1) \mid i \in \mathbb{Z}\}.$$

For each $i \in \mathbb{Z}$ put

$$\Upsilon_i := \operatorname{Im}(\varphi_i^{i-1}) + \operatorname{Im}(\varphi_i^{i+1}) \quad \text{and} \quad \Sigma_i := \operatorname{Ker}(\varphi_{i-1}^i) + \operatorname{Ker}(\varphi_{i+1}^i).$$

First of all, let $s_{i,1}, \ldots, s_{i,\ell_i}$ be liftings of linearly independent vectors in the quotient of V_i by Σ_i for each $i \in \mathbb{Z}$. We claim first that the $s_{i,j}|_{V_m}$ are linearly independent for each $m \in \mathbb{Z}$. As an immediate consequence, we obtain Statement 1 as well as that Statement b) implies Statement a).

Indeed, suppose not. Among all the zero nontrivial linear combinations $\sum c_{i,j} s_{i,j}|_{V_m}$, pick m and the $c_{i,j}$ for which the minimum number of the $c_{i,j}$ are nonzero. Clearly, since $s_{i,j}|_{V_m} \in \Sigma_m$ for all $i \neq m$ and all j, the coefficients $c_{m,j}$ are zero. Thus

$$\sum_{i<m} \sum_j c_{i,j} s_{i,j}|_{V_m} = -\sum_{i>m} \sum_j c_{i,j} s_{i,j}|_{V_m}. \tag{5.1}$$

Also, there is an integer $i < m$ such that $c_{i,j} \neq 0$ for some j, or there is an integer $i > m$ such that $c_{i,j} \neq 0$ for some j.

If both occurred then, applying φ_{m-1}^m to both sides of (5.1), we would obtain the zero nontrivial linear combination

$$\sum_{i>m} \sum_j c_{i,j} s_{i,j}|_{V_{m-1}} = 0$$

which involves fewer nonzero coefficients. We may thus assume without loss of generality that $c_{i,j} = 0$ for all $i > m$ and all j. Then

$$\sum_{i<m} \sum_{j} c_{i,j} s_{i,j}{}_{|V_m} = 0.$$

Let p be maximum integer such that $p < m$ and $c_{p.j} \neq 0$ for some j. Then

$$\sum_{i<m} \sum_{j} c_{i,j} s_{i,j}{}_{|V_m} = \varphi_m^{p+1} \varphi_{p+1}^{p} \big(\sum_{i \leq p} \sum_{j} c_{i,j} s_{i,j}{}_{|V_p} \big),$$

whence

$$\sum_{i \leq p} \sum_{j} c_{i,j} s_{i,j}{}_{|V_p} \in \mathrm{Ker}(\varphi_{p+1}^{p}) \subseteq \Sigma_p.$$

On the other hand,

$$\sum_{i<p} \sum_{j} c_{i,j} s_{i,j}{}_{|V_p} \in \mathrm{Im}(\varphi_p^{p-1}) \subseteq \Upsilon_p \subseteq \Sigma_p.$$

It follows that $\sum_j c_{p,j} s_{p,j} \in \Sigma_p$, contradicting the choice of the $s_{p,j}$, as $c_{p.j} \neq 0$ for some j.

Consider now Statement 2. If $\mathfrak{g}$ has finite support, say in the interval $[-i, i]$, then φ_j^i and and φ_{-j}^{-i} are isomorphisms for each $j \geq i$, whence $\Upsilon_j = V_j$ for each j with $|j| > i$ and thus $\mathcal{V}(\mathfrak{g})$ has finite dimension.

Furthermore, let $s_{i,1}, \ldots, s_{i,\ell_i}$ be a lifting of a basis of the quotient of V_i by Υ_i for each $i \in \mathbb{Z}$. There are a finite number of $s_{i,j}$, equal to $\dim \mathcal{V}(\mathfrak{g})$. We claim that the $s_{i,j}{}_{|V_m}$ generate V_m for each $m \in \mathbb{Z}$. As an immediate consequence, we obtain Statement 2 as well as that Statement (c) implies Statement (a).

Indeed, suppose not. Let $m \in \mathbb{Z}$ and $s \in V_m$ which is not a linear combination of the $s_{i,j}{}_{|V_m}$. We may assume that $s \in \Upsilon_m$. Suppose by induction that are $s_1 \in V_p$ and $s_2 \in V_q$ for $p < m$ and $q > m$ such that $s := \varphi_m^p(s_1) + \varphi_m^q(s_2)$ is not a linear combination of the $s_{i,j}{}_{|V_m}$. We may assume that $s_1 = \varphi_p^{p-1}(s_{1,1}) + \varphi_p^{p+1}(s_{1,2})$ for certain $s_{1,1} \in V_{p-1}$ and $s_{1,2} \in V_{p+1}$. Since $\varphi_m^p \varphi_p^{p+1} = 0$, we have that $\varphi_m^p(s_1) = \varphi_m^{p-1}(s_{1,1})$. Likewise, we may assume that $\varphi_m^q(s_2) = \varphi_m^{q+1}(s_{2,2})$ for a certain $s_{2,2} \in V_{q+1}$. By induction, for each $p < m$ and $q > m$, there is $s \in \mathrm{Im}(\varphi_m^p) + \mathrm{Im}(\varphi_m^q)$ which is not a linear combination of the $s_{i,j}{}_{|V_m}$. But for $p << 0$ and $q >> 0$ we have that both φ_m^p and φ_m^q are zero, reaching a contradiction.

Proposition 5.1 yields that Statement a) implies Statement d), whereas

Proposition 5.2 yields that Statement a) implies Statement c). Clearly, Statement d) implies that $\mathcal{V}(\mathfrak{g}) = \mathcal{W}(\mathfrak{g})$. Now, Statements 1 and 2 yield

$$\dim \mathcal{V}(\mathfrak{g}) \leq r \leq \mathcal{W}(\mathfrak{g}).$$

The equalities thus hold, yielding Statements b) and c). □

On Table 8.3 on Page 127 there is an example of a non-exact linked net of vector spaces $\mathfrak{g}$ of dimension 2 for which dim $\mathcal{W}(\mathfrak{g}) = 0$.

Chapter 6

The scheme $\mathbb{P}(\mathfrak{g})$

In this chapter we will study the scheme $\mathbb{P}(\mathfrak{g})$, where $\mathfrak{g}$ is a linked net of vector spaces over a $\mathbb{Z}^n$-quiver, defined as a certain subscheme of the product $\prod \mathbb{P}(V_{w_i})$ for $w_0, \dots, w_n$ a certain collection of vertices of the quiver.

The main goal of this chapter is to develop an algorithm to compute the Chow class of $\mathbb{P}(\mathfrak{g})$ inside $\mathbb{P}^r \times \mathbb{P}^r \times \mathbb{P}^r$, in the case $n = 2$. (As the class is invariant under the product of the automorphisms groups of the $\mathbb{P}(V_{w_i})$, we may choose identifications of these with $\mathbb{P}^r$, where $r+1$ is the dimension of $\mathfrak{g}$.) The computation will be based on the behavior of the numbers $p_I^v = \dim \operatorname{Ker}(\varphi_I^v)$.

Let $\mathfrak{g}$ be a linked net of vector spaces of dimension $r+1$ over a $\mathbb{Z}^n$-quiver Q. Assume that $\mathfrak{g}$ has finite support on a set of vertices H. For each $i = 0, \dots, n$, let w_i be the i-th bound of H. Recall that w_i is the vertex of the cone $C_{J_i}(w_i)$ of vertices u of Q for which every admissible path γ from every vertex $z \in H$ to u satisfies $\gamma(i) = 0$. For each $z \in H$, if all the linear maps $\varphi_{w_i}^z \colon V_z \longrightarrow V_{w_i}$ are nonzero, they induce a rational map

$$\mathbb{P}(V_z) \dashrightarrow \mathbb{P}(V_{w_0}) \times \cdots \times \mathbb{P}(V_{w_n}) =: \Pi(\mathfrak{g}). \tag{6.1}$$

We let $\mathbb{P}(\mathfrak{g})_z$ denote the closure of the image of this map if all the linear maps $\varphi_{w_i}^z \colon V_z \longrightarrow V_{w_i}$ are nonzero, and put $\mathbb{P}(\mathfrak{g})_z := \emptyset$ otherwise. Let $\mathbb{P}(\mathfrak{g})$ be the union of the $\mathbb{P}(\mathfrak{g})_z$ for $z \in H$.

The choice of H does not change the nature of $\mathbb{P}(\mathfrak{g})$, as the next proposition implies.

Proposition 6.1. Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver Q. Let H' be a finite collection of vertices and $H \subseteq H'$. Assume that $\mathfrak{g}$ has support on H. For each $i = 0, \ldots, n$, let w_i and w_i' be the i-th bounds of H and H', respectively. Then the following statements hold:

1. $\varphi^z_{w_i'} = \varphi^{w_i}_{w_i'}\varphi^z_{w_i}$ for each $z \in H$ and each $i = 0, \ldots, n$.

2. $\varphi^{w_i}_{w_i'}$ is an isomorphism for each $i = 0, \ldots, n$.

3. Given $z \in H'$ for which $\varphi^z_{w_i'} \neq 0$ for all i, there is $u \in H$ such that φ^u_z is an isomorphism and $\varphi^z_{w_i'}\varphi^u_z = \varphi^{w_i}_{w_i'}\varphi^u_{w_i}$ for each $i = 0, \ldots, n$.

Proof. Fix $i \in \{0, \ldots, n\}$. Since w_i' is the ith bound of H' and $H \subseteq H'$, there is a path γ connecting w_i to w_i' with $\gamma(i) = 0$. Also, there is also a path μ connecting each $z \in H$ to w_i with $\mu(i) = 0$, and hence

$$\varphi^z_{w_i'} = \varphi^{w_i}_{w_i'}\varphi^z_{w_i}$$

for each $z \in H$, proving Statement 1.

Now, since $\mathfrak{g}$ has support on H, there is $z \in H$ such that $\varphi^z_{w_i'}$ is an isomorphism. It thus follows from Statement 1 that $\varphi^{w_i}_{w_i'}$ is an isomorphism, proving Statement 2.

Let now $z \in H'$. Since $\mathfrak{g}$ has support on H, there is $u \in H$ such that φ^u_z is an isomorphism. If $\varphi^z_{w_i'} \neq 0$, also $\varphi^z_{w_i'}\varphi^u_z \neq 0$. It follows that there is an admissible path connecting u to w_i' through z, and hence $\varphi^z_{w_i'}\varphi^u_z = \varphi^u_{w_i'}$. Statement 3 follows now from Statement 1. $\square$

This scheme $\mathbb{P}(\mathfrak{g})$ appeared in [1], in a context of compact type curves with two components, as explained in Chapter 1. The authors proved that if the limit linear series $\mathfrak{g}$ is exact, then $\mathbb{P}(\mathfrak{g})$ is reduced, connected, Cohen-Macaulay of pure dimension r = rank of $\mathfrak{g}$ and his Hilbert polynomial is the same of the diagonal.

Muñoz studied this scheme in the context of compact type curves with three components. In his article [7] he proved that if $\mathfrak{g}$ is an exact limit linear series which is the unique exact extension of a refined limit linear series (review Definition 1.2), then $\mathbb{P}(\mathfrak{g})$ has the same Hilbert polynomial of the diagonal. The hypothesis of being a unique extension of a refined limit linear series can not be dropped. In section 8.2 we present a example of a exact limit linear series $\mathfrak{g}$ such that the associated scheme $\mathbb{P}(\mathfrak{g})$ is not a deformation of the diagonal. This $\mathfrak{g}$ does not admit a simple basis.

In this Chapter, we are concerned with determining the class $[\mathbb{P}(\mathfrak{g})]$ of $\mathbb{P}(\mathfrak{g})$ in the Chow ring of $\Pi(\mathfrak{g})$, which we identify with $\prod_{i=0}^n \mathbb{P}^r$. Recall this Chow ring is generated by the classes $h_0, h_1, \ldots, h_n$, where h_i is the pullback to $\Pi(\mathfrak{g})$ of the hyperplane class of the i-th factor. The indeterminacy locus of (6.1) is the union of the subspaces $\mathbb{P}(K^z_{w_i})$ for $i = 0, \ldots, n$. If $\mathbb{P}(\mathfrak{g})_z$ is not empty, then (6.1) induces a birational map $\mathbb{P}(V_z) \dashrightarrow \mathbb{P}(\mathfrak{g})_z$. Thus $\mathbb{P}(\mathfrak{g})$ has pure codimension nr in $\Pi(\mathfrak{g})$, and hence its class is a linear combination of the monomials of the form $h_0^i h_1^j h_2^k$, where $i + j + k = nr$ and $0 \leq i, j, k \leq r$. Each non-empty $\mathbb{P}(\mathfrak{g})_v$ contributes with certain of these monomials.

We will now restrict our attention to $\mathbb{Z}^2$-quivers. Let thus $\mathfrak{g}$ be a linked net of vector spaces of dimension $r + 1$ of finite support over a $\mathbb{Z}^2$-quiver Q. As in Chapter 4, we assume that Q is the standard $\mathbb{Z}^2$-quiver, with vertex set $\mathbb{Z}^2$ and generated by the vectors $v_0 = (-1, 0)$, $v_1 = (1, 1)$ and $v_2 = (0, -1)$. We assume as well that $\mathfrak{g}$ has support on

$$\mathbb{N}^2(\leq d) := \{(i, j) \in \mathbb{Z}^2 \mid 0 \leq i, j \leq i + j \leq d\}$$

for some positive integer d. Recall as well from Chapter 4 all the alternate notation used in this case.

In this case the i-th bounds are $w_0 = (d, 0)$, $w_1 = (0, 0)$ and $w_2 = (0, d)$. Thus, for each $\underline{d} \in \mathbb{N}^2(\leq d)$ we are considering the natural map

$$\mathbb{P}(V_{\underline{d}}) \dashrightarrow \mathbb{P}(V_{(d,0)}) \times \mathbb{P}(V_{(0,0)}) \times \mathbb{P}(V_{(0,d)}) =: \Pi(\mathfrak{g}). \tag{6.2}$$

Its indeterminacy locus is $\mathbb{P}(K^{\underline{d}}_{X+Y}) \cup \mathbb{P}(K^{\underline{d}}_{X+Z}) \cup \mathbb{P}(K^{\underline{d}}_{Y+Z})$.

Indeed, if $\underline{d} \neq (0, 0)$ then $K^{\underline{d}}_{(0,0)} = K^{\underline{d}}_{X+Z}$ by Lemma 3.2, and if $\underline{d} = (0, 0)$ then $K^{\underline{d}}_{(0,0)} = 0$ but also $K^{\underline{d}}_{X+Z} = 0$. In any case, the indeterminacy locus of the map $\mathbb{P}(V_{\underline{d}}) \dashrightarrow \mathbb{P}(V_{(0,0)})$ is $\mathbb{P}(K^{\underline{d}}_{X+Z})$. An analogous argument yields the indeterminacy loci of the other maps composing (6.2).

The next result tells us exactly what the contribution to the Chow class of $\mathbb{P}(\mathfrak{g})$ of each $\mathbb{P}(\mathfrak{g})_{\underline{d}}$ is when $\mathfrak{g}$ is exact.

Theorem 6.1. Let $\mathfrak{g}$ be an exact linked net of vector spaces of dimension $r+1$ over the standard $\mathbb{Z}^2$-quiver with finite support on $\mathbb{N}^2(\leq d)$. Let $\underline{d} \in \mathbb{N}^2(\leq d)$. Let i, j, k be integers satisfying $i + j + k = 2r$ and $0 \leq i, j, k \leq r$. Then the term $h_0^i h_1^j h_2^k$ appears in the expression of the class $\left[\mathbb{P}(\mathfrak{g})_{\underline{d}}\right]$ in the Chow ring of $\mathbb{P}^r \times \mathbb{P}^r \times \mathbb{P}^r$ (and then with coefficient 1) if and only if the following relations are true:

$$\begin{aligned} p^{\underline{d}}_{Y+Z} &\leq i, & p^{\underline{d}}_{X} &\leq r-i, \\ p^{\underline{d}}_{X+Z} &\leq j, & p^{\underline{d}}_{Y} &\leq r-j, \\ p^{\underline{d}}_{X+Y} &\leq k, & p^{\underline{d}}_{Z} &\leq r-k. \end{aligned}$$

Proof. Suppose the numerical conditions hold. To find the coefficient of $\left[\mathbb{P}(\mathfrak{g})_{\underline{d}}\right]$ with respect to the monomial $h_0^i h_1^j h_2^k$, we may compute its intersection with the complementary class $h_0^{r-i} h_1^{r-j} h_2^{r-k}$. Let $H_X \subset V_{(d,0)}$, $H_Y \subset V_{(0,0)}$ and $H_Z \subset V_{(0,d)}$ be general subspaces of dimension $i+1, j+1$ and $k+1$ respectively. Denote by $\rho_X : V_{\underline{d}} \longrightarrow V_{(d,0)}$, $\rho_Y : V_{\underline{d}} \longrightarrow V_{(0,0)}$ and $\rho_Z : V_{\underline{d}} \longrightarrow V_{(0,d)}$ the maps $\varphi^{\underline{d}}_{(d,0)}$, $\varphi^{\underline{d}}_{(0,0)}$ and $\varphi^{\underline{d}}_{(0,d)}$, respectively.

From now on, to avoid unnecessary notation we will drop the multi-index $\underline{d}$ from the maps.

Now, observe that $\mathrm{Ker}(\rho_X) = \mathrm{Ker}(\varphi_{Y+Z})$, and analogously for the other two maps. We want to prove that, for general subspaces H_X, H_Y and H_Z, the triple intersection

$$\rho_X^{-1}(H_X) \cap \rho_Y^{-1}(H_Y) \cap \rho_Z^{-1}(H_Z)$$

has dimension 1, and hence corresponds to a point on $\mathbb{P}(V_{\underline{d}})$. This will imply, by Kleiman's Transversality Theorem, that the coefficient of $h_0^i h_1^j h_2^k$ in $[\mathbb{P}(\mathfrak{g})_{\underline{d}}]$ is equal to 1.

Since $p_{Y+Z} \leq i$:

$$\dim \mathrm{Im}(\rho_X) + \dim H_X = r + 1 - p_{Y+Z} + i + 1 \geq r + 2 > r + 1.$$

Hence

$$\mathrm{Im}(\rho_X) \cap H_X \neq (0)$$

and the intersection is transversal, since we are taking a general H_X.

We want now to guarantee that the intersection

$$\rho_Y(\rho_X^{-1}(H_X)) \cap H_Y$$

is nonzero and transversal. It is enough to show that

$$\dim \rho_Y(\rho_X^{-1}(H_X)) > \mathrm{codim}\ H_Y.$$

First:

$$\dim \rho_Y(\rho_X^{-1}(H_X)) = \dim \rho_X^{-1}(H_X) - \dim \left(\rho_X^{-1}(H_X) \cap K_{X+Z}\right).$$

But since the intersection $\mathrm{Im}(\rho_X) \cap H_X$ is transversal:

$$\dim \rho_X^{-1}(H_X) = i + 1.$$

And so we obtain:

$$\dim \rho_Y(\rho_X^{-1}(H_X)) = i + 1 - \dim \left(\rho_X^{-1}(H_X) \cap K_{X+Z}\right). \tag{6.3}$$

Now we analyze two cases:

Case 1: $\rho_X(K_{X+Z}) \cap H_X = 0$.

