Instituto De Matemática Pura E Aplicada

Doctoral Thesis

under the supervision of **Hubert LACOIN**

Mixing Time for Interface Models and Particle System

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This thesis is based on the following works:

1. **S. Yang,** Cutoff for polymer pinning dynamics in the repulsive phase
   Arxiv:1909.04635 to appear in AIHP Probabilités et Statistiques
2. **H. Lacoin, S. Yang,** Metastability for expanding bubbles on a sticky substrate
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3. **H. Lacoin, S. Yang,** Mixing time of the asymmetric simple exclusion process in random environment Arxiv:2102.02606 submitted for publication

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Contents

Summary

Markov chain is the simplest stochastic process describing the evolution of random phenomenon such that given the present, the future of process does not depend on the past, which was first studied by Markov [Mar06] in 1906. Since Markov chain plays an important and ubiquitous role in statistical mechanics, population dynamics, Monte Carlo simulation etc., nowadays it is still very active and widely studied in mathematics ([Ald83b, Ald83a, BD92]), physics ([DS87, SZ92, MO94]) and computer science ([JVV86, JSV04, Sin93]). In this thesis, we focus on Markov chain in the perspective of out-of-equilibrium statistical mechanics.

Statistical mechanics aims at explaining the laws governing the macroscopic observable of a physical system (temperature, pressure, magnetization, etc...) via a probabilistic representation at the microscopic scale. The state of a system at equilibrium is given by a probability distribution on a finite state space, which assigns to each possible configuration a probability proportional to a Gibbs weight defined by a Hamiltonian functional describing the energy of the system and an inverse temperature. This equilibrium distribution is called the Gibbs state of the system. In out-of-equilibrium statistical mechanics, the time evolution of a system is modeled by a Markov chain for which the Gibbs state is an invariant measure.

In this thesis we are interested in dynamics of the heat-bath type. Considering an initial configuration which differs from the equilibrium measure, we let the system evolve dynamically as follows: given the current configuration, the state of every site is updated at a constant rate, and the updated state of a site is sampled from the equilibrium measure conditioning on the states of all the other sites unchanged. The central problem in the thesis is how long the dynamics needs to relax to equilibrium, which is referred to as *mixing time*. The first chapter of the thesis is devoted to an introduction to mixing for continuous-time Markov chains. Each of the next three chapters presents original research concerning the study of a specific Markov chain. More precisely, a brief introduction of these three chapters is listed as follows:

- Our second chapter focuses on a dynamical version of the directed random polymer pinning model which considers the paths of the one-dimensional nearest-neighbor simple random walk interacting with an impenetrable defected line. The statics of the model have been well studied, and we refer to [Gia07] for reviews. In the static aspect, the model exhibits a delocalized phase where the polymer fluctuates freely except for obeying the positive constraint due to the impenetrable line and a localized phase where the polymers get localized on the defected line. Caputo et al. in [CMT08] introduced the dynamical version of the model, and studied the relaxation of the heat-bath dynamics to equilibrium in the delocalized/localized phases.

  In the delocalized phase, we show that the heat-bath dynamics suddenly changes from being poorly mixed to being well mixed when looking at the proper time scale, which improved a previous result by Caputo et al. This chapter is a preprint in Arxiv:1909.04635, which will appear in the journal of Annales de l'Institut Henri Poincaré, Probabilités et Statistiques.
- In the third chapter, we consider a variant of the polymer pinning model where the polymers are also subjected to another external force pulling the polymer surfaces away from the impenetrable defected line. We give a full phase diagram for the statics identifying the localized/delocalized phases. Concerning the heat-bath dynamics of the model, we derive a full phase diagram separating the rapidly/slowly mixing phases where the system relaxes to equilibrium in polynomial/superpolynomial time in terms of the size of the interacting defected line. Whereas in the slowly mixing phase, the dynamics mixes in exponential time and also exhibits metastablility. More precisely, there are two local wells in the dynamics with one deeper than the other. If the dynamics starts from

the shallow well, it thermalizes in the local well in polynomial time and takes another exponential time to pass the tunnel to enter the deeper well where the dynamics gets absorbed. This chapter is a preprint in Arxiv:2007.07832 (submitted for publication), which is a joint work with Hubert Lacoin.

- The fourth and final chapter of this thesis is devoted to the study of the simple exclusion process in a random environment. The simple exclusion process is one of the simplest particle system. It is a very simplified lattice model for a gas of colliding particles, and we refer to [Lig12, Chapter VIII.6] for a historical introduction. In this process, particles perform independent continuous-time random walks on the lattice subject to the exclusion rule, that is, each lattice site can be occupied by at most one particle.

  The problem of mixing time for the simple exclusion process on the line segment has been extensively studied in the case where the jump rates of the underlying random walks are homogeneous, see [Wil04, Lac16b] for the symmetric case, [BBHM05, LL19] for asymmetric exclusion and [LP16, LL20] for the weakly asymmetric (that is asymmetry tending to zero with the size of the system). Much more recently the problem has been considered for the case of spatially inhomogeneous environment in [Sch19]. Assuming that the particles have a tendency to move to the right (or left), we show that the dynamics mixes in polynomial time. This chapter is a preprint in Arxiv:2102.02606 (submitted for publication), collaborated with Hubert Lacoin.

CHAPTER 1

An introduction to mixing for continuous-time Markov chains

The aim of this chapter is to introduce the main topic of the thesis to the readers: mixing times for continuous-time Markov chains. We only cover the basic notions which are necessary to the understanding of our research results and refer to [Nor98, LP17] for a more complete pedagogical approach to continuous-time Markov chains and mixing times respectively.

This introduction is organized as follows: In Section 1, we provide an axiomatic definition of a continuous-time Markov chain on a finite state space (defined via a few properties), provide an explicit construction of the process, and discuss the existence and uniqueness of its stationary probability measure. In Section 2, we introduce the mixing times of continuous-time Markov chains, which are of central interests in this thesis. In Section 3, we focus on reversible Markov chains and discuss about the eigenvalues and eigenfunctions of the semi-group. In Section 4, we introduce the cutoff phenomena of continuous-time Markov chains, which are in the focus of Chapter 2. In Section 5, we give a picture about the next three chapters.

1. Definition of continuous-time Markov chains

A continuous-time Markov chain on a finite discrete space Ω is defined by its generator. The generator $\mathcal{L} = (r(x,y))_{x,y\in\Omega}$ of a Markov chain is an $\Omega \times \Omega$ matrix whose elements satisfies

$$\begin{cases} r(x,y) \geq 0, & \text{if } x \neq y \in \Omega, \\ \sum_{y\in\Omega} r(x,y) = 0, & \forall x \in \Omega. \end{cases} \tag{1.1}$$

The second line implies that $r(x,x) = -\sum_{y:y\neq x} r(x,y) =: -r(x)$. The generator $\mathcal{L}$ can be identified with the following homeomorphism on $\mathbb{R}^\Omega$

$$(\mathcal{L}f)(x) := \sum_{y\in\Omega} r(x,y)\left(f(y) - f(x)\right). \tag{1.2}$$

Intuitively, the Markov chain with generator $\mathcal{L}$ is an $\Omega-$valued stochastic process, which when located at state x, jumps to state y at a rate $r(x,y)$ independently from the past. To give a rigorous definition which corresponds to this informal description, we first introduce the Markov semi-group corresponding to $\mathcal{L}$.

Proposition 1.1. *We define Markov semi-group associated with $\mathcal{L}$ to be*

$$P_t := e^{t\mathcal{L}} = \sum_{k=0}^{\infty} \frac{(t\mathcal{L})^k}{k!}, \tag{1.3}$$

using the convention $\mathcal{L}^0 := \mathrm{Id}$ and $0! := 1$. For all $t \geq 0$, the matrix $P_t = e^{t\mathcal{L}}$ is well-defined satisfying for all $s,t \geq 0$

$$P_{t+s} = P_t P_s. \tag{1.4}$$

Moreover, P_t is a stochastic matrix for all $t \geq 0$. That is to say, P_t satisfies

$$\begin{aligned} P_t(x,y) \geq 0, \quad \forall x, y \in \Omega; \\ \sum_{y\in\Omega} P_t(x,y) = 1, \quad \forall x \in \Omega. \end{aligned} \tag{1.5}$$

Furthermore, for all $t \geq 0$, the matrix P_t satisfies $P_0 = \mathrm{Id}$ and the following equations:

$$\begin{aligned} \frac{\mathrm{d}P_t}{\mathrm{d}t} &= \mathcal{L}P_t \quad \textit{(the backward equation)}, \\ \frac{\mathrm{d}P_t}{\mathrm{d}t} &= P_t\mathcal{L} \quad \textit{(the forward equation)}, \end{aligned} \tag{1.6}$$

and

$$e^{-t\mathcal{L}}P_t = P_t e^{-t\mathcal{L}} = \mathrm{Id}. \tag{1.7}$$

Proof. To show that P_t is well-defined, we show that the series (1.3) is convergent for the ℓ_2 operator norm. We define the ℓ_2 norm of $f \in \mathbb{R}^\Omega$ as

$$\|f\|_2 := \left(\sum_{x\in\Omega} f(x)^2\right)^{1/2},$$

and then for any $|\Omega| \times |\Omega|$ matrix $A : \Omega \times \Omega \mapsto \mathbb{R}$ the ℓ_2 operator norm by

$$\|A\| := \sup\left\{\frac{\|Af\|_2}{\|f\|_2} : \|f\|_2 \neq 0\right\}.$$

It satisfies

$$\begin{aligned} \|AB\| &\leq \|A\| \cdot \|B\|, \\ \|A+B\| &\leq \|A\| + \|B\|. \end{aligned} \tag{1.8}$$

Thus since $\|\mathcal{L}\| \leq |\Omega| \max_{x\in\Omega} r(x)$, we know that

$$\|e^{t\mathcal{L}}\| \leq \sum_{k=0}^{\infty} \frac{t^k \|\mathcal{L}\|^k}{k!} = e^{t\|\mathcal{L}\|} < \infty. \tag{1.9}$$

Therefore $P_t = e^{t\mathcal{L}}$ is well-defined for all $t \geq 0$, and

$$P_t P_s = \sum_{k=0}^{\infty} \frac{t^k\mathcal{L}^k}{k!} \sum_{j=0}^{\infty} \frac{t^j\mathcal{L}^j}{j!} = \sum_{n=0}^{\infty} \mathcal{L}^n \sum_{\substack{k,j \\ k+j=n}} \frac{t^k s^j}{k!j!} = P_{t+s}. \tag{1.10}$$

We postpone the proof of (1.5) to the end of Subsection 1.1 where we rely on a graphical construction. Now we move to prove (1.6) and (1.7). From the definition

$$P_t = e^{t\mathcal{L}} = \sum_{k=0}^{\infty} \frac{t^k\mathcal{L}^k}{k!}$$

and by the Dominate Convergence Theorem, we obtain (1.6). Moreover, since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} e^{-t\mathcal{L}}P_t &= -e^{-t\mathcal{L}}\mathcal{L}P_t + e^{-t\mathcal{L}}\mathcal{L}P_t = 0, \\ \frac{\mathrm{d}}{\mathrm{d}t} P_t e^{-t\mathcal{L}} &= -P_t\mathcal{L}e^{-t\mathcal{L}} + P_t\mathcal{L}e^{-t\mathcal{L}} = 0, \end{aligned} \tag{1.11}$$

we obtain (1.7).

□

Before defining a continuous-time Markov chain, we let $\mathcal{D}(\mathbb{R}_+, \Omega)$ be the space of càdlàg functions mapping $\mathbb{R}_+$ to Ω, and

$$\mathcal{B} := \sigma\left(\bigcap_{i=1}^{n}\{f \in \mathcal{D}(\mathbb{R}_+, \Omega) \ : \ f(t_i) = x_i\}\, ; x_i \in \Omega, t_i \in \mathbb{R}_+, n \in \mathbb{N}\right) \tag{1.12}$$

is the $\sigma-$algebra generated by the finite dimensional cylinder sets of $\mathcal{D}(\mathbb{R}_+, \Omega)$.

DEFINITION 1.2 (Continuous-time Markov chain). *Let ν be a probability measure on Ω. The random process $(X_t)_{t\geq 0}$ taking values in $\mathcal{D}(\mathbb{R}_+, \Omega)$ with probability distribution $\mathbf{P}$ is a continuous-time Markov chain with generator $\mathcal{L}$ and initial distribution ν if*

(1) *we have*

$$\mathbf{P}\left[X_0 = x\right] = \nu(x), \forall x \in \Omega; \tag{1.13}$$

(2) *the process $(X_t)_{t\geq 0}$ satisfies Markov property which is for $0 \leq t_1 < \cdots < t_n < s < s+t$*

$$\mathbf{P}\left[X_{s+t} = y | X_s = x; X_{t_k} = z_k, \forall k \leq n\right] = \mathbf{P}\left[X_{s+t} = y | X_s = x\right] = P_t(x, y) \tag{1.14}$$

where P_t is the Markov semi-group associated with $\mathcal{L}$.

The following statement ensures that Markov chain is well defined.

THEOREM 1.3. *Given ν and $\mathcal{L}$, there exists a unique probability law $\mathbf{P}$ on $\mathcal{D}(\mathbb{R}_+, \Omega)$ satisfying the conditions stated in Definition 1.2.*

We prove the existence via a graphical construction in the following subsection.

PROOF OF UNIQUENESS. The conditions in (1.13) and (1.14) together provide the finite dimensional distributions of $\mathbf{P}$, *i.e.* such that for $0 =: t_0 \leq t_1 < t_2 < \cdots < t_n$

$$\mathbf{P}\left[f \in \mathcal{D}(\mathbb{R}_+, \Omega) : \ f(t_i) = x_i, 1 \leq i \leq n\right] = \sum_{x_0 \in \Omega} \nu(x_0) \prod_{i=1}^{n} P_{t_i - t_{i-1}}(x_{i-1}, x_i), \tag{1.15}$$

where ν is the initial distribution and P_t is defined in (1.3). The uniqueness of $\mathbf{P}$ is an immediate application of the $\pi - \lambda$ Theorem. $\square$

1.1. Existence of Markov chains via a graphical construction. In this subsection, we present a graphical construction of a Markov chain. A graphical construction is the construction of a stochastic processes by the means of auxiliary variables (typically Poisson clock processes and uniform variables). It is an important tool in the study of Markov process and is often used to provide coupling between Markov chains with different initial condition, for instance in Chapter 2 (Section 2.1) and Chapter 4 (Section 3.2).

To construct a càdlàg process $(\widetilde{X}_t)_{t\geq 0}$ whose distribution will be shown to satisfy the conditions of Definition 1.2, we associate a Poisson clock process and uniform variables on $[0, 1)$ with each $x \in \Omega$, denoted respectively by $\mathcal{T}_x = (\mathcal{T}_x(n))_{n\geq 0}$ and $\mathcal{U}_x = (U_x(n))_{n\geq 1}$ where $\mathcal{T}_x(0) = 0$ and

$$(\mathcal{T}_x(n) - \mathcal{T}_x(n-1))_{n\geq 1}$$

is a field of i.i.d. exponential random variables with mean $1/r(x)$. In addition, let $U(0)$ be a uniform variable on $[0, 1)$, and all the random variables $(\mathcal{T}_x, \mathcal{U}_x)_{x\in\Omega}$ and $U(0)$ are independent whose common law is denoted by $\mathbb{P}$. Given $(\mathcal{T}_x, \mathcal{U}_x)_{x\in\Omega}$ and $U(0)$, we construct $(\widetilde{X}_t)_{t\geq 0}$ in a deterministic manner, and for convenience of the statement we fix a labeling for all the elements of Ω as $(y_i)_{1\leq i\leq |\Omega|}$. If

$$\sum_{j<i} \nu(y_j) \leq U(0) < \sum_{j\leq i} \nu(y_j),$$

we set $\widetilde{X}_0 = y_i$. When the clock process $\mathcal{T}_x$ rings at time $t = \mathcal{T}_x(n)$ for $n \geq 1$ and $\widetilde{X}_{t^-} = x$, we update $\widetilde{X}_{t^-}$ as follows:

(1) for $r(x) = 0$, we set $\widetilde{X}_s = x$ for all $s \geq t$;
(2) for $r(x) > 0$, if
$$r(x)^{-1} \sum_{\substack{j<i \\ y_j \neq x}} r(x, y_j) \leq U_x(n) < r(x)^{-1} \sum_{\substack{j \leq i \\ y_j \neq x}} r(x, y_j),$$
we set $\widetilde{X}_t = y_i$.

From the graphical construction above, it is clear that the trajectory $(\widetilde{X}_t)_{t \geq 0}$ is càdlàg and
$$\mathbb{P}\left[\widetilde{X}_0 = x\right] = \nu(x), \forall x \in \Omega. \tag{1.16}$$
When we emphasize the initial distribution of $(\widetilde{X}_t)_{t \geq 0}$ as ν, we write $(\widetilde{X}^\nu_t)_{t \geq 0}$. In particular, when $\nu = \delta_x$, we write $(\widetilde{X}^x_t)_{t \geq 0}$. By the memoryless property of the Poisson clocks, we can adapt the proof in [Lan18, Proposition 1.4], whose the details are left for the readers, to show that $\widetilde{P}_t(x, y) := \widetilde{P}\left[\widetilde{X}^x_t = y\right]$ is well-defined and $\widetilde{P}_t$ is a transition matrix, and that for $0 \leq t_1 < \cdots < t_n < s$ and $t > 0$,
$$\mathbb{P}\left[\widetilde{X}_{s+t} = y \mid \widetilde{X}_s = x; \widetilde{X}_{t_k} = x_k, \forall k \leq n\right] = \mathbb{P}\left[\widetilde{X}_{s+t} = y \mid \widetilde{X}_s = x\right] = \widetilde{P}_t(x, y) \tag{1.17}$$
where the leftmost hand-side is only defined for the case $\mathbb{P}\left[\widetilde{X}_s = x; \widetilde{X}_{t_k} = x_k, \forall k \leq n\right] > 0$. Our next step is to show that $\widetilde{P}_t$ satisfies the forward equation in the following proposition, and then by the uniqueness of the solution to the forward equation we have $P_t = \widetilde{P}_t$. Therefore, the law of $(\widetilde{X}_t)_{t \geq 0}$ is $\mathbf{P}$.

Proposition 1.4 (The forward equation). *We have*
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \widetilde{P}_t &= \widetilde{P}_t \mathcal{L}, \\ \widetilde{P}_0 &= \mathrm{Id}. \end{aligned} \tag{1.18}$$
Therefore, for all $t \geq 0$ we have $P_t = \widetilde{P}_t$.

Proof. From the graphical construction, it is clear that $\widetilde{P}_0 = \mathrm{Id}$, and then we focus on $\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{P}_t(x, y)$. For all $h > 0$ sufficiently small, and $x, y \in \Omega$ (including the case $x = y$), by (1.17) we decompose the possible values of $\widetilde{X}^x_t$ to obtain
$$\begin{aligned} &\widetilde{P}_{t+h}(x, y) \\ =&\mathbb{P}\left[\widetilde{X}^x_t = y\right] \mathbb{P}\left[\widetilde{X}^y_h = y\right] + \sum_{z : z \neq y} \mathbb{P}\left[\widetilde{X}^x_t = z\right] \mathbb{P}\left[\widetilde{X}^z_h = y\right] \\ =&\mathbb{P}\left[\widetilde{X}^x_t = y\right] (1 - r(y)h + o(h)) + \sum_{z : z \neq y} \mathbb{P}\left[\widetilde{X}^x_t = z\right] \left(r(z)h \cdot \frac{r(z, y)}{r(z)} + o(h) \right). \end{aligned} \tag{1.19}$$
By (1.19), we obtain the right derivative of $\widetilde{P}_t(x, y)$, *i.e.*
$$\begin{aligned} &\lim_{h \to 0^+} \frac{1}{h} \left(\widetilde{P}_{t+h}(x, y) - \widetilde{P}_t(x, y) \right) \\ =& -\widetilde{P}_t(x, y) r(y) + \sum_{z : z \neq y} \widetilde{P}_t(x, z) r(z, y). \end{aligned} \tag{1.20}$$

Now we turn to the left derivative of $\widetilde{P}_t(x,y)$. Similar to (1.19), we can replace $t-h$ by t, and decompose the possible values of $\widetilde{X}^x_{t-h}$ to obtain

$$\begin{aligned}&\widetilde{P}_t(x,y)\\ =&\mathbb{P}\left[\widetilde{X}^x_{t-h}=y\right]\mathbb{P}\left[\widetilde{X}^y_h=y\right]+\sum_{z:z\neq y}\mathbb{P}\left[\widetilde{X}^x_{t-h}=z\right]\mathbb{P}\left[\widetilde{X}^z_h=y\right]\\ =&\mathbb{P}\left[\widetilde{X}^x_{t-h}=y\right](1-r(y)h+o(h))+\sum_{z:z\neq y}\mathbb{P}\left[\widetilde{X}^x_{t-h}=z\right]\left(r(z)h\cdot\frac{r(z,y)}{r(z)}+o(h)\right),\end{aligned} \tag{1.21}$$

and then

$$\lim_{h\to 0^+}|\widetilde{P}_t(x,y)-\widetilde{P}_{t-h}(x,y)|=0. \tag{1.22}$$

By (1.21) and (1.22), we obtain the left derivative of $\widetilde{P}_t(x,y)$, *i.e.*

$$\begin{aligned}&\lim_{h\to 0^+}\frac{1}{h}\left(\widetilde{P}_t(x,y)-\widetilde{P}_{t-h}(x,y)\right)\\ =&-\widetilde{P}_t(x,y)r(y)+\sum_{z:z\neq y}\widetilde{P}_t(x,z)r(z,y).\end{aligned} \tag{1.23}$$

Hence, writing in the matrix language, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{P}_t=\widetilde{P}_t\mathcal{L}. \tag{1.24}$$

By the uniqueness of the solution to the forward equation with given initial condition as Id (stated in (1.6) and (1.18)), for all $t\geq 0$ we have

$$P_t=\widetilde{P}_t.$$

□

PROOF OF (1.5). We know that $\widetilde{P}_t$ satisfies

$$\begin{aligned}&\widetilde{P}_t(x,y)\geq 0,\quad \forall x,y\in\Omega;\\ &\sum_{y\in\Omega}\widetilde{P}_t(x,y)=1,\quad \forall x\in\Omega.\end{aligned} \tag{1.25}$$

By Proposition 1.4, we conclude the proof for (1.5).

□

1.2. The invariant probability measure. In this subsection, we discuss the existence and uniqueness of invariant probability measures for a continuous-time Markov chain.

A probability measure μ on Ω is said to be invariant for the Markov chain associated with the generator $\mathcal{L}$ if for all $t>0$,

$$\mu P_t=\mu. \tag{1.26}$$

In other words, for all $y\in\Omega$,

$$\sum_{x\in\Omega}\mu(x)P_t(x,y)=\mu(y). \tag{1.27}$$

However, due to the complexity of the matrix P_t, it is difficult to verify the condition (1.27). But the following lemma provides a feasible method.

LEMMA 1.5. *The following two conditions are equivalent*

$$(\forall t\geq 0,\mu P_t=\mu)\Leftrightarrow(\mu\mathcal{L}=0). \tag{1.28}$$

Proof. First assuming $\mu P_t = \mu$, by the backward equation (1.6) we obtain

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\mu = \frac{\mathrm{d}}{\mathrm{d}t}\mu P_t = \mu \mathcal{L} P_t. \tag{1.29}$$

Multiplied by $e^{-t\mathcal{L}}$ in both sides above, by (1.7) we conclude the proof.

Now we assume $\mu\mathcal{L} = 0$, and by the backward equation (1.6) we obtain that for all $t \geq 0$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu P_t - \mu\right) &= \mu \mathcal{L} P_t = 0, \\ \mu P_0 - \mu &= 0. \end{aligned} \tag{1.30}$$

Therefore, for all $t \geq 0$, $\mu P_t = \mu$. □

Detailed balance condition. The probability measure μ is said to satisfy the detailed balance condition for the generator $\mathcal{L}$, if for all $x, y \in \Omega$

$$\mu(x) r(x,y) = \mu(y) r(y,x). \tag{1.31}$$

Lemma 1.6. *If μ satisfies the detailed balance condition, μ is an invariant probability measure for the system $(\Omega, \mathcal{L})$.*

Proof. By Lemma 1.5, we just need to check $\mu\mathcal{L} = 0$. Equivalently, that is for all $y \in \Omega$

$$\sum_{x\in\Omega} \mu(x) r(x,y) = \sum_{x:x\neq y} \mu(y) r(y,x) - \mu(y) r(y) = 0 \tag{1.32}$$

where the first equality uses the detailed balance condition and the last inequality uses $r(y) = \sum_{x:x\neq y} r(y,x)$. □

Existence and uniqueness. Before dealing with the uniqueness and existence of invariant probability measures, we introduce the irreducibility which intuitively means that a Markov chain can move between any two states.

Definition 1.7. *The system $(\Omega, \mathcal{L})$ is said to be irreducible if for all $x \neq y \in \Omega$ there exists a path with vertices in Ω: $\Gamma_{xy} = (z_0, z_1, \cdots, z_\ell)$ with $z_0 = x, z_\ell = y$ and $r(z_{k-1}, z_k) > 0$ for all $1 \leq k \leq \ell(x,y)$.*

Theorem 1.8. *If the system $(\Omega, \mathcal{L})$ is irreducible, there exists a unique invariant probability measure μ for the Markov chain $(X_t)_{t\geq 0}$ associated with $\mathcal{L}$.*

Proof. We reduce the issues of existence and uniqueness of the invariant probability measure of a continuous-time Markov chain to that of an associated discrete-time Markov chain. First we define the jumping instants of $(X_t)_{t\geq 0}$ by $S_0 := 0$, and for $n \geq 1$

$$S_{n+1} := \inf\left\{t \geq S_n : X_t \neq X_{S_n}\right\}, \tag{1.33}$$

and then define for all $n \geq 0$,

$$Y_n := X_{S_n}. \tag{1.34}$$

From the graphical construction, we know that $(Y_n)_{n\geq 0}$ is a discrete-time Markov chain and for $x \neq y \in \Omega$

$$\mathbb{P}\left[Y_{n+1} = y | Y_n = x\right] = r(x,y)/r(x). \tag{1.35}$$

Let μ be an invariant probability measure for $(X_t)_{t\geq 0}$, and by Lemma 1.5 we have for all $y \in \Omega$,

$$\sum_{x:x\neq y} \mu(x) r(x,y) = \mu(y) r(y) \tag{1.36}$$

which is the same as

$$\sum_{x:x\neq y} \mu(x)r(x)\frac{r(x,y)}{r(x)} = \mu(y)r(y). \tag{1.37}$$

Therefore, the probability measure $\widetilde{\mu}$ on Ω defined by

$$\forall y \in \Omega, \quad \widetilde{\mu}(y) := \frac{\mu(y)r(y)}{\sum_{x\in\Omega}\mu(x)r(x)} \tag{1.38}$$

is invariant for $(Y_n)_{n\geq 0}$. Moreover, we can also write μ in terms of $\widetilde{\mu}$, *i.e.* for all $y \in \Omega$

$$\mu(y) = \frac{\widetilde{\mu}(y)}{r(y)} \Big/ \sum_{x\in\Omega} \frac{\widetilde{\mu}(x)}{r(x)}. \tag{1.39}$$

By Definition 1.7, we know that the discrete-time chain $(Y_n)_{n\geq 0}$ is irreducible whose invariant probability measure exists and is unique (c.f. [LP17, Proposition 1.14 and Corollary 1.17]). By (1.38) and (1.39), we know that there exists a unique invariant probability measure for the continuous-time chain $(X_t)_{t\geq 0}$.

□

Convergence to equilibrium. Now we are concerned about the following convergence theorem.

THEOREM 1.9. *If the system $(\Omega, \mathcal{L})$ is irreducible, the distribution of $(X_t)_{t\geq 0}$ converges to its unique invariant probability measure μ, i.e.*

$$\lim_{t\to\infty} \sum_{y\in\Omega} \Big| \mathbb{P}\left[X_t = y\right] - \mu(y) \Big| = 0. \tag{1.40}$$

We postpone the proof of Theorem 1.9 in Section 2 where we will use coupling provided by the graphical construction and the following lemma.

LEMMA 1.10. *Assuming that the system $(\Omega, \mathcal{L})$ is irreducible, for all $x, y \in \Omega$ (including $x = y$) and all $t > 0$ we have*

$$P_t(x,y) > 0. \tag{1.41}$$

PROOF. By $P_t = (P_{t/n})^n$ for all $n \in \mathbb{N}$, we just need to prove (1.41) for all $t > 0$ sufficiently small. By the definition of P_t, we know that for $t \geq 0$ sufficiently small,

$$P_t = \mathrm{Id} + t\mathcal{L} + O(t^2). \tag{1.42}$$

Therefore, for $t > 0$ sufficiently small and $x \neq y$ satisfying $\mathcal{L}(x,y) > 0$, we have

$$P_t(x,x) > 0 \text{ and } P_t(x,y) > 0. \tag{1.43}$$

For $x \neq y$ with $\mathcal{L}(x,y) = 0$, let Γ_{xy} be a shortest path connecting x with y and $\ell(x,y) := |\Gamma_{xy}|$, and then $(\mathcal{L})^\ell(x,y) > 0$ and $(\mathcal{L})^k(x,y) = 0$ for all $k < \ell$. Since for $t > 0$ sufficiently small

$$P_t = \mathrm{Id} + \sum_{k=1}^{\ell} \frac{t^k\mathcal{L}^k}{k!} + O(t^{\ell+1}), \tag{1.44}$$

we have $P_t(x,y) > 0$. As $|\Omega| < \infty$, for all $t > 0$ sufficiently small we have $P_t(x,y) > 0$ for all $x, y \in \Omega$. □

2. Markov chain mixing

From now on, we always assume that the system $(\Omega, \mathcal{L})$ is irreducible. As we have seen in last section, an irreducible continuous-time Markov chain in finite state space will converge to its equilibrium measure. We are interested in how long the chain takes to approximate its equilibrium measure with prescribe distance, which is *mixing time* and the center topic of the thesis. In this section, we introduce the total variation mixing time, and give a relation between mixing times at different prescribed distances.

As mentioned above, we need to assign a distance between the equilibrium measure μ and $\mathbb{P}[X_t^\nu = \cdot]$, *i.e.* the distribution of the chain $(X_t^\nu)_{t\geq 0}$ at time t. Generally speaking, for two probability measures α, β on Ω, we define the distance between them to be

$$\|\alpha - \beta\|_{\mathrm{TV}} := \sup_{A\subset\Omega} |\alpha(A) - \beta(A)|\,. \tag{2.1}$$

This is the the total variation distance which measures the largest possible difference between two probability measures assigned to the same event. As we want to sample the observables and spatial properties of a system from the dynamics, the total variation distance tells the confidence interval. That is why the total variation distance is widely used in Markov chain mixing. Note that

$$\|\alpha - \beta\|_{\mathrm{TV}} = \frac{1}{2}\sum_{x\in\Omega} |\alpha(x) - \beta(x)| \tag{2.2}$$

where the last equality is by $\alpha(A) - \beta(A) = -\left(\alpha(A^{\complement}) - \beta(A^{\complement})\right)$ and

$$\sup_{A\subset\Omega} (\alpha(A) - \beta(A)) = \sum_{x:\alpha(x)\geq\beta(x)} (\alpha(x) - \beta(x)) = -\sum_{x:\alpha(x)<\beta(x)} (\alpha(x) - \beta(x))\,. \tag{2.3}$$

For another probability measure γ on Ω, due to the triangle inequality

$$|\alpha(x) - \beta(x)| \leq |\alpha(x) - \gamma(x)| + |\gamma(x) - \beta(x)|,$$

we have

$$\|\alpha - \beta\|_{\mathrm{TV}} \leq \|\alpha - \gamma\|_{\mathrm{TV}} + \|\gamma - \beta\|_{\mathrm{TV}}.$$

That is to say, the total variation distance is a metric. The following proposition says that the total variation distance measures how well we can couple two random variables with distribution laws α and β respectively. We say that ϑ is a coupling of α and β, if ϑ is a probability measure on $\Omega \times \Omega$ such that $\vartheta(x \times \Omega) = \alpha(x)$ and $\vartheta(\Omega \times y) = \beta(y)$ for any elements $x, y \in \Omega$. We refer to [LP17, Proposition 4.7] for a complete proof.

PROPOSITION 2.1. *Let α and β be two probability distributions on Ω, and then*

$$\|\alpha - \beta\|_{\mathrm{TV}} = \inf\Big\{\vartheta\left(\{(x,y) : x \neq y\}\right) : \ \vartheta \text{ is a coupling of } \alpha \text{ and } \beta.\Big\}.$$

Moreover, there exists a coupling which attains the infimum above.

Furthermore, the operator P_t, defined in (1.14), is contractive which is the following lemma.

LEMMA 2.2. *We have*

$$\|\alpha P_t - \beta P_t\|_{\mathrm{TV}} \leq \|\alpha - \beta\|_{\mathrm{TV}}, \tag{2.4}$$

and then

$$\|\alpha P_t - \mu\|_{\mathrm{TV}} \leq \|\alpha - \mu\|_{\mathrm{TV}}. \tag{2.5}$$

PROOF. Note that

$$\begin{aligned} 2\|\alpha P_t - \beta P_t\|_{\mathrm{TV}} &= \sum_{y\in\Omega} |(\alpha P_t)(y) - (\beta P_t)(y)| \\ &= \sum_{y\in\Omega} \left| \sum_{x\in\Omega} (\alpha(x) - \beta(x))\, P_t(x,y) \right| \\ &\le \sum_{y\in\Omega} \sum_{x\in\Omega} |(\alpha(x) - \beta(x))|\, P_t(x,y) \\ &= 2\|\alpha - \beta\|_{\mathrm{TV}}, \end{aligned} \tag{2.6}$$

where we use the triangle inequality in the inequality, and interchange the sum orders of x, y. Concerning (2.5), by $\mu P_t = \mu$ we take $\beta = \mu$ in (2.4) to conclude the proof. □

By Theorem 1.9 and Lemma 2.2, the total variation distance between the distribution of a continuous-time chain and its equilibrium is decreasing. Since a chain can start with any initial distribution, we define the distance to equilibrium as

$$\begin{aligned} d(t) &:= \sup \{\|\nu P_t - \mu\|_{\mathrm{TV}} : \nu \text{ is a probability measure on } \Omega\} \\ &= \max_{x\in\Omega} \|P_t(x,\cdot) - \mu\|_{\mathrm{TV}}, \end{aligned} \tag{2.7}$$

where we have used

$$\|\nu P_t - \mu\|_{\mathrm{TV}} = \left\| \sum_{x\in\Omega} \nu(x) P_t(x,\cdot) - \mu \right\|_{\mathrm{TV}} \le \sum_{x\in\Omega} \nu(x) \|P_t(x,\cdot) - \mu\|_{\mathrm{TV}}.$$

By Lemma 2.2, the function $d(t)$ is decreasing. Furthermore, we are interested in how long the Markov chain with the worst initial distribution needs to be within a prescribed distance to equilibrium. That is, for given $\varepsilon \in (0,1)$, the $\varepsilon-$mixing-time is defined to be

$$t_{\mathrm{mix}}(\varepsilon) := \inf \{t \ge 0 : d(t) \le \varepsilon\}. \tag{2.8}$$

For simplicity of notation, we write $t_{\mathrm{mix}} := t_{\mathrm{mix}}(1/4)$.

In the study of mixing, we define another distance to equilibrium as

$$\bar{d}(t) := \sup_{x,y\in\Omega} \|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{TV}}, \tag{2.9}$$

and we have

$$d(t) \le \bar{d}(t) \le 2d(t) \tag{2.10}$$

which is a corollary of $\mu P_t = \mu$ and the triangle inequality. With (2.10), now we are ready for the proof of Theorem 1.9.

PROOF OF THEOREM 1.9. Note that the graphical construction in Subsection 1.1 provides a coupling such that all the dynamics $\{(X_t^x)_{t\ge 0} : \forall x \in \Omega\}$ live in one common probability space. For the two dynamics $(X_t^x)_{t\ge 0}$ and $(X_t^y)_{t\ge 0}$ starting from $x, y \in \Omega$ respectively, we define their coalesce time to be

$$\tau_{xy} := \inf \{t \ge 0 : X_t^x = X_t^y\}. \tag{2.11}$$

For all $t \ge \tau_{xy}$, from the graphical construction we know that

$$X_t^x = X_t^y.$$

Therefore, by Proposition 2.1 we have

$$d(t) \le \bar{d}(t) \le \sup_{x,y} \mathbb{P}[X_t^x \ne X_t^y] = \sup_{x,y} \mathbb{P}[\tau_{xy} > t]. \tag{2.12}$$

For fixed $t_0 > 0$, by Lemma 1.10 we have

$$\lambda := \min_{x,y} P_{t_0}(x, y) > 0, \tag{2.13}$$

so that two Markov chains starting from any two initial states will coalesce before t_0 with probability at least λ. Then by Markov property, for any positive integer n we have

$$\mathbb{P}\left[\tau_{xy} > nt_0\right] \leq (1-\lambda)^n. \tag{2.14}$$

Since the function $d(t)$ is decreasing, by (2.12) and (2.14) we obtain

$$\lim_{t\to\infty} d(t) = 0.$$

□

The remaining of this section is concerned with the question: what is the decay rate of the distance to equilibrium? We answer this question in terms of t_{mix}.

2.1. The decay rate of the distance to equilibrium in terms of t_{mix}. We are concerned with the decay rate of the distance to equilibrium in terms of t_{mix}, which is the following proposition.

PROPOSITION 2.3. *For all $t \geq 0$, we have*

$$d(t) \leq 2^{-\lfloor t/t_{\text{mix}} \rfloor}, \tag{2.15}$$

and for $\varepsilon \in (0, 1)$

$$t_{\text{mix}}(\varepsilon) \leq \left\lceil \frac{-\log \varepsilon}{\log 2} \right\rceil t_{\text{mix}}. \tag{2.16}$$

For the proof of Proposition 2.3, with the inequality (2.10), we still need to show that the function $\bar{d}(t)$ is submultiplicative, which is in the following lemma.

LEMMA 2.4. *The function $\bar{d}(t)$ is submultiplicative, i.e. for $s, t \geq 0$,*

$$\bar{d}(s+t) \leq \bar{d}(s)\bar{d}(t). \tag{2.17}$$

PROOF. Recall that $(X_t^x)_{t\geq 0}$ and $(X_t^y)_{t\geq 0}$ are two continuous-time Markov chains starting from x, y respectively, and their marginal distributions at time instant s are $P_s(x, \cdot)$ and $P_s(y, \cdot)$ respectively. By Proposition 2.1, there exists a coupling probability measure ϑ on $\Omega \times \Omega$ such that their marginal distributions are $P_s(x, \cdot)$ and $P_s(y, \cdot)$ respectively and

$$\|P_s(x, \cdot) - P_s(y, \cdot)\|_{\text{TV}} = \vartheta\left[X_s^x \neq X_s^y\right]. \tag{2.18}$$

We observe that

$$P_{s+t}(x, z) = \sum_{w\in\Omega} P_s(x, w)P_t(w, z) = \sum_{w\in\Omega} \vartheta(X_s^x = w)P_t(w, z) = \mathbb{E}_\vartheta\left[P_t(X_s^x, z)\right] \tag{2.19}$$

where $\mathbb{E}_\vartheta$ denotes the expectation with respect to ϑ. Similarly, we have

$$P_{s+t}(y, z) = \mathbb{E}_\vartheta\left[P_t(X_s^y, z)\right]. \tag{2.20}$$

Moreover,

$$
\begin{aligned}
\|P_{s+t}(x,\cdot)-P_{s+t}(y,\cdot)\|_{\mathrm{TV}} &= \frac{1}{2}\sum_{z\in\Omega}|P_{s+t}(x,z)-P_{s+t}(y,z)| \\
&= \frac{1}{2}\sum_{z\in\Omega}|\mathbb{E}_{\vartheta}\left[P_t(X_s^x,z)\right]-\mathbb{E}_{\vartheta}\left[P_t(X_s^y,z)\right]| \\
&= \mathbb{E}_{\vartheta}\left[\frac{1}{2}\sum_{z\in\Omega}|P_t(X_s^x,z)-P_t(X_s^y,z)|\right] \\
&\le \mathbb{E}_{\vartheta}\left[\bar{d}(t)\mathbf{1}_{\{X_s^x\neq X_s^y\}}\right] \le \bar{d}(t)\bar{d}(s).
\end{aligned}
\tag{2.21}
$$

Since $x,y\in\Omega$ are arbitrary, we conclude (2.17).

□

Proof of Proposition 2.3. By (2.10) and Lemma 2.4, we have

$$
d\,(nt_{\mathrm{mix}}) \le \bar{d}\,(nt_{\mathrm{mix}}) \le \bar{d}\,(t_{\mathrm{mix}})^n \le \frac{1}{2^n}. \tag{2.22}
$$

We take $n\in\mathbb{N}$ such that the rightmost hand side of (2.22) is smaller than or equal to ε to conclude the proof.

□

3. Eigenvalues

The fact that the distribution of X_t^ν is νP_t and $P_t = e^{t\mathcal{L}}$ suggests us to investigate the eigenvalues and eigenfunctions of $\mathcal{L}$ to understand how the function νP_t evolves. Therefore we can tell how long the dynamics $(X_t^\nu)_{t\ge 0}$ needs to relax to equilibrium. In this section, we assume that the invariant measure μ for the irreducible system $(\Omega,\mathcal{L})$ satisfies the detailed balance condition in (1.31), so that we can introduce the eigenvalues (including spectral gap) and eigenfunctions of the generator $\mathcal{L}$. Furthermore, we study the relation between mixing time and spectral gap.

3.1. Eigenvalues and eigenfunctions of $\mathcal{L}$. To prepare the statement for the following proposition, we introduce an inner product denoted by $\langle\cdot,\cdot\rangle_\mu$ on $\mathbb{R}^\Omega$ defined by (for $f,g\in\mathbb{R}^\Omega$)

$$
\langle f,g\rangle_\mu := \sum_{x\in\Omega}\mu(x)f(x)g(x).
$$

Proposition 3.1. *Under the assumption of irreducibility and detailed balance condition, the inner product space $(\mathbb{R}^\Omega,\langle\cdot,\cdot\rangle_\mu)$ has an othonormal basis $(\Phi_i)_{i=1}^{|\Omega|}$ which are eigenfunctions of $\mathcal{L}$ corresponding to real eigenvalues $\{-\lambda_i\}_{i=1}^{|\Omega|}$. More precisely,*

$$
\begin{aligned}
\mathcal{L}\Phi_i &= -\lambda_i\Phi_i, \\
\langle\Phi_i,\Phi_j\rangle_\mu &= \delta_{ij},
\end{aligned}
\tag{3.1}
$$

where δ_{ij} represents the delta of Kronecker and $0=-\lambda_1>-\lambda_2\ge-\lambda_3\ge\cdots\ge-\lambda_{|\Omega|}$. Moreover, for the semi-group $P_t=e^{t\mathcal{L}}$, we have

$$
P_t\Phi_i = e^{-\lambda_i t}\Phi_i. \tag{3.2}
$$

PROOF. To study eigenvalues and eigenfunctions of $\mathcal{L}$, we turn to that of a symmetric matrix A such that $A(x,y) = A(y,x)$ for all $x, y \in \Omega$. With the assumption (1.31), we define the matrix A as: for all $x, y \in \Omega$

$$A(x,y) := \mu(x)^{1/2} \cdot r(x,y) \cdot \mu(y)^{-1/2} = \mu(y)^{1/2} \cdot r(y,x) \cdot \mu(x)^{-1/2} = A(y,x), \tag{3.3}$$

so that the matrix $A : \Omega \times \Omega \mapsto \mathbb{R}$ is symmetric. We introduce the usual inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}$ given by (for $f, g \in \mathbb{R}^\Omega$)

$$\langle f, g \rangle := \sum_{x \in \Omega} f(x) g(x).$$

From the spectral theorem about symmetric matrix, there is an orthonormal basis $(\phi_i)_{i=1}^{|\Omega|}$ such that ϕ_i is an eigenfunction with real eigenvalue $-\lambda_i$, *i.e.* $A\phi_i = -\lambda_i \phi_j$ and for all $1 \le i, j \le |\Omega|$

$$\langle \phi_i, \phi_j \rangle = \delta_{ij}.$$

Moreover, the function $\sqrt{\mu} = (\sqrt{\mu(x)})_{x \in \Omega}$ is an eigenfunction with eigenvalue 0, since for all $x \in \Omega$, $\sum_{y \in \Omega} r(x,y) = 0$.

Now we translate the eigenvalues and eigenfunctions of A to that of $\mathcal{L}$. Let D_μ be a diagonal matrix with diagonal entry at (x,x) be $\mu(x)$, and then $A = D_\mu^{1/2} \mathcal{L} D_\mu^{-1/2}$. Define $\Phi_i := D_\mu^{-1/2} \phi_i$ for all $1 \le i \le |\Omega|$, and then

$$\begin{aligned} \mathcal{L}\Phi_i &= -\lambda_i \Phi_i, \\ P_t \Phi_i &= e^{t\mathcal{L}} \Phi_i = e^{-\lambda_i t} \Phi_i, \\ \langle \Phi_i, \Phi_j \rangle_\mu &= \langle D_\mu^{1/2} \Phi_i, D_\mu^{1/2} \Phi_j \rangle = \delta_{ij}. \end{aligned} \tag{3.4}$$

That is to say, $(\Phi_i)_{1 \le i \le |\Omega|}$ is an orthonormal basis of the space $(\mathbb{R}^\Omega, \langle \cdot, \cdot \rangle_\mu)$.

We order the eigenvalues as $-\lambda_1 \ge -\lambda_2 \ge \cdots \ge -\lambda_{|\Omega|}$, and claim $\lambda_1 = 0$. Since P_t is a stochastic matrix, for any function $f \in \mathbb{R}^\Omega$ we have

$$\max_{x \in \Omega} |(P_t f)(x)| \le \max_{x \in \Omega} |f(x)| \tag{3.5}$$

and then $-\lambda_1 \le 0$. Moreover, the eigenfunction $\Phi_1 = D_\mu^{-1/2} \sqrt{\mu} = \mathbf{1}$ corresponds to the eigenvalue 0. Therefore $\lambda_1 = 0$.

We now argue that $-\lambda_2 < 0$ as we assume the Markov chain on Ω with generator $\mathcal{L}$ is irreducible. We argue by contradiction, supposing that $-\lambda_2 = 0$. Then we have

$$\begin{aligned} P_t \Phi_2 &= \Phi_2, \\ \langle \phi_1, \phi_2 \rangle &= \langle \sqrt{\mu}, D_\mu^{1/2} \Phi_2 \rangle = \sum_{x \in \Omega} \mu(x) \Phi_2(x) = 0. \end{aligned} \tag{3.6}$$

By irreducible assumption and (1.41), we investigate the equality $P_t \Phi_2 = \Phi_2$ at the coordinate $y \in \Omega$ such that $\Phi_2(y) = \max_{x \in \Omega} \Phi_2(x)$. We can see that Φ_2 is a constant function, which can not satisfy the second equality in (3.6). Therefore, we have $-\lambda_2 < 0$. ☐

DEFINITION 3.2. *Assuming the irreducibility and the detailed balance condition, we define the spectral gap of the Markov chain to be*

$$\text{gap} := \lambda_2, \tag{3.7}$$

which is also referred to as the minimal nonzero eigenvalue of $-\mathcal{L}$, and define the relaxation time as the inverse of the spectral gap, i.e.

$$t_{\text{rel}} := \text{gap}^{-1} = \lambda_2^{-1}. \tag{3.8}$$

To state a variational formula for the spectral gap we rely on the Dirichlet form defined by (for $f \in \mathbb{R}^{\Omega}$)

$$
\begin{aligned}
\mathcal{E}(f) := -\langle f, \mathcal{L} f\rangle_\mu &= \frac{1}{2}\sum_{x,y\in\Omega} \mu(x) r(x,y) f(x) \left(f(x) - f(y)\right) \\
&+ \frac{1}{2}\sum_{x,y\in\Omega} \mu(y) r(y,x) f(y) \left(f(y) - f(x)\right) \\
&= \frac{1}{2}\sum_{x,y\in\Omega} \mu(x) r(x,y) \left(f(x) - f(y)\right)^2
\end{aligned}
\tag{3.9}
$$

where we have used $\mu(x)r(x,y) = \mu(y)r(y,x)$ in the last equality. The variational formula is in the following lemma.

Lemma 3.3. *Let* $\mathrm{Var}_\mu(f) := \langle f, f\rangle_\mu - \langle f, \mathbf{1}\rangle_\mu^2$*, and then we have*

$$
\mathrm{gap} = \inf_{\mathrm{Var}_\mu(f)>0} \frac{-\langle f, \mathcal{L} f\rangle_\mu}{\mathrm{Var}_\mu(f)}. \tag{3.10}
$$

Proof. Since $(\Phi_i)_{i=1}^{|\Omega|}$ is an orthonormal basis of the inner product space $(\mathbb{R}^\Omega, \langle\cdot,\cdot\rangle_\mu)$, we can write $f \in \mathbb{R}^\Omega$ in terms of $(\Phi_i)_{i=1}^{|\Omega|}$ as follows:

$$
f = \sum_{i=1}^{|\Omega|} \langle f, \Phi_i\rangle_\mu \Phi_i. \tag{3.11}
$$

Therefore, we have

$$
\mathcal{E}(f) = -\langle f, \mathcal{L} f\rangle_\mu = -\left\langle \sum_{i=1}^{|\Omega|} \langle f, \Phi_i\rangle_\mu \Phi_i, \sum_{i=1}^{|\Omega|} \langle f, \Phi_i\rangle_\mu \mathcal{L}\Phi_i \right\rangle_\mu = \sum_{i=2}^{|\Omega|} \lambda_i \langle f, \Phi_i\rangle^2. \tag{3.12}
$$

where we have used $\langle \Phi_i, \Phi_j\rangle_\mu = \delta_{ij}$ and $\mathcal{L}\Phi_1 = \mathcal{L}\mathbf{1} = 0$. Similarly, we have

$$
\mathrm{Var}_\mu(f) = \sum_{i=2}^{|\Omega|} \langle f, \Phi_i\rangle_\mu^2 \tag{3.13}
$$

where we have used $\Phi_1 = \mathbf{1}$. For $f \in \mathbb{R}^\Omega$ satisfying $\mathrm{Var}_\mu(f) > 0$, there exists $i_0 \geq 2$ such that

$$
\langle f, \Phi_{i_0}\rangle_\mu \neq 0.
$$

Furthermore, we recall that $0 = -\lambda_1 > -\lambda_2 \geq gl_3 \geq \cdots \geq -\lambda_{|\Omega|}$ and then we have

$$
\frac{\mathcal{E}(f)}{\mathrm{Var}_\mu(f)} = \frac{\sum_{i=2}^{|\Omega|} \lambda_i \langle f, \Phi_i\rangle^2}{\sum_{i=2}^{|\Omega|} \langle f, \Phi_i\rangle_\mu^2} \geq \lambda_2. \tag{3.14}
$$

Moreover, when we take $f = \Phi_2$, we attain the equality in (3.14). Therefore, we conclude the proof. □

Analogously, we can repeat the proof in Lemma 3.3 to obtain for $f \in \mathbb{R}^\Omega$

$$
\mathrm{Var}_\mu(P_t f) \leq e^{-2t\cdot\mathrm{gap}} \mathrm{Var}_\mu(f) \tag{3.15}
$$

where the equality is attainable for $f = \Phi_2$. Therefore, the spectral gap tells the decay rate of the function $P_t f$. Furthermore, we will show a similar inequality for νP_t which is the distribution of X_t^ν, and we will see that the spectral gap is deeply related with the time for the dynamics $(X_t^\nu)_{t\geq 0}$ to relax to equilibrium.

3.2. The decay rate of the distance to equilibrium in terms of the relaxation time. In the following, we mimic the proof in [LP17, Theorem 12.4] to tell the decay rate of the distance to equilibrium in terms of the relaxation time.

THEOREM 3.4. *Assume that the system* $(\Omega, \mathcal{L})$ *is irreducible and reversible with respect to* μ*, and let* $\mu_{\min} := \min_{x\in\Omega} \mu(x)$*. For* $\varepsilon \in (0,1)$*, we have*

$$t_{\mathrm{mix}}(\varepsilon) \le t_{\mathrm{rel}} \log \frac{1}{2\varepsilon\mu_{\min}}, \tag{3.16}$$

$$t_{\mathrm{mix}}(\varepsilon) \ge t_{\mathrm{rel}} \log \frac{1}{2\varepsilon}, \tag{3.17}$$

$$\lim_{t\to\infty} \frac{1}{t} \log d(t) = -\mathrm{gap}, \tag{3.18}$$

where t_{rel} *is the relaxation time defined in* (3.8).

PROOF. We first deal with (3.16) by observing

$$2\|P_t(x,\cdot)\|_{\mathrm{TV}} = \sum_{y\in\Omega} \left| \frac{P_t(x,y)}{\mu(y)} - 1 \right| \mu(y). \tag{3.19}$$

We write $P_t(x,y) = \left(P_t \mathbf{1}_{\{y\}}\right)(x)$ in term of the eigenfunctions $(\Phi_i)_{1\le i\le|\Omega|}$ stated in Proposition 3.1. Since

$$\mathbf{1}_{\{y\}} = \sum_{i=1}^{|\Omega|} \langle \mathbf{1}_{\{y\}}, \Phi_i\rangle_\mu \Phi_i = \mu(y) \sum_{i=1}^{|\Omega|} \Phi_i(y)\Phi_i, \tag{3.20}$$

we have

$$P_t(x,y) = \mu(y) \sum_{i=1}^{|\Omega|} \Phi_i(y)(P_t\Phi_i)(x) = \mu(y) \sum_{i=1}^{|\Omega|} \Phi_i(y) e^{-\lambda_i t}\Phi_i(x). \tag{3.21}$$

Recalling $\Phi_1 = \mathbf{1}$ and $\lambda_1 = 0$, we have

$$\begin{aligned} \left| \frac{P_t(x,y)}{\mu(y)} - 1 \right| &= \left| \sum_{i=2}^{|\Omega|} \Phi_i(y) e^{-\lambda_i t} \Phi_i(x) \right| \\ &\le e^{-\lambda_2 t} \left(\sum_{i=2}^{|\Omega|} \Phi_i(y)^2 \right)^{1/2} \left(\sum_{i=2}^{|\Omega|} \Phi_i(x)^2 \right)^{1/2}, \end{aligned} \tag{3.22}$$

where we have used Cauchy-Schwarz inequality in the inequality above. In order to give an upper bound on the right-hand side of (3.22), we rely on (3.20) to obtain

$$\sum_{i=2}^{|\Omega|} \Phi_i(y)^2 \le \frac{1}{\mu(y)} \le \frac{1}{\mu_{\min}}. \tag{3.23}$$

Therefore, we have

$$\|P_t(x,\cdot) - \mu\|_{\mathrm{TV}} = \frac{1}{2} \sum_{y\in\Omega} \mu(y) \left| \frac{P_t(x,y)}{\mu(y)} - 1 \right| \le e^{-\lambda_2 t} \frac{1}{2\mu_{\min}}, \tag{3.24}$$

where we take the right-hand side smaller than ε to obtain (3.16).

We now turn to (3.17). We take the eigenfunction Φ_2 to obtain

$$\begin{aligned}\left|e^{-\lambda_2 t}\Phi_2(x)\right| = |(P_t\Phi_2)(x)| &= \left|\sum_{y\in\Omega} P_t(x,y)\Phi_2(y) - \sum_{y\in\Omega}\mu(y)\Phi_2(y)\right| \\ &\le 2d(t)\|\Phi_2\|_\infty\end{aligned} \tag{3.25}$$

where $\|f\|_\infty := \max_{x\in\Omega}|f(x)|$ for $f \in \mathbb{R}^\Omega$, and we have used

$$\langle \mathbf{1}, \Phi_2\rangle_\mu = \sum_{y\in\Omega}\mu(y)\Phi_2(y) = 0,$$

$$\sup_{f\in\mathbb{R}^\Omega:\|f\|_\infty\le 1}\left|\sum_{y\in\Omega} P_t(x,y)\Phi_2(y) - \sum_{y\in\Omega}\mu(y)f(y)\right| = 2\left\|P_t(x,\cdot)-\mu\right\|_{\mathrm{TV}}.$$

We take $x_0 \in \Omega$ satisfying $|\Phi_2(x_0)| = \|\Phi_2\|_\infty$ in (3.25) to obtain (3.17).

Now we head to (3.18). By (3.24), we have

$$\limsup_{t\to\infty}\frac{1}{t}\log d(t) \le -\lambda_2 = -\mathrm{gap}. \tag{3.26}$$

From (3.25), we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\log d(t) \ge -\lambda_2 = -\mathrm{gap}. \tag{3.27}$$

Combining (3.26) with (3.27), we conclude the proof for (3.18).

□

4. Cutoff phenomenon

In out-of-equilibrium of statistical mechanics, to model the dynamical evolution of a physical system we consider a sequence of Markov chains with state space size going to infinity, and the sequence of chains will be naturally indexed by the systems size. That is to say, let $(\Omega_n, \mathcal{L}_n)$ be a system with a unique invariant probability measure μ_n, and let $t^{(n)}_{\mathrm{mix}}(\varepsilon)$ be the associated $\varepsilon-$mixing-time. The central topic in the thesis is to study how the function $t^{(n)}_{\mathrm{mix}}(\varepsilon)$ grows in terms of n and ε.

We say that the sequence of Markov chains $(\Omega_n, \mathcal{L}_n)_{n\in\mathbb{N}}$ has a cutoff if for all $\epsilon \in (0,1)$,

$$\lim_{n\to\infty}\frac{t^{(n)}_{\mathrm{mix}}(\epsilon)}{t^{(n)}_{\mathrm{mix}}(1-\epsilon)} = 1. \tag{4.1}$$

Moreover, there is another equivalent definition as follow:

$$\lim_{n\to\infty} d_n\left(ct^{(n)}_{\mathrm{mix}}\right) = \begin{cases}1 & \text{if } c<1,\\ 0 & \text{if } c>1.\end{cases} \tag{4.2}$$

In other words, the sequence of Markov chains $(\Omega_n, \mathcal{L}_n)_{n\in\mathbb{N}}$ suddenly transitions from being poorly mixed to being well mixed. The cutoff phenomenon is first discovered in [DS81] and surveyed in the seminal paper [Dia96], and we refer to [LP17, Chapter 18] for more information. There are many Markov chains exhibiting the cutoff phenomenon for example: the riffle shuffle of a deck of n cards [BD92], the lazy random biased random walk on a line segment [LP17, Theorem 18.2], the symmetric simple random walk in the hypercube [LP17, Theorem 18.3]. In [Dia96, Section 5], Diaconis wrote:

> At present writing, proof of a cutoff is a difficult, delicate affair, requiring detailed knowledge of the chain, such as all eigenvalues and eigenvectors.

A step toward cutoff is to prove a precutoff for the sequence of Markov chains, that is,

$$\sup_{\varepsilon\in(0,\frac{1}{2})} \limsup_{n\to\infty} \frac{t^{(n)}_{\text{mix}}(\varepsilon)}{t^{(n)}_{\text{mix}}(1-\varepsilon)} < \infty. \tag{4.3}$$

By the definitions of cutoff and precutoff, we can see that cutoff implies precutoff. However, there exists an example in which precutoff holds but cutoff does not, and we refer to [LP17, Notes in Chapter 18] for such an example due to Aldous. Furthermore, there are many Markov chains for which only precutoff has been proved and for which cutoff is predicted such as the symmetry simple exclusion process on a line segment with open boundaries [GNS20]. Indeed, there also exist examples where there are no precutoff phenomena, and we refer to [LP17, Examples 18.5, 18.6].

