Enumerative geometry of rank loci

and (a selective retelling of my experience) learning from Steven Kleiman

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Six papers that spoke to me

[KL1] S.L. Kleiman and D. Laksov, *Schubert calculus*, Amer. Math. Monthly **79** (1972), no. 10, 1061–1082.

[KL2] S.L. Kleiman and D. Laksov, *On the existence of special divisors*, Amer. J. Math. **93** (1972), 431–436.

[KeL] G. Kempf and D. Laksov, *The determinantal formula of Schubert calculus*, Acta. Math. **132** (1974), 153–162.

[K1] S.L. Kleiman, *r-special subschemes and an argument of Severi's*, Adv. Math. **22** (1976), 1–31.

[K2] S.L. Kleiman, *The enumerative theory of singularities*, in "Real and complex singularities (Oslo 1976)", P. Holm (ed.), Sijthoff and Noordhoff (1977), 297–396.

[JK] T. Johnsen and S.L. Kleiman, *Rational curves of degree at most 9 on a general quintic threefold*, Comm. Alg. **25** (1996), no. 8, 2721–2753.

Rank loci

Steve Kleiman's work in intersection theory is often connected with the following basic paradigm:

Given a map between vector bundles $\varphi : E \to F$ over a variety X, we'd like to *"count" the number of points $x \in X$ over which φ_x has rank at most k*, for fixed $k \in \mathbb{N}$.

These *rank loci* are ubiquitous, and their (Chow or cohomology) classes are fundamental.

[KL1], a reprint of which Steve gave me in his office one afternoon at MIT, played a crucial role in shaping my understanding of these concepts as they relate to Grassmannians.

It is an entirely self-contained and comprehensive introduction to Schubert calculus, i.e., intersection theory on Grassmannians G, from Schubert's point of view (though cohomology, and the tautological sub- and quotient bundles on G are added) whose last section includes a discussion of active research (much of which remains active).

Porteous' formula for rank loci, and curve geometry

After I completed my undergrad studies at MIT (which involved writing a senior thesis under Steve), I did a master's degree in Paris, and attended Kleiman 60 (my first international conference).

My Ph.D thesis, under Joe Harris, is in Brill–Noether theory, the study of linear series on algebraic curves. As such, it was strongly informed by [KeL], [K1], and [KL2].

The article [KeL] of Kempf and Laksov, who were Steve's students, is a proof of what is typically referred to as *Porteous' formula* for the cohomology class of a rank locus, whenever the codimension is as expected. It realizes the class as an explicit determinant in Chern classes of the domain and target bundles.

The main result of [KL2] is a class calculation for the Brill–Noether locus G^r_d of linear series of fixed degree d and ambient projective dimension r on a nonsingular algebraic curve X. Kleiman and Laksov used the Porteous formula to realize $[G^r_d]$ as an explicit positive multiple of a power of the theta class on the Jacobian of X. The upshot is that G^r_d is always non-empty.

Revisiting and reworking classical mathematics

Anyone who has interacted with Steve in a serious way will have been impressed by his absolute integrity and careful scholarship, part of which is embodied by his emphasis on reading (or at least properly crediting) and writing mathematics.

The article [K2], which Steve dedicated to the memory of Norman Levinson, "who understood life and appreciated mathematics with classical roots", is a lovely example of this. In it he reduces the *Brill–Noether theorem* on the (nonemptiness and) dimension of G^r_d over a general curve in moduli to a dimensional transversality statement for $(d-r-1)$-planes intersecting g general secant lines of a rational normal curve in $\mathbb{P}^d$. In doing so, he builds on previous work of Severi, who had presented an incomplete "proof" of Brill–Noether.

Ultimately, Griffiths and Harris completed the proof of Brill–Noether by proving dimensional transversality.

Building connections

Steve has been instrumental in building bridges between mathematical communities in Norway, Brazil, and the US.

My undergraduate thesis with Steve built on [JK], whose focus is *Clemens' conjecture* about the finiteness of rational curves in every fixed degree on a general quintic threefold in $\mathbb{P}^4$.