This is equivalent to $K_{X+Z} \cap \rho_X^{-1}(H_X) = K_Z$, since $K_{Y+Z} \cap K_{X+Z} = K_Z$ and $K_{Y+Z} = \mathrm{Ker}(\rho_X)$. Equation (6.3) becomes:

$$\dim \rho_Y(\rho_X^{-1}(H_X)) = i + 1 - p_Z \geq i + 1 - r + k.$$

Since $i + j + k = 2r$, we finally get:

$$\dim \rho_Y(\rho_X^{-1}(H_X)) \geq r - j + 1 > \mathrm{codim}\ H_Y.$$

Case 2: $\rho_X(K_{X+Z}) \cap H_X \neq 0$.

In this case, since the intersection is transversal, we get

$$\dim \left(\rho_X^{-1}(H_X) \cap K_{X+Z}\right) = \dim K_{X+Z} - \mathrm{codim}\ H_X.$$

Equation (6.3) becomes:

$$\dim \rho_Y(\rho_X^{-1}(H_X)) = i + 1 - p_{X+Z} + r - i = r + 1 - p_{X+Z}.$$

By hypothesis, $p_{X+Z} \leq j$, so:

$$\dim \rho_Y(\rho_X^{-1}(H_X)) \geq r + 1 - j > \mathrm{codim}\ H_Y.$$

Therefore, $\rho_Y(\rho_X^{-1}(H_X)) \cap H_Y \neq (0)$ and hence $\rho_X^{-1}(H_X) \cap \rho_X^{-1}(H_Y) \neq 0$.

The intersection is transversal, so it has the expected codimension, i.e.,

$$\begin{aligned}\operatorname{codim}(\rho_X^{-1}(H_X) \cap \rho_X^{-1}(H_Y)) &= \operatorname{codim}(\rho_X^{-1}(H_X)) + \operatorname{codim}(\rho_X^{-1}(H_X)) \\ &= \operatorname{codim}(H_X) + \operatorname{codim}(H_Y) \\ &= 2r - (i+j) = k\end{aligned}$$

Hence, $\dim(\rho_X^{-1}(H_X) \cap \rho_X^{-1}(H_Y)) = r - k + 1$.

Finally, we want show that the triple intersection

$$\rho_X^{-1}(H_X) \cap \rho_Y^{-1}(H_Y) \cap \rho_Z^{-1}(H_Z)$$

is transversal and has dimension 1. For this, it is enough to show that:

$$\dim \rho_Z(\rho_X^{-1}(H_X) \cap \rho_Y^{-1}(H_Y)) > \operatorname{codim} H_Z.$$

We know that:

$$\begin{aligned}\dim \rho_Z(\rho_X^{-1}(H_X) \cap \rho_Y^{-1}(H_Y)) = &\dim\left(\rho_X^{-1}(H_X) \cap \rho_Y^{-1}(H_Y)\right) \\ &- \dim\left(K_{X+Y} \cap \rho_X^{-1}(H_X) \cap \rho_Y^{-1}(H_Y)\right) \\ = &\, r - k + 1 \\ &- \dim\left(K_{X+Y} \cap \rho_X^{-1}(H_X) \cap \rho_Y^{-1}(H_Y)\right).\end{aligned}$$

Again, we will analyze several cases.

Case 1: $\rho_X(K_{X+Y}) \cap H_X = 0$ and $\rho_Y(K_{X+Y}) \cap H_Y = 0$.

This is equivalent to:

$$K_{X+Y} \cap \rho_X^{-1}(H_X) = K_Y \text{ and } K_{X+Y} \cap \rho_Y^{-1}(H_Y) = K_X.$$

So we get:

$$K_{X+Y} \cap \rho_X^{-1}(H_X) \cap \rho_Y^{-1}(H_Y) \subset K_X \cap K_Y = 0.$$

And then:

$$\dim \rho_Z(\rho_X^{-1}(H_X) \cap \rho_Y^{-1}(H_Y)) = r - k + 1 > \operatorname{codim} H_Z.$$

Case 2: $\rho_X(K_{X+Y}) \cap H_X \neq 0$.

This implies

$$\dim K_{X+Y} \cap \rho_X^{-1}(H_X) = p_{X+Y} - (r-i).$$

This case splits out in some subcases:

Subcase 2.1: $\rho_Y(K_{X+Y} \cap \rho_X^{-1}(H_X)) \cap H_Y = 0$. Then:

$$\begin{aligned} K_{X+Y} \cap \rho_X^{-1}(H_X) \cap \rho_Y^{-1}(H_Y) =& K_{X+Y} \cap \rho_X^{-1}(H_X) \cap K_{X+Z} \\ =& K_X \cap \rho_X^{-1}(H_X). \end{aligned}$$

Subcase 2.1 (a): $\rho_X(K_X) \cap H_X = 0$.

Then $K_X \cap \rho_X^{-1}(H_X) = 0$, and so:

$$\dim \rho_Z(\rho_X^{-1}(H_X) \cap \rho_Y^{-1}(H_Y)) = r - k + 1 > \text{codim } H_Z.$$

Subcase 2.1 (b): $\rho_X(K_X) \cap H_X \neq 0$.

This implies:

$$\dim \left(K_X \cap \rho_X^{-1}(H_X)\right) = p_X - (r-i).$$

And then:

$$\begin{aligned} \dim \rho_Z(\rho_X^{-1}(H_X) \cap \rho_Y^{-1}(H_Y)) =& r - k + 1 - p_X + r - i \\ \geq& r - k + 1 - (r-i) + r - i \\ =& r - k + 1 \\ >& \text{ codim } H_Z. \end{aligned}$$

Subcase 2.2: $\rho_Y(K_{X+Y} \cap \rho_X^{-1}(H_X)) \cap H_Y \neq 0$.

In this case:

$$\dim \left(K_{X+Y} \cap \rho_X^{-1}(H_X) \cap \rho_Y^{-1}(H_Y)\right) = p_{X+Y} - (r-i) - (r-j) = p_{X+Y} - k.$$

Then:

$$\begin{aligned}\dim \rho_Z(\rho_X^{-1}(H_X) \cap \rho_Y^{-1}(H_Y)) =& r - k + 1 - (p_{X+Y} - k) \\ =& r + 1 - p_{X+Y} \\ \geq& r + 1 - k \\ >& \text{ codim } H_Z.\end{aligned}$$

Case 3: $\rho_Y(K_{X+Y}) \cap H_Y \neq 0$.

It is analogous to Case 2.

That finishes the proof that $\dim \rho_Z(\rho_X^{-1}(H_X) \cap \rho_Y^{-1}(H_Y)) >$ codim H_Z. So, the intersection:

$$\rho_X^{-1}(H_X) \cap \rho_Y^{-1}(H_Y) \cap \rho_Z^{-1}(H_Z)$$

is transversal. Its codimension is $(r - i) + (r - j) + (r - k) = r$. Hence, it has dimension 1.

For the converse, if we suppose the monomial $h_0^i h_1^j h_2^k$ appears in the expression of $\left[\mathbb{P}(\mathfrak{g})_{\underline{d}}\right]$, then, for a general subspace $H_X \subset V_{(d,0)}$ of dimension $i+1$, we must have:

$$\dim \operatorname{Im}(\rho_X) + \dim H_X \geq r + 2$$

and therefore

$$p_{Y+Z}^d \leq i.$$

Analogous arguments prove that $p_{X+Z}^d \leq j$ and $p_{X+Y}^d \leq k$.

We must also have $\dim \rho_Y(\rho_X^{-1}(H_X)) >$ codim H_Y for a general $H_Y \subset V_{(0,0)}$. Since we have already proven that $p_{Y+Z}^d \leq i$, the intersection

$$\operatorname{Im}(\rho_X) \cap H_X$$

is transversal, so $\dim \rho_X^{-1}(H_X) = i + 1$. Therefore,

$$\dim \rho_Y(\rho_X^{-1}(H_X)) = i + 1 - \dim(\rho_X^{-1}(H_X) \cap K_{X+Z}).$$

Here we have two cases, as we have already seen in the first part of the proof. The first case is when $\rho_X(K_{X+Z}) \cap H_X = 0$, which is equivalent to $K_{X+Z} \cap \rho_X^{-1}(H_X) = K_Z$. Then we have

$$\dim \rho_Y(\rho_X^{-1}(H_X)) = i + 1 - p_Z > \text{codim } H_Y = r - j,$$

and therefore, since $i + j + k = 2r$:

$$p_Z \leq r - k.$$

The second case is when $\rho_X(K_{X+Z}) \cap H_X \neq 0$. Since H_X is general, this is only possible if

$$\dim \rho_X(K_{X+Z}) > \text{codim } H_X$$

By the rank nullity theorem,

$$\dim \rho_X(K_{X+Z}) = p_{X+Z} - \dim \operatorname{Ker}\left(\rho_X|_{K_{X+Z}}\right)$$

But $\operatorname{Ker}\left(\rho_X|_{K_{X+Z}}\right) = K_Z$. Hence:

$$p_{X+Z} - p_Z = \dim \rho_X(K_{X+Z}) > \text{codim } H_X = r - i.$$

Finally,

$$p_Z < p_{X+Z} - r + i \leq i + j - r = r - k.$$

Analogous arguments prove that $p_X \leq r - i$ and $p_Y \leq r - j$. □

Corollary 6.1. Assuming the setup of the theorem:

(a) If $p_X^{\underline{d}} + p_Y^{\underline{d}} + p_Z^{\underline{d}} = r + 1$, then $\left[\mathbb{P}(\mathfrak{g})_{\underline{d}}\right] = 0$.

(b) If $p_{X+Y}^{\underline{d}} + p_Z^{\underline{d}} = r + 1$, then $\left[\mathbb{P}(\mathfrak{g})_{\underline{d}}\right] = 0$.

(c) If the monomial $h_0^i h_1^j h_2^k$ does not appear in $\left[\mathbb{P}(\mathfrak{g})_{\underline{d}}\right]$, then it does not appear in $\left[\mathbb{P}(\mathfrak{g})_{\underline{e}}\right]$ for some $\underline{e} \in N(\underline{d})$.

(d) If the monomial $h_0^i h_1^j h_2^k$ appears in $\left[\mathbb{P}(\mathfrak{g})_{\underline{d}}\right]$, then it does not appear in $\left[\mathbb{P}(\mathfrak{g})_{\underline{e}}\right]$ for any $\underline{e} \in N(\underline{d})$.

Proof. If we keep in mind that $i+j+k = 2r$, then from Theorem 6.1, if $h_0^i h_1^j h_2^k$ appears in the expression of $[\mathbb{P}(\mathfrak{g})_{\underline{d}}]$, we get

$$p_X + p_Y + p_Z \leq 3r - (i + j + k) = r.$$

So if $p_X + p_Y + p_Z = r + 1$, then $\left[\mathbb{P}(\mathfrak{g})_{\underline{d}}\right] = 0$.

Since $p_{X+Y} \leq k \leq r - p_Z$, we have $\left[\mathbb{P}(\mathfrak{g})_{\underline{d}}\right] = 0$ if $p_{X+Y} + p_Z = r + 1$.

Now, recall Lemma 4.1 on Page 56. By its item (f), if $p_X^{\underline{d}} \geq r + 1 - k$ then $p_X^{\underline{e}} \geq r+1-k$, for $\underline{e} \in \{Y \cdot \underline{d}, Z \cdot \underline{d}, (Y+Z) \cdot \underline{d}\}$. And by item (g), if $p_{X+Y}^{\underline{d}} \geq i+1$

then $p^{\underline{e}}_{X+Y} \geq i+1$, for $\underline{e} = Z \cdot \underline{d}$. Hence, if the term $h_0^i h_1^j h_2^k$ does not appear in $\left[\mathbb{P}(\mathfrak{g})_{\underline{d}}\right]$ then it does not appear in some "neighboring" $\left[\mathbb{P}(\mathfrak{g})_{\underline{e}}\right]$.

By Lemma 4.1, item (h), $p^{\underline{d}}_X \leq r - i$ if and only if $p^{X \cdot \underline{d}}_{Y+Z} \geq i+1$. Also, $p^{\underline{d}}_{Y+Z} \leq i$ if and only if $p^{(Y+Z)\cdot \underline{d}}_X \geq r - i + 1$. Therefore, if the term $h_0^i h_1^j h_2^k$ appears in $\left[\mathbb{P}(\mathfrak{g})_{\underline{d}}\right]$ then it does not appear in any neighboring $\left[\mathbb{P}(\mathfrak{g})_{\underline{e}}\right]$. □

We have now a practical algorithm to compute the Chow class of $\mathbb{P}(\mathfrak{g})$. Let's see how this works with an example.

Example 6.1. Let's consider the linked net $\mathfrak{g}$ of vector spaces of dimension 3 of Example 1 on Table 8.2 on Page 123. See Section 8.1 for the notation. The vertices of the $\mathbb{Z}^2$-quiver are the triples $\underline{d} = (i, j, k)$ of integers with $i+j+k = 3$. The net $\mathfrak{g}$ is supported on the set of effective $\underline{d} = (i, j, k)$, that is, for $i, j, k \geq 0$. Also, the bounds are $w_0 = (3, 0, 0)$, $w_1 = (0, 3, 0)$ and $w_2 = (0, 0, 3)$.

In light of Corollary 6.1, the only indices $\underline{d}$ that contribute for nonzero classes are $(0, 2, 1), (0, 3, 0), (1, 0, 2), (1, 2, 0)$ and $(2, 1, 0)$. Let's see the contribution of each of them.

For $\underline{d} = (0, 2, 1)$, $\mathbb{P}(\mathfrak{g})_{\underline{d}}$ contributes to the class of $\mathbb{P}(\mathfrak{g})$ with the monomial $h_0^i h_1^j h_2^k$ if, and only if:

$$\begin{aligned} 2 \leq i \leq 2, \\ 1 \leq j \leq 1, \\ 1 \leq k \leq 1. \end{aligned}$$

The only possible monomial is $h_0^2 h_1 h_2$.

For $\underline{d} = (0, 3, 0)$, $\mathbb{P}(\mathfrak{g})_{\underline{d}}$ contributes to the class of $\mathbb{P}(\mathfrak{g})$ with the monomial $h_0^i h_1^j h_2^k$ if, and only if:

$$\begin{aligned} 2 \leq i \leq 2, \\ 0 \leq j \leq 1, \\ 2 \leq k \leq 2. \end{aligned}$$

The only possible monomial is $h_0^2 h_2^2$.

For $\underline{d} = (1, 0, 2)$, $\mathbb{P}(\mathfrak{g})_{\underline{d}}$ contributes to the class of $\mathbb{P}(\mathfrak{g})$ with the monomial

$h_0^i h_1^j h_2^k$ if, and only if:

$$
\begin{aligned}
1 \leq i \leq 2, \\
1 \leq j \leq 2, \\
0 \leq k \leq 1.
\end{aligned}
$$

The only possible monomials are $h_0 h_1^2 h_2$, $h_0^2 h_1 h_2$ and $h_0^2 h_1^2$.

For $\underline{d} = (1, 2, 0)$, $\mathbb{P}(\mathfrak{g})_{\underline{d}}$ contributes to the class of $\mathbb{P}(\mathfrak{g})$ with the monomial $h_0^i h_1^j h_2^k$ if, and only if:

$$
\begin{aligned}
1 \leq i \leq 1, \\
1 \leq j \leq 2, \\
2 \leq k \leq 2.
\end{aligned}
$$

The only possible monomial is $h_0 h_1 h_2^2$.

For $\underline{d} = (2, 1, 0)$, $\mathbb{P}(\mathfrak{g})_{\underline{d}}$ contributes to the class of $\mathbb{P}(\mathfrak{g})$ with the monomial $h_0^i h_1^j h_2^k$ if, and only if:

$$
\begin{aligned}
0 \leq i \leq 0, \\
2 \leq j \leq 2, \\
2 \leq k \leq 2.
\end{aligned}
$$

The only possible monomial is $h_1^2 h_2^2$.

The Chow class of $\mathbb{P}(\mathfrak{g})$ is the sum of all the monomials that appear in each $\mathbb{P}(\mathfrak{g})_{\underline{d}}$. In this case,

$$
\begin{aligned}
[\mathbb{P}(\mathfrak{g})] &= \left[\mathbb{P}(\mathfrak{g})_{(0,2,1)}\right] + \left[\mathbb{P}(\mathfrak{g})_{(0,3,0)}\right] + \left[\mathbb{P}(\mathfrak{g})_{(1,0,2)}\right] + \left[\mathbb{P}(\mathfrak{g})_{1,2,0)}\right] + \left[\mathbb{P}(\mathfrak{g})_{(2,1,0)}\right] \\
&= h_0^2 h_1 h_2 + h_0^2 h_2^2 + h_0 h_1^2 h_2 + h_0^2 h_1 h_2 + h_0^2 h_1^2 + h_0 h_1 h_2^2 + h_1^2 h_2^2.
\end{aligned}
$$

It is precisely the class of the diagonal.

We observe that, in the above example, no monomial appeared in the expressions of the classes of two different $\mathbb{P}(\mathfrak{g})_{\underline{d}}$ and $\mathbb{P}(\mathfrak{g})_{\underline{e}}$. In fact, this is always true under the hypothesis of a simple basis. To prove this, we will need the next lemma:

Lemma 6.1. Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver Q. Assume $\mathfrak{g}$ admits a simple basis. Let γ be a nontrivial admissible path in Q, and u and

v its initial and terminal vertices. Let $m := \max(\gamma(i))$ and $I_0 := \{i \mid \gamma(i) < m\}$. Then, for each $I \supseteq I_0$,

$$p_I^v \geq p_I^u.$$

Proof. Let $w_1, \ldots, w_r$ be vertices of Q and $s_i \in V_{w_i}$ for $i = 1, \ldots, r$ forming a simple basis for $\mathfrak{g}$. To prove the statement of the lemma, it is enough to prove that if

$$\varphi_I^v\left(s_{i|_{V_v}}\right) \neq 0 \tag{6.4}$$

then

$$\varphi_I^u\left(s_{i|_{V_u}}\right) \neq 0.$$

Assume (6.4). Let μ be a path leaving v such that $\mu(i) + \gamma(i) = m$ for $i = 0, \ldots, n$. Then μ is admissible and its final point is u. Also, $I_0 = \{i \mid \mu(i) > 0\}$. Since $I \supseteq I_0$, we have that $\mathrm{Ker}(\varphi_\mu) = K_{I_0}^v \subseteq K_I^v$. It thus follows from (6.4) that $\varphi_\mu(s_{i|_{V_v}}) \neq 0$, and hence

$$\varphi_\mu\left(s_{i|_{V_v}}\right) = s_{i|_{V_u}}.$$

Set $z := I(v)$ and let w be the endpoint of a path ν leaving z such that $\nu(i) = \mu(i)$ for each $i = 0, \ldots, n$. Then

$$\begin{aligned}\varphi_I^u\left(s_{i|_{V_u}}\right) &= \varphi_I^u\left(\varphi_\mu\left(s_{i|_{V_v}}\right)\right)\\ &= \varphi_\nu\left(\varphi_I^v\left(s_{i|_{V_v}}\right)\right) \neq 0,\end{aligned}$$

where the last equality follows from the inequality, and the inequality from the fact that $I \supseteq I_0$, and thus

$$\mathrm{Ker}\left(\varphi_\nu \varphi_I^v\right) = \mathrm{Ker}\left(\varphi_I^v\right)$$

by Lemma 3.2. ∎

Proposition 6.2. Let $\mathfrak{g}$ be a linked net of vector spaces over the standard $\mathbb{Z}^2$-quiver with finite support on $\mathbb{N}^2(\leq d)$. Assume $\mathfrak{g}$ admits a simple basis. Then, if a monomial $h_0^i h_1^j h_2^k$ appears in the class $\left[\mathbb{P}(\mathfrak{g})_{\underline{d}}\right]$ for some $\underline{d} \in \mathbb{N}^2(\leq d)$, it does not appear in the class $\left[\mathbb{P}(\mathfrak{g})_{\underline{e}}\right]$ for any $\underline{e} \in \mathbb{N}^2(\leq d)$ distinct from $\underline{d}$.