5. What is done in the thesis?

In this section, we give macroscopic pictures and main ideas of the following three chapters where the last two chapters are based on joint works with Hubert Lacoin.

5.1. Chapter 2: Cutoff for polymer pinning dynamics in the repulsive phase.

Background. The study of effective interface models is a large field in statistical mechanics, in particular for the problem of wetting of a random walk which dates back to the seminal paper of [Fis84]. Several variants and generalizations of the model have been considered since then, and we refer to [Gia07, Gia11] for recent reviews. A polymer is a substance composed of many monomers, and between any two consecutive monomers there is a bond connecting them. A polymer interacts with a colloid or an attractive (repulsive) substrate, gaining rewards or penalty from the interaction.

Model. We model polymers in the framework of statistical mechanics in a simplest setting.

- We represent the colloid (substrate) by an impenetrable half-space.
- The polymers are modeled by the paths of the one-dimensional nearest-neighbor simple random walk, stretching in one direction to avoid the self-interaction of the polymers.

For simplification, we choose the state space to be

$$\Omega_L := \left\{\xi \in \mathbb{Z}_+^{L+1} \ : \ \xi_0 = \xi_L = 0 \ ; \forall x \in [\![1, L]\!], |\xi_x - \xi_{x-1}| = 1\right\}, \tag{5.1}$$

which is the set of nonnegative integer-value one-dimensional nearest-neighbor paths, starting at 0 and ending at 0 after L steps where $L \in 2\mathbb{N}$. Each path $\xi \in \Omega_L$ is given a probability proportional to $\lambda^{\mathcal{N}(\xi)}$, where $\lambda \geq 0$ is a parameter modeling the intensity of the interaction between the polymer and the $x-$axis (the impenetrable wall) and $\mathcal{N}(\xi)$ is the amount of contacts with the wall given by

$$\mathcal{N}(\xi) := \sum_{x=1}^{L-1} \mathbf{1}_{\{\xi_x=0\}}. \tag{5.2}$$

That is to say, we define a probability measure μ_L^λ on Ω_L as

$$\mu_L^\lambda(\xi) := \frac{\lambda^{\mathcal{N}(\xi)}}{Z_L(\lambda)} \tag{5.3}$$

where $\xi \in \Omega_L$ and $Z_L(\lambda)$ is the partition function given by

$$Z_L(\lambda) := \sum_{\xi'\in\Omega_L} \lambda^{\mathcal{N}(\xi')}. \tag{5.4}$$

The partition function encodes some information concerning the equilibrium behavior of the polymer such as the contact fraction

$$\frac{1}{L}\frac{\mathrm{d}\log Z_L(\lambda)}{\mathrm{d}\log\lambda} = \frac{1}{L}\frac{\sum_{\xi\in\Omega_L}\mathcal{N}(\xi)\lambda^{\mathcal{N}(\xi)}}{Z_L(\lambda)} = \mu_L^\lambda\left(\frac{\mathcal{N}(\xi)}{L}\right). \tag{5.5}$$

In the static aspect, we have the following detailed asymptotics for the partition function (cf. [Gia07, Theorem 2.2])

$$2^{-L}Z_L(\lambda) = \begin{cases}(1+o(1))C_\lambda L^{-3/2} & \text{if } \lambda\in[0,2),\\ (1+o(1))C_2 L^{-1/2} & \text{if } \lambda = 2,\\ (1+o(1))C_\lambda e^{LF(\lambda)} & \text{if } \lambda>2,\end{cases} \tag{5.6}$$

where $F(\lambda) = \mathbf{1}_{\{\lambda>2\}}\log\frac{\lambda}{2\sqrt{\lambda-1}}$. By Hölder inequality, for $\theta\in[0,1]$ and $\lambda_1,\lambda_2>0$, we have

$$Z_L(\lambda_1^\theta\lambda_2^{1-\theta}) \le Z_L(\lambda_1)^\theta Z_L(\lambda_2)^{1-\theta},$$

so that the function $F(\lambda) = \lim_{L\to\infty}\frac{1}{L}\log Z_L(\lambda) - \log 2$ is a convex function of $\log\lambda$ and differentiable with respect to $\log\lambda$. Therefore, we can interchange the positions of limit and derivative to obtain

$$\frac{\mathrm{d}F(\lambda)}{\mathrm{d}\log\lambda} = \lim_{L\to\infty}\mu_L^\lambda\left(\frac{\mathcal{N}(\xi)}{L}\right)\begin{cases}=0, & \text{if } \lambda\le 2,\\ >0, & \text{if } \lambda>2.\end{cases} \tag{5.7}$$

We can see that this model displays a transition from a delocalized phase to a localized phase (see [CMT08, Section 1]): (a) if $0\le\lambda<2$, the expected number of contacts $\mu_L^\lambda(\mathcal{N}(\xi))$ is uniformly bounded in L and the height of the middle point $\xi_{L/2}$ is typically of order $\sqrt{L}$; (b) if $\lambda>2$, the amount of contacts with the $x-$axis of typical paths is of order L and the distribution of the height of the middle point $\xi_{L/2}$ is (exponentially) tight in L. These two phases are referred to as the delocalized/localized phase respectively, at the critical point $\lambda=2$ the system displays an intermediate behavior.

We are interested in the classical heat-bath dynamics to equilibrium, which equilibrates the value of each ξ_x at rate one via corner-flip. It was first studied by Caputo et al. in [CMT08] . To clarify how to flip a corner, for $\xi\in\Omega_L$ and $x\in[\![1,L-1]\!]$ we define $\xi^x\in\Omega_L$ by

$$\xi^x_y := \begin{cases}\xi_y & \text{if } y\ne x,\\ (\xi_{x-1}+\xi_{x+1})-\xi_x & \text{if } y=x \text{ and } \xi_{x-1}=\xi_{x+1}\ge 1 \text{ or } \xi_{x-1}\ne\xi_{x+1},\\ \xi_x & \text{if } y=x \text{ and } \xi_{x-1}=\xi_{x+1}=0.\end{cases} \tag{5.8}$$

When $\xi_{x-1}=\xi_{x+1}$, ξ displays a local extremum at x and we obtain ξ^x by flipping the corner of ξ at the coordinate x, provided that $\xi^x\in\Omega_L$ (this excludes corner-flipping when $\xi_{x-1}=\xi_{x+1}=0$.). See Figure 1 for a graphical representation. Moreover, a generator $\mathcal{L}$ of a continuous-time Markov chain on Ω_L is given by (for $f\colon\Omega_L\to\mathbb{R}$)

$$(\mathcal{L}f)(\xi) := \sum_{x=1}^{L-1}R_x(\xi)\big[f(\xi^x)-f(\xi)\big], \tag{5.9}$$

and

$$R_x(\xi) := \begin{cases}\frac{1}{2} & \text{if } \xi_{x-1}=\xi_{x+1}>1,\\ \frac{\lambda}{1+\lambda} & \text{if } (\xi_{x-1},\xi_x,\xi_{x+1})=(1,2,1),\\ \frac{1}{1+\lambda} & \text{if } (\xi_{x-1},\xi_x,\xi_{x+1})=(1,0,1),\\ 0 & \text{if } \xi_{x-1}\ne\xi_{x+1} \text{ or } \xi_{x-1}=\xi_{x+1}=0.\end{cases}$$

The reader can check that μ_L^λ satisfies the detailed balance condition *i.e.*

$$\mu_L^\lambda(\xi)R_x(\xi) = \mu_L^\lambda(\xi^x)R_x(\xi^x),$$

and thus μ_L^λ is the invariant probability measure for the system $(\Omega_L, \mathcal{L})$. Our work focuses on the study of the mixing time for this dynamics (and small variations of it). Let us introduce first the results that have been previously obtained for comparison.

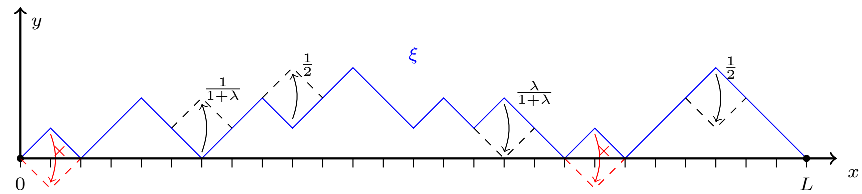


FIGURE 1. A graphical representation of the jump rates for the system pinned at $(0,0)$ and $(L,0)$. A transition of the dynamics corresponds to flipping a corner, whose rate depends on how it changes the number of contact points with the x-axis. The rates are chosen in a manner such that the dynamics is reversible with respect to μ_L^λ. The two red dashed corners are not available and labeled with $\times$, because of the nonnegative restriction of the state space Ω_L. Note that not all the possible transitions are shown in the figure.

Previous results. As in (2.8), let $t_{\text{mix}}^{L,\lambda}(\varepsilon)$ denote the $\varepsilon-$mixing-time of the system $(\Omega_L, \mathcal{L})$ with parameter λ. The following has been proved in [CMT08]:

- if $0 \le \lambda < 2$, then for all $\varepsilon \in (0,1)$
$$\frac{1}{2\pi^2} \le \liminf_{L\to\infty} \frac{t_{\text{mix}}^{L,\lambda}(\varepsilon)}{L^2 \log L} \le \limsup_{L\to\infty} \frac{t_{\text{mix}}^{L,\lambda}(\varepsilon)}{L^2 \log L} \le \frac{6}{\pi^2} \tag{5.10}$$
and there exists a positive constant $C > 0$ independent of λ such that
$$1 - \cos\left(\frac{\pi}{L}\right) \le \text{gap}_{L,\lambda} \le CL^{-2}. \tag{5.11}$$
Equation (5.10) implies that the dynamic displays pre-cutoff (see (4.3) above) for $\lambda \in [0,2)$. For $\lambda = 2$, $t_{\text{mix}}^{L,2}$ and $\text{gap}_{L,2}$ are conjectured to behave as (5.10) and (5.11) respectively, we refer to [CMT08] for more details.
- if $\lambda > 2$, Caputo et al. showed that there exists a positive constant $C > 0$ independent of λ such that
$$\liminf_{L\to\infty} \frac{t_{\text{mix}}^{L,\lambda}}{L^2} \ge C, \tag{5.12}$$
and
$$\text{gap}_{L,\lambda} \ge c(\lambda)L^{-1}. \tag{5.13}$$
In [CMT08], they believed that the lower bounds about $t_{\text{mix}}^{L,\lambda}$ and $\text{gap}_{L,\lambda}$ are sharp up to a constant factor.

Our results. In Chapter 2, we obtain sharper estimates concerning the mixing time in the regime $\lambda \in [0,2)$. First we prove that cutoff holds when $\lambda \in [0,1]$ by improving both the lower bound and the upper bound (by a factor 2 and 6 respectively).

THEOREM 5.1 (Theorem 1.1 of Chapter 2). *For $\lambda \in [0,1]$ and all $\varepsilon \in (0,1)$, we have*

$$\lim_{L\to\infty} \frac{\pi^2 t_{\text{mix}}^{L,\lambda}(\varepsilon)}{L^2 \log L} = 1. \tag{5.14}$$

For $\lambda \in (1,2)$, we provide a partial result similar to (5.14). Namely, we prove that cutoff holds for the mixing time of Markov chains starting from *extremal conditions*. Let $\wedge$ and $\vee$ respectively denote the highest and lowest configuration in Ω_L (see the definition (1.17) in Chapter 2 for a definition). We define

$$\check{t}^{L,\lambda}_{\mathrm{mix}}(\varepsilon) := \inf\left\{t \geq 0 : d^{\vee,\wedge}_L(t) \leq \varepsilon\right\} \tag{5.15}$$

where (recall that P_t is the semi-group associated with the dynamics)

$$d^{\vee,\wedge}_L(t) = \max\Big(\|P_t(\wedge,\cdot) - \mu^\lambda_L\|_{\mathrm{TV}}, \|P_t(\vee,\cdot) - \mu^\lambda_L\|_{\mathrm{TV}}\Big). \tag{5.16}$$

We prove the following:

THEOREM 5.2 (Theorem 1.1 of Chapter 2). *For $\lambda \in (1,2)$ and all $\varepsilon \in (0,1)$, we have*

$$\lim_{L\to\infty} \frac{\pi^2 \check{t}^{L,\lambda}_{\mathrm{mix}}(\varepsilon)}{L^2 \log L} = 1. \tag{5.17}$$

Let us conclude the presentation by giving a short insight about the techniques used in Chapter 2.

Main idea for the lower bound on the mixing time for $\lambda \in [0,2)$. As in [CMT08] we investigate the time evolution of the weighted area function $\Phi\colon \Omega_L \to \mathbb{R}$ defined by

$$\Phi(\xi) := \sum_{x=1}^{L-1} \xi_x \sin\left(\frac{\pi x}{L}\right), \tag{5.18}$$

which is almost the area enclosed by the x-axis and the path $\xi \in \Omega_L$. The function Φ was introduced in [Wil04, Equation (1)], which is the eigenfunction corresponding to the principle mode of the heat equation. Under equilibrium μ^λ_L, Φ is at most of order $L^{3/2}$, while for the dynamics $(\sigma^\wedge_t)_{t\geq 0}$ starting from the highest path, Φ is initially of order L^2. We show that the time needed for $\Phi(\sigma^\wedge_t)$ to become of order $L^{3/2}$ is at least $(1-o(1))\frac{1}{\pi^2}L^2\log L$. Our improvement w.r.t. to the approach in [CMT08], which allows to gain a factor 2 in the lower bound estimate, is that instead of simply controlling the first and second moments, we use more refined estimates involving martingale techniques (more precisely, the control of the bracket of a martingale closely related to $\Phi(\sigma^\wedge_t)$).

Main idea for the upper bound on the mixing time when $\lambda \in [0,1]$. We can reduce the problem to that of the coupling of a dynamic starting from the highest path $(\sigma^\wedge_t)_{t\geq 0}$ and a dynamic $(\sigma^\xi_t)_{t\geq 0}$ starting from an arbitrary initial configuration. For this coupling we use a specific graphical construction which conserves the order. We need to bound the coalescing time

$$\tau := \inf\{t > 0 \ : \ \sigma^\wedge_t = \sigma^\mu_t\}.$$

Inspired by [Wil04, Equation (1)], we define a function $\overline{\Phi}\colon \Omega_L \to [0,\infty)$ given by

$$\overline{\Phi}(\xi) := \sum_{x=1}^{L-1} \xi_x \cos\left(\frac{\beta(x-\frac{L}{2})}{L}\right)$$

where $\beta < \pi$ and β is chosen sufficiently close to π. When $\lambda \in [0,1]$, the process $A_t := \overline{\Phi}(\sigma^\wedge_t) - \overline{\Phi}(\sigma^\mu_t)$ is a nonnegative supermartingale (positivity comes the fact that our coupling is order-preserving) and τ corresponds to the hitting time of 0 by A_t. We use Wilson's argument to show that the decay rate of $\mathbb{E}[A_t]$ is at least $1-\cos(\beta/L)$ so that at time $t_0 = \frac{1+\delta}{\pi^2}L^2\log L$, A_t is much smaller than $L^{3/2}$. Then for $t \geq t_0$, we rely on a refined study of the variations of A_t, borrowing techniques from [LL20], to show that it only takes an extra amount of time of order L^2 for A_t to shrink from $L^{3/2}$ to zero.

Main idea for the upper bound on the mixing time when $\lambda \in (1,2)$. When $\lambda \in (1,2)$, the above technique breaks down since in that case A_t is not a super-martingale. In that case, we use a different approach based on the Peres-Winkler censoring inequality [PW13, Theorem 1.1]. This approach heavily depends on starting from an initial condition which is extremal, hence our partial result.

5.2. Chapter 3: Metastability for expanding bubbles on a sticky substrate.

Model. The model we consider is a close relative of the polymer pinning model considered in Chapter 2, but this time we consider a polymer which is also subjected to another external force, and this force pulls the polymer interface away from the wall. Since the two chapters have slightly different notation conventions, let us reintroduce the state space setting

$$\Omega_N := \Big\{\xi \in \mathbb{Z}_+^{2N+1} \;:\; \xi_0 = \xi_{2N} = 0 \;; \forall x \in [\![1,2N]\!], |\xi_x - \xi_{x-1}| = 1\Big\}. \tag{5.19}$$

For $\xi \in \Omega_N$, we denote by H and A respectively the number of zeros and the (algebraic) area between the path and the horizontal axis

$$H(\xi) := \sum_{x=1}^{2N-1} \mathbf{1}_{\{\xi_x=0\}} \quad \text{and} \quad A(\xi) := \sum_{x=1}^{2N} \xi_x.$$

We define a probability measure on Ω_N using a Gibbs weight constructed from an Hamiltonian which is the sum of two terms, one proportional to the area and another one proportional to the number of contacts. We rescale the area by a factor N so that these two effects play on the same scale. Given $\lambda \geq 0$ and $\sigma \in \mathbb{R}$, we define $\mu_N^{\lambda,\sigma}$ on Ω_N by

$$\mu_N^{\lambda,\sigma}(\xi) := \frac{2^{-2N}\lambda^{H(\xi)} \exp\left(\frac{\sigma}{N}A(\xi)\right)}{Z_N(\lambda,\sigma)} \tag{5.20}$$

where $Z_N(\lambda,\sigma)$ is the partition function, given by

$$Z_N(\lambda,\sigma) := 2^{-2N} \sum_{\xi' \in \Omega_N} \lambda^{H(\xi')} \exp\left(\tfrac{\sigma}{N}A(\xi')\right). \tag{5.21}$$

By convention, $0^0 := 1$ and $0^k := 0$ for any positive integer $k \geq 1$. The factor 2^{-2N} is irrelevant for the definition of $\mu_N^{\lambda,\sigma}$ but is convenient for statements about the partition function. When it is clear from the context, we omit the indices λ and σ in $\mu_N^{\lambda,\sigma}$. The graph of ξ depicts the spatial configuration of an interface (see Figure 2).

The equilibrium measure $\mu_N^{\lambda,\sigma}$ describes the static behavior of the polymers. Furthermore, we are interested in the heat-bath dynamics which equilibrates the value of each ξ_x at a constant rate one via corner-flip. If a polymer ξ presents a corner at x, a new polymer ξ^x is obtained by flipping the corner at x of ξ provided that $\xi^x \in \Omega_N$, which is defined as in (5.8). The generator of this heat-bath dynamics is defined by (for $f \in \mathbb{R}^{\Omega_N}$)

$$(\mathcal{L}_N f)(\xi) := \sum_{\xi' \in \Omega_N} r_N(\xi,\xi')\big[f(\xi') - f(\xi)\big] = \sum_{x=1}^{2N-1} r_N(\xi,\xi^x)\big[f(\xi^x) - f(\xi)\big], \tag{5.22}$$

where

$$r_N(\xi,\xi^x) := \begin{cases} \frac{\exp(\frac{2\sigma}{N})}{1+\exp(\frac{2\sigma}{N})} & \text{if } \xi_{x-1}=\xi_{x+1}>\xi_x\geq 1, \\ \frac{1}{1+\exp(\frac{2\sigma}{N})} & \text{if } \xi_x>\xi_{x-1}=\xi_{x+1}>1, \\ \frac{\lambda}{\lambda+\exp(\frac{2\sigma}{N})} & \text{if } (\xi_{x-1},\xi_x,\xi_{x+1})=(1,2,1), \\ \frac{\exp(\frac{2\sigma}{N})}{\lambda+\exp(\frac{2\sigma}{N})} & \text{if } (\xi_{x-1},\xi_x,\xi_{x+1})=(1,0,1), \\ 0 & \text{if } \xi_{x-1}\neq\xi_{x+1} \text{ or } \xi_{x-1}=\xi_{x+1}=0, \end{cases} \tag{5.23}$$

and the other transition rates $r_N(\xi,\xi')$ when ξ' is not one of the ξ^xs are equal to zero. We refer to Figure 2 for a graphical description. The reader can check that the measure $\mu_N^{\lambda,\sigma}$ satisfies the detailed balance condition and then it is the unique invariant probability measure for the heat-bath dynamics. We are interested in how the mixing time $t_{\text{mix}}^{N,\lambda,\sigma}$ grows in terms of N, λ and σ. Let us introduce first related results that have been previously obtained for comparison.

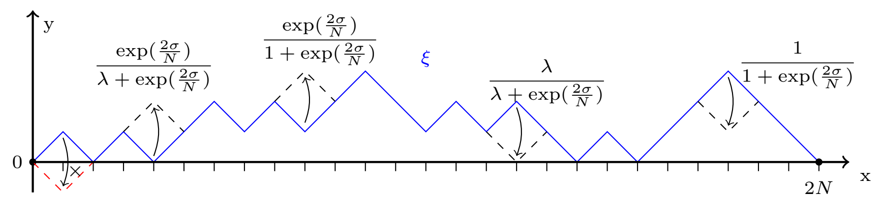


FIGURE 2. A graphical representation of the jump rates for the system. A transition of the chain corresponds to flipping a corner, whose rate is chosen such that the chain is reversible with respect to $\mu_N^{\lambda,\sigma}$. The red dashed corner is not available, due to the nonnegative restriction of the state space Ω_N. Note that not all of the possible transitions are shown in the figure.

Related previous results. To study the mixing time, a first step is to understand the equilibrium properties of the system, and in particular the asymptotic of the partition function. We mention now results which have been obtained for special cases of the model:

- When $\sigma = 0$, the model considered here is the polymer pinning model whose partition function and the properties of typical paths are described in (5.6) and below (5.6). Moreover, it is proved in [CMT08] that the mixing time is at most of order $N^2 \log N$.
- Another case is $\lambda = 1$ and $\sigma > 0$ for the state space without positive constraint, which corresponds to the height profile of the weakly asymmetric simple exclusion process (or WASEP) on the line segment $[\![1,2N]\!]$ with N particles, *i.e.*

$$\widetilde{\Omega}_N := \left\{\xi \in \mathbb{Z}^{2N+1} \ : \ \xi_0=\xi_{2N}=0\ ; \forall x\in[\![1,2N]\!], |\xi_x-\xi_{x-1}|=1\right\}, \tag{5.24}$$

and its corresponding partition function is

$$\widetilde{Z}_N(\sigma) := 2^{-2N}\sum_{\xi\in\widetilde{\Omega}_N}\exp\left(\tfrac{\sigma}{N}A(\xi)\right). \tag{5.25}$$

By [Lab18, Proposition 3 and Lemma 11], we know

$$\widetilde{Z}_N(\sigma) = (1+o(1))C_\sigma N^{-1/2}e^{2NG(\sigma)}, \tag{5.26}$$

where

$$G(\sigma) := \int_0^1 \log\cosh\left(\sigma(1-2x)\right)\mathrm{d}x. \tag{5.27}$$

Furthermore, it is shown in [LP16, LL20] that the mixing time is of order $N^2 \log N$.

Our equilibrium results. We identify the free energy when both pinning and area tilt are present, and identify the right order asymptotic. Before stating the results, we recall that $F(\lambda) = \mathbf{1}_{\{\lambda>2\}} \log \frac{\lambda}{2\sqrt{\lambda-1}}$.

PROPOSITION 5.3 (Proposition 2.1 in Chapter 3). *We have for any $\lambda \geq 0$ and $\sigma \geq 0$*

$$\lim_{N\to\infty} \frac{1}{2N} \log Z_N(\lambda,\sigma) = F(\lambda) \vee G(\sigma). \tag{5.28}$$

More precisely there exists a constant $C_1(\lambda,\sigma) > 0$ such that:

(1) *If $G(\sigma) > F(\lambda)$, then for all $N \geq 1$ we have*

$$\frac{1}{C_1(\lambda,\sigma)} \leq \frac{\sqrt{N} Z_N(\lambda,\sigma)}{\exp(2NG(\sigma))} \leq C_1(\lambda,\sigma); \tag{5.29}$$

(2) *If $G(\sigma) \leq F(\lambda)$ and $\lambda > 2$, then for all $N \geq 1$ we have*

$$\frac{1}{C_1(\lambda,\sigma)} \leq \frac{Z_N(\lambda,\sigma)}{\exp(2NF(\lambda))} \leq C_1(\lambda,\sigma). \tag{5.30}$$

REMARK 5.4. *The above result shows that the two effect of area tilt and pinning do not combine and that only the stronger of the two prevails. When $F(\lambda) > G(\sigma)$ the pinning effect dominates, while when $F(\lambda) < G(\sigma)$ the effect of area tilt dominates. In the case of a tie between $F(\lambda)$ and $G(\sigma)$, the estimates* (5.29)-(5.30) *entails that the pinning has a stronger effect. This is illustrated in Theorem 5.5 below.*

We derive the full phase diagram for the free energy $\mathrm{F}(\lambda,\sigma) := \lim_{N\to\infty} \frac{1}{2N} \log Z_N(\lambda,\sigma)$ in λ and σ of this model, and identify the critical line (see Figure 3) separating the localized/delocalized phases.

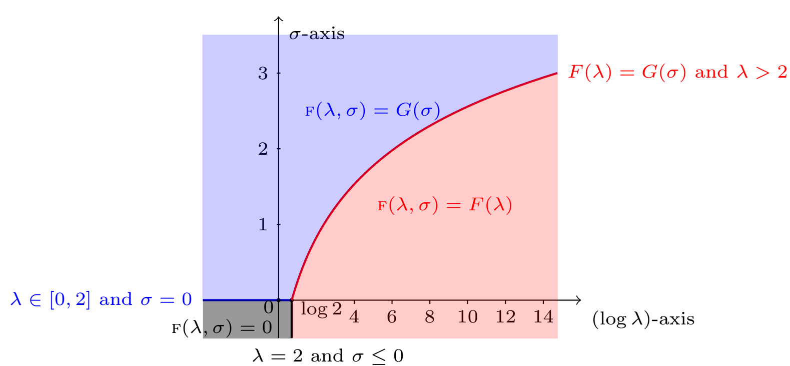


FIGURE 3. The statics phase diagram for the free energy $\mathrm{F}(\lambda,\sigma)$ where the red curve is $F(\lambda) = G(\sigma)$ and $\lambda > 2$, the black line is $\lambda = 2$ and $\sigma \leq 0$, and the blue line is $\lambda \in [0,2]$ and $\sigma = 0$.

Moreover, during the proof for the partition function, we obtain a detailed description for the typical behavior of ξ under $\mu_N^{\lambda,\sigma}$, and we refer to Figure 4 for a graphical presentation. Let us define

$$M_\sigma(u) := \int_0^u \tanh(\sigma(1-x))\mathrm{d}x = \frac{1}{\sigma} \log\left(\frac{\cosh(\sigma)}{\cosh(\sigma(1-u))}\right). \tag{5.31}$$

THEOREM 5.5 (Theorem 2.4 in Chapter 3). *For $\lambda \geq 0$, $\sigma > 0$, we have*

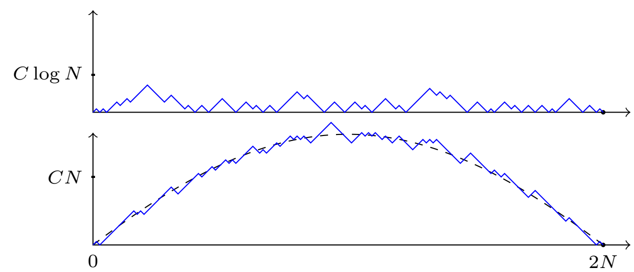


FIGURE 4. The macroscopic shape of the substrate in equilibrium when $F(\lambda) \geq G(\sigma)$ (at the top) and $F(\lambda) < G(\sigma)$ (at the bottom). The dotted line illustrates the macroscopic shape, which is the scaling limit when $N \to \infty$ (the dotted line in the top figure coincides with the $x-$axis).

1. *if $G(\sigma) > F(\lambda)$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all N sufficiently large,*

$$\mu_N \left(\sup_{u \in [0,2]} \left| \frac{1}{N} \xi_{\lceil uN \rceil} - M_\sigma(u) \right| > \varepsilon \right) \leq e^{-\delta N}; \tag{5.32}$$

2. *if $G(\sigma) < F(\lambda)$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all N sufficiently large,*

$$\mu_N \left(\sup_{x \in [\![0,2N]\!]} \xi_x > \varepsilon N \right) \leq e^{-\delta N}; \tag{5.33}$$

3. *if $G(\sigma) = F(\lambda)$, then for every $\varepsilon > 0$ and all N sufficiently large,*

$$\frac{1}{C\sqrt{N}} \leq \mu_N \left(\sup_{x \in [\![0,2N]\!]} \xi_x > \varepsilon N \right) \leq \frac{C}{\sqrt{N}}, \tag{5.34}$$

and furthermore there exists $\delta > 0$ such that

$$\mu_N \left(\sup_{x \in [\![0,2N]\!]} \xi_x > \varepsilon N \text{ and } \sup_{u \in [0,2]} \left| \frac{1}{N} \xi_{\lceil uN \rceil} - M_\sigma(u) \right| > \varepsilon \right) \leq e^{-\delta N}. \tag{5.35}$$

Our dynamical results. We are interested in the mixing time $T_{\text{mix}}^{N,\lambda,\sigma}(\varepsilon)$. More precisely, we want to know when (for fixed λ and σ) it behaves asymptotically like a power of N and when it grows exponentially in N. By Theorem 3.4, the mixing time can be compared to the relaxation time $T_{\text{rel}}^N(\lambda, \sigma)$ as follows

$$T_{\text{rel}}^N(\lambda, \sigma) \log \frac{1}{2\epsilon} \leq T_{\text{mix}}^{N,\lambda,\sigma}(\epsilon) \leq T_{\text{rel}}^N(\lambda, \sigma) \log \frac{1}{\varepsilon \mu_N^*} \tag{5.36}$$

where $\mu_N^* := \min_{\xi \in \Omega_N} \mu_N^{\lambda;\sigma}(\xi)$. By Proposition 5.3 about the partition function $Z_N(\lambda, \sigma)$, it is immediate to check that in our case $\log \mu_N^*$ is of order N (with a prefactor depending on λ and σ). Since this factor N is irrelevant for the kind of result we look for, we can focus on the relaxation time $T_{\text{rel}}^N(\lambda, \sigma)$.

Before stating our dynamical result, we provide a first heuristic for the cases where the relaxation time grows polynomially/exponentially. Let $\beta \in [0,1]$ denote the fraction of the polymer length which is unpinned. By Theorem 5.5, the contribution (to the partition function) of those polymers which macroscopically have only one unpinned bubble with length $2\beta N$ is $e^{-2NV(\beta)}$ where

$$V \ : \ \beta \mapsto -\beta G(\beta\sigma) - (1-\beta) F(\lambda).$$

The idea is that the unpinned fraction should look like a stochastic diffusion on the segment, with a potential $2NV(\cdot)$. The relaxation time corresponds to the time required for such a diffusion to overcome the energy barrier between the two local mimina of $V(\beta)$ (at 0 and 1 see Figure 5).

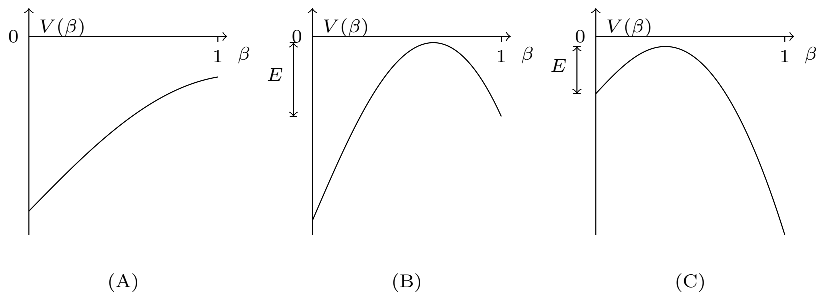


FIGURE 5. The shape of V in three cases.

(A) If $G(\sigma) + \sigma G'(\sigma) \leq F(\lambda)$, then the unpinned region can shrink down without barrier and the system should mix in polynomial time.
(B) If $G(\sigma) \leq F(\lambda) < G(\sigma) + \sigma G'(\sigma)$, then the system starting from the fully unpinned state needs to overcome the energy barrier to reach the fully pinned equilibrium state, which takes an exponential time.
(C) If $F(\lambda) < G(\sigma)$, then the system starting from the fully pinned state needs to overcome the energy barrier to reach the fully unpinned equilibrium state, which takes an exponential time.

The size of the effective potential barrier to be overcome in case (B) and (C) is equal to

$$E(\lambda, \sigma) := F(\lambda) \wedge G(\sigma) - [(1 - \beta^*)F(\lambda) + \beta^* G(\beta^* \sigma)]$$

with β^* such that $V(\beta^*) = \max_{\beta \in [0,1]} V(\beta)$.

THEOREM 5.6 (Theorem 2.7 in Chapter 3). *For all $\lambda > 2$ and all $\sigma > 0$, we have*

$$\lim_{N \to \infty} \frac{1}{2N} \log T_{\text{rel}}^N(\lambda, \sigma) = E(\lambda, \sigma). \tag{5.37}$$

When $E(\lambda, \sigma) = 0$, there exist constants $C(\lambda, \sigma) > 0$ and $C(\lambda) > 0$ such that for all $N \geq 1$,

$$C(\lambda, \sigma)^{-1} N \leq T_{\text{rel}}^N(\lambda, \sigma) \leq C(\lambda, \sigma) N^{C(\lambda)}. \tag{5.38}$$

When $E(\lambda, \sigma) > 0$, there exists constants $C(\lambda, \sigma) > 0$ and $C'(\lambda, \sigma) > 0$ such that

$$C'(\lambda, \sigma)^{-1} N^{-2} \leq T_{\text{rel}}^N(\lambda, \sigma) e^{-2NE(\lambda, \sigma)} \leq C'(\lambda, \sigma) N^{C(\lambda, \sigma)}.$$

When $E(\lambda, \sigma) > 0$, the state space display two distinct wells of potential, for which we refer to Figure 6 for a graphical description. In this case, let $\mathcal{H}_N$ denote the domain of attraction of the unstable local equilibrium of the dynamics

$$\mathcal{H}_N := \begin{cases} \{\xi \in \Omega_N : L_{\max}(\xi) > \beta^* N\}, & \text{if } G(\sigma) \leq F(\lambda) < G(\sigma) + \sigma G'(\sigma), \\ \{\xi \in \Omega_N : L_{\max}(\xi) \leq \beta^* N\} & \text{if } F(\lambda) < G(\sigma), \end{cases} \tag{5.39}$$

where

$$L_{\max}(\xi) := \max\{y - x \ : \ \xi_{2x} = 0, \xi_{2y} = 0, \forall z \in [\![x, y]\!], \xi_{2z} > 0\}.$$

When $E(\lambda, \sigma) > 0$ and the dynamics starts from a configuration $\xi \in \mathcal{H}_N$, the system should quickly thermalize in $\mathcal{H}_N$ (within a time which is polynomial in N) and then takes a time of order $\exp(2NE(\lambda, \sigma))$ to jump from $\mathcal{H}_N$ to $\Omega_N \setminus \mathcal{H}_N$ and reaches equilibrium in another polynomial

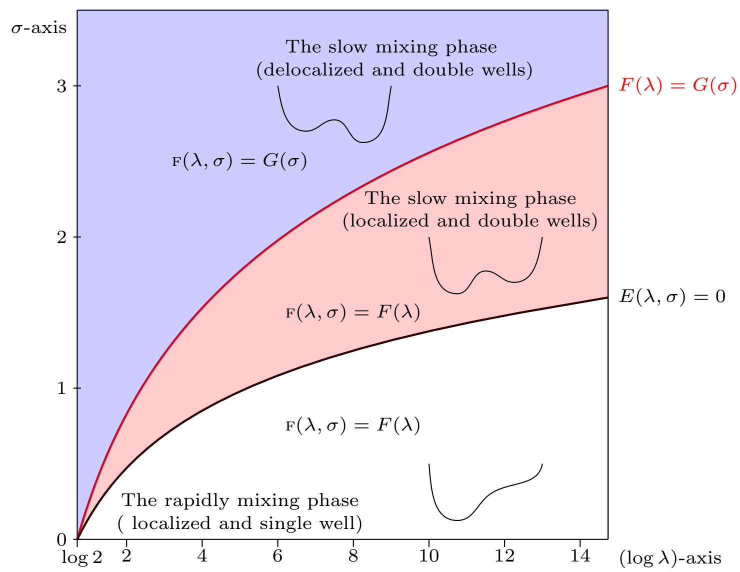


FIGURE 6. The dynamical phase diagram in the regime $\lambda > 2$ and $\sigma > 0$: The line $F(\lambda) = G(\sigma)$ separates the localized phase from the delocalized phase, while the line $E(\lambda, \sigma) = 0$ separates the rapidly mixing phase from the slow mixing phase.

time in terms of N. Moreover, the properly rescaled time for jumping from $\mathcal{H}_N$ to $\Omega_N \setminus \mathcal{H}_N$ should converge to an exponential random variable which is the following theorem.

THEOREM 5.7 (Theorem 2.8 in Chapter 3). *When $E(\lambda, \sigma) > 0$, we have*

$$\lim_{N \to \infty} \mathbb{P}_{\mu_N(\cdot|\mathcal{H}_N)} \left(\eta_{tT^N_{\mathrm{rel}}(\lambda,\sigma)} \in \mathcal{H}_N \right) = \exp(-t)$$

where $\mathbb{P}_{\mu_N(\cdot|\mathcal{H}_N)}$ denotes the law of the Markov chain $(\eta_t)_{t \geq 0}$ starting with η_0 distributed as $\mu_N(\cdot|\mathcal{H}_N)$.

Let us conclude the presentation by giving a short insight about the ideas in Chapter 3.

Idea for the proof about statics. Since the proof for Theorem 5.5 is a refinement of that for Proposition 5.3, we are mainly concerned with the idea for Proposition 5.3. We decompose the paths into excursions away from the x-axis by factorizing, and apply the renewal approach to show that

(A) When $F(\lambda) \geq G(\sigma)$, the main contribution to the partition function $Z_N(\lambda, \sigma)$ comes from those paths with $L_{\max}(\xi) \leq C \log N$ and typical height at constant order. Therefore, we have $Z_N(\lambda, \sigma) \asymp Z_N(\lambda, 0)$.

(B) When $F(\lambda) < G(\sigma)$, the main contribution to the partition function $Z_N(\lambda, \sigma)$ comes from those paths with $L_{\max}(\xi) \geq (1 - \varepsilon)N$ and typical height of order N. Therefore, we have $Z_N(\lambda, \sigma) \asymp \widetilde{Z}_N(\sigma)$.

Idea for the proof about dynamics. To obtain the exponential lower bound on the mixing time we use bottleneck techniques (see [LP17, Chapter 7.2]). The analysis of the equilibrium system allows to identify that the worst bottleneck should be between $\mathcal{H}_N$ and $\mathcal{H}_N^{\complement}$ and the problem reduces to a (technical) estimate of the corresponding bottleneck ratio.

To get the corresponding upper-bound, or a polynomial upper-bound when $E(\lambda, \sigma) = 0$, a significant obstacle is the large dimension of the system. We rely on [JSTV04, Theorem 1] which, very roughly speaking, is an inequality which allows to compare the mixing time of a Markov chain to that of a reduced version, which lives on a smaller and simplified state space. Successive use of this result (each of them involves heavy computation) allows to reduce the study of the chain to that of the evolution of only the largest excursions of the chain η_t^ξ away from the $x-$axis.

5.3. Chapter 4: Mixing time of the asymmetric simple exclusion process in a random environment.

Model. The simple exclusion process is a reasonable toy model to describe the relaxation of a low density gas, for which we refer to [Lig12, Chapter VIII.6] for a historical introduction. Due to the impurity of the environment, we are interested in the case where the jump rates of the particles are spatially inhomogeneous. Given a sequence $\omega = (\omega_x)_{x\in[\![1,N]\!]}$ taking values in $(0,1)$, the exclusion process with k particles in the segment $[\![1,N]\!]$ with environment ω is a Markov process which can be informally described as follow: (see Figure 7 for a graphical description)

(A) Each site is occupied by at most one particle (we refer to this constraint as *the exclusion rule)*. Therefore at all time there are k occupied sites and $N-k$ empty sites.
(B) Each of the k particles perform a random walk such that a particle at site $x \in [\![1,N]\!]$ jumps to site $x-1$ (if $x\geq 2$) at rate $1-\omega_x$ and to site $x+1$ (if $x \leq N-1$) at rate ω_x if the target site is not occupied.

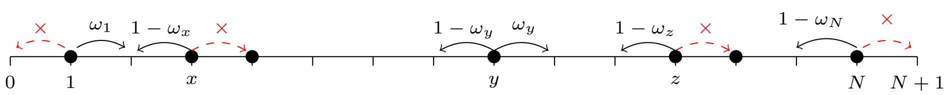


FIGURE 7. A graphical representation of the simple exclusion process in environment $(\omega_x)_{1\leq x\leq N}$: a bold circle represents a particle, and the number above every arrow represents the jump rate while a red " $\times$ " represents an inadmissible jump.

We denote its state space by

$$\Omega_{N,k} := \left\{ \xi \in \{0,1\}^N : \sum_{x=1}^N \xi(x) = k \right\}, \tag{5.40}$$

where the 1s' are denoting particles while 0s' correspond to empty sites. To describe transition rates, let $\xi^{x,y}$ denote the configuration obtained by swapping the values of ξ at sites x and y of the configuration ξ, and transition rates are given by

$$\begin{aligned} r^\omega(\xi,\xi^{x,x+1}) &:= \begin{cases} \omega_x & \text{if } \xi(x)=1 \text{ and } \xi(x+1)=0, \\ 1-\omega_{x+1} & \text{if } \xi(x+1)=1 \text{ and } \xi(x)=0, \end{cases} \quad \text{for } x\in[\![1,N-1]\!] \\ r^\omega(\xi,\xi') &:= 0 \quad \text{in all other cases.} \end{aligned} \tag{5.41}$$

The simple exclusion process with k particles in the segment $[\![1,N]\!]$ and environment ω is the Markov process whose generator is defined by (for $f:\Omega_{N,k}\to\mathbb{R}$)

$$\mathcal{L}^\omega_{N,k}(f)(\xi) := \sum_{x=1}^{N-1} r^\omega(\xi,\xi^{x,x+1})\left[f(\xi^{x,x+1})-f(\xi)\right]. \tag{5.42}$$

The chain is ergodic and reversible. In order to give a nice expression for the equilibrium measure, let us introduce the random potential $V^\omega:\mathbb{N}\to\mathbb{R}$ defined as follows, $V^\omega(1):=0$ and for $x\geq 2$

$$V^\omega(x) := \sum_{y=2}^x \log\left(\frac{1-\omega_y}{\omega_{y-1}}\right). \tag{5.43}$$

With a small abuse of notation, we extend V^ω to a function of $\Omega_{N,k}$ by summing the value of V^ω among the positions of the particles in the configuration ξ:

$$V^\omega(\xi) := \sum_{x=1}^{N} V^\omega(x)\xi(x). \tag{5.44}$$

We consider the probability measure $\pi^\omega_{N,k}$ defined by

$$\pi^\omega_{N,k}(\xi) := \frac{1}{Z^\omega_{N,k}} e^{-V^\omega(\xi)} \quad \text{with} \quad Z^\omega_{N,k} = \sum_{\xi\in\Omega_{N,k}} e^{-V^\omega(\xi)}. \tag{5.45}$$

We can check by inspection that $\pi^\omega_{N,k}$ satisfies the detailed balance condition for $\mathcal{L}^\omega_{N,k}$, and thus that it is the unique invariant probability measure on $\Omega_{N,k}$.

We are interested in the case where the environment $\omega = (\omega_x)_{1\le x\le N}$ is independently sampled from a common law denoted by $\mathbb{P}$ (and its expectation denoted by $\mathbb{E}$), and set $\rho_i := (1-\omega_i)/\omega_i$. Due to the symmetry between particles and empty sites, between left and right to which direction particles tend to move, we assume that

$$1 \le k \le N/2 \quad \text{and } \mathbb{E}\left[\log \rho_1\right] < 0, \tag{5.46}$$

so that by [Sol75] we know that particles have a tendency to move to right. Moreover, due to some technical issue we assume that there exists $\alpha \in (0, 1/2)$ such that

$$\mathbb{P}\left(\omega_1 \in [\alpha, 1-\alpha]\right) = 1, \tag{5.47}$$

which is referred to as the uniform ellipticity condition. Let $t^{N,k,\omega}_{\text{mix}}$ be the mixing time for the process in a fixed realization of ω, and we are interested in how the function $t^{N,k,\omega}_{\text{mix}}$ grows in terms of N and k for typical realization of ω.

Let us introduce first the results that have been previously obtained for comparison.

Related previous results about random walk in a random environment. Given $\omega = (\omega_x)_{x\in\mathbb{Z}}$, a random walk (of only one particle) on $\mathbb{Z}$ in the environment ω is first studied in [Sol75]. Let $(X_t)_{t\ge 0}$ be a continuous-time random walk starting at 0 with jump rates given by

$$\begin{cases} q^\omega(x, x+1) = \omega_x, \\ q^\omega(x, x-1) = 1-\omega_x, \\ q^\omega(x, y) = 0 & \text{if } |x-y| \neq 1, \end{cases} \tag{5.48}$$

and we let Q^ω denote the corresponding law. Solomon in [Sol75] showed that

$$\begin{cases} \mathbb{E}[\log \rho_1] = 0 \Rightarrow X_t \text{ is recurrent under } Q^\omega \text{ , } \mathbb{P}\text{-a.s.,} \\ \mathbb{E}[\log \rho_1] < 0 \Rightarrow \lim_{t\to\infty} X_t = \infty \text{ under } Q^\omega \text{ , } \mathbb{P}\text{-a.s.,} \\ \mathbb{E}[\log \rho_1] > 0 \Rightarrow \lim_{t\to\infty} X_t = -\infty \text{ under } Q^\omega \text{ , } \mathbb{P}\text{-a.s.} \end{cases} \tag{5.49}$$

From above, if $\mathbb{E}[\log \rho_1] \neq 0$, the random walk $(X_t)_{t\ge0}$ is transient. Moreover, the rate at which X_t goes to infinity has also been identified in [KKS75]. It can be expressed in terms of a simple parameter of the distribution ω, yielding in particular a necessary and sufficient condition for balisticity. Let us assume that $\mathbb{E}[\log \rho_1] < 0$, and set

$$\lambda = \lambda_{\mathbb{P}} := \inf\{s > 0, \mathbb{E}[\rho_1^s] \ge 1\} \in (0, \infty].$$

It has been proved in [KKS75] that if $\lambda > 1$ then there exists $\vartheta_{\mathbb{P}} > 0$ such that

$$\lim_{t\to\infty} \frac{X_t}{t} = \vartheta \tag{5.50}$$

and that if $\lambda \in (0,1]$ then

$$\lim_{t\to\infty} \frac{\log(X_t)}{\log t} = \lambda. \tag{5.51}$$

Related previous results about $t_{\rm mix}^{N,k,\omega}$ when $\omega \equiv p$. The mixing time of the exclusion process on the line segment has been extensively studied in the case where the sequence ω is constant, *i.e.* $\omega \equiv p$. In that case, not only the right order of magnitude has been identified for the mixing time, but also the sharp asymptotic equivalent.

- When $p = 1/2$, this is the symmetric simple exclusion process. It was shown in [Ald83a] that the mixing time for the exclusion on the segment is of order at least N^2 and at most $N^2(\log N)^2$. It was later established (see [Wil04] for the lower bound and [Lac16b] for the upper bound) that if k_N satisfies $\liminf_{N\to\infty} K_N = \infty$, we have

$$t_{\rm mix}^{N,k_N}(\varepsilon) = [1+o(1)]\frac{1}{\pi^2}N^2 \log k_N. \tag{5.52}$$

- When $p \neq 1/2$, this is the asymmetric simple exclusion process. It was shown in [BBHM05] that the mixing time is of order N. This result was refined in [LL19] by identifying the proportionality constant, showing that if k_N satisfies $\lim_{N\to\infty} k_N/N = \theta \in (0,1)$, then

$$t_{\rm mix}^{N,k_N}(\varepsilon) = [1+o(1)]\frac{(\sqrt{\theta}+\sqrt{1-\theta})^2}{|2p-1|}N. \tag{5.53}$$

- when $p = \frac{1}{2} \pm \varepsilon_N$ with $\lim_{N\to\infty} \varepsilon_N = 0$, the mixing time was investigated in [LP16, LL20] where its order of magnitude and its sharp asymptotic were respectively determined.

Related previous results about $t_{\rm mix}^{N,1,\omega}$. The mixing time for a random walk in the segment with a transient random environment (*i.e.* $k=1$ in the setting) was investigated in [GK13]. It is shown that whenever $\lambda_{\mathbb{P}} > 1$ then

$$t_{\rm mix}^{N,1,\omega}(\varepsilon) = [1+o(1)]N\mathbb{E}\left[Q^\omega\left[T_1^\omega\right]\right], \tag{5.54}$$

where T_1^ω is the first hitting time of 1 for the random walk in a random environment ω starting from 0 (the result in [GK13] is slightly more precise and the assumption is more general than (5.47)). When $\lambda_{\mathbb{P}} < 1$, it is shown that the mixing time is of a much larger magnitude but that cutoff does not hold. More precisely, for $\lambda_{\mathbb{P}} \leq 1$ we have

$$\lim_{N\to\infty} \frac{\log t_{\rm mix}^{N,1,\omega}(\varepsilon)}{\log N} = \frac{1}{\lambda_{\mathbb{P}}}. \tag{5.55}$$

The asymptotic $N^{1/\lambda_{\mathbb{P}}+o(1)}$ corresponds to the time that is required to overcome the largest potential barrier present in the system, whose height is of order $(1/\lambda)\log N$.

Related previous results about $t_{\rm mix}^{N,k,\omega}$ in ballistic environment. In [Sch19], the mixing time $t_{\rm mix}^{N,k_N,\omega}$ were investigated under the assumption that $\lim_{N\to\infty} k_N/N = \theta \in (0,1/2]$ and $\lambda_{\mathbb{P}} > 1$. Three different cases are considered.

- When $\operatorname{ess\,inf} \omega_1 > 1/2$, it is shown that the mixing $t_{\rm mix}^{N,k_N,\omega}$ is of order N, by a simple comparison with the case of homogeneous asymmetric environment.
- When $\operatorname{ess\,inf} \omega_1 < 1/2$, it is shown that there exists a positive δ such that the mixing time satisfies $t_{\rm mix}^{N,k_N,\omega} \geq N^{1+\delta}$.
- When $\operatorname{ess\,inf} \omega_1 = 1/2$, it is shown that

$$\liminf_{N\to\infty} t_{\rm mix}^{N,k_N,\omega}(\varepsilon)/N = \infty \quad \text{and} \quad t_{\rm mix}^{N,k_N,\omega}(\varepsilon) \leq CN(\log N)^3, \tag{5.56}$$

together with a quantitative lower bound if $\mathbb{P}[\omega_1 = 1/2] > 0$.

Our results. The main object of Chapter 4 is the study of the mixing time for the exclusion process in an i.i.d. environment.

THEOREM 5.8. *Under the assumptions* (5.47)-(5.46) *and assuming further that* $\lambda_{\mathbb{P}} < \infty$, *there exists a positive constant* $c(\alpha, \mathbb{P})$ *such that with high probability we have for every* N *and* $k \in [\![1, N/2]\!]$

$$t_{\text{mix}}^{N,k,\omega} \geq c \max\left\{N, N^{\frac{1}{\lambda}}(\log N)^{-\frac{2}{\lambda}}, kN^{\frac{1}{2\lambda}}(\log N)^{-2(1+\frac{1}{\lambda})}\right\}. \tag{5.57}$$

In order to describe our explicit upper bound, we need to introduce the function F which is the log-Laplace transform of $\log \rho_1$ that is

$$F(u) := \log \mathbb{E}\left[\rho_1^u\right]. \tag{5.58}$$

Moreover, F is strictly convex and satisfies $F(0) = F(\lambda) = 0$. We let u_0 be defined by

$$F(u_0) = \min_{u \in \mathbb{R}} F(u) < 0.$$

THEOREM 5.9. *Under the assumptions* (5.46) *and* (5.47), *then with high probability we have*

$$t_{\text{mix}}^{N,k,\omega} \leq 80kN\alpha^{-1}\left(\frac{3u_0+2}{|F(u_0)|}\log N\right)^4 N^{\frac{3u_0+2}{|F(u_0)|}\left(2\log\frac{1-\alpha}{\alpha}+4\log 4-3\log 3\right)}. \tag{5.59}$$

Let us conclude the presentation by providing a first heuristic in Chapter 4.

Idea for the lower bound on the mixing time. For typical environment ω, under the equilibrium measure $\pi_{N,k}^{\omega}$ the distance between the rightmost empty site and the leftmost particle is at constant order. Then we consider the time for the dynamics $(\sigma_t^{\min})_{t\geq 0}$ starting with the configuration $\xi_{\min} := \mathbf{1}_{\{1\leq x\leq k\}}$ to reach equilibrium. Each of the three terms in Theorem 5.8 corresponds to a different mechanism listed as follow:

- *Mass transport cannot be faster than ballistic:* The speed of the rightmost particle is at most linear. Indeed, particles cannot move faster than balistically for all realization of ω, so that the time required to transport the mass of particles to equilibrium has to be at least of order N. This idea is already present in [BBHM05].
- *The leftmost particle may be blocked by traps in the potential profile:* As soon as $\operatorname{ess\,inf} \omega_1 < 1/2$, the potential profile V is non-monotone and will create energy barriers. It is known that $\max_{1\leq x<y\leq N} V(y) - V(x) \sim (1/\lambda)\log N$, and we consider the time for the leftmost particle to cross the deepest trap in the segment $[\![1, N/4]\!]$ by comparing with the random walk. It takes time roughly $N^{1/\lambda}$.
- *Potential barrier may also create bottleneck for the flow of particles:* When the particles flow through the deepest trap, the particle are "filling" half of the potential well, so that the remaining potential barrier to be crossed is halved. Therefore, the time for a particle to flow out of the trap is roughly $N^{1/(2\lambda)}$.

Idea for the upper bound on the mixing time. We provide a canonical coupling which keeps the monotonicity, and study the hitting time

$$\tau := \inf\left\{t \geq 0 : \sigma_t^{\min} = \xi_{\max}\right\} \tag{5.60}$$

where $\xi_{\max} := \mathbf{1}_{\{N-k+1\leq x\leq N\}}$ and $(\sigma_t^{\min})_{t\geq 0}$ is the dynamics starting from $\xi_{\min}$. We turn to study the hitting time τ, and by Markov property

$$\mathbf{P}\left[\tau > nt\right] \leq \mathbf{P}\left[\tau > t\right]^n \leq \left(1 - \mathbf{P}\left[\sigma_t^{\min} = \xi_{\max}\right]\right)^n. \tag{5.61}$$

To give a lower bound on $\mathbf{P}\left[\sigma_t^{\min} = \xi_{\max}\right]$, we apply the Peres-Winkler inequality [PW13, Theorem 1.1] to guide the particles to the right, for which we use flow method to show that the spectral gap satisfies $\text{gap}_{N,k}^{\omega} \geq \exp(-C(\alpha)N)$ for all realization of ω.

CHAPTER 2

Cutoff for polymer pinning dynamics in the repulsive phase

Abstract: In this chapter, we consider the Glauber dynamics for model of polymer interacting with a substrate or wall. The state space is the set of one-dimensional nearest-neighbor paths on $\mathbb{Z}$ with nonnegative integer coordinates, starting at 0 and coming back to 0 after L ($L \in 2\mathbb{N}$) steps and the Gibbs weight of a path $\xi = (\xi_x)_{x=0}^L$ is given by $\lambda^{\mathcal{N}(\xi)}$, where $\lambda \geq 0$ is a parameter which models the intensity of the interaction with the substrate and $\mathcal{N}(\xi)$ is the number of zeros in ξ. The dynamics proceeds by updating ξ_x with rate one for each $x = 1, \dots, L-1$, in a heat-bath fashion. This model was introduced in [CMT08] with the aim of studying the relaxation to equilibrium of the system.
We present new results concerning the total variation mixing time for this dynamics when $\lambda < 2$, which corresponds to the phase where the effects of the wall's entropic repulsion dominates. For $\lambda \in [0, 1]$, we prove that the total variation distance to equilibrium drops abruptly from 1 to 0 at time $(L^2 \log L)(1 + o(1))/\pi^2$. For $\lambda \in (1, 2)$, we prove that the system also exhibits cutoff at time $(L^2 \log L)(1 + o(1))/\pi^2$ when considering mixing time from "extremal conditions" (that is, either the highest or lowest initial path for the natural order on the set of paths). Our results improve both previously proved upper and lower bounds in [CMT08].

1. Introduction

1.1. The random walk pinning model. Consider the set of all one-dimensional nearest-neighbor paths on $\mathbb{Z}$ with nonnegative integer coordinates, starting at 0 and coming back to 0 after L steps, *i.e.*

$$\Omega_L := \left\{ \xi \in \mathbb{Z}^{L+1} : \xi_0 = \xi_L = 0; |\xi_{x+1} - \xi_x| = 1, \forall x \in [\![0, L-1]\!]; \xi_x \geq 0, \forall x \in [\![0, L]\!] \right\},$$

where $L \in 2\mathbb{N}$, and $[\![a, b]\!] := \mathbb{Z} \cap [a, b]$ for all $a, b \in \mathbb{R}$ with $a < b$. We study the polymer pinning model. This model is obtained by assigning to each path $\xi \in \Omega_L$ a weight $\lambda^{\mathcal{N}(\xi)}$, in which $\lambda \geq 0$ is the pinning parameter and

$$\mathcal{N}(\xi) := \sum_{x=1}^{L-1} \mathbf{1}_{\{\xi_x = 0\}} \tag{1.1}$$

is the number of contact points with the x-axis. By convention, $0^0 := 1$ and $0^n := 0$ for any positive integer $n \geq 1$. Normalizing the weights, we obtain a Gibbs probability measure μ_L^λ on Ω_L, defined by

$$\mu_L^\lambda(\xi) := \frac{\lambda^{\mathcal{N}(\xi)}}{Z_L(\lambda)} \tag{1.2}$$

where $\xi \in \Omega_L$ and

$$Z_L(\lambda) := \sum_{\xi' \in \Omega_L} \lambda^{\mathcal{N}(\xi')}. \tag{1.3}$$

The graph of ξ represents the spatial conformation of the polymer and λ models the energetic interaction with an impenetrable substrate which fills the lower half plane ($\lambda < 1$ corresponding to a repulsive interaction, $\lambda > 1$ to an attractive one). Since $\xi_x \geq 0$ for any $\xi \in \Omega_L$ and any

$x \in [\![0, L]\!]$, we say that the polymers interact with an impenetrable substrate. When there is no confusion, we drop the indices λ and L in μ_L^λ.

The random walk pinning model was introduced in the seminal paper [Fis84] several decades ago, and its various derivative models have been studied since. We refer to [Gia07, Gia11] for recent reviews, and mention [Gia07, Chapter 2] and references therein for more details. This model displays a transition from a delocalized phase to a localized phase (see [CMT08, Section 1]): (a) if $0 \le \lambda < 2$, the expected number of contacts $\mu_L^\lambda(\mathcal{N}(\xi))$ is uniformly bounded in L and the height of the middle point $\xi_{L/2}$ is typically of order $\sqrt{L}$; (b) if $\lambda > 2$, the amount of contacts with the $x-$axis of typical paths is of order L and the distribution of the height of the middle point $\xi_{L/2}$ is (exponentially) tight in L. These two phases are referred to as the delocalized and localized phase respectively, at the critical point $\lambda = 2$ the system displays an intermediate behavior.