A subsidiary theme in [JK] is that of counting conditions on linear series imposed by singularities of particular types, which relates to [K2] (though in [K2] the emphasis is on counting in the usual zero-dimensional setting).

Reconsidering and reworking the counts of Johnsen and Kleiman using valuation-theoretic tools has led, in turn, to a number of papers with Renato Vidal Martins and his/our Ph.D students over the past nine years. At issue is: to what extent does the local geometry of curve singularity of fixed delta-invariant g recapitulate the global geometry of a smooth curve of genus g?

Rank loci over arbitrary fields

Another exciting line of inquiry is to look for enumerative formulae for rank loci over arbitrary base fields k.

This is possible within the context of $\mathbb{A}^1$-homotopy theory, an algebraic topological theory of Morel and Voevodsky which has been repurposed to spectacular effect in algebraic geometry by Kass, Wickelgren, and Levine. Formulae are valued in the *Grothendieck–Witt group* $\mathrm{GW}(k)$ of quadratic forms over k, which has a standard presentation in terms of generators and relations.

Ex: $\mathrm{GW}(\mathbb{C}) = \mathbb{Z}$, as every quadratic form is determined up to isomorphism by its rank. Similarly, $\mathrm{GW}(\mathbb{R}) = \mathbb{Z} \oplus \mathbb{Z}$ via (rank; signature) and $\mathrm{GW}(\mathbb{F}_q) = \mathbb{Z} \oplus \mathbb{Z}/2$ via (rank; discriminant, modulo squares) whenever q is not a power of 2. So enumerative formulae valued in $\mathrm{GW}(k)$ are "decorated" analogues of classical formulae.

Technical caveat: We usually require 1) the rank locus to be realizable as a 0-dimensional "Euler class", i.e., as the zero locus of a section of a bundle $\mathcal{V}$ of rank equal to the dimension of the underlying variety X; and 2) $\mathcal{V}$ to be *relatively orientable*.

Curve geometry over arbitrary fields

We now apply this machinery to produce decorated versions of classical enumerative formulae for linear series on curves.

Prototype: *Plücker's formula* for total inflection of a g^r_d (L, V) on a curve X of genus g. Over $\mathbb{C}$, this is an explicit polynomial $\psi(g,r,d)$, equal to the determinant of "jet bundle" $J^{r+1}(L)$ with fiber $H^0(L/L(-(r+1)p))$ over $p \in X$. This is because the inflection locus is the rank-r locus of the evaluation map

$$V \otimes \mathcal{O} \to J^{r+1}(L).$$

Thm (C.-Darago-Han, `arXiv:2010.01714`, to appear in Math. Nachrichten): $\mathbb{A}^1$-analogue of Plücker's formula for arbitrary multiples of the g^1_2 on a hyperelliptic curve over a field not of characteristic 2.

The global $\mathbb{A}^1$-analogue of $\psi(g,r,d)$ is $\frac{\psi(g,r,d)}{2}\mathbb{H}$, where $\mathbb{H} = \langle 1 \rangle + \langle -1 \rangle$. We also give explicit formulae for *local* $\mathbb{A}^1$-inflectionary indices.

Happy birthday, Steve, and thank you for always being such a generous mentor and friend.

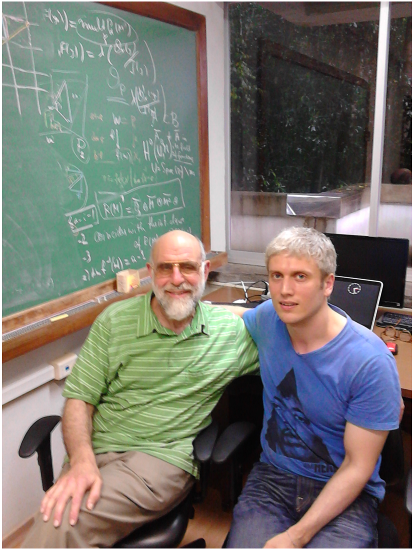

Figure: At Impa, July 2015 (photo credit: A. Rangachev)