Proof. In light of Theorem 6.1, we have to show that if the $p_I^{\underline{d}}$ satisfy all numerical conditions required in the theorem for some $\underline{d} \in \mathbb{N}^2(\leq d)$, then for every other $\underline{e} \neq \underline{d}$, at least one of the p_I^e does not satisfy the corresponding numerical conditions.

Write $\underline{d} = (x, y)$. It is marked as the blue point in Figure 6.1. Let $r+1$ be the dimension of $\mathfrak{g}$

Using Lemma 4.1, item (h), the first and second column of inequalities in Theorem 6.1 imply the second and first column of inequalities below:

$$\begin{aligned} p_{Y+Z}^{(x-1,y)} &\geq i+1, & p_X^{(x+1,y)} &\geq r-i+1, \\ p_{X+Z}^{(x+1,y+1)} &\geq j+1, & p_Y^{(x-1,y-1)} &\geq r-j+1, \\ p_{X+Y}^{(x,y-1)} &\geq k+1, & p_Z^{(x,y+1)} &\geq r-k+1. \end{aligned}$$

Then, by using Lemma 6.1, we conclude that all the vertices $\underline{e} \neq \underline{d}$ do not satisfy the numerical conditions of Theorem 6.1, as highlighted in Figure 6.1.

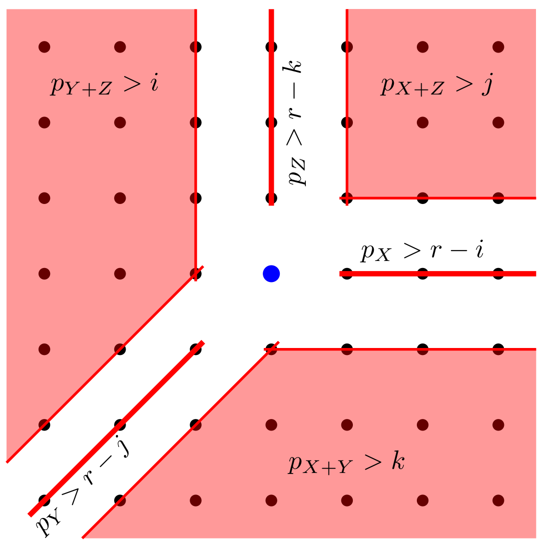

Figure 6.1: Proof of Proposition 6.2

□

The next theorem is a direct consequence of the previous results:

Theorem 6.2. Let $\mathfrak{g}$ be a linked net of vector spaces of dimension $r+1$ with finite support over a $\mathbb{Z}^2$-quiver. If $\mathfrak{g}$ admits a simple basis, then

$$[\mathbb{P}(\mathfrak{g})] = \sum_{i+j+k=2r} \delta_{ijk} h_0^i h_1^j h_2^k$$

where $\delta_{ijk} \in \{0,1\}$.

But to conclude that the scheme $\mathbb{P}(\mathfrak{g})$ has the same class as the diagonal, for $\mathfrak{g}$ admitting a simple basis, we must prove that every possible monomial $h_0^i h_1^j h_2^k$ with $i+j+k=2r$ actually appears in the expression of $[\mathbb{P}(\mathfrak{g})_v]$ for some vertex v in the support of $\mathfrak{g}$. This is still a work in progress.

The example discussed in Section 8.2 on Page 127 shows us that there exist exact linked nets of vectors spaces $\mathfrak{g}$ with no simple basis and with Chow class different from that of the diagonal. However, we conjecture that, if $\mathfrak{g}$ admits a simple basis, then $\mathbb{P}(\mathfrak{g})$ has the same Chow class as that of the diagonal.

Chapter 7

Linked projective spaces

This chapter is dedicated to the study of $\mathbb{LP}(\mathfrak{g})$, where $\mathfrak{g}$ is a linked net of vector spaces over a $\mathbb{Z}^2$-quiver.

We determine the structure of $\mathbb{LP}(\mathfrak{g})$, in the same spirit of the work by Santana in [2]. In summary, in Theorem 7.1 we prove that the exact points of $\mathbb{LP}(\mathfrak{g})$ form an open set $\mathbb{LP}(\mathfrak{g})^*$, which is equal to the nonsingular locus. Therefore, the non-exact points are all singular and we know how to classify them: they are precisely the five types described in Chapter 4 (see Theorem 4.3).

In the third section we investigate the Hilbert polynomial of $\mathbb{LP}(\mathfrak{g})$. We conjecture it is the same as that of the diagonal. We prove this fact for linked nets of dimension 2, using a characterization by Cartwright and Sturmfels [8].

7.1 $\mathbb{LP}(\mathfrak{g})$

Let $\mathfrak{g}$ be a linked net of vector spaces of dimension $r+1$ over a $\mathbb{Z}^n$-quiver Q. Assume $\mathfrak{g}$ has finite support. Let H be a finite set of vertices of Q supporting $\mathfrak{g}$. Define $\mathbb{LP}(\mathfrak{g})_H$ as the quiver Grassmannian of pure dimension 1 subrepresentations of the representation restricted to the full subquiver of Q supported on H. Thus:

$$\mathbb{LP}(\mathfrak{g})_H = \left\{ ([s_v] \mid v \in H) \in \prod_{v \in H} \mathbb{P}(V_v) \mid \varphi_w^v(s_v) \wedge s_w = 0 \text{ for all } v, w \in H \right\}.$$

Observe that there is a natural injection between the quiver Grassmannian $\mathbb{LP}(\mathfrak{g})$ of pure dimension 1 and subrepresentations of the representation $\mathfrak{g}$ over the whole quiver Q and $\mathbb{LP}(\mathfrak{g})_H$, induced by restriction.

Proposition 7.1. If $H = P(H)$ the restriction map $\mathbb{LP}(\mathfrak{g}) \to \mathbb{LP}(\mathfrak{g})_H$ is bijective.

Proof. Indeed, suppose $P(H) = H$. Then, for each vertex v of Q there is a unique $w_v \in H$ such that for each $z \in H$ there is an admissible path connecting z to v through w_v (see Proposition 3.5). Notice that $w_v = v$ if $v \in H$. Since $\mathfrak{g}$ is supported in H there is $z \in H$ such that φ_v^z is an isomorphism. Since

$$\varphi_v^z = \varphi_v^{w_v} \varphi_{w_v}^z,$$

we have that $\varphi_v^{w_v}$ is an isomorphism.

Given $\mathfrak{I} := ([s_v] \,|\, v \in H) \in \mathbb{LP}(\mathfrak{g})_H$, for each vertex v of Q let $t_v := \varphi_v^{w_v}(s_{w_v})$. Since $s_{w_v} \neq 0$ and $\varphi_v^{w_v}$ is an isomorphism, also $t_v \neq 0$ for every vertex v of Q. Furthermore, the t_v define a point of the quiver Grassmannian of dimension 1 subspaces of the representation $\mathfrak{g}$ over the whole Q. In other words, they satisfy the equations

$$\varphi_v^u(t_u) \wedge t_v = 0 \quad \text{for each two vertices } u \text{ and } v \text{ of } Q,$$

The above equations are clearly satisfied if $u, v \in H$ and if $u = w_v$ for each vertex v of Q. We will see that these equations imply the remaining ones.

Indeed, if $u \in H$ but $v \notin H$, there is an admissible path connecting u to v passing through w_u, whence $\varphi_v^u = \varphi_v^{w_v} \varphi_{w_v}^u$. Applying $\varphi_v^{w_v}$ to the equation $\varphi_{w_v}^u(t_u) \wedge t_{w_v} = 0$ we get $\varphi_v^u(t_u) \wedge \varphi_{w_v}^v(t_{w_v}) = 0$. But, since $t_v \wedge \varphi_v^{w_v}(t_{w_v}) = 0$, and since $t_{w_v} \neq 0$ and $\varphi_v^{w_v}$ is an isomorphism, it follows that $\varphi_v^u(t_u) \wedge t_v = 0$.

In addition, if $u, v \notin H$, there is an admissible path μ connecting w_u to v passing through w_v, whence $\varphi_v^{w_u} = \varphi_v^{w_v} \varphi_{w_v}^{w_u}$. If φ_v^u is nonzero, so is the composition $\varphi_v^u \varphi_u^{w_u}$, and hence there is an admissible path ν connecting w_u to v passing through u. As $\varphi_\mu = \varphi_\nu$, we have that

$$\varphi_v^{w_v} \varphi_{w_v}^{w_u} = \varphi_v^u \varphi_u^{w_u}.$$

Apply $\varphi_v^{w_v}$ to the equation $\varphi_{w_v}^{w_u}(t_{w_u}) \wedge t_{w_v} = 0$ to obtain

$$\varphi_v^u(\varphi_u^{w_u}(t_{w_u})) \wedge \varphi_v^{w_v}(t_{w_v}) = 0. \tag{7.1}$$

Since

$$\varphi_u^{w_u}(t_{w_u}) \wedge t_u = 0 \quad \text{and} \quad \varphi_v^{w_v}(t_{w_v}) \wedge t_v = 0,$$

and since t_{w_u} and t_{w_v} are nonzero, and $\varphi_u^{w_u}$ and $\varphi_v^{w_v}$ are isomorphisms, Equation (7.1) implies $\varphi_v^u(t_u) \wedge t_v = 0$. □

If $\mathfrak{g}$ has finite support, then there is a finite set of vertices H supporting $\mathfrak{g}$ with $P(H) = H$; use Proposition 3.4. The set H is not unique but even though the structure of $\mathbb{LP}(\mathfrak{g})_H$ depends on the choice of H, many of its properties do not, for instance, the intrinsic structure, at least if $P(H) = H$.

Indeed, let H' be a larger subset of vertices of Q satisfying $P(H') = H'$. Consider the map

$$\Psi\colon \prod_{v\in H} \mathbb{P}(V_v) \longrightarrow \prod_{v\in H'-H} \mathbb{P}(V_v)$$

taking $([s_v] \,|\, v \in H)$ to $([t_v] \,|\, v \in H')$ where $t_v := \varphi_v^{w_v}(s_{w_v})$ for each $v \in H'$. Then, by what we have just proved above $(1, \Psi)$ restricts to an isomorphism from $\mathbb{LP}(\mathfrak{g})_H$ to $\mathbb{LP}(\mathfrak{g})_{H'}$.

Thus, not only are $\mathbb{LP}(\mathfrak{g})_H$ and $\mathbb{LP}(\mathfrak{g})_{H'}$ isomorphic as abstract schemes, but also the multivariate Hilbert polynomial of the latter, $\mathrm{Hilb}_{\mathbb{LP}(\mathfrak{g})_{H'}}(n_v \,|\, v \in H')$, is obtained from that of the former, $\mathrm{Hilb}_{\mathbb{LP}(\mathfrak{g})_H}(n_u \,|\, u \in H)$, by replacing each n_u for $u \in H$ by the sum of the n_v for all $v \in H'$ such that $w_v = u$. Thus, the multivariate Hilbert polynomial of $\mathbb{LP}(\mathfrak{g})_H$ is that of the diagonal if and only if so is the multivariate Hilbert polynomial of $\mathbb{LP}(\mathfrak{g})_{H'}$.

With all of the above statements in mind, for what we will do here, we will simplify the notation and identify $\mathbb{LP}(\mathfrak{g})$ with $\mathbb{LP}(\mathfrak{g})_H$ for each finite set H of vertices supporting $\mathfrak{g}$ with $P(H) = H$.

We will often describe a point of $\mathbb{LP}(\mathfrak{g})$ as a collection of subspaces I_v of dimension 1 of the V_v, instead of the $[s_v]$, the relation being that I_v is generated by s_v for each v. We will also view a point on $\mathbb{LP}(\mathfrak{g})$ as a linked net of one-dimensional vector spaces over Q, thus attributing to the point properties we attribute to those nets.

Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver $Q = (G, A)$. For each vertex v of Q define

$$\mathbb{LP}(\mathfrak{g})_v^* = \{(I_w \,|\, w \in G) \in \mathbb{LP}(\mathfrak{g}) \mid \varphi_w^v(I_v) = I_w\}.$$

Thus $\mathbb{LP}(\mathfrak{g})_v^*$ is the set of one-dimensional linked subnets of $\mathfrak{g}$ which have a simple basis at v. They are all exact. And in fact, it follows from Theorem 3.1

that, if $\mathfrak{g}$ has finite support, then all exact points of $\mathbb{LP}(\mathfrak{g})$ are of this form.

If $\mathfrak{g}$ is supported in a subset $H \subseteq G$ then $\mathbb{LP}(\mathfrak{g})^*_v$ is empty if $v \notin H$. Indeed, if $v \notin H$ then there is $z \in H$ such that φ^z_v is an isomorphism; but then $\varphi^v_z = 0$. Also, $\varphi^v_w(I_v) = I_w$ for every $w \in G$ if and only if $\varphi^v_w(I_v) = I_w$ for every $w \in H$. Indeed, if the latter holds then for each $w \in G$ there is $z \in H$ such that φ^z_w is an isomorphism and thus $\varphi^v_z(I_v) = I_z$ and $\varphi^z_w(I_z) = I_w$; but then $\varphi^z_w\varphi^v_z(I_v) = I_w$ and thus $\varphi^v_w = \varphi^z_w\varphi^v_z$ and $\varphi^v_w(I_v) = I_w$. Thus, if $\mathfrak{g}$ has finite support then $\mathbb{LP}(\mathfrak{g})^*_v$ is a Zariski open subset of $\mathbb{LP}(\mathfrak{g})$. Define

$$\mathbb{LP}(\mathfrak{g})_v := \overline{\mathbb{LP}(\mathfrak{g})^*_v},$$

the Zariski closure (in $\mathbb{LP}(\mathfrak{g})$).

Proposition 7.2. Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver $Q = (G, A)$. Let $H \subseteq G$ be a set of vertices supporting $\mathfrak{g}$. Let $v \in G$. Then the following three statements are equivalent:

1. $\mathbb{LP}(\mathfrak{g})^*_v$ is non-empty.

2. φ^v_w is nonzero for each $w \in G$.

3. $v \in H$ and φ^v_w is nonzero for each $w \in H$

Furthermore, if $\mathfrak{g}$ has finite support and $\mathbb{LP}(\mathfrak{g})^*_v$ is non-empty then $\mathbb{LP}(\mathfrak{g})^*_v$ is nonsingular, irreducible and there is a birational map

$$\mathbb{P}(V_v) \dashrightarrow \mathbb{LP}(\mathfrak{g})_v$$

whose composition with the projection $\mathbb{LP}(\mathfrak{g})_v \to \mathbb{P}(V_w)$ is the rational map $\mathbb{P}(V_v) \dashrightarrow \mathbb{P}(V_w)$ induced by φ^v_w for each $w \in G$.

Proof. Statement 1 clearly implies Statement 2. Conversely, assume φ^v_w is nonzero, or equivalently, $K^v_w \neq V_v$ for each $w \in G$. It follows from Lemma 3.2 that $K^v_w = K^v_z$ for a neighbor z of v. As v has finitely many neighbors,

$$U := \mathbb{P}(V_v) - \bigcup_{w \in G} \mathbb{P}(K^v_w)$$

is non-empty. Let $I_v \subseteq V_v$ be a subspace parameterized by U. Then $I_w := \varphi^v_w(I_v)$ is nonzero for each $w \in G$. For each $w, z \in G$ we have that $\varphi^w_z(I_w) = \varphi^w_z\varphi^v_w(I_v)$, which is either equal to $\varphi^v_z(I_v)$, if there is an admissible path from

v to z through w, or zero, in any case contained in I_z. Thus the I_w define an element $\mathfrak{I} \in \mathbb{LP}(\mathfrak{g})^*_v$. We have just defined a map $U \to \mathbb{LP}(\mathfrak{g})^*_v$, which is clearly a bijection. In particular, $\mathbb{LP}(\mathfrak{g})^*_v$ is non-empty.

We have already proved that Statement 1 implies $v \in H$, whence Statements 1 and 2 imply Statement 3. And we have also seen that Statement 3 implies Statement 2.

As for the additional statement, we may assume H is finite and $P(H) = H$. The rational map $\mathbb{P}(V_v) \dashrightarrow \mathbb{LP}(\mathfrak{g})_v$ is that defined on the open subscheme $U \subseteq \mathbb{P}(V_v)$ above, clearly a morphism. It has a natural inverse, induced by projection. Also, its composition with the projection $\mathbb{LP}(\mathfrak{g})_v \to \mathbb{P}(V_w)$ is the map $U \longrightarrow \mathbb{P}(V_w)$ induced by taking I_v to $I_w = \varphi^v_w(I_v)$. $\square$

Remark 7.1. If $\mathbb{LP}(\mathfrak{g})^*_v \cap \mathbb{LP}(\mathfrak{g})^*_w \neq \emptyset$ then $v = w$.

Let $\mathfrak{g}$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver Q. Assume that $\mathfrak{g}$ is exact. Given two neighboring vertices u and v of Q, we say that $\varphi^u_v \colon V_u \longrightarrow V_v$ is a *liaison map* if either $\mathrm{Ker}(\varphi^u_v) = 0$ or $\mathrm{Ker}(\varphi^u_v) = V_u$, in other words, if φ^u_v has maximum rank or rank zero. Observe that φ^u_v is liaison if and only if φ^v_u is liaison.

We extend liaison to a relation. Two (not necessarily neighboring) vertices u and v of Q are said to be $\mathfrak{g}$-*linked* or simply *linked* if and only if they can be connected by a sequence of liaison maps. In symbols:

$$\begin{aligned} u \sim v \iff & \text{There exist } w_0, \dots, w_m \text{ with } w_0 = u \text{ and } w_m = v \text{ such that} \\ & w_{i+1} \in N(w_i) \text{ and } \varphi^{w_i}_{w_{i+1}} \text{ is a liaison map } \forall\, i = 0, \dots, m-1. \end{aligned}$$

This is clearly an equivalence relation.