A dynamical version of this model was introduced more recently by Caputo *et al.* in [CMT08]. The corner-flip Glauber dynamics is a continuous-time reversible Markov chain on Ω_L with μ_L^λ as the unique invariant probability measure, whose transitions are given by the updates of local coordinates. We refer to Figure 1 for a graphical description of the jump rates for the system. The dynamics is studied to understand how the system relaxes to equilibrium. Caputo *et al.* in [CMT08, Theorems 3.1 and 3.2] proved that for $\lambda \in [0, 2)$, the mixing time of the dynamics in Ω_L is of order $L^2 \log L$, with non-matching constant prefactors for the upper and lower bounds.

The goal of this chapter is to improve both the upper and lower bounds proved in [CMT08] and to show that the mixing time of the system is exactly $(1 + o(1))(L^2 \log L)/\pi^2$ for $\lambda \in [0, 2)$. We prove the result for the worst initial condition mixing time when $\lambda \in [0, 1]$. When $\lambda \in (1, 2)$, our result is valid only for the mixing time starting from either the lowest or highest initial condition but we believe that this is only a technical restriction.

1.2. The dynamics. For $\xi \in \Omega_L$ and $x \in [\![1, L-1]\!]$, we define $\xi^x \in \Omega_L$ by

$$\xi^x_y := \begin{cases} \xi_y & \text{if } y \neq x, \\ (\xi_{x-1} + \xi_{x+1}) - \xi_x & \text{if } y = x \text{ and } \xi_{x-1} = \xi_{x+1} \ge 1 \text{ or } \xi_{x-1} \neq \xi_{x+1}, \\ \xi_x & \text{if } y = x \text{ and } \xi_{x-1} = \xi_{x+1} = 0. \end{cases} \tag{1.4}$$

When $\xi_{x-1} = \xi_{x+1}$, ξ displays a local extremum at x and we obtain ξ^x by flipping the corner of ξ at the coordinate x, provided that the path obtained by flipping the corner is in Ω_L (this excludes corner-flipping when $\xi_{x-1} = \xi_{x+1} = 0$.). See Figure 1 for a graphical representation. Given the probability measure μ_L^λ defined in (1.2), we construct a continuous-time Markov chain whose generator $\mathcal{L}$ is given by its action on the functions $\mathbb{R}^{\Omega_L}$. It can be written explicitly as

$$(\mathcal{L}f)(\xi) := \sum_{x=1}^{L-1} R_x(\xi)\big[f(\xi^x) - f(\xi)\big], \tag{1.5}$$

where $f\colon \Omega_L \to \mathbb{R}$ is a function, and

$$R_x(\xi) := \begin{cases} \frac{1}{2} & \text{if } \xi_{x-1} = \xi_{x+1} > 1, \\ \frac{\lambda}{1+\lambda} & \text{if } (\xi_{x-1}, \xi_x, \xi_{x+1}) = (1, 2, 1), \\ \frac{1}{1+\lambda} & \text{if } (\xi_{x-1}, \xi_x, \xi_{x+1}) = (1, 0, 1), \\ 0 & \text{if } \xi_{x-1} \neq \xi_{x+1} \text{ or } \xi_{x-1} = \xi_{x+1} = 0. \end{cases}$$

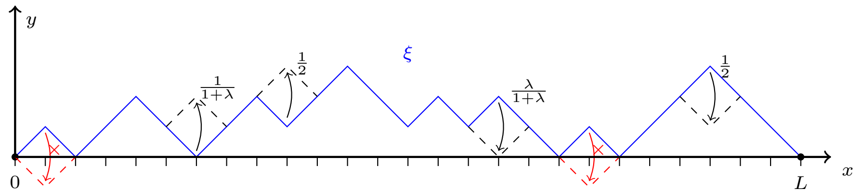


FIGURE 1. A graphical representation of the jump rates for the system pinned at $(0,0)$ and $(L,0)$. A transition of the dynamics corresponds to flipping a corner, whose rate depends on how it changes the number of contact points with the x-axis. The rates are chosen in a manner such that the dynamics is reversible with respect to μ_L^λ. The two red dashed corners are not available and labeled with $\times$, because of the nonnegative restriction of the state space Ω_L. Note that not all the possible transitions are shown in the figure.

Equivalently, we can rewrite the generator as

$$(\mathcal{L}f)(\xi) = \sum_{x=1}^{L-1} \Big[Q_x(f)(\xi) - f(\xi) \Big], \tag{1.6}$$

and

$$Q_x(f)(\xi) := \mu_L^\lambda \Big(f \Big| (\xi_y)_{y \neq x} \Big).$$

Let $(\sigma_t^{\xi,\lambda})_{t\geq 0}$ be the trajectory of the Markov chain with initial condition $\sigma_0^{\xi,\lambda} = \xi$ and parameter λ, and let $P_t^{\xi,\lambda}$ be the law of distribution of the time marginal $\sigma_t^{\xi,\lambda}$. Since $\mu_L^\lambda(\xi)R_x(\xi) = \mu_L^\lambda(\xi^x)R_x(\xi^x)$, the continuous-time chain is reversible with respect to the probability measure μ_L^λ. This chain is called the Glauber dynamics. Because the Markov chain is irreducible, by [Nor98, Theorem 3.5.2] we know that for all $\xi \in \Omega_L$, $P_t^{\xi,\lambda}$ converges to μ_L^λ in the discrete topology as t tends to infinity. We ask a quantitative question: how long does it take for $P_t^{\xi,\lambda}$ to converge to μ_L^λ, especially for the worst initial starting path $\xi \in \Omega_L$?

Let us state the aforementioned question in a mathematical framework. If α and β are two probability measures on the space $(\Omega_L, 2^{\Omega_L})$, the total variation distance between α and β is

$$\|\alpha - \beta\|_{\mathrm{TV}} := \frac{1}{2} \sum_{\xi \in \Omega_L} |\alpha(\xi) - \beta(\xi)| = \sup_{\mathcal{A} \subset \Omega_L} \left(\alpha(\mathcal{A}) - \beta(\mathcal{A}) \right). \tag{1.7}$$

We define the distance to equilibrium at time t by

$$d^{L,\lambda}(t) := \max_{\xi \in \Omega_L} \|P_t^{\xi,\lambda} - \mu_L^\lambda\|_{\mathrm{TV}}. \tag{1.8}$$

For any given $\epsilon \in (0,1)$, let the ϵ-mixing-time be

$$T_{\mathrm{mix}}^{L,\lambda}(\epsilon) := \inf \big\{ t \geq 0 : d^{L,\lambda}(t) \leq \epsilon \big\}. \tag{1.9}$$

We say that this sequence of Markov chains has a cutoff, if for all $\epsilon \in (0,1)$,

$$\lim_{L\to\infty} \frac{T_{\mathrm{mix}}^{L,\lambda}(\epsilon)}{T_{\mathrm{mix}}^{L,\lambda}(1-\epsilon)} = 1. \tag{1.10}$$

The cutoff phenomenon is surveyed in the seminal paper [Dia96], and we refer to [LP17, Chapter 18] for more information. In [CMT08, Theorems 3.1 and 3.2], for all $\lambda \in [0,2)$, Caputo *et al.* proved that for all $\delta > 0$ and all $\epsilon \in (0,1)$, if L is sufficiently large, we have

$$\frac{1-\delta}{2\pi^2} L^2 \log L \leq T_{\mathrm{mix}}^{L,\lambda}(\epsilon) \leq \frac{6+\delta}{\pi^2} L^2 \log L. \tag{1.11}$$

Moreover, the spectral gap, denoted by $\mathrm{gap}_{L,\lambda}$, is the minimal positive eigenvalue of $-\mathcal{L}$ and the relaxation time $T_{\mathrm{rel}}^{L,\lambda}$ is its inverse. That is

$$T_{\mathrm{rel}}^{L,\lambda} := \sup_{f\,:\,\mathrm{Var}_L(f)>0} -\frac{\mathrm{Var}_L(f)}{\mu_L^\lambda(f\mathcal{L}f)} = \mathrm{gap}_{L,\lambda}^{-1}, \tag{1.12}$$

where $\mathrm{Var}_L(f) := \mu_L^\lambda(f^2) - \mu_L^\lambda(f)^2$ with $\mu_L^\lambda(f) := \sum_{\xi\in\Omega_L}\mu_L^\lambda(\xi)f(\xi)$. There is no explicit eigenfunction of the generator $\mathcal{L}$ due to the effect of the impenetrable wall (*i.e.* the $x-$axis), but Caputo *et al.* adapted the idea in [Wil04, Lemma 1] to find a function (defined in (3.2) below) which is almost an eigenfunction. In [CMT08, Theorems 3.1 and 3.2], they showed that for all $\lambda \in [0, 2)$, there exists a universal constant $C > 0$ independent of L and λ, such that

$$C^{-1}L^2 \le T_{\mathrm{rel}}^{L,\lambda} \le CL^2, \tag{1.13}$$

which together with (1.11) implies $T_{\mathrm{rel}}^{L,\lambda} \ll T_{\mathrm{mix}}^{L,\lambda}(\frac14)$ and then strongly indicates that Equation (1.10) should hold. Note that the condition $T_{\mathrm{rel}}^{L,\lambda} \ll T_{\mathrm{mix}}^{L,\lambda}(\frac14)$ is not sufficient to imply the cutoff phenomenon, and we refer to [LP17, Notes in Chapter 18] for an example.

1.3. Main results. We find that the mixing time is $\pi^{-2}\big(1+o(1)\big)(L^2\log L)$ for all $\lambda \in [0, 1]$, improving both the lower and upper bounds in [CMT08]. That is the following theorem.

THEOREM 1.1. *For all $\epsilon \in (0, 1)$ and all $\lambda \in [0, 1]$, we have*

$$\lim_{L\to\infty} \frac{\pi^2 T_{\mathrm{mix}}^{L,\lambda}(\epsilon)}{L^2\log L} = 1. \tag{1.14}$$

Therefore, there is a cutoff phenomenon in the Glauber dynamics for $\lambda \in [0, 1]$. The reason why we include the result for $\lambda = 0$ is the need for the mixing time about the dynamics when $\lambda \in (1, 2)$.

REMARK 1.2. *Theorem 1.1 about $\lambda = 0$ is the same as the case $\lambda = 1$ by the following identification. Let*

$$\Omega_L^+ := \Big\{\xi \in \Omega_L : \mathcal{N}(\xi) = 0\Big\} \tag{1.15}$$

where $\mathcal{N}(\xi)$ is defined in (1.1), and identify Ω_L^+ with Ω_{L-2} by lifting the x-axis up by distance one in Ω_L. Precisely, the identification is as follows: $\xi = (\xi_x)_{0\le x\le L} \in \Omega_L^+$ is identified with $\varsigma = (\varsigma_x)_{0\le x\le L-2} \in \Omega_{L-2}$, if $\varsigma_x = \xi_{x+1} - 1$ for all $x \in [\![0, L-2]\!]$. We can see:

(a) *μ_L^0 is the same as the probability measure μ_{L-2}^1;*
(b) *the dynamics $(\sigma_t^{\xi,0})_{t\ge0}$—living in the space Ω_L^+—is the same as the dynamics $(\sigma_t^{\varsigma,1})_{t\ge0}$ living in the space Ω_{L-2}, where $\xi \in \Omega_L^+$ is identified with $\varsigma \in \Omega_{L-2}$.*

Therefore, we only need to prove Theorem 1.1 for $\lambda \in (0, 1]$. In addition, we have a partial result for $\lambda \in (1, 2)$. Let us state the framework. We introduce a natural partial order " $\le$ " on Ω_L as follows

$$\Big(\xi \le \xi'\Big) \Leftrightarrow \Big(\forall x \in [\![0, L]\!], \xi_x \le \xi'_x\Big). \tag{1.16}$$

In other words, if $\xi \le \xi'$, the path ξ lies below the path ξ'. Then the maximal path $\wedge$ and the minimal path $\vee$ are respectively given by

$$\wedge_x := \min\big(x, -x+L\big), \quad \forall x \in [\![0, L]\!]; \tag{1.17}$$

$$\vee_x := x - 2\lfloor x/2\rfloor, \quad \forall x \in [\![0, L]\!]. \tag{1.18}$$

where $\lfloor x/2 \rfloor := \sup\left\{n \in \mathbb{Z} : n \leq x/2\right\}$. Define

$$T_{\text{mix}}^{L,\wedge}(\epsilon) := \inf\left\{t \geq 0 : \|P_t^{\wedge,\lambda} - \mu_L^\lambda\|_{\text{TV}} \leq \epsilon\right\},$$
$$T_{\text{mix}}^{L,\vee}(\epsilon) := \inf\left\{t \geq 0 : \|P_t^{\vee,\lambda} - \mu_L^\lambda\|_{\text{TV}} \leq \epsilon\right\},$$

and

$$\breve{T}_{\text{mix}}^{L}(\epsilon) := \max\left(T_{\text{mix}}^{L,\wedge}, T_{\text{mix}}^{L,\vee}\right). \tag{1.19}$$

For $\lambda \in (1,2)$, applying Peres-Winkler censoring inequality in [PW13, Theorem 1.1], we discover that the mixing time is also $(1+o(1))(L^2 \log L)/\pi^2$ for the dynamics starting with the two extremal paths. That is the following theorem.

THEOREM 1.3. *For all $\epsilon \in (0,1)$ and $\lambda \in (1,2)$, we have*

$$\lim_{L\to\infty} \frac{\pi^2 \breve{T}_{\text{mix}}^{L}(\epsilon)}{L^2 \log L} = 1. \tag{1.20}$$

1.4. Other values of λ. Our analysis excludes the case $\lambda > 2$, let us just mention that the convergence to equilibrium follows a different pattern in this case. While the relaxation time and the mixing time are of order L^2 and $L^2 \log L$ respectively in the repulsive phase $\lambda < 2$, it is believed that they become of order L and L^2 respectively in the attractive phase $\lambda > 2$. Rigorous lower bound has been proved in [CMT08, Theorem 3.2], but matching order upper bound has only been shown when $\lambda = \infty$ ([CMT08, Proposition 5.6] for the mixing time). Furthermore in [Lac14, Theorem 2.7], it is shown that in this last case the mixing time is equal to $L^2/4(1+o(1))$. When $\lambda \in (2,\infty)$, the conjecture in [Lac14, Section 2.7] seems to indicate that the mixing time should be of order $C(\lambda)L^2(1+o(1))$ for some explicit $C(\lambda)$.

At the critical value $\lambda = 2$, we believe that the mixing time continues to be $\frac{L^2}{\pi^2}(\log L)(1+o(1))$ but our techniques do not allow to treat this case.

1.5. Organization of the chapter. Section 2 introduces a grand coupling for the dynamics corresponding to different ξ and λ, and some useful reclaimed results.

Section 3 is dedicated to the lower bound on the mixing time for $\lambda \in (0,2)$.

Section 4 supplies the upper bound on the mixing time for $\lambda \in (0,1]$.

Section 5 is about the upper bound on the mixing time for the dynamics starting with the two extremal paths when $\lambda \in (1,2)$, applying censoring inequality.

1.6. Notation. We use " := " to define a new quantity on the left-hand side, and use " =: " in some cases when the quantity is defined on the right-hand side.

We let $(C_n(\lambda))_{n\in\mathbb{N}}$ and $(c_n(\lambda))_{n\in\mathbb{N}}$ be some positive constants, which are only dependent on λ. Additionally, we let $(c_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$ be some positive and universal constants.

2. Technical preliminaries

To use the monotonicity of the Glauber dynamics, we provide a graphical construction of the Markov chain such that all dynamics, *i.e.* $\left\{(\sigma_t^{\xi,\lambda})_{t\geq 0} : \forall \xi \in \Omega_L, \forall \lambda \in [0,\infty)\right\}$, live in one common probability space. This construction appears in [Lac16b, Section 8.1], which provides more independent flippable corners than the coupling in [CMT08, Subsection 2.2.1].

2.1. A graphical construction. We set the exponential clocks and independent "coins" in the centers of the squares formed by all the possible corners and their counterparts. Let

$$\Theta := \Big\{(x,z) : x \in [\![2, L-2]\!], z \in [\![1, L/2-1-|x-L/2|]\!]; x+z \in 2\mathbb{N}+1\Big\}, \tag{2.1}$$

and let $\mathcal{T}^{\uparrow}$ and $\mathcal{T}^{\downarrow}$ be two independent rate-one exponential clock processes indexed by Θ. That is to say, for every $(x,z) \in \Theta$ and $n \geq 0$, we have $\mathcal{T}^{\uparrow}_{(x,z)}(0) = 0$, and

$$\Big(\mathcal{T}^{\uparrow}_{(x,z)}(n) - \mathcal{T}^{\uparrow}_{(x,z)}(n-1)\Big)_{n\geq 1}$$

is a field of i.i.d. exponential random variables with mean 1. Similarly, this holds for $\mathcal{T}^{\downarrow}_{(x,z)}$. Moreover, let $\mathcal{U}^{\uparrow} = \Big(U^{\uparrow}_{(x,z)}(n)\Big)_{(x,z)\in\Theta, n\geq 1}$ and $\mathcal{U}^{\downarrow} = \Big(U^{\downarrow}_{(x,z)}(n)\Big)_{(x,z)\in\Theta, n\geq 1}$ be two independent fields of i.i.d. random variables uniformly distributed in $[0,1]$, which are independent of $\mathcal{T}^{\uparrow}$ and $\mathcal{T}^{\downarrow}$. Given $\mathcal{T}^{\uparrow}$, $\mathcal{T}^{\downarrow}$, $\mathcal{U}^{\uparrow}$ and $\mathcal{U}^{\downarrow}$, we construct, in a deterministic way, $(\sigma^{\xi,\lambda}_t)_{t\geq 0}$ the trajectory of the Markov chain with parameter λ and starting with $\xi \in \Omega_L$, *i.e.* $\sigma^{\xi,\lambda}_0 = \xi$.

When the clock process $\mathcal{T}^{\uparrow}_{(x,z)}$ rings at time $t = \mathcal{T}^{\uparrow}_{(x,z)}(n)$ for $n \geq 1$ and $\sigma^{\xi,\lambda}_{t^-}(x) = z-1$, we update $\sigma^{\xi,\lambda}_{t^-}$ as follows:

- if $\sigma^{\xi,\lambda}_{t^-}(x-1) = \sigma^{\xi,\lambda}_{t^-}(x+1) = z \geq 2$ and $U_{(x,z)}(n)^{\uparrow} \leq \frac{1}{2}$, let $\sigma^{\xi,\lambda}_t(x) = z+1$ and the other coordinates remain unchanged;
- if $\sigma^{\xi,\lambda}_{t^-}(x-1) = \sigma^{\xi,\lambda}_{t^-}(x+1) = z = 1$ and $U^{\uparrow}_{(x,z)}(n) \leq \frac{1}{1+\lambda}$, let $\sigma^{\xi,\lambda}_t(x) = 2$ and the other coordinates remain unchanged.

If neither of these two aforementioned conditions is satisfied, we do nothing.

When the clock process $\mathcal{T}^{\downarrow}_{(x,z)}$ rings at time $t = \mathcal{T}^{\downarrow}_{(x,z)}(n)$ for $n \geq 1$ and $\sigma^{\xi,\lambda}_{t^-}(x) = z+1$, we update $\sigma^{\xi,\lambda}_{t^-}$ as follows:

- if $\sigma^{\xi,\lambda}_{t^-}(x-1) = \sigma^{\xi,\lambda}_{t^-}(x+1) = z \geq 2$ and $U^{\downarrow}_{(x,z)}(n) \leq \frac{1}{2}$, let $\sigma^{\xi,\lambda}_t(x) = z-1$ and the other coordinates remain unchanged;
- if $\sigma^{\xi,\lambda}_{t^-}(x-1) = \sigma^{\xi,\lambda}_{t^-}(x+1) = z-1 = 0$ and $U^{\downarrow}_{(x,z)}(n) \leq \frac{\lambda}{1+\lambda}$, let $\sigma^{\xi,\lambda}_t(x) = 0$ and the other coordinates remain unchanged.

If neither of these two aforementioned conditions is satisfied, we do nothing.

Let $\mathbb{P}$ or $\mathbb{E}$ stand for the probability law corresponding to $\mathcal{T}^{\uparrow}$, $\mathcal{T}^{\downarrow}$, $\mathcal{U}^{\uparrow}$ and $\mathcal{U}^{\downarrow}$. Recall that μ^{λ}_L is the stationary probability measure for the dynamics. The dynamics $(\sigma^{\mu,\lambda}_t)_{t\geq 0}$ is constructed by first taking the initial path ξ sampling from μ at $t=0$ and then using the graphical construction with parameter λ for $t > 0$. This sampling is independent of $\mathbb{P}$. Define $P^{\mu,\lambda}_t(\cdot) := \mathbb{P}(\sigma^{\mu,\lambda}_t = \cdot)$, and likewise $P^{\mu,\lambda}_t(\mathcal{A}) := \mathbb{P}[\sigma^{\mu,\lambda}_t \in \mathcal{A}]$ for $\mathcal{A} \subset \Omega_L$. When it is clear in the context, we use the notations $(\sigma^{\mu}_t)_{t\geq 0}$ and P^{μ}_t, ignoring the parameter λ.

This graphical construction allows us to construct all the trajectories $(\sigma^{\xi,\lambda}_t)_{t\geq 0}$ starting from all $\xi \in \Omega_L$ and all parameters $\lambda \in [0,\infty)$ simultaneously. It preserves the order, affirmed in the following proposition. The proof of this proposition, which we omit, is almost identical to that of [Lac16b, Proposition 3.1].

Proposition 2.1. *Let ξ and ξ' be two elements of Ω_L satisfying $\xi \leq \xi'$, and $0 \leq \lambda \leq \lambda'$. With the graphical construction above, we have*

$$\begin{aligned} &\mathbb{P}\Big[\forall t \in [0,\infty) : \sigma^{\xi,\lambda}_t \leq \sigma^{\xi',\lambda}_t\Big] = 1,\\ &\mathbb{P}\Big[\forall t \in [0,\infty) : \sigma^{\xi,\lambda'}_t \leq \sigma^{\xi,\lambda}_t\Big] = 1. \end{aligned} \tag{2.2}$$

2.2. Useful reclaimed results. We have the asymptotic information about $Z_L(\lambda)$, which is:

THEOREM 2.2 (Theorem 2.1 in [CMT08]). *For every $\lambda \in [0,2)$, we have*

$$\lim_{L\to\infty} \frac{Z_L(\lambda)}{2^L L^{-3/2}} = C(\lambda), \tag{2.3}$$

where $C(\lambda) > 0$ is a constant, only dependent on λ.

Furthermore, to understand the Glauber dynamics, it is important to understand how the generator $\mathcal{L}$ acts on the paths in Ω_L. Let us introduce the settings. For a function $g\colon [\![0, L]\!] \to \mathbb{R}$, the discrete Laplace operator Δ is defined as follows: for any $x \in [\![1, L-1]\!]$,

$$(\Delta g)_x := \frac{1}{2}\Big(g(x-1) + g(x+1)\Big) - g(x).$$

Besides, we define a function $f\colon \Omega_L \mapsto \mathbb{R}$ to be $f(\xi) := \xi_x$, and let $\mathcal{L}\xi_x := (\mathcal{L}f)(\xi)$ for $x \in [\![1, L-1]\!]$. Considering (1.6), we know that $\mathcal{L}\xi_x = \mu_L^\lambda(\xi_x|\xi_{x-1}, \xi_{x+1}) - \xi_x$, and a calculation yields the following identity which we recall as a lemma.

LEMMA 2.3 (Lemma 2.3 in [CMT08]). *For every $\lambda > 0$ and every $x \in [\![1, L-1]\!]$, we have*

$$\mathcal{L}\xi_x = (\Delta\xi)_x + \mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=0\}} - \left(\frac{\lambda-1}{\lambda+1}\right)\mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=1\}}. \tag{2.4}$$

3. Lower bound on the mixing time for $\lambda \in (0,2)$

This section is devoted to provide a lower bound on the mixing time of the Glauber dynamics for $\lambda \in (0,2)$, which is the following proposition.

PROPOSITION 3.1. *For all $\lambda \in (0,2)$ and all $\epsilon \in (0,1)$, we have*

$$T_{\text{mix}}^{L,\lambda}(\epsilon) \geq \tfrac{1}{\pi^2} L^2 \log L - C(\lambda,\epsilon)L^2 =: t_{C(\lambda,\epsilon)}, \tag{3.1}$$

where $C(\lambda,\epsilon) > 0$ is a constant, only dependent on λ and ϵ.

Before we start the proof of Proposition 3.1, let us explain the idea. Note that the function $\Phi(\xi)$, defined in (3.2) below, is almost the area enclosed by the x-axis and the path $\xi \in \Omega_L$. Intuitively, $\Phi(\wedge)$ is of order L^2, while at equilibrium $\Phi(\xi)$ is of order $L^{3/2}$. The second moment method in [CMT08, Theorem 3.2] does not supply a sharp lower bound on the mixing time. We adapt the idea in [CMT08, Theorem 3.2] to provide the lower bound in (3.1) by proving the following.

(i) While the expected equilibrium value $\mu(\Phi)$ is at most of order $L^{3/2}$, $\mathbb{E}[\Phi(\sigma_t^\wedge)]$ is much bigger than $L^{3/2}$ for all $t \leq t_{C(\lambda,\epsilon)}$;
(ii) On the one hand $\Phi(\sigma_t^\mu)$ is fairly close to its mean $\mu(\Phi)$ by Markov's inequality, and on the other hand $\Phi(\sigma_t^\wedge)$ is well concentrated around $\mathbb{E}[\Phi(\sigma_t^\wedge)]$ by controlling its fluctuation through martingale approach.

Subsection 3.1 prepares all the ingredients for the first step of this strategy, and Subsection 3.2 is dedicated to the second step of the strategy, giving the proof of Proposition 3.1.

3.1. Ingredients for the lower bound of the mixing time. Inspired by [Wil04, Equation (1)], Caputo *et al.* in [CMT08, Equation (2.39)] defined the weighted area function $\Phi\colon \Omega_L \to \mathbb{R}$ by

$$\Phi(\xi) := \sum_{x=1}^{L-1} \xi_x \overline{\sin}(x), \tag{3.2}$$

where $\overline{\sin}(x) := \sin(\frac{\pi x}{L})$ and $\xi \in \Omega_L$. As [CMT08, Equation (4.3)], we use Lemma 2.3 and summation by part to obtain

$$(\mathcal{L}\Phi)(\xi) = \sum_{x=1}^{L-1} \overline{\sin}(x)\mathcal{L}\xi_x = -\kappa_L \Phi(\xi) + \Psi(\xi), \tag{3.3}$$

where $\kappa_L := 1 - \cos(\frac{\pi}{L})$ and

$$\Psi(\xi) := \sum_{x=1}^{L-1} \overline{\sin}(x)\left[\mathbf{1}_{\{\xi_{x-1}=\xi_{x=1}=0\}} - \left(\frac{\lambda-1}{\lambda+1}\right)\mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=1\}}\right]. \tag{3.4}$$

Since $\overline{\sin}(x) \geq 0$ for all $x \in [\![0, L]\!]$, we have

$$|\Psi(\xi)| \leq \sum_{x=1}^{L-1} \overline{\sin}(x)\left[\mathbf{1}_{\{\xi_{x-1}=\xi_{x=1}=0\}} + \left|\frac{\lambda-1}{\lambda+1}\right|\mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=1\}}\right] =: \overline{\Psi}(\xi). \tag{3.5}$$

Caputo *et al.* gave an upper bound on $\mu_L^\lambda(\Phi)$. In [CMT08, Equation (5.15)], they used coupling and monotonicity to obtain that for every positive integer k,

$$\sup_{\lambda\geq 0, L\in 2\mathbb{N}} \sup_{x\in[\![1,L-1]\!]} \mu_L^\lambda\left(\frac{(\xi_x)^k}{L^{k/2}}\right) < \infty.$$

Consequently, using $k=1$ and $\overline{\sin}(x) \leq 1$, we have

$$\mu_L^\lambda(\Phi) \leq \sum_{x=1}^{L-1} \mu_L^\lambda(\xi_x) \leq cL^{3/2}, \tag{3.6}$$

where $c > 0$ does not depend on λ. In addition, Caputo *et al.* also gave a lower bound on $\mathbb{E}[\Phi(\sigma_t^\wedge)]$, which we recall as a lemma below.

LEMMA 3.2 (Equation (5.24) in [CMT08]). *For all $\lambda \in (0,2)$, all $t \geq 0$, all $L \geq 2$ and some constant $c(\lambda) > 0$, we have*

$$\mathbb{E}[\Phi(\sigma_t^\wedge)] \geq \Phi(\sigma_0^\wedge)e^{-\kappa_L t} - c(\lambda)L^{3/2}.$$

In view of (3.5), we need an upper bound on $\mathbb{P}\big[\sigma_t^\wedge(x-1) = \sigma_t^\wedge(x+1) \in \{0,1\}\big]$ for $x \in [\![1, L-1]\!]$, which is the following lemma.

LEMMA 3.3. *For all $t \geq 0$, all $x \in [\![1, L-1]\!]$ and all $L \geq 2$, we have*

$$\mathbb{P}\big[\sigma_t^\wedge(x-1) = \sigma_t^\wedge(x+1) \in \{0,1\}\big] \leq C_1(\lambda)\frac{L^{3/2}}{x^{3/2}(L-x)^{3/2}}. \tag{3.7}$$

PROOF. Since $\sigma_t^\wedge \geq \sigma_t^\mu$ for all $t \geq 0$, we know that for all $x \in [\![1, L-1]\!]$,

$$\begin{aligned}\mathbb{P}\big[\sigma_t^\wedge(x-1) = \sigma_t^\wedge(x+1) \in \{0,1\}\big] &\leq \mathbb{P}\big[\sigma_t^\mu(x-1) = \sigma_t^\mu(x+1) \in \{0,1\}\big]\\ &= \mu_L^\lambda\big(\xi_{x-1} = \xi_{x+1} \in \{0,1\}\big).\end{aligned}$$

For all $\lambda \in (0,2)$, all $x \in [\![1, L-1]\!] \cap 2\mathbb{N}$ and all $L \geq 2$, applying Theorem 2.2, we obtain

$$\mu_L^\lambda(\xi_x = 0) = \lambda \frac{Z_x(\lambda) Z_{L-x}(\lambda)}{Z_L(\lambda)} \leq C_2(\lambda) \frac{L^{3/2}}{x^{3/2}(L-x)^{3/2}}, \tag{3.8}$$

and

$$\mu_L^\lambda(\xi_{x-1} = \xi_{x+1} = 0) = \lambda^2 \frac{Z_{x-1}(\lambda) Z_{L-x-1}(\lambda)}{Z_L(\lambda)} \leq C_2(\lambda) \frac{L^{3/2}}{x^{3/2}(L-x)^{3/2}}. \tag{3.9}$$

With the same conditions about λ, x and L above, as [CMT08, Equation (5.23)] we have

$$\mu_L^\lambda(\xi_{x-1} = \xi_{x+1} = 1) = \frac{1+\lambda}{\lambda} \mu_L^\lambda(\xi_x = 0). \tag{3.10}$$

Therefore, by (3.8), (3.9) and (3.10), we obtain (3.7). □

3.2. Proof of the lower bound on the mixing time. Let us detail the second step of the aforementioned strategy. To prove that $\Phi(\sigma_t^\wedge)$ is well concentrated around its mean $\mathbb{E}[\Phi(\sigma_t^\wedge)]$, we do the following.

(i) For a fixed time t_0, we use the function $F(t,\xi) = \exp(\kappa_L(t-t_0))\Phi(\xi)$ to construct a Dynkin's martingale M (see [KL99, Lemma 5.1 in Appendix 1]).

(ii) To estimate the fluctuation of $F(t_0, \sigma_{t_0}^\wedge) = \Phi(\sigma_{t_0}^\wedge)$, we control the martingale bracket $\langle M.\rangle$ and the mean of $(\partial_t + \mathcal{L})F(t, \sigma_t^\wedge)$, which comes from the construction of Dynkin's martingale.

While $\Phi(\sigma_t^\mu)$ is at most of order $L^{3/2}$, $\Phi(\sigma_{t_0}^\wedge)$ is much bigger than $L^{3/2}$ for all $t_0 \leq t_{C(\lambda,\epsilon)}$. This property of Φ about σ_t^μ and $\sigma_{t_0}^\wedge$ can be used to provide a lower bound on the distance between μ and $P_{t_0}^\wedge$.

Proof of Proposition 3.1. We adapt the approach in [CMT08, Proposition 5.3]. For $C \in (0,\infty)$, define

$$\mathcal{A}_C := \left\{\xi \in \Omega_L : \Phi(\xi) \leq CL^{3/2}\right\}. \tag{3.11}$$

Using Markov's inequality and (3.6), we obtain

$$1 - \mu(\mathcal{A}_C) = \mu(\Phi > CL^{3/2}) \leq \frac{\mu(\Phi)}{CL^{3/2}} \leq \frac{c}{C}, \tag{3.12}$$

where the rightmost term is smaller than or equal to $\epsilon/2$ for $C \geq 2c/\epsilon$. Our next step is to prove that for any given $\epsilon > 0$, if $t_0 \leq t_{C(\lambda,\epsilon)}$, we have

$$P_{t_0}^\wedge(\mathcal{A}_C) \leq \epsilon/2.$$

In order to obtain such an upper bound, we construct a Dynkin's martingale and control its fluctuation. Let t_0 be a fixed time, we define a function $F\colon [0,t_0] \times \Omega_L \to \mathbb{R}$ by

$$F(t,\xi) := e^{\kappa_L(t-t_0)}\Phi(\xi).$$

We recall that $\sigma_t^\wedge$, defined in Subsection 2.1, is the dynamics at time t starting with the maximal path $\wedge$. Further, we define a Dynkin's martingale by

$$M_t := F(t, \sigma_t^\wedge) - F(0, \sigma_0^\wedge) - \int_0^t (\partial_s + \mathcal{L})F(s, \sigma_s^\wedge)\mathrm{d}s. \tag{3.13}$$

Applying $(\mathcal{L}\Phi)(\xi) = -\kappa_L\Phi(\xi) + \Psi(\xi)$ in (3.3), we obtain

$$(\partial_t + \mathcal{L})F(t, \sigma_t^\wedge) = e^{\kappa_L(t-t_0)}\Psi(\sigma_t^\wedge). \tag{3.14}$$

For simplicity of notation, set

$$B(t) := \int_0^t e^{\kappa_L(s-t_0)}\Psi(\sigma_s^\wedge)\mathrm{d}s. \tag{3.15}$$

Now we give an upper bound on $\mathbb{E}[M_t^2]$ by controlling the martingale bracket $\langle M.\rangle$, which is such that the process $\big(M_t^2 - \langle M.\rangle_t\big)_{t\geq 0}$ is a martingale with respect to its natural filtration. Since there is at most one transition at each coordinate and each transition can change the value of M_t in absolute value by at most $2e^{\kappa_L(t-t_0)}$, we have

$$\partial_t \langle M.\rangle_t \leq \sum_{x=1}^{L-1} 4e^{2\kappa_L(t-t_0)} \leq 4Le^{2\kappa_L(t-t_0)}.$$

As $M_0 = 0$ and $\kappa_L = 1-\cos\left(\frac{\pi}{L}\right) \geq \frac{\pi^2}{4L^2}$ for all $L \geq 4$, we obtain

$$\mathbb{E}[M_{t_0}^2] = \mathbb{E}[\langle M.\rangle_{t_0}] \leq \int_0^{t_0} 4Le^{2\kappa_L(t-t_0)}dt \leq \frac{8L^3}{\pi^2}. \tag{3.16}$$

Furthermore, we give an upper bound for the mean of $B(t_0)$, defined in (3.15). Recalling the definitions of Ψ and $\overline{\Psi}$ in (3.4) and (3.5) respectively, we have

$$\begin{aligned}
\mathbb{E}[|B(t_0)|] &\leq \mathbb{E}\left[\int_0^{t_0} e^{\kappa_L(t-t_0)}\overline{\Psi}(\sigma_t^{\wedge})dt\right] \\
&\leq \mathbb{E}\left[\int_0^{t_0} e^{\kappa_L(t-t_0)}\overline{\Psi}(\sigma_t^{\mu})dt\right] \\
&\leq C_3(\lambda)\kappa_L^{-1}\sum_{x=1}^{L-1}\overline{\sin}(x)\frac{L^{3/2}}{x^{3/2}(L-x)^{3/2}} \\
&\leq C_4(\lambda)L^{3/2}.
\end{aligned} \tag{3.17}$$

The first inequality uses $|\Psi(\xi)| \leq \overline{\Psi}(\xi)$ for all $\xi \in \Omega_L$. The second inequality is due to two facts: (1) $\overline{\Psi}(\xi) \leq \overline{\Psi}(\xi')$ for $\xi \leq \xi'$; and (2) $\sigma_t^{\wedge} \geq \sigma_t^{\mu}$. In the third inequality, we use Fubini's Theorem to interchange the orders of integration and expectation, and use Lemma 3.3 to give an upper bound for $\mathbb{E}[\overline{\Psi}(\sigma_t^{\mu})]$. In the last inequality, we use the following inequality:

$$\overline{\sin}(x) = \sin\left(\frac{\pi x}{L}\right) \leq \frac{\min\big(x, L-x\big)\pi}{L}.$$

Here and now, we try to find the suitable small t_0 such that $\Phi(\sigma_{t_0}^{\wedge})$ is much larger than $L^{3/2}$ with high probability. We note that $\Phi(\sigma_0^{\wedge}) \geq \frac{1}{36}L^2$ and $\kappa_L \leq \frac{\pi^2}{2L^2}$ for all $L \geq 2$. Let $C \geq 1$, and define

$$t_0 := \frac{1}{\pi^2}L^2\log L - CL^2.$$

If $t_0 \leq 0$, nothing needs to be done (for $L \leq 4$, $t_0 \leq 0$). In the remaining of this subsection, we assume $t_0 > 0$. Then for all $L \geq 2$, $t_0\kappa_L \leq \frac{1}{2}\log L - C$. Moreover, there exists a universal constant $C_0 \geq 1$ such that if $C \geq C_0$, we have

$$\frac{1}{36}e^C \geq 3C.$$

By Lemma 3.2, for all $C \geq \max\big(C_0, c(\lambda)\big)$, we have

$$\mathbb{E}[\Phi(\sigma_{t_0}^{\wedge})] \geq 3CL^{3/2} - c(\lambda)L^{3/2} \geq 2CL^{3/2}.$$

Then, if $\Phi(\sigma_{t_0}^{\wedge}) \leq CL^{3/2}$ (*i.e.* $\sigma_{t_0}^{\wedge} \in \mathcal{A}_C$, defined in (3.11)), it implies

$$\big|\Phi(\sigma_{t_0}^{\wedge}) - \mathbb{E}\big[\Phi(\sigma_{t_0}^{\wedge})\big]\big| \geq CL^{3/2}$$

and

$$P_{t_0}^{\wedge}(\mathcal{A}_C) \leq \mathbb{P}\big[|\Phi(\sigma_{t_0}^{\wedge}) - \mathbb{E}[\Phi(\sigma_{t_0}^{\wedge})]| \geq CL^{3/2}\big]. \tag{3.18}$$

In addition, recalling $\Phi(\sigma^{\wedge}_{t_0}) = F(t_0, \sigma^{\wedge}_{t_0}) = M_{t_0} + F(0, \sigma^{\wedge}_0) + B(t_0)$ in (3.13) and using Markov's inequality, we obtain

$$\begin{aligned}
&\mathbb{P}\Big[\big|\Phi(\sigma^{\wedge}_{t_0}) - \mathbb{E}[\Phi(\sigma^{\wedge}_{t_0})]\big| \geq CL^{3/2}\Big] \\
=&\mathbb{P}\Big[\big|M_{t_0} + B(t_0) - \mathbb{E}[B(t_0)]\big| \geq CL^{3/2}\Big] \\
\leq&\mathbb{P}\Big[\big|M_{t_0}\big| \geq \frac{1}{3}CL^{3/2}\Big] + \mathbb{P}\Big[\big|B(t_0)\big| \geq \frac{1}{3}CL^{3/2}\Big] \\
\leq&\frac{9\mathbb{E}[M^2_{t_0}]}{C^2L^3} + \frac{3\mathbb{E}[|B(t_0)|]}{CL^{3/2}},
\end{aligned} \tag{3.19}$$

where the second last inequality holds for $C > 3C_4(\lambda)$ by $\mathbb{E}[|B(t_0)|] \leq C_4(\lambda)L^{3/2}$ in (3.17). The last term in (3.19) is smaller than or equal to $\epsilon/2$ for $C \geq \max\big(\frac{18}{\pi\sqrt{\epsilon}}, \frac{12C_4(\lambda)}{\epsilon}\big)$, on account of $\mathbb{E}[M^2_{t_0}] \leq \frac{8L^3}{\pi^2}$ in (3.16) and (3.17). Combining (3.12), (3.18) and (3.19), we know that

$$\|P^{\wedge}_{t_0} - \mu\|_{\mathrm{TV}} \geq \mu(A_C) - P^{\wedge}_{t_0}(A_C) \geq 1 - \epsilon, \tag{3.20}$$

which holds for $C \geq \max\{\frac{2c}{\epsilon}, C_0, 3C_4(\lambda), \frac{18}{\pi\sqrt{\epsilon}}, \frac{12C_4(\lambda)}{\epsilon}\} =: C(\lambda,\epsilon)$. Therefore, for $C \geq C(\lambda,\epsilon)$, we have

$$T^{L,\lambda}_{\mathrm{mix}}(\epsilon) \geq \frac{1}{\pi^2}L^2\log L - CL^2.$$

□

4. Upper bound on the mixing time for $\lambda \in (0,1]$

This section is devoted to providing an upper bound on the mixing time of the dynamics for the regime $\lambda \in (0,1]$. For any $\xi \in \Omega_L$, by the triangle inequality, we have

$$\|P^{\xi}_t - P^{\mu}_t\|_{\mathrm{TV}} \leq \sum_{\xi' \in \Omega_L} \mu(\xi')\|P^{\xi}_t - P^{\xi'}_t\|_{\mathrm{TV}} \leq \max_{\xi' \in \Omega_L} \|P^{\xi}_t - P^{\xi'}_t\|_{\mathrm{TV}}. \tag{4.1}$$

To give an upper bound for the term in the rightmost hand side above, we use the following characterization of total variation distance. Let α and β be two probability measures on Ω_L. We say that ϑ is a coupling of α and β, if ϑ is a probability measure on $\Omega_L \times \Omega_L$ such that $\vartheta(\xi \times \Omega_L) = \alpha(\xi)$ and $\vartheta(\Omega_L \times \xi') = \beta(\xi')$ for any elements $\xi, \xi' \in \Omega_L$. The following proposition says that the total variation distance measures how well we can couple two random variables with distribution laws α and β respectively.

Proposition 4.1 (Proposition 4.7 [LP17]). *Let α and β be two probability distributions on Ω_L. Then*

$$\|\alpha - \beta\|_{\mathrm{TV}} = \inf\Big\{\vartheta\left(\{(\xi,\xi') : \xi \neq \xi'\}\right) : \ \vartheta \textit{ is a coupling of } \alpha \textit{ and } \beta\Big\}.$$

The graphical construction in Subsection 2.1 provides a coupling between P^{ξ}_t and $P^{\xi'}_t$, which preserves the monotonicity asserted in Proposition 2.1. Therefore, σ^{ξ}_t lies between $\sigma^{\vee}_t$ and $\sigma^{\wedge}_t$ for any $\xi \in \Omega_L$. Applying Proposition 4.1, we obtain

$$\|P^{\xi}_t - P^{\xi'}_t\|_{\mathrm{TV}} \leq \mathbb{P}[\sigma^{\xi}_t \neq \sigma^{\xi'}_t] \leq \mathbb{P}[\sigma^{\wedge}_t \neq \sigma^{\vee}_t], \tag{4.2}$$

where the last inequality is due to the fact that after the dynamics starting from the two extremal paths have coalesced, we must have $\sigma^{\wedge}_t = \sigma^{\xi}_t = \sigma^{\vee}_t$ for any $\xi \in \Omega_L$. This argument was used in [CMT08, Theorem 3.1] to obtain an upper bound on the mixing time. Comparing with the coupling in [CMT08, Subsection 2.2.1], the graphical construction in Subsection 2.1 provides more independent flippable corners and maximizes the fluctuation of the area enclosed by $\sigma^{\wedge}_t$

and $\sigma_t^{\vee}$. Adapting the approach in [LL20, Section 7], we use a supermartingale approach to have a good control of the fluctuation of the area enclosed by $\sigma_t^{\wedge}$ and $\sigma_t^{\vee}$ to obtain a sharp upper bound on the mixing time. Let the coalescing time τ be

$$\tau := \inf\{t \geq 0 : \sigma_t^{\wedge} = \sigma_t^{\vee}\},$$

which is the first instant when the dynamics starting from the two extremal paths coalesce. By (4.1) and (4.2), we obtain

$$d^{L,\lambda}(t) \leq \mathbb{P}\left[\sigma_t^{\wedge} \neq \sigma_t^{\vee}\right] = \mathbb{P}\left[\tau > t\right]. \tag{4.3}$$

In this section, our goal is to show that for any given $\delta > 0$ and all L sufficiently large, with high probability, we have

$$\tau \leq \frac{1+\delta}{\pi^2} L^2 \log L.$$

We adapt the approach in [LL20, Section 7] to achieve this goal. In practice, it is more feasible to couple two dynamics when, at least, one of them is at equilibrium. Let

$$\begin{aligned} \tau_1 &: = \inf\{t \geq 0, \sigma_t^{\wedge} = \sigma_t^{\mu}\}, \\ \tau_2 &: = \inf\{t \geq 0, \sigma_t^{\vee} = \sigma_t^{\mu}\}, \end{aligned} \tag{4.4}$$

where we recall that the dynamics $(\sigma_t^{\mu})_{t\geq 0}$ is constructed by first taking the initial path ξ by sampling μ at $t = 0$ and then using the graphical construction for $t > 0$. By the definition of τ, we know that

$$\tau = \max\big(\tau_1, \tau_2\big).$$

For this goal, it is sufficient to prove the following proposition.

Proposition 4.2. *For $i \in \{1, 2\}$, any given $\lambda \in (0, 1]$ and $\delta > 0$, we have*

$$\lim_{L\to\infty} \mathbb{P}\Big[\tau_i \leq (1+\delta)\frac{1}{\pi^2} L^2 \log L\Big] = 1. \tag{4.5}$$

Theorem 1.1 is proved as a combination of Proposition 3.1 and Proposition 4.2. Therefore, there is a cutoff in the Markov chains for $\lambda \in (0, 1]$. Since the proofs about τ_1 and τ_2 in Proposition 4.2 are similar, we only give the proof of (4.5) for τ_1. For any given $\delta > 0$, set

$$t_\delta := (1+\delta)\frac{1}{\pi^2} L^2 \log L.$$

We outline the idea for the proof. We define a weighted area function A_t in (4.9) below, which is almost the area enclosed by the paths $\sigma_t^{\wedge}$ and σ_t^{μ} at time t. Moreover, $(A_t)_{t\geq 0}$ is a supermartingale when $\lambda \in (0, 1]$. Due to this, we obtain that at time $t_{\delta/2} = (1 + \frac{\delta}{2})\frac{1}{\pi^2} L^2 \log L$, $A_{t_{\delta/2}}$ is close to equilibrium. After time $t_{\delta/2}$, we estimate the fluctuation of $(A_t)_{t\geq t_{\delta/2}}$ by the supermartingale approach applying [LL20, Proposition 29], and then relate the time interval with the fluctuation to obtain (4.5).

4.1. A weighted area function. In this subsection, we define an area function A_t. First, inspired by [Wil04, Equation (1)], we define a function $\overline{\Phi}_\beta \colon \Omega_L \to [0, \infty)$ given by

$$\overline{\Phi}_\beta(\xi) := \sum_{x=1}^{L-1} \xi_x \overline{\cos}_\beta(x),$$

where $\overline{\cos}_\beta(x) := \cos\left(\frac{\beta(x-L/2)}{L}\right)$, and β is a constant in $(2\pi/3, \pi)$. The constant β is only dependent on δ and sufficiently close to π, which will be chosen in the proof of Lemma 4.4 below. We can see that $\overline{\Phi}_\beta(\xi)$ is approximately the area enclosed by the x-axis and the path $\xi \in \Omega_L$.

Throughout this chapter, we omit the index β in $\overline{\Phi}_\beta$ and $\overline{\cos}_\beta$ as much as possible. Observe that if ξ and ξ' are two elements of Ω_L satisfying $\xi \le \xi'$, then

$$\overline{\Phi}(\xi) \le \overline{\Phi}(\xi'). \tag{4.6}$$

The minimal increment of the function $\overline{\Phi}$ is

$$\delta_{\min} := \min_{\xi \le \xi', \xi \ne \xi'} \Big(\overline{\Phi}(\xi') - \overline{\Phi}(\xi)\Big) = 2\cos\Big(\frac{\beta(L/2-1)}{L}\Big), \tag{4.7}$$

and

$$2\cos\Big(\frac{\beta(L/2-1)}{L}\Big) \ge \frac{1}{2}(\pi - \beta) \tag{4.8}$$

for $L \ge 6$ and $\beta \in (2\pi/3, \pi)$, where we use the inequality $\cos(\pi/2 - x) = \sin x \ge x/2$ for $x \in [0, \pi/3]$. Let the weighted area function $A\colon [0,\infty) \mapsto [0,\infty)$ be

$$A_t := \frac{\overline{\Phi}(\sigma_t^\wedge) - \overline{\Phi}(\sigma_t^\mu)}{\delta_{\min}}. \tag{4.9}$$

We observe that τ_1, defined in (4.4), is the first time at which A_t reaches zero. Moreover, A_t equals to zero if and only if $\sigma_t^\wedge$ equals to σ_t^μ. If $\tau_1 \le t_{\delta/2}$, we are done. In the rest of this section, we assume $\tau_1 > t_{\delta/2}$.

Take $\eta > 0$ and sufficiently small, and $K := \lceil 1/(2\eta) \rceil$. We define a sequence of successive stopping times $(\mathcal{T}_i)_{i=2}^K$ by

$$\mathcal{T}_2 := \inf\Big\{t \ge t_{\delta/2} : A_t \le L^{\frac{3}{2}-2\eta}\Big\},$$

and for $3 \le i \le K$,

$$\mathcal{T}_i := \inf\Big\{t \ge \mathcal{T}_{i-1} : A_t \le L^{\frac{3}{2}-i\eta}\Big\}.$$

For consistency of notations, we set $\mathcal{T}_\infty := \max\big(\tau_1, t_{\delta/2}\big)$. The remaining of this section is devoted to proving the following proposition.

Proposition 4.3. *Given $\delta > 0$, if η is chosen to be a sufficiently small positive constant with $K = \lceil 1/(2\eta) \rceil > 1/(2\eta)$, we have*

$$\lim_{L\to\infty} \mathbb{P}\Bigg[\{\mathcal{T}_2 = t_{\delta/2}\} \cap \Bigg(\bigcap_{i=3}^K \{\Delta\mathcal{T}_i \le 2^{-i}L^2\}\Bigg) \cap \{\mathcal{T}_\infty - \mathcal{T}_K \le L^2\}\Bigg] = 1,$$

where $\Delta\mathcal{T}_i := \mathcal{T}_i - \mathcal{T}_{i-1}$ for $3 \le i \le K$.

If Proposition 4.3 holds, for L sufficiently large, we have

$$\tau_1 = \mathcal{T}_\infty \le t_{\delta/2} + \sum_{i=3}^K 2^{-i}L^2 + L^2 \le (1+\delta)\frac{1}{\pi^2}L^2\log L.$$

Then Proposition 4.2 is proved. The idea for Proposition 4.3 is from [LL20, Section 7] as follows:

1. We first show that the decay rate of A_t is at least $1 - \cos(\frac{\pi}{L})$, and then we obtain $\mathcal{T}_2 = t_{\delta/2}$ with high probability.
2. During the time interval $[\mathcal{T}_{i-1}, \mathcal{T}_i]$ for $3 \le i \le K$, we apply the supermartingale approach ([LL20, Proposition 29]) to show that with high probability

$$\langle A.\rangle_{\mathcal{T}_i} - \langle A.\rangle_{\mathcal{T}_{i-1}} \le L^{3-2(i-1)\eta+\frac{1}{2}\eta}.$$

   Similarly for the time interval $[\mathcal{T}_K, \mathcal{T}_\infty]$, we apply [LL20, Proposition 29] to show that with high probability

$$\langle A.\rangle_{\mathcal{T}_\infty} - \langle A.\rangle_{\mathcal{T}_K} \le L^2.$$

3. We compare $\mathcal{T}_\infty - \mathcal{T}_K$ with $\langle A.\rangle_{\mathcal{T}_\infty} - \langle A.\rangle_{\mathcal{T}_K}$. As $\partial_t\langle A.\rangle \geq 1$ (justified in Subsection 4.3) for all $t < \mathcal{T}_\infty$, we have
$$\mathcal{T}_\infty - \mathcal{T}_K \leq \int_{\mathcal{T}_K}^{\mathcal{T}_\infty} \partial_t\langle A.\rangle \mathrm{d}t = \langle A.\rangle_{\mathcal{T}_\infty} - \langle A.\rangle_{\mathcal{T}_K}.$$
For $3 \leq i \leq K$, to compare $\langle A.\rangle_{\mathcal{T}_i} - \langle A.\rangle_{\mathcal{T}_{i-1}}$ with $\mathcal{T}_i - \mathcal{T}_{i-1}$, we provide a better lower bound on $\partial_t\langle A.\rangle$ in terms of the highest point of $\sigma_t^\wedge$ and the maximal length of the monotone segments of σ_t^μ in Lemma 4.7.
4. We use induction method to show that $\mathcal{T}_i - \mathcal{T}_{i-1} \leq 2^{-i}L^2$ for all $i \in [\![3, K]\!]$, arguing by contradiction.

4.2. The proof of $\mathcal{T}_2 = t_{\delta/2}$. The main task of this subsection is to prove that the function A_t has a contraction property, due to which we obtain $\mathcal{T}_2 = t_{\delta/2}$ with high probability. Above all, we want to understand how the generator $\mathcal{L}$ acts on the function $\overline{\Phi}$. We have
$$(\mathcal{L}\overline{\Phi})(\xi) = \sum_{x=1}^{L-1} \overline{\cos}(x)\mathcal{L}\xi_x.$$
We recall Lemma 2.3: for any $\xi \in \Omega_L$,
$$\mathcal{L}\xi_x = (\Delta\xi)_x + \mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=0\}} + \left(\frac{1-\lambda}{1+\lambda}\right)\mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=1\}}.$$
For $\xi, \xi' \in \Omega_L$, we have
$$\sum_{x=1}^{L-1} \overline{\cos}(x)\Big((\Delta\xi')_x - (\Delta\xi)_x\Big) = -\Big(1 - \cos\Big(\frac{\beta}{L}\Big)\Big) \sum_{x=1}^{L-1} \overline{\cos}(x)(\xi'_x - \xi_x). \tag{4.10}$$
Considering
$$\mathcal{L}\xi_x - (\Delta\xi)_x = \mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=0\}} + \left(\frac{1-\lambda}{1+\lambda}\right)\mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=1\}},$$
we see that both terms in the right-hand side are nonnegative and monotonically decreasing in ξ for $\lambda \in (0, 1]$. Hence, if $\xi \leq \xi'$, we know that
$$\mathcal{L}\xi_x - (\Delta\xi)_x \geq \mathcal{L}\xi'_x - (\Delta\xi')_x. \tag{4.11}$$
For simplicity of notation, we set
$$\gamma = \gamma_{L,\beta} := 1 - \cos(\beta/L).$$
On the grounds of Lemma 2.3, (4.10) and (4.11), if $\xi \leq \xi'$, we obtain
$$\begin{aligned}(\mathcal{L}\overline{\Phi})(\xi') - (\mathcal{L}\overline{\Phi})(\xi) &= \sum_{x=1}^{L-1} \overline{\cos}(x)\Big((\Delta\xi')_x - (\Delta\xi)_x + (\mathcal{L}\xi'_x - (\Delta\xi')_x) - (\mathcal{L}\xi_x - (\Delta\xi)_x)\Big)\\ &\leq \sum_{x=1}^{L-1} \overline{\cos}(x)\Big((\Delta\xi)'_x - (\Delta\xi)_x\Big)\\ &\leq -\gamma\Big(\overline{\Phi}(\xi') - \overline{\Phi}(\xi)\Big).\end{aligned} \tag{4.12}$$
Now we are ready to prove that $\mathcal{T}_2 = t_{\delta/2}$ with high probability.