Given an equivalence class C of linked vertices, a *generator* is a vertex $u \in C$ such that for each $v \in C$ there exist vertices $w_0, \dots, w_m$ of Q with $w_0 = u$ and $w_m = v$ such that $\varphi^{w_i}_{w_{i+1}}$ has maximum rank for each $i = 0, \dots, m-1$. Clearly, the w_i must belong to C as well. Equivalently, $u \in C$ is a generator if and only if φ^u_v is an isomorphism for each $v \in C$.

Proposition 7.3. Let $\mathfrak{g}$ be an exact linked net of vector spaces over a $\mathbb{Z}^n$-quiver Q. Let u be a vertex of Q. Then $\mathbb{LP}(\mathfrak{g})^*_u$ is non-empty if and only if u is a generator of its liaison class. Furthermore, if $\mathfrak{g}$ has finite support and $\mathbb{LP}(\mathfrak{g})^*_u$ is non-empty then the natural projection $\mathbb{LP}(\mathfrak{g})_u \longrightarrow \mathbb{P}(V_v)$ is surjective if and only if u and v are in the same liaison class.

Proof. Let C be the liaison class of u. Suppose u is not a generator of it. Then there are $w \in C$ such that φ_w^u is not an isomorphism. Among them, consider those $w \in C$ and those sequences of vertices $z_0, \dots, z_m$ with $z_0 = u$ and $z_m = w$, and with $z_i \in N(z_{i-1})$ and $\varphi_{z_i}^{z_{i-1}}$ being a liaison map for $i = 1, \dots, m$, where the number of zero maps among the $\varphi_{z_i}^{z_{i-1}}$ is minimum, certainly positive because φ_w^u is not an isomorphism. For one of those w and one of those sequences z_i there is a minimum $p > 0$ such that $\varphi_{z_p}^{z_{p-1}}$ is zero. By minimality of v and the sequence (z_i), we have that $\varphi_{z_p}^u$ is not an isomorphism. Since $\mathfrak{g}$ is exact, $\varphi_{z_{p-1}}^{z_p}$ is an isomorphism. Since $\varphi_{z_{p-1}}^u$ is an isomorphism as well, it follows that there is no admissible path from u to z_{p-1} passing through z_p. Hence there is one admissible path from u to z_p through z_{p-1}, by Lemma 3.1. Thus $\varphi_{z_p}^u = \varphi_{z_p}^{z_{p-1}} \varphi_{z_{p-1}}^u = 0$, and hence $\mathbb{LP}(\mathfrak{g})_u^*$ is empty.

Assume from now on that u is a generator of its liaison class. Suppose $\mathbb{LP}(\mathfrak{g})_u^*$ is empty by contradiction. Then there is a vertex w of Q such that $\varphi_w^u = 0$. Then $\varphi_z^u = 0$ for a certain vertex z of Q connected to u by a simple admissible path, by Lemma 3.2. Thus u and z are neighbors and φ_z^u is liaison. It follows that $z \in C$, and since u generates C, we have that φ_z^u is an isomorphism, a contradiction.

Finally, assume that $\mathfrak{g}$ is supported on a finite set of vertices H with $P(H) = H$. Then $u \in H$. Let v be any other vertex of Q. Since $\mathbb{LP}(\mathfrak{g})_u$ is non-empty, by Proposition 7.2 there is a birational map $\mathbb{P}(V_u) \dashrightarrow \mathbb{LP}(\mathfrak{g})_u$, in particular dominant. Furthermore, the composition

$$\mathbb{P}(V_u) \dashrightarrow \mathbb{LP}(\mathfrak{g})_u \longrightarrow \mathbb{P}(V_v)$$

is the rational map $\mathbb{P}(V_u) \dashrightarrow \mathbb{P}(V_v)$ induced by $\varphi_v^u \colon V_u \longrightarrow V_v$. Since $\mathbb{LP}(\mathfrak{g})_u$ is a projective variety, the projection $\mathbb{LP}(\mathfrak{g})_u \longrightarrow \mathbb{P}(V_v)$ is surjective if and only if it is dominant, if and only if the composition $\mathbb{P}(V_u) \dashrightarrow \mathbb{P}(V_v)$ is dominant, if and only if φ_v^u is an isomorphism. This implies that $v \in C$. But if $v \in C$, since u generates C, we must have that φ_v^u is an isomorphism. Thus $\mathbb{LP}(\mathfrak{g})_u \longrightarrow \mathbb{P}(V_v)$ is surjective if and only if $v \in C$. $\square$

Proposition 7.4. Let $\mathfrak{g}$ be an exact linked net of vector spaces of finite support over a $\mathbb{Z}^n$-quiver. Then each liaison equivalence class has a generator.

Proof. Let C be a liaison equivalence class. Let $u \in C$. We have that u is a generator of C if and only if $\mathbb{LP}(\mathfrak{g})_u^*$ is non-empty by Proposition 7.3, whence if and only if φ_v^u is nonzero for each vertex v of Q by Proposition 7.2. Suppose u

does not generate C. Then $\varphi_v^u = 0$ for a certain vertex v of Q. We may assume v is a neighbor of u by Lemma 3.2. Then φ_u^v is an isomorphism because $\mathfrak{g}$ is exact and $v \in C$. Put $u_1 := v$ and repeat. Unless we obtain a generator of C we end up with an infinite sequence of vertices $u_0, u_1, u_2, \dots$ of C such that $\varphi_{u_{i+1}}^{u_i}$ is zero and $\varphi_{u_i}^{u_{i+1}}$ is an isomorphism for each i.

However, $\mathfrak{g}$ has support in a finite set of vertices H. We have $H \cap C \neq \emptyset$. Indeed, given $w \in C - H$ there is $z \in H$ such that φ_w^z is an isomorphism, and thus clearly $z \in C$. We could have thus started with $u \in C \cap H$. Furthermore, having chosen u_1 as above, there is $z_1 \in H$ such that $\varphi_{u_1}^{z_1}$ is an isomorphism and thus $\varphi_u^{z_1}$ is an isomorphism, the composition $\varphi_u^{u_1}\varphi_{u_1}^{z_1}$. In particular, $z_1 \in C$. Notice that $z_1 \neq u$ because $u_1 \neq u$. Thus $\varphi_{z_1}^u = 0$. We could have thus replaced u_1 by z_1, and assumed $u_1 \in C \cap H$. But H is finite and thus there are $i < j$ such that $u_i = u_j$. Since $\varphi_{u_{i+1}}^{u_i}$ is zero, $u_{i+1} \neq u_i$. Then the composition of isomorphisms $\varphi_{u_{j-1}}^{u_j} \cdots \varphi_{u_i}^{u_{i+1}}$ is the map associated to a non-admissible path and thus should be zero, an absurd. $\square$

In the next section we will show that the generator is unique if $n = 2$.

7.2 $\mathbb{LP}(\mathfrak{g})$ over $\mathbb{Z}^2$-quivers

In this section we will focus on $\mathbb{LP}(\mathfrak{g})$ over $\mathbb{Z}^2$-quivers.

We prove first that each liaison equivalence class has a unique generator. To achieve this, we need a lemma first.

For the statement of the next lemma and the proof of the following proposition we will assume that Q is the standard $\mathbb{Z}^2$-quiver and we will use the suggestive notation first employed in Chapter 4.

Lemma 7.1. Let $\mathfrak{g}$ an exact linked net of vector spaces over the standard $\mathbb{Z}^2$-quiver. Let $\underline{a}, \underline{b} \in \mathbb{Z}^2$ such that $\underline{b} = R \cdot \underline{a}$ for some $R \in \{X, Y, Z\}$. Let $\underline{c} := (R + S) \cdot \underline{a}$ and $\underline{d} := (R + S) \cdot \underline{b}$ for $S \in \{X, Y, Z\} - \{R\}$. If $\varphi_{\underline{b}}^{\underline{a}}$ is an isomorphism, then so is $\varphi_{\underline{d}}^{\underline{c}}$.

Proof. Let $s \in \ker(\varphi_R^{\underline{c}})$. Since $\mathfrak{g}$ is exact, there exist $s' \in V_{\underline{d}}$ such that $s = \varphi_{S+T}^{\underline{d}}(s')$, where $\{R, S, T\} = \{X, Y, Z\}$. Define $t := \varphi_T^{\underline{d}}(s') \in V_{\underline{b}}$. Then $\varphi_{S+T}^{\underline{b}}(t) = \varphi_T^{\underline{c}}(s)$. If $\varphi_{\underline{b}}^{\underline{a}}$ is an isomorphism then $\varphi_{S+T}^{\underline{b}}$ is the zero map. Thus $\varphi_{S+T}^{\underline{b}}(t) = 0$. Since $t = \varphi_T^{\underline{d}}(s')$, it follows that $\varphi_{S+T}^{\underline{d}}(s') = 0$, that is $s = 0$. As this is true for every s, we have that $\varphi_{\underline{d}}^{\underline{c}}$ is an isomorphism. $\square$

Proposition 7.5. Let $\mathfrak{g}$ be an exact linked net of vector spaces of finite support over a $\mathbb{Z}^2$-quiver. Then each liaison equivalence class has a unique generator.

Proof. We assume that Q is the standard $\mathbb{Z}^2$-quiver and use the suggestive notation first employed in Chapter 4.

Let C be a liaison equivalence class. That there is a generator for C follows from Proposition 7.4. Suppose there are two generators $\underline{a}, \underline{b}$ of C. Since $\mathbb{LP}(\mathfrak{g})_{\underline{a}}$ and $\mathbb{LP}(\mathfrak{g})_{\underline{b}}$ are both non-empty by Proposition 7.3, it follows from Proposition 7.2 that both $\underline{a}$ and $\underline{b}$ satisfy property (P): We will say that a vertex $\underline{v} \in \mathbb{Z}^2$ *satisfies property* (P) if $\varphi^{\underline{v}}_{\underline{w}}$ is nonzero for each $\underline{w} \in \mathbb{Z}^2$.

Let now $\underline{a}, \underline{b} \in \mathbb{Z}^2$ be two distinct vertices satisfying property (P). We will end the argument by proving that $\underline{a}$ and $\underline{b}$ can not be in the same liaison equivalence class, by proving they can not be connected by a sequence of liason maps. Without loss of generality, we may assume that $\underline{a} = (0,0)$ and $\underline{b} \in C_{Y+Z}(\underline{a})$. We may also assume $\underline{b} = (p, -q)$, for $p, q \geq 0$. Both $\underline{a}$ and $\underline{b}$ are represented as orange points in Figure 7.1.

Consider the following set of maps:

$$
\begin{aligned}
\varphi_Z \colon V_{(-i,0)} &\longrightarrow V_{(-i,-1)}, \quad \forall\, i \geq 0,\\
\varphi_{X+Z} \colon V_{(-i,0)} &\longrightarrow V_{(-i-1,-1)}, \quad \forall\, i \geq 0,\\
\varphi_{Y+Z} \colon V_{(i,i)} &\longrightarrow V_{(i+1,i)}, \quad \forall\, i \geq 0,\\
\varphi_Z \colon V_{(i,i)} &\longrightarrow V_{(i,i-1)}, \quad \forall\, i \geq 0.
\end{aligned}
$$

Suppose by contradiction that $\varphi_Z : V_{(-i,0)} \longrightarrow V_{(-i,-1)}$ is an isomorphism for some $i \geq 0$, represented by the filled blue arrow in Figure 7.1. Then $\varphi_Z^{(-i,-j)}$ is an isomorphism for all $j > 1$, represented by the dashed blue arrows; in particular, $\varphi_Z \colon V_{(-i,-q+1)} \longrightarrow V_{(-i,-q)}$ is an isomorphism. Applying repeatedly Lemma 7.1 with $R = Z$ and $S = Y$ we conclude that $\varphi_Z^{(k,-q+1)}$ is an isomorphism for all $k > -i$ (represented by the dashed cyan arrows). In particular, the map $\varphi_Z \colon V_{(p,-q+1)} \longrightarrow V_{(p,-q)}$ is an isomorphism. But this is a contradiction, because $\underline{b} = (p, -q)$ satisfies property (P). Hence, $\varphi_Z \colon V_{(-i,0)} \longrightarrow V_{(-i,-1)}$ can not be an isomorphism for any $i \geq 0$.

Using a similar argument, we conclude that $\varphi_Z \colon V_{(i,i)} \longrightarrow V_{(i,i-1)}$ can not be an isomorphism for any $i \geq 0$. Finally, if $\varphi_{X+Z} \colon V_{(-i,0)} \longrightarrow V_{(-i-1,-1)}$ is an isomorphism for some $i \geq 0$, then $\varphi_Z \colon V_{(-i-1,0)} \longrightarrow V_{(-i-1,-1)}$ is an isomorphism, which we have already proved not possible. The same goes for the map $\varphi_{Y+Z} \colon V_{(i,i)} \longrightarrow V_{(i+1,i)}$ if $i \geq 0$.

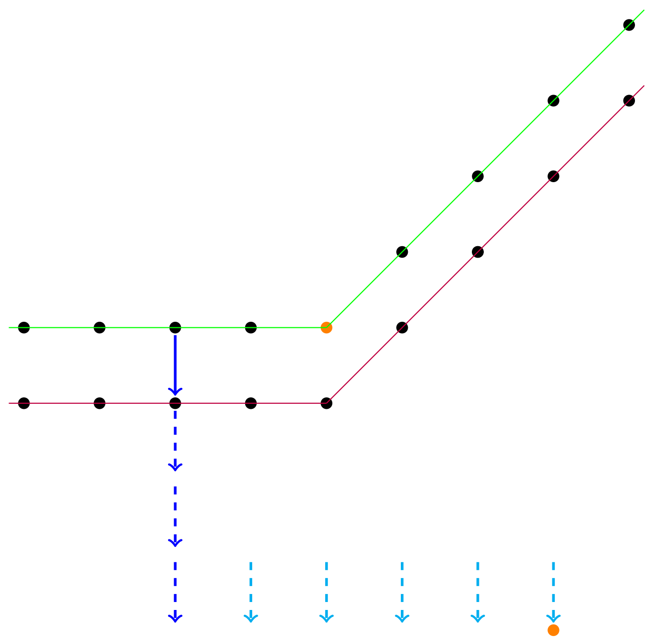

Figure 7.1: Proof of Proposition 7.5, no vertex along the green lines is in the same class of some vertex in the purple lines

So none of the maps in the set can be an isomorphism.

Using similar arguments, we can also prove that none of the maps in the set

$$\begin{aligned}
\varphi_{X+Y} \colon V_{(-i,-1)} &\longrightarrow V_{(-i,0)}, \quad \forall\, i \geq 0,\\
\varphi_{Y} \colon V_{(-i-1,-1)} &\longrightarrow V_{(-i,0)}, \quad \forall\, i \geq 0,\\
\varphi_{X} \colon V_{(i+1,i)} &\longrightarrow V_{(i,i)}, \quad \forall\, i \geq 0,\\
\varphi_{X+Y} \colon V_{(i,i-1)} &\longrightarrow V_{(i,i)}, \quad \forall\, i \geq 0
\end{aligned}$$

can be an isomorphism, as otherwise property (P) for $\underline{a}$ would be violated.

So there is no way $\underline{a}$ can be linked to $\underline{b}$. Indeed, $\underline{a}$ lies on the green broken line in Figure 7.1, whereas $\underline{b}$ lies on the purple broken line or below. A sequence $\underline{w}_0, \ldots, \underline{w}_m$ of vertices connecting $\underline{a}$ to $\underline{b}$ such that $\underline{w}_i$ is a neighbor of w_{i-1} for $i = 1, \ldots, m$ would have to feature w_{i-1} on the green broken line and w_i on the purple broken line for some $i > 0$. But we have seen that for every two such neighboring vertices, one on each broken line, none of the two maps given by $\mathfrak{g}$ can not be an isomorphism, so can not be a liaison map. $\square$

We will now describe the structure of the scheme $\mathbb{LP}(\mathfrak{g})$ over a $\mathbb{Z}^2$-quiver.

Proposition 7.6. Let $\mathfrak{g}$ be an exact linked net of dimension $r+1$ with finite support over a $\mathbb{Z}^2$-quiver. Let u, v, w be the vertices of a triangle. Then the following two statements hold:

1. The $\mathfrak{I} \in \mathbb{LP}(\mathfrak{g})$ which are minimally generated by the segment $\{u, v\}$ form a dense open subset of $\mathbb{LP}(\mathfrak{g})_u \cap \mathbb{LP}(\mathfrak{g})_v$.
2. The $\mathfrak{I} \in \mathbb{LP}(\mathfrak{g})$ which are minimally generated by the triangle $\{u, v, w\}$ form the whole $\mathbb{LP}(\mathfrak{g})_u \cap \mathbb{LP}(\mathfrak{g})_v \cap \mathbb{LP}(\mathfrak{g})_w$.

Proof. Let us first show that the $\mathfrak{I}$ with the property stated above form a subset of the given intersection in each case. Indeed, let $\mathfrak{I} = (I_v \,|\, v \in H) \in \mathbb{LP}(\mathfrak{g})$.

Suppose $\mathfrak{I}$ is minimally generated by $\{u, v\}$. Choose generators $x \in V_u$ and $y \in V_v$ for I_u and I_v. Because of the minimality, $\varphi_v^u(x) = 0$ and $\varphi_u^v(y) = 0$. Since $\mathfrak{g}$ is exact, and u and v are neighbors, there is $z \in V_v$ such that $x = \varphi_u^v(z)$. Now, for a general $t \in k$ we can define a new point $\mathfrak{I}^t := (I_a^t \,|\, a \in H) \in \mathbb{LP}(\mathfrak{g})$ by letting I_a^t be generated by $\varphi_a^v(y+tz)$ for each $a \in H$. Indeed, for all $t \in k$ we have $\varphi_u^v(y+tz) = tx$. Thus, since $\varphi_a^u(x)$ and $\varphi_a^v(y)$ generate a one-dimensional subspace of V_a for each $a \in H$, so does $\varphi_a^v(y+tz)$ for general t. Clearly, $\mathfrak{I}^t \in \mathbb{LP}(\mathfrak{g})_v^*$ for general t. Then $\mathfrak{I} \in \mathbb{LP}(\mathfrak{g})_v$. An analogous argument shows that $\mathfrak{I} \in \mathbb{LP}(\mathfrak{g})_u$ as well.