Lemma 4.4. *For all $\epsilon > 0$, all sufficiently small $\delta > 0$ and $0 < \eta < \delta/10$, if L is sufficiently large, we have*
$$\mathbb{P}\big[\mathcal{T}_2 > t_{\delta/2}\big] \leq \epsilon.$$

PROOF. By $\sigma_t^{\wedge} \geq \sigma_t^{\mu}$ and (4.12), we obtain

$$\begin{aligned} \frac{d}{dt}\mathbb{E}\Big[\overline{\Phi}(\sigma_t^{\wedge}) - \overline{\Phi}(\sigma_t^{\mu})\Big] &= \mathbb{E}\Big[(\mathcal{L}\overline{\Phi})(\sigma_t^{\wedge}) - (\mathcal{L}\overline{\Phi})(\sigma_t^{\mu})\Big] \\ &\leq -\gamma\mathbb{E}\Big[\overline{\Phi}(\sigma_t^{\wedge}) - \overline{\Phi}(\sigma_t^{\mu})\Big]. \end{aligned} \tag{4.13}$$

Using (4.13), $\overline{\Phi}(\sigma_0^{\wedge}) \leq \frac{1}{2}L^2$ and $\overline{\Phi}(\xi) \geq 0$ for all $\xi \in \Omega_L$, we obtain

$$\mathbb{E}\Big[\overline{\Phi}(\sigma_t^{\wedge}) - \overline{\Phi}(\sigma_t^{\mu})\Big] \leq e^{-\gamma t}\Big(\overline{\Phi}(\sigma_0^{\wedge}) - \overline{\Phi}(\sigma_0^{\mu})\Big) \leq \frac{1}{2}L^2 e^{-\gamma t}. \tag{4.14}$$

Thus, applying Markov's inequality, we achieve

$$\begin{aligned} \mathbb{P}[\mathcal{T}_2 > t_{\delta/2}] &= \mathbb{P}[A_{t_{\delta/2}} > L^{\frac{3}{2}-2\eta}] \\ &\leq \frac{1}{2\delta_{\min}} L^{2\eta+\frac{1}{2}} e^{-\gamma t_{\delta/2}}, \end{aligned} \tag{4.15}$$

where the last inequality uses (4.14) and the definition of A_t in (4.9). Recalling $\gamma = 1-\cos(\beta/L)$ and using the inequality $1-\cos x \geq \frac{1}{2}x^2 - \frac{1}{24}x^4$ for all $x \geq 0$, we have

$$\gamma t_{\delta/2} \geq \frac{\beta^2}{2\pi^2}(1+\frac{\delta}{2})\log L - \frac{\beta^4}{24L^2}(1+\frac{\delta}{2})\log L.$$

For $\delta > 0$ sufficiently small and $0 < \eta < \delta/10$, we choose

$$\beta = \pi\sqrt{\frac{1+\frac{9}{20}\delta}{1+\frac{\delta}{2}}} \in (2\pi/3, \pi)$$

which satisfies

$$\frac{1}{2}(1+\frac{\delta}{2})\frac{\beta^2}{\pi^2} = \frac{1}{2} + \frac{9}{40}\delta > \frac{1}{2} + 2\eta.$$

With this choice of β, the rightmost term of (4.15) vanishes as L tends to infinity. □

4.3. The estimation of $\langle A.\rangle_{\mathcal{T}_i} - \langle A.\rangle_{\mathcal{T}_{i-1}}$. Due to Dynkin's martingale formula, we know that

$$A_t - A_0 - \int_0^t \mathcal{L}A_s \mathrm{d}s$$

is a martingale. Moreover, we let $\langle A.\rangle_t$ represent the predictable bracket associated with this martingale. The objective of this subsection is to show that $\langle A.\rangle_{\mathcal{T}_i} - \langle A.\rangle_{\mathcal{T}_{i-1}}$ is small for all $i \in [\![3, K]\!]$. For any $i \in [\![3, K]\!]$, let

$$\Delta_i\langle A\rangle := \langle A.\rangle_{\mathcal{T}_i} - \langle A.\rangle_{\mathcal{T}_{i-1}}, \tag{4.16}$$

and let

$$\Delta_\infty\langle A\rangle := \langle A.\rangle_{\mathcal{T}_\infty} - \langle A.\rangle_{\mathcal{T}_K}. \tag{4.17}$$

We have $\mathcal{L}A_s \leq 0$, according to (4.12), $\sigma_t^{\wedge} \geq \sigma_t^{\mu}$, and the monotonicity of the function $\overline{\Phi}$ stated in (4.6). Then, A_t is a supermartingale for $\lambda \in (0,1]$. Its jump amplitudes in absolute value are bounded below by 1 for $t < \tau_1$ where the absorption time τ_1 is defined in (4.4). Moreover, for $t < \tau_1$ we can always find one flippable corner in $\sigma_t^{\wedge}$ and one in σ_t^{μ} which can change the value of A_t, and the total rates of these two corners are at least 1. Therefore, the jump rates of A_t are least 1 for $t < \tau_1$. We refer to Figure 2 for illustration: those flippable corners in σ_t^{μ} and $\sigma_t^{\wedge}$ which are not totally colored black can change the value of A_t, and the total rates of these corners are at least 1. Now, we are in the setting to apply [LL20, Proposition 29] which, under some condition, allows to control hitting times of supermartingales in terms of the martingale bracket.

PROPOSITION 4.5 (Proposition 29 in [LL20]). *Let $(\mathbf{M}_t)_{t\geq 0}$ be a pure-jump supermartingale with bounded jump rates and jump amplitudes, and $\mathbf{M}_0 \leq a$ almost surely. Let $\langle \mathbf{M}.\rangle$, with an abuse of notation, denote the predictable bracket associated with the martingale $\overline{\mathbf{M}}_t = \mathbf{M}_t - I_t$ where I is the compensator of $\mathbf{M}$. Given $b \in \mathbb{R}$ and $b \leq a$, we set*

$$\tau_b := \inf\{t \geq 0 : \mathbf{M}_t \leq b\}.$$

If the amplitudes of the jumps of $(\mathbf{M}_t)_{t\geq 0}$ are bounded above by $a - b$, for any $u \geq 0$, we have

$$\mathbb{P}\big[\langle \mathbf{M}.\rangle_{\tau_b} \geq (a-b)^2 u\big] \leq 8u^{-1/2}. \tag{4.18}$$

Now we apply Proposition 4.5 to prove that the event

$$\mathcal{A}_L := \Big\{\forall i \in [\![3, K]\!], \Delta_i\langle A\rangle \leq L^{3-2(i-1)\eta+\frac{1}{2}\eta}\Big\} \bigcap \Big\{\Delta_\infty\langle A\rangle \leq L^2\Big\}$$

has almost the full mass, which is the following lemma.

LEMMA 4.6. *We have*

$$\lim_{L\to\infty} \mathbb{P}[\mathcal{A}_L] = 1. \tag{4.19}$$

PROOF. We just need to show that the probability of its complement $\mathcal{A}_L^{\complement}$ is almost zero. We apply Proposition 4.5 to $(A_{t+\mathcal{T}_{i-1}})_{t\geq 0}$ with $a_i = L^{\frac{3}{2}-(i-1)\eta}$ and $b_i = L^{\frac{3}{2}-i\eta}$. For every $i \in [\![3, K]\!]$, we obtain

$$\mathbb{P}\Big[\Delta_i\langle A\rangle \geq (L^{\frac{3}{2}-(i-1)\eta} - L^{\frac{3}{2}-i\eta})^2 u_i\Big] \leq 8u_i^{-\frac{1}{2}}, \tag{4.20}$$

where we choose $u_i = L^{\frac{1}{2}\eta}(1 - L^{-\eta})^{-2}$, satisfying

$$\Big(L^{\frac{3}{2}-(i-1)\eta} - L^{\frac{3}{2}-i\eta}\Big)^2 u_i = L^{3-2(i-1)\eta+\frac{1}{2}\eta}.$$

We see that u_i tends to infinity as L tends to infinity. Accordingly, the rightmost term in (4.20) vanishes as L tends to infinity.

We apply Proposition 4.5 to $(A_{t+\mathcal{T}_K})_{t\geq 0}$ with $a_\infty = L^{\frac{3}{2}-K\eta}$ and $b_\infty = 0$. We choose u_∞ such that $(a_\infty - b_\infty)^2 u_\infty = L^2$, *i.e.*

$$u_\infty = L^{-1+2K\eta},$$

which tends to infinity due to $K = \lceil 1/(2\eta)\rceil > 1/(2\eta)$. Thus $\mathbb{P}[\Delta_\infty\langle A\rangle \geq L^2]$ tends to zero as L tends to infinity. Since K is a constant, we have

$$\lim_{L\to\infty} \mathbb{P}\Big[\mathcal{A}_L^{\complement}\Big] = 0.$$

□

4.4. The comparison of $\mathcal{T}_i - \mathcal{T}_{i-1}$ to $\Delta_i\langle A\rangle$. As explained in Subsection 4.3, we have $\partial_t\langle A.\rangle \geq 1$ for all $t < \mathcal{T}_\infty$. Therefore, we obtain

$$\Delta_\infty\langle A\rangle = \int_{\mathcal{T}_K}^{\mathcal{T}_\infty} \partial_t\langle A.\rangle dt \geq \int_{\mathcal{T}_K}^{\mathcal{T}_\infty} 1 dt = \mathcal{T}_\infty - \mathcal{T}_K.$$

Hence, when the event $\mathcal{A}_L$ holds, we obtain

$$\mathcal{T}_\infty - \mathcal{T}_K \leq \Delta_\infty\langle A\rangle \leq L^2.$$

Now we control the intermediate increment $\mathcal{T}_i - \mathcal{T}_{i-1}$ for $3 \leq i \leq K$. To do that, we compare $\mathcal{T}_i - \mathcal{T}_{i-1}$ with $\langle A.\rangle_{\mathcal{T}_i} - \langle A.\rangle_{\mathcal{T}_{i-1}} = \Delta_i\langle A.\rangle$. First, we give a lower bound on $\partial_t\langle A.\rangle$, which is related with: (a) the maximal contribution among all the coordinates $x \in [\![0, L]\!]$ in the definition of A_t;

and (b) the amount of flippable corners in σ_t^μ or $\sigma_t^\wedge$ that can change the value of A_t. Considering the definition of A_t in (4.9), set

$$H(t) := \max_{x \in [\![0,L]\!]} \sigma_t^\wedge(x). \tag{4.21}$$

For a lower bound on the quantity mentioned in (b), we need the maximal length of the monotone segment of σ_t^μ. For $\xi \in \Omega_L$, we define

$$Q_1(\xi) := \max\{n \geq 1, \exists i \in [\![0, L-n]\!], \forall x \in [\![i+1, i+n]\!], \xi_x - \xi_{x-1} = 1\},$$
$$Q_2(\xi) := \max\{n \geq 1, \exists i \in [\![0, L-n]\!], \forall x \in [\![i+1, i+n]\!], \xi_x - \xi_{x-1} = -1\},$$

and

$$Q(\xi) := \max\big(Q_1(\xi), Q_2(\xi)\big). \tag{4.22}$$

Using these two quantities $H(t)$ and $Q(\sigma_t^\mu)$, we obtain a lower bound for $\partial_t\langle A.\rangle$, which is the following lemma.

Lemma 4.7. *We have*

$$\partial_t\langle A.\rangle \geq \max\left(1, \frac{\lambda \delta_{\min} A_t}{3(1+\lambda)H(t)Q(\sigma_t^\mu)}\right). \tag{4.23}$$

Proof. We observe that A_t displays a jump whenever either σ_t^μ or $\sigma_t^\wedge$ flips a corner. Note that by (4.9) and (4.7), any jump amplitude in absolute value of A is at least 1. Since any flippable corner is flipped with rate at least

$$\min\left\{\frac{1}{2}, \frac{1}{1+\lambda}, \frac{\lambda}{1+\lambda}\right\} = \frac{\lambda}{1+\lambda},$$

we obtain

$$\partial_t\langle A.\rangle_t \geq \frac{\lambda}{1+\lambda} \#\big\{x \in \mathcal{B}_t : \Delta\sigma_t^\mu(x) \neq 0\big\}$$

where $\mathcal{B}_t := \big\{x \in [\![1, L-1]\!] : \exists y \in [\![x-1, x+1]\!], \sigma_t^\wedge(y) \neq \sigma_t^\mu(y)\big\}$. For simplicity of notation, set $\mathcal{D}_t := \big\{x \in \mathcal{B}_t : \Delta\sigma_t^\mu(x) \neq 0\big\}$. Let $[\![a,b]\!]$ denote the horizontal coordinates of a maximal connected component of $\mathcal{B}_t$, for which we refer to Figure 2 for illustration. Since σ_t^μ can not be monotone in the entire domain $[\![a,b]\!]$, we know that

$$\#(\mathcal{D}_t \cap [\![a,b]\!]) \geq 1.$$

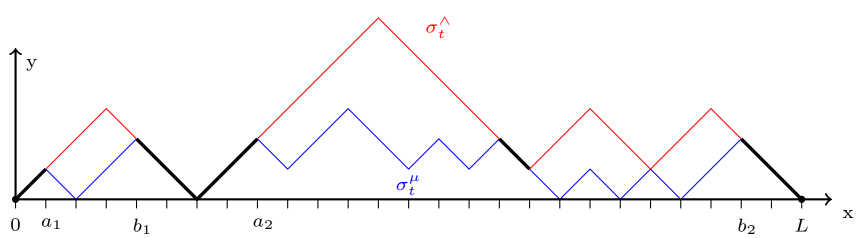


Figure 2. In this figure, $\sigma_t^\wedge$ consists of the red line segments and black thick line segments, while σ_t^μ consists of the blue line segments and black thick line segments. Moreover, $\mathcal{B}_t = [\![a_1, b_1]\!] \cup [\![a_2, b_2]\!]$, $\#(\mathcal{D}_t \cap [\![a_1, b_1]\!]) = 3$, and $\#(\mathcal{D}_t \cap [\![a_2, b_2]\!]) = 13$. In $[\![a_2, b_2]\!]$, the monotone segments of σ_t^μ are $[\![a_2, a_2+1]\!]$, $[\![a_2+1, a_2+3]\!]$, $[\![a_2+3, a_2+5]\!]$, and so on as shown in the figure.

In $\mathcal{B}_t$, we decompose the path associated with σ_t^μ into consecutive maximal monotone segments. Then we know that in $\mathcal{B}_t$ every two consecutive components correspond to one flippable corner, which is a point in $\mathcal{D}_t$. As any maximal monotone component is at most of length $Q(\sigma_t^\mu)$ defined in (4.22), we obtain

$$\#(\mathcal{D}_t \cap [\![a,b]\!]) \geq \frac{1}{2}\left\lfloor \frac{b-a}{Q(\sigma_t^\mu)} \right\rfloor \geq \frac{1}{3}\frac{b-a}{Q(\sigma_t^\mu)}. \tag{4.24}$$

In addition, we observe that

$$\sum_{x=a}^{b} \frac{(\sigma_t^\wedge(x) - \sigma_t^\mu(x))\overline{\cos}(x)}{\delta_{\min}} \leq (b-a)\frac{H(t)}{\delta_{\min}}, \tag{4.25}$$

where $H(t)$ is defined in (4.21). Summing up all such intervals $[\![a,b]\!]$ and using (4.24) and (4.25), we obtain

$$A_t \leq \frac{3}{\delta_{\min}} H(t) Q(\sigma_t^\mu) \# \mathcal{D}_t.$$

Therefore, we have

$$\partial_t \langle A.\rangle \geq \frac{\lambda}{1+\lambda}\#\mathcal{D}_t \geq \frac{\lambda\delta_{\min}}{3(1+\lambda)}\frac{A_t}{H(t)Q(\sigma_t^\mu)}.$$

This yields the desired result. □

To give a good lower bound for $\partial_t\langle A.\rangle$, we need to control $Q(\sigma_t^\mu)$ and $H(t)$. Our next step is to give an upper bound on $Q(\sigma_t^\mu)$, which is the following lemma. We recall the notation

$$t_\delta = (1+\delta)\frac{1}{\pi^2}L^2 \log L.$$

Lemma 4.8. *We have*

$$\lim_{L\to\infty} \mathbb{P}\Big[\exists t \in [0, t_\delta] : Q(\sigma_t^\mu) > (\log L)^2\Big] = 0. \tag{4.26}$$

Proof. Firstly, we prove that there exists a constant $C(\lambda) > 0$ such that for all $L \geq 2$

$$\mu\big(Q(\xi) > (\log L)^2\big) \leq 2C(\lambda)L^{5/2}2^{-(\log L)^2}. \tag{4.27}$$

Since there are at most L starting positions for a monotone segments either monotonically increasing or decreasing, we have

$$\#\{\xi \in \Omega_L : Q(\xi) > (\log L)^2\} \leq L2^{1+L-(\log L)^2}.$$

Moreover, as $\lambda^{\mathcal{N}(\xi)} \leq 1$ for $\lambda \in (0,1]$ and any $\xi \in \Omega_L$, we obtain

$$\mu\big(Q(\xi) > (\log L)^2\big) \leq C_5(\lambda)\frac{L2^{1+L-(\log L)^2}}{2^L L^{-3/2}} = 2C_5(\lambda)L^{5/2}2^{-(\log L)^2}, \tag{4.28}$$

where we use the inequality $Z_L(\lambda) \geq C_5(\lambda)^{-1}2^L L^{-3/2}$ for all $L \geq 2$ and some $C_5(\lambda) > 0$ by Theorem 2.2. Secondly, since there are at most L corners in any path $\xi \in \Omega_L$, we have

$$\sum_{x=1}^{L-1} R_x(\xi) \leq L,$$

where $R_x(\xi)$ is defined in (1.5). Therefore, for any subset $\mathcal{A} \subset \Omega_L$ and $s \geq 0$,

$$\mathbb{P}\big[\forall t \in [s, s+L^{-1}] : \sigma_t^\mu \in \mathcal{A} \mid \sigma_s^\mu \in \mathcal{A}\big] \geq e^{-1}. \tag{4.29}$$

Taking $\mathcal{A} := \{\xi \in \Omega_L : Q(\xi) > (\log L)^2\}$, we define the occupation time to be

$$u(t) := \int_0^t \mathbf{1}_{\mathcal{A}}(\sigma_s^\mu)\mathrm{d}s. \tag{4.30}$$

By Fubini's Theorem, we obtain

$$\mathbb{E}[u(2t_\delta)] = 2t_\delta\mu(\mathcal{A}). \tag{4.31}$$

Using (4.29) and strong Markov property, we give a lower bound for $\mathbb{E}[u(2t_\delta)]$:

$$\mathbb{E}[u(2t_\delta)] \geq e^{-1}L^{-1}\mathbb{P}[\exists t \in [0,t_\delta] : \sigma_t^\mu \in \mathcal{A}]. \tag{4.32}$$

By (4.31), (4.32) and (4.27), we have

$$\mathbb{P}\big[\exists t \in [0,t_\delta] : \sigma_t^\mu \in \mathcal{A}\big] \leq 2eLt_\delta\mu(\mathcal{A}) \leq 4eC_5(\lambda)L^{7/2}t_\delta 2^{-(\log L)^2}, \tag{4.33}$$

which vanishes as L tends to infinity. Therefore, we conclude the proof. □

The last ingredient for the proof of Proposition 4.3 is to control $H(t)$, defined in (4.21). Recall that $t_\delta = (1+\delta)\frac{1}{\pi^2}L^2\log L$.

LEMMA 4.9. *We have*

$$\lim_{L\to\infty} \sup_{t\in[t_{\delta/2},t_\delta]} \mathbb{P}\Big[H(t) \geq 2L^{\frac{1}{2}}(\log L)^2\Big] = 0. \tag{4.34}$$

Intuitively, for $\lambda \in (0,2)$, $\big(\frac{\xi_{[xL]}}{\sqrt{L}}\big)_{x\in[0,1]}$ under μ_L^λ converges to Brownian excursion. (A rough argument for the intuition goes as follows. By Equation (3.8), we have

$$\mu_L^\lambda\Big(\exists x \in [\![L^{1/3}, L-L^{1/3}]\!] : \xi_x = 0\Big) \leq 2c(\lambda)\sum_{x=L^{1/3}}^{L/2} \frac{L^{3/2}}{x^{3/2}(L-x)^{3/2}} \leq c'(\lambda)L^{-1/6}.$$

For $\xi \in \Omega_L$, define

$$\mathbf{L}(\xi) := \sup\{x \leq L/2 : \xi_x = 0\},$$
$$\mathbf{R}(\xi) := \inf\{x \geq L/2 : \xi_x = 0\},$$

and we observe that

$$\mu_L^\lambda(\cdot|\mathbf{L}=\ell,\mathbf{R}=r) = \mu_\ell^\lambda \otimes \mathbf{P}(\cdot|\min_{1\leq i<r-\ell} S_i > 0; S_{r-\ell}=0) \otimes \mu_{L-r}^\lambda,$$

where $\mathbf{P}$ denotes the law of the symmetric nearest-neighbor simple random walk on $\mathbb{Z}$. As $\big(\frac{S_{[x(r-\ell)]}}{\sqrt{r-\ell}}\big)_{x\in[0,1]}$ under the law $\mathbf{P}(\cdot|\min_{1\leq i<r-\ell} S_i > 0; S_{r-\ell}=0)$ converges to the Brownian excursion, we conclude the proof.) Therefore, the dynamics $(\sigma_t^\wedge)_{t\geq 0}$ is like the simple exclusion process, and we can apply [Lac16b, Theorem 2.4] to obtain Lemma 4.9. We postpone the proof in Appendix 2.A. Now, we are ready to prove Proposition 4.3.

PROOF OF PROPOSITION 4.3. We define the event $\mathcal{H}_L$ where the highest point of $\sigma_t^\wedge$ is not too high and there are a lot of flippable corners in σ_t^μ during the time interval $[t_{\delta/2}, t_{\delta/2}+L^2]$,

$$\mathcal{H}_L = \left\{\int_{t_{\delta/2}}^{t_{\delta/2}+L^2} \mathbf{1}_{\{H(t)\leq 2L^{\frac{1}{2}}(\log L)^2\}\bigcap\{Q(\sigma_t^\mu)\leq(\log L)^2\}}dt \geq L^2\Big(1-2^{-(K+1)}\Big)\right\}.$$

First, we show that $\mathcal{H}_L$ holds with high probability. We have

$$\begin{aligned}\mathbb{P}\Big[\mathcal{H}_L^\complement\Big] &= \mathbb{P}\Bigg[\int_{t_{\delta/2}}^{t_{\delta/2}+L^2} \mathbf{1}_{\{H(t)>2L^{\frac{1}{2}}(\log L)^2\}\bigcup\{Q(\sigma_t^\mu)>(\log L)^2\}}dt \geq L^2 2^{-(K+1)}\Bigg]\\ &\leq \mathbb{P}\Bigg[\int_{t_{\delta/2}}^{t_{\delta/2}+L^2} \mathbf{1}_{\{H(t)>2L^{\frac{1}{2}}(\log L)^2\}}dt \geq L^2 2^{-(K+2)}\Bigg]\\ &+ \mathbb{P}\Bigg[\int_{t_{\delta/2}}^{t_{\delta/2}+L^2} \mathbf{1}_{\{Q(\sigma_t^\mu)>(\log L)^2\}}dt \geq L^2 2^{-(K+2)}\Bigg],\end{aligned} \tag{4.35}$$

which vanishes as L tends to infinity, grounded on Markov's inequality, Lemma 4.8, Lemma 4.9 and the fact that K is a constant.

From now on, we assume the event $\mathcal{A}_L \cap \mathcal{H}_L \cap \{\mathcal{T}_2 = t_{\delta/2}\}$. Based on (4.35), Lemma 4.6 and Lemma 4.8, we have

$$\lim_{L\to\infty} \mathbb{P}\Big[\mathcal{A}_L \cap \mathcal{H}_L \cap \{\mathcal{T}_2 = t_{\delta/2}\}\Big] = 1.$$

By induction, we show that $\Delta\mathcal{T}_j = \mathcal{T}_j - \mathcal{T}_{j-1} \le 2^{-j}L^2$ for all $j \in [\![3, K]\!]$. We argue by contradiction: let i_0 be the smallest integer satisfying

$$\Delta\mathcal{T}_{i_0} > 2^{-i_0}L^2.$$

We know that

$$\Delta_{i_0}\langle A\rangle \ge \int_{\mathcal{T}_{i_0-1}}^{\mathcal{T}_{i_0-1}+2^{-i_0}L^2} \partial_t\langle A.\rangle \mathbf{1}_{\{H(t)\le 2L^{\frac12}(\log L)^2\}\cap\{Q(\sigma_t^\mu)\le(\log L)^2\}} dt. \tag{4.36}$$

According to Lemmas 4.7, 4.8 and 4.9, we have a lower bound for $\partial_t\langle A.\rangle$ when the indicator function equals to 1. That bound is

$$\partial_t\langle A.\rangle \ge \frac{\lambda\delta_{\min}}{3(1+\lambda)}\frac{A_t}{H(t)Q(\sigma_t^\mu)} \ge \frac{\lambda\delta_{\min}}{6(1+\lambda)}\frac{A_t}{L^{\frac12}(\log L)^4}. \tag{4.37}$$

Since $\mathcal{T}_2 = t_{\delta/2}$ and $\Delta\mathcal{T}_j = \mathcal{T}_j - \mathcal{T}_{j-1} \le 2^{-j}L^2$ for $j < i_0$, we know that

$$\mathcal{T}_{i_0-1} \le t_{\delta/2} + L^2\sum_{j=3}^{i_0-1} 2^{-j} \le t_{\delta/2} + (1-2^{-(i_0-1)})L^2,$$

and then $\mathcal{T}_{i_0-1} + 2^{-i_0}L^2 \le t_{\delta/2} + L^2$. Moreover, when the assumption $\mathcal{H}_L$ holds, the indicator function

$$\mathbf{1}_{\{H(t)\le 2L^{\frac12}(\log L)^2\}\cap\{Q(\sigma_t^\mu)\le(\log L)^2\}}$$

is equal to 1 on a set, which is of Lebesgue measure at least

$$(2^{-i_0} - 2^{-(K+1)})L^2 \ge 2^{-(K+1)}L^2. \tag{4.38}$$

Combining (4.36), (4.37) and (4.38), we obtain

$$\Delta_{i_0}\langle A\rangle \ge 2^{-(K+1)}L^2\frac{\lambda\delta_{\min}}{6(1+\lambda)}\frac{A_t}{L^{\frac12}(\log L)^4} \ge 2^{-(K+1)}\frac{\lambda\delta_{\min}}{6(1+\lambda)}L^{3-i_0\eta}(\log L)^{-4}, \tag{4.39}$$

where the last inequality uses the fact that $A_t > L^{\frac32 - i_0\eta}$, for $t < \mathcal{T}_{i_0}$. In addition, since we are in $\mathcal{A}_L$, we know that

$$\Delta_{i_0}\langle A\rangle \le L^{3-2(i_0-1)\eta+\frac12\eta}. \tag{4.40}$$

However, as $i_0 \ge 3$, we have

$$3 - 2(i_0-1)\eta + \frac12\eta < 3 - i_0\eta.$$

Therefore, there is a contradiction between (4.39) and (4.40), as long as L is large enough. □

5. Upper bound on the dynamics starting from the extremal paths for $\lambda \in (1, 2)$

For the pinning model without positive constraint (see [CMT08, Section 1]), the critical value is $\lambda_c = 1$, while the critical value is $\lambda_c = 2$ for the pinning model with positive constraint. Due to the repulsion effect of the impenetrable wall, the process $(A_t)_{t\ge0}$ defined in Subsection 4.3 is not a supermartingale for $\lambda \in (1, 2)$. But there is still monotonicity in the dynamics starting with the maximal (or minimal) path for $\lambda \in (1, 2)$, which can be exploited to provide an upper bound on the mixing time by applying the censoring inequality in [PW13, Theorem 1.1]. This inequality says that canceling some prescribed updates slows down the mixing of the Glauber dynamics starting from the maximal (or minimal) configuration of a monotone spin system.

Let us state the setting for applying the censoring inequality. A censoring scheme is a càdlàg function defined by

$$\mathcal{C}\colon \mathbb{R}^+ \to \mathcal{P}(\Theta),$$

where Θ is defined in (2.1) and $\mathcal{P}(\Theta)$ is the set of all subsets of Θ. The censored dynamics with a censoring scheme $\mathcal{C}$ is the dynamics obtained from the graphical construction in Subsection 2.1, except that the update at time t is canceled if and only if it is an element of $\mathcal{C}(t)$. In other words, we construct the dynamics by using the graphical construction in Subsection 2.1 with one extra rule: if $\mathcal{T}^{\uparrow}_{(x,z)}$ or $\mathcal{T}^{\downarrow}_{(x,z)}$ rings at time t, the update is performed if and only if $(x,z) \notin \mathcal{C}(t)$. Let $(\sigma_t^{\xi,\mathcal{C}})_{t\geq 0}$ denote the trajectory of the censored dynamics with a censoring scheme $\mathcal{C}$ and starting from the path $\xi \in \Omega_L$, and let $P_t^{\xi,\mathcal{C}}$ denote the law of distribution of the time marginal $\sigma_t^{\xi,\mathcal{C}}$.

The Glauber dynamics of this polymer pinning model is a monotone spin system in the sense of [PW13, Subsection 1.1] (detailed in Appendix 2.B), and we refer to Figure 5 in Appendix 2.B for a quick look. The following proposition follows directly from [PW13, Theorem 1.1].

Proposition 5.1. *For any prescribed censoring scheme $\mathcal{C}$, for all $\lambda \in [0,\infty)$, all $t \geq 0$ and $\xi \in \{\wedge, \vee\}$, we have*

$$\|P_t^{\xi} - \mu\|_{\mathrm{TV}} \leq \|P_t^{\xi,\mathcal{C}} - \mu\|_{\mathrm{TV}}. \tag{5.1}$$

Besides Proposition 5.1, we need the two following results in the proof of the upper bound on the mixing time. Firstly, by [LP17, Lemmas 20.5 and 20.11], we know that the asymptotic rate of convergence to equilibrium of this reversible Markov chain is

$$\lim_{t\to\infty} t^{-1} \log d^{L,\lambda}(t) = -\mathrm{gap}_{L,\lambda}, \tag{5.2}$$

where $\mathrm{gap}_{L,\lambda} > 0$ is the spectral gap defined in (1.12). By monotonicity of the Glauber dynamics and (4.3), for all $\lambda > 0$ we have

$$d^{L,\lambda}(t) \leq \mathbb{P}\Big(\sigma_t^{\wedge} \neq \sigma_t^{\vee}\Big) = \mathbb{P}\Big(\Phi(\sigma_t^{\wedge}) - \Phi(\sigma_t^{\vee}) \geq 2\sin(\tfrac{\pi}{L})\Big), \tag{5.3}$$

where $\Phi(\xi)$ is defined in (3.2). Moreover, for all $\lambda > 0$, by [CMT08, Equation (4.1)] we have

$$\mathbb{E}\big[\Phi(\sigma_t^{\wedge})\big] - \mathbb{E}\big[\Phi(\sigma_t^{\vee})\big] \leq \tfrac{L^2}{2} e^{-t\kappa_L}.$$

Applying Markov's inequality, we reclaim the useful result in [CMT08].

Lemma 5.2. *For all $\lambda > 0$, we have*

$$d^{L,\lambda}(t) \leq \frac{L^2 e^{-\kappa_L t}}{4\sin(\frac{\pi}{L})}. \tag{5.4}$$

Plugging this into (5.2)*, we obtain*

$$\mathrm{gap}_{L,\lambda} \geq \kappa_L = 1 - \cos\Big(\frac{\pi}{L}\Big). \tag{5.5}$$

Secondly, the following lemma is an application of the Cauchy-Schwarz inequality and the reversibility of the Markov chain. For reference, we mention [CLM$^+$12, Equation (2.6)].

Lemma 5.3. *For any probability distribution ν on Ω_L, we have*

$$\|\nu P_t - \mu\|_{\mathrm{TV}} \leq \frac{1}{2} e^{-t\cdot \mathrm{gap}_{L,\lambda}} \sqrt{\mathrm{Var}_{\mu}(\rho)}, \tag{5.6}$$

where $\rho := \frac{d\nu}{d\mu}$ and $\mathrm{Var}_{\mu}(\rho) := \mu(\rho^2) - \mu(\rho)^2$.

We define

$$G_L := \{(x,1) : x \in [\![2, L-2]\!] \cap 2\mathbb{N}\} \tag{5.7}$$

where $\mathcal{T}^{\uparrow}_{(x,1)}$ or $\mathcal{T}^{\downarrow}_{(x,1)}$ rings, the update—in the graphical construction of Subsection 2.1—changes the number of contact points $\mathcal{N}$, defined in (1.1). Moreover, G_L corresponds to the centers of the green squares shown in Figure 5. Before we start the proof of the upper bound on the mixing time for the dynamics starting with the maximal path $\wedge$, we outline the idea for Proposition 5.4 with $\lambda \in (1,2)$.

- (i) We elaborate a censoring scheme $\mathcal{C}$, where $\mathcal{C}(t) = G_L$ for $t < t_{\delta/2}$ and $\mathcal{C}(t) = \varnothing$ for $t \geq t_{\delta/2}$. Therefore the dynamics $(\sigma_t^{\wedge,\mathcal{C}})_{0 \leq t < t_{\delta/2}}$ does not touch the $x-$axis except at the two coordinates $x = 0, L$.
- (ii) By Remark 1.2 and Theorem 1.1, the distribution of $\sigma^{\wedge,\mathcal{C}}_{t_{\delta/2}}$ is close to μ_L^0 in total variation distance.
- (iii) As the Radon-Nikodym derivative of μ_L^0 with respect to μ_L^λ is bounded by a constant, we apply Lemma 5.3 and use (5.5) to conclude the proof.

PROPOSITION 5.4. *For any $\lambda \in (1,2)$, any $\epsilon > 0$ and any $\delta > 0$, if L is sufficiently large, we have*

$$T^{L,\wedge}_{\text{mix}}(\epsilon) \leq \frac{1+\delta}{\pi^2} L^2 \log L. \tag{5.8}$$

PROOF. Recall that $\mathcal{N}$ is the number of contact points, defined in (1.1). We run the dynamics starting from the maximal path $\wedge$, censoring those updates which change the value of contact points $\mathcal{N}$ for $t < t_{\delta/2}$. More precisely, recalling $t_\delta = (1+\delta)\frac{1}{\pi^2} L^2 \log L$, we present a censoring scheme $\mathcal{C} \colon \mathbb{R}^+ \to \mathcal{P}(\Theta)$, defined by

$$\mathcal{C}(t) := \begin{cases} G_L & \text{if } t \in [0, t_{\delta/2}), \\ \varnothing & \text{if } t \in [t_{\delta/2}, \infty). \end{cases}$$

We recall that $\sigma_t^{\wedge,\mathcal{C}}$ is the dynamics constructed by using the graphical construction with one extra rule: when the clock process $\mathcal{T}^{\uparrow}_{(x,1)}$ or $\mathcal{T}^{\downarrow}_{(x,1)}$ rings for any $x \in [\![1, L-1]\!] \cap 2\mathbb{N}$ and all $t < t_{\delta/2}$, we do not update. We refer to Figure 3 for illustration. While $t \geq t_{\delta/2}$, $(\sigma_t^{\wedge,\mathcal{C}})_{t \geq t_{\delta/2}}$ is constructed by the graphical construction in Subsection 2.1 without censoring.

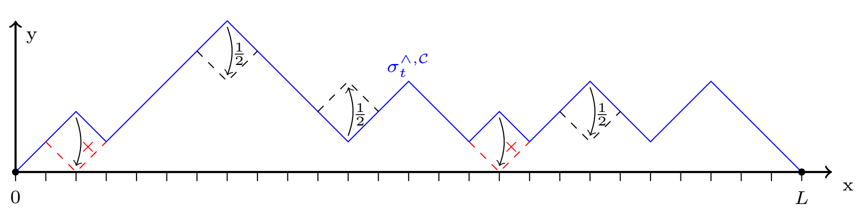


FIGURE 3. A graphical representation of the jump rates for the dynamics $\sigma_t^{\wedge,\mathcal{C}}$ when $t < t_{\delta/2}$. Those red dashed corners are not available and labeled with $\times$, while the other corners are flippable with rate 1/2.

Now we show that $P^{\wedge,\mathcal{C}}_{t_{\delta/2}}$ is close to μ_L^0. By Remark 1.2, applying Theorem 1.1, for all $\lambda \in (1,2)$, all $\delta > 0$ and all $\epsilon > 0$, if L is sufficiently large, we have

$$\|P^{\wedge,\mathcal{C}}_{t_{\delta/2}} - \mu_L^0\|_{\text{TV}} \leq \epsilon/2. \tag{5.9}$$

For any $\xi \in \Omega_L$, define

$$\rho(\xi) := \frac{d\mu_L^0}{d\mu_L^\lambda}(\xi),$$

and we want to show that ρ is bounded above uniformly for $\xi \in \Omega_L$. For any $\xi \in \Omega_L \setminus \Omega_L^+$—recalling $\Omega_L^+ = \{\xi \in \Omega_L : \mathcal{N}(\xi) = 0\}$, since $\mu_L^0(\xi) = 0$,

$$\rho(\xi) = \frac{\mu_L^0(\xi)}{\mu_L^\lambda(\xi)} = 0.$$

While for any $\xi \in \Omega_L^+$, applying Theorem 2.2, for all $L \geq 4$ we have

$$\rho(\xi) = \frac{d\mu_L^0}{d\mu_L^\lambda}(\xi) = \frac{\mu_L^0(\xi)}{\mu_L^\lambda(\xi)} = \frac{1/Z_{L-2}(1)}{1/Z_L(\lambda)} \leq C_5(\lambda),$$

where $C_5(\lambda) > 0$ is a suitable constant and only depends on λ. By Lemma 5.3 and (5.5), for any given $\delta > 0$, we have

$$\lim_{L\to\infty} \|\mu_L^0 P_{\frac{\delta}{2}L^2\log L} - \mu_L^\lambda\|_{\mathrm{TV}} = 0. \tag{5.10}$$

At this moment, we are ready to show that $P_{t_\delta}^{\wedge,\mathcal{C}}$—the distribution of the censored dynamics at t_δ—is close to the stationary measure μ_L^λ. By the definition of $\mathcal{C}$, we have

$$\begin{aligned}
\|P_{t_\delta}^{\wedge,\mathcal{C}} - \mu_L^\lambda\|_{\mathrm{TV}} &= \|P_{t_{\delta/2}}^{\wedge,\mathcal{C}} P_{t_\delta - t_{\delta/2}} - \mu_L^\lambda\|_{\mathrm{TV}} \\
&\leq \|P_{t_{\delta/2}}^{\wedge,\mathcal{C}} P_{t_\delta - t_{\delta/2}} - \mu_L^0 P_{t_\delta - t_{\delta/2}}\|_{\mathrm{TV}} + \|\mu_L^0 P_{t_\delta - t_{\delta/2}} - \mu_L^\lambda\|_{\mathrm{TV}} \\
&\leq \|P_{t_{\delta/2}}^{\wedge,\mathcal{C}} - \mu_L^0\|_{\mathrm{TV}} + \|\mu_L^0 P_{t_\delta - t_{\delta/2}} - \mu_L^\lambda\|_{\mathrm{TV}}.
\end{aligned} \tag{5.11}$$

Here the first inequality uses the triangle inequality. The second inequality is based on the fact that $\|\alpha P_t - \beta P_t\|_{\mathrm{TV}} \leq \|\alpha - \beta\|_{\mathrm{TV}}$ for any two probability measures α, β on Ω_L, and P_t is a transition matrix on Ω_L. The first term in (5.11) is not bigger than $\epsilon/2$ by (5.9) for L sufficiently large. The second term in (5.11) is smaller than or equal to $\epsilon/2$ by (5.10) for L sufficiently large.

Recall that $P_t^\wedge$ is the distribution of $\sigma_t^\wedge$ without censoring. By Proposition 5.1, for any $t \geq 0$, we have

$$\|P_t^\wedge - \mu_L^\lambda\|_{\mathrm{TV}} \leq \|P_t^{\wedge,\mathcal{C}} - \mu_L^\lambda\|_{\mathrm{TV}}. \tag{5.12}$$

Combining (5.11) and (5.12), we conclude the proof. □

Our next task is to provide an upper bound on the mixing time for the dynamics starting from the minimal path.

Proposition 5.5. *For any $\lambda \in (1,2)$, any $\epsilon > 0$ and any $\delta > 0$, if L is sufficiently large, we have*

$$T_{\mathrm{mix}}^{L,\vee}(\epsilon) \leq \frac{1+\delta}{\pi^2} L^2 \log L. \tag{5.13}$$

The idea for the proof of Proposition 5.5 for $\lambda \in (1,2)$ is similar to Proposition 5.4:

(i) We first show that under $P_{s_0(L)}^\vee$ with $s_0(L) := 10L^{16/9}\log L$ which is the marginal distribution of $\sigma_{s_0(L)}^\vee$, with high probability $\sigma_{s_0(L)}^\vee$ does not touch the $x-$axis in the interval $[\![M, L-M]\!]$ for some M sufficiently large.

(ii) For the time interval $[s_0(L), s_0(L)+t_{\delta/2})$, let $(\sigma_t^{\vee,\mathcal{C}})_{s_0(L)\leq t<s_0(L)+t_{\delta/2}}$ denote the dynamics censoring those updates which can change the number of contact points.

(iii) By Remark 1.2 and Theorem 1.1, roughly speaking, the distribution of $\sigma_{s_0(L)+t_{\delta/2}}^{\vee,\mathcal{C}}$ is close to μ_L^0 in total variation distance. Then we repeat the (iii) step stated above Proposition 5.4 to conclude the proof.

LEMMA 5.6. *For any given $\epsilon > 0$ and $\lambda \in (1,2)$, let $M = M(\lambda, \epsilon)$ be a positive integer, and*

$$\mathcal{E}_{L,M} := \Big\{ \xi \in \Omega_L : \xi_x \geq 1, \forall x \in [\![M, L-M]\!] \Big\}. \tag{5.14}$$

For all $L \geq 2M$, we have

$$\mathbb{P}\Big[\sigma_{s_0}^{\vee} \in \mathcal{E}_{L,M} \Big] \geq 1 - \epsilon/2. \tag{5.15}$$

PROOF. Let m and n be two positive integers, and $n < m < L/2$. Observe that in the graphical construction, if we run the dynamics $(\sigma_t^{\vee})_{t \geq 0}$ with the points $(2n, 0)$ and $(2m, 0)$ fixed for all $t \geq 0$, denoted as $(\overline{\sigma}_t^{\vee})_{t\geq 0}$ with $\overline{\sigma}_t^{\vee}(2n) \equiv \overline{\sigma}_t^{\vee}(2m) \equiv 0$ for all $t \geq 0$, we have

$$\forall t \geq 0, \ \overline{\sigma}_t^{\vee} \leq \sigma_t^{\vee}. \tag{5.16}$$

By symmetry, to give an upper bound on $\mathbb{P}[\sigma_t^{\vee}(x) = 0]$, we only need to consider $x \in [\![0, L/2]\!]$. For all $M \leq x \leq L/2$ and $x \in 2\mathbb{N}$, let $\bar{x} := 2\lfloor x^{8/9}/2 \rfloor$ and $\bar{L} := 2\lfloor L^{8/9}/2 \rfloor$. For all L sufficiently large, by (5.4) we obtain

$$T_{\text{mix}}^{L,\lambda}(L^{-3/2}) \leq \frac{18}{\pi^2} L^2 \log L. \tag{5.17}$$

Therefore, the quantity s_0 satisfies

$$T_{\text{mix}}^{\bar{L},\lambda}\left(\bar{L}^{-3/2} \right) \leq s_0.$$

Using (5.16), (5.17) and (3.8) respectively, for all $t \geq s_0$, we take $2n := x - \overline{x}$ and $2m := x + \overline{x}$ in (5.16) to obtain

$$\begin{aligned} \mathbb{P}\Big[\sigma_t^{\vee}(x) = 0 \Big] &\leq \mathbb{P}\Big[\overline{\sigma}_t^{\vee}(x) = 0 \Big] \\ &\leq \mu_{2\bar{x}}^{\lambda}(\xi_{\bar{x}} = 0) + \| P_t^{\vee} - \mu_{2\bar{x}}^{\lambda} \|_{\text{TV}} \leq C_6(\lambda) x^{-4/3}, \end{aligned} \tag{5.18}$$

where $C_6(\lambda) > 0$ only depends on λ. In the second inequality, there is an abuse of notation—$P_t^{\vee}$ denotes the distribution of $\sigma_t^{\vee}$ starting with the minimal path $\vee$ of $\Omega_{2\bar{x}}$. Therefore, due to symmetry and (5.18), we obtain

$$\begin{aligned} \sum_{x=M}^{L-M} \mathbb{P}[\sigma_{s_0}^{\vee}(x) = 0] &= 2 \sum_{x=M}^{L/2} \mathbb{P}[\sigma_{s_0}^{\vee}(x) = 0] \\ &\leq 2C_7(\lambda) M^{-1/3}. \end{aligned} \tag{5.19}$$

Let $C(\lambda, \epsilon) > 0$ be a constant such that the right-hand side is smaller than $\epsilon/2$, if $M \geq C(\lambda, \epsilon)$. Applying Markov's inequality and (5.19), we obtain

$$\mathbb{P}\left[\sigma_{s_0}^{\vee} \notin \mathcal{E}_{L,M} \right] = \mathbb{P}\left[\sum_{x=M}^{L-M} \mathbf{1}_{\{\sigma_{s_0}^{\vee}(x) = 0\}} \geq 1 \right] \leq \epsilon/2. \tag{5.20}$$

□

For the dynamics starting from $\xi \in \mathcal{E}_{L,M}$, we censor the updates that change the number of the contact points until time $t_{\delta/2}$. Then we show that its distribution at time $t_{3\delta/4}$ is close to μ_L^{λ} in total variation distance.

LEMMA 5.7. *Let $\xi \in \mathcal{E}_{L,M}$, and let $(\sigma_t^{\xi,\mathcal{C}})_{t\geq 0}$ be a censored dynamics with the censoring scheme $\mathcal{C} \colon \mathbb{R}^+ \to \mathcal{P}(\Theta)$ defined by*

$$\mathcal{C}(t) := \begin{cases} G_L & \text{if } t \in [0, t_{\delta/2}), \\ \varnothing & \text{if } t \in [t_{\delta/2}, \infty). \end{cases}$$

where G_L is defined in (5.7)*. For any given $\epsilon > 0$, for all L sufficiently large, we have*

$$\|P^{\xi,\mathcal{C}}_{t_{3\delta/4}} - \mu^\lambda_L\|_{\mathrm{TV}} < \epsilon/2, \tag{5.21}$$

where we recall that $t_\delta = (1+\delta)\pi^{-2}L^2\log L$ and $P^{\xi,\mathcal{C}}_t$ denotes the marginal distribution of the censored dynamics $(\sigma^{\xi,\mathcal{C}}_t)_{t\geq 0}$ at time t.

With Lemma 5.7 at hand, we are ready to prove Proposition 5.5. Combining Lemma 5.6, Lemma 5.7 and Proposition 5.1, we conclude the proof of Proposition 5.5, since $s_0 + t_{3\delta/4} \leq t_\delta$.

Proof of Lemma 5.7. For $\xi \in \mathcal{E}_{L,M}$, set

$$\begin{aligned} \ell(\xi) &:= \sup\{x \leq M : \xi_x = 0\}, \\ r(\xi) &:= \inf\{x \geq L-M : \xi_x = 0\}. \end{aligned} \tag{5.22}$$

Observe that the censored dynamics $(\sigma^{\xi,\mathcal{C}}_t)_{0\leq t<t_{\delta/2}}$ restricted in the intervals $[\![0,\ell]\!]$, $[\![\ell,r]\!]$ and $[\![r,L]\!]$ respectively are independent. Let the marginal distribution restricted in these three intervals be denoted by $P^{\xi,\mathcal{C}}_{t,\ell}$, $P^{\xi,\mathcal{C}}_{t,r-\ell}$, $P^{\xi,\mathcal{C}}_{t,L-r}$ respectively. We refer to Figure 4 for illustration.

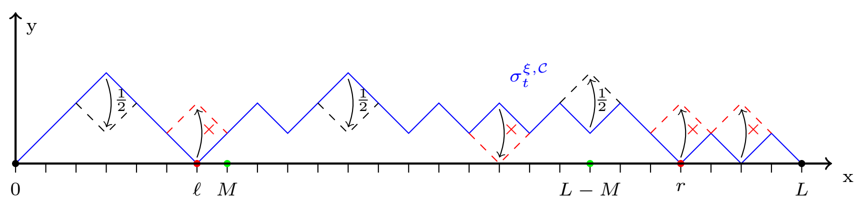


Figure 4. A graphical representation of the jump rates for the censored dynamics $(\sigma^{\xi,\mathcal{C}}_t)_{0\leq t<t_{\delta/2}}$ starting from $\xi \in \mathcal{E}_{L,M}$. The red dashed corners are not available corners, labeled with $\times$. To the left hand side of the green point $(M,0)$, the red point $(\ell,0)$ is the first contact point with the x-axis at time $t=0$. Moreover, the corner at $(\ell,0)$ is fixed for $t \in [0,t_{\delta/2})$. Likewise, the same phenomenon holds for the green point $(L-M,0)$ and the red point $(r,0)$. In the time interval $[0,t_{\delta/2})$, the censored dynamics $(\sigma^{\xi,\mathcal{C}}_t)_{0\leq t<t_{\delta/2}}$ does not touch the $x-$axis in the interval $[\![\ell+1,r-1]\!]$.

Let the censored dynamics restricted in the interval $[\![\ell,r]\!]$ be denoted by $(\widetilde{\sigma}^{\xi}_t)_{t<t_{\delta/2}}$, whose invariant probability measure is $\mu^0_{r-\ell}$ defined in (1.2). By Theorem 1.1 and Remark 1.2, for given $\delta > 0$ and $\epsilon > 0$, for all L sufficiently large, we have

$$\|P^{\xi,\mathcal{C}}_{t_{\delta/2},r-\ell} - \mu^0_{r-\ell}\|_{\mathrm{TV}} \leq \epsilon/4. \tag{5.23}$$

Note that the upper bound in (5.23) does not depend on the value of (ℓ,r). Moreover, observe that for any $\xi' \in \Omega_L$, the product distribution $P^{\xi,\mathcal{C}}_{t_{\delta/2},l} \otimes \mu^0_{r-\ell} \otimes P^{\xi,\mathcal{C}}_{t_{\delta/2},L-r}$ satisfies

$$\Big(P^{\xi,\mathcal{C}}_{t_{\delta/2},\ell} \otimes \mu^0_{r-\ell} \otimes P^{\xi,\mathcal{C}}_{t_{\delta/2},L-r}\Big)(\xi') \leq \frac{1}{Z_{r-\ell}(0)}, \tag{5.24}$$

while $\mu^\lambda_L(\xi') \geq 1/Z_L(\lambda)$ since $\lambda \in (1,2)$. Therefore, for all $L > 2M$ and for any $\xi' \in \Omega_L$, we have

$$\frac{\mathrm{d}P^{\xi,\mathcal{C}}_{t_{\delta/2},\ell} \otimes \mu^0_{r-\ell} \otimes P^{\xi,\mathcal{C}}_{t_{\delta/2},L-r}}{\mathrm{d}\mu^\lambda_L}(\xi') \leq C_8(\lambda)2^{2M}, \tag{5.25}$$

where the last inequality uses Theorem 2.2 and $r-\ell \geq L-2M$, since $\xi \in \mathcal{E}_{L,M}$. Note that the right-most hand side in (5.25) does not depend on the value of (ℓ, r), and the distribution of $\sigma^{\xi,\mathcal{C}}_{t_{\delta/2}}$ is

$$P^{\xi,\mathcal{C}}_{t_{\delta/2}} = P^{\xi,\mathcal{C}}_{t_{\delta/2},\ell} \otimes P^{\xi,\mathcal{C}}_{t_{\delta/2},r-\ell} \otimes P^{\xi,\mathcal{C}}_{t_{\delta/2},L-r}, \tag{5.26}$$

instead of $P^{\xi,\mathcal{C}}_{t_{\delta/2},\ell} \otimes \mu^0_{r-\ell} \otimes P^{\xi,\mathcal{C}}_{t_{\delta/2},L-r}$. Due to (5.23), we repeat the same procedure in (5.11) to obtain

$$\begin{aligned} \|P^{\xi,\mathcal{C}}_{t_{3\delta/4}} - \mu^\lambda_L\| &= \|P^{\xi,\mathcal{C}}_{t_{\delta/2}} P_{t_{3\delta/4}-t_{\delta/2}} - \mu^\lambda_L\| \\ &\leq \|P^{\xi,\mathcal{C}}_{t_{\delta/2}} P_{t_{3\delta/4}-t_{\delta/2}} - (P^{\xi,\mathcal{C}}_{t_{\delta/2},\ell} \otimes \mu^0_{r-\ell} \otimes P^{\xi,\mathcal{C}}_{t_{\delta/2},L-r}) P_{t_{3\delta/4}-t_{\delta/2}}\| \\ &\quad + \|(P^{\xi,\mathcal{C}}_{t_{\delta/2},\ell} \otimes \mu^0_{r-\ell} \otimes P^{\xi,\mathcal{C}}_{t_{\delta/2},L-r}) P_{t_{3\delta/4}-t_{\delta/2}} - \mu^\lambda_L\|. \end{aligned} \tag{5.27}$$

Moreover, we have

$$\begin{aligned} &\|P^{\xi,\mathcal{C}}_{t_{\delta/2}} P_{t_{3\delta/4}-t_{\delta/2}} - (P^{\xi,\mathcal{C}}_{t_{\delta/2},\ell} \otimes \mu^0_{r-\ell} \otimes P^{\xi,\mathcal{C}}_{t_{\delta/2},L-r}) P_{t_{3\delta/4}-t_{\delta/2}}\| \\ \leq &\|P^{\xi,\mathcal{C}}_{t_{\delta/2}} - P^{\xi,\mathcal{C}}_{t_{\delta/2},\ell} \otimes \mu^0_{r-\ell} \otimes P^{\xi,\mathcal{C}}_{t_{\delta/2},L-r}\| = \|P^{\xi,\mathcal{C}}_{t_{\delta/2},r-\ell} - \mu^0_{r-\ell}\| \leq \varepsilon/4 \end{aligned} \tag{5.28}$$

where we have used (5.26) in the equality and (5.23) in the last inequality. While for the last term in (5.27), by (5.25) and Lemma 5.3, for all L sufficiently large we have

$$\|(P^{\xi,\mathcal{C}}_{t_{\delta/2},\ell} \otimes \mu^0_{r-\ell} \otimes P^{\xi,\mathcal{C}}_{t_{\delta/2},L-r}) P_{t_{3\delta/4}-t_{\delta/2}} - \mu^\lambda_L\| \leq \varepsilon/4. \tag{5.29}$$

Combining (5.28) with (5.29), we conclude the proof. □

Theorem 1.3 is a combination of Proposition 3.1, Proposition 5.4, and Proposition 5.5.

2.A. Proof of Lemma 4.9.

We lift the maximal path $\wedge$ up by a height $L^{1/2}(\log L)^2$. To be precise, define $\overline{\wedge} := \wedge + m$, *i.e.* $\overline{\wedge}_x = \wedge_x + m$ for all $x \in [\![0, L]\!]$, where $m := 2\lceil L^{1/2}(\log L)^2/2\rceil$. The graphical construction in Subsection 2.1, with Θ changed to be

$$\Theta' := \Big\{(x,z) : x \in [\![1, L-1]\!], z \in [\![1, m + L/2 - 1 - |x - L/2|]\!], x+z \in 2\mathbb{N}+1\Big\},$$

allows us to couple the three dynamics $(\sigma^{\wedge,\lambda}_t)_{t\geq 0}$, $(\sigma^{\overline{\wedge},\lambda}_t)_{t\geq 0}$ and $(\sigma^{\overline{\wedge},0}_t)_{t\geq 0}$, starting from $\wedge$, $\overline{\wedge}$ and $\overline{\wedge}$ respectively, with parameter λ, λ and 0 respectively. By the monotonicity of the starting paths and the parameters λ in the dynamics, asserted in Proposition 2.1, we have

$$\begin{aligned} \sigma^{\wedge,\lambda}_t &\leq \sigma^{\overline{\wedge},\lambda}_t, \\ \sigma^{\overline{\wedge},\lambda}_t &\leq \sigma^{\overline{\wedge},0}_t. \end{aligned}$$

Set

$$\overline{H}(t) := \max_{x \in [\![0,L]\!]} \sigma^{\overline{\wedge},0}_t(x).$$

Since $\overline{H}(t) \geq H(t)$, it is enough to prove that

$$\lim_{L\to\infty} \mathbb{P}\Big[\exists t \in [t_{\delta/2}, t_\delta] : \overline{H}(t) \geq 2L^{1/2}(\log L)^2\Big] = 0, \tag{2.A.1}$$

where we recall that $t_\delta = (1+\delta)\frac{1}{\pi^2}L^2 \log L$. We obtain such an upper bound in (2.A.1) by comparing $(\sigma^{\overline{\wedge},0}_t)_{t\geq 0}$ with the symmetric simple exclusion process.

2.A.1. Simple exclusion process. Define

$$\mathcal{S}_L := \Big\{\zeta \in \mathbb{Z}^{L+1} : \zeta_0 = \zeta_L = m; |\xi_{x+1} - \xi_x| = 1, \forall x \in [\![0, L-1]\!]\Big\}, \tag{2.A.2}$$

and

$$\mathcal{S}_L^+ := \Big\{\zeta \in \mathcal{S}_L : \zeta_x \geq 1, \forall x \in [\![0, L]\!]\Big\}.$$

We define a Markov chain on $\mathcal{S}_L$ by specifying its generator $\mathfrak{L}$. The generator $\mathfrak{L}$ is defined by its action on the functions $\mathbb{R}^{\mathcal{S}_L}$,

$$(\mathfrak{L}f)(\zeta) := \frac{1}{2}\sum_{x=1}^{L-1}\Big(f(\zeta^x) - f(\zeta)\Big), \tag{2.A.3}$$

where $\zeta^x \in \mathcal{S}_L$ is defined by

$$\zeta^x_y := \begin{cases} \zeta_y & \text{if } y \neq x, \\ \zeta_{x-1} + \zeta_{x+1} - \zeta_x & \text{if } y = x. \end{cases}$$

When $\zeta_{x-1} = \zeta_{x+1}$, ζ displays a local extremum at x and we obtain ζ^x by flipping the corner of ξ at the coordinate x. Let U_L denote the uniform probability measure on $\mathcal{S}_L$. We can see that this Markov chain is reversible with respect to the uniform measure U_L. Therefore, U_L is the invariant probability measure for this Markov chain. The Markov chain starting with the maximal path $\overline{\wedge}$ is denoted by $(\eta_t^{\overline{\wedge}})_{t\geq 0}$. Likewise, let $(\eta_t^{U_L})_{t\geq 0}$ denote the Markov chain with generator $\mathfrak{L}$ and starting path chosen by sampling U_L. There is a one-one correspondence between this Markov chain and the symmetric simple exclusion process, for which we refer to [Lac16b, Section 2.3] for more information. Under the measure U_L, typical path $\zeta \in \mathcal{S}_L$ does not touch the x-axis, which is the following lemma.

LEMMA 2.A.1. *For all L sufficiently large, we have*

$$U_L(\mathcal{S}_L \setminus \mathcal{S}_L^+) \leq e^{-\frac{1}{2}(\log L)^2}. \tag{2.A.4}$$

PROOF. Let $\mathbf{P}$ be the law of the nearest-neighbor symmetric simple random walk on $\mathbb{Z}$, and $(S_i)_{i\in\mathbb{N}}$ be its trajectory with $S_0 = 0$. Since any trajectory of this simple random walk has the same mass, we have

$$\begin{aligned} U_L(\mathcal{S}_L \setminus \mathcal{S}_L^+) &= \mathbf{P}\Big[\exists i \in [\![0, L]\!] : S_i + m \leq 0 | S_L = 0\Big] \\ &\leq L^{\frac{1}{2}}\mathbf{P}\Big[\min_{i\in[\![0,L]\!]} S_i \leq -m, S_L = 0\Big] \\ &\leq 2L^{\frac{1}{2}}\mathbf{P}[S_L \leq -m] \\ &\leq e^{-\frac{1}{2}(\log L)^2}, \end{aligned} \tag{2.A.5}$$

which vanishes as L tends to infinity. The first inequality uses $\mathbf{P}[S_L = 0] \geq L^{-1/2}$, for all L sufficiently large. The second inequality uses

$$\mathbf{P}\Big[\min_{i\in[\![0,L]\!]} S_i \leq -m, S_L = 0\Big] \leq 2\mathbf{P}\Big[S_L \leq -m\Big].$$

In the last inequality, we use the inequality, $\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n} \leq n! \leq en^{n+\frac{1}{2}}e^{-n}$ for all $n \geq 1$, to obtain

$$\mathbf{P}[S_L \leq -m] \leq (L - m + 1)\binom{L}{\frac{L+m}{2}}2^{-L} \leq (L - m + 1)e^{-(\log L)^2}.$$

□

2.A.2. Compare the polymer pinning dynamics to simple exclusion process. There is a graphical construction similar to that mentioned at the beginning of Appendix 2.A, allowing to couple the three dynamics $(\sigma_t^{\overline{\wedge},0})_{t\geq 0}$, $(\eta_t^{\overline{\wedge}})_{t\geq 0}$ and $(\eta_t^{U_L})_{t\geq 0}$ such that for all $t \geq 0$,

$$\sigma_t^{\overline{\wedge},0} \geq \eta_t^{\overline{\wedge}} \geq \eta_t^{U_L}. \tag{2.A.6}$$

Let $P_t^{\overline{\wedge},-}(\cdot) := \mathbb{P}(\eta_t^{\overline{\wedge}} = \cdot)$ and $P_t^{\overline{\wedge},0}(\cdot) := \mathbb{P}(\sigma_t^{\overline{\wedge},0} = \cdot)$. Intuitively, the distribution of $\sigma_t^{\overline{\wedge},0}$ is close to that of $\eta_t^{\overline{\wedge}}$ for all $t \geq 0$.