Suppose now that $\mathfrak{I}$ is minimally generated by $\{u, v, w\}$. Choose generators $x \in V_u$, $y \in V_v$ and $z \in V_w$ generating I_u, I_v and I_w. We may suppose that v is connected to u, that u is to w and w is to v by three distinct arrows. Because of the minimality, $\varphi_u^v(y) = 0$ and $\varphi_w^v(y) = 0$. Also, $\varphi_w^u(x) = 0$ and $\varphi_v^w(z) = 0$. Since v is neighbor to u and w, and $\mathfrak{g}$ is exact, there are thus $s_1, s_2 \in V_v$ such that $\varphi_u^v(s_1) = x$ and $\varphi_w^v(s_2) = z$. As before, for a general $t \in k$ we can define a new point $\mathfrak{I}^t := (I_a^t \,|\, a \in H) \in \mathbb{LP}(\mathfrak{g})$ by letting I_a^t be generated by $\varphi_a^v(y + ts_1 + t^2 s_2)$ for each $a \in H$. Indeed, $\varphi_u^v(y + ts_1 + t^2 s_2) = tx + t^2 \varphi_u^v(s_2)$ and

$$\varphi_w^v(y + ts_1 + t^2 s_2) = \varphi_w^u \varphi_u^v(y + ts_1 + t^2 s_2) = t^2 z.$$

Thus, since $\varphi_a^u(x)$, $\varphi_a^v(y)$ and $\varphi_a^w(z)$ generate a one-dimensional subspace of V_a for each $a \in H$, so does $\varphi_a^v(y + ts_1 + t^2 s_2)$ for general t. Clearly, $\mathfrak{I}^t \in \mathbb{LP}(\mathfrak{g})_v^*$ for general t. Then $\mathfrak{I} \in \mathbb{LP}(\mathfrak{g})_v$. An analogous argument shows that $\mathfrak{I} \in \mathbb{LP}(\mathfrak{g})_u \cap \mathbb{LP}(\mathfrak{g})_w$ as well.

Let $\mathfrak{I} = (I_v \,|\, v \in H) \in \mathbb{LP}(\mathfrak{g})$. It follows from Theorem 4.1 that $\mathfrak{I}$ is generated by a triangle $\{i, j, \ell\}$. If $\mathfrak{I} \in \mathbb{LP}(\mathfrak{g})_u^*$ then $\varphi_u^z(I_z) = 0$ for each $z \in H - \{u\}$. Thus, if $\mathfrak{I} \in \mathbb{LP}(\mathfrak{g})_u \cap \mathbb{LP}(\mathfrak{g})_v$, then $\varphi_u^z(I_z) = 0$ for each $z \in H - \{u\}$ and $\varphi_v^z(I_z) = 0$ for each $z \in H - \{v\}$. In particular, $\{u, v\} \subseteq \{i, j, \ell\}$.

As a consequence, if $\mathfrak{I} \in \mathbb{LP}(\mathfrak{g})_u \cap \mathbb{LP}(\mathfrak{g})_v \cap \mathbb{LP}(\mathfrak{g})_w$ then $\{u, v, w\} = \{i, j, \ell\}$, and hence $\mathfrak{I}$ is generated by $\{u, v, w\}$. It cannot be generated by $\{u, v\}$ as $\varphi_w^u(I_u) = \varphi_w^v(I_v) = 0$. Analogously, it cannot be generated by any segment in $\{u, v, w\}$. The proof of Statement 2 is complete.

If $\mathfrak{I} \in \mathbb{LP}(\mathfrak{g})_u \cap \mathbb{LP}(\mathfrak{g})_v$, then $\mathfrak{I}$ is minimally generated by the segment $\{u, v\}$ if and only if it is generated by the segment $\{u, v\}$, or equivalently, for each $a \in H$, we must have:

$$\varphi_a^u(I_u) \neq 0 \quad \text{ or } \quad \varphi_a^v(I_v) \neq 0 \tag{7.2}$$

an open condition. Indeed, clearly (7.2) is the condition that $\mathfrak{I}$ be generated by $\{u, v\}$. But $\mathfrak{I}$ cannot be generated by either u or v because $\varphi_u^v(I_v) = 0$ and $\varphi_v^u(I_u) = 0$.

It remains to show that those $\mathfrak{I}$ minimally generated by $\{u, v\}$ form a dense subset of $\mathbb{LP}(\mathfrak{g})_u \cap \mathbb{LP}(\mathfrak{g})_v$. Suppose thus that $\mathfrak{I} \in \mathbb{LP}(\mathfrak{g})_u \cap \mathbb{LP}(\mathfrak{g})_v$, and assume that $\mathfrak{I}$ is not minimally generated, thus not generated by $\{u, v\}$. We have seen that $\mathfrak{I}$ is generated by a triangle containing $\{u, v\}$, whence we may assume w is the other vertex of the triangle, with v connected to u, with u to w and w to v by three distinct arrows. Since $\varphi_u^w(I_w) = \varphi_u^v(I_v) = 0$, we have that $\mathfrak{I}$ is not generated by $\{v, w\}$. Similarly, $\mathfrak{I}$ is not generated by $\{u, w\}$. Thus $\mathfrak{I}$ is minimally generated by $\{u, v, w\}$, or $\mathfrak{I} \in \mathbb{LP}(\mathfrak{g})_u \cap \mathbb{LP}(\mathfrak{g})_v \cap \mathbb{LP}(\mathfrak{g})_w$ by Statement 2.

In particular, $\mathbb{LP}(\mathfrak{g})_u^*$ and $\mathbb{LP}(\mathfrak{g})_w^*$ are nonempty. Let

$$\mathfrak{J} = (J_a \,|\, a \in H) \in \mathbb{LP}(\mathfrak{g})_u^*, \quad \text{ and } \quad \mathfrak{L} = (L_a \,|\, a \in H) \in \mathbb{LP}(\mathfrak{g})_w^*.$$

Choose generators f' and g' for L_w and J_u respectively. Put $f := \varphi_u^w(f')$ and $g := \varphi_v^u(g')$.

Choose generators x, y and z for I_u, I_v and I_w, respectively. Since $\varphi_v^w(z) = 0$, and since $\mathfrak{g}$ is exact, there is $p' \in V_v$ such that $z = \varphi_w^v(p')$. Put $p := \varphi_u^v(p')$.

Observe that

$$\varphi_v^u(x + tp + t^2 f) = \varphi_v^u(x) + t\varphi_v^u\varphi_u^v(p') + t^2\varphi_v^u\varphi_u^w(f') = 0 \tag{7.3}$$

and

$$\varphi_u^v(y + tg) = \varphi_u^v(y) + t\varphi_u^v\varphi_v^u(g') = 0. \tag{7.4}$$

For general t and each $a \in H$, consider the subspace I_a^t:

$$I_a^t = \langle \varphi_a^u(x + tp + t^2 f), \varphi_a^v(y + tg) \rangle.$$

We claim that I_a^t has dimension 1. Moreover, we claim that if there is an admissible path from u to a through v then $\varphi_a^u(x + tp + t^2 f) = 0$ and $\varphi_a^v(y + tg) \neq 0$, whereas if there is an admissible path from v to a through u then $\varphi_a^u(x + tp + t^2 f) \neq 0$ and $\varphi_a^v(y + tg) = 0$. It follows from Lemma 3.1 that either one or the other case occurs in the second claim, never both, whence the first claim.

As for the second claim, assume first that there is an admissible path from u to a through v, whence $\varphi_a^u = \varphi_a^v \varphi_v^u$. Then $\varphi_a^u(x + tp + t^2 f)$ because of (7.3), whereas $\varphi_a^v(y + tg)$ is nonzero because

$$\varphi_a^v(g) = \varphi_a^v \varphi_v^u(g') = \varphi_a^u(g'),$$

which is nonzero since $\mathfrak{J} \in \mathbb{LP}(\mathfrak{g})_u^*$.

Assume now that there is an admissible path from v to a through u, whence $\varphi_a^v = \varphi_a^u \varphi_u^v$. Since $\varphi_u^v(y) = 0$, we have that $\varphi_a^v(y) = 0$ and thus

$$\varphi_a^v(y + tg) = t\varphi_a^u \varphi_u^v \varphi_v^u(g') = 0.$$

In addition, $\varphi_a^u(x + tp + t^2 f)$ is nonzero. Indeed, assume $\varphi_a^u(x) = 0$. Since $\mathfrak{I}$ is generated by $\{u, v, w\}$, we have $\varphi_a^w(z) \neq 0$. Now, either there is an admissible path from w to a through u, in which case

$$\varphi_a^u(f) = \varphi_a^u \varphi_u^w(f') = \varphi_a^w(f'),$$

which is nonzero because $\mathfrak{L} \in \mathbb{LP}(\mathfrak{g})_w^*$, or there is an admissible path from u to a through w, in which case

$$\varphi_a^u(p) = \varphi_a^w \varphi_w^u \varphi_u^v(p') = \varphi_a^w \varphi_w^v(p') = \varphi_a^w(z),$$

which is nonzero.

We claim now that $\varphi_{a_2}^{a_1}(I_{a_1}^t) \subseteq I_{a_2}^t$ for each $a_1, a_2 \in H$. Indeed, either there is an admissible path from u to a_1 through v or one from v to a_1 through u. In the first case, $\varphi_{a_2}^{a_1}(I_{a_1}^t)$ is generated by $\varphi_{a_2}^{a_1} \varphi_{a_1}^v(y + tg)$, which is either equal to $\varphi_{a_2}^v(y + tg)$ or to zero, and hence is contained in $I_{a_2}^t$. An analogous argument yields the same conclusion in the second case. It follows that $\mathfrak{I}^t := (I_a^t \mid a \in H) \in \mathbb{LP}(\mathfrak{g})$.

From the definition, it is clear that I_u^t is generated by $x + tp + t^2 f$ and I_v^t is generated by $y + tg$. Hence $\mathfrak{I}^t$ is generated by $\{u, v\}$. It is minimally generated because of (7.3) and (7.4).

Finally, for each $a \in H$, the limit of I_a^t as t approaches zero is the space generated by $\varphi_a^u(x)$ and $\varphi_a^v(y)$, whence I_a, if either is nonzero. If $\varphi_a^u(x) = \varphi_a^v(y) = 0$ then $\varphi_a^w(z) \neq 0$. In particular, there is no admissible path from w to a through v, and hence there is an admissible path from v to a passing through u and then w. It follows that I_a^t is generated by $\varphi_a^u(x + tp + t^2 f)$. But $\varphi_a^u(x) = 0$. Also, since there is an admissible path from u to a through w, we have $\varphi_a^u(p) = \varphi_a^w(z)$, as seen before. It follows that the limit of I_a^t as t approaches zero is the space generated by $\varphi_a^w(z)$, whence I_a.

In either case, $\mathfrak{I}$ is the limit as t approaches zero of $\mathfrak{I}^t$, which is minimally generated by $\{u, v\}$ for general t. The proof is complete. □

Theorem 7.1. Let $\mathfrak{g}$ be an exact linked net of dimension $r+1$ with finite support over a $\mathbb{Z}^2$-quiver. Then the scheme $\mathbb{LP}(\mathfrak{g})$ is of pure dimension r, and all of its irreducible components are rational. More precisely, the components are the non-empty schemes $\mathbb{LP}(\mathfrak{g})_v$ and there is a one to one correspondence between the components of $\mathbb{LP}(\mathfrak{g})$ and the equivalence classes of $\mathfrak{g}$. Furthermore, the set of exact points of $\mathbb{LP}(\mathfrak{g})$ is its nonsingular locus.

Proof. It follows from Theorem 3.1 that the exact points of $\mathbb{LP}(\mathfrak{g})$ lie on the union $\bigcup \mathbb{LP}(\mathfrak{g})_v^*$, which is contained in the nonsingular locus of $\mathbb{LP}(\mathfrak{g})$ by Proposition 7.2. Furthermore, it follows from Theorem 4.1 and Proposition 7.6 that the remaining points lies on the intersection of two or three of the $\mathbb{LP}(\mathfrak{g})_v$. Then

$$\mathbb{LP}(\mathfrak{g}) = \bigcup_{v \in H} \mathbb{LP}(\mathfrak{g})_v$$

and thus the non-empty $\mathbb{LP}(\mathfrak{g})_v$ are the irreducible components of $\mathbb{LP}(\mathfrak{g})$. Each $\mathbb{LP}(\mathfrak{g})_v$ is irreducible and rational of dimension r by Proposition 7.2, thus $\mathbb{LP}(\mathfrak{g})$ has pure dimension r. Also, Proposition 7.5 garantees the one to one correspondence between the non-empty $\mathbb{LP}(\mathfrak{g})_{\underline{v}}$ and the classes of the equivalence relation on $\mathfrak{g}$. The non-exact points of $\mathbb{LP}(\mathfrak{g})$ lie on the intersection of at least two components, thus lie on the singular locus of $\mathbb{LP}(\mathfrak{g})$. □

7.3 The Hilbert polynomial of $\mathbb{LP}(\mathfrak{g})$

7.3.1 General setup

Given a linked net of vector spaces $\mathfrak{g}$ of dimension $r+1$ over a $\mathbb{Z}^2$-quiver Q supported in a finite set H there is a natural injective map

$$\Phi\colon \mathbb{LP}(\mathfrak{g}) \longrightarrow \prod_{v\in H} \mathbb{P}(V_v).$$

As in the case of $\mathbb{P}(\mathfrak{g})$, we can ask: what is the Hilbert polynomial (or the Chow class) of $\mathbb{LP}(\mathfrak{g})$ inside $\prod \mathbb{P}(V_v)$? In this section we say some words on what we know about this subject.

As in Chapter 4, even though we may make general statements, we will assume in our argument that Q is the standard $\mathbb{Z}^2$-quiver, with vertex set $\mathbb{Z}^2$ and generated by the vectors $v_0 = (-1,0)$, $v_1 = (1,1)$ and $v_2 = (0,-1)$. We assume as well that

$$H = \mathbb{N}^2(\leq d) := \{(i,j) \in \mathbb{Z}^2 \mid 0 \leq i, j \leq i+j \leq d\}$$

for some positive integer d. Recall as well from Chapter 4 all the alternate notation used in this case.

Also, in this first moment, we will work under the hypothesis that $\mathfrak{g}$ admits a simple basis. This allows us to put a system of coordinates in $\prod \mathbb{P}(V_{\underline{d}})$. In fact, let $\underline{d}_0, \dots, \underline{d}_r \in H$ and $s_i \in V_{\underline{d}_i}$ for $i = 0, \dots, r$ forming a simple basis for $\mathfrak{g}$. Given $p = (\sigma_{\underline{d}}(p) \mid \underline{H})) \in \mathbb{LP}(\mathfrak{g})$ we write

$$\sigma_{\underline{d}}(p) = \sum_{i=0}^{r} x_i^{\underline{d}}(p) s_{i|_{V_{\underline{d}}}}.$$

In other words, the simple basis induces coordinates $(x_0^{\underline{d}} : \cdots : x_r^{\underline{d}})$ on $\mathbb{P}(V_{\underline{d}})$ for each $\underline{d} \in H$. In these coordinates, the map $\Phi : \mathbb{LP}(\mathfrak{g}) \longrightarrow \prod \mathbb{P}^r$ is expressed:

$$\Phi(p) = \prod_{\underline{d}} (x_0^{\underline{d}}(p) : \cdots : x_r^{\underline{d}}(p)).$$

The existence of a simple basis also simplifies the matrix representations of the maps $\varphi_{\underline{e}}^{\underline{d}}$. Since $\{s_{0|_{V_{\underline{d}}}}, \dots, s_{r|_{V_{\underline{d}}}}\}$ is a basis for $V_{\underline{d}}$, the map $\varphi_{\underline{e}}^{\underline{d}} : V_{\underline{d}} \longrightarrow V_{\underline{e}}$, for $\underline{d}$ and $\underline{e}$ being neighbors, is completely determined by

$$\varphi_{\underline{e}}^{\underline{d}}(s_{0}{}_{|V_{\underline{d}}}), \ldots, \varphi_{\underline{e}}^{\underline{d}}(s_{r}{}_{|V_{\underline{d}}}) \in V_{\underline{e}}.$$

But

$$\varphi_{\underline{e}}^{\underline{d}}(s_{i}{}_{|V_{\underline{d}}}) = \begin{cases} s_{i}{}_{|V_{\underline{e}}} & \text{if } s_{i}{}_{|V_{\underline{d}}} \notin \operatorname{Ker}(\varphi_{\underline{e}}^{\underline{d}} \colon V_{\underline{d}} \longrightarrow V_{\underline{e}}), \\ 0 & \text{otherwise.} \end{cases}$$

So, the map $V_{\underline{d}} \longrightarrow V_{\underline{e}}$ is represented by a diagonal matrix:

$$\begin{bmatrix} \epsilon_0^{\underline{d},\underline{e}} & 0 & \ldots & 0 \\ 0 & \epsilon_1^{\underline{d},\underline{e}} & \ldots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \ldots & \epsilon_r^{\underline{d},\underline{e}} \end{bmatrix}$$

where

$$\epsilon_i^{\underline{d},\underline{e}} = \begin{cases} 1 & \text{if } s_{i}{}_{|V_{\underline{d}}} \notin \operatorname{Ker}(\varphi_{\underline{e}}^{\underline{d}} \colon V_{\underline{d}} \longrightarrow V_{\underline{e}}), \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\varphi_{\underline{e}}^{\underline{d}}(\sigma_{\underline{d}}(p)) = \varphi_{\underline{e}}^{\underline{d}}\left(\sum_i x_i^{\underline{d}}(p) s_{i}{}_{|V_{\underline{d}}}\right) = \sum_i \epsilon_i^{\underline{d},\underline{e}} x_i^{\underline{d}}(p) s_{i}{}_{|V_{\underline{e}}},$$

the vectors:

$$(\epsilon_0^{\underline{d},\underline{e}} x_0^{\underline{d}}(p), \ldots, \epsilon_r^{\underline{d},\underline{e}} x_r^{\underline{d}}(p)) \quad \text{and} \quad (x_0^{\underline{e}}(p), \ldots, x_r^{\underline{e}}(p))$$

are linearly dependent. Therefore, inside the product $\prod \mathbb{P}^r$, the scheme $\mathbb{LP}(\mathfrak{g})$ is defined by the equations

$$\epsilon_i^{\underline{d},\underline{e}} x_i^{\underline{d}} x_j^{\underline{e}} = \epsilon_j^{\underline{d},\underline{e}} x_j^{\underline{d}} x_i^{\underline{e}} \tag{7.5}$$

for all $\underline{d}, \underline{e} \in H$ and all $i, j \in \{0, \ldots, r\}$.

Observe that if either $\varphi_{\underline{e}}^{\underline{d}} \longrightarrow V_{\underline{d}} \longrightarrow V_{\underline{e}}$ or $\varphi_{\underline{d}}^{\underline{e}} \longrightarrow V_{\underline{e}} \longrightarrow V_{\underline{d}}$ is the zero map, then Equations (7.5) for the pair $(\underline{d}, \underline{e})$ are

$$x_i^{\underline{d}} x_j^{\underline{e}} = x_j^{\underline{d}} x_i^{\underline{e}} \qquad \forall i, j = 0, \ldots, r \tag{7.6}$$

which are the equations defining the diagonal inside $\mathbb{P}(V_{\underline{d}}) \times \mathbb{P}(V_{\underline{e}})$. In this case,

$\mathbb{LP}(\mathfrak{g})$ is isomorphic to its image under the projection

$$\prod_{\underline{b}\in H}\mathbb{P}(V_{\underline{b}}) \longrightarrow \prod_{\underline{b}\in H-\{\underline{a}\}}\mathbb{P}(V_{\underline{b}})$$

for $\underline{a}=\underline{d}$ or $\underline{a}=\underline{e}$. The equations of the projection are those obtained from the equations of $\mathbb{LP}(\mathfrak{g})$ by replacing $x_i^{\underline{d}}$ and $x_i^{\underline{e}}$ by $x_i^{\underline{a}}$ for each $i=0,\ldots,r$. In other words $\underline{d}$ and $\underline{e}$ are in the same equivalence class.