LEMMA 2.A.2. *For any given $\epsilon > 0$ and all L sufficiently large, we have*

$$\sup_{0\leq t\leq t_\delta} \|P_t^{\overline{\wedge},0} - P_t^{\overline{\wedge},-}\|_{\mathrm{TV}} \leq \epsilon. \tag{2.A.7}$$

PROOF. By Proposition 4.1 and the monotonicity in (2.A.6), we obtain

$$\begin{aligned}\sup_{0\leq t\leq t_\delta} \|P_t^{\overline{\wedge},0} - P_t^{\overline{\wedge},-}\|_{\mathrm{TV}} &\leq \mathbb{P}\Big[\exists t\in[0,t_\delta] : \sigma_t^{\overline{\wedge},0} \neq \eta_t^{\overline{\wedge}}\Big] \\ &\leq \mathbb{P}\Big[\exists t\in[0,t_\delta] : \min_{x\in[\![0,L]\!]} \eta_t^{\overline{\wedge}}(x) \leq 0\Big] \\ &\leq \mathbb{P}\Big[\exists t\in[0,t_\delta] : \min_{x\in[\![0,L]\!]} \eta_t^{U_L}(x) \leq 0\Big]. \end{aligned}\tag{2.A.8}$$

The second inequality is based on the fact that in the coupling if $\sigma_t^{\overline{\wedge},0} \neq \eta_t^{\overline{\wedge}}$, there must exist $x \in [\![0,L]\!]$ satisfying $\eta_s^{\overline{\wedge}}(x) = 0$ for some $s \in [0,t]$. The third inequality uses the monotonicity of the dynamics, *i.e.* $\eta_t^{\overline{\wedge}} \geq \eta_t^{U_L}$ for all $t \geq 0$. The last term in (2.A.8) vanishes as L tends to infinity, which follows exactly as that in (4.33) of Lemma 4.8, using occupation time (4.30), strong Markov property and Lemma 2.A.1. □

Furthermore, by [Lac16b, Theorem 2.4], for any given $\epsilon > 0$ and $t \geq t_{\delta/2}$, if L is sufficiently large, we have

$$\|P_t^{\overline{\wedge},-} - U_L\|_{\mathrm{TV}} \leq \epsilon. \tag{2.A.9}$$

Then we use the information of U_L to give an upper bound for the highest point of $\sigma_t^{\overline{\wedge},0}$.

PROOF OF LEMMA 4.9. By triangle inequality, Lemma 2.A.2 and (2.A.9), for $t \in [t_{\delta/2}, t_\delta]$, if L is sufficiently large, we have

$$\|P_t^{\overline{\wedge},0} - U_L\|_{\mathrm{TV}} \leq 2\epsilon. \tag{2.A.10}$$

By (2.A.10), for every $t \in [t_{\delta/2}, t_\delta]$ and L sufficiently large, we obtain

$$\mathbb{P}\Big[\overline{H}(t) \geq 2L^{\frac{1}{2}}(\log L)^2\Big] \leq U_L\bigg(\sup_{x\in[\![0,L]\!]} \zeta_x \geq 2L^{\frac{1}{2}}(\log L)^2, \zeta\in\mathcal{S}_L\bigg) + \|P_t^{\overline{\wedge},0} - U_L\|_{\mathrm{TV}} \leq 3\epsilon,$$

where the first term in the right hand side vanishes as L tends to infinity, whose proof is the same as Lemma 2.A.1. Since $\epsilon > 0$ is arbitrary, we finish the proof. □

2.B. Spin system.

To deduce Proposition 5.1 from [PW13, Theorem 1.1], we construct a monotone system $\langle \Omega_L^*, S, V_L, \mu_L^* \rangle$ which is the same as the Glauber dynamics of the polymer pinning model.

For $(x,z) \in \mathbb{N}^2$, a square with four vertices $\{(x-1,z),(x+1,z),(x,z-1),(x,z+1)\}$ is denoted as $Sq(x,z)$. Recalling Θ defined in (2.1), let $S := \{\oplus, \ominus\}$ denote the spins, and $V_L := \{Sq(x,z) : \forall (x,z) \in \Theta\}$ denote the set of all sites, which consists of all green or white squares shown Figure 5. Each square of V_L is endowed with $\oplus$ or $\ominus$. Moreover, we give a natural order for the spins, say, $\ominus \leq \oplus$. For any given $\xi \in \Omega_L$, every square $Sq(x,z)$ lying under the path ξ is endowed with $\oplus$, while every square $Sq(x,z)$ lying above ξ is endowed with $\ominus$. This spin configuration is denoted as ξ^*. For $\xi, \xi' \in \Omega_L$, $\xi \le \xi'$ if and only if $\xi^* \le \xi'^*$. Let $\Omega^*_L := \{\xi^*, \xi \in \Omega_L\}$ and $\mu^*_L(\xi^*) := \mu(\xi)$.

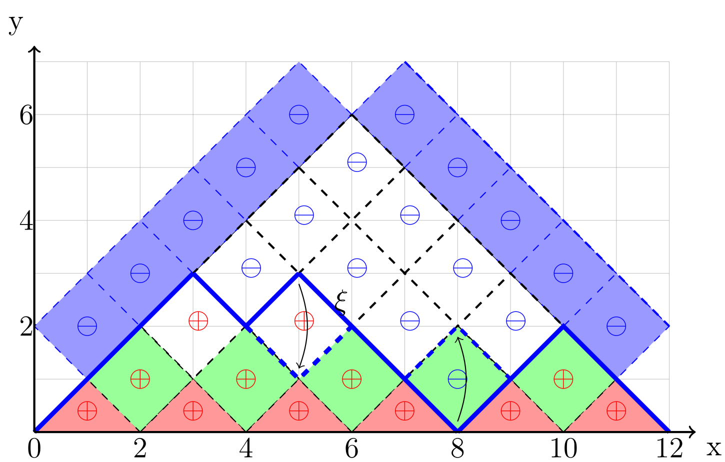


FIGURE 5. An example shows the equivalence between the polymer pinning model and the spin system with $L = 12$. The blue path ξ is an element of Ω_L. This configuration in the spin system is denoted as ξ^*, and its probability measure is $\mu(\xi)$. The corner at $x = 8$ of thick blue path ξ flips up with rate $1/(1+\lambda)$ to the dashed blue corner, while the spin $\ominus$ at the green square centered at $(8,1)$ flips to $\oplus$ with rate $1/(1+\lambda)$. The corner at $x = 5$ of thick blue path ξ flips down with rate $1/2$ to the dashed blue corner, while the spin $\oplus$ at the white square centered at $(5,2)$ flips to $\ominus$ with rate $1/2$. Note that not all the correspondence between the flipping of the corners of ξ and that of the spins of ξ^* are shown in the picture.

For convenience of describing the Glauber dynamics of spin system, we introduce two fixed boundary conditions. We assign a negative spin $\ominus$ to each square $Sq(x,z)$ where

$$\Big\{(x,z) : x \in [\![1, L/2-1]\!] \cup [\![L/2+1, L-1]\!], z = L/2 + 1 - |x - L/2|\Big\}.$$

These are the blue squares shown in Figure 5. In addition, we also introduce a positive boundary condition. A triangle with three vertices $\{(x-1,z),(x+1,z),(x,z+1)\}$ is denoted as $Tr(x,z)$ for $(x,z) \in \mathbb{N}^2$. We assign a positive spin $\oplus$ to each triangle $Tr(x,0)$ for all $x \in [\![1, L-1]\!] \setminus 2\mathbb{N}$. These are the red triangles shown in Figure 5. We say that two spins are neighbors if the squares or triangles on which they lie share an edge. We use the same exponential clocks and uniform coins $\mathcal{T}^\uparrow$, $\mathcal{T}^\downarrow$, $\mathcal{U}^\uparrow$, and $\mathcal{U}^\downarrow$ define in Subsection 2.1 to describe the dynamics of the spin system.

Given $\mathcal{T}^\uparrow$, $\mathcal{T}^\downarrow$, $\mathcal{U}^\uparrow$ and $\mathcal{U}^\downarrow$, we construct, in a deterministic way, $(\sigma^{\xi^*}_t)_{t\ge 0}$ the Glauber dynamics of the spin system starting with ξ^* with parameter λ. The trajectory $(\sigma^{\xi^*}_t)_{t\ge0}$ is càdlàg with $\sigma^{\xi^*}_0 = \xi^*$ and is constant in the intervals, where the clock processes are silent.

When the clock process $\mathcal{T}^\uparrow_{(x,z)}$ rings at time $t = \mathcal{T}^\uparrow_{(x,z)}(n)$ for $n \ge 1$, we update the configuration $\sigma^{\xi^*}_{t^-}$ as follows:

- if the spin in the square $Sq(x,z)$ is $\ominus$, and has two neighbors with $\oplus$ spins, and $z = 1$, and $\mathcal{U}^\uparrow_{(x,z)}(n) \le \frac{1}{1+\lambda}$, we let the spin in the square $Sq(x,z)$ change to $\oplus$ at time t, and the other spins remain unchanged;
- if the spin in the square $Sq(x,z)$ is $\ominus$, and has two neighbors with $\oplus$ spins, and $z \ge 2$, and $\mathcal{U}^\uparrow_{(x,z)}(n) \le 1/2$, we let the spin in the square $Sq(x,z)$ change to $\oplus$.

If these two conditions aforementioned are not satisfied, we do nothing.

When the clock process $\mathcal{T}^{\downarrow}_{(x,z)}$ rings at time $t = \mathcal{T}^{\downarrow}_{(x,z)}(n)$ for $n \geq 1$, we update the configuration $\sigma^{\xi^*}_{t-}$ as follows:

- if the spin in the square $Sq(x, z)$ is $\oplus$, and has two neighbors with $\ominus$ spins, and $z = 1$, and $\mathcal{U}^{\downarrow}_{(x,z)}(n) \leq \frac{\lambda}{1+\lambda}$, we let the spin in the square $Sq(x, z)$ change to $\ominus$ at time t, and the other spins remain unchanged;
- if the spin in the square $Sq(x, z)$ is $\oplus$, and has two neighbors with $\ominus$ spins, and $z \geq 2$, and $\mathcal{U}^{\downarrow}_{(x,z)}(n) \leq 1/2$, we let the spin in the square $Sq(x, z)$ change to $\ominus$ at time t, and the other spins remain unchanged.

If these two conditions aforementioned are not satisfied, we do nothing.

We can see that $\langle \Omega^*, S, V, \mu^* \rangle$ is a monotone system in the sense of [PW13, Section 1.1], whose Glauber dynamics is the same as that of the polymer pinning model.

CHAPTER 3

Metastability for expanding bubbles on a sticky substrate

Abstract: In this chapter, we study the dynamical behavior of a one dimensional interface interacting with a sticky impenetrable substrate or wall. The interface is subject to two effects going in opposite directions. Contact between the interface and the substrate are given an energetic bonus while an external force with constant intensity pulls the interface away from the wall. Our interface is modeled by the graph of a one-dimensional nearest-neighbor path on $\mathbb{Z}_+$, starting at 0 and ending at 0 after $2N$ steps, the wall corresponding to level-zero the horizontal axis. At equilibrium each path $\xi = (\xi_x)_{x=0}^{2N}$, is given a probability proportional to $\lambda^{H(\xi)} \exp(\frac{\sigma}{N} A(\xi))$, where $H(\xi) := \#\{x \ : \xi_x = 0\}$ and $A(\xi)$ is the area enclosed between the path ξ and the x-axis. We then consider the classical heat-bath dynamics which equilibrates the value of each ξ_x at a constant rate via corner-flip.

Investigating the statics of the model, we derive the full phase diagram in λ and σ of this model, and identify the critical line which separates a localized phase where the pinning force sticks the interface to the wall and a delocalized one, for which the external force stabilizes ξ around a deterministic shape at a macroscopic distance of the wall. On the dynamical side, we identify a second critical line, which separates a rapidly mixing phase (for which the system mixes in polynomial time) to a slow phase where the mixing time grows exponentially. In this slowly mixing regime we obtain a sharp estimate of the mixing time on the log scale, and provide evidences of a metastable behavior.

1. Introduction

The present manuscript investigates the dynamical behavior for a discrete interface model in the vicinity of an impenetrable substrate or wall. We assume that the interface is subject to:

(A) An interaction with the wall, modelized by an energetic reward or penalty for each contact.
(B) An homogeneous external force field, which drives away the interface from the wall which translates into adding a potential energy proportional to the interface heights.

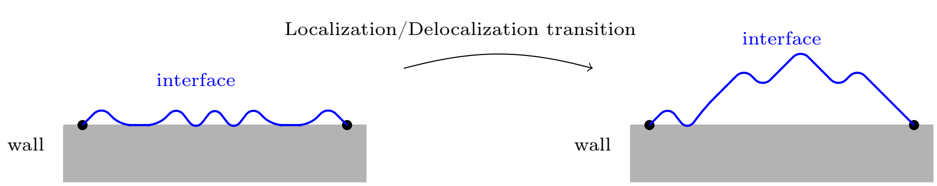


Figure 1. The typical behavior of the interface changes when the external force field passes a certain threshold from a localized phase to a delocalized phase.

We want to understand in depth how these two competing effects can affect the mixing properties of the system. We consider the simplest possible setup. Our interface is modeled by the graph of a one dimensional simple random walk, with a configuration space given by

$$\Omega_N := \Big\{ \xi \in \mathbb{Z}_+^{2N+1} \ : \ \xi_0 = \xi_{2N} = 0 \ ; \forall x \in [\![1, 2N]\!], |\xi_x - \xi_{x-1}| = 1 \Big\}.$$

We are going to consider a reversible Markov chain $(\eta_t)_t \geq 0$ on Ω_N with transition rules which reflect the two driving forces described in (A) and (B) (see (2.5) below for an explicit description of the Markov chain and (2.2) for the corresponding reversible probability).

The study of effective interface models is a large field of study both in mathematics and physics. The problem of wetting of a random walk (which is the study of effect (A) alone) dates back to the seminal paper of [Fis84]. Several variants and generalizations of the model have been considered since (with a particular interest for the disordered model see [Gia11, Gia07] for a review). Interest in the dynamics associated to this model and its mixing properties came later [CMT08, CLM+12, Yan19].

Interfaces subjected to an external force (effect (B)), on the other hand, have been studied in an infinite volume, both because it is a natural model for growth and because of its connection with the asymmetric simple exclusion process, mostly in the infinite volume setup (see e.g. [Ros81, Rez91, DMPS89, Gä87] for early references dealing with hydrodynamics with total, partial and weak asymmetry). The model on the segment is slightly different, since in particular the boundary condition makes the dynamics reversible, and its static and dynamical properties were investigated [BBHM05, Lab18, LL19, LL20, LP16] (see also [GNS20, Sch19] for variants with open boundaries and random environment).

As can be seen in the above references, under the effect of (A) or (B) alone, the system mixes fast. By this we mean that the mixing time (whose definition is recalled in Section 2 below) grows only like a power of the size of the system.

In the chapter, we show that this state of fact changes dramatically when (A) and (B) are combined, at least for some choices of parameters. To take full advantage of the effect (A) or (B), the interface must adopt two very different strategies. To get the best of the energetic bonus awarded for contacts with the wall, the interface wants to locally optimize the contact fraction which implies staying very close to the wall (see [Gia07, Theorem 2.4] and Figure 3). On the other hand the pulling force, when considered alone, makes the interface stabilize around a macroscopic profile which optimizes the competition between the energetic reward given by the pulling force field and the large deviation cost for the one dimensional random walk (see [Lab18, Theorem 4] and Figure 3). When both the attraction to the wall and the external field are turned on, there is no efficient way to combine the two above strategies. As a result the equilibrium state of the system is simply determined by comparing which of the two effects is dominant. In particular we have an abrupt phase transition when the external field grows, from a localized phase where the interface sticks to the wall, to a delocalized one, where the interface is repelled at a macroscopic distance away from it. As a first result in our chapter, we give a detailed description of the equilibrium phase diagram of the system, which includes the identification of the free-energy and a description of the interface behavior on the critical line.

The more important contribution is the study of the dynamics. We establish that depending on the value of the parameters which tune the intensity of effects (A) and (B) the system either mixes in polynomial time or takes an exponential time to reach its equilibrium state. We also identify the critical line which separates the slow and fast mixing phases, which does not coincide with the line delimiting the static phase transition. We will show that when the wall is attractive and the external force is sufficiently large, then the mixing time becomes exponentially large in the size of the system. Moreover we identify the critical line which separates the fast-mixing regime from the slow-mixing regime, which differs from the one appearing on the equilibrium phase diagram.

The slow mixing phase displays a metastable behavior. In that regime, the two strategies which maximize the benefits of contact with the wall and the external force field respectively correspond heuristically two distinct local equilibrium states for the dynamics. The mixing time then corresponds to the typical time needed to travel from the thermodynamically less favorable state (corresponding the the less beneficial strategy) to the point of equilibrium. We prove that properly rescaled, the traveling time for leaving the thermodynamically unstable local equilibrium rescales to an exponential random variable.

This metastable picture is present in many systems of statistical mechanics and has been the object of an extensive mathematical attention in the past two decades (see [BL15, BDH16] and references therein). In the specific realm of pinning model, our picture is reminiscent of the Cassie-Baxter/Wenzel transition observed for wetting of irregular substrate (see [GCMC12] and references therein for a review and studies of the phenomenon and [DCDH11, LT15] for the mathematical treatement of a simplified model accounting for it).

2. Model and results

2.1. The setup.

The static model. Let us now introduce a simple statistical mechanics model which combines the substrate interaction and the external force-field effect. Consider the set of nonnegative integer-valued one-dimensional nearest-neighbor paths which start at 0 and end at 0 after $2N$ steps, that is

$$\Omega_N := \Big\{ \xi \in \mathbb{Z}_+^{2N+1} \ : \ \xi_0 = \xi_{2N} = 0 \ ; \forall x \in [\![1, 2N]\!], |\xi_x - \xi_{x-1}| = 1 \Big\}, \tag{2.1}$$

where $N \in \mathbb{N}$, and $[\![a, b]\!] := [a, b] \cap \mathbb{Z}$ for $a, b \in \mathbb{R}$ with $a < b$. For $\xi \in \Omega_N$, we denote by H and A respectively the number of zeros and the (algebraic) area between the path and the horizontal axis

$$H(\xi) := \sum_{x=1}^{2N-1} \mathbf{1}_{\{\xi_x = 0\}} \quad \text{and} \quad A(\xi) := \sum_{x=1}^{2N} \xi_x.$$

We define a probability measure on Ω_N using a Gibbs weight constructed from an Hamiltonian which is the sum of two terms, one proportional to the area and another one proportional to the number of contacts. We rescale the area by a factor N so that these two effects play on the same scale. Given $\lambda \geq 0$ and $\sigma \in \mathbb{R}$, we define $\mu_N^{\lambda,\sigma}$ on Ω_N by

$$\mu_N^{\lambda,\sigma}(\xi) := \frac{2^{-2N} \lambda^{H(\xi)} \exp\left(\frac{\sigma}{N} A(\xi)\right)}{Z_N(\lambda, \sigma)} \tag{2.2}$$

where $Z_N(\lambda, \sigma)$ is the partition function, given by

$$Z_N(\lambda, \sigma) := 2^{-2N} \sum_{\xi' \in \Omega_N} \lambda^{H(\xi')} \exp\left(\frac{\sigma}{N} A(\xi')\right). \tag{2.3}$$

By convention, $0^0 := 1$ and $0^k := 0$ for any positive integer $k \geq 1$. The factor 2^{-2N} is irrelevant for the definition of $\mu_N^{\lambda,\sigma}$ but is convenient for the partition function. When it is clear from the context, we omit the indices λ and σ in $\mu_N^{\lambda,\sigma}$. The graph of ξ depicts the spatial configuration of an interface (see Figure 1).

The dynamics. The object of this chapter is to investigate the relaxation property of the Glauber dynamics associated with the equilibrium measure $\mu_N^{\lambda,\sigma}$. This is a continuous-time reversible Markov chain on Ω_N, which proceeds by flipping the corners in the path $\xi \in \Omega_N$. For $\xi \in \Omega_N$ and $x \in [\![1, 2N-1]\!]$, we define ξ^x by

$$\xi^x_y := \begin{cases} \xi_y & \text{if } y \neq x, \\ (\xi_{x-1} + \xi_{x+1}) - \xi_x & \text{if } y = x \text{ and } \xi_{x-1} = \xi_{x+1} \geq 1 \text{ or } \xi_{x-1} \neq \xi_{x+1}, \\ \xi_x & \text{if } y = x \text{ and } \xi_{x-1} = \xi_{x+1} = 0. \end{cases} \tag{2.4}$$

In other words, if $\xi_{x-1} = \xi_{x+1}$, ξ presents a local extremum at x and ξ^x is obtained by flipping the corner at the coordinate x provided that $\xi^x \in \Omega_N$ (see Figure 2). The rates at which each corner is flipped is specified by the following rates

$$r_N(\xi, \xi^x) := \begin{cases} \frac{\exp(\frac{2\sigma}{N})}{1+\exp(\frac{2\sigma}{N})} & \text{if} \xi_{x-1} = \xi_{x+1} > \xi_x \geq 1, \\ \frac{1}{1+\exp(\frac{2\sigma}{N})} & \text{if } \xi_x > \xi_{x-1} = \xi_{x+1} > 1, \\ \frac{\lambda}{\lambda+\exp(\frac{2\sigma}{N})} & \text{if } (\xi_{x-1}, \xi_x, \xi_{x+1}) = (1,2,1), \\ \frac{\exp(\frac{2\sigma}{N})}{\lambda+\exp(\frac{2\sigma}{N})} & \text{if} (\xi_{x-1}, \xi_x, \xi_{x+1}) = (1,0,1), \\ 0 & \text{if } \xi_{x-1} \neq \xi_{x+1} \text{ or } \xi_{x-1} = \xi_{x+1} = 0. \end{cases} \tag{2.5}$$

The other transition rates $r_N(\xi, \xi')$ when ξ is not one of the ξ^xs are equal to zero. The generator $\mathcal{L}_N$ of the Markov chain is thus given (for $f : \Omega_N \to \mathbb{R}$) by

$$(\mathcal{L}_N f)(\xi) := \sum_{\xi' \in \Omega_N} r_N(\xi, \xi')\big[f(\xi') - f(\xi)\big] = \sum_{x=1}^{2N-1} r_N(\xi, \xi^x)\big[f(\xi^x) - f(\xi)\big]. \tag{2.6}$$

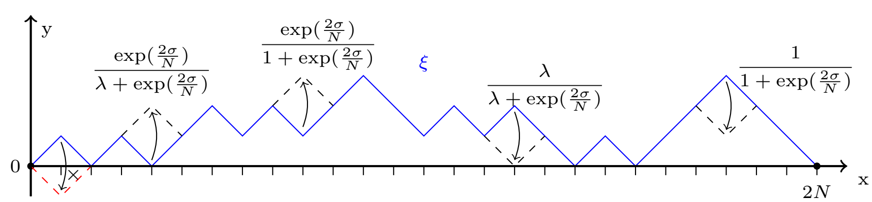


FIGURE 2. A graphical representation of the jump rates for the system. A transition of the chain corresponds to flipping a corner, whose rate is chosen such that the chain is reversible with respect to $\mu_N^{\lambda,\sigma}$. The red dashed corner is not available, due to the nonnegative restriction of the state space Ω_N. Note that not all of the possible transitions are shown in the figure.

An interpretation of $\mathcal{L}_N$ is that for each x, the coordinate ξ_x is resampled with respect to the conditional equilibrium measure $\mu_N(\cdot \mid (\xi_y)_{y \neq x})$. Indeed the generator can be rewritten as

$$(\mathcal{L}_N f)(\xi) = \sum_{x=1}^{2N-1} \Big[Q_x(f)(\xi) - f(\xi)\Big],$$

where Q_x is the following operator

$$Q_x(f)(\xi) := \mu_N\left(f(\xi) \mid (\xi_y)_{y \neq x}\right).$$

Here and in what follows $\nu(f)$ is used to denote the expectation of f with respect to ν and similar convention is used for conditional expectation. The chain is irreducible, and since the rates r_N satisfy the detailed balance condition for the measure μ_N, it is also reversible. We are interested in the speed relaxation to equilibrium of the above dynamics which is encoded by the spectral gap of the generator $\mathcal{L}_N$. In our context the spectral gap can be defined as the minimal positive eigenvalue of $-\mathcal{L}_N$. It can be characterized using the Dirichlet form associated with the dynamic defined by

$$\mathcal{E}(f) := -\langle f, \mathcal{L}_N f\rangle_{\mu_N} = \sum_{x=1}^{2N-1} \mu_N\left((Q_x f - f)^2\right),$$

where $\langle f, g\rangle_{\mu_N} := \sum_{\xi\in\Omega_N} \mu_N(\xi) f(\xi) g(\xi)$ denotes the usual inner-product in $L^2(\mu_N)$. Moreover, the spectral gap, denoted by $\mathrm{gap}_N(\lambda,\sigma)$, is the minimal positive eigenvalue of $-\mathcal{L}_N$ and the relaxation time is its inverse. That is

$$T_{\mathrm{rel}}^N(\lambda,\sigma) := \sup_{f\,:\,\mathrm{Var}_{\mu_N}(f)>0} \frac{\mathrm{Var}_{\mu_N}(f)}{\mathcal{E}(f)} = \mathrm{gap}_N^{-1}(\lambda,\sigma), \tag{2.7}$$

where $\mathrm{Var}_{\mu_N}(f) := \langle f, f\rangle_{\mu_N} - \langle f, 1\rangle_{\mu_N}^2$.

2.2. Equilibrium results. While our main result concerns the dynamics, our first task is to understand the properties of the model at equilibrium, and in particular the asymptotic behavior of the partition function. Our result is obtained via comparison with two previously studied models.

The Random walk pinning model. The case $\sigma = 0$ is very well understood, since in that case the model is the classical random walk pinning model in [Fis84]. We refer to [Gia07] (see also [CMT08, Yan19] for studies of the dynamics). The model undergoes a phase transition at $\lambda = 2$: when $\lambda < 2$, our random interfaces typically have a finite number of contact points with the $x-$axis and typical heights are of order $\sqrt{N}$ while when $\lambda > 2$, we have a positive density of contact points with the $x-$axis and the largest height is of order $\log N$.

This transition is encoded in the free energy of the model defined by

$$F(\lambda) := \lim_{N\to\infty} \frac{1}{2N} \log Z_N(\lambda, 0).$$

From [Gia07, Proposition 1.1] the free energy can be computed explicitly and we have (see [LT15, Equation (1.5)]),

$$F(\lambda) = \log\left(\frac{\lambda}{2\sqrt{\lambda-1}}\right)\mathbf{1}_{\{\lambda>2\}}. \tag{2.8}$$

Furthermore we have the following, more detailed asymptotics for the partition function (cf. [Gia07, Theorem 2.2],

$$Z_N(\lambda,0) = \begin{cases} (1+o(1))C_\lambda N^{-3/2} & \text{if } \lambda\in[0,2), \\ (1+o(1))C_2 N^{-1/2} & \text{if } \lambda = 2, \\ (1+o(1))C_\lambda e^{2NF(\lambda)} & \text{if } \lambda > 2. \end{cases} \tag{2.9}$$

Our aim is to derive similar precise asymptotics when $\sigma > 0$.

The weakly asymmetric simple exclusion process on the segment. Another case for which details on the partition function have been obtained is that when $\lambda = 1$, $\sigma > 0$, and no half-space constraint is given (meaning that we allow for $\xi_x < 0$). In that case the model corresponds to the equilibrium height profile of the weakly asymmetric simple exclusion process (or WASEP) on the line segment $[\![1, 2N]\!]$ with N particles. Its equilibrium properties have been investigated in details in [Lab18, Section 2] (also with the objective of studying the dynamics) with some attention given to the asymptotic behavior the corresponding partition function, namely

$$\widetilde{Z}_N(\sigma) := 2^{-2N} \sum_{\xi \in \widetilde{\Omega}_N} \exp\left(\tfrac{\sigma}{N} A(\xi)\right), \tag{2.10}$$

where

$$\widetilde{\Omega}_N := \left\{ \xi \in \mathbb{Z}^{2N+1} \ : \ \xi_0 = \xi_{2N} = 0 \ ; \forall x \in [\![1, 2N]\!], |\xi_x - \xi_{x-1}| = 1 \right\}. \tag{2.11}$$

In particular by [Lab18, Proposition 3] the limit

$$\lim_{N\to\infty} \frac{1}{2N} \log \widetilde{Z}_N(\sigma) := G(\sigma),$$

exists and is given by

$$G(\sigma) = \int_0^1 L\left(\sigma(1-2x)\right) \mathrm{d}x \quad \text{ where } \quad L(x) := \log \cosh x. \tag{2.12}$$

Furthermore we have (from [Lab18, Lemma 11] in the case $k = 1$, $\alpha = 1$, see also (3.7)-(3.9) below)

$$\widetilde{Z}_N(\sigma) = (1 + o(1)) C_\sigma N^{-1/2} e^{2NG(\sigma)}. \tag{2.13}$$

The hybrid model. In the present chapter, we identify the free energy when both pinning and area tilt are present, and identify (up to a constant) the right order asymptotic.

PROPOSITION 2.1. *We have for any $\lambda \geq 0$ and $\sigma \geq 0$*

$$\lim_{N\to\infty} \frac{1}{2N} \log Z_N(\lambda, \sigma) = F(\lambda) \vee G(\sigma). \tag{2.14}$$

More precisely there exists a constant $C_1(\lambda, \sigma) > 0$ such that:

(1) *If $G(\sigma) > F(\lambda)$, then for all $N \geq 1$ we have*

$$\frac{1}{C_1(\lambda,\sigma)} \leq \frac{\sqrt{N} Z_N(\lambda,\sigma)}{\exp\left(2NG(\sigma)\right)} \leq C_1(\lambda, \sigma); \tag{2.15}$$

(2) *If $G(\sigma) \leq F(\lambda)$ and $\lambda > 2$, then for all $N \geq 1$ we have*

$$\frac{1}{C_1(\lambda,\sigma)} \leq \frac{Z_N(\lambda,\sigma)}{\exp\left(2NF(\lambda)\right)} \leq C_1(\lambda, \sigma). \tag{2.16}$$

The above result confirms that the two effect of area tilt and pinning do not combine and that only the stronger of the two (which is determined by the comparison of $F(\lambda)$ and $G(\sigma)$) prevails. In the case of a tie between $F(\lambda)$ and $G(\sigma)$, the estimates (2.15)-(2.16) entails that the pinning has a stronger effect. This is illustrated in Theorem 2.4 below.

REMARK 2.2. *In the result above, we do not identify the asymptotic equivalent of the partition function in* (2.15)-(2.16) *and leave unmatching constants for the upper and lower bounds. This is mostly to avoid lengthier computation and because the estimates*(2.15)-(2.16) *are sufficient to prove our results about the dynamics.*

REMARK 2.3. *We excluded the case $\sigma < 0$ from the analysis. Little efforts would be necessary to show that we have in that case also*

$$\lim_{N\to\infty} \frac{1}{2N} \log Z_N(\lambda, \sigma) = F(\lambda), \tag{2.17}$$

and that (2.16) *also holds. The case $\lambda < 2$ and $\sigma < 0$ should correspond to a different regime where*

$$-C_1(\lambda,\sigma) N^{1/3} \leq \log Z_N(\lambda, \sigma) \leq -\frac{1}{C_1(\lambda,\sigma)} N^{1/3}. \tag{2.18}$$

This is reminiscent of the behavior observe in [FS05] *for a Brownian motion in presence of a curved barrier (see also references therein for numerous occurrences of $N^{1/3}$ fluctuation). This is in any case out of the focus of this chapter.*

The information we gathered about the partition function allows for a detailed description the typical behavior of ξ under $\mu_N^{\lambda,\sigma}$. Let us define

$$M_\sigma(u) := \int_0^u \tanh(\sigma(1-x))\mathrm{d}x = \frac{1}{\sigma}\log\left(\frac{\cosh(\sigma)}{\cosh(\sigma(1-u))}\right). \tag{2.19}$$

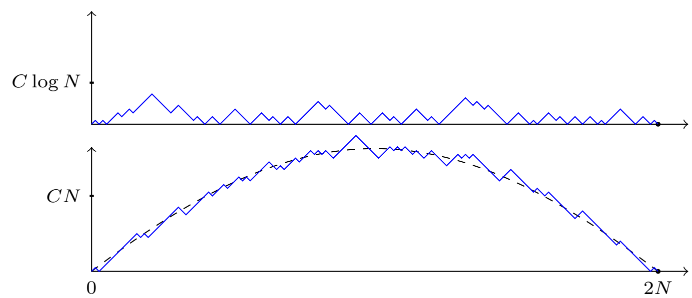


FIGURE 3. The macroscopic shape of the substrate in equilibrium when $F(\lambda) \geq G(\sigma)$ (at the top) and $F(\lambda) < G(\sigma)$ (at the bottom). The dotted line illustrates the macroscopic shape, which is the scaling limit when $N \to \infty$ (The dotted line in the top figure coincides with the $x-$axis.).

THEOREM 2.4. *For $\lambda \geq 0$, $\sigma > 0$, we have*

1. *if $G(\sigma) > F(\lambda)$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all N sufficiently large,*
$$\mu_N\left(\sup_{u\in[0,2]}\left|\frac{1}{N}\xi_{\lceil uN\rceil} - M_\sigma(u)\right| > \varepsilon\right) \leq e^{-\delta N}; \tag{2.20}$$
2. *if $G(\sigma) < F(\lambda)$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all N sufficiently large,*
$$\mu_N\left(\sup_{x\in[\![0,2N]\!]}\xi_x > \varepsilon N\right) \leq e^{-\delta N}; \tag{2.21}$$
3. *if $G(\sigma) = F(\lambda)$, then for every $\varepsilon > 0$ and all N sufficiently large,*
$$\frac{1}{C\sqrt{N}} \leq \mu_N\left(\sup_{x\in[\![0,2N]\!]}\xi_x > \varepsilon N\right) \leq \frac{C}{\sqrt{N}}, \tag{2.22}$$
and furthermore there exists $\delta > 0$ such that
$$\mu_N\left(\sup_{x\in[\![0,2N]\!]}\xi_x > \varepsilon N \text{ and } \sup_{u\in[0,2]}\left|\frac{1}{N}\xi_{\lceil uN\rceil} - M_\sigma(u)\right| > \varepsilon\right) \leq e^{-\delta N}. \tag{2.23}$$

REMARK 2.5. *Note that the corresponding shape result in the case of pure pinning ($\sigma = 0$) can be deduced from* [Gia07, Chapter 2] *while that for WASEP interfaces (corresponding to (2.10)) can be extracted from the results in* [Lab18].

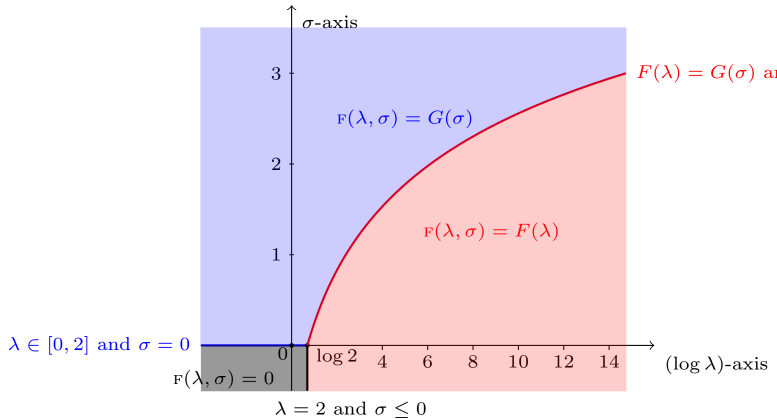


FIGURE 4. The statics phase diagram for the free energy $\mathrm{F}(\lambda,\sigma)$: the red curve is $F(\lambda) = G(\sigma)$ and $\lambda > 2$, the black line is $\lambda = 2$ and $\sigma \leq 0$, and the blue line is $\lambda \in [0,2]$ and $\sigma = 0$.

REMARK 2.6. *Looking at* (2.14) *we see that the free energy of our model defined by*

$$\mathrm{F}(\lambda,\sigma) := \lim_{N\to\infty} \frac{1}{2N} \log Z_N(\lambda,\sigma), \tag{2.24}$$

is real-analytic in λ and σ, except on the curve $\{(\lambda,\sigma) \ : \ \lambda \geq 2,\ F(\lambda) = G(\lambda)\}$, on the half line line $\{(\lambda,\sigma) \ : \ \lambda = 2, \sigma \leq 0\}$ and the segment $\{(\lambda,\sigma) \ : \ \lambda \in [0,2], \sigma = 0\}$ (see Figure 4). The partial derivatives of $\mathrm{F}(\lambda,\sigma)$ (corresponding to the asymptotic contact fraction and rescaled area respectively) are discontinuous across the line, indicating that the corresponding phase transition is of first order.

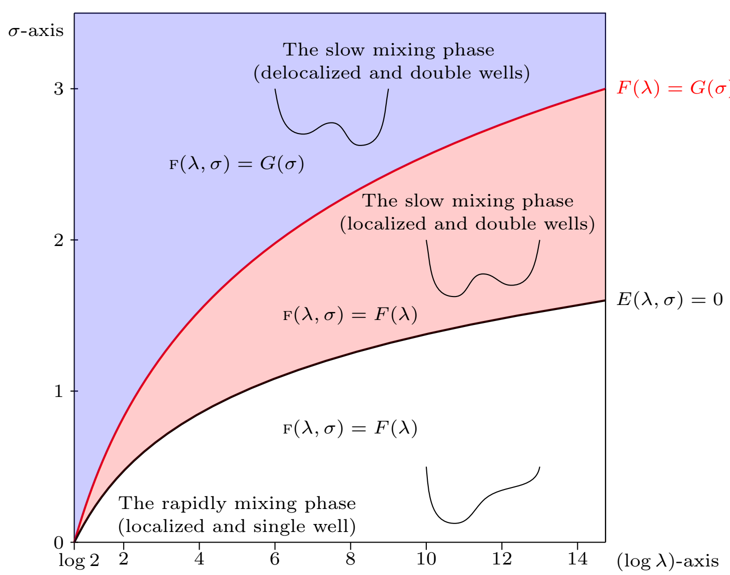


FIGURE 5. The dynamical phase diagram in the regime $\lambda > 2$ and $\sigma > 0$: The line $F(\lambda) = G(\sigma)$ separates the localized phase from the delocalized phase, while the line $E(\lambda,\sigma) = 0$ separates the rapidly mixing phase from the slow mixing phase.

2.3. Dynamics results. As the main result for our chapter we manage to identify two regimes for the dynamics, one where the system relaxes in polynomial time and one where the relaxation time grows exponentially with the size of the system. To state our result, we need to introduce a new quantity. We define the activation energy of the system by

$$E(\lambda,\sigma) = G(\sigma) \wedge F(\lambda) - \inf_{\beta\in[0,1]} \left(\beta G(\beta\sigma) + (1-\beta)F(\lambda)\right). \tag{2.25}$$

Note that $E(\lambda,\sigma) \geq 0$ and that $E(\lambda,\sigma) > 0$ if and only if the equation

$$G(\beta\sigma) + \sigma\beta G'(\beta\sigma) - F(\lambda) = 0 \tag{2.26}$$

admits a solution in $(0,1)$. This condition is equivalent to $G(\sigma) + \sigma G'(\sigma) > F(\lambda) > 0$.

The main result. We show that the system relaxation to equilibrium is "fast", that is, polynomial in N when $E(\lambda,\sigma) = 0$ while it is exponentially slow when $E(\lambda,\sigma) > 0$.

THEOREM 2.7. *For all $\lambda > 2$ and all $\sigma > 0$, we have*

$$\lim_{N\to\infty} \frac{1}{2N} \log T^N_{\mathrm{rel}}(\lambda,\sigma) = E(\lambda,\sigma). \tag{2.27}$$

When $E(\lambda,\sigma) = 0$, there exist constants $C(\lambda,\sigma) > 0$ and $C(\lambda) > 0$ such that for all $N \geq 1$,

$$C(\lambda,\sigma)^{-1} N \leq T^N_{\mathrm{rel}}(\lambda,\sigma) \leq C(\lambda,\sigma) N^{C(\lambda)}. \tag{2.28}$$

When $E(\lambda,\sigma) > 0$, there exists constants $C(\lambda,\sigma) > 0$ and $C'(\lambda,\sigma) > 0$ such that

$$C'(\lambda,\sigma)^{-1} N^{-2} \leq T^N_{\mathrm{rel}}(\lambda,\sigma) e^{-2NE(\lambda,\sigma)} \leq C'(\lambda,\sigma) N^{C(\lambda,\sigma)}.$$

The curve $\{(\lambda,\sigma) \ : \ \sigma > 0\ ,\ G(\sigma) + \sigma G'(\sigma) = F(\lambda)\}$ delimits a second phase transition (the first transition being the wetting transition materialized by the curve $F(\lambda) = G(\sigma)$ see Figure 5) from a slow mixing regime to a fast mixing regime. This transition is not visible in the phase diagram of the static model and appears when considering the dynamics.

Mixing time. For the sake of completeness, let us mention how our result translates for the mixing time of the Markov chain (see [LP17] for a full review of the topic). We let $(\eta^\xi_t)_{t\geq 0}$ denote the Markov chain with generator $\mathcal{L}_N$ (2.6) starting with initial condition $\xi \in \Omega_N$, and let P^ξ_t denote its marginal distribution at time t. For all $\epsilon \in (0,1)$, the $\epsilon-$mixing time for the dynamics is

$$T^{N,\lambda,\sigma}_{\mathrm{mix}}(\epsilon) := \inf\left\{ t \geq 0 : \sup_{\xi\in\Omega_N} \|P^\xi_t - \mu_N\|_{\mathrm{TV}} \leq \epsilon \right\}, \tag{2.29}$$

where $\|\pi_1 - \pi_2\|_{\mathrm{TV}} := \frac{1}{2}\sum_{\xi\in\Omega_N} |\pi_1(\xi) - \pi_2(\xi)|$ denotes the total variation distance. By [LP17, Lemma 20.11, Theorem 12.3], the mixing time can be compared to the relaxation time as follows

$$T^N_{\mathrm{rel}}(\lambda,\sigma) \log \frac{1}{2\epsilon} \leq T^{N,\lambda,\sigma}_{\mathrm{mix}}(\epsilon) \leq T^N_{\mathrm{rel}}(\lambda,\sigma) \log \frac{1}{\varepsilon \mu^*_N}, \tag{2.30}$$

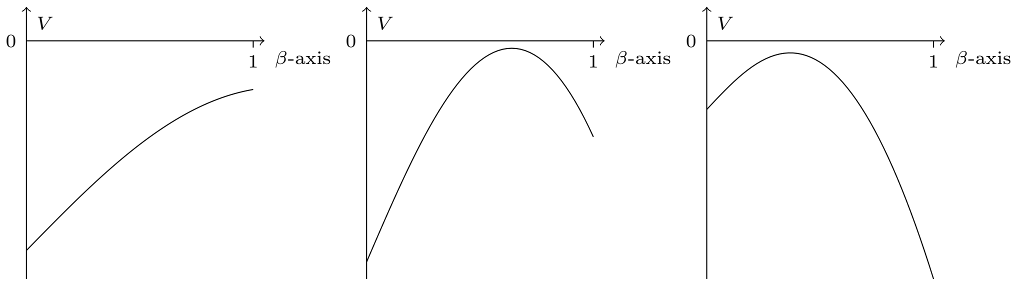
where $\mu^*_N := \min_{\xi\in\Omega_N} \mu_N(\xi)$. It is almost immediate to check that in our case $\log \mu^*_N$ is of order N (with a prefactor depending on λ and σ). Thus Theorem 2.7 remains essentially if one replaces $T^N_{\mathrm{rel}}(\lambda,\sigma)$ by $T^{N,\lambda,\sigma}_{\mathrm{mix}}(\epsilon)$.

A first heuristic. Let us try to give a first explanation for the slower relaxation time when $E(\lambda,\sigma) > 0$ (additional elements will be brought in the course of the proof see the discussion in Section 4.1). In that case, the state space displays two distinct "wells of potential" for the effective energy functional

$$V \ : \ \beta \mapsto -\beta G(\beta\sigma) - (1-\beta)F(\lambda).$$

The parameter $\beta \in [0,1]$ above corresponds to the fraction of the polymer length which is unpinned and the functional corresponds to the contribution to the partition function (on the exponential scale) of the polymer configurations which are macroscopically unpinned on a fraction β of their length. The idea is that the unpinned fraction should look like a stochastic diffusion on the segment, with a potential $2NV(\cdot)$.

The time $e^{2NE(\lambda,\sigma)}$ corresponds to the time required for such a diffusion to overcome the energy barrier between the two local mimina of $V(\beta)$ (at 0 and 1 see Figure 6).



(a) $G(\sigma)+\sigma G'(\sigma)\le F(\lambda)$ where $E(\lambda,\sigma)=0$;

(b) $G(\sigma)+\sigma G'(\sigma)> F(\lambda)$ and $F(\lambda)\ge G(\sigma)$ where $E(\lambda,\sigma)>0$;

(c) $G(\sigma)>F(\lambda)$ where $E(\lambda,\sigma)>0$.

FIGURE 6. The shapes of the functional $V(\beta):=-\beta G(\beta\sigma)-(1-\beta)F(\lambda)$ for three phases: (a) $G(\sigma)+\sigma G'(\sigma)\le F(\lambda)$; (b) $G(\sigma)+\sigma G'(\sigma)>F(\lambda)$ and $F(\lambda)\ge G(\sigma)$; (c) $G(\sigma)>F(\lambda)$.

We obtain more detailed information concerning the tunnelling time between the higher local minimum of V (which corresponds to a locally stable, or *metastable* state) and the absolute minimum which corresponds to the equilibrium state. For $\xi\in\Omega_N$, we define the (half) length of the largest excursion of ξ to be

$$L_{\max}(\xi)=\sup\left\{\ell\in[\![1,N]\!]\ :\exists x\in[\![0,2N]\!],\ \xi_x=\xi_{x+2\ell}=0,\ \forall y\in[\![1,2\ell-1]\!],\ \xi_{x+y}>0\right\}. \tag{2.31}$$

Assuming that $E(\lambda,\sigma)>0$, we let $\beta^*\in(0,1)$ denote the unique solution of (2.26) and let $\mathcal{E}_N^i$, $i=1,2$ be the domains of attraction of the two local minima of V

$$\begin{aligned}\mathcal{E}_N^1&:=\{\xi\in\Omega_N : L_{\max}(\xi)\le\beta^*N\},\\ \mathcal{E}_N^2&:=\{\xi\in\Omega_N : L_{\max}(\xi)>\beta^*N\}.\end{aligned} \tag{2.32}$$

We let $\mathcal{H}_N$ denote the domain of attraction of the higher of these two minima, that is

$$\mathcal{H}_N:=\begin{cases}\mathcal{E}_N^2 & \text{if } G(\sigma)\le F(\lambda),\\ \mathcal{E}_N^1 & \text{if } G(\sigma)>F(\lambda).\end{cases} \tag{2.33}$$

Our choice for breaking the tie when $G(\sigma)=F(\lambda)$ is not arbitrary at all and comes from the estimates for the partition function beyond the exponential scale obtained in Proposition 2.1.

According to our heuristic analysis, the behavior of the dynamics when $E(\lambda,\sigma)>0$ should be the following: If starting from a configuration $\xi\in\mathcal{H}_N$, the system should quickly thermalize in $\mathcal{H}_N$ (within a time which is polynomial in N) and then take a time of order $\exp(2NE(\lambda,\sigma))$ to jump from $\mathcal{H}_N$ to $\Omega_N\setminus\mathcal{H}_N$ and reach equilibrium. Moreover, when properly rescaled the time for jumping from $\mathcal{H}_N$ to $\Omega_N\setminus\mathcal{H}_N$ should converge to an exponential random variable.

These features (existence of different time scales, and loss of memory from one time scale to another) are the signature of metastable behavior of the system. We refer to [BDH16, Lan19] for an introduction to the phenomenon and a review of the literature.

Given ν a probability on Ω_N we let $\mathbb{P}_\nu$ denote the law of the Markov chain $(\eta_t)_{t\ge0}$ starting with η_0 distributed as ν. Our last result establishes the metastability of our system in the sense that it shows that the dynamics starting from $\mathcal{H}_N$ exits it at an exponential rate which is given by the relaxation time of the dynamics.

THEOREM 2.8. *We have*

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N(\cdot|\mathcal{H}_N)}\left(\eta_{tT^N_{\mathrm{rel}}(\lambda,\sigma)} \in \mathcal{H}_N\right) = \exp(-t),$$

and the finite-dimensional distributions of the process $\mathbf{1}_{\mathcal{H}_N}(\eta_{tT^N_{\mathrm{rel}}(\lambda,\sigma)})$ *(under* $\mathbb{P}_{\mu_N(\cdot|\mathcal{H}_N)}$*) converges to that of a Markov process which starts at one and jumps, at rate one, to zero where it is absorbed.*

REMARK 2.9. *We have chosen to present the result in the above form because it comes as an easy consequence of the analysis needed to prove Theorem 2.7 and of a general criterion established in* [BL15]*. Pushing the analysis further and following the ideas developed in* [CLM$^+$12, Section 1.3] *for monotone system, one can most likely get a more detailed picture of the metastable behavior (convergence profile to equilibrium starting from extremal conditions, exponential hitting times for the potential wells etc...).*

2.4. Organization of the chapter. In Section 3, we gather most of the technical estimates on the partition function $Z_N(\lambda, \sigma)$. This contains in particular the proof of Proposition 2.1 and Theorem 2.4 but also some of the estimates needed in the following sections to estimate the relaxation time.

In Section 4, we derive the lower bound on the relaxation time in Theorem 2.7. This is the easier of the two bounds, but perhaps the more important since the proof allows to identify exactly what slows down the relaxation to equilibrium, which is a single bottleneck in the space of configuration.

In Section 5, we prove almost matching upper bound (up to correction of polynomial order). Our proof relies on the combination of several techniques (induction, chain reduction, path/flow methods...). While these techniques now became part of the classic toolbox to study mixing time, their combination and implementation to this case required an insightful understanding of the relaxation mechanism of this particular system. This is the most technical part of the chapter.

In Section 6, we show that the estimates proved in previous sections are sufficient to check all the conditions needed to apply the general metastability results from [BL15].

About notation. In order to make the proof more readable we avoid writing integer parts and write in many instances $\sum_{i=1}^{t}$ for $\sum_{i=1}^{\lfloor t\rfloor}$. The constants used in the proof are not numbered the same C can assume different values in different equations. We tried to underline the dependence in the parameter by writing $C(\lambda)$ and $C(\lambda, \sigma)$ when it has some importance, with a particular care for the dependence in σ since some parts of the proof crucially rely on it.

3. Equilibrium behavior and partition function asymptotics

Let us expose here our general strategy to understand the equilibrium measure, and obtain not only the asymptotics for the partition functions contained in Proposition 2.1 but also a variety going to be required to analyse the dynamics and prove Theorem 2.7. Our starting point is the observation that decomposing the path into excursions away from the x-axis and factorizing we obtain

$$Z_N(\lambda, \sigma) := \sum_{k\geq 1} \sum_{\substack{n_1,\dots,n_k \\ \sum_{i=1}^k n_k = N}} \lambda^{k-1} \prod_{i=1}^{k} Z_{n_i}\left(0, \frac{\sigma n_i}{N}\right). \tag{3.1}$$

Hence our first task is going to be to understand the detailed behavior of $Z_N(0,\sigma)$ for a large range of σ and then use it in the above decomposition.

3.1. The case $\lambda = 0$. This case is first treated separately. It then plays an important role to obtain estimates both for $\lambda \le 2$ and $\lambda > 2$. The statement is actually more precise than what is required for Proposition 2.1 (in the sense that it is uniform in σ). This precision is necessary for some of the spectral gap estimates in Section 5.

PROPOSITION 3.1. *For all $K > 0$, there exists a constant $C = C_K > 0$, such that for all $N \ge 1$, and all $\sigma \in [0, K]$*

$$\frac{1}{C\sqrt{N}}\left(N^{-1/2} \vee \sigma\right)^2 \le \frac{Z_N(0,\sigma)}{\exp\left(2NG(\sigma)\right)} \le \frac{C}{\sqrt{N}}\left(N^{-1/2} \vee \sigma\right)^2 \tag{3.2}$$

where $G(\sigma)$ is defined in (2.12)*. Moreover, given $\varepsilon, K > 0$ then, there exists $\delta = \delta(\varepsilon) > 0$ such that we have for all $N \ge N_0(\varepsilon, K)$, and $\sigma \in [0, K]$*

$$\mu_N^{0,\sigma}\left(\sup_{u\in[0,2]}\left|\frac{1}{N}\xi_{\lceil uN\rceil} - M_\sigma(u)\right| > \varepsilon\right) \le e^{-\delta N}. \tag{3.3}$$

PROOF. Our proof follows the mainline of [Lab18, Proposition 3] with an additional care needed to deal with the positivity constraint. Hence the first step is to reduce the statement to the estimate of the probability of a given event. We let $\mathbf{P}$ denote the distribution of the nearest-neighbor symmetric simple random walk in $\mathbb{Z}$ starting from 0. Given a simple random walk trajectory we define $A_N(S) := \sum_{n=1}^{2N-1} S_n + \frac{S_{2N}}{2}$, to be the algebraic area between the graph of $S = (S_n)_{n=1}^{2N}$ and the x-axis. We have (the tilt by $-\sigma S_{2N}$ having no effect)

$$Z_N(0,\sigma) = \mathbf{E}\left[e^{\frac{\sigma A_N(S)}{N} - \sigma S_{2N}}\mathbf{1}_{\{S_{2N}=0\ ;\ \forall n\in[\![1,2N-1]\!],\ S_n>0\}}\right]. \tag{3.4}$$

We introduce ν_N a probability which is absolutely continuous with respect to $\mathbf{P}$ with density given by

$$\frac{\mathrm{d}\nu_N}{\mathrm{d}\mathbf{P}}(S) := \frac{e^{\frac{\sigma A_N(S)}{N} - \sigma S_{2N}}}{\mathbf{E}\left[e^{\frac{\sigma A_N(S)}{N} - \sigma S_{2N}}\right]}. \tag{3.5}$$

The tilt by $-\sigma S_{2N}$ has the effect of recentering the distribution of S_{2N} and to make the event $\{S_{2N} = 0\}$ typical under ν_N. Indeed let $(X_k)_{1\le k\le 2N}$ denote the increments of our random walk, and we have

$$\frac{\sigma A_N(S)}{N} - \sigma S_{2N} = \sum_{k=1}^{2N} h_k^N X_k \quad \text{where} \quad h_k^N := \frac{\sigma}{N}\left(N - k + \frac{1}{2}\right). \tag{3.6}$$

We have

$$Z_N(0,\sigma) = \mathbf{E}\left[e^{\frac{\sigma A_N(S)}{N} - \sigma S_{2N}}\right]\nu_N\left(S_{2N} = 0\ ;\ \forall n \in [\![1, 2N-1]\!],\ S_n > 0\right). \tag{3.7}$$

Recalling the definition of L in (2.12) we have

$$\mathbf{E}\left[e^{\frac{\sigma A_N(S)}{N} - \sigma S_{2N}}\right] = \exp\left(\sum_{k=1}^{2N} L(h_k^N)\right). \tag{3.8}$$

By the approximation of Riemann integral and the Taylor-Lagrange inequality (we have $L''(x) = 1 - \tanh^2(x) \in [0,1]$) we obtain

$$\Big|\sum_{k=1}^{2N} L(h_k^N) - 2N\int_0^1 L\left(\sigma(1-2x)\right)dx\Big| \le \frac{\sigma^2}{4N}, \tag{3.9}$$

and hence that

$$\Big|\log \mathbf{E}\left[e^{\frac{\sigma A_N(S)}{N} - \sigma S_{2N}}\right] - 2NG(\sigma)\Big| \le \frac{\sigma^2}{4N}. \tag{3.10}$$

The first term in the r.h.s. in (3.7) can be replaced by $e^{2NG(\sigma)}$ to obtain an asymptotic equivalent. The asymptotic equivalent of the second term $\nu_N(\cdots)$ is the object of Proposition 3.2 which allows to conclude the proof of (3.2).

Let us now prove (3.3). The rewriting of $Z_N(0,\sigma)$ in (3.7) can be performed for the partition function integrated against an arbitrary event A yields $S_{2N} = 0$

$$\mu_N^{0,\sigma}(A) = \nu_N\left(A \mid S_{2N} = 0\ ;\ \forall n \in [\![1, 2N-1]\!],\ S_n > 0\right) \le C_K N^{3/2}\nu_N(A). \tag{3.11}$$

where for the last inequality we used Proposition 3.2 below. Hence it is sufficient for us to show that

$$\nu_N\left(\sup_{u\in[0,2]}\left|\frac{1}{N}\xi_{\lceil uN\rceil} - M_\sigma(u)\right| > \varepsilon\right) \le 2Ne^{-2\delta N}. \tag{3.12}$$

Since M_σ is $1-$Lipschitz, by union bound it is sufficient to check that that

$$\sup_{n\in[\![0,2N]\!]} \nu_N\left(|\xi_n - NM_\sigma(n/N)| > N\varepsilon/2\right) \le e^{-2\delta N}, \tag{3.13}$$

where $\delta = \varepsilon^2/130$ for all $N \ge N_0(\varepsilon, K)$. This is a simple consequence of Hoeffding's inequality (see e.g. [Pet19, Proposition 1.8]) for a sum of bounded independent variables. The only thing to check is that $NM_\sigma(n/N)$ approximates well the expectation of ξ_n (that is, that the difference is of a smaller order than N). By Riemann sum approximation we have

$$|\nu_N[\xi_n] - NM_\sigma(n/N)| = \left|\sum_{k=1}^{n}\tanh(h_k^N) - NM_\sigma(n/N)\right| \le \frac{\sigma^2}{N}, \tag{3.14}$$

which allows to conclude.

□

Proposition 3.2. *With the definitions above, there exists a constant $C = C_K$ such that for every $N \ge 1$ and $\sigma \in [0,K]$*

$$\frac{1}{C\sqrt{N}}(\sigma \vee N^{-1/2})^2 \le \nu_N\left(S_{2N} = 0\ ;\ \forall n \in [\![1, 2N-1]\!],\ S_n > 0\right) \le \frac{C}{\sqrt{N}}(\sigma \vee N^{-1/2})^2. \tag{3.15}$$

Proof. First we show that we can find a constant C such that for every $\sigma \in [0,K]$

$$\frac{1}{C}N^{-1/2} \le \nu_N(S_{2N} = 0) \le CN^{-1/2}. \tag{3.16}$$

This follows from the proof of [Lab18, Lemma 11], a quick way to check is via Fourier transform. Grouping the increments of S_{2N} with opposite drifts we obtain (since $S_{2N} \in 2\mathbb{Z}$ we only need to average over an interval of length π)

$$\nu_N(S_{2N} = 0) = \frac{1}{\pi}\int_{[-\pi/2,\pi/2]\nu_N[e^{i\xi S_{2N}}]\mathrm{d}\xi} = \frac{1}{\pi}\int_{[-\pi/2,\pi/2]}\prod_{k=1}^{N}\left(1 - \alpha_{k,N}(1-\cos(2\xi))\right)\mathrm{d}\xi. \tag{3.17}$$

where

$$\alpha_{k,N} = 1 - \nu_N[X_k + X_{2N-k+1} = 0] = \frac{1}{2}\left(1 - \tanh^2(h_k^N)\right).$$

This shows that (3.17) is increasing in σ and we can obtain the upper and lower bounds by considering the cases $\sigma = K$ and $\sigma = 0$ respectively. This is then a standard computation to check that there exists a constant C (depending on K) such that for every $\xi \in [-\pi/2, \pi/2]$

$$e^{-CN|\xi|^2} \le \nu_N[e^{i\xi S_{2N}}] \le e^{-\frac{N}{C}|\xi|^2}, \tag{3.18}$$

and conclude. Another thing we can deduce from the above computation and using the fact that $S_{2N} - S_N$ is independent from S_N and has the same distribution as $-S_N$ is that

$$|\nu_N[e^{i\xi S_N}]|^2 = \nu_N[e^{i\xi S_{2N}}] \le e^{-\frac{N}{C}|\xi|^2}, \tag{3.19}$$

and thus we have

$$\nu_N(S_N = x) = \frac{1}{\pi}\int_{[-\pi/2,\pi/2]} \nu_N[e^{i\xi(S_N - x)}]\mathrm{d}\xi \le \frac{1}{\pi}\int_{[-\pi/2,\pi/2]} |\nu_N[e^{i\xi(S_N - x)}]|\mathrm{d}\xi \le CN^{-1/2}. \tag{3.20}$$

Our second observation uses the FKG inequality (cf. [Lac16b, Lemma 3.3]) for the measure $\mathbf{P}(\cdot \mid S_N = x)$. Note that for every $\sigma > 0$, the density of ν_N with respect to $\mathbf{P}$ is an increasing function for the natural partial order on S. Hence from the FKG inequality we have

$$\begin{aligned}\nu_N&\left(\forall n \in [\![1, 2N-1]\!],\ S_n > 0 \mid S_{2N} = 0\right)\\ &\ge \mathbf{P}\left(\forall n \in [\![1, 2N-1]\!],\ S_n > 0 \mid S_{2N} = 0\right) = \frac{1}{2(2N-1)}.\end{aligned} \tag{3.21}$$

where the last equality is easily obtained combining the reflection principle and some basic combinatorics (see e.g. [Dur10, Theorem 4.3.1]). Thus there is a constant for which for every $\sigma \in [0, K]$

$$\nu_N\left(S_{2N} = 0\ ;\ \forall n \in [\![1, 2N-1]\!],\ S_n > 0\right) \ge CN^{-3/2}. \tag{3.22}$$

As a consequence, we have to prove the lower bound in (3.15) only when $\sigma\sqrt{N}$ is large. Let $\widetilde{S}_n := S_{2N-n} - S_{2N}$. Note that $(\widetilde{S}_n)_{n=1}^N$ and $(S_n)_{n=1}^N$ are independent and identically distributed. Hence we have

$$\begin{aligned}\nu_N&\left(S_{2N} = 0\ ;\ \forall n \in [\![1, 2N-1]\!],\ S_n > 0\right)\\ &= \nu_N\left(S_N = \widetilde{S}_N\ ;\ \forall n \in [\![1, N]\!],\ S_n, \widetilde{S}_n > 0\right)\\ &= \sum_{x=1}^{N} \nu_N\left(S_N = x\ ;\ \forall n \in [\![1, N-1]\!],\ S_n > 0\right)^2.\end{aligned} \tag{3.23}$$

To obtain a lower-bound, the FKG inequality applied to the measure $\mathbf{P}(\cdot \mid S_N = x)$ yields

$$\nu_N\left(\forall n \in [\![1, N-1]\!],\ S_n > 0 \mid S_N = x\right) \ge \mathbf{P}\left(\forall n \in [\![1, N-1]\!],\ S_n > 0 \mid S_N = x\right) = \frac{x}{N}, \tag{3.24}$$

where the last equality is the ballot theorem. Now as we have for all $\sigma \in [0, K]$

$$\nu_N(S_N) = \sum_{k=1}^{N} \tanh(h_k^N) \ge c\sigma N \quad \text{and} \quad \mathrm{Var}_{\nu_N}(S_N) \le N. \tag{3.25}$$

and thus we obtain that

$$\nu_N(S_N \in \{|S_N - \nu_N(S_N)| \le \sqrt{2N}\}) \ge 1/2. \tag{3.26}$$

Hence assuming that $c\sigma N \geq 2\sqrt{2N}$ and using Cauchy-Schwartz inequality we have

$$\begin{aligned}\nu_N\left(S_{2N}=0\ ;\ \forall n\in [\![1,2N-1]\!],\ S_n>0\right) &\geq N^{-2}\sum_{|x-\nu_N(S_N)|\leq\sqrt{2N}}\nu_N(S_N=x)^2x^2\\ &\geq N^{-2}(c\sigma N-\sqrt{2N})^2\sum_{|x-\nu_N(S_N)|\leq\sqrt{2N}}\nu_N(S_N=x)^2\geq c'N^{-1/2}\sigma^2, \end{aligned}\tag{3.27}$$

which is the desired lower bound. For the upper-bound, we can assume that $\sigma\leq 1/20$ since in all other cases (3.16) is sufficient to conclude. Our aim is to prove that for every $x\geq 0$

$$\nu_N\left(\forall n\in[\![1,N-1]\!],\ S_n>0\mid S_N=x\right)\leq 10\left(\frac{x+2\sqrt{N}}{N}+\sigma\right).\tag{3.28}$$

This is trivial when $x\geq N/10$, so we may assume that $x\leq N/10$. We let ν_N^x the measure defined by adding an extra tilt at the end point setting

$$\frac{\mathrm{d}\nu_N^x}{\mathrm{d}\nu_N}(S)=\frac{1}{J_{x,N}}e^{\frac{3(x+\sqrt{N})S_N}{N}}\quad\text{ with } J_{x,N}=\nu_N\left(e^{\frac{3(x+\sqrt{N})S_N}{N}}\right).\tag{3.29}$$

The average of S_N under this alternative measure is given by

$$\nu_N^x(S_N)=\sum_{k=1}^N\tanh\left(h_k^N+\frac{3(x+\sqrt{N})}{N}\right)\geq\frac{\sigma N}{4}+2(x+\sqrt{N}).\tag{3.30}$$

Since the variance is smaller than N we have in particular $\nu_N^x(S_N\geq x)\geq 1/2$ and hence

$$\begin{aligned}\nu_N\left(\forall n\in[\![1,N-1]\!],\ S_n>0\mid S_N=x\right)&\leq\nu_N^x\left(\forall n\in[\![1,N-1]\!],\ S_n>0\mid S_N=x\right)\\ &\leq\nu_N^x\left(\forall n\in[\![1,N-1]\!],\ S_n>0\mid S_N\geq x\right)\leq 2\nu_N^x\left(\forall n\in[\![1,N]\!],\ S_n>0\right).\end{aligned}\tag{3.31}$$

To bound the last estimate, we can compare ν_N^x with $\mathbf{Q}_{N,x,\sigma}$ under which S is a simple random walk with constant tilt equal to $\frac{3(x+\sqrt{N})}{N}+\sigma$, that is, increments are IID and

$$\mathbf{Q}_{N,x,\sigma}(S_1=\pm1)=\frac{e^{\pm\left(\frac{3(x+\sqrt{N})}{N}+\sigma\right)}}{2\cosh\left(\frac{3(x+\sqrt{N})}{N}+\sigma\right)}.$$

We have

$$\nu_N^x\left(\forall n\in[\![1,N]\!],\ S_n>0\right)\leq\mathbf{Q}_{N,x,\sigma}\left(\forall n\in[\![1,N]\!],\ S_n>0\right)=\frac{1}{N}\mathbf{Q}_{N,x,\sigma}(S_N\vee 0).\tag{3.32}$$

The equality above is simply a consequence of the fact that by the ballot Theorem, for every $y\geq 0$

$$\mathbf{Q}_{N,x,\sigma}\left(\forall n\in[\![1,N]\!],\ S_n>0\mid S_N=y\right)=\frac{y}{N}.$$

Now we have (using Cauchy-Schwartz inequality, the inequality $\sqrt{a+b}\leq\sqrt{a}+\sqrt{b}$ and bounding the variance by N)

$$\begin{aligned}\mathbf{Q}_{N,x,\sigma}(S_N\vee 0)\leq\left(\mathbf{Q}_{N,x,\sigma}(S_N^2)\right)^{1/2}&\leq\mathbf{Q}_{N,x,\sigma}(S_N)+\sqrt{\mathrm{Var}_{\mathbf{Q}_{N,x,\sigma}}(S_N)}\\ &\leq N\tanh\left(\frac{3(x+\sqrt{N})}{N}+\sigma\right)+\sqrt{N}.\end{aligned}\tag{3.33}$$

The inequality (3.28) follows by combining (3.31) and (3.32). We are now ready to conclude our upper bound proof. Recall (3.23), and from (3.20) we have

$$\sum_{x=1}^{N} \nu_N\left(S_N = x\ ;\ \forall n \in [\![1, N-1]\!],\ S_n > 0\right)^2$$
$$\leq CN^{-1/2} \sum_{x=1}^{N} \nu_N\left(S_N = x\right) \nu_N\left(\forall n \in [\![1, N-1]\!],\ S_n > 0 \mid S_N = x\right)^2$$
$$\leq CN^{-1/2} \nu_N\left[\left(\frac{S_N + 2\sqrt{N}}{N} + \sigma\right)^2\right]. \quad (3.34)$$

where the second inequality is a direct consequence of (3.28). The upper bound in (3.15) then follows from our estimates on variance of S_N (3.25) and that on the expectation since from the explicit expression in (3.25) we can deduce that $\nu_N(S_N) \leq \sigma N$.

□

Now it remains to provide an upper bound on the partition function valid for every $\sigma > 0$ and $\lambda > 0$. We treat separately the cases $F(\lambda) \geq G(\sigma)$ and $G(\sigma) > F(\lambda)$.

3.2. The case when $F(\lambda) \geq G(\sigma)$. This subsection is devoted to the proof of the upper bound on the partition function when $F(\lambda) \geq G(\sigma)$, that is

Proposition 3.3. *When $G(\sigma) \leq F(\lambda)$ and $\lambda > 2$, there exists a constant $C(\lambda) > 0$, such that for all $N \geq 1$,*

$$Z_N(\lambda, \sigma) \leq C(\lambda) \exp\left(2NF(\lambda)\right). \quad (3.35)$$

Moreover when $G(\sigma) < F(\lambda)$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all N sufficiently large,

$$\mu_N\left(L_{\max}(\xi) \geq \varepsilon N\right) \leq e^{-\delta N}. \quad (3.36)$$

When $G(\sigma) = F(\lambda)$, for all $N \geq N_0(\varepsilon)$ sufficiently large we have

$$\begin{aligned} &\mu_N\left(L_{\max}(\xi) \in [\varepsilon N, (1-\varepsilon)N]\right) \leq e^{-\delta N}, \\ &\frac{1}{C(\lambda)\sqrt{N}} \leq \mu_N\left(L_{\max}(\xi) > (1-\varepsilon)N\right) \leq \frac{C(\lambda)}{\sqrt{N}}. \end{aligned} \quad (3.37)$$

We provide a proof for Proposition 3.3 from the viewpoint of renewal process. For simplicity of notations, for each $n \in [\![1, N]\!]$, set

$$\begin{aligned} K(n) &:= \mathbf{P}\left(S_{2n} = 0; \forall k \in [\![1, 2n-1]\!], S_k > 0\right), \\ \widetilde{K}(n) &:= \lambda e^{-2nF(\lambda)} Z_n\left(0, \frac{n\sigma}{N}\right). \end{aligned} \quad (3.38)$$

Note that with this definition, we have from (3.1)

$$\lambda e^{-2NF(\lambda)} Z_N(\lambda, \sigma) = \sum_{k=1}^{N} \sum_{\substack{(n_1, \ldots, n_k) \\ \sum_{i=1}^k n_i = N}} \prod_{i=1}^{k} \widetilde{K}(n_i). \quad (3.39)$$

The key point here is that with our assumption, $\widetilde{K}(n)$ almost sums to 1 and thus can be interpreted as the interarrival law of a renewal process.

LEMMA 3.4. *When $G(\sigma) < F(\lambda)$, there exists a constant $C(\lambda,\sigma) > 0$ such that for all $N \geq 1$,*

$$\sum_{n=1}^{N} \widetilde{K}(n) \leq 1 + \frac{C(\lambda,\sigma)}{N}. \tag{3.40}$$

When $0 < G(\sigma) = F(\lambda)$, for every given $\varepsilon \in (0, \frac{1}{2})$ there exists a constant $C(\lambda)$ such that for all $N \geq N_0(\varepsilon)$ sufficiently large,

$$\sum_{n=1}^{(1-\varepsilon)N} \widetilde{K}(n) \leq 1 + \frac{C(\lambda)}{N}. \tag{3.41}$$

PROOF OF PROPOSITION 3.3 FROM LEMMA 3.4. By monotonicity in σ, it is sufficient to treat the case $G(\sigma) = F(\lambda)$. For pedagogical reason however we start with the easier case $G(\sigma) < F(\lambda)$ (and a slightly weak-statement see below). We set

$$\widehat{K}(n) := \widetilde{K}(n) / \sum_{m=1}^{N} \widetilde{K}(m) \tag{3.42}$$

and let $\widehat{\mathbf{P}}$ denote the law of a renewal process τ starting from zero with interarrival law $\widehat{K}$. That is a increasing sequence $(\tau_k)_{k\geq 0}$ with IID increments whose distribution is given by $\widehat{K}(n)$. We also consider τ as a subset of $\mathbb{N}$ and write $\{N \in \tau\}$ for $\{\exists k \geq 0, \tau_k = N\}$. We have from (3.39)

$$\begin{aligned}\lambda e^{-2NF(\lambda)} Z_N(\lambda,\sigma) &= \sum_{k=1}^{N} \left(\sum_{m=1}^{N} \widetilde{K}(m) \right)^k \sum_{\substack{(n_1,\dots,n_k) \\ \sum_{i=1}^k n_i = N}} \prod_{i=1}^{k} \widehat{K}(n_i) \\ &\leq \left(1 \vee \sum_{m=1}^{N} \widetilde{K}(m) \right)^N \widehat{\mathbf{P}}\left(N \in \tau\right) \leq e^{C(\lambda,\sigma)} \end{aligned} \tag{3.43}$$

where the last inequality uses Lemma 3.4 (and the fact that a probability is always smaller than one). Note that this does not provide a full proof of (3.35) since the constant in the upper bound *does depend* on σ.

Let us now treat the case $G(\sigma) = F(\lambda)$. For a given $\varepsilon \in (0, \frac{1}{2})$, we redefine

$$\widehat{K}(n) := \widetilde{K}(n) \mathbf{1}_{\{n \leq (1-\varepsilon)N\}} / \left(\sum_{1 \leq m \leq (1-\varepsilon)N} \widetilde{K}(m) \right). \tag{3.44}$$

and update the definition of $\widehat{\mathbf{P}}$ accordingly. Now we can make a computation similar to (3.43) but including possibly one long jump. We obtain (we have put in the factor term $e^{C(\lambda)}$ which accounts for the fact that the $\widetilde{K}$ do not sum to one.)

$$\begin{aligned}\lambda e^{-2NF(\lambda)} Z_N(\lambda,\sigma) &\leq e^{C(\lambda)} \left[\widehat{\mathbf{P}}\left(N \in \tau\right) + \sum_{\substack{a,b \in [\![0,N]\!] \\ b-a > (1-\varepsilon)N}} \widehat{\mathbf{P}}\left(a \in \tau\right) \widetilde{K}(b-a) \widehat{\mathbf{P}}\left(N-b \in \tau\right) \right] \\ &\leq e^{C(\lambda)} \left(1 + \sum_{\substack{a,b \in [\![0,N]\!] \\ b-a > (1-\varepsilon)N}} \widetilde{K}(b-a) \right) \leq e^{C(\lambda)} \left(1 + \frac{C'(\lambda)}{\sqrt{N}} \right). \end{aligned} \tag{3.45}$$

To obtain the last inequality, note that as $G(\sigma)=F(\lambda)$, by Proposition 3.1 we have

$$\widetilde{K}(n) \leq \frac{\lambda C}{\sqrt{N}} e^{2n(G(n\sigma/N)-G(\sigma))} \leq \frac{\lambda C}{\sqrt{N}} e^{-\frac{2n(N-n)}{N}G(\sigma)},$$

where the last inequality follows by convexity of G. Summed over a and b this yields the adequate $C'(\lambda)/\sqrt{N}$ term (since we are on the critical line, σ is a function of λ).

Let us now turn to the proof the statements concerning the length of the largest excursion $L_{\max}$. When $F(\lambda)>G(\sigma)$, repeating (3.39) but summing over ξ displaying a large jump we have

$$\mu_N(L_{\max}(\xi)\geq \varepsilon N) \leq \frac{C(\lambda,\sigma)\widehat{\mathbf{P}}(L_{\max}(\tau)\geq \varepsilon N;\ N\in\tau)}{e^{-2NF(\lambda)}Z_N(\lambda,\sigma)}, \tag{3.46}$$

where $L_{\max}(\tau) := \max\{|\tau_k-\tau_{k-1}| \ : \tau_k \leq N\}$ is the largest inter-arrival before N in the renewal sequence. The denominator in the r.h.s. in (3.46) is larger than $e^{-2NF(\lambda)}Z_N(\lambda,0)$ which according to (2.9) is of constant order. It remains to show that the denominator is exponentially small. We have

$$\widehat{\mathbf{P}}(L_{\max}(\tau)\geq \varepsilon N;\ N\in\tau) \leq N\widehat{\mathbf{P}}(\tau_1\geq \varepsilon N) \leq \frac{N}{\widetilde{K}(1)}\sum_{n=\varepsilon N}^{N}\widetilde{K}(n). \tag{3.47}$$

Now from (3.7)-(3.9) and the definition of $\widetilde{K}$, we have

$$\widetilde{K}(n) \leq \lambda e^{2n\left(G(\frac{\sigma n}{N})-F(\lambda)\right)+\frac{\sigma^2 n}{4N^2}} \leq C(\lambda,\sigma)e^{2n(G(\sigma)-F(\lambda))} \tag{3.48}$$

and hence it decays exponentially, and so does the sum in (3.47). When $F(\lambda)=G(\sigma)$, we proceed similarly and we only have to show that (for the renewal defined in (3.44))

$$\widehat{\mathbf{P}}(L_{\max}(\tau)\in[\varepsilon N,(1-\varepsilon)N]\ ;\ N\in\tau) \leq e^{-\delta N}. \tag{3.49}$$

We use (3.48) and $G(\frac{n}{N}\sigma)\leq G((1-\varepsilon)\sigma)$ for all $n\leq(1-\varepsilon)N$ to obtain

$$\widehat{\mathbf{P}}(\tau_1\in[\varepsilon N,(1-\varepsilon)N]) \leq \frac{1}{\widetilde{K}(1)}\sum_{\varepsilon N\leq n\leq(1-\varepsilon)N}\widetilde{K}(n) \leq C(\lambda,\sigma)e^{-2\varepsilon N(F(\lambda)-G((1-\varepsilon)\sigma))}. \tag{3.50}$$

Finally to estimate (from above and below) the probability of having long jumps when $F(\lambda)=G(\sigma)$ (in that case the value of σ is determined by that of λ) we first observe that from Proposition 3.1 and (3.35) we have

$$\mu_N(L_{\max}(\xi)=N) = \frac{Z_N(0,\sigma)}{Z_N(\lambda,\sigma)} \geq \frac{1}{C(\lambda)\sqrt{N}}.$$

For the upper-bound, we observe that in (3.45), the contribution of jumps larger than $(1-\varepsilon)N$ is given by the sum over a and b and this readily implies that for all $N\geq N_0(\varepsilon)$

$$\mu_N(L_{\max}(\xi)>(1-\varepsilon)N) \leq \frac{C(\lambda)}{\sqrt{N}}. \tag{3.51}$$

□

Proof of Lemma 3.4. Recall the notations $K(n)$ and $\widetilde{K}(n)$ in (3.38). By [Gia07, Equation (1.6)] we know that

$$\sum_{n=1}^{\infty}\lambda K(n)e^{-2nF(\lambda)} = 1. \tag{3.52}$$

Moreover, there exists a universal constant $C_0>0$ such that for all $n\geq 1$,

$$C_0^{-1}n^{-3/2} \leq K(n) \leq C_0 n^{-3/2}. \tag{3.53}$$

We are going to use different estimates for $\widetilde{K}(n)$ depending on whether n is small or large. We adopt the same notation as in the proof of Proposition 3.1, S being a simple random walk and A_n being the area between its graph and the x axis (see Equation (3.4) and above). For small values of n, we observe that since $A_n(S) \le n^2$ when $S_{2n} = 0$ we have

$$\widetilde{K}(n) = \lambda e^{-2nF(\lambda)} \mathbf{E}\left[e^{\frac{\sigma A_n(S)}{N}} \mathbf{1}_{\{S_1>0,\cdots,S_{2n-1}>0,S_{2n}=0\}}\right] \le \lambda e^{-2nF(\lambda)} e^{\frac{\sigma n^2}{N}} K(n). \tag{3.54}$$

Using (3.52), and the bounds $K(n) \le 1$ and $e^u - 1 \le 2u$ for $u \le 1$ we obtain for large values of N

$$\begin{aligned}\sum_{n=1}^{\sqrt{N/\sigma}} \widetilde{K}(n) - 1 &\le \sum_{n=1}^{\sqrt{N/\sigma}} \left(\widetilde{K}(n) - \lambda e^{-2nF(\lambda)} K(n)\right) \\ &\le \sum_{n=1}^{\sqrt{N/\sigma}} \lambda K(n) e^{-2nF(\lambda)} \left(e^{\frac{\sigma n^2}{N}} - 1\right) \le \lambda \sum_{n=1}^{\sqrt{N/\sigma}} e^{-2nF(\lambda)} \frac{2\sigma n^2}{N} \le \frac{\sigma C(\lambda)}{N}. \end{aligned} \tag{3.55}$$

For large values of n we rely on (3.48). When $G(\sigma) < F(\lambda)$, we bound $G(\frac{\sigma n}{N})$ by $G(\sigma)$. Using this we obtain

$$\sum_{n=\sqrt{N/\sigma}+1}^{N} \widetilde{K}(n) \le \sum_{n \ge \sqrt{N/\sigma}+1} C\lambda e^{2n(G(\sigma)-F(\lambda))} \le C'(\lambda,\sigma) e^{-2\sqrt{N/\sigma}(F(\lambda)-G(\sigma))} \le \frac{C'}{N}. \tag{3.56}$$

When $G(\sigma) = F(\lambda)$, we bound $G(\frac{n}{N}\sigma)$ by $G((1-\varepsilon)\sigma)$ for $n \le (1-\varepsilon)N$ which is sufficient to conclude. □

3.3. The case $G(\sigma) > F(\lambda)$.

Our objective in this section is to prove:

PROPOSITION 3.5. *If $G(\sigma) > F(\lambda)$, then there exists a constant $C(\lambda, \sigma)$ such that for every N we have*

$$Z_N(\lambda,\sigma) \le \frac{C(\lambda,\sigma)}{\sqrt{N}} e^{2NG(\sigma)}. \tag{3.57}$$

On top of this, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that for all N sufficiently large we have

$$\mu_N^{\lambda,\sigma}(L_{\max}(\xi) \le (1-\varepsilon)N) \le e^{-\delta N}. \tag{3.58}$$

PROOF. Observe that if $0 \le \lambda \le \lambda'$, we have

$$\begin{aligned} Z_N(\lambda,\sigma) &\le Z_N(\lambda',\sigma), \\ \mu_N^{\lambda,\sigma}(L_{\max}(\xi) \le (1-\varepsilon)N) &\le \mu_N^{\lambda',\sigma}(L_{\max}(\xi) \le (1-\varepsilon)N), \end{aligned} \tag{3.59}$$

where the last inequality can be proved by FKG inequality (cf. [Lac16b, Lemma 3.3]). Therefore, it is sufficient to prove the statements for $\lambda > 2$. To prove (3.57), we fix $\varepsilon_0 := \varepsilon_0(\lambda,\sigma) > 0$ (not related to the ε in (3.58)) sufficiently small such that

$$F(\lambda) \le G((1-\varepsilon_0)\sigma). \tag{3.60}$$

From Lemma 3.4, we have

$$\sum_{m=1}^{\varepsilon_0 N} \widetilde{K}(m) \le 1 + \frac{C(\lambda,\sigma)}{N}. \tag{3.61}$$

In order to estimate the partition function we are going to split the trajectories according to where the position of jumps larger than $\varepsilon_0 N$ away from the x-axis are located. Starting with (3.1), letting $\mathbf{l} = (l_1, \ldots, l_k)$ denote the length of those jumps and $\mathbf{m} = (m_0, \ldots, m_k)$ the space between those jumps, we have (similarly to (3.45))

$$Z_N(\lambda,\sigma) \le \left(1 \vee \sum_{m=1}^{\varepsilon_0 N} \widetilde{K}(m)\right)^N \sum_{k=0}^{\infty} \lambda^{k-1} \sum_{(\mathbf{l},\mathbf{m})\in\mathcal{A}_{N,k}^{(\varepsilon_0)}} \prod_{i=0}^{k} e^{2m_i F(\lambda)} \widehat{\mathbf{P}}(m_i \in \tau) \prod_{j=1}^{k} Z_{l_j}\left(0, \frac{\sigma l_j}{N}\right). \quad (3.62)$$

where

$$\mathcal{A}_{N,k}^{(\varepsilon_0)} := \left\{ [(l_j)_{j=1}^k, (m_i)_{i=0}^k] \in \mathbb{Z}_+^{2k+1} : \forall j \in [\![1,k]\!],\ l_j \ge \varepsilon_0 N \text{ and } \sum_{i=0}^k m_i + \sum_{j=1}^k l_j = N \right\}. \quad (3.63)$$

Bounding above the probabilites by 1, and using the fact only $k \le \varepsilon_0^{-1}$ are positive, we obtain that

$$Z_N(\lambda,\sigma) \le e^{C(\lambda,\sigma)} \sum_{k=0}^{\varepsilon_0^{-1}} \lambda^{k-1} \sum_{(\mathbf{l},\mathbf{m})\in\mathcal{A}_{N,k}^{(\varepsilon_0)}} e^{\sum_{i=0}^k 2m_i F(\lambda)} \prod_{j=1}^{k} Z_{l_j}\left(0, \frac{\sigma l_j}{N}\right) =: e^{C(\lambda,\sigma)} \sum_{k=0}^{\varepsilon_0^{-1}} \lambda^{k-1} Z_{N,k}. \quad (3.64)$$

We are going to show first that the contribution of $k = 0$ and $k \ge 2$ in the above sum are small. We have $Z_{N,0} = e^{2NF(\lambda)}$. For $k \ge 2$, we simply use the fact that $\#\mathcal{A}_{N,k} \le N^{2k+1}$ and (3.15) to obtain that

$$Z_{N,k} \le C_\sigma^k N^{2k+1} e^{\sum_{i=0}^k 2m_i F(\lambda) + \sum_{j=1}^k 2l_j G\left(\frac{l_j\sigma}{N}\right)} \le C_\sigma^k N^{2k+1} e^{2NG((1-\varepsilon_0)\sigma)}, \quad (3.65)$$

where the second inequality uses only the fact that $l_j/N \le (1-\varepsilon_0)$ and the assumption in (3.60). Finally for the case $k = 1$ we have

$$\begin{aligned} Z_{N,k} \le & \sum_{\substack{m_0,m_1 \\ m_0+m_1<N(1-\varepsilon_0)}} e^{2(m_0+m_1)F(\lambda)} Z_{N-m_0-m_1}(0,\sigma) \\ & \le Ce^{2NG(\sigma)} \sum_{\substack{m_0,m_1 \\ m_0+m_1<N(1-\varepsilon_0)}} \frac{e^{-2(m_0+m_1)[G(\sigma)-F(\lambda)]}}{\sqrt{N-m_0-m_1}}, \quad (3.66) \end{aligned}$$

and we conclude that the last sum is bounded above by $CN^{-1/2}$ since $F(\lambda) < G(\sigma)$.

Now we move to provide an upper bound on $\mu_N(L_{\max}(\xi) \le (1-\varepsilon)N)$. We need to estimate $Z_N(\lambda,\sigma)\mu_N(L_{\max}(\xi) \le (1-\varepsilon)N)$. Using the decomposition above with $Z_{N,0} = e^{2NF(\lambda)}$ and (3.65) to bound the contribution of $k \ge 2$, it remains to to estimate the contribution corresponds to case $k = 1$ and $\varepsilon N \le (m_0 + m_1) \le (1-\varepsilon_0)N$,

$$\begin{aligned} \sum_{\substack{m_0,m_1 \\ \varepsilon N \le (m_0+m_1) \le (1-\varepsilon_0)N}} & e^{2(m_0+m_1)F(\lambda)} Z_{N-m_0-m_1}\left(0, \sigma\frac{N-(m_0+m_1)}{N}\right) \\ & \le C_\sigma N^2 \exp\left(2NG((1-\varepsilon\wedge\varepsilon_0)\sigma)\right), \quad (3.67) \end{aligned}$$

where we use the assumption (3.60) and bound $Z_n(0, \sigma\frac{n}{N})$ by $Z_n(0,(1-\varepsilon)\sigma)$ for all $n \le (1-\varepsilon)N$. The above inequality together with $Z_N(\lambda,\sigma) \ge Z_N(0,\sigma)$ and the lower-bound in (3.2) allows to conclude.

□

3.4. Proof of Proposition 2.1 and Theorem 2.4. Let us first check that the combination of the previous statements yield Proposition 2.1. Proposition 3.3 and Proposition 3.5 give the desired upper bound on the partition function. Concerning the lower bound, we have by monotonicity for every $\lambda, \sigma \geq 0$

$$Z_N(\lambda,\sigma) \geq \max\left(Z_N(\lambda,0), Z_N(0,\sigma)\right), \tag{3.68}$$

and thus the lower bounds in (2.15)-(2.16) are a direct consequence of Proposition 3.1 and (2.9).

Let us now turn to Theorem 2.4 which requires a bit more work. The statements in (2.21) and (2.22) are proved in Proposition 3.3 and we are left with the proof of (2.20) and (2.23). We focus on (2.20), the proof of (2.23) follows along the same line, and we leave it to the reader. Since we have already proven the statement in the case $\sigma = 0$, our strategy is to reduce ourselves to this case, by conditioning on the size of the unpinned region appearing in bulk of the system (which we have proved to be of size $N(1-o(1))$ (cf. Proposition 3.1 and Proposition 3.5). Let us set

$$\begin{aligned} L(\xi) &:= \sup\left\{k \leq N : \xi_k = 0\right\}, \\ R(\xi) &:= \inf\left\{k \geq N : \xi_k = 0\right\}. \end{aligned} \tag{3.69}$$

We fix $\varepsilon' > 0$ sufficiently small in a way that depends on ε and not on N (we will mention the requirement along the proof). We have

$$\begin{aligned} &\mu_N^{\lambda,\sigma}\left(\sup_{u\in[0,2]}\left|\frac{1}{N}\xi_{\lceil uN\rceil} - M_\sigma(u)\right| > \varepsilon\right) \\ &\qquad \leq \max_{\substack{\ell,r\in[\![0,2N]\!] \\ r-\ell\geq 2N(1-\varepsilon')}} \mu_N^{\lambda,\sigma}\left(\sup_{u\in[0,2]}\left|\frac{1}{N}\xi_{\lceil uN\rceil} - M_\sigma(u)\right| > \varepsilon \;\middle|\; L(\xi)=\ell, R(\xi)=r\right) \\ &\qquad\qquad + \mu_N^{\lambda,\sigma}\left(L_{\max}(\xi) \leq (1-\varepsilon')N\right). \end{aligned} \tag{3.70}$$

The second term is exponentially small by Proposition 3.5. Concerning the first term in the r.h.s. of (3.70), we observe that for ε' sufficiently small we have with probability one

$$\forall s \notin [0,2\ell]\cup[2r,2N], \quad \left|\frac{1}{N}\xi_s - M_\sigma(s/N)\right| \leq \varepsilon,$$

simply because both functions are $1/N$-Lipschitz. Setting $\bar N = (r-\ell)/2$ and $\bar\sigma := \frac{r-\ell}{2N}\sigma$ we only have to look at the middle part of the path which after conditioning has distribution $\mu_{\bar N}^{0,\bar\sigma}$. Hence we need to estimate

$$\mu_{\bar N}^{0,\bar\sigma}\left(\sup_{u\in[0,2]}\left|\frac{1}{\bar N}\xi_{\lceil u\bar N\rceil} - \frac{N}{\bar N}M_\sigma\left(\frac{\ell}{2\bar N} + \frac{u\bar N}{N}\right)\right| > \frac{\varepsilon N}{\bar N}\right). \tag{3.71}$$

Choosing ε' small we can ensure that

$$\sup_{u\in[0,2]}\left|\frac{N}{\bar N}M_\sigma\left(\frac{\ell}{\bar N} + \frac{u\bar N}{N}\right) - M_{\bar\sigma}(u)\right| \leq \varepsilon/2 \tag{3.72}$$

and we obtain that the term in the max in the r.h.s of (3.70) is smaller than

$$\mu_{\bar N}^{0,\bar\sigma}\left(\sup_{u\in[0,2]}\left|\frac{1}{\bar N}\xi_{\lceil u\bar N\rceil} - M_{\bar\sigma}(u)\right| > \varepsilon/2\right) \tag{3.73}$$

which is exponentially small from Proposition 3.1 (recall that $\bar N \geq N/2$).

□

4. Bottleneck identification and lower bound on the relaxation time

4.1. Heuristics. In order to understand Theorem 2.7, let us explain heuristically what makes the systems mixing slowly when $E(\lambda,\sigma) > 0$. For this we have to describe the most likely pattern that the system uses to relax to equilibrium.

In the case where $F(\lambda) \geq G(\sigma)$ which might be the more illustrative. Since at equilibrium the interface is pinned, the configuration which is the further away from the x-axis (that is $\xi^{\max}_x = x\wedge(2N-x)$) should be the furthest away from equilibrium. In order to reach equilibrium, ξ needs to pin itself entirely on the wall, and the most likely way to do so is to shrink the unpinned region, "continuously" (that is, in a way that appears continuous in the large N limit) moving the extremities of the unpinned region inwards. When $G(\sigma) > F(\lambda)$ the pattern should be simply the opposite: we start from the bottommost configuration and try to grow an unpinned bubble from the bulk of the interface until it reaches one of the extremities.

Following this strategy, for any $\beta \in (0,1)$ the dynamics must display at some point an unpinned region of length $2\beta N(1+o(1))$ and a pinned region of length $2(1-\beta)N(1+o(1))$. From Proposition 3.1, we can heuristically infer that the contribution to the partition function of configurations with an unpinned proportion β is, on the exponential scale, of order

$$\exp\left(2N[\beta G(\beta\sigma) + (1-\beta)F(\lambda)]\right).$$

Hence in order to understand relaxation to equilibrium, we need to study the function

$$\beta \mapsto -\beta G(\beta\sigma) - (1-\beta)F(\lambda)$$

corresponding to the effective energy for a system constrained on having a large unpinned region of relative size $2\beta N$. This function admits a local maximum inside the interval $[0,1]$ if and only if the equation $G(\beta\sigma)+\beta\sigma G'(\beta\sigma) = F(\lambda)$ admits a solution in $(0,1)$ which in turn occurs if and only if $G(\sigma)+\sigma G'(\sigma) > F(\lambda)$.

When $G(\sigma)+\sigma G'(\sigma) \leq F(\lambda)$, when diminishing β from 1 to 0, the effective energy $-\beta G(\beta\sigma)-(1-\beta)F(\lambda)$ only decreases (see Figure 6) indicating that the system should mix rapidly.

When $G(\sigma)+\sigma G'(\sigma) > F(\lambda)$, on the contrary in order to from β to go from 1 to 0 (if $F(\lambda) \geq G(\sigma)$) or 0 to 1 (if $F(\lambda) < G(\sigma)$), it needs to overcome an energy barrier. The height of the energy barrier to overcome is exactly $2NE(\lambda,\sigma)$ (see Figure 6) which yields a heuristic justification for having a mixing time of order $e^{2NE(\lambda,\sigma)}$.

Transforming this heuristic into a rigorous lower-bound on the mixing time is the easier part of the argument. Indeed the value β^* which maximizes the effective energy should correspond to a bottleneck in the system in the sense given in [LP17, Section 7.2]. Getting a lower bound on the mixing time from the bottleneck ratio is then a very standard and direct computation (cf. [LP17, Theorem 7.4]).

The upper-bound is more delicate. The strategy above assumes that only one unpinned region is formed and that the size of that unpinned region is the only relevant parameter for the estimate of the relaxation time. In order to obtain an upper bound, without proving these claim directly, we will use a set of techniques (induction, chain reduction, path-method...) which allows to circumvent these issues.

4.2. Lower bound on the relaxation time. The goal of this subsection is to prove the following result.

PROPOSITION 4.1. *Let us assume that $\sigma > 0$. Then if $E(\lambda,\sigma) > 0$, then for all $N \geq 1$, we have*

$$T_{\mathrm{rel}}^N(\lambda,\sigma) \geq \frac{c(\lambda,\sigma)}{N^2}\exp(2NE(\lambda,\sigma)), \tag{4.1}$$

where $E(\lambda,\sigma)$ is defined in (2.25). *Moreover, if $E(\lambda,\sigma) = 0$, then*

$$T_{\mathrm{rel}}^N(\lambda,\sigma) \geq c(\lambda,\sigma)N. \tag{4.2}$$

To obtain (4.1), we simply evaluate the minimized quantity (2.7) for a function f which is the indicator of our bottleneck event $f := \mathbf{1}_{\mathcal{E}_N^1}$ where $\mathcal{E}_N^1$ is defined in (2.32). To estimate the Dirichlet form of this function we need to introduce the internal boundary of $\mathcal{E}_N^1$ defined by

$$\partial\mathcal{E}_N^1 := \left\{\xi \in \mathcal{E}_N^1 : \exists x \in [\![1, 2N-1]\!], \xi^x \notin \mathcal{E}_N^1\right\}, \tag{4.3}$$

and set for any event $B \subset \Omega_N$

$$\mathbf{Z}(B) = \mathbf{Z}_{\lambda,\sigma}(B) := \mu_N(B)Z_N(\lambda,\sigma) = \sum_{\xi\in B} 2^{-2N}\lambda^{H(\xi)}\exp\left(\tfrac{\sigma}{N}A(\xi)\right). \tag{4.4}$$

The more important computation in this section is the estimate of the relative weight of each of the $\mathcal{E}_N^i$ and of the boundary separating them.

PROPOSITION 4.2. *If $E(\lambda,\sigma) > 0$, then there exists a constant $C = C(\lambda,\sigma)$ such that for every $N \geq 1$*

$$\begin{aligned} C^{-1} \leq& \mathbf{Z}\left(\mathcal{E}_N^1\right)e^{-2NF(\lambda)} \leq C,\\ C^{-1} \leq& N^{1/2}\mathbf{Z}\left(\mathcal{E}_N^2\right)e^{-2NG(\sigma)} \leq C. \end{aligned} \tag{4.5}$$

Furthermore, we have

$$\frac{1}{C} \leq \frac{\mathbf{Z}\left(\partial\mathcal{E}_N^1\right)}{\sqrt{N}e^{2\beta^*NG(\beta^*\sigma)+2N(1-\beta^*)F(\lambda)}} \leq C. \tag{4.6}$$

PROOF OF PROPOSITION 4.1. We first deal with the case $E(\lambda,\sigma) > 0$. By definition, we know that $\mathrm{Var}_{\mu_N}(f) = \mu_N\left(\mathcal{E}_N^1\right)\mu_N\left(\mathcal{E}_N^2\right)$ and $\mathcal{E}(f) \leq 2N\mu_N\left(\partial\mathcal{E}_N^1\right)$, where the last inequality uses the fact that $\sum_{\xi'\in\Omega_N} r_N(\xi,\xi') \leq 2N$ for all $\xi \in \Omega_N$. Thus, we have

$$T_{\mathrm{rel}}^N(\lambda,\sigma) \geq \frac{\mu_N\left(\mathcal{E}_N^1\right)\mu_N\left(\mathcal{E}_N^2\right)}{2N\mu_N\left(\partial\mathcal{E}_N^1\right)} = \frac{\mathbf{Z}\left(\mathcal{E}_N^1\right)\mathbf{Z}\left(\mathcal{E}_N^2\right)}{2N\mathbf{Z}\left(\partial\mathcal{E}_N^1\right)Z_N(\lambda,\sigma)}. \tag{4.7}$$

Therefore, by Proposition 4.2 and Proposition 2.1 we have

$$T_{\mathrm{rel}}^N(\lambda,\sigma) \geq \frac{1}{CN^2}e^{2NE(\lambda,\sigma)}. \tag{4.8}$$

We move to the case $E(\lambda,\sigma) = 0$ and adopt the strategy of [CMT08, Proposition 5.1]. We plug the test function $f_a(\xi) = \exp(\frac{a}{N}\sum_{x=1}^{2N}\xi_x)$ with $a > 0$ in (2.7) and estimate the Dirichlet form for f_a. Since $|Q_x(f_a) - f_a| \leq \frac{C}{N}f_a$ for all $x \in [\![1, 2N]\!]$, we have

$$\mathcal{E}(f_a) \leq \frac{2C^2}{N}\mu_N(f_a^2),$$

and then

$$T_{\mathrm{rel}}^N(\lambda,\sigma) \geq \frac{\mu_N(f_a^2) - \mu_N(f_a)^2}{\frac{2C^2}{N}\mu_N(f_a^2)} = \frac{N}{2C^2}\left(1 - \frac{Z_N(\lambda,\sigma+a)^2}{Z_N(\lambda,\sigma)Z_N(\lambda,\sigma+2a)}\right). \tag{4.9}$$

By Proposition 2.1, we choose the constant a such that $G(\sigma+a) \leq F(\lambda) < G(\sigma+2a)$, and then the r.h.s. of (4.9) is larger than or equal to

$$\frac{N}{2C^2}\left(1-\exp(-cN)\right),$$

which allows us to conclude.

□

PROOF OF PROPOSITION 4.2. Recalling that β^* is the unique solution of (2.26), we have $G(\sigma\beta^*) < F(\lambda)$. Using this observation, using the definition (3.38) we have from the proof of Lemma 3.4 that for every $N \geq \sigma$

$$\sum_{n=1}^{\beta^* N} \widetilde{K}(n) \leq 1 + \frac{\sigma C(\lambda)}{N}. \tag{4.10}$$

Indeed (3.55) yields the right-bound for the summation over $1 \leq n \leq \sqrt{N/\sigma}$, it is then sufficient to replace N by $\beta^* N$ in (3.56) and use the first inequality in (3.48) to obtain

$$\sum_{n=\sqrt{N/\sigma}+1}^{\beta^* N} \widetilde{K}(n) \leq \sum_{n=\sqrt{N/\sigma}+1}^{\beta^* N} \lambda e^{2n\left[(G(\beta^*\sigma)-F(\lambda))+\frac{\sigma^2}{N^2}\right]} \leq C'(\lambda) e^{-c(\lambda)\sqrt{N/\sigma}}. \tag{4.11}$$

For the last inequality above, we simply have observed that $\sigma\beta^*$ depends only on λ. Now we start with a decomposition in (3.1) and proceed as in the proof of Proposition 3.3 to obtain

$$\mathbf{Z}(\mathcal{E}_N^1) = \sum_{k\geq 1} \sum_{\substack{n_1,\dots,n_k \\ \sum_{i=1}^k n_i = N \\ n_i \leq \beta^* N}} \lambda^{k-1} \prod_{i=1}^k Z_{n_i}\left(0, \frac{\sigma n_i}{N}\right) \leq \lambda^{-1} e^{2NF(\lambda)} \left(1+\frac{C}{N}\right)^N \widehat{\mathbf{P}}[N \in \widehat{\tau}] \leq C' e^{2NF(\lambda)}, \tag{4.12}$$

where $\widehat{\tau}$ is a renewal with interarrival law

$$\widehat{K}(n) = \widetilde{K}(n) \mathbf{1}_{\{n \leq \beta^* N\}} \Big/ \left(\sum_{m=1}^{\beta^* N} \widetilde{K}(m)\right). \tag{4.13}$$

For the lower bound, observe that by monotonicity for any $\varepsilon > 0$ (hence in particular for $\varepsilon = \beta^*(\lambda,\sigma)$)

$$\mathbf{Z}_{\lambda,\sigma}(L_{\max} \leq \varepsilon N) \geq \mathbf{Z}_{\lambda,0}(L_{\max} \leq \varepsilon N) = \mu_N^{\lambda,0}(L_{\max} \leq \varepsilon N) Z_N(\lambda, 0),$$

and we can then use (2.9) and (2.21) (in the easier case $\sigma = 0$) to conclude.

For $\mathbf{Z}(\mathcal{E}_N^2)$ we first notice that by Proposition 3.1, we have

$$\mathbf{Z}(\mathcal{E}_N^2) \geq Z_N(0,\sigma) \geq \frac{1}{C_\sigma \sqrt{N}} e^{2NG(\sigma)}. \tag{4.14}$$

and thus we can focus on the proof of the upper bound.

We proceed as for (3.62), but with a threshold at size $\beta^* N$ for big jumps. We have

$$\mathbf{Z}(\mathcal{E}_N^2) \leq \left(1+\frac{C}{N}\right)^N \sum_{k=1}^{\infty} \lambda^{k-1} \sum_{(\mathbf{l},\mathbf{m}) \in \mathcal{A}_{N,k}^{(\beta^*)}} \prod_{i=0}^k e^{2m_i F(\lambda)} \widehat{\mathbf{P}}(m_i \in \tau) \prod_{j=1}^k Z_{l_j}\left(0, \frac{\sigma l_j}{N}\right) \tag{4.15}$$

with $\mathcal{A}_{N,k}^{(\beta^*)}$ defined in (3.63). Let us first control the contribution to the sum of the $k=1$ term. Using (3.15) it is bounded above by

$$C_\sigma (N\beta^*)^{-1/2} \sum_{\substack{m_0,m_1\\ m_0+m_1\le N(1-\beta^*)}} e^{2(N-m_0-m_1)G\left(\sigma\left(1-\frac{m_0+m_1}{N}\right)\right)+2(m_0+m_1)F(\lambda)} \le C(\lambda,\sigma)N^{-1/2}e^{2NG(\sigma)} \tag{4.16}$$

where the last inequality is a consequence of the fact that when $m_0+m_1\le N(1-\beta^*)$ then

$$(N-m_0-m_1)G\left(\sigma\left(1-\frac{m_0+m_1}{N}\right)\right)+(m_0+m_1)F(\lambda) \le NG(\sigma)-(m_0+m_1)\frac{G(\sigma)-\beta^*G(\sigma\beta^*)-(1-\beta^*)F(\lambda)}{1-\beta^*}, \tag{4.17}$$

which itself derives from convexity (in $\mathbb{R}_+$) of $u\mapsto uG(\sigma u)+(1-u)F(\lambda)$. For any $k\ge 2$ (and smaller than $(\beta^*)^{-1}$) a similar computation gives us that the k-th term in the inequality is smaller than

$$N^{2k}e^{2N\bar{G}(\sigma,k)} \quad \text{with} \quad \bar{G}(\sigma,k):=\sup_{\substack{\beta_1,\dots,\beta_k\in(\beta^*,1)\\ \sum\beta_i\le 1}}\left(\sum_{i=1}^k \beta_i G(\sigma\beta_i)+(1-\sum_{i=1}^k\beta_i)F(\lambda)\right).$$

The result then follows from the fact that $\bar{G}(\sigma,k)<G(\sigma)$.

Now let us move to the case of $\mathbf{Z}(\partial\mathcal{E}_N^1)$. If $\xi\in\partial\mathcal{E}_N^1$, then it means that there is $x\in[\![0,N]\!]$ such that $\xi_{2x}=0$ and $\xi^{2x}\in\mathcal{E}_N^2$. Hence if a and b are such that $a<x<b$ and, $\xi_{2a}=\xi_{2b}=0$ and $\xi_{2y}>0$ for $y\in[\![a,b]\!]\setminus\{x\}$ then one must have

$$\max(b-x,x-a)\le N\beta^* \quad \text{and} \quad b-a>N\beta^*. \tag{4.18}$$

Decomposing over all possible values for a, b and x we find

$$\mathbf{Z}(\partial\mathcal{E}_N^1)\le\lambda^3 \sum_{\substack{a,b\in[\![0,N]\!]\\ \beta^*N<b-a\le 2\beta^*N}} \sum_{x=b-\beta*N}^{a+\beta^*N} \times \overline{Z}_a^{(N)}(\lambda,\sigma)Z_{x-a}\left(0,\frac{(x-a)\sigma}{N}\right)Z_{b-x}\left(0,\frac{(b-x)\sigma}{N}\right)\overline{Z}_{N-b}^{(N)}(\lambda,\sigma), \tag{4.19}$$

where $\overline{Z}_m^{(N)}(\sigma,\lambda)$ corresponds to a partition function with a constraint of having no large jumps:

$$\overline{Z}_m^{(N)}(\lambda,\sigma):=\sum_{k\ge1}\sum_{\substack{n_1,\dots,n_k\\ \sum_{i=1}^k n_k=m\\ n_i\le\beta^*N}}\lambda^{k-1}\prod_{i=1}^k Z_n\left(0,\frac{\sigma n_i}{N}\right). \tag{4.20}$$

From the upper bound on $\mathbf{Z}(\mathcal{E}_N^1)$, we have $\overline{Z}_m^{(N)}(\sigma,\lambda)\le Ce^{2mF(\lambda)}$. Using the upper bound in (3.15) and observing that at least one of the two length $(x-a)$ or $(b-x)$ is of order N we obtain that

$$\sum_{x=b-\beta^*N}^{a+\beta^*N} Z_{x-a}\left(0,\frac{(x-a)\sigma}{N}\right) Z_{b-x}\left(0,\frac{(b-x)\sigma}{N}\right)$$
$$\le CN^{-1/2}\sum_{y=0}^{2\beta^*N-b+a} e^{2(\beta^*N-y)G\left(\sigma\left(\beta^*-\frac{y}{N}\right)\right)+2(b-a-\beta^*N+y)G\left(\sigma\left(\frac{(b-a+y)}{N}-\beta^*\right)\right)}$$
$$\le 2CN^{-1/2}e^{2\beta^*NG(\sigma\beta^*)+2(b-a-N\beta^*)G\left(\sigma\left(\frac{b-a}{N}-\beta^*\right)\right)}$$
$$\sum_{y=0}^{(2\beta^*N-b+a)/2} e^{\frac{4y}{(2\beta^*N-b+a)}\left[(b-a)G\left(\frac{\sigma(b-a)}{2N}\right)-\beta^*NG(\sigma\beta^*)-(b-a-N\beta^*)G\left(\sigma\left(\frac{b-a}{N}-\beta^*\right)\right)\right]} \tag{4.21}$$

where in the last inequality we used the fact that second half of the sum is equal to the first half and the convexity of the function

$$u\mapsto(\beta^*-u)G(\sigma(\beta^*-u)+\left(\frac{b-a}{N}-\beta^*+u\right)G\left(\sigma\left(\frac{b-a}{N}-\beta^*+u\right)\right)$$

on $[0,(2\beta^*N-b+a)/2N]$. Now if $(b-a)\le 3\beta^*N/2$, the sum in the last line of (4.21) is bounded above by a constant (since we are summing something smaller than $e^{-c(\lambda,\sigma)y}$). If $(b-a)>3\beta^*N/2$, we bound the sum above by N. Going back to (4.19), we obtain altogether that

$$\frac{\mathbf{Z}(\partial\mathcal{E}_N^1)}{e^{2\beta^*NG(\beta^*\sigma)+2N(1-\beta^*)F(\lambda)}}$$
$$\le CN^{-1/2}\sum_{\substack{a,b\in[\![0,N]\!]\\ \beta^*N<b-a\le 2\beta^*N}} e^{2N\left[\left(\frac{b-a}{N}-\beta^*\right)\left(G\left(\left(\frac{(b-a)}{N}-\beta^*\right)\sigma\right)-F(\lambda)\right)\right]+(\log N)\mathbf{1}_{\{(b-a)>3\beta^*N/2\}}}$$
$$\le C\sqrt{N}\sum_{k=1}^{\beta^*N} e^{2k\left(G\left(\frac{k\sigma}{N}\right)-F(\lambda)\right)+(\log N)\mathbf{1}_{\{k>3\beta^*N/2\}}}\le C'\sqrt{N}, \tag{4.22}$$

where the last inequality follows from the fact that $G(\beta^*\sigma)-F(\lambda)<0$. To obtain the convert bound, we just need to consider the contribution to the sum of a,b,x such that $x=a+\beta^*N$ and $b=x+1$, and to avoid double counting, we impose the constraint that there is no jump of size larger than $N\beta^*/2$ outside of (a,b). Therefore, let $a':=(1-\beta^*)N-a-1$ and we have

$$\mathbf{Z}(\partial\mathcal{E}_N^1)\ge Z_{\beta^*N}(0,\sigma\beta^*)\sum_{a=0}^{(1-\beta^*)N-1} Z_a(\lambda,0)\mu_a^{\lambda,0}\left(L_{\max}\le\frac{\beta^*N}{2}\right)Z_{a'}(\lambda,0)\mu_{a'}^{\lambda,0}\left(L_{\max}\le\frac{\beta^*N}{2}\right)$$
$$\ge\frac{1}{C}\sqrt{N}e^{2N(\beta^*G(\sigma\beta^*)+(1-\beta^*)F(\lambda))}, \tag{4.23}$$

where the last inequality follows from Proposition 2.1 and (2.21).

□

5. Upper bounds on the relaxation time

5.1. Stating the results. Let us state here the two main statements that we are going to prove in this section and which, together with Proposition 4.1, provides a complete proof of Theorem 2.7. The proof of these propositions will also provide most of the ingredients required to prove the metastable behavior of the system when $E(\lambda,\sigma)>0$, that is Theorem 2.8.

We first prove that the system mixes in polynomial time when the activation energy is zero.

PROPOSITION 5.1. *Given $\lambda > 2$ there exists a constants $C(\lambda)$ and $\widetilde{C}(\lambda)$ such that for all σ satisfying $E(\lambda,\sigma) = 0$, for all $N \geq 1$ we have*

$$T^N_{\rm rel}(\lambda,\sigma) \leq C(\lambda) N^{\widetilde{C}(\lambda)}. \tag{5.1}$$

The second result of this section shows that when the activation energy of the system $E(\lambda,\sigma)$ is positive the lower bound proved in the previous section (that is, Proposition 4.1) is sharp up to polynomial correction.

PROPOSITION 5.2. *If $E(\lambda,\sigma) > 0$, for all $N \geq 1$ we have*

$$T^N_{\rm rel}(\lambda,\sigma) \leq C(\lambda,\sigma) N^{\widetilde{C}(\lambda,\sigma)} \exp(2NE(\lambda,\sigma)). \tag{5.2}$$

5.2. The chain decomposition strategy. In order to obtain upper bounds on the relaxation times $T^N_{\rm rel}(\lambda,\sigma)$, we are going to rely repeatedly on a decomposition technique developed in [JSTV04]. Let us state here this decomposition in a general framework. We consider a generic continuous-time reversible and irreducible Markov chain on a finite state space S, with generator $\mathcal{L}$ given by

$$(\mathcal{L}\varphi)(x) := \sum_{y\in\Omega} r(x,y)\left(\varphi(y) - \varphi(x)\right), \tag{5.3}$$

where r are the transition rates. We let π and gap denote respectively the equilibrium measure and the spectral gap associated with this Markov chain.

We consider also $(S_i)_{i\in I}$ a partition of S indexed by an arbitrary index set I and let $\mathcal{L}_i$ to be the generator of the *restricted chain* with state space S_i (it corresponds to the original chain conditioned to remain in S_i at all time). It is defined by

$$(\mathcal{L}_i f)(x) := \sum_{y\in S_i} r(x,y)\left(f(y) - f(x)\right). \tag{5.4}$$

for $f : S_i \to \mathbb{R}$ and $x \in S_i$. We let gap_i denote the spectral gap associated with $\mathcal{L}_i$. Note that the probability measure π_i defined by $\pi_i(A) = \pi(A)/\pi(S_i)$ for $A \subset S_i$ is reversible for $\mathcal{L}_i$. We let gap_i denote the spectral gap of $\mathcal{L}_i$. Finally we define the *reduced chain* on I with generator $\bar{\mathcal{L}}$ given by (for $\phi : I \to \mathbb{R}$)

$$(\bar{\mathcal{L}}\phi)(i) := \sum_{j\in I} \bar{r}(i,j)\left(\varphi(j) - \varphi(i)\right), \quad \text{where} \;\; \bar{r}(i,j) := \sum_{x\in S_i, y\in S_j} \pi_i(x) r(x,y), \;\; i,j \in I. \tag{5.5}$$

The probability $\bar{\pi}(i) = \pi(S_i)$ for all $i \in I$ is reversible for $\bar{\mathcal{L}}$. We let $\overline{\mathrm{gap}}$ denote its spectral gap. Note that the reduced chain does not correspond to the projection of the original chain on I (which is in general a non-Markovian process) but to the projection of a modified process that would be resampled using the probability π_i between any two consecutive steps. Finally we let

$$\bar{\gamma} := \max_{i\in I} \max_{x\in S_i} \sum_{y\in S\setminus S_i} r(x,y) \tag{5.6}$$

denote the maximal exit rate from one of the S_is. The following proposition is the continuous time adaptation of [JSTV04, Theorem 1]. How it allows to control the spectral gap of $\mathcal{L}$ is one can control that of the reduced chain and those of the restricted chains.

PROPOSITION 5.3. [CLM+12, Proposition 2.1] *With the notation introduced above we have*

$$\mathrm{gap} \geq \min\left(\frac{\overline{\mathrm{gap}}}{3}, \frac{\overline{\mathrm{gap}} \min_{i\in I} \mathrm{gap}_i}{\overline{\mathrm{gap}} + 3\bar{\gamma}}\right). \tag{5.7}$$

5.3. The induction strategy. The main idea of the proof here is to use a decomposition strategy, where the partition of the states is done according to the position of $L(\xi)$ and $R(\xi)$ whose definition (given in (3.69)) we recall

$$\begin{aligned} L(\xi) &:= \sup\{k \le N : \xi_k = 0\}, \\ R(\xi) &:= \inf\{k \ge N : \xi_k = 0\}. \end{aligned} \tag{5.8}$$

We want to apply Proposition 5.3 with the partition of Ω_N given by $\Omega_N = \sqcup_{(x,y)\in\Upsilon_N}\Omega_{(x,y)}$

$$\begin{aligned} \Upsilon_N &:= \{(x,y) \ : x,y \in [\![0,N]\!], 2x \le N \le 2y\}, \\ \Omega_{(x,y)} &:= \{\xi \in \Omega_N \ : \ L(\xi) = 2x \text{ and } R(\xi) = 2y\}. \end{aligned} \tag{5.9}$$

We need to estimate the spectral gap for the reduced chain on Υ_N and for each of the restricted chain on $\Omega_{(x,y)}$. Roughly speaking, the idea is that when $G(\sigma) + \sigma G'(\sigma) < F(\lambda)$, both $L(\xi)$ and $R(\xi)$ display a uniform drift towards the center and this makes the spectral gap bounded away from below (like for a random walk with drift). The very sharp equilibrium estimates proved in Section 3 allows us to make this rigorous in Proposition 5.6. Now the chain restricted to $\Omega_{(x,y)}$ is in fact a product chain since the respective restrictions of η_t to the intervals $[\![0,2x]\!]$, $[\![2x,2y]\!]$ and $[\![2y,2N]\!]$ are independent Markov chains. The spectral gap $\mathrm{gap}_{(x,y)}$ of the restricted chain is thus given by the minimum of these three chains.