Let's take a look at some examples.

Example 7.1. In Example 1 on Table 8.1 on Page 122, taking the parameters $\lambda_1 := 0$ and $\lambda_2 := 1$, we get the following linked net (see Section 8.1 for the notation):

$V_{(0,0,2)} =< (0,0,z),(0,0,z^2) >$	$V_{(1,0,1)} =< (x,y,z),(0,0,z^2) >$
$V_{(0,1,1)} =< (0,y,z),(0,0,z^2) >$	$V_{(1,1,0)} =< (x,y,0),(x^2,y^2,z^2) >$
$V_{(0,2,0)} =< (0,y,0),(0,y^2,z^2) >$	$V_{(2,0,0)} =< (x,0,0),(x^2,0,0) >$

Table 7.1: A linked net of dimension 2 and "degree" 2

It has simple basis is $\{(x,y,z),(x^2,y^2,z^2)\}$. We can simplify the notation, by using coordinates $x^{\underline{d}}$ and $y^{\underline{d}}$ instead of $x_0^{\underline{d}}$ and $x_1^{\underline{d}}$ on $\mathbb{P}(V_{\underline{d}})$. Then Equations (7.5) become, in this example:

$x^{011}y^{002}=y^{011}x^{002}$	$x^{101}y^{011}=y^{101}x^{011}$	$x^{110}y^{020}=y^{110}x^{020}$
$x^{011}y^{020}=0$	$x^{101}y^{110}=0$	$x^{110}y^{200}=y^{110}x^{200}$
$x^{101}y^{020}=0$		

Table 7.2: Equations of $\mathbb{LP}(\mathfrak{g})$

Since the maps $\varphi_{(0,0,2)}^{(0,1,1)}$ and $\varphi_{(0,1,1)}^{(1,0,1)}$ are isomorphisms,

$$\mathbb{LP}(\mathfrak{g})_{(0,1,1)} = \mathbb{LP}(\mathfrak{g})_{(0,0,2)} = \emptyset.$$

Also, since $\varphi_{(0,2,0)}^{(1,1,0)}$ and $\varphi_{(2,0,0)}^{(1,1,0)}$ are isomorphisms,

$$\mathbb{LP}(\mathfrak{g})_{(0,2,0)} = \mathbb{LP}(\mathfrak{g})_{(2,0,0)} = \emptyset.$$

Hence $\mathbb{LP}(\mathfrak{g})$ has two components, each a copy of $\mathbb{P}^1$:

$$\mathbb{LP}(\mathfrak{g})_{101} = \{(0:1)\times(0:1)\times(x:y)\times(0:1)\times(x:y)\times(x:y) \mid (x:y)\in\mathbb{P}^1\},$$
$$\mathbb{LP}(\mathfrak{g})_{110} = \{(x:y)\times(x:y)\times(1:0)\times(x:y)\times(1:0)\times(1:0) \mid (x:y)\in\mathbb{P}^1\}.$$

Observe that $\mathbb{LP}(\mathfrak{g})$ is connected and for each factor of $\prod\mathbb{P}(V_{\underline{d}})$ there is one and only one component of $\mathbb{LP}(\mathfrak{g})$ that projects isomorphically onto that factor. By a theorem by Cartwright and Sturmfels (Propostion 7.7 below), $\mathbb{LP}(\mathfrak{g})$ is a deformation of the diagonal.

What is the deformation? For the general case, we can define for each t the subscheme $\mathbb{LP}(\mathfrak{g}_t)$ of $\prod\mathbb{P}(V_{\underline{d}})$ by the equations:

$$t^{(1-\epsilon_i^{\underline{d},\underline{e}})}x_i^{\underline{d}}x_j^{\underline{e}} = t^{(1-\epsilon_j^{\underline{d},\underline{e}})}x_j^{\underline{d}}x_i^{\underline{e}}.$$

For $t\neq 0$, the scheme $\mathbb{LP}(\mathfrak{g}_t)$ is obviously the diagonal, up to a certain automorphism. The scheme we obtain when we set $t=0$ is equal to our original $\mathbb{LP}(\mathfrak{g})$. But is the family a flat deformation of the diagonal, i.e., is $\mathbb{LP}(\mathfrak{g})$ the limit of the family $\mathbb{LP}(\mathfrak{g}_t)$? According to Cartwright and Sturmfels [8], this is true for linked nets $\mathfrak{g}$ of dimension 2, since the equations are the exact same equations that appear in the proof of Theorem 4.1 in loc. cit..

In our example, the deformation is given by the following equations: It's

$x^{011}y^{002} = y^{011}x^{002}$	$x^{101}y^{011} = y^{101}x^{011}$	$x^{110}y^{020} = y^{110}x^{020}$
$x^{011}y^{020} = ty^{011}x^{020}$	$x^{101}y^{110} = ty^{101}x^{110}$	$x^{110}y^{200} = y^{110}x^{200}$
$x^{101}y^{020} = ty^{101}x^{020}$		

Table 7.3: Equations of $\mathbb{LP}(\mathfrak{g}_t)$

easy to see that $\mathbb{LP}(\mathfrak{g}_t)$ is, for $t\neq 0$, the image of the map

$$\phi_t\colon \mathbb{P}^1 \longrightarrow \prod\mathbb{P}(V_{\underline{d}})$$

$$\phi_t(x:y) = ((x:y),(x:y),(x:ty),(x:y),(x:ty),(x:ty)).$$

As mentioned above, we will see that $\mathbb{LP}(\mathfrak{g})$ is the "limit of the diagonal" for $\mathfrak{g}$ of dimension 2. This will follow from a nice description of the scheme $\mathbb{LP}(\mathfrak{g})$: its components are isomorphic to certain of their projections. If the dimension is bigger than 2, things get a little more complicated: the components may no longer be isomorphic to their projections, as the following example shows.

Example 7.2. Let's look at Example 4 on Table 8.2, Page 123. This linked net has a simple basis $\{s_0, s_1, s_2\}$, where

$$s_0 \in V_{(030)}, s_1 \in V_{(102)}, s_2 \in V_{(201)}.$$

The equivalence classes of vertices are:

$$\{(003),(012),(102)\},\{(030)\},\{(201),(210),(300)\},\{(021),(111),(120)\}$$

For these classes, we will associate a set of coordinates: X, Y, Z, W, respectively. In these coordinates, the equations for $\mathbb{LP}(\mathfrak{g})$ are:

$X_1W_0 = 0$	$X_0W_2 = X_2W_0$	$X_1W_2 = 0$
$Y_0W_1 = 0$	$Y_0W_2 = 0$	$Y_1W_2 = Y_2W_1$
$W_0Z_1 = W_1Z_0$	$W_0Z_2 = 0$	$W_1Z_2 = 0$

Table 7.4: Equations for $\mathbb{LP}(\mathfrak{g})$ of example 7.2

Each equivalence class gives rise to an irreducible component of $\mathbb{LP}(\mathfrak{g})$. In this case, the components are:

$$\begin{aligned}
\mathbb{LP}(\mathfrak{g})_{003} &= \{(x_0 : x_1 : x_2) \times (0:1:0) \times (0:1:0) \times (0:1:0)\} \cong \mathbb{P}^2 \\
\mathbb{LP}(\mathfrak{g})_{030} &= \{(1:0:0) \times (y_0, y_1, y_2) \times (1:0:0) \times (1:0:0)\} \cong \mathbb{P}^2 \\
\mathbb{LP}(\mathfrak{g})_{201} &= \{(0:0:1) \times (0:0:1) \times (z_0, z_1, z_2) \times (0:0:1)\} \cong \mathbb{P}^2 \\
\mathbb{LP}(\mathfrak{g})_{111} &= \overline{\{(w_0 : 0 : w_2) \times (0 : w_1 : w_2) \times (w_0 : w_1 : 0) \times (w_0 : w_1 : w_2)\}}
\end{aligned}$$

We see that $\mathbb{LP}(\mathfrak{g})$ is an $\mathbb{P}^2$ blown up at three points $P_0 = (1:0:0)$,$P_1 = (0:1:0)$ and $P_2 = (0:0:1)$, with exceptional divisors E_0, E_1, E_2, and such that for each E_i there is another $\mathbb{P}^2$ attached to it.

7.3.2 Two dimensional case

We will prove that for a linked net of vector spaces $\mathfrak{g}$ of dimension 2 over a $\mathbb{Z}^2$-quiver, the scheme $\mathbb{LP}(\mathfrak{g})$ is a deformation of the diagonal, even without the assumption of simple basis. For that, we will need a result about degenerations of the diagonal in a product of projective spaces.

Proposition 7.7 (Cartwright and Sturmfels, 2009)**.** Let $Y \subset (\mathbb{P}^1)^n$ be a union of projective lines satisfying:

1. Y is connected;

2. For each factor of $(\mathbb{P}^1)^n$ there is a unique line in Y which projects isomorphically onto that factor.

Then Y is a degeneration of the diagonal. In particular, its multivariate Hilbert polynomial is

$$P(t_1, \ldots, t_n) = 1 + t_1 + \cdots + t_n.$$

Proof. Proposition 4.4. of [8]. $\square$

Theorem 7.2. Let $\mathfrak{g}$ be a linked net of vector spaces of dimension 2 over a $\mathbb{Z}^2$-quiver. Let H be a finite set of vertices supporting $\mathfrak{g}$ such that $P(H) = H$. Then

$$\mathbb{LP}(\mathfrak{g}) \subset \prod_{v \in H} \mathbb{P}(V_v)$$

is a connected union of projective lines, $Z_1 \cup \cdots \cup Z_m$, with the following property: for each $v \in H$ there is a unique $j \in \{1, \ldots, m\}$ such that the projection $Z_j \to \mathbb{P}(V_v)$ is an isomorphism. Furthermore, $\mathbb{LP}(\mathfrak{g})$ is a degeneration of the diagonal and the multivariate Hilbert polynomial of $\mathbb{LP}(\mathfrak{g})$ is

$$P(t_1, \ldots, t_N) = 1 + t_1 + \cdots + t_N,$$

where $N := \#H$.

Proof. From Proposition 7.3 we know that for each $v \in H$ there is a unique vertex u of the quiver such that $\mathbb{LP}(\mathfrak{g})_u$ is non-empty and the projection $\mathbb{LP}(\mathfrak{g})_u \longrightarrow \mathbb{P}(V_v)$ is surjective, the generator of the liaison class of v. So $\mathbb{LP}(\mathfrak{g})$ satisfies the hypothesis of Proposition 7.7. Hence $\mathbb{LP}(\mathfrak{g})$ is a degeneration of the diagonal and the multivariate Hilbert polynomial of $\mathbb{LP}(\mathfrak{g})$ is

$$P(t_1, \ldots, t_N) = 1 + t_1 + \cdots + t_N.$$

$\square$

Chapter 8

Examples

In this chapter we summarize a collection of examples that have been cited in the previous chapters.

8.1 Tree of projective lines

In this section we focus on degenerations of linear series for a particular type of nodal curve C. Namely, our curve C will be the union of three projective lines, $C = X \cup Y \cup Z$, where $X \simeq Y \simeq Z \simeq \mathbb{P}^1$. Their non-empty intersections, $X \cap Y$ and $Y \cap Z$, consist of one point each only, which we denote by P and Q, respectively. They are thus the the only singularities of C. We use coordinates $x, y, z \in \mathbb{C} \cup \{\infty\}$ for the lines X, Y, Z respectively, in such way that on P we have $x = 0, y = 0$ and on Q we have $y = \infty, z = 0$.

Since the components are $\mathbb{P}^1$, every invertible sheaf of multi-degree (i, j, k) over C is isomorphic to $\mathcal{O}(i, j, k)$, the unique invertible sheaf satisfying

$$\mathcal{O}(i,j,k)_{|X} \simeq \mathcal{O}_X(i), \quad \mathcal{O}(i,j,k)_{|Y} \simeq \mathcal{O}_Y(j) \quad \text{and} \quad \mathcal{O}(i,j,k)_{|Z} \simeq \mathcal{O}_Z(k).$$

We may describe a global section of this sheaf, when $\underline{d} = (i, j, k)$ is effective, that is, i, j and k are non-negative, with sum $i + j + k = d$, as a triple of global sections of

$$\mathcal{O}_X(d) \oplus \mathcal{O}_Y(d) \oplus \mathcal{O}_Z(d),$$

the first vanishing at P with order at least $d - i$, the second vanishing at P with order at least i and at Q with order at least k (notice that $i + k = d - j$), and the third vanishing at Q with order at least $d-k$, whence as a triple of polynomials of degree d on x, y and z of the form:

$$\left(\sum_{l=d-i}^{d} a_l x^l, \sum_{l=i}^{d-k} b_l y^l, \sum_{l=d-k}^{d} c_l z^l\right) \quad \text{where } b_i = a_{d-i} \text{ and } b_{d-k} = c_{d-k}. \tag{8.1}$$

A linked net of vector spaces arising from a degeneration of linear series of rank r and degree d to C is a linked net $\mathfrak{g}$ of vector spaces over a $\mathbb{Z}^2$ quiver Q as follows:

First of all, the vertex set of Q is the set G of multidegrees $\underline{d} = (i, j, k)$ of total degree $i + j + k = d$. The arrow set A is the disjoint union of the subsets $A_\ell \subseteq G \times G$ for $\ell = 0, 1, 2$, where $(\underline{d}, \underline{e}) \in A_\ell$ if and only if $\underline{e} - \underline{d}$ is equal to $(-1, 1, 0)$ for $\ell = 0$, to $(1, -2, 1)$ for $\ell = 1$, and to $(0, 1, -1)$ for $\ell = 2$.

Second, the $V_{(i,j,k)}$ are subspaces of the $H^0(\mathcal{O}(i, j, k))$ of dimension $r + 1$, whence spaces of triples of polynomials of the form as in (8.1). As such the map $\varphi^{\underline{d}}_{\underline{e}} \colon V_{\underline{d}} \longrightarrow V_{\underline{e}}$ corresponding to an arrow in A_ℓ connecting effective $\underline{d}, \underline{e} \in G$ is the one that sends a triple of polynomials to the same triple, except that the ℓ-th polynomial is changed to 0. The net $\mathfrak{g}$ is supported on the set of effective $\underline{d} \in G$.

In the remaining of this chapter we study linked nets $\mathfrak{g}$ of vector spaces of dimension $r + 1$ over $\mathbb{Z}^2$-quivers Q as described in the above two paragraphs, for low degree d. As in past chapters we will use the suggestive notation

$$\varphi^{\underline{d}}_X,\ \varphi^{\underline{d}}_Y,\ \varphi^{\underline{d}}_Z,\ \varphi^{\underline{d}}_{X+Y},\ \varphi^{\underline{d}}_{X+Z},\ \varphi^{\underline{d}}_{Y+Z}$$

for the maps associated to the arrows leaving each effective $\underline{d}$ in A_0, A_1 and A_2 and the maps leaving $\underline{d}$ associated to simple length 2 paths with arrows in $A_0 \cup A_1$, in $A_0 \cup A_2$ and $A_1 \cup A_2$, in that order,

$$K^{\underline{d}}_X,\ K^{\underline{d}}_Y,\ K^{\underline{d}}_Z,\ K^{\underline{d}}_{X+Y},\ K^{\underline{d}}_{X+Z},\ K^{\underline{d}}_{Y+Z},$$

for their respective kernels, and

$$p^{\underline{d}}_X,\ p^{\underline{d}}_Y,\ p^{\underline{d}}_Z,\ p^{\underline{d}}_{X+Y},\ p^{\underline{d}}_{X+Z},\ p^{\underline{d}}_{Y+Z}$$

for their respective kernel dimensions. We will drop the superscript $\underline{d}$ when easily inferred from the context.

For $d = 2$, we have:

$$
\begin{aligned}
V_{(0,0,2)} &\subset \{(a_2x^2, a_2, a_2 + c_1z + c_2z^2) \mid a_2, c_1, c_2 \in k\},\\
V_{(0,1,1)} &\subset \{(a_2x^2, a_2 + b_1y, b_1z + c_2z^2) \mid a_2, b_1, c_2 \in k\},\\
V_{(0,2,0)} &\subset \{(a_2x^2, a_2 + b_1y + b_2y^2, b_2z^2) \mid a_2, b_1, b_2 \in k\},\\
V_{(1,0,1)} &\subset \{(a_1x + a_2x^2, a_1y, a_1z + c_2z^2) \mid a_1, a_2, c_2 \in k\},\\
V_{(1,1,0)} &\subset \{(a_1x + a_2x^2, a_1y + b_2y^2, b_2z^2) \mid a_1, a_2, b_2 \in k\},\\
V_{(2,0,0)} &\subset \{(a_0 + a_1x + a_2x^2, a_0y^2, a_0z^2) \mid a_0, a_1, a_2 \in k\}.
\end{aligned}
$$

Using the explicit description given above, it is easy to see, for example, that:

$$
\begin{aligned}
p^{020}_{X+Y} &\leq 2,\\
p^{020}_{X+Z} &= 0,\\
p^{020}_{Y+Z} &\leq 2,\\
p^{020}_{X} &= 0,\\
p^{020}_{Y} &\leq 1,\\
p^{020}_{Z} &= 0.
\end{aligned}
$$

Likewise, it is also true that:

$$
\begin{aligned}
p^{110}_{X} &\leq 1,\\
p^{011}_{Z} &\leq 1.
\end{aligned}
$$

So, by the item (h) of Corollary 4.1 on page 56, we get:

$$
\begin{aligned}
p^{020}_{Y+Z} &\geq 1,\\
p^{020}_{X+Y} &\geq 1.
\end{aligned}
$$

That reduces the possibilities for the numbers $p^{020}_{\bullet}$. One such possibility is $p_{Y+Z} = 2, p_Y = p_{X+Y} = 1, p_X = p_Z = p_{X+Z} = 0$. These numbers completely determine the space $V_{(0,2,0)}$. In fact, since $p_{Y+Z} = 2$, we have

$$V_{(0,2,0)} = K^{020}_{Y+Z} = \{(a_2x^2, a_2{+}b_1y{+}b_2y^2, b_2z^2) \mid a_2 = 0\} = \langle (0, y, 0), (0, y^2, z^2)\rangle.$$

With this, we can give an example of an exact linked net. It is just the process of choosing the right generators for the spaces $V_{\underline{d}}$ in a kind of inductive way. Meaning: we construct a $V_{\underline{d}}$, then use the images of the maps $\varphi^{\underline{d}}_{\bullet}$ to get some of the generators of the neighboring spaces, and so on. In our example, $V_{(0,2,0)}$ having been chosen as above, we get:

$$\mathrm{Im}(\varphi_Y^{020}) = K_{X+Z}^{101} = \langle (0,0,z^2)\rangle.$$

But $p_Z^{101} \leq 1$ and $\varphi_Z^{101}(0,0,z^2) = 0$, so $K_Z^{101} = \langle (0,0,z^2)\rangle$. Also, $K_Z = K_{X+Y} \cap K_{Y+Z}$, and since $p_{Y+Z}^{101} \leq 1$, we have:

$$K_Z^{101} = K_{X+Z}^{101} = K_{Y+Z}^{101} = \langle (0,0,z^2)\rangle.$$

Since $K_X^{101} \subset K_{X+Z}^{101} = \langle (0,0,z^2)\rangle$ and $\varphi_X^{101}(0,0,z^2) = (0,0,z^2) \neq 0$, we conclude that

$$K_X^{101} = (0).$$

Likewise,

$$K_Y^{101} = (0).$$

Now, take a vector

$$\mathbf{v} = (a_1x + a_2x^2, a_1y, a_1z + c_2z^2) \in V_{(1,0,1)} \backslash \langle (0,0,z^2)\rangle.$$

We can actually assume that $\mathbf{v} = (a_1x + a_2x^2, a_1y, a_1z)$. Since $\varphi_{X+Z}^{101}(\mathbf{v}) = (0, a_1y, 0) \neq 0$, we have $a_1 \neq 0$. So, there exists $\lambda_1 \in k$ such that

$$V_{(1,0,1)} = \langle (0,0,z^2), (x + \lambda_1x^2, y, z)\rangle.$$

Also,

$$V_{(0,0,1)} = \mathrm{Im}(\varphi_X^{101}) = \langle (0,0,z^2), (0,y,z)\rangle$$

and

$$V_{(0,0,2)} = \mathrm{Im}(\varphi_{X+Y}^{011}) = \langle (0,0,z), (0,0,z^2)\rangle.$$

Now, "going down in the graph," we have

$$K_{X+Y}^{110} = \mathrm{Im}(\varphi_Z^{101}) = \langle (x + \lambda_1x^2, y, 0)\rangle.$$

Take a vector:

$$\mathbf{w} = (a_1x + a_2x^2, a_1y + b_2y^2, b_2z^2) \in V_{(1,1,0)} \backslash \langle (x + \lambda_1x^2, y, 0)\rangle.$$

As before, we may assume that $\mathbf{w} = (a_2x^2, b_2y^2, b_2z^2)$ and $b_2 \neq 0$, since

$\varphi_{X+Y}(\mathbf{w}) = (0,0,b_2z^2) \neq 0$. So, $\mathbf{w} = (\lambda_2x^2, y^2, z^2)$ for some $\lambda_2 \in k$.