The restriction the interval $[\![2x,2y]\!]$ is a variant of the weakly asymmetric exclusion process whose mixing properties have been studied in details in [LL20]. Its spectral gap is well understood and scales like $(y-x+1)^{-2}$ (see Proposition 5.5 below). The restrictions to $[\![0,2x]\!]$ and $[\![2y,2N]\!]$ on the other hand are simply the same as the original chain but on a smaller interval. This forces us to proceed by induction. Our main task is going to be the proof of the following statement. We let $\sigma_0(\lambda)$ be such that

$$G(\sigma_0) + \sigma_0 G'(\sigma_0) = F(\lambda). \tag{5.10}$$

PROPOSITION 5.4. *For any $\sigma_1 < \sigma_0$ there exists a constant $c(\lambda,\sigma_1)$ such that for any $\sigma \le \sigma_1$ and any $N \ge 2$ we have*

$$\mathrm{gap}_N(\lambda,\sigma) \ge c(\lambda,\sigma_1) \min_{n\le N/2}\left(\mathrm{gap}_n\left(\lambda, \frac{n\sigma}{N}\right), (N/2)^{-2}\right). \tag{5.11}$$

We also have for all $\sigma \le \sigma_0$

$$\mathrm{gap}_N(\lambda,\sigma) \ge c(\lambda) N^{-4} \min_{n\le N/2}\left(\mathrm{gap}_n\left(\lambda, \frac{n\sigma}{N}\right), (N/2)^{-2}\right). \tag{5.12}$$

PROOF OF PROPOSITION 5.1 USING PROPOSITION 5.4. We start by setting (using the constant $c(\lambda,\sigma_0/2)$ given by Proposition 5.4)

$$\widetilde{C}(\lambda) := 2 \vee \log_2\left(\frac{1}{c(\lambda,\sigma_0/2)}\right) + 4. \tag{5.13}$$

We are going to prove by induction that for every $N \ge 2$ the property $\mathcal{U}_N$ defined as

$$\forall \sigma \in [0,\sigma_0/2], \quad \mathrm{gap}_N(\lambda,\sigma) \ge N^{-\widetilde{C}(\lambda)+4} \tag{5.14}$$

is satisfied. When $N=2$, we can see that $\#\Omega_2 = 2$, and $\mathrm{gap}_2(\lambda,\sigma) = 1$ for all $\sigma \in [0,\sigma_1]$ using (2.7). Now given $N \ge 3$ and assuming that $\mathcal{U}_n$ is valid for all $n \le N-1$, we want to prove $\mathcal{U}_N$. Therefore, by (5.11) and the induction hypothesis, we have

$$\mathrm{gap}_N(\lambda,\sigma) \ge c(\lambda,\sigma_0/2)\left(\frac{N}{2}\right)^{-\widetilde{C}(\lambda)+4} \ge N^{-\widetilde{C}(\lambda)+4}, \tag{5.15}$$

which concludes the induction proof. Now when $\sigma \in (\sigma_0/2, \sigma_0]$ we apply (5.12) to obtain

$$\mathrm{gap}_N(\lambda,\sigma) \geq c(\lambda)(N/2)^{-\widetilde{C}(\lambda)} \tag{5.16}$$

and this concludes our proof.

□

5.4. Proof of proposition 5.4. As discussed above the key point here is to apply Proposition 5.3. However, if we apply it directly the factor (5.6) corresponding to the partition $\Omega_N = \sqcup_{(x,y)\in\Upsilon_N}\Omega_{(x,y)}$ is much too large. More specifically it is of order N, and applying Proposition 5.3 directly would make us lose a factor N in (5.11) which, after the induction, would turn into a factor $\exp((\log N)^2)$ in Proposition 5.1. Hence we perform a small modification to the chain which is crucial to obtain a polynomial bound on the relaxation time.

Our modification simply constrains $L(\xi)$ and $R(\xi)$ to make only nearest neighbor move. Recalling the definition of r_N in (2.5), this corresponds to consider the Markov chain with generator

$$\mathcal{L}^*_N(f)(\xi) := \sum_{\xi'\in\Omega_N} r^*_N(\xi,\xi')(f(\xi')-f(\xi))$$

where

$$r^*_N(\xi,\xi') := r_N(\xi,\xi')\mathbf{1}_{\{|L(\xi)-L(\xi')|\leq 2 \text{ and } |R(\xi)-R(\xi')|\leq 2\}}. \tag{5.17}$$

Note that $\mathcal{L}^*_N$ is irreducible and reversible with respect to the same measure $\mu_N^{\lambda,\sigma}$ and thus for this reason has a smaller spectral gap than the original chain. Letting gap^*_N be the spectral gap associated with this chain, we are going to prove that for $\sigma \leq \sigma_1$

$$\mathrm{gap}^*_N(\lambda,\sigma) \geq c(\lambda,\sigma_1)\min_{n\leq N/2}\left(\mathrm{gap}_n\left(\lambda,\frac{n\sigma}{N}\right),N^{-2}\right). \tag{5.18}$$

and similarly for (5.12).

We apply Proposition 5.3 for $\mathcal{L}^*_N$ with the partition $\Omega_N = \sqcup_{(x,y)\in\Upsilon_N}\Omega_{(x,y)}$. We let $\mathrm{gap}_{(x,y)}(\lambda,\sigma)$ and $\overline{\mathrm{gap}}_N(\lambda,\sigma)$ be the spectral gaps of the corresponding restricted and reduced chains. Now note that for our modified chain there are (at most) 4 transitions that change the value of $L(\xi)$ or $R(\xi)$ and thus we have

$$\max_{(x,y)\in\Upsilon_N}\max_{\xi\in\Omega_{(x,y)}}\sum_{\xi'\in\Omega_N\setminus\Omega_{(x,y)}} r^*_N(\xi,\xi') \leq 4. \tag{5.19}$$

As a consequence we have

$$\mathrm{gap}^*_N(\lambda,\sigma) \geq \min\left(\frac{\overline{\mathrm{gap}}_N(\lambda,\sigma)}{3}, \frac{\overline{\mathrm{gap}}_N(\lambda,\sigma)\min_{\Upsilon_N}\mathrm{gap}_{(x,y)}(\lambda,\sigma)}{\overline{\mathrm{gap}}_N(\lambda,\sigma)+12}\right). \tag{5.20}$$

Now from the discussion of the previous section we have

$$\mathrm{gap}_{(x,y)}(\lambda,\sigma) = \mathrm{gap}_x\left(\lambda,\frac{x\sigma}{N}\right)\wedge \mathrm{gap}_{N-y}\left(\lambda,\frac{\sigma(N-y)}{N}\right)\wedge \mathrm{gap}_{y-x}\left(0,\frac{(y-x)\sigma}{N}\right), \tag{5.21}$$

and as a consequence

$$\min_{\Upsilon_N}\mathrm{gap}_{(x,y)}(\lambda,\sigma) \geq \left(\min_{n\leq N}\mathrm{gap}_n\left(0,\frac{n\sigma}{N}\right)\right)\wedge\left(\min_{n\leq N/2}\mathrm{gap}_n\left(\lambda,\frac{n\sigma}{N}\right)\right). \tag{5.22}$$

To conclude the proof we need to rely on two estimates. The first one concerns the spectral gap of the unpinned dynamics, and can be obtained via a simple comparision with the unconstrained ASEP (see [LL19, Theorem 1] for the identification of the spectral gap in this case). The proof is included in Appendix 3.A for completeness.

PROPOSITION 5.5. *For any $n \le N$ and for any $\sigma > 0$ we have*

$$\mathrm{gap}_n(0,\sigma) \ge 2\sin\left(\frac{\pi}{4N}\right)^2. \tag{5.23}$$

The second one concerns the reduced chain. This chain informally can be thought as describing the evolution of a large unpinned zone present in the middle of the system. As remarked in Section 4.1, when $E(\lambda,\sigma) = 0$, the corresponding effective potential does not display several local minima, and thus avoids any bottlenecking. Combining this fact with the relatively simple geometry of Υ_N we obtain the following estimates.

PROPOSITION 5.6. *We recall the definition of σ_0 in (5.10). For $\sigma_1 < \sigma_0$, There exists a constant $C(\lambda,\sigma_1)$ such that for every N, every $\sigma \in [0,\sigma_1]$*

$$\overline{\mathrm{gap}}_N(\lambda,\sigma) \ge C(\lambda,\sigma_1). \tag{5.24}$$

Also there exists an constant $C(\lambda)$ such that for all $\sigma \le \sigma_0$

$$\overline{\mathrm{gap}}_N(\lambda,\sigma) \ge C(\lambda)N^{-4}. \tag{5.25}$$

REMARK 5.7. *The exponent 4 appearing in (5.25) is not optimal and a closer analysis would show that the spectral gap is of order N^{-1} in that case. We have choosen to aim for a simpler proof since we do not aim for an explicit exponent in Proposition 5.1.*

PROOF OF PROPOSITION 5.6. Consider the order on Υ_N which is induced by the inclusion order for the interval $[x,y]$ that is

$$(x',y') \succcurlyeq (x,y) \quad \text{if } x' \le x \text{ and } y' \ge y.$$

We are in fact going to prove a lower bound on the Cheeger constant associated with the dynamics, which is defined by

$$\chi := \min_{A\subset\Upsilon_N \,:\, \bar\pi(A)\le 1/2} \frac{\sum_{(x,y)\in A,(x',y')\in A^{\complement}} \bar\pi(x',y')\bar r_N[(x',y'),(x,y)]}{\bar\pi(A)}. \tag{5.26}$$

In fact we are going to prove a lower bound on

$$\chi' := \min_{A\subset\Upsilon_N \,:\, (x_0,y_0)\notin A} \frac{\sum_{(x,y)\in A,(x',y')\in A^{\complement}} \bar\pi(x',y')\bar r_N[(x',y'),(x,y)]}{\bar\pi(A)}. \tag{5.27}$$

where (x_0,y_0) is the minimal element with positive propability in Υ_N (which is either $(N/2,N/2)$ or $((N-1)/2,(N+1)/2)$) for the order considered above. It is easy to check that $\chi \ge \chi'$ since the numerator of the minimized quantity is unchanged when A is replaced by $A^{\complement}$. Now from the above observation and [LP17, Theorem 13.10] we have

$$\overline{\mathrm{gap}}_N(\lambda,\sigma) \ge (\chi')^2/2. \tag{5.28}$$

We are going to use an approximation for $\bar\pi$. We set

$$\bar p(x,y) := e^{-2(y-x)F(\lambda)+2(y-x)G\left(\frac{\sigma(y-x)}{N}\right)}(y-x+1)^{-3/2}\left(\frac{\sigma^2(y-x+1)3}{N^2}\vee 1\right). \tag{5.29}$$

We have by Propositions 3.1 and 3.3 that for some constant $C_1(\lambda)$

$$C_1(\lambda)^{-1} \le \frac{\bar\pi((x,y))}{\bar p((x,y))} \le C_1(\lambda). \tag{5.30}$$

Since we also have

$$\inf_{x,y} \bar r_N\left((x,y),(x\pm 1,y\pm 1)\right) \ge r^*(\lambda,\sigma_1) > 0, \tag{5.31}$$

this implies that

$$\chi' \geq r^* C_1^{-2} \min_{A \subset \Upsilon_N \,:\, (x_0,y_0)\notin A} \frac{\sum_{(x,y)\in A, (x',y')\in A^{\complement}} \bar{p}(x',y') \mathbf{1}_{\{|x-x'|+|y-y'|=1\}}}{\bar{p}(A)}. \tag{5.32}$$

Now for every x and y

$$\min\left[\log\left(\frac{\bar{p}(x+1,y)}{\bar{p}(x,y)}\right), \log\left(\frac{\bar{p}(x,y-1)}{\bar{p}(x,y)}\right)\right] \geq 2\left[F(\lambda) - \sigma_1 G'(\sigma_1) - G(\sigma_1)\right] =: \gamma(\lambda,\sigma_1). \tag{5.33}$$

Hence we have

$$\sum_{(x',y')\succcurlyeq(x,y)} \bar{p}(x',y') \leq (1-e^{-\gamma})^{-2} \bar{p}(x,y). \tag{5.34}$$

Now given A such that $(x_0,y_0) \notin A$. We let A' denote the set of points which are immediate inferior neighbor of a point in A,

$$A' := \{(x,y) \in A^{\complement} : \{(x-1,y),(x,y+1)\} \cap A \neq \emptyset\}. \tag{5.35}$$

Since (by immediate induction) for $(x,y) \in A$ there is $(x',y') \in A'$ such that $(x',y') \preccurlyeq (x,y)$, then (5.34) implies that

$$\bar{p}(A) \leq (1-e^{-\gamma})^{-2} \bar{p}(A'). \tag{5.36}$$

On the other hand we have

$$\sum_{(x,y)\in A, (x',y')\in A^{\complement}} \bar{p}(x',y') \mathbf{1}_{\{|x-x'|+|y-y'|=1\}} \geq \bar{p}(A'). \tag{5.37}$$

In view of (5.32) and (5.28) this implies that

$$\overline{\text{gap}}_N(\lambda,\sigma) \geq (C_1[1-e^{-\gamma}])^{-4} (r^*)^2/2.$$

In the case where $G(\sigma) + \sigma G'(\sigma) = F(\lambda)$, then we simply need to replace $(1-e^{\gamma})^{-2}$ by N^2 in (5.34) and we obtain that

$$\overline{\text{gap}}_N(\lambda,\sigma) \geq (C_1 N)^{-4} (r^*)^2/2.$$

□

5.5. Proof of Proposition 5.2. Let us now prove that the lower bound proved in Proposition 4.1 using a simple bottleneck argument is sharp up to polynomial correction. Our starting point is to apply Proposition 5.3 considering this time the partition in two $\Omega_N = \mathcal{E}_N^1 \sqcup \mathcal{E}_N^2$. We let $\text{gap}_{N,i}$ be the spectral gap of the Markov chain restricted to $\mathcal{E}_N^i$ for $i = 1,2$ and and let $\overline{\text{gap}}_{1,2}$ denote the spectral gap of the reduced chain on $\{1,2\}$. Using the fact that for every $\xi \in \Omega_N$,

$$\sum_{\xi'\in\Omega_N} r_N(\xi,\xi') \leq 2N, \tag{5.38}$$

we have

$$\text{gap}_N(\lambda,\sigma) \geq \min\left(\frac{1}{3}\overline{\text{gap}}_{1,2}, \frac{\overline{\text{gap}}_{1,2} \min_{i\in\{1,2\}} \text{gap}_{N,i}}{\overline{\text{gap}}_{1,2} + 6N}\right). \tag{5.39}$$

The quantity $\overline{\text{gap}}_{1,2}$ corresponds exactly to $\mathcal{E}(f)/\text{Var}_{\mu_N}(f)$ with $f = \mathbf{1}_{\mathcal{E}_N^1}$, which was estimated in Equation 4.7. The main task in our proof is thus to show that $\text{gap}_{N,i}$ decays only like a power of N, or in other words, that the chains restricted to each of the potential wells mix rapidly. This corresponds to the following two propositions:

PROPOSITION 5.8. *There exists $c(\lambda) > 0$ and $C(\lambda,\sigma)$ such that for all $N \geq 2$, we have*

$$\text{gap}_{N,1} \geq c(\lambda) N^{-C(\lambda,\sigma)}. \tag{5.40}$$

Moreover $C(\lambda,\sigma)$ can be chosen to be increasing in σ.

Proposition 5.9. *There exists $c(\lambda,\sigma)>0$ such that for all $N\ge 2$, we have*

$$\mathrm{gap}_{N,2}\ge c(\lambda,\sigma)N^{-C(\lambda,\sigma)}. \tag{5.41}$$

To prove these results, our strategy will be to use again the chain reduction to simplify the geometry of the state space.

Proof of Proposition 5.2 from Proposition 5.8 and 5.9. Let $\bar r$ and $\bar\pi$ denote the rates associated to the reduced chain. By the variational formula (2.7) , we have

$$\overline{\mathrm{gap}}_{1,2}=\frac{\bar r(1,2)}{\bar\pi(2)}=\frac{\sum_{\xi\in\mathcal{E}^1_N,\xi'\in\mathcal{E}^2_N}\mu_N(\xi)r_N(\xi,\xi')}{\mu_N(\mathcal{E}^1_N)\mu_N\left(\mathcal{E}^2_N\right)}\ge\frac{\exp(\frac{2\sigma}{N})}{\lambda+\exp(\frac{2\sigma}{N})}\frac{\mu_N\left(\partial\mathcal{E}^1_N\right)}{\mu_N\left(\mathcal{E}^2_N\right)\mu_N(\mathcal{E}^1_N)}.$$

The last inequality comes from the fact that for every $\xi\in\partial\mathcal{E}^1_N$ there is at least one transition to $\mathcal{E}^2_N$, and has rate $\frac{\exp(\frac{2\sigma}{N})}{\lambda+\exp(\frac{2\sigma}{N})}$. Hence from Proposition 4.2 we have

$$\overline{\mathrm{gap}}_{1,2}\ge c(\lambda,\sigma)\sqrt{N}\exp(-2NE(\lambda,\sigma)). \tag{5.42}$$

To conclude, we use (5.42) together with the results of Propositions 5.8 and 5.9 in (5.39).

□

5.6. Proof of Proposition 5.8. Let us assume by convention that if $E(\lambda,\sigma)=0$ then $\mathcal{E}^1_N=\Omega_N$ and $\mathrm{gap}_{N,1}(\lambda,\sigma)=\mathrm{gap}_(\lambda,\sigma)$. Since our proof proceeds by an iterative structure similar to that of Proposition 5.1, we are going to proceed by by induction. Recall the definition (5.10), we are going to prove the following statement (for the constant $\widetilde{C}(\lambda)$ given in Proposition 5.1) (which we refer to as $\mathcal{U}_k$) is valid for all $k\ge 0$ (for a sequence $C_k(\lambda)$ that will be specified in the course of the proof)

$$\forall N\ge 1,\quad \forall\sigma\le 2^k\sigma_0,\quad \mathrm{gap}_{N,1}(\lambda,\sigma)\ge C_k(\lambda)N^{-\widetilde{C}(\lambda)-4k}. \tag{5.43}$$

The statement for $k=0$ is exactly Proposition 5.1, so there is nothing to prove to start the induction. Now assuming $\mathcal{U}_k$ let us prove $\mathcal{U}_{k+1}$.

Again we replace the rate by restricting the transitions of L and R to nearest neighbor as in (5.17). We apply Proposition 5.3 to this modified chain with the partition of $\mathcal{E}^1_N$ given by $\mathcal{E}^1_N=\sqcup_{(x,y)\in\Upsilon'_N}\Omega'_{(x,y)}$ where

$$\begin{aligned}\Omega'_{(x,y)}&:=\{\xi\in\mathcal{E}^1_N\ :\ L(\xi)=2x\text{ and }R(\xi)=2y\},\\ \Upsilon'_N&:=\{(x,y)\ :x,y\in[\![0,N]\!],2x\le N\le 2y\text{ and }y-x\le\beta^*N\}.\end{aligned}\tag{5.44}$$

We let $\mathrm{gap}'_{(x,y)}$ be the spectral gap associated with the Markov chain restricted to $\Omega'_{(x,y)}$ and let $\overline{\mathrm{gap}}_{N,1}$ denote the spectral gap associated with the reduced chain on Υ'_N (whose transition are only $(x,y\pm1)$ and $(x\pm1,y)$). Applying Proposition 5.3 we obtain that

$$\mathrm{gap}_{N,1}\ge\min\left(\frac{\overline{\mathrm{gap}}_{N,1}}{3},\frac{\overline{\mathrm{gap}}_{N,1}\min_{\Upsilon_N}\mathrm{gap}'_{(x,y)}(\lambda,\sigma)}{\overline{\mathrm{gap}}_{N,1}+12}\right). \tag{5.45}$$

To provide a lower bound on $\overline{\mathrm{gap}}_{N,1}$, we can repeat the proof of (5.25) in Proposition 5.6. The important point here is that the probability distribution for the reduced chain is given by

$$\bar\pi_1(x,y):=\frac{Z_x(\lambda,\frac{\sigma x}{N})\mu_x^{\lambda,\frac{\sigma x}{N}}(L_{\max}\le\beta^*N)Z_{y-x}(0,\frac{\sigma(y-x)}{N})Z_{N-y}(\lambda,\frac{\sigma(N-y)}{N})\mu_{N-y}^{\lambda,\frac{\sigma N-y}{N}}(L_{\max}\le\beta^*N)}{\mathbf{Z}(\mathcal{E}^1_N)}.$$

Now we have by a variant Proposition 4.2 (the estimate for $\mathbf{Z}(\mathcal{E}_1)$) we have

$$\frac{1}{C(\lambda)}e^{2xF(\lambda)} \le Z_x\left(\lambda, \frac{\sigma x}{N}\right)\mu_x^{\lambda,\frac{\sigma x}{N}}(L_{\max} \le \beta^* N) \le C(\lambda,\sigma)e^{2xF(\lambda)}. \tag{5.46}$$

One needs to check within the proof of Proposition 4.2 that the bounding constant C does not depend on x. The lower bound is easy and is obtained by replacing σ by 0. For upper bound on the other hand, one only needs to apply the bound (4.10) (which depends on σ but not on x since $N\beta^*(\sigma) = x\beta^*(\frac{\sigma x}{N})$). Using a similar bound for $\mu_{N-y}^{\lambda,\frac{\sigma N-y}{N}}(L_{\max} \le \beta^* N)$ we obtain that $\bar{\pi}_1(x,y)$ can be replaced by $\bar{p}(x,y)$ as in the proof of Proposition 5.6 and proceed similarly (here the restriction $y - x \le \beta^* N$ plays a crucial role) to obtain

$$\forall \sigma \le 2^{k+1}\sigma_0, \quad \overline{\mathrm{gap}}_{N,1} \ge C'_k(\lambda)N^{-4}. \tag{5.47}$$

(the constant depend on σ but can be made uniform in the range $\sigma \le 2^{k+1}\sigma_0$). Let us now turn to $\mathrm{gap}'_{(x,y)}$. As in the proof of Proposition 5.4, the dynamic restricted to $\Omega_{(x,y)}$ consists in three independent part and thus we have

$$\mathrm{gap}'_{(x,y)} = \mathrm{gap}_{x,1}\left(\lambda, \frac{x\sigma}{N}\right) \wedge \mathrm{gap}_{y-x}\left(0, \frac{(y-x)\sigma}{N}\right) \wedge \mathrm{gap}_{N-y,1}\left(\lambda, \frac{(N-y)\sigma}{N}\right). \tag{5.48}$$

where we recall that $\mathrm{gap}_{x,1}\left(\lambda, \frac{x\sigma}{N}\right)$ is the spectral gap of the chain restricted to $\{\xi \in \Omega_x : L_{\max}(\xi) \le \beta^* N\}$ (here it is important to notice that $N\beta^*(\sigma) = x\beta^*(\frac{\sigma x}{N})$). Now $\frac{x\sigma}{N}, \frac{(N-y)\sigma}{N} \le 2^k\sigma_0$ so that one can apply the induction hypothesis to them. Combining this with Proposition 5.5 we have for every $x, y \in \Upsilon'_{(x,y)}$

$$\mathrm{gap}'_{(x,y)} \ge C_k(\lambda)N^{-\widetilde{C}(\lambda)-4k}. \tag{5.49}$$

Finally we can conclude that $\mathcal{U}_{k+1}$ holds combining (5.49) and (5.47) and (5.45).

□

5.7. Proof of Proposition 5.9. While still relying on the chain decomposition method, the proof of this result requires a new partition of the state space. This time we need to trace the location of all the the excursions of size larger than $\beta^* N$. We define thus

$$\Psi_N := \{[k, (\ell_i, r_i)_{i=1}^k] \ : \ k \ge 1\ ; \forall i \in [\![1,k]\!], r_i - \ell_i > \beta^* N, \text{ and } \ell_{i+1} \ge r_i\}. \tag{5.50}$$

Now given $\xi \in \mathcal{E}_N^2$ we define $k(\xi)$ and $(\ell_i(\xi), r_i(\xi))_{i=1}^{k(\xi)}$ as the number and position of excursions of size larger than $\beta^* N$. Moreover, ℓ_i and r_i are the unique increasing sequences that satisfy

$$\begin{cases} \forall i \in [\![1,k]\!],\ r_i(\xi) - \ell_i(\xi) > \beta^* N, \\ \forall i \in [\![1,k]\!],\ \xi_{2\ell_i} = \xi_{2r_i} = 0 \text{ and } \forall x \in [\![2\ell_i+1, 2r_i-1]\!],\ \xi_x > 0, \\ \forall x \in [\![0, N-1]\!] \setminus \{(\ell_i, r_i)\}_{i=1}^k, \exists y \in [\![x+1, (x+\beta^* N)\wedge N]\!],\ \xi_{2y} = 0. \end{cases} \tag{5.51}$$

We also define

$$\Omega_{[k,(\ell_i,r_i)_{i=1}^k]} := \left\{\xi \in \mathcal{E}_N^2 \ : \ [k(\xi), (\ell_i(\xi), r_i(\xi))_{i=1}^{k(\xi)}] = [k, (\ell_i, r_i)_{i=1}^k]\right\}. \tag{5.52}$$

We use the letter ψ to denote a generic element of Ψ_N. In addition, let gap_ψ denote the spectral gap associated with the Markov chain restricted to Ω_ψ, and let $\overline{\mathrm{gap}}_{N,2}$ denote the spectral gap associated with the reduced chain on Ψ_N. Our result easily follows from the following estimates for the restricted and reduced chains respectively.

Proposition 5.10. *There exist constants $c(\lambda) > 0$ and $C(\lambda,\sigma) > 0$ such that for all $N \ge 1$,*

$$\min_{\psi \in \Psi_N} \mathrm{gap}_\psi \ge c(\lambda)N^{-C(\lambda,\sigma)}.$$

Proposition 5.11. *For all $N \geq 1$, we have*

$$\overline{\text{gap}}_{N,2} \geq c(\lambda,\sigma) N^{-3}.$$

Proof of Proposition 5.9 using Propositions 5.10 and 5.11. Applying Proposition 5.3 together with the fact that $\sum_{\xi' \in \Omega_N} r_N(\xi,\xi') \leq 2N$ for all $\xi \in \Omega_N$, we have

$$\text{gap}_{N,2} \geq \min\left(\frac{\overline{\text{gap}}_{N,2}}{3}, \frac{\overline{\text{gap}}_{N,2} \min_{\psi\in\Psi_N} \text{gap}_\psi}{\overline{\text{gap}}_{N,2} + 6N}\right) \geq c'(\lambda,\sigma) N^{-C'(\lambda,\sigma)}. \tag{5.53}$$

□

Proof of Proposition 5.10. Note that the chain restricted to Ω_ψ is indeed a product chain since the respective restrictions of η_t to the intervals $(\llbracket 2\ell_i, 2r_i\rrbracket)_{i=1}^k$ and $(\llbracket 2r_i, 2\ell_{i+1}\rrbracket)_{i=0}^k$ are independent Markov chains where $r_0 := 0$ and $\ell_{k+1} := N$. The spectral gap $\text{gap}_{[k,(\ell_i,r_i)_{i=1}^k]}$ associated with this restricted chain is thus given by the minimum of these chains. Furthermore, the spectral gap of the restricted chain in the interval $\llbracket 2\ell_i, 2r_i\rrbracket$ is $\text{gap}_{r_i-\ell_i}(0, \sigma\frac{r_i-\ell_i}{N})$, and the spectral gap of the restricted chain in the interval $\llbracket 2r_i, 2\ell_{i+1}\rrbracket$ is $\text{gap}_{\ell_{i+1}-r_i,1}(\lambda, \sigma\frac{\ell_{i+1}-r_i}{N})$. Using Propositions 5.5 and 5.8, we obtain

$$\text{gap}_\psi \geq \min\left(c(\lambda) N^{-C(\lambda,\sigma)}, N^{-2}\right) = c(\lambda) N^{-C(\lambda,\sigma)}. \tag{5.54}$$

□

Proof of Proposition 5.11. In this proof we let $\bar{r}$ and $\bar{\pi}$ denote the rates and invariant measure associated to the reduced chain respectively. Additionally, define the edge set E and the edge flows Q respectively by

$$\begin{aligned} E &:= \left\{\{\psi,\psi'\} : \bar{r}(\psi,\psi') > 0\right\}, \\ Q\left(\psi,\psi'\right) &:= \bar{\pi}\left(\psi\right)\bar{r}\left(\psi,\psi'\right) = \bar{\pi}\left(\psi'\right)\bar{r}\left(\psi',\psi\right). \end{aligned} \tag{5.55}$$

In order to get our bound for the spectral gap we are going to rely on the so called "path method" (see [LP17, Chapter 13] for an introduction to the method and bibliographical remarks). For two distinct elements ψ and ψ' of Ψ_N we construct a path from ψ to ψ' denoted by $\Gamma(\psi,\psi')$. Our paths (whose explicit algorithmic construction is given below) are sequences $(\psi_0,\psi_1,\dots,\psi_{|\Gamma|})$ elements such that $\psi_0 = \psi$, $\psi_{|\Gamma|} = \psi'$ and any two consecutive elements forms an edge in E. We say that $e \in \Gamma$ if there exists $j \leq |\Gamma|$ such that $\{\psi_{j-1},\psi_j\} = e$. For $e \in E$, we define the congestion ratio over the edge e as

$$B(e) := \frac{1}{Q(e)} \sum_{\substack{\psi,\psi'\in\Psi_N \\ e\in\Gamma(\psi,\psi')}} \bar{\pi}\left(\psi\right)\bar{\pi}\left(\psi'\right). \tag{5.56}$$

By [LP17, Corollary 13.21], we have

$$\overline{\text{gap}}_{N,2} \geq \left(\max_{e\in E}(B(e)) \max_{\psi,\psi'\in\Psi_N} |\mathit{\Gamma}(\psi,\psi')|\right)^{-1}. \tag{5.57}$$

Since we aim for a polynomial bound and the cardinal of Ψ_N is a power of N, the length of the path will not be an issue. Our construction must thus aim minimizing the congestion ratio.

To construct a path from ψ to ψ', we construct in fact a path from ψ to $[1,(0,N)]$ and from ψ' to $[1,(0,N)]$ and concatenate these two paths (taking the second path in reverse order) to get our full path whose length is at most $2N$.

To construct the finite sequence $[k(j),(\ell_i(j),r_i(j))_{i=1}^{k(j)}]_{j=0}^J$ from ψ to $[1,(0,N)]$ we proceed as follows:

- We set $[k(0),(\ell_i(0),r_i(0))_{i=1}^{k(0)}]=\psi$.
- If $\ell_1(j)>0$ then $\ell_1(j+1)=\ell_1(j)-1$ and the other coordinates are unchanged.
- If $\ell_1(j)=0$ and $r_1(j)<\ell_2(j)$ (or $r_1(j)<N$ if $k(j)=1$) then $r_1(j+1)=r_1(j)+1$ and the other coordinates are unchanged.
- If $\ell_1(j)=0$ and $\ell_2(j)=r_1(j)$ then $k(j+1)=k(j)-1$, $r_1(j+1)=r_2(j)$ and $r_i(j+1)=r_{i+1}(j)$, $\ell_i(j+1)=\ell_{i+1}(j)$ for $i\in[\![2,k(j)-1]\!]$.
- We stop the algorithm when one reaches $[1,(0,N)]$.

By construction the length of the path satisfies $|\Gamma(\psi,\psi')|\le 2N$ for any ψ and ψ'. Now we provide an upper bound on $\max_{e\in E}B(e)$ using the precise estimates in Section 3 and Section 4. By symmetry, given e at the cost of multiplicative factor 2, we can only sum over paths for which e belongs to the "first-half" of the paths (that linking ψ to $[1,(0,N)]$ let us call it $\Gamma_1(\psi)$). Summing over all possible end points ψ' we obtain that

$$B(e)=\frac{2}{Q(e)}\bar\pi\left(\{\psi\ :\ e\in\Gamma_1(\psi)\}\right). \tag{5.58}$$

To control the above quantity we need an explicit description of the set $\Psi(e):=\{\psi\ :\ e\in\Gamma_1(\psi)\}$. Let us say that $e=\{[m,(x_i,y_i)_{i=1}^m],[m',(x_i',y_i')_{i=1}^{m'}]\}$ and that $[m,(x_i,y_i)_{i=1}^m]$ is the first state visited on the path to $\Gamma_1(\psi)$ (with our algorithm which state is visited first does not depend on ψ). We are going to prove the two following inequalities

$$\begin{aligned} Q(e)&\ge\frac{1}{C(\lambda,\sigma)}\frac{\mathbf{Z}(\Omega_{[m,(x_i,y_i)_{i=1}^m]})}{\mathbf{Z}(\mathcal{E}_N^2)}\\ \bar\pi\left(\Psi(e)\right)&\le C(\lambda,\sigma)\frac{N^2\mathbf{Z}(\Omega_{[m,(x_i,y_i)_{i=1}^m]})}{\mathbf{Z}(\mathcal{E}_N^2)},\end{aligned} \tag{5.59}$$

which are then sufficient to conclude using (5.57) and the bound we have for the path length. For the first one, we just have to check that the rate $\bar r([m,(x_i,y_i)_{i=1}^m],[m',(x_i',y_i')_{i=1}^{m'}])$ is bounded away from zero (even though it is slightly improper since edges are not oriented, we use the shorthand notation $\bar r(e)$ for the rate). There are two cases to treat: either the transition e merges two excursions or it enlarges the first one. In the first case we have $\bar r(e)=\frac{\exp(\frac{2\sigma}{N})}{\lambda+\exp(\frac{2\sigma}{N})}$. In the second case, let us assume that $x_1'=x_1-1$ (the case $y_1'=y_1+1$ being identical) we have

$$\bar r(e)=\frac{\exp(\frac{2\sigma}{N})}{\lambda+\exp(\frac{2\sigma}{N})}\mu_N(\xi_{2(x_1-1)}=0\mid\xi\in\Omega_{[m,(x_i,y_i)_{i=1}^m]})=\frac{\exp(\frac{2\sigma}{N})}{\lambda+\exp(\frac{2\sigma}{N})}\frac{\lambda\mathcal{Z}_{x_1-1}\mathcal{Z}_1}{\mathcal{Z}_{x_1}}, \tag{5.60}$$

where we have used the notation

$$\mathcal{Z}_n:=Z_n\left(\lambda,\sigma\frac{n}{N}\right)\mu_n^{\lambda,\sigma\frac{n}{N}}(L_{\max}\le\beta^*N) \tag{5.61}$$

for $n\in[\![1,N]\!]$ and $\mathcal{Z}_0=1$. Recalling (5.46) we have

$$C(\lambda)^{-1}\le e^{-2nF(\lambda)}\mathcal{Z}_n\le C(\lambda,\sigma), \tag{5.62}$$

and thus we have the desired uniform lower bound for $\bar r(e)$. Now let us prove the second estimate in (5.59). Note that

$$\Psi(e)\subset\left\{[n+m-1,(x_j'',y_j'')_{j=1}^n\cup(x_i,y_i)_{i=2}^m]\in\Psi_N: n\ge1, x_1''\ge x_1\text{ and }y_n''\le y_1\right\}. \tag{5.63}$$

Now we can partition $\Psi(e)$ according to the value of x_1'' and y_n'' (let us call them ℓ and r respectively. Now for any element of this set we have

$$\frac{\bar{\pi}\left(\Psi(e)\right)\mathbf{Z}(\mathcal{E}_N^2)}{\mathbf{Z}(\Omega_{[m,(x_i,y_i)_{i=1}^m]})} = \sum_{\psi\in\Psi(e)} \frac{\mathbf{Z}(\Omega_\psi)}{\mathbf{Z}(\Omega_{[m,(x_i,y_i)_{i=1}^m]})}$$
$$\leq \sum_{\substack{\ell\geq x_1, r\leq y_1\\ r-\ell\geq\beta^* N}} \frac{\mathcal{Z}_\ell Z_{r-\ell}\left(\lambda,\frac{r-\ell}{N}\sigma\right)\mu_{r-\ell}^{\lambda,\frac{r-\ell}{N}\sigma}\left(L_{\max}>\beta^* N\right)\mathcal{Z}_{x_2-r}}{\mathcal{Z}_{x_1} Z_{y_1-x_1}\left(0,\sigma\frac{y_1-x_1}{N}\right)\mathcal{Z}_{x_2-y_1}}. \quad (5.64)$$

We can apply Proposition 4.2 to obtain that for any $n\in[\beta^* N, N]$

$$Z_n\left(\lambda,\sigma\frac{n}{N}\right)\mu_n^{\lambda,\sigma\frac{n}{N}}\left(L_{\max}>\beta^* N\right)\leq \frac{C(\lambda,\sigma)}{\sqrt{N}} e^{2nG(\frac{n}{N}\sigma)}. \quad (5.65)$$

We can use (5.62) and Proposition 3.1 to estimate the other terms. We obtain then (for a difference constant)

$$\frac{\mathcal{Z}_\ell Z_{r-\ell}\left(\lambda,\frac{r-\ell}{N}\sigma\right)\mu_{r-\ell}^{\lambda,\frac{r-\ell}{N}\sigma}\left(L_{\max}>\beta^* N\right)\mathcal{Z}_{x_2-r}}{\mathcal{Z}_{x_1} Z_{y_1-x_1}\left(0,\sigma\frac{y_1-x_1}{N}\right)\mathcal{Z}_{x_2-y_1}}$$
$$\leq C(\lambda,\sigma)e^{2(\ell-x_1+y_1-r)F(\lambda)+2(r-\ell)G(\frac{r-\ell}{N}\sigma)-2(y_1-x_1)G(\frac{y_1-x_1}{N}\sigma)}\leq C(\lambda,\sigma), \quad (5.66)$$

where in the last inequality is simply due to the monotonicity of the functional $\beta\mapsto\beta G(\beta\sigma)-\beta F(\lambda)=H(\beta)$ on the interval $\left[\frac{r-\ell}{N},\frac{y_1-x_1}{N}\right]\subset[\beta^*,1]$. Indeed the quantity in the exponent is equal to $2N[H\left(\frac{r-\ell}{N}\right)-H\left(\frac{y_1-x_1}{N}\right)]$. Summing over ℓ and r we obtain the desired bound.

□

6. Metastability proof of Theorem 2.8

For the proof of Theorem 2.8, we simply have to use the previously proved estimates and use a general result proved in [BL15]. We more specifically need a slightly modified version of the statement which we cite from [LT15].

THEOREM 6.1 (Theorem 5.1 in [LT15]). *We consider a sequence of irreducible reversible Markov chains in the state space Ω_N, $\mathcal{H}_N$ a subset of Ω_N and set $\mathcal{H}_N^{\complement}:=\Omega_N\setminus\mathcal{H}_N$. We let μ_N denote the reversible measure of the chain,* gap_N *the spectral gap of the chain, and* $\mathrm{gap}_{N,\mathcal{H}_N}$, $\mathrm{gap}_{N,\mathcal{H}_N^{\complement}}$ *the spectral gap of the corresponding restricted chains. Let $\mathbb{P}_{\mu_N(\cdot|\mathcal{H}_N)}$ denote the distribution of the Markov chain (η_t) with initial distribution $\mu_N(\cdot|\mathcal{H}_N)$. Let us assume that*

(1) $\lim_{N\to\infty}\mu_N(\mathcal{H}_N)=0$.
(2) $\lim_{N\to\infty}\frac{\mathrm{gap}_N}{\min(\mathrm{gap}_{\mathcal{H}_N},\mathrm{gap}_{\mathcal{H}_N^{\complement}})}=0$.

Then under $\mathbb{P}_{\mu_N(\cdot|\mathcal{H}_N)}$ the finite dimensional distribution of the process $\mathbf{1}_{\mathcal{H}_N}(\eta_{tT_{\mathrm{rel}}^N})$ converges to that of a Markov chain which starts at 1 and jumps, at rate one, to 0 where it is absorbed.

The first condition in Theorem 6.1 says that all the mass is concentrated in $\mathcal{H}_N^{\complement}$, and the second condition says that the time for the dynamics restricted to $\mathcal{H}_N$ (or $\mathcal{H}_N^{\complement}$) to relax to local equilibrium is much shorter than that for the dynamics in Ω_N to relax to global equilibrium. Now we collect all for ingredients for verifying the assumptions in Theorem 6.1 to prove Theorem 2.8.

Proof of Theorem 2.8. We recall the definition of $\mathcal{H}_N$ in (2.33). We first check the case $G(\sigma) \le F(\lambda)$ where $\mathcal{H}_N = \mathcal{E}_N^2$. By (3.36) and (3.37) respectively, we have

$$\begin{cases} \mu_N(\mathcal{E}_N^2) \le e^{-CN}, & \text{if } G(\sigma) < F(\lambda); \\ \mu_N(\mathcal{E}_N^2) \le \frac{C}{\sqrt{N}}, & \text{if } G(\sigma) = F(\lambda). \end{cases} \tag{6.1}$$

Now we turn to the case $G(\sigma) > F(\lambda)$ where $\mathcal{H}_N = \mathcal{E}_N^1$. By (3.58), we have

$$\mu_N(\mathcal{E}_N^1) \le e^{-CN}. \tag{6.2}$$

We have thus checked the first assumption in Theorem 6.1 in every case. Now we turn to verify the second assumption. By Proposition 5.8 and Proposition 5.9, we have

$$\min\left(\mathrm{gap}_{\mathcal{H}_N}, \mathrm{gap}_{\mathcal{H}_N^{\complement}}\right) = \min\left(\mathrm{gap}_{N,1}, \mathrm{gap}_{N,2}\right) \ge c(\lambda,\sigma) N^{-C(\lambda,\sigma)}.$$

Moreover, by Proposition 4.1 we have

$$\mathrm{gap}_N \le C(\lambda,\sigma) N^2 \exp\left(-2NE(\lambda,\sigma)\right), \tag{6.3}$$

which allows us to verify the second assumption in Theorem 6.1. We apply Theorem 6.1 to conclude the proof. □

3.A. Proof of Proposition 5.5

Since $\mathrm{gap}_n(0,\sigma) = \mathrm{gap}_{n-1}(1, \sigma\frac{n-1}{n})$ and it is more convenient to deal with $\mathrm{gap}_n(1,\sigma)$, we focus on the lower bound on $\mathrm{gap}_n(1,\sigma)$ combining the ideas in [LL19] and [CMT08] (since this is not a new argument, our proof while complete, keeps the level of details at minimum, we refer the readers to [LL19, section 3.3] and [CMT08, Section 4] for more details in the computation and intuition). For $x \in [\![1, 2n-1]\!]$ and $f : [\![0, 2n]\!] \to \mathbb{R}$, set $(\Delta f)(x) := f(x+1) + f(x-1) - 2f(x)$ and

$$p := \frac{\exp(\frac{2\sigma}{n})}{1 + \exp(\frac{2\sigma}{n})}, \quad q := 1 - p.$$

For $\xi \in \Omega_n$ and $x \in [\![1, 2n-1]\!]$ with $f_\xi(x) := (\frac{q}{p})^{\frac{1}{2}\xi_x}$, a direct computation yields

$$(\mathcal{L}f_\cdot(x))(\xi) = \sqrt{pq}(\Delta f_\xi)(x) - (\sqrt{p} - \sqrt{q})^2 f_\xi(x) - (2p-1)\sqrt{\frac{q}{p}}\mathbf{1}_{\{\xi_{x-1} = \xi_{x+1} = 0\}}. \tag{3.A.1}$$

In view of (3.A.1) and [LL19, Subsection 3.3], for $\xi \in \Omega_n$ we define

$$\begin{aligned} h_n(\xi) &:= -\sum_{x=1}^{2n-1} \left(\frac{q}{p}\right)^{\frac{1}{2}\xi_x} \sin\left(\frac{\pi x}{2n}\right), \\ \Psi(\xi) &:= (2p-1)\sqrt{\frac{q}{p}} \sum_{x=1}^{2n-1} \sin\left(\frac{\pi x}{2n}\right) \mathbf{1}_{\{\xi_{x-1} = \xi_{x+1} = 0\}}. \end{aligned} \tag{3.A.2}$$

Moreover, we introduce a natural partial order on $\Omega_n \times \Omega_n$ as follows

$$\left(\xi \le \xi'\right) \quad \Leftrightarrow \quad \left(\forall x \in [\![1, 2n]\!], \quad \xi_x \le \xi'_x\right),$$

and there is a maximal element and a minimal element in Ω_n. If $\xi \le \xi'$, then

$$h_n(\xi) \le h_n(\xi') \quad \text{and} \quad \Psi(\xi) \ge \Psi(\xi').$$

If $\xi \leq \xi'$, by (3.A.1) we have

$$\begin{aligned}(\mathcal{L}h_n)(\xi') - (\mathcal{L}h_n)(\xi) &= -\left[4\sqrt{pq}\sin^2\left(\frac{\pi}{4n}\right) + (\sqrt{p}-\sqrt{q})^2\right]\big(h_n(\xi') - h_n(\xi)\big) + \Psi(\xi') - \Psi(\xi) \\ &\leq -\left[4\sqrt{pq}\sin^2\left(\frac{\pi}{4n}\right) + (\sqrt{p}-\sqrt{q})^2\right]\big(h_n(\xi') - h_n(\xi)\big)\,,\end{aligned} \tag{3.A.3}$$

where we have used summation by part in the equality. Let $(\eta_t^\xi)_{t\geq 0}$ denote the dynamics starting from $\xi \in \Omega_n$, and there exists a canonical coupling (c.f. [LL20, Appendix A] with the positive constraint) such that

$$\big(\xi \leq \xi'\big) \Rightarrow \left(\forall t \geq 0, \eta_t^\xi \leq \eta_t^{\xi'}\right).$$

Therefore, by [Wil04, Proposition 3] and the fact that

$$\min_{\xi\leq\xi', \xi\neq\xi'} h_n(\xi') - h_n(\xi) > 0, \tag{3.A.4}$$

we have

$$\mathrm{gap}_n(1,\sigma) \geq 4\sqrt{pq}\sin^2\left(\frac{\pi}{4n}\right) + (\sqrt{p}-\sqrt{q})^2 = 1 - 2\sqrt{pq}\left[1 - 2\sin^2\left(\frac{\pi}{4n}\right)\right] \geq 2\sin^2\left(\frac{\pi}{4n}\right), \tag{3.A.5}$$

where we have used $2\sqrt{pq} \leq 1$ in the last inequality. □

CHAPTER 4

Mixing time of the asymmetric simple exclusion process in a random environment

Abstract: We consider the simple exclusion process in the integer segment $[\![1,N]\!]$ with $k \le N/2$ particles and spatially inhomogenous jumping rates. A particle at site $x \in [\![1,N]\!]$ jumps to site $x-1$ (if $x \ge 2$) at rate $1-\omega_x$ and to site $x+1$ (if $x \le N-1$) at rate ω_x if the target site is not occupied. The sequence $\omega = (\omega_x)_{x\in\mathbb{Z}}$ is chosen by IID sampling from a probability law whose support is bounded away from zero and one (in other words the random environment satisfies the uniform ellipticity condition). We further assume $\mathbb{E}[\log \rho_1] < 0$ where $\rho_1 := (1-\omega_1)/\omega_1$, which implies that our particles have a tendency to move to the right. We prove that the mixing time of the exclusion process in this setup grows like a power of N. More precisely, for the exclusion process with $N^{\beta+o(1)}$ particles where $\beta \in [0,1)$, we have in the large N asymptotic

$$N^{\max\left(1,\frac{1}{\lambda},\beta+\frac{1}{2\lambda}\right)+o(1)} \le t_{\mathrm{mix}}^{N,k} \le N^{C+o(1)}$$

where $\lambda > 0$ is such that $\mathbb{E}[\rho_1^\lambda] = 1$ ($\lambda = \infty$ if the equation has no positive root) and C is a constant which depends on the distribution of ω. We conjecture that our lower bound is sharp up to sub-polynomial correction.

1. Introduction

1.1. Overview. From the viewpoint of Probability and Statistical Mechanics, the simple exclusion process is one of the simplest interacting particle system. It is a reasonable toy model to describe the relaxation of a low density gas and we refer to [Lig12, Chapter VIII.6] for a historical introduction. Its relaxation to equilibrium has been the object of extensive study under a variety of perspective: Hydrodynamic limits [Ros81, KOV89, Rez91], Relaxation Time [DSC93, Qua92] log-Sobolev inequalites [Yau97] and Mixing Time [BBHM05, Mor06] (the list of references is very far from exhaustive).

All the above mentioned works are concerned with the exclusion in an *homogeneous* medium and a small modification of this setup can lead to a drastic change of the pattern of relaxation, see for instance [FGS16, FN17] (and references therein) for the phenomenology induced by the change of the jump rate on a single bond. The *disordered* setup, where the jump rate of the particles is random and varies in space fostered interest only more recently, see for instance [Fag08, FRS19, Sch19].

In the present paper, we are interested in the case IID site disorder on a one dimensional segment, in particular in the case where the local drift felt by particles has a non-constant sign. For the system to reach equilibrium, individual particles need to travel on macroscopic distances and in particular have to fight against drift in some regions. This phenomenon, also present in the case of the random walk in a random environment [GK13, KKS75], induces a slower mixing than in the constant nonzero bias case, as was proved in [Sch19]. Our objective is to quantify further this slow down of the mixing time.

In order to estimate the mixing time of the disordered exclusion process, we need to understand in details how these regions with unfavorable drift – which we refer to as *traps* – affect the pattern of relaxation to equilibrium. We make two important steps toward this objective:

- We prove that the mixing time grows at most like a power of N (the upper bound we prove displays a non optimal exponent).
- We obtain a lower bound on the mixing time, which we conjecture to be optimal, and which allows to identify, depending on the parameters of the system, which is the main factor that slows down the mixing.

More precisely, our proof of the lower bound shows that the mixing time can be bounded from below by three different mechanisms:

(i) Particles cannot move faster than ballistically, so that the mixing time is at least of order N which is the length of the system.

(ii) The particles may remain trapped in potential wells which are created by the environment (see the definition 2.11), so that the mixing time is at least of order $e^{\Delta V}$ where ΔV is the height of the worse potential well in the system.

(iii) The potential wells also limit the flow of particles through the system which is at most of order $e^{-\Delta V/2}$. For this last reason, the mixing time is at least of order $ke^{\Delta V/2}$ when k is the number of particles in the system.

While the two first limitations (i) and (ii) follow from early studies of one dimensional random walk in a random environment [KKS75] and have already been used to determine its mixing time [GK13]. The third limitation is specific to systems with many particles, and to our knowledge, had not been identified so far. It creates a third phase in the conjectured mixing time diagram (see Figure 5).

1.2. The exclusion process in a random environment. Let us introduce formally the random process whose study is the object of this paper. The exclusion process on the segment $[\![1, N]\!]$ with k particles and $1 \le k \le N/2$ is a Markov process that can informally be described as follows.

(A) Each site is occupied by at most one particle (we refer to this constraint as *the exclusion rule)*. Therefore at all time there are k occupied sites and $N - k$ empty sites.

(B) Each of the k particles performs a random walk on the segment, independently of the others, except that any jump that violates the exclusion rule is cancelled.

More precisely, we want to consider the case exclusion process in a *random environement* where the jump rates of the particles are specified by sampling an IID sequence of random variables $\omega = (\omega_x)_{x\in\mathbb{Z}}$, and the transition rates are given by

$$\begin{cases} q_N^\omega(x, x+1) = \omega_x \mathbf{1}_{\{x \le N-1\}}, \\ q_N^\omega(x, x-1) = (1-\omega_x)\mathbf{1}_{\{x\ge 2\}}, \\ q_N^\omega(x, y) = 0 & \text{if } y \notin \{x-1, x+1\}. \end{cases} \tag{1.1}$$

The random walk with transitions q_N^ω which corresponds to the case $k = 1$ is an extensively studied process, usually referred to as Random Walk in a Random Environment (RWRE). The RWRE on the full line $\mathbb{Z}$ was first studied by Solomon in [Sol75] who established a criterion for recurrence/transience. The limit law of the random walk in a random environment is studied by Kesten *et al.* in [KKS75] when the random walk is transient, and by Sinai in [Sin82] when the random walk is recurrent (we refer to [Szn04, Zei04] for complete introductions to this research field).

We are interested in the following quantitative question: How long does the system need to relax to equilibrium, forgetting the information of its initial configuration in the sense of

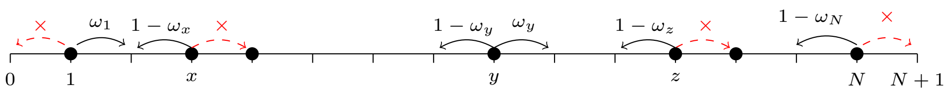


FIGURE 1. A graphical representation of the simple exclusion process in the segment $[\![1, N]\!]$ and environment $\omega = (\omega_x)_{x\in\mathbb{Z}}$: a bold circle represents a particle, and the number above every arrow represents the jump rate while a red " × " represents an inadmissible jump.

total-variation distance? More precisely we are interested in the asymptotic in the limit when $k, N \to \infty$ of this *total-variation mixing time*. This question has been extensively studied in the case where the sequence $\omega = (\omega_x)_{x\in\mathbb{Z}}$ is constant, which we refer to as the *homogeneous environment* case:

(1) When $\omega_x \equiv \frac{1}{2}$, Wilson in [Wil04] showed that the system takes time of order $N^2 \log \min(k, N-k)$ and later Lacoin in [Lac16b] proved that the lower bound in [Wil04] is sharp.
(2) When $\omega_x \equiv p \neq \frac{1}{2}$, Benjamini *et al.* in [BBHM05] told that the system takes time of order N, and later Labbé and Lacoin in [LL19] provided the exact constant.
(3) The case $\omega_x \equiv p_N = \frac{1}{2} + \varepsilon_N$ with $\lim_{N\to\infty} \varepsilon_N = 0$ is studied by Levins and Peres in [LP16], Labbé and Lacoin in [LL20].

From the results mentioned above, for homogeneous environment the system takes time at least of order N and most of order $N^2 \log N$ to relax to equilibrium. However, when the sequence $\omega = (\omega_x)_{x\in\mathbb{Z}}$ is chosen by independently sampling a nondegenerate common law, the system can exhibit a very different behavior because the random environment can create wells of potential which trap particles (see Equation (2.11) below for a definition of the potential associated to ω).

Gantert and Kochler has studied the mixing time problem when $k = 1$ (and transient environment) in [GK13] for random environment and identified the mixing time, which is related with the depth of the deepest trap and may be much larger than $N^2 \log N$. Schmid [Sch19] studied the question in the case of a positive density of particles, when the environment is ballistic to the right, (that is, when the random walk is transient with positive speed) and provided bounds for the mixing time, showing in particular that the mixing time is of order larger than N as soon as the local drift (which is equal to $2\omega_x - 1$) is not uniformly bounded from below by a positive constant and is larger than $N^{1+\delta}$ for some $\delta > 0$ when some sites can display negative drift ($\mathbb{P}[\omega_x < 1/2] > 0$).

In our study we focus on the case of random environments which are such that the random walk is transient (the case of recurrent environment is quite different and should be considered separately). In that setup, the results in [Sch19] leave several questions open, among which the following ones:

(A) Is the mixing time always bounded from above by a power of N?
(B) If this is the case, for the exclusion process with $k_N = N^\beta$ particles and $\beta \in (0, 1)$, can one identify an exponent $\nu > 0$ (depending on β and the distribution) which is such that the mixing time is of order N^ν?

We provide a positive answer to question (A) by proving an upper bound on the mixing time which grows like a power of N. This upper bound is achieved by using a censoring procedure which allows to transport particles one by one to their equilibrium positions. Concerning question (B), we provide a new lower bound on the mixing time which we believe to be optimal and provide a conjecture concerning the value of ν. The bound is based on an analysis of the effect of the deepest trap on the particle flow through the system. Significant technical obstacles prevented us from obtaining a matching upper bound.

2. Model and result

2.1. An introduction to Random Walk in a Random Environment ω. Let us briefly recall the definition for random walk in a random environment. Given $\omega = (\omega_x)_{x\in\mathbb{Z}}$ a sequence with values in $(0,1)$, the random walk in the environment ω is the continuous time Markov chain on $\mathbb{Z}$ whose transition rates are given by

$$\begin{cases} q^\omega(x,x+1) = \omega_x, \\ q^\omega(x,x-1) = 1-\omega_x, \\ q^\omega(x,y) = 0 & \text{if } |x-y| \neq 1. \end{cases} \tag{2.1}$$

We let $(X_t)_{t\geq 0}$ denote the random walk in environment ω and initial condition 0 (we let Q^ω denote the corresponding law). This process has been extensively studied in the case where $\omega = (\omega_x)_{x\in\mathbb{Z}}$ is (the fixed realization of) a sequence of IID random variables (we will use $\mathbb{P}$ and $\mathbb{E}$ denote the associated law and expectation respectively), and we refer to [Szn04, Zei04] for recent reviews.

Simple criteria have been derived on the distribution of ω as necessary and/or sufficient conditions for recurrence/transience, ballisticity etc... Even though most of the results are valid in a more general setup, for the sake of simplicity let us assume in the discussion that the variables $(\omega_x)_{x\in\mathbb{Z}}$ are bounded away from 0 and 1, that is, for some $\alpha \in (0,1/2)$ we have

$$\mathbb{P}(\omega_1 \in [\alpha, 1-\alpha]) = 1. \tag{2.2}$$

Setting $\rho_x := (1-\omega_x)/\omega_x$, it has been proved in [Sol75] that

$$\begin{cases} \mathbb{E}[\log \rho_1] = 0 \Rightarrow X_t \text{ is reccurent under } Q^\omega \text{ , } \mathbb{P}\text{-a.s.,} \\ \mathbb{E}[\log \rho_1] \neq 0 \Rightarrow X_t \text{ is transient under } Q^\omega \text{ , } \mathbb{P}\text{-a.s.} \end{cases} \tag{2.3}$$

More precisely in the second case we have with probability one $\lim_{t\to\infty} X_t = \infty$ (resp. $-\infty$) if $\mathbb{E}[\log \rho_1] < 0$ (resp. $\mathbb{E}[\log \rho_1] > 0$).