Here there are two situations, depending on whether λ_2 is zero or not.

Case 1: $\lambda_2 \neq 0$

In this case,

$$V_{(1,1,0)} = \langle (x+\lambda_1x^2, y, 0), (\lambda_2x^2, y^2, z^2)\rangle,\ \lambda_2 \neq 0$$

and

$$V_{(2,0,0)} = \mathrm{Im}(\varphi^{110}_{Y+Z}) = \langle (x+\lambda_1x^2, 0, 0), (\lambda_2x^2, 0, 0)\rangle = \langle (x,0,0), (x^2,0,0)\rangle.$$

This configuration is Example 1 on Table 8.1.

Case 2: $\lambda_2 = 0$

Then

$$V_{(1,1,0)} = \langle (x+\lambda_1x^2, y, 0), (0, y^2, z^2)\rangle$$

and

$$K^{200}_X = \mathrm{Im}(\varphi^{110}_{Y+Z}) = \langle (x+\lambda_1x^2, 0, 0)\rangle.$$

Take a vector

$$\mathbf{u} = (a_0 + a_2x^2, a_0y^2, a_0z^2) \in V_{(2,0,0)} \backslash \langle (x+\lambda_1x^2, 0, 0)\rangle.$$

Since $\varphi^{200}_X(\mathbf{u}) = (0, a_0y^2, a_0z^2) \neq 0$, we have $a_0 \neq 0$. So, for some $\lambda_2 \in k$, we get

$$V_{(2,0,0)} = \langle (x+\lambda_1x^2, 0, 0), (1+\lambda_2x^2, y^2, z^2)\rangle.$$

This configuration is Example 2 on Table 8.1 on page 122.

Actually, Table 8.1 shows all the exact linked nets (in our study) of dimension 2 and "degree" $d = 2$. Table 8.2 gives certain exact linked nets of dimension 3 and "degree" $d = 3$. We note that all these nets admit simple bases. For instance, in Example 1 on Table 8.1, the simple basis is

$$s_0 = (x+\lambda_1x^2, y, z) \in V_{(1,0,1)} \quad \text{and} \quad s_1 = (\lambda_2x^2, y^2, z^2) \in V_{(1,1,0)}.$$

If we change a little bit the example above, we will obtain the non-exact linked net $\mathfrak{g}$ shown on Table 8.3. In this case, $\mathcal{W}(\mathfrak{g}) = 0$ and $\mathbb{P}(\mathfrak{g}) = \emptyset$.

Table 8.1: Exact linked nets of dimension 2 and "degree" 2

Example 1	
$V_{(0,0,2)}$	$\langle(0,0,z),(0,0,z^2)\rangle$
$V_{(0,1,1)}$	$\langle(0,y,z),(0,0,z^2)\rangle$
$V_{(0,2,0)}$	$\langle(0,y,0),(0,y^2,z^2)\rangle$
$V_{(1,0,1)}$	$\langle(x+\lambda_1x^2,y,z),(0,0,z^2)\rangle$
$V_{(1,1,0)}$	$\langle(x+\lambda_1x^2,y,0),(\lambda_2x^2,y^2,z^2)\rangle,\ \lambda_2\neq 0$
$V_{(2,0,0)}$	$\langle(x,0,0),(x^2,0,0)\rangle$
Example 2	
$V_{(0,0,2)}$	$\langle(0,0,z),(0,0,z^2)\rangle$
$V_{(0,1,1)}$	$\langle(0,y,z),(0,0,z^2)$
$V_{(0,2,0)}$	$\langle(0,y,0),(0,y^2,z^2)\rangle$
$V_{(1,0,1)}$	$\langle(x+\lambda_1x^2,y,z),(0,0,z^2)\rangle$
$V_{(1,1,0)}$	$\langle(x+\lambda_1x^2,y,0),(0,y^2,z^2)\rangle$
$V_{(2,0,0)}$	$\langle(x+\lambda_1x^2,0,0),(1+\lambda_2x^2,y^2,z^2)\rangle$
Example 3	
$V_{(0,0,2)}$	$\langle(0,0,z),(0,0,z^2)\rangle$
$V_{(0,1,1)}$	$\langle(x^2,1+\lambda_1y,\lambda_1z),(0,0,z^2)\rangle$
$V_{(0,2,0)}$	$\langle(x^2,1+\lambda_1y,0),(0,\lambda_2y+y^2,z^2)\rangle$
$V_{(1,0,1)}$	$\langle(x^2,0,0),(0,0,z^2)\rangle$
$V_{(1,1,0)}$	$\langle(x^2,0,0),(\lambda_2x,\lambda_2y+y^2,z^2)\rangle\ \lambda_2\neq 0$
$V_{(2,0,0)}$	$\langle(x,0,0),(x^2,0,0)\rangle$
Example 4	
$V_{(0,0,2)}$	$\langle(0,0,z),(0,0,z^2)\rangle$
$V_{(0,1,1)}$	$\langle(0,y,z+\lambda_2z^2),(0,0,z^2)$
$V_{(0,2,0)}$	$\langle(0,y,0),(x^2,1+\lambda_1y^2,\lambda_1z^2)\rangle,\ \lambda_1\neq 0$
$V_{(1,0,1)}$	$\langle(x^2,0,\lambda_1z^2)(x,y,z+\lambda_2z^2)\rangle$
$V_{(1,1,0)}$	$\langle(x^2,0,0)(x,y,0)\rangle$
$V_{(2,0,0)}$	$\langle(x,0,0),(x^2,0,0)\rangle$
Example 5	
$V_{(0,0,2)}$	$\langle(0,0,z),(0,0,z^2)\rangle$
$V_{(0,1,1)}$	$\langle(0,y,z+\lambda_1z^2),(x^2,1,\lambda_2z^2)\rangle,\ \lambda_2\neq 0$
$V_{(0,2,0)}$	$\langle(0,y,0),(x^2,1,0)\rangle$
$V_{(1,0,1)}$	$\langle(x,y,z+\lambda_1z^2),(x^2,0,0)\rangle$

(cont.)	
$V_{(1,1,0)}$	$\langle (x,y,0),(x^2,0,0)\rangle$
$V_{(2,0,0)}$	$\langle (x,0,0),(x^2,0,0)\rangle$
Example 6	
$V_{(0,0,2)}$	$\langle (0,0,z+\lambda_1 z^2),(x^2,1,1+\lambda_2 z^2)\rangle$
$V_{(0,1,1)}$	$\langle (0,y,z+\lambda_1 z^2),(x^2,1,0)\rangle$
$V_{(0,2,0)}$	$\langle (0,y,0),(x^2,1,0)\rangle$
$V_{(1,0,1)}$	$\langle (x,y,z+\lambda_1 z^2),(x^2,0,0)\rangle$
$V_{(1,1,0)}$	$\langle (x,y,0),(x^2,0,0)\rangle$
$V_{(2,0,0)}$	$\langle (x,0,0),(x^2,0,0)\rangle$
Example 7	
$V_{(0,0,2)}$	$\langle (0,0,z),(0,0,z^2)\rangle$
$V_{(0,1,1)}$	$\langle (0,y,z),(0,0,z^2)\rangle$
$V_{(0,2,0)}$	$\langle (0,y,0),(x^2,1+\lambda_1 y^2,\lambda_1 z^2)\rangle$, $\lambda_1 \neq 0$
$V_{(1,0,1)}$	$\langle (x,y,z+\lambda_2 z^2),(x^2,0,\lambda_1 z^2)\rangle$
$V_{(1,1,0)}$	$\langle (x,y,0),(x^2,0,0)\rangle$
$V_{(2,0,0)}$	$\langle (x,0,0),(x^2,0,0)\rangle$
End of the table	

Table 8.2: Exact linked nets of dimension 3 and "degree" 3

Exact linked nets of dimension 3 and "degree" 3	
Example 1	
$V_{(0,0,3)}$	$\langle (0,0,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(0,1,2)}$	$\langle (0,y,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(0,2,1)}$	$\langle (0,y,0),(x^3,1+\lambda_1 y^2,\lambda_1 z^2),(0,0,z^3)\rangle$, $\lambda_1 \neq 0$
$V_{(0,3,0)}$	$\langle (0,y,0),(x^3,1+\lambda_1 y^2,0),(0,\lambda_3 y^2+y^3,z^3)\rangle$, $\lambda_3 \neq 0$
$V_{(1,0,2)}$	$\langle (x^2,y,z+\lambda_2 z^2),(x^3,0,\lambda_1 z^2),(0,0,z^3)\rangle$
$V_{(1,1,1)}$	$\langle (x^2,y,0),(x^3,0,0),(0,0,z^3)\rangle$
$V_{(1,2,0)}$	$\langle (x^2,y,0),(x^3,0,0),(0,\lambda_3 y^2+y^3,z^3)\rangle$
$V_{(2,0,1)}$	$\langle (x^2,0,0),(x^3,0,0),(0,0,z^3)\rangle$
$V_{(2,1,0)}$	$\langle (x^2,0,0),(x^3,0,0),(\lambda_3 x,\lambda_3 y^2+y^3,z^3)\rangle$
$V_{(3,0,0)}$	$\langle (x^2,0,0),(x^3,0,0),(x,0,0)\rangle$
Example 2	
$V_{(0,0,3)}$	$\langle (0,0,z),(0,0,z^2),(0,0,z^3)\rangle$

Continuation of Table	
$V_{(0,1,2)}$	$\langle(0,y,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(0,2,1)}$	$\langle(0,y,0),(x^3,1+\lambda_1 y^2,\lambda_1 z^2),(0,0,z^3)\rangle,\ \lambda_1\neq 0$
$V_{(0,3,0)}$	$\langle(0,y,0),(x^3,1+\lambda_1 y^2,0),(0,y^3,z^3)\rangle,\ \lambda_3\neq 0$
$V_{(1,0,2)}$	$\langle(x^2,y,z+\lambda_2 z^2),(x^3,0,\lambda_1 z^2),(0,0,z^3)\rangle$
$V_{(1,1,1)}$	$\langle(x^2,y,0),(x^3,0,0),(0,0,z^3)\rangle$
$V_{(1,2,0)}$	$\langle(x^2,y,0),(x^3,0,0),(0,y^3,z^3)\rangle$
$V_{(2,0,1)}$	$\langle(x^2,0,0),(x^3,0,0),(0,0,z^3)\rangle$
$V_{(2,1,0)}$	$\langle(x^2,0,0),(x^3,0,0),(0,y^3,z^3)\rangle$
$V_{(3,0,0)}$	$\langle(x^2,0,0),(x^3,0,0),(1+\lambda_3 x,y^3,z^3)\rangle$
Example 3	
$V_{(0,0,3)}$	$\langle(0,0,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(0,1,2)}$	$\langle(0,y,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(0,2,1)}$	$\langle(0,y,0),(0,y^2,z^2),(0,0,z^3)\rangle$
$V_{(0,3,0)}$	$\langle(0,y,0),(0,y^2,0),(\lambda_2 x^3,\lambda_2+y^3,z^3),\ \lambda_2\neq 0$
$V_{(1,0,2)}$	$\langle(x^2+\lambda_1 x^3,y,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(1,1,1)}$	$\langle(x^2+\lambda_1 x^3,y,0),(\lambda_2 x^3,0,z^3),(0,y^2,z^2+\lambda_3 z^3)\rangle$
$V_{(1,2,0)}$	$\langle(x^2+\lambda_1 x^3,y,0),(x^3,0,0),(0,y^2,0)\rangle$
$V_{(2,0,1)}$	$\langle(x^2+\lambda_1 x^3,0,0),(x^3,0,0),(x,y^2,z^2+\lambda_3 z^3)\rangle$
$V_{(2,1,0)}$	$\langle(x^2+\lambda_1 x^3,0,0),(x^3,0,0),(x,y^2,0)\rangle$
$V_{(3,0,0)}$	$\langle(x^2+\lambda_1 x^3,0,0),(x^3,0,0),(x,0,0)\rangle$
Example 4	
$V_{(0,0,3)}$	$\langle(0,0,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(0,1,2)}$	$\langle(0,y,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(0,2,1)}$	$\langle(0,y,0),(0,y^2,z^2),(0,0,z^3)\rangle$
$V_{(0,3,0)}$	$\langle(0,y,0),(0,y^2,0),(0,y^3,z^3)\rangle$
$V_{(1,0,2)}$	$\langle(x^2+\lambda_1 x^3,y,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(1,1,1)}$	$\langle(x^2+\lambda_1 x^3,y,0),(\lambda_2 x^3,y^2,z^2),(0,0,z^3)\rangle,\ \lambda_2\neq 0$
$V_{(1,2,0)}$	$\langle(x^2+\lambda_1 x^3,y,0),(\lambda_2 x^3,y^2,0),(0,\lambda_3 y^2+y^3,z^3)\rangle,\ \lambda_3\neq 0$
$V_{(2,0,1)}$	$\langle(x^2+\lambda_1 x^3,0,0),(x^3,0,0),(0,0,z^3)\rangle$
$V_{(2,1,0)}$	$\langle(x^2+\lambda_1 x^3,0,0),(x^3,0,0),(\lambda_3 x,\lambda_3 y^2+y^3,z^3)\rangle$
$V_{(3,0,0)}$	$\langle(x^2+\lambda_1 x^3,0,0),(x^3,0,0),(x,0,0)\rangle$
Example 5	
$V_{(0,0,3)}$	$\langle(0,0,z),(0,0,z^2)(0,0,z^3)\rangle$
$V_{(0,1,2)}$	$\langle(0,y,z),(0,0,z^2),(0,0,z^3)\rangle$

Continuation of Table	
$V_{(0,2,1)}$	$\langle(0,y,0),(0,y^2,z^2),(0,0,z^3)\rangle$
$V_{(0,3,0)}$	$\langle(0,y,0),(0,y^2,0),(0,y^3,z^3)\rangle$
$V_{(1,0,2)}$	$\langle(x^2+\lambda_1x^3,y,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(1,1,1)}$	$\langle(x^2+\lambda_1x^3,y,0),(\lambda_2x^3,y^2,z^2),(0,0,z^3)\rangle,\ \lambda_2\neq 0$
$V_{(1,2,0)}$	$\langle(x^2+\lambda_1x^3,y,0),(\lambda_2x^3,y^2,0),(0,y^3,z^3)\rangle$
$V_{(2,0,1)}$	$\langle(x^2+\lambda_1x^3,0,0),(x^3,0,0),(0,0,z^3)\rangle$
$V_{(2,1,0}$	$\langle(x^2+\lambda_1x^3,0,0),(x^3,0,0),(0,y^3,z^3)\rangle$
$V_{(3,0,0)}$	$\langle(x^2+\lambda_1x^3,0,0),(x^3,0,0),(1+\lambda_3x,y^3,z^3)\rangle$
Example 6	
$V_{(0,0,3)}$	$\langle(0,0,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(0,1,2)}$	$\langle(0,y,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(0,2,1)}$	$\langle(0,y,0),(0,y^2,z^2),(0,0,z^3)\rangle$
$V_{(0,3,0)}$	$\langle(0,y,0),(0,y^2,0),(0,y^3,z^3)\rangle$
$V_{(1,0,2)}$	$\langle(x^2+\lambda_1x^3,y,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(1,1,1)}$	$\langle(x^2+\lambda_1x^3,y,0),(0,y^2,z^2),(0,0,z^3)\rangle$
$V_{(1,2,0)}$	$\langle(x^2+\lambda_1x^3,y,0),(0,y^2,0),(\lambda_2x^3,y^3,z^3)\rangle,\ \lambda_2\neq 0$
$V_{(2,0,1)}$	$\langle(x^2+\lambda_1x^3,0,0),(x,y^2,z^2+\lambda_3z^3),(\lambda_2x^3,0,z^3)\rangle$
$V_{(2,1,0)}$	$\langle(x^2+\lambda_1x^3,0,0),(x,y^2,0),(x^3,0,0)\rangle$
$V_{(3,0,0)}$	$\langle(x^2+\lambda_1x^3,0,0),(x,0,0),(x^3,0,0)\rangle$
Example 7	
$V_{(0,0,3)}$	$\langle(0,0,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(0,1,2)}$	$\langle(0,y,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(0,2,1)}$	$\langle(0,y,0),(0,y^2,z^2),(0,0,z^3)\rangle$
$V_{(0,3,0)}$	$\langle(0,y,0),(0,y^2,0),(0,y^3,z^3)\rangle$
$V_{(1,0,2)}$	$\langle(x^2+\lambda_1x^3,y,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(1,1,1)}$	$\langle(x^2+\lambda_1x^3,y,0),(0,y^2,z^2),(0,0,z^3)\rangle$
$V_{(1,2,0)}$	$\langle(x^2+\lambda_1x^3,y,0),(0,y^2,0),(0,y^3,z^3)\rangle$
$V_{(2,0,1)}$	$\langle(x^2+\lambda_1x^3,0,0),(x+\lambda_2x^3,y^2,z^2),(0,0,z^3)\rangle$
$V_{(2,1,0)}$	$\langle(x^2+\lambda_1x^3,0,0),(x+\lambda_2x^3,y^2,0),(\lambda_3x^3,y^3,z^3)\rangle,\ \lambda_3\neq 0$
$V_{(3,0,0)}$	$\langle(x^2+\lambda_1x^3,0,0),(x+\lambda_2x^3,0,0),(x^3,0,0)\rangle$
Example 8	
$V_{(0,0,3)}$	$\langle(0,0,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(0,1,2)}$	$\langle(0,y,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(0,2,1)}$	$\langle(0,y,0),(0,y^2,z^2),(0,0,z^3)\rangle$

Continuation of Table	
$V_{(0,3,0)}$	$\langle (0,y,0),(0,y^2,0),(0,y^3,z^3)\rangle$
$V_{(1,0,2)}$	$\langle (x^2+\lambda_1 x^3,y,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(1,1,1)}$	$\langle (x^2+\lambda_1 x^3,y,0),(0,y^2,z^2),(0,0,z^3)\rangle$
$V_{(1,2,0)}$	$\langle (x^2+\lambda_1 x^3,y,0),(0,y^2,0),(0,y^3,z^3)\rangle$
$V_{(2,0,1)}$	$\langle (x^2+\lambda_1 x^3,0,0),(x+\lambda_2 x^3,y^2,z^2),(0,0,z^3)\rangle$
$V_{(2,1,0)}$	$\langle (x^2+\lambda_1 x^3,0,0),(x+\lambda_2 x^3,y^2,0),(0,y^3,z^3)\rangle$
$V_{(3,0,0)}$	$\langle (x^2+\lambda_1 x^3,0,0),(x+\lambda_2 x^3,0,0),(1+\lambda_3 x^3,y^3,z^3)\rangle$
Example 9	
$V_{(0,0,3)}$	$\langle (0,0,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(0,1,2)}$	$\langle (0,y,z+\lambda_1 z^2),(x^3,1,\lambda_2 z^2),(0,0,z^3)\rangle,\ \lambda_2\neq 0$
$V_{(0,2,1)}$	$\langle (0,y,0),(x^3,1,0),(0,0,z^3)\rangle$
$V_{(0,3,0)}$	$\langle (0,y,0),(x^3,1,0),(0,y^3,z^3)\rangle$
$V_{(1,0,2)}$	$\langle (x^2,y,z+\lambda_1 z^2),(x^3,0,0),(0,0,z^3)\rangle$
$V_{(1,1,1)}$	$\langle (x^2,y,0),(x^3,0,0),(0,0,z^3)\rangle$
$V_{(1,2,0)}$	$\langle (x^2,y,0),(x^3,0,0),(0,y^3,z^3)\rangle$
$V_{(2,0,1)}$	$\langle (x^2,0,0),(x^3,0,0),(0,0,z^3)\rangle$
$V_{(2,1,0)}$	$\langle (x^2,0,0),(x^3,0,0),(0,y^3,z^3)\rangle$
$V_{(3,0,0)}$	$\langle (x^2,0,0,),(x^3,0,0),(1+\lambda_3 x,y^3,z^3)\rangle$
Example 10	
$V_{(0,0,3)}$	$\langle (0,0,z),(0,0,z^2),(0,0,z^3)\rangle$
$V_{(0,1,2)}$	$\langle (0,y,z+\lambda_1 z^2),(x^3,1,\lambda_2 z^2),(0,0,z^3)\rangle,\ \lambda_2\neq 0$
$V_{(0,2,1)}$	$\langle (0,y,0),(x^3,1,0),(0,0,z^3)\rangle$
$V_{(0,3,0)}$	$\langle (0,y,0),(x^3,1,0),(0,\lambda_3 y^2+y^3,z^3)\rangle,\ \lambda_3\neq 0$
$V_{(1,0,2)}$	$\langle (x^2,y,z+\lambda_1 z^2),(x^3,0,0),(0,0,z^3)\rangle$
$V_{(1,1,1)}$	$\langle (x^2,y,0),(x^3,0,0),(0,0,z^3)\rangle$
$V_{(1,2,0)}$	$\langle (x^2,y,0),(x^3,0,0),(0,\lambda_3 y^2+y^3,z^3)\rangle$
$V_{(2,0,1)}$	$\langle (x^2,0,0),(x^3,0,0),(0,0,z^3)\rangle$
$V_{(2,1,0)}$	$\langle (x^2,0,0),(x^3,0,0),(\lambda_3 x,\lambda_3 y^2+y^3,z^3)\rangle$
$V_{(3,0,0)}$	$\langle (x^2,0,0),(x^3,0,0),(x,0,0)\rangle$
Example 11	
$V_{(0,0,3)}$	$\langle (0,0,z+\lambda_1 z^2),(x^3,1,1+\lambda_2 z^2),(0,0,z^3)$
$V_{(0,1,2)}$	$\langle (0,y,z+\lambda_1 z^2),(x^3,1,0),(0,0,z^3)\rangle$
$V_{(0,2,1)}$	$\langle (0,y,0),(x^3,1,0),(0,0,z^3)\rangle$
$V_{(0,3,0)}$	$\langle (0,y,0),(x^3,1,0),(0,y^3,z^3)\rangle$

Continuation of Table	
$V_{(1,0,2)}$	$\langle (x^2, y, z+\lambda_1 z^2), (x^3, 0, 0), (0, 0, z^3) \rangle$
$V_{(1,1,1)}$	$\langle (x^2, y, 0), (x^3, 0, 0), (0, 0, z^3) \rangle$
$V_{(1,2,0)}$	$\langle (x^2, y, 0), (x^3, 0, 0), (0, y^3, z^3) \rangle$
$V_{(2,0,1)}$	$\langle (x^2, 0, 0), (x^3, 0, 0), (0, 0, z^3) \rangle$
$V_{(2,1,0)}$	$\langle (x^2, 0, 0), (x^3, 0, 0), (0, y^3, z^3) \rangle$
$V_{(3,0,0)}$	$\langle (x^2, 0, 0), (x^3, 0, 0), (1+\lambda_3 x, y^3, z^3) \rangle$
Example 12	
$V_{(0,0,3)}$	$\langle (0, 0, z+\lambda_1 z^2), (x^3, 1, 1+\lambda_2 z^2), (0, 0, z^3) \rangle$
$V_{(0,1,2)}$	$\langle (0, y, z+\lambda_1 z^2), (x^3, 1, 0), (0, 0, z^3) \rangle$
$V_{(0,2,1)}$	$\langle (0, y, 0), (x^3, 1, 0), (0, 0, z^3) \rangle$
$V_{(0,3,0)}$	$\langle (0, y, 0), (x^3, 1, 0), (0, \lambda_3 y^2 + y^3, z^3) \rangle$
$V_{(1,0,2)}$	$\langle (x^2, y, z+\lambda_1 z^2), (x^3, 0, 0), (0, 0, z^3) \rangle$
$V_{(1,1,1)}$	$\langle (x^2, y, 0), (x^3, 0, 0), (0, 0, z^3) \rangle$
$V_{(1,2,0)}$	$\langle (x^2, y, 0), (x^3, 0, 0), (0, \lambda_3 y^2 + y^3, z^3) \rangle$
$V_{(2,0,1)}$	$\langle (x^2, 0, 0), (x^3, 0, 0), (0, 0, z^3) \rangle$
$V_{(2,1,0)}$	$\langle (x^2, 0, 0), (x^3, 0, 0), (\lambda_3 x, \lambda_3 y^2 + y^3, z^3) \rangle$
$V_{(3,0,0)}$	$\langle (x^2, 0, 0), (x^3, 0, 0), (x, 0, 0) \rangle$
end of table	

Non-exact linked net	
$V_{(0,0,2)}$	$\langle (0, 0, z), (0, 0, z^2) \rangle$
$V_{(0,1,1)}$	$\langle (0, y, 0), (0, 0, z^2) \rangle$
$V_{(0,2,0)}$	$\langle (0, y, 0), (0, y^2, z^2) \rangle$
$V_{(1,0,1)}$	$\langle (x^2, 0, 0), (0, 0, z^2) \rangle$
$V_{(1,1,0)}$	$\langle (x, y, 0), (x^2, 0, 0) \rangle$
$V_{(2,0,0)}$	$\langle (x, 0, 0), (x^2, 0, 0) \rangle$

Table 8.3: Non-exact linked net

8.2 An exact linked net of vector spaces with no simple basis

For several months we believed that exactness and the existence of a simple basis were equivalent properties for a limit linear series. In fact, they are equivalent in the case of curves with two components.

To our surprise, Vidal [10] came up with an example of a limit linear series over a non-compact type curve which is exact but does not admit a simple basis. Inspired by the type of obstruction he got in his counterexample, we tried to pursue a counterexample for compact type curves. We've found the following one, in degree 4 and dimension 2.

Exact example with degree 4, with no simple basis	
$V_{(0,0,4))}$	$\langle (0,0,z^4),(0,0,z^2)\rangle$
$V_{(0,1,3)}$	$\langle (0,0,z^4),(0,0,z^2)\rangle$
$V_{(0,2,2)}$	$\langle (0,0,z^4),(0,y^2,z^2)\rangle$
$V_{(0,3,1)}$	$\langle (0,0,z^4),(0,y^2,-z^4)\rangle$
$V_{(0,4,0)}$	$\langle (x^4,1,z^4),(0,y^2,0)\rangle$
$V_{(1,0,3)}$	$\langle (0,0,z^4),(0,0,z^2)\rangle$
$V_{(1,1,2)}$	$\langle (0,0,z^4),(x^4,y^2,z^2)\rangle$
$V_{(1,2,1)}$	$\langle (x^4,0,z^4),(0,y^2,-z^4)\rangle$
$V_{(1,3,0)}$	$\langle (x^4,0,0),(0,y^2,0)\rangle$
$V_{(2,0,2)}$	$\langle (x^4,0,z^4),(0,0,-z^4)\rangle$
$V_{(2,1,1)}$	$\langle (x^4,0,0),(x^2,y^2,-z^4)\rangle$
$V_{(2,2,0)}$	$\langle (x^4,0,0),(x^2,y^2,0)\rangle$
$V_{(3,0,1)}$	$\langle (x^4,0,0),(x^2,0,0)\rangle$
$V_{(3,1,0)}$	$\langle (x^4,0,0),(x^2,0,0)\rangle$
$V_{(4,0,0)}$	$\langle (x^4,0,0),(x^2,0,0)\rangle$

Table 8.4: An exact $\mathfrak{g}$ that does not admit a simple basis

Recall certain concepts defined previously:

$$\Sigma_{\underline{d}} = \sum_{\underline{e}\in N(\underline{d})} \operatorname{Ker}\left(\varphi^{\underline{d}}_{\underline{e}} \colon V_{\underline{d}} \longrightarrow V_{\underline{e}}\right) \subset V_{\underline{d}}.$$

If $\{s_0,\dots,s_r\}$ is a simple basis for $\mathfrak{g}$ and $s_i \in V_{\underline{d}}$ then $s_i \notin \Sigma_{\underline{d}}$. Actually, the space

$$\mathcal{W}(\mathfrak{g}) = \bigoplus_{\underline{d}} \left(\frac{V_{\underline{d}}}{\Sigma_{\underline{d}}}\right)$$

has dimension $r+1$ and $\{\overline{s_0},\dots,\overline{s_r}\}$ is a basis for it, where $\overline{s_i}$ is the class of s_i in ${}^{V_{\underline{d}}}/_{\Sigma_{\underline{d}}}$; see Proposition 5.2.

In our example, $\Sigma_{\underline{d}} = V_{\underline{d}}$ for all $\underline{d}$, except that $\sum_{(0,4,0)} = \langle (0,y^2,0)\rangle$, that $\sum_{(1,1,2)} = \langle (0,0,z^4)\rangle$ and $\sum_{(2,1,1)} = \langle (x^4,0,0)\rangle$. Therefore

$$\dim \mathcal{W}(\mathfrak{g}) = 3 > 2 = r+1$$

Hence, $\mathfrak{g}$ cannot admit a simple basis.

Let's look more closely to the maps $\varphi_D^{\underline{d}} : V_{\underline{d}} \longrightarrow V_{\underline{e}}$, where $\underline{e} = D \cdot \underline{d}$. The key point is to understand what is happening at the vertex $\underline{d} = (1,2,1)$. At it, the maps are given by the following matrices:

$$[\varphi_X^{(1,2,1)}] = [\varphi_Z^{(1,2,1)}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\varphi_X^{(2,1,1)}] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\varphi_Z^{(1,1,2)}] = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$[\varphi_Y^{(0,4,0)}] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

And then we can see that:

$$\langle (x^4, 0, z^4) \rangle = (\mathrm{Im}(\varphi_X^{(2,1,1)}) + \mathrm{Im}(\varphi_Z^{(1,1,2)})) \cap \mathrm{Im}(\varphi_Y^{(0,4,0)}) \neq K_X^{(1,2,1)} + K_Z^{(1,1,2)} = (0).$$

If there were a simple basis, the above equation would be an equality. That is precisely the condition found by Vidal, which his counterexample also fail. At the vertex $\underline{d} = (1,2,1)$ there is an obstruction to the existence of a simple basis in $\mathfrak{g}$.

We now want to study the scheme $\mathbb{LP}(\mathfrak{g})$ associated to this linked net of vector spaces. Observe that in this case, there are four equivalence classes in $\mathbb{N}^3(\leq 4)$:

$$\begin{aligned}
C_X &= \{(0,0,4),(0,1,3),(0,2,2),(1,0,3),(1,1,2)\},\\
C_Y &= \{(0,4,0)\},\\
C_Z &= \{(2,1,1),(2,2,0),(3,0,1),(3,1,0),(4,0,0)\},\\
C_W &= \{(1,2,1),(0,3,1),(1,3,0),(2,0,2)\}.
\end{aligned}$$

They will correspond to variables X_i, Y_i, Z_i, W_i in $\mathbb{P}^1$, respectively. We can consider

$$\mathbb{LP}(\mathfrak{g}) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

where the projective lines have coordinates X_i, Y_i, Z_i, W_i, respectively. By Theorem 7.2, $\mathbb{LP}(\mathfrak{g})$ has the Hilbert polynomial of the diagonal. The scheme $\mathbb{LP}(\mathfrak{g})$ has four components, one for each equivalence class, which are the following:

$$\begin{aligned}
\mathbb{LP}(\mathfrak{g})_{(1,1,2)} &= \{(x_0 : x_1) \times (0:1) \times (1:0) \times (1:0)\} \cong \mathbb{P}^1, \\
\mathbb{LP}(\mathfrak{g})_{(0,4,0)} &= \{(1:0) \times (y_0 : y_1) \times (1:0) \times (1:0)\} \cong \mathbb{P}^1, \\
\mathbb{LP}(\mathfrak{g})_{(2,1,1)} &= \{(0:1) \times (0:1) \times (z_0 : z_1) \times (0:1)\} \cong \mathbb{P}^1, \\
\mathbb{LP}(\mathfrak{g})_{(1,2,1)} &= \{(1:0) \times (0:1) \times (1:0) \times (w_0 : w_1)\} \cong \mathbb{P}^1.
\end{aligned}$$

But for this linked net of vector spaces, the Hilbert polynomial of $\mathbb{P}(\mathfrak{g})$ is not the Hilbert polynomial of the diagonal. In fact, its Chow class in $\mathbb{PV}$ is not the class of diagonal. We can use Theorem 6.1 to calculate the class $[\mathbb{P}(\mathfrak{g})]$. The only indices $\underline{d}$ that contribute are $(0,4,0), (1,1,2)$ and $(2,1,1)$.

For $\underline{d} = (0,4,0)$, the monomial $h_0^i h_1^j h_2^k$ appears in $\left[\mathbb{P}(\mathfrak{g})_{\underline{d}}\right]$ if and only if:

$$\begin{aligned}
1 \leq i \leq 1, \\
0 \leq j \leq 0, \\
1 \leq k \leq 1.
\end{aligned}$$

The only possible monomial is $h_0 h_2$.

For $\underline{d} = (1,1,2)$, the monomial $h_0^i h_1^j h_2^k$ appears in $\left[\mathbb{P}(\mathfrak{g})_{\underline{d}}\right]$ if and only if:

$$\begin{aligned}
1 \leq i \leq 1, \\
1 \leq j \leq 1, \\
0 \leq k \leq 0.
\end{aligned}$$

The only possible monomial is $h_0 h_1$.

For $\underline{d} = (2,1,1)$, the monomial $h_0^i h_1^j h_2^k$ appears in $\left[\mathbb{P}(\mathfrak{g})_{\underline{d}}\right]$ if and only if :

$$\begin{aligned}
0 \leq i \leq 0, \\
1 \leq j \leq 1, \\
1 \leq k \leq 1.
\end{aligned}$$

The only possible monomial is $h_1 h_2$.

Therefore, the class of $\mathbb{P}(\mathfrak{g})$ is:

$$\begin{aligned}[\mathbb{P}(\mathfrak{g})] &= \left[\mathbb{P}(\mathfrak{g})_{(0,4,0)}\right] + \left[\mathbb{P}(\mathfrak{g})_{(1,1,2)}\right] + \left[\mathbb{P}(\mathfrak{g})_{(2,1,1)}\right] \\ &= h_0h_1 + h_0h_2 + h_1h_2.\end{aligned}$$

This is not the class of the diagonal. So the main result in [7] (Theorem 4.6) is not true for more general limit linear series.

Bibliography

[1] Eduardo Esteves and Brian Osserman. Abel maps and limit linear series. *Rend. Circ. Mat. Palermo*, 62, 02 2011.

[2] Daniel Santana. *Abel maps in the Hilbert scheme of curves and limit linear series*. PhD thesis, IMPA, preprint.

[3] David Eisenbud and Joe Harris. Limit linear series: Basic theory. *Inventiones Mathematicae*, 85:337–371, 06 1986.

[4] Brian Osserman. A limit linear series moduli scheme. *Annales de l'institut Fourier*, 56, 08 2004.

[5] David Helm and Brian Osserman. Flatness of the linked grassmannian. *Proceedings of the American Mathematical Society*, 136, 06 2006.

[6] Pedro Rizzo. *Level-delta and stable limit linear series on singular curves*. PhD thesis, IMPA, 2013.

[7] Gabriel Muñoz. Abel maps and limit linear series for curves of compact type with three irreducible components. *Bull Braz Math Soc*, 49, 2018.

[8] Dustin Cartwright and Bernd Sturmfels. The hilbert scheme of the diagonal in a product of projective spaces. *International Mathematics Research Notices*, 2010, 12 2009.

[9] Eduardo Esteves. Limit linear series, an introduction. *Kyoto University Research Information Repository*, 1490:84–108, 05 2006.

[10] Eduardo Vital. *Limit linear series in three components*. PhD thesis, IMPA, to appear.