When transience holds, the rate at which X_t goes to infinity has also been identified in [KKS75]. It can be expressed in terms of a simple parameter of the distribution ω, yielding in particular a necessary and sufficient condition for ballisticity. Let us assume that $\mathbb{E}[\log \rho_1] < 0$, and set

$$\lambda = \lambda_{\mathbb{P}} := \inf\{s > 0, \mathbb{E}[\rho_1^s] \geq 1\} \in (0,\infty].$$

It has been proved in [KKS75] that if $\lambda > 1$ then there exists $\vartheta_{\mathbb{P}} > 0$ such that

$$\lim_{t\to\infty} \frac{X_t}{t} = \vartheta \tag{2.4}$$

and that if $\lambda \in (0,1]$ then

$$\lim_{t\to\infty} \frac{\log(X_t)}{\log t} = \lambda. \tag{2.5}$$

2.2. The Simple Exclusion process in an environment ω.

Definition. Given a sequence $\omega = (\omega_x)_{x\in\mathbb{Z}}$ taking values in $(0,1)$, $N \geq 2$ and $1 \leq k \leq N-1$, the simple exclusion process in a random environment on the line segment $[\![1,N]\!]$ (we use the notation $[\![a,b]\!] := [a,b] \cap \mathbb{Z}$) with k particles is a Markov process on the space

$$\Omega_{N,k} := \left\{ \xi \in \{0,1\}^N : \sum_{x=1}^{N} \xi(x) = k \right\}. \tag{2.6}$$

The 1s' are denoting particles while 0s' correspond to empty sites. It can be informally described as follows: each of the k particles performs independently a random walk with transitions given by q^ω in (2.1), with the constraints that particles must remain in the segment and each site can be occupied by at most one particle. All transitions that would make this constraint violated (that is, a particle tries to jump either on 0, $N+1$ or an already occupied site) are cancelled.

More formally we let $\xi^{x,y}$ is the configuration obtained by swapping the values of ξ at sites x and y of the configuration ξ, more formally defined by

$$\forall z \in [\![1,N]\!], \quad \xi^{x,y}(z) = \xi(z)\mathbf{1}_{[\![1,N]\!]\setminus\{x,y\}} + \xi(x)\mathbf{1}_{\{y\}} + \xi(y)\mathbf{1}_{\{x\}}. \tag{2.7}$$

The simple exclusion process in environment ω is the Markov process with transition rates given by

$$\begin{aligned} r^\omega(\xi,\xi^{x,x+1}) &:= \begin{cases} \omega_x & \text{if } \xi(x)=1 \text{ and } \xi(x+1)=0, \\ 1-\omega_{x+1} & \text{if } \xi(x+1)=1 \text{ and } \xi(x)=0, \end{cases} \quad \text{for } x\in[\![1,N-1]\!] \\ r^\omega(\xi,\xi') &:= 0 \quad \text{in all other cases.} \end{aligned} \tag{2.8}$$

Equivalently the generator of the process is defined for $f:\Omega_{N,k}\to\mathbb{R}$ by

$$\mathcal{L}^\omega_{N,k}(f)(\xi) := \sum_{x=1}^{N-1} r^\omega(\xi,\xi^{x,x+1})\big[f(\xi^{x,x+1})-f(\xi)\big]. \tag{2.9}$$

The chain is ergodic and reversible. In order to give a simple compact expression for the equilibrium measure, let us introduce the random potential $V^\omega:\mathbb{N}\to\mathbb{R}$ defined as follows, $V^\omega(1):=0$ and for $x\geq 2$

$$V^\omega(x) := \sum_{y=2}^{x} \log\left(\frac{1-\omega_y}{\omega_{y-1}}\right). \tag{2.10}$$

With a small abuse of notation, we extend V^ω to a function of $\Omega_{N,k}$. This extension is obtained by summing the value of V^ω among the positions of the particles in the configuration ξ:

$$V^\omega(\xi) := \sum_{x=1}^{N} V^\omega(x)\xi(x). \tag{2.11}$$

We consider the probability measure $\pi^\omega_{N,k}$ defined by

$$\pi^\omega_{N,k}(\xi) := \frac{1}{Z^\omega_{N,k}} e^{-V^\omega(\xi)} \quad \text{with} \quad Z^\omega_{N,k} = \sum_{\xi\in\Omega_{N,k}} e^{-V^\omega(\xi)}. \tag{2.12}$$

It is immediate to check by inspection that $\pi^\omega_{N,k}$ satisfies the detailed balance condition for $\mathcal{L}^\omega_{N,k}$, and thus that it is the unique invariant probability measure on $\Omega_{N,k}$.

If $\xi\in\Omega_{N,k}$, we let $(\sigma^\xi_t)_{t\geq 0}$ denote the Markov chain with initial condition ξ. We are going to provide a construction $(\sigma^\xi_t)_{t\geq 0}$ for all $\xi\in\Omega_{N,k}$ on a common probability space in Section 3.2, and we use $\mathbf{P}$ and $\mathbf{E}$ for the corresponding probability law and expectation respectively. We let $(P_t)_{t\geq 0}$ (the dependence in ω, N, k is omitted in the notation to keep it light) denote the corresponding Markov semi-group and set $P^\xi_t := \mathbf{P}(\sigma^\xi_t\in\cdot) = P_t(\xi,\cdot)$ to be the marginal distribution of $(\sigma^\xi_t)_{t\geq 0}$ at time t.

Mixing time and spectral gap. In a standard fashion, we set the total variation-distance to equilibrium at time t to be

$$d^\omega_{N,k}(t) := \max_{\xi\in\Omega_{N,k}} \|P^\xi_t - \pi^\omega_{N,k}\|_{\mathrm{TV}} \tag{2.13}$$

where $\|\nu_1 - \nu_2\|_{\mathrm{TV}} := \sup_{A \subset \Omega_{N,k}} |\nu_1(A) - \nu_2(A)|$ denotes the total variation between two probability measures ν_1, ν_2 on $\Omega_{N,k}$. Since the Markov chain is irreducible, we know that (cf. [LP17, Theorem 4.9])

$$\lim_{t \to \infty} d^{\omega}_{N,k}(t) = 0. \tag{2.14}$$

We are interested in having quantitative statements related to the convergence (2.14), and for this reason we want to evaluate the mixing time and spectral gap of the chain (see [LP17] for a motivated and thorough introduction to these notions). For $\epsilon \in (0,1)$, let the ε-mixing time of the chain be defined by

$$t^{N,k,\omega}_{\mathrm{mix}}(\epsilon) := \inf \left\{ t \geq 0 : d^{\omega}_{N,k}(t) \leq \epsilon \right\}. \tag{2.15}$$

By convention, we simply write $t^{N,k,\omega}_{\mathrm{mix}}$ when $\varepsilon = 1/4$. The spectral gap of the chain $\mathrm{gap}^{\omega}_{N,k}$, in our context, can be defined as the smallest non-zero eigenvalue of $-\mathcal{L}^{\omega}_{N,k}$. It can be shown using a spectral decomposition (see for instance [LP17, Corollary 12.7]) to determine the asymptotic rate of convergence of $d^{\omega}_{N,k}$ as

$$\lim_{t \to \infty} \frac{1}{t} \log d^{\omega}_{N,k}(t) = -\mathrm{gap}^{\omega}_{N,k}. \tag{2.16}$$

The mixing time and spectral gap are related to one another by the following relation valid for $\varepsilon \in (0, 1/2)$ (cf. [LP17, Theorems 12.4 and 12.5])

$$\frac{1}{\mathrm{gap}^{\omega}_{N,k}} \log\left(\frac{1}{2\varepsilon}\right) \leq t^{N,k,\omega}_{\mathrm{mix}}(\epsilon) \leq \frac{1}{\mathrm{gap}^{\omega}_{N,k}} \log\left(\frac{1}{\varepsilon \pi_{\min}}\right) \tag{2.17}$$

where

$$\pi_{\min} = \min_{\xi \in \Omega_{N,k}} \pi^{\omega}_{N,k}(\xi).$$

2.3. Results. The main object of the chapter is the study of the exclusion process in an IID environment. On the way to our main result, we also prove bounds on the mixing time which are valid for any realization of ω, and which we present first.

Universal bounds for the mixing time on the exclusion process. We assume without loss of generality (by symmetry) that $k \leq N/2$. We prove that the mixing time grows at least linearly with the size of the system and at most exponentially. Both results are in a sense optimal (see the discussion in Section 2.4. below).

PROPOSITION 2.1. *For any $k \in [\![1, N/2]\!]$ and $N \geq 2$, for any $(\omega_x)_{x \in \mathbb{Z}}$ we have*

$$t^{N,k,\omega}_{\mathrm{mix}} \geq \frac{1}{16} N. \tag{2.18}$$

Furthermore, if k_N is a sequence such that

$$k_N \leq N/2 \text{ and } \lim_{N \to \infty} k_N = \infty, \tag{2.19}$$

we have for any $\varepsilon > 0$, for $N \geq N_0(\varepsilon)$ sufficiently large for any $(\omega_x)_{x \in \mathbb{Z}}$

$$t^{N,k_N,\omega}_{\mathrm{mix}}(1-\varepsilon) \geq \frac{1}{30} N. \tag{2.20}$$

For the upper bound, we require an assumption similar to (2.2), that is

$$\forall x \in \mathbb{Z}, \quad \omega_x \in [\alpha, 1-\alpha]. \tag{2.21}$$

PROPOSITION 2.2. *For any sequence* $(\omega_x)_{x\in\mathbb{Z}}$ *satisfying* (2.21) *all* $N \geq 2$ *and all* $k \in [\![1, N/2]\!]$, *we have*

$$\mathrm{gap}^{\omega}_{N,k} \geq \alpha N^{-2} |\Omega_{N,k}|^{-1} \left(\frac{1-\alpha}{\alpha}\right)^{-N/2}, \tag{2.22}$$

and as a consequence for all $\varepsilon \in (0, 1/2)$

$$t^{N,k,\omega}_{\mathrm{mix}}(\varepsilon) \leq \alpha^{-1} N^2 |\Omega_{N,k}| \left(\frac{1-\alpha}{\alpha}\right)^{N/2} \left(\log|\Omega_{N,k}| + Nk \log\frac{1-\alpha}{\alpha} - \log\varepsilon\right). \tag{2.23}$$

Mixing time for the exclusion process in a random environment. Let us now introduce our main results concerning the exclusion process in a random environment. We assume that (2.2) holds,

$$\mathbb{E}[\log\rho_1] < 0 \quad \text{and} \quad 1 \leq k \leq N/2. \tag{2.24}$$

Using the various symmetries of the the system (between left and right, particles and empty sites...), assumption (2.24) entails almost no-loss of generality, and the only case being left aside is that of a recurrent environment (that is $\mathbb{E}[\log\rho_1] = 0$). We are also going to consider that $\lambda_{\mathbb{P}} < \infty$, this corresponds to saying that $\mathbb{P}[\,\omega_1 < 1/2] > 0$ (the case $\mathbb{P}[\,\omega_1 \geq 1/2] = 1$ is discussed in the next section).

In order to bet a better intuition on the result, let us provide a description of the equilibrium measure. We introduce the event $\mathcal{A}_r \subset \Omega_{N,k}$ that the leftmost particle and rightmost empty site are at a distance smaller than $2r$ of their respective maximal and minimal possible values:

$$\mathcal{A}_r := \{\xi \in \Omega_{N,k} \ : \ \forall x \in [\![1, N-k-r]\!],\ \xi(x) = 0\ ;\ \forall x \geq N-k+r, \xi(x) = 1\}. \tag{2.25}$$

The following result tells us that the mass of π_{N,k_N} is essentially concentrated at a finite distance of the configuration $\xi_{\max}$ with all k particles packed to the right (see (3.3)).

LEMMA 2.3. *Under the assumptions* (2.21) *and* (2.24), *for all* N *sufficiently large we have*

$$\lim_{r\to\infty} \inf_{\substack{N\geq 1\\ k\in[\![1,N/2]\!]}} \mathbb{E}\left[\pi^{\omega}_{N,k_N}(\mathcal{A}_r)\right] = 1. \tag{2.26}$$

Our first main result is that if the environment satisfies the assumptions (2.2) and (2.24), the system relaxes to equilibrium in polynomial time, or in other words that $t^{N,k,\omega}_{\mathrm{mix}}$ grows like a power of N with an explicit upper bound on the growth exponent. In order to describe our explicit bound, we need to introduce the function F which is the log-Laplace transform of $\log\rho_1$ that is

$$F(u) := \log\mathbb{E}\left[\rho_1^u\right]. \tag{2.27}$$

Since V^ω is, up to a small modification, a sum of IID variables with the same distribution

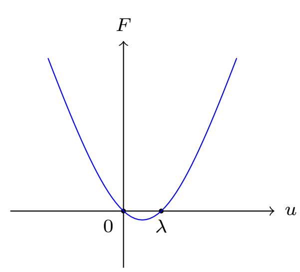


FIGURE 2. A graphical description of the function $F(u)$ with only two zeros at $u = 0$ and $u = \lambda$.

as $\log\rho_1$, the function F is used to compute the large deviations of V^ω, and in particular to determine the geometry of the deepest potential wells. It is strictly convex and satisfies $F(0) = F(\lambda) = 0$ (see Figure 2). We let u_0 be defined by

$$F(u_0) = \min_{u\in\mathbb{R}} F(u) < 0.$$

Given a sequence of events $(A_N)_{N\geq 1}$, we say that A_N holds *with high probability* (which we sometimes abbreviate as w.h.p.) if $\lim_{N\to\infty} \mathbb{P}[A_N] = 1$. Given a sequence $(B_{N,k})_{N\geq 1, k\in[\![1,N/2]\!]}$, we say that $B_{N,k}$ holds *with high probability* if

$$\lim_{N\to\infty} \inf_{k\in[\![1,N/2]\!]} \mathbb{P}[B_{N,k}] = 1.$$

We are now ready to state the result.

THEOREM 2.4. *Under the assumptions* (2.2) *and*(2.24)*, then with high probability we have*

$$t_{\mathrm{mix}}^{N,k,\omega} \leq 80kN\alpha^{-1}\left(\frac{3u_0+2}{|F(u_0)|}\log N\right)^4 N^{\frac{3u_0+2}{|F(u_0)|}\left(2\log\frac{1-\alpha}{\alpha}+4\log 4-3\log 3\right)}. \tag{2.28}$$

Our second result provides a lower bound for the mixing time which depends both on N and k.

THEOREM 2.5. *Under the assumptions* (2.2)-(2.24) *and assuming further that $\lambda_{\mathbb{P}} < \infty$, there exists a positive constant $c(\alpha,\mathbb{P})$ such that w.h.p. we have for every N and $k \in [\![1, N/2]\!]$*

$$t_{\mathrm{mix}}^{N,k,\omega} \geq c\max\left\{N, N^{\frac{1}{\lambda}}(\log N)^{-\frac{2}{\lambda}}, kN^{\frac{1}{2\lambda}}(\log N)^{-2(1+\frac{1}{\lambda})}\right\}. \tag{2.29}$$

2.4. Related work. Let us provide now a short review of related results present in the literature.

Mixing time for the exclusion process in a homogeneous environment. The mixing time of the exclusion process on the line segment has been extensively studied in the case where the sequence ω is constant, *i.e.* $\omega \equiv p$. In that case, not only the right order of magnitude has been identified for the mixing time, but also the sharp asymptotic equivalent. The case of the exclusion with no bias that is $p = 1/2$ (the simple symmetric exclusion process), it was shown in [Ald83a] that the mixing time for the exclusion process on the segment is of order at least N^2 and at most $N^2(\log N)^2$. It was later established (see [Wil04] for the lower bound and [Lac16b] for the upper bound) that if k_N satisfies (2.19), we have

$$t_{\mathrm{mix}}^{N,k_N}(\varepsilon) = \frac{(1+o(1))}{\pi^2}N^2\log k_N. \tag{2.30}$$

In the case where the walk presents a bias, that is $p \neq 1/2$, it was shown in [BBHM05] that the mixing time is of order N. This result was refined in [LL19] by identifying the proportionality constant, showing that if k_N satisfies $\lim_{N\to\infty} k_N/N = \theta$, then

$$t_{\mathrm{mix}}^{N,k_N}(\varepsilon) = [1+o(1)]\frac{(\sqrt{\theta}+\sqrt{1-\theta})^2}{|2p-1|}N. \tag{2.31}$$

The case where p is allowed to depend on N was investigated in [LP16, LL20] where the order of magnitude and the sharp asymptotic of the mixing time were respectively determined. Note that in (2.30) and (2.31) the asymptotic behavior of $t_{\mathrm{mix}}^{N,k_N}(\varepsilon)$ does not display any dependence on ε at first order. This implies that $d_{N,k_N}(t)$ abruptly drops from 1 to 0 on the time scale $N^2\log k_N$ and N respectively. This phenomenon, called cutoff, is expected to hold for a large class of Markov chains, we refer to [LP17, Chapter 18] for an introduction.

Let us also mention that the mixing time for the one-dimensional exclusion process has also been investigated for a variety of different boundary conditions. We refer to [Lac16a] for a sharp estimate of the convergence profile to equilibrium for the periodic boundary condition in the symmetric case and to [GNS20] (and references therein) for the study of a variety of boundary conditions, with or without bias. The case of higher dimension has also been considered, see e.g. [Mor06] where the order of magnitude of the mixing time is determined up to a constant.

Mixing time for the random walk in a random environment. In [GK13], the case of the mixing time for a random walk in the segment with a transient random environment (which corresponds to the case $k = 1$ in the present chapter) was investigated. It is shown that whenever $\lambda_{\mathbb{P}} > 1$ then

$$t_{\mathrm{mix}}^{N,1,\omega}(\varepsilon) = [1 + o(1)] \frac{N}{\mathbb{E}\left[Q^{\omega}[T_1^{\omega}]\right]}, \tag{2.32}$$

where T_1^{ω} is the first hitting time of 1 for the random walk in a random environment ω starting from 0 (the result in [GK13] is slightly more precise and the assumption is more general than (2.2)). When $\lambda_{\mathbb{P}} < 1$, it is shown that the mixing time is of a much larger magnitude but that cutoff does not hold. More precisely, for $\lambda_{\mathbb{P}} \leq 1$ we have

$$\lim_{N\to\infty} \frac{\log t_{\mathrm{mix}}^{N,1,\omega}(\varepsilon)}{\log N} = \frac{1}{\lambda_{\mathbb{P}}}. \tag{2.33}$$

The asymptotic $N^{1/\lambda_{\mathbb{P}}+o(1)}$ corresponds to the time that is required to overcome the largest potential barrier present in the system, whose height is of order $(1/\lambda)\log N$.

Mixing time for the exclusion in a ballistic environment. In [Sch19], the mixing time $t_{\mathrm{mix}}^{N,k_N,\omega}$ were investigated under the assumption that $\lim_{N\to\infty} k_N/N = \theta \in (0, 1/2]$ and $\lambda_{\mathbb{P}} > 1$. Three different cases are considered.

- When $\operatorname{ess\,inf} \omega_1 > 1/2$, it is shown that the mixing $t_{\mathrm{mix}}^{N,k_N,\omega}$ is of order N, by a simple comparison with the case of homogeneous asymmetric environment.
- When $\operatorname{ess\,inf} \omega_1 < 1/2$, it is shown that there exists a positive δ such that the mixing time satisfies $t_{\mathrm{mix}}^{N,k_N,\omega} \geq N^{1+\delta}$.
- When $\operatorname{ess\,inf} \omega_1 = 1/2$, it is shown that

$$\liminf_{N\to\infty} t_{\mathrm{mix}}^{N,k_N,\omega}(\varepsilon)/N = \infty \quad \text{and} \quad t_{\mathrm{mix}}^{N,k_N,\omega}(\varepsilon) \leq CN(\log N)^3, \tag{2.34}$$

together with a quantitative lower bound if $\mathbb{P}[\omega_1 = 1/2] > 0$.

Other perspectives concerning the exclusion process and random environments. The exclusion process with other types of random environment has also been considered in the literature. One possibility is to consider a random environment on bonds instead of sites. A particular choice which makes the uniform measure on $\mathbb{Z}$ reversible for the random walk is the model of random conductance. In that case the mixing property of the system strongly differs from model considered here: the equilibrium measure is uniform on $\Omega_{N,k}$ so that there is no trapping by potential. It is expected that for a large class of environment in that case the mixing properties are very similar to that of the homogeneous system. The hydrodynamic limit of exclusion processes with bond–dependent random transition rates have been studied in [Fag08, Jar11] (see also [Fag20] for a recent work going slightly beyond the random conductance model).

Another corpus of work has been considering the (homogeneous) exclusion process itself as a dynamical random environment, which determines the transition probability of the random walk. The asymptotic behavior of a random walker in this setup is studied in [HKT20, HS15], and the hydrodynamic limit for the exclusion process as seen by this walker is studied in [AFJV15]. In a more general setup for the jump rates of the walker, an invariance principle about the random walk when the exclusion process starts from equilibrium is studied in [JM20].

2.5. Interpretation of our results and conjectures.

Comments on Propositions 2.1 and 2.2. The asymptotic for the mixing time for ASEP in homogeneous environment (2.31) shows that the lower bound of Proposition 2.1 is sharp up to a constant factor. An important observation is also that (2.20) is not true without the assumption that k_N goes to infinity, even if $1/30$ is replaced by an arbitrarily small constant provided that it is not allowed to depend on ε.

However the constant in our bounds (2.18) and (2.20) are clearly not optimal. Let us state now a natural conjecture. We believe that if $\lim_{N\to\infty} k_N/N = \theta \in (0,1/2]$, and $\omega_x \in [\alpha, 1-\alpha]$ for all $x \in \mathbb{Z}$ (with the possibility of having $\alpha = 0$) then we should have

$$\liminf_{N\to\infty} \frac{1}{N} t_{\mathrm{mix}}^{N,k_N}(1-\varepsilon) \geq \frac{(\sqrt{\theta}+\sqrt{1-\theta})^2}{1-2\alpha}. \tag{2.35}$$

One can obtain counter examples to (2.35) in the zero density case by considering the case $\omega_x = 1-\alpha$ in the first half of the segment $[\![1,N]\!]$ and $\omega = \alpha$ in the second half of the segment, and k_N diverging to infinity such that $\lim_{N\to\infty} k_N/(\log N) = 0$. In that case, one can with some minor efforts, show that the mixing time is asymptotically equivalent $\frac{N}{2-4\alpha}$ (which is half of the lower bound in (2.35)).

Proposition 2.2 can also be shown to be sharp within constant in the sense that there exists a constant C_α, and for given N and k it is always possible to construct an environment ω such that

$$\mathrm{gap}_{N,k}^{\omega} \geq e^{-C_\alpha N}. \tag{2.36}$$

We conjecture that the best possible lower bound on the spectral gap when $\lim_{N\to\infty} k_N/(\log N) = \theta \in (0,1/2]$ is the following

$$\liminf_{N\to\infty} \frac{\log \mathrm{gap}_{N,k_N}^{\omega}}{N} = -\frac{(1-\theta)}{2} \log\left(\frac{1-\alpha}{\alpha}\right). \tag{2.37}$$

The lim inf is reached asymptotically by the environment

$$\begin{cases} \omega_x = \alpha & \text{if } 1 \leq x \leq \frac{(1-\theta)N}{2}, \\ \omega_x = 1-\alpha & \text{if } \frac{(1-\theta)N}{2} < x \leq N. \end{cases} \tag{2.38}$$

Comments on Theorems 2.4 and 2.4. Our chapter brings a complement to the results in [Sch19], in the case when $\operatorname{ess\,inf} \omega_1 < 1/2$. Firstly it provides a complementary upper bound result, which shows that the mixing time in transient environment always scales like a power of N, even in the non-ballistic case $\lambda_{\mathbb{P}} \leq 1$.

Secondly, it provides a more quantitative lower bound. In (2.29) the mixing time is bounded by the maximum of three quantities. Each of them corresponds to a different mechanism which prevents the mixing time to be lower than a certain value.

- *Mass transport cannot be faster than ballistic:* Which is explored in Proposition 2.1 is that particle cannot move faster than ballistically (and this is independent of the choice of ω), so that the time required to transport the mass of particles to equilibrium has to be at least of order N. This idea is already present in [BBHM05].
- *Individual particles may be blocked by traps in the potential profile:* As soon as $\operatorname{ess\,inf} \omega_1 < 1/2$, the potential profile V is non-monotone and will create energy barriers. It is known since [KKS75] that these energy barriers can slow down particles to subballistic speed in $\lambda_{\mathbb{P}} \leq 1$ by creating traps that will require a long time to be crossed. This is the mechanism that was used to identify the mixing time in case of a single particle in [GK13] (recall (2.33)), and it corresponds to the time needed to cross the largest trap in the potential. This yields the second term in (2.29).

- *Potential barrier may also create bottleneck for the flow of particles:* The third mechanism which was partially identified in [Sch19] is that potential barrier may also limit the flow of particles throughout the system. The limitation on the flow does not correspond to the inverse of the time that a particle needs to cross the trap, but rather to the square root of this inverse. The reason for this is that when particles are flowing through the system, the particle are "filling" half of the potential well, so that the remaining potential barrier to be crossed is halved. This reasoning yields the third term in (2.29).

We believe that the three mechanism described above are the only possible limiting factor to mixing, and thus that the lower bound give in Theorem 2.5 is sharp as far as the exponent is concerned. Let us formulate this as a conjecture. Let us assume that k_N satisfies

$$\lim_{N\to\infty} \frac{\log k_N}{\log N} = \beta,$$

and then we should have the following convergence w.h.p.

$$\lim_{N\to\infty} \frac{t_{\mathrm{mix}}^{N,k_N}}{\log N} = \max\left(1, \frac{1}{\lambda}, \frac{1}{2\lambda} + \beta\right). \tag{2.39}$$

We refer to Figure 3 for the phase diagram concerning the conjectured exponent of the mixing time.

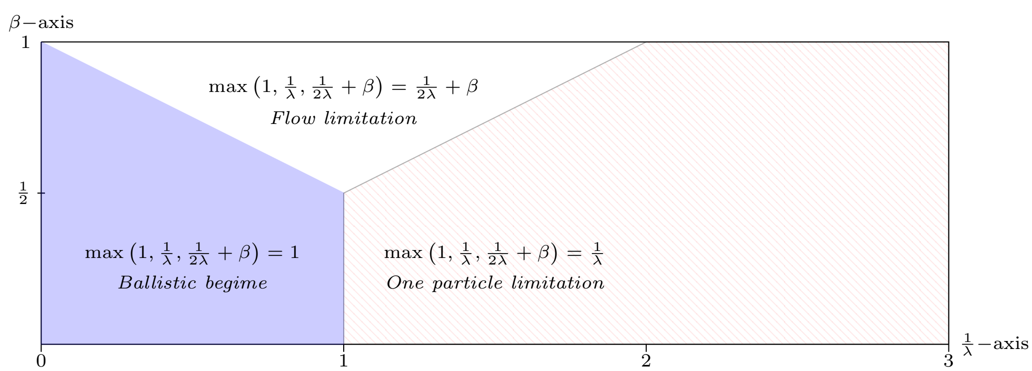


FIGURE 3. The phase diagram for the exponent of the mixing time (the lower bound is proved rigorously and the upper bound is only conjectured). The transition between the blue and red (hatched) regions of the diagram corresponds to the transition of the RWRE from the ballistic phase to the transient-with-zero-speed phase. A third phase represented by the white region appears when one considers a large number of particles, in this phase the main limitation to mixing is the flow of particle through the deepest trap.

In particular this means that when $\beta \le 1/(2\lambda)$ then the mixing time of the exclusion process on the segment coincides (as far as the exponent is concerned) with that of the random walk in the segment.

Organization of the chapter. Section 3 is devoted to some technical preliminaries including the particle description, equilibrium estimates, partial order, a graphical construction and a composed censoring inequality.

Section 4 is devoted to universal lower and upper bounds on the mixing time for all random environments, that is, the proof of Propositions 2.1 and Proposition 2.2.

Section 5 is devoted to lower bounds on the mixing time, that is Theorem 2.5. There are three bounds to prove, one of them is a consequence of Proposition 2.1, the other two are presented as two distinct results (Proposition 5.1 and Proposition 5.2) and proved in separate subsections. The first bound rely on controlling the displacement of the leftmost particle while the other is based on a control of the particle flow.

Section 6 is concerned with the upper bound on the mixing time (Theorem 2.4). The proof is based on application of the censoring inequality and of our upper bound from Proposition 2.2: blocking the transition along carefully chosen edges (in a way that varies throught time) we guide all particlesto the right of the segment (where they are typically located at equilibrium) in polynomial time.

Notation. We use $c(\alpha,\mathbb{P})$ and $C(\alpha,\mathbb{P})$ to stress that the constants c and C depend on α and the law of the random environment ω. Moreover, we use " := " (or " =: ") to define a new quantity on the left-hand (right-hand, resp.) side, and $[\![a,b]\!] := [a,b]\cap\mathbb{Z}$. Furthermore, we let

$$\Omega_{[a,b],k} := \left\{ \xi\in\{0,1\}^{[\![a,b]\!]} : \sum_{x\in[\![a,b]\!]} \xi(x) = k \right\} \tag{2.40}$$

denote the state space of k particles performing exclusion process restricted in the interval $[\![a,b]\!]$ and environment ω, and let $\pi^{\omega}_{[a,b],k}$ denote the corresponding equilibrium probability measure.

3. Technical preliminaries

3.1. Partial order on $\Omega_{N,k}$. Given $\xi\in\Omega_{N,k}$ we define $\bar{\xi}\ :\ [\![1,k]\!]\to[\![1,N]\!]$ as an increasing function which provides the positions of the particles of ξ from left to right:

$$\{\bar{\xi}(i)=x\} \quad\Longleftrightarrow\quad \left\{\xi(x)=1 \text{ and } \sum_{y=1}^{x}\xi(x)=i\right\}. \tag{3.1}$$

We introduce a natural partial order relation "$\le$" on $\Omega_{N,k}\times\Omega_{N,k}$ as follows

$$(\xi\le\eta) \ \Leftrightarrow \left(\forall i\in[\![1,k]\!],\quad \bar{\xi}(i)\le\bar{\eta}(i)\right). \tag{3.2}$$

Informally $\xi\le\eta$ means that the particles in the configuration η are located "more to the right" than those of ξ. Let $\xi_{\max}$ and $\xi_{\min}$ denote the maximal and minimal configurations of $(\Omega_{N,k}, \text{“}\le\text{”})$ respectively, given by

$$\xi_{\max} := \mathbf{1}_{\{N-k+1\le x\le N\}} \quad\text{ and }\quad \xi_{\min} := \mathbf{1}_{\{1\le x\le k\}}. \tag{3.3}$$

This order plays a special role for our dynamic $(\sigma^{\xi}_t)_{t\ge 0}$, and the next two subsections provide tools to exploit this link.

3.2. Canonical coupling via graphical construction. Let us present a construction of a grand coupling for the exclusion process on the segment $[\![1,N]\!]$ which has the property of conserving the order defined above.

To each site $x\in[\![1,N]\!]$ we associate an independent rate 1 Poisson clock process $(T^{(x)}_i)_{i\ge1}$ (the increments of the sequence $(T^{(x)}_i)_{i\ge1}$ are independent exponential variables of parameter 1) and an independent sequence of IID variables $(U^{(x)}_i)_{i\ge1}$ with uniform distribution on $[0,1]$. These variables are independent of the environment $\omega=(\omega_x)_{x\in\mathbb{Z}}$, and the trajectory $(\sigma^{\xi}_t)_{t\ge0}$ for each ξ is a deterministic function of $(T^{(x)}_i, U^{(x)}_i)_{i\ge1, x\in[\![1,N]\!]}$. In the remainder of the paper, $\mathbf{P}$ denote the joint law of $(T^{(x)}_i, U^{(x)}_i)_{i\ge1, x\in[\![1,N]\!]}$, and $\mathbf{E}$ denotes the corresponding expectation. Let us also introduce a natural filtration $(\mathcal{F}_t)_{t\ge0}$ in this probability space setting

$$i_0(x,t) := \max\{i\ge1\ :\ T^{(x)}_i\le t\} \tag{3.4}$$

with the convention that $\max\emptyset = 0$ and set

$$\mathcal{F}_t := \sigma\left(T_i^{(x)}, U_i^{(x)}, x \in \mathbb{Z}, i \le i_0(x,t)\right). \tag{3.5}$$

Now, given $1 \le k \le N-1$ and an initial configuration $\xi \in \Omega_{N,k}$, we construct the trajectory $(\sigma_t^\xi)_{t\ge 0}$ as follows:

(1) $(\sigma_t^\xi)_{t\ge 0}$ is càdlàg and may change its value only at times $T_i^{(x)}$, $x \in [\![1,N]\!]$ and $i \ge 1$.

(2) We construct the trajectory starting with $\sigma_0^\xi = \xi$ and modifying it sequentially at the update times $(T_i^{(x)})_{i\ge 1, x\in[\![1,N]\!]}$. For instance if $t = T_i^{(x)}$ we obtain σ_{t-}^ξ from σ_t^ξ as follows:

(A) If $U_i^{(x)} \le \omega_x$, $x \le N-1$, $\sigma_{t_-}^\xi(x) = 1$ and $\sigma_{t_-}^\xi(x+1) = 0$, then $\sigma_t^\xi(x+1) = 1$ and $\sigma_t^\xi(x) = 0$ (and $\sigma_t^\xi(y) = \sigma_{t_-}^\xi(y)$ for $y \notin \{x, x+1\}$).

(B) If $U_i^{(x)} > \omega_x$, $x \ge 2$, $\sigma_{t_-}^\xi(x) = 1$ and $\sigma_{t_-}^\xi(x-1) = 0$, then $\sigma_t^\xi(x-1) = 1$ and $\sigma_t^\xi(x) = 0$ (and $\sigma_t^\xi(y) = \sigma_{t_-}^\xi(y)$ for $y \notin \{x-1, x\}$).

(C) In all other cases $\sigma_t^\xi = \sigma_{t_-}^\xi$.

It is immediate by inspection to check that the above construction corresponds indeed to the Markov chain with generator $\mathcal{L}_{N,k}^\omega$. Note also that our process is adapted and Markov with respect to the filtration $(\mathcal{F}_t)_{t\ge 0}$. In the same manner, the reader can check that it preserves the order in the following sense.

Proposition 3.1. *For the coupling constructed above, we have for all $\xi, \xi' \in \Omega_{N,k}$*

$$\xi \le \xi' \Rightarrow \mathbf{P}\left[\forall t \ge 0, \quad \sigma_t^\xi \le \sigma_t^{\xi'}\right] = 1. \tag{3.6}$$

3.3. Composed censoring inequality. We are going to use a variant of the censoring inequality introduced by Peres and Winckler [PW13]. Let $E_N = \{\{n, n+1\} : n \in [\![1, N-1]\!]\}$ be the set of edges in $[\![1,N]\!]$, and a censoring scheme $\mathcal{C} : [0,\infty) \to \mathcal{P}(E_N)$ is a deterministic càdlàg function where $\mathcal{P}(E_N)$ is the set of all subsets of E_N.

The censored chain $(\sigma_t^{\xi,\mathcal{C}})_{t\ge 0}$ is a time inhomegenous Markov chain, with a generator obtained by cancelling the transition using edges in $\mathcal{C}(t)$

$$\mathcal{L}_{N,k}^{\mathcal{C},t}(f)(\xi) := \sum_{x=1}^{N-1} r_{N,k}^\omega(\xi, \xi^{x,x+1}) \mathbf{1}_{\{\{x,x+1\}\notin\mathcal{C}(t)\}}\left[f(\xi^{x,x+1}) - f(\xi)\right], \tag{3.7}$$

where $r_{N,k}^\omega(\xi, \xi^{x,x+1})$ is defined in (2.8). We let $P_t^{\mathcal{C}}$ be the associated semigroup (the solution of $\partial_t P_t = P_t \mathcal{L}_{N,k}^{\mathcal{C},t}$ with initial condition given by the identity). We will use the following corollary of the censoring inequality [PW13, Theorem 1] (recall (3.3)).

Proposition 3.2. *For any $\xi \in \Omega_{N,k}$ and any censoring scheme $\mathcal{C}$, we have*

$$P_t(\xi, \xi_{\max}) \ge P_t^{\mathcal{C}}(\xi_{\min}, \xi_{\max}) \tag{3.8}$$

Sketch of proof. Proposition 3.1 implies that $P_t(\xi, \xi_{\max}) \ge P_t(\xi_{\min}, \xi_{\max})$. To compare $P_t(\xi_{\min}, \xi_{\max})$ with $P_t^{\mathcal{C}}(\xi_{\min}, \xi_{\max})$, we rely on the censoring inequality [PW13, Theorem 1] (to see that the exclusion process fits the setup in [PW13], one uses the height function representation see e.g. [Lac16b, Section A.2]) which implies that $P_t(\xi_{\min}, \cdot)$ stochastically dominates $P_t^{\mathcal{C}}(\xi_{\min}, \cdot)$. □

We consider a modified censored dynamics, where on top of censoring, at fixed time, we replace the current configuration by one which is lower for the order $\ge$ by moving some particles to the left. For the application we have in mind, we can consider that these replacements are performed deterministically (although the result would hold also for random replacements).

Let $(s_i)_{i=1}^I$ be an increasing time sequence tending to infinity and let $(Q_i)_{i=1}^I$ be a sequence of stochastic matrices on $\Omega_{N,k}$ such that for all ξ in $\Omega_{N,k}$ there exists ξ' (depending on ξ and i) such that

$$\begin{cases} \xi' \leq \xi, \\ Q_i(\xi,\xi') = 1, \\ Q_i(\xi,\xi'') = 0, \text{ when } \xi'' \neq \xi'. \end{cases} \tag{3.9}$$

We consider $\widetilde{P}_t$ the semigroup defined by

$$\begin{cases} \widetilde{P}_0 = \mathrm{Id}, \\ \partial_t \widetilde{P}_t = \widetilde{P}_t \mathcal{L}^{\mathcal{C},t} \text{ if } t \notin \{s_i\}_{i=1}^I, \\ \widetilde{P}_{s_i} = \widetilde{P}_{(s_i)_-} Q_i. \end{cases} \tag{3.10}$$

Proposition 3.3. *For any choice of $(s_i)_{i=1}^I$, $(Q_i)_{i=1}^I$ and $\mathcal{C}$, we have for all $t \geq 0$*

$$P_t^{\mathcal{C}}(\xi_{\min},\xi_{\max}) \geq \widetilde{P}_t(\xi_{\min},\xi_{\max}). \tag{3.11}$$

Proof. We construct both $(\widetilde{\sigma}_t^{\min})_{t\geq 0}$ with transition probability $\widetilde{P}_t$ with initial condition $\xi_{\min}$ and $(\sigma_t^{\min,\mathcal{C}})_{t\geq 0}$ the censored dynamics with the same initial condition on the same probability space, using the variables $(T_i^{(x)}, U_i^{(x)})_{i\geq 1, x\in [\![1,N]\!]}$.

For $(\sigma_t^{\min,\mathcal{C}})_{t\geq 0}$ we use the same procedure as for $(\sigma_t^{\xi})_{t\geq 0}$ (for $\xi = \xi_{\min}$) with the following added requirement for the transitions: $\{x,x+1\} \notin \mathcal{C}(t)$ in the case (A) and $\{x,x-1\} \notin \mathcal{C}(t)$ in the case (B).

For $(\widetilde{\sigma}_t^{\min})_{t\geq 0}$ we use the same procedure as for $(\sigma_t^{\min,\mathcal{C}})_{t\geq 0}$ but with the addition of new deterministic jumps in the trajectories at times $(s_i)_{i\in I}$. More precisely if $t = s_i$, $\widetilde{\sigma}_t^{\min}$ is determined from $\widetilde{\sigma}_{t_-}^{\min}$ as the unique element of $\Omega_{N,k}$ such that

$$Q_i(\widetilde{\sigma}_{t_-}^{\min}, \widetilde{\sigma}_t^{\min}) = 1. \tag{3.12}$$

We have by definition $\widetilde{\sigma}_0^{\min} = \sigma_0^{\min,\mathcal{C}}$, and it can be checked by inspection that all the transitions are order preserving (this is a property of the graphical construction when $t \notin \{s_i\}_{i=1}^I$ and a consequence of (3.9) for the special values $t \in \{s_i\}_{i=1}^I$).

□

3.4. Equilibrium estimates. Recalling (2.27) let us define

$$\kappa := F'(\lambda) = \mathbb{E}\left[\rho_1^{\lambda}\log(\rho_1)\right] > 0, \tag{3.13}$$

and set

$$\Delta V_{\max}^{\omega,N} = \max_{1\leq x\leq y\leq N} \left(V(y) - V(x)\right). \tag{3.14}$$

The literature on the subject of random walks in a random environment contains very sharp information concerning $\Delta V_{\max}^{\omega,N}$, and the length of the corresponding trap (see [GK13]). In particular it is known under quite general assumptions that $|\Delta V_{\max}^{\omega,N} - \frac{1}{\lambda}\log N|$ displays random fluctuations of order 1 and that the corresponding traps are of a length $\frac{1}{\lambda\kappa}\log N$ at first order.

For the sake of completeness we include a short proof of the following non-optimal result which is sufficient to our purpose. Set

$$q_N := \frac{3u_0 + 2}{|F(u_0)|}\log N, \tag{3.15}$$

where u_0 is the point at which F attains its minimum.

PROPOSITION 3.4. *We have*

$$\lim_{N\to\infty} \mathbb{P}\left[-\left(\frac{1+\varepsilon}{\lambda}\right)\log\log N \le \Delta V^{\omega,N}_{\max} - \frac{1}{\lambda}\log N \le \frac{\varepsilon}{\lambda}\log\log N\right] = 1. \tag{3.16}$$

Furthermore we have

$$\lim_{N\to\infty} \mathbb{P}\left[\max_{\substack{1\le x\le y\le N\\ y-x\ge q_N}} (V(y)-V(x)) \ge -3\log N\right] = 0. \tag{3.17}$$

In particular, with high probability we have

$$\forall x,y \in [\![1,N]\!],\ \left\{V(y)-V(x) = \Delta V^{\omega,N}_{\max}\right\} \quad \Rightarrow \{(y-x)\le q_N\}.$$

PROOF. At the cost of an additive constant on our bounds (which we omit in the proof for readability), using our uniform ellipticity assumption we can replace $V(y)-V(x)$ in the definition of (3.14) by a sum of IID random variables, setting $\bar V(1)=0$ and

$$\sum_{z=x+1}^{y} \log\rho_z := \bar V(y) - \bar V(x). \tag{3.18}$$

By definition of λ, $M_n = \left(\prod_{x=1}^n (\rho_x)^\lambda\right)_{n\ge 1}$ is a martingale for the filtration $\mathcal{G}_n := \sigma(\omega_x, x \in [\![1,n]\!])$. Using the optional stopping theorem at $T_A := \inf\{n, M_n \ge A\}$ and using that

$$\begin{cases} A \le M_{T_A} \le A\left(\frac{1-\alpha}{\alpha}\right)^\lambda, \\ \lim_{n\to\infty} M_n = 0, \end{cases} \tag{3.19}$$

we have for any A

$$\frac{1}{A}\left(\frac{\alpha}{1-\alpha}\right)^\lambda \le \mathbb{P}\left[\max_{n\ge1}\prod_{x=1}^n (\rho_x)^\lambda \ge A\right] \le \frac{1}{A}. \tag{3.20}$$

The bound above can be used to obtain the upper bound on $\Delta V^{\omega,N}_{\max}$ via a union bound using translation invariance

$$\begin{aligned} \mathbb{P}&\left[\max_{1\le x\le y\le N} \bar V(y)-\bar V(x) \ge \frac{1}{\lambda}\log N + \frac{\varepsilon}{\lambda}\log\log N\right] \\ &\le \sum_{x=1}^N \mathbb{P}\left[\max_{y\ge x} \bar V(y)-\bar V(x) \ge \frac{1}{\lambda}\log N + \frac{\varepsilon}{\lambda}\log\log N\right] \\ &\le N\mathbb{P}\left[\max_{n\ge1}\prod_{x=1}^n (\rho_x)^\lambda \ge N(\log N)^\varepsilon\right] \le (\log N)^{-\varepsilon}. \end{aligned} \tag{3.21}$$

Before proving the corresponding lower bound, let us move to the proof of (3.17). Again using translation invariance and union bound, it is sufficient to show that

$$\lim_{N\to\infty} N\mathbb{P}\left[\max_{n\ge q_N}\sum_{x=1}^n \log\rho_x \ge -3\log N\right] = 0. \tag{3.22}$$

We use Doob's maximal inequality for the martingale $e^{-nF(u_0)}\prod_{x=1}^n(\rho_x)^{u_0}$. Since $F(u_0)<0$, we have

$$\mathbb{P}\left[\max_{n\geq q_N}\prod_{x=1}^n(\rho_x)^{u_0}\geq N^{-3u_0}\right]\leq \mathbb{P}\left[\max_{n\geq 1}e^{-nF(u_0)}\prod_{x=1}^n(\rho_x)^{u_0}\geq N^{-3u_0}e^{-q_NF(u_0)}\right]$$
$$\leq N^{3u_0}e^{q_NF(u_0)}\leq N^{-2}. \quad (3.23)$$

This is sufficient to conclude the proof of (3.17). Note that as a consequence by (3.20) and (3.23), we have for N sufficiently large

$$\mathbb{P}\left[\max_{1\leq n\leq q_N}\prod_{x=1}^n(\rho_x)^{\lambda}\geq N(\log N)^{-(1+\varepsilon)}\right]\geq \frac{1}{2}\left(\frac{\alpha}{1-\alpha}\right)^{\lambda}N^{-1}(\log N)^{1+\varepsilon}. \quad (3.24)$$

As a consequence of independence we have

$$\mathbb{P}\left[\forall(i,j)\in [\![1,\lfloor N/q_N\rfloor-1]\!]\times[\![1,q_N]\!],\ \bar V(iq_N+j)-\bar V(iq_N)\leq \frac{\log N-(1+\varepsilon)\log\log N}{\lambda}\right]$$
$$\leq\left(1-\frac{1}{2}\left(\frac{\alpha}{1-\alpha}\right)^{\lambda}N^{-1}(\log N)^{(1+\varepsilon)}\right)^{\lfloor N/q_N\rfloor-1}\leq e^{-c(\log N)^{\varepsilon}} \quad (3.25)$$

This yields the lower bound in (3.16).

□

Proof of Lemma 2.3. For $\xi\in\Omega_{N,k}$, we define the positions of its leftmost particle and rightmost empty site to be respectively

$$\begin{aligned}L_{N,k}(\xi)&:=\inf\{x\in[\![1,N]\!]:\ \xi(x)=1\},\\ R_{N,k}(\xi)&:=\sup\{x\in[\![1,N]\!]:\ \xi(x)=0\}.\end{aligned} \quad (3.26)$$

Then

$$\pi^{\omega}_{N,k}\left(\mathcal{A}_r^{\complement}\right)\leq \pi^{\omega}_{N,k}\left(L_{N,k}(\xi)\leq N-k-r\right)+\pi^{\omega}_{N,k}\left(R_{N,k}(\xi)\geq N-k+r\right).$$

Let us bound the second term, the first one can be treated in a symmetric manner. Moreover, we have

$$\pi^{\omega}_{N,k}\left(R_{N,k}(\xi)\geq N-k+r\right)=\sum_{\substack{x\in[\![1,N-k]\!]\\ y\in[\![N-k+r,N]\!]}}\pi^{\omega}_{N,k}\left(L_{N,k}=x,R_{N,k}=y\right) \quad (3.27)$$

Furthermore, we recall that $\xi^{x,y}$, defined in (2.7), denotes the configuration obtained by swapping the values at sites x,y of the configuration ξ, and observe that the map $\xi\to\xi^{x,y}$ is injective from $\{\xi\in\Omega_{N,k}:L_{N,k}(\xi)=x,R_{N,k}(\xi)=y\}$ to $\Omega_{N,k}$ defined by $\xi\mapsto\xi^{x,y}$. Then we have

$$\pi^{\omega}_{N,k}\left(L_{N,k}=x,R_{N,k}=y\right)=\sum_{\{\xi:\ L_{N,k}(\xi)=x,R_{N,k}(\xi)=y\}}\pi^{\omega}_{N,k}(\xi^{x,y})e^{V^{\omega}(y)-V^{\omega}(x)}$$
$$\leq e^{V^{\omega}(y)-V^{\omega}(x)}\leq Ce^{\bar V^{\omega}(y)-\bar V^{\omega}(x)}. \quad (3.28)$$

Now from the law of large number applied to sum of IID variables, we have

$$\lim_{r\to\infty}\inf_{\substack{N\geq 1\\ k\in[\![1,N/2]\!]}}\mathbb{P}\left[\forall(x,y)\in[\![1,N-k]\!]\times[\![N-k+r,N]\!],\ \bar V^{\omega}(y)-\bar V^{\omega}(x)\leq\frac{(y-x)\mathbb{E}[\log\rho_1]}{2}\right]=1. \quad (3.29)$$

Moreover, since

$$\sum_{\substack{x\in[\![1,N-k]\!]\\ y\in[\![N-k+r,N]\!]}} e^{\frac{\mathbb{E}[\log\rho_1](y-x)}{2}} \leq \frac{e^{\mathbb{E}[\log\rho_1]r/2}}{(1-e^{\mathbb{E}[\log\rho_1]/2})^2}$$

we have

$$\lim_{r\to\infty} \inf_{\substack{N\geq 1\\ k\in[\![1,N/2]\!]}} \mathbb{P}\left[\pi^{\omega}_{N,k}\left(R_{N,k}(\xi)\geq N-k+r\right) \leq \left(1-e^{\frac{\mathbb{E}[\log\rho_1]}{2}}\right)^{-2} e^{\frac{\mathbb{E}[\log\rho_1]r}{2}}\right] = 1, \tag{3.30}$$

which concludes the proof.

□

4. Bounds for the mixing time with arbitrary environments

4.1. Proof of Proposition 2.1. We look at the variable

$$m(\xi) := \sum_{x=1}^{N} x\xi(x).$$

Note that $m(\xi)\in\left[\frac{k(k+1)}{2},\frac{k(2N-k+1)}{2}\right]$. We assume that

$$\pi^{\omega}_{N,k}\left(m(\xi)\geq\frac{k(N+1)}{2}\right)\geq 1/2$$

(the other case can be treated symmetrically). Now, since at all time each particle jumps to right with a rate which is at most one, starting from $\xi_{\min}$ (we write $\sigma^{\min}_t$ for $\sigma^{\xi_{\min}}_t$ to lighten the notation) we have

$$\mathbf{E}\left[m(\sigma^{\min}_t)\right]\leq\frac{k(k+1)}{2}+kt. \tag{4.1}$$

As a consequence of Markov's inequality, we have

$$\mathbf{P}\left[m(\sigma^{\min}_t)\geq\frac{k(N+1)}{2}\right]=\mathbf{P}\left[m(\sigma^{\min}_t)-\frac{k(k+1)}{2}\geq\frac{k(N-k)}{2}\right]\leq\frac{2t}{(N-k)}, \tag{4.2}$$

which is smaller than $1/4$ if $t\leq N/16$.

When the number of particles goes to infinity, we use the same kind of reasoning but adding concentration estimates for $m(\xi)$, under the equilibrium measure $\pi^{\omega}_{N,k}$ (which is denoted simply by π in this proof for readability). Let us prove that

$$\mathrm{Var}_{\pi}[m(\xi)]\leq N^2k. \tag{4.3}$$

To this end we introduce the filtration $(\mathcal{G}_i)_{i=1}^N$ defined by $\mathcal{G}_i:=\sigma(\xi(x),x\in[\![1,i]\!])$, and consider the martingale

$$M_i := E_{\pi}\left[m(\xi)\mid\mathcal{G}_i\right] \tag{4.4}$$

where $E_{\pi}[\cdot\mid\mathcal{G}_i]$ denotes the conditional expectation under π . We have by construction

$$\mathrm{Var}_{\pi}[m(\xi)]=\sum_{i=1}^{N}\mathrm{Var}(M_i-M_{i-1}) \tag{4.5}$$

Now, we are going to show that

$$\mathrm{Var}(M_i-M_{i-1})\leq\pi(\xi_i=1)(N-i)^2 \tag{4.6}$$

which implies (4.3). To prove (4.6) we are going to show that for any $\chi \in \{0,1\}^{i-1}$ with at most $k-1$ ones and at most $N-k-1$ zeros, the quantity

$$\Delta_i(\chi) = E_\pi\left[m(\xi) \mid \xi_{[\![1,i-1]\!]} = \chi, \xi(i) = 0\right] - E_\pi\left[m(\xi) \mid \xi_{[\![1,i-1]\!]} = \chi, \xi(i) = 1\right] \tag{4.7}$$

satisfies

$$0 \le \Delta_i(\chi) \le N - i. \tag{4.8}$$

Note that we have

$$E_\pi\left[m(\xi) \mid \xi_{[\![1,i-1]\!]} = \chi\right] = \sum_{x=1}^{i-1} x\chi(x) + \pi^\omega_{[\![i,N]\!],k-\sum_{x=1}^{i-1}\chi(x)}\left(\sum_{x=i}^{N} x\xi(x)\right), \tag{4.9}$$

where if I is a segment on $\mathbb{Z}$ and $k' \le |I|$, $\pi^\omega_{I,k'}$ denotes the equilibrium measure for exclusion process on I with k' particles and environment ω. For this reason it is sufficient to prove (4.7) for $i = 1$, and arbitrary k (not necessarily assuming $k \le N/2$). Hence we need to prove that for $N \ge 1$ and $k \in [\![1, N-1]\!]$ we have

$$0 \le E_\pi\left[m(\xi) \mid \xi(1) = 0\right] - E_\pi\left[m(\xi) \mid \xi(1) = 1\right] \le N - 1. \tag{4.10}$$

To prove this we observe that there exists a probability Π on $\Omega^2_{N,k}$ with marginals $\pi(\cdot \mid \xi(1) = 0)$ and $\pi(\cdot \mid \xi(1) = 1)$ such that

$$\Pi\left(\sum_{x=1}^{N} \mathbf{1}_{\{\xi^1(x)\neq\xi^2(x)\}} = 2\right) = 1 \tag{4.11}$$

(meaning that $\xi^1(x) = \xi^2(x)$ except at two sites, 1 and another random site). With this coupling we have

$$E_\pi\left[m(\xi) \mid \xi(1) = 0\right] - E_\pi\left[m(\xi) \mid \xi(1) = 1\right] = \Pi\left[\sum_{x=1}^{N} x(\xi^1(x) - \xi^2(x))\right], \tag{4.12}$$

which yields (4.10). The coupling Π can be achieved using the graphical construction: we define (ξ^1_t) and (ξ^2_t) starting with initial configuration $\mathbf{1}_{[\![2,k+1]\!]}$ and $\mathbf{1}_{[\![1,k]\!]}$ respectively and evolving using the graphical construction with the edge $\{1,2\}$ censored (recall Section 3.3). The dynamic conserves the number of discrepancy and $\pi(\cdot| \ \xi(1) = 0)$ and $\pi(\cdot| \ \xi(1) = 1)$ are the respective equilibrium distribution of the marginals, so that any limit point of $\mathbf{P}\left[(\xi^1_t, \xi^2_t) \in \cdot\right]$ (existence is ensured by compactness) provides a coupling satisfying (4.11).

Now to see that (4.7) implies (4.6), we simply observe that, conditionned to the state of the first $i = 1$ vertices of the segment, $(M_i - M_{i-1})$ can only assume two values which differ by an amount $\Delta_i(\chi)$ (cf. (4.7)). The corresponding conditioned variance is equal to $\Delta_i(\chi)^2$ times that of the corresponding Bernoulli variable that is

$$\begin{aligned} E_\pi[(M_i - M_{i-1})^2 \mid \xi_{[\![1,i-1]\!]} = \chi] &= \pi(\xi(i) = 1 \mid \xi_{[\![1,i-1]\!]} = \chi)\pi(\xi(i) = 0 \mid \xi_{[\![1,i-1]\!]} = \chi)\Delta_i(\chi)^2 \\ &\le \pi(\xi(i) = 1 \mid \xi_{[\![1,i-1]\!]} = \chi)(N-i)^2. \end{aligned} \tag{4.13}$$

We then consider the average the inequality with respect to $\xi_{[\![1,i-1]\!]}$ to conclude. Now using (4.3) we can assume that for any ε there exists $N_0(\varepsilon)$ such that for $N \ge N_0(\varepsilon)$ we have

$$\min\left[\pi^\omega_{N,k}(m(\xi) \le Nk/3), \pi^\omega_{N,k}(m(\xi) \ge 2Nk/3)\right] \le \varepsilon/2. \tag{4.14}$$

Let us assume that the first of these two terms is smaller (the other case is treated symmetrically). To conclude, we must show that for $t = \frac{N}{30}$ we have

$$\mathbf{P}\left(m(\sigma^{\min}_t) > Nk/3\right) \le \varepsilon/2. \tag{4.15}$$

To check this we observe that

$$m(\sigma_t^{\min}) \le \frac{k(k+1)}{2} + \mathcal{N}_t \tag{4.16}$$

where $\mathcal{N}_t$ is the total number of particle jumps to the right. Since each particle jumps at most with rate one, we have for N sufficiently large

$$\mathbf{P}\left[\mathcal{N}_t \ge 2kt\right] \le \varepsilon/2, \tag{4.17}$$

which allows to conclude.

□

4.2. Proof of Proposition 2.2. For the proof of Proposition 2.2, we apply the so-called flow method (see [LP17, Chapter 13.4]). A path Γ is a sequence of configurations $(\xi_0, \dots, \xi_{|\Gamma|})$ which is such that $r^\omega(\xi_{i-1}, \xi_i) > 0$ for $i \in [\![1, |\Gamma|]\!]$. For any given ordered pair $(\xi, \xi') \in \Omega_{N,k} \times \Omega_{N,k}$, we assign a path $\Gamma_{\xi,\xi'}$, whose starting point is ξ and ending point is ξ'.

Using [LP17, Corollary 13.21], the spectral gap of the chain can be controlled by a simple quantity depending on the functional $(\xi, \xi') \mapsto \Gamma_{\xi,\xi'}$. We say that an unordered pair $e = \{\xi, \xi'\} \subset \Omega_{N,k}$ is an edge if $q(e) := \pi^\omega_{N,k}(\xi) r(\xi, \xi') > 0$ (note that by reversibility $q(e)$ does not depend on the orientation). We write $e \in \Gamma = (\xi_0, \dots, \xi_{|\Gamma|})$ if there exists $i \in [\![1, |\Gamma|]\!]$ such that $e = \{\xi_{i-1}, \xi_i\}$. We have then (the factor 1/2 is irrelevant but appears because we are considering unoriented edges rather than oriented one)

$$\mathrm{gap}^\omega_{N,k} \ge \left(\max_e \frac{1}{2q(e)} \sum_{(\xi,\xi') \in \Omega_{Nk} \times \Omega_{N,k} \,:\, e \in \Gamma_{\xi,\xi'}} \pi^\omega_{N,k}(\xi) \pi^\omega_{N,k}(\xi') |\Gamma_{\xi,\xi'}| \right)^{-1}. \tag{4.18}$$

In the proof we describe a choice for $\Gamma_{\xi,\xi'}$ which yields a relevant bound for the spectral gap. Let us fix a state $\xi^* \in \Omega_{N,k}$ which has maximal probability, that is such that

$$V^\omega(\xi^*) = \min_{\xi \in \Omega_{N,k}} V^\omega(\xi) \tag{4.19}$$

(we make an arbitrary choice if there are several minimizers). Now to build the path $\Gamma_{\xi,\xi'}$ we are going to build first a path from ξ to ξ^* and then one from ξ^* to ξ' and then concatenate the two.

We can thus focus on the construction of Γ_{ξ,ξ^*}. Let

$$m := d_H(\xi, \xi^*) := \frac{1}{2} \sum_{x=1}^{N} |\xi(x) - \xi^*(x)|$$

denote the Hamming distance between ξ and ξ^*. Our first step is to build a sequence $\xi^{(0)}, \dots, \xi^{(m)}$ which reduces the Hamming distance in incremental steps that is such that

$$\begin{cases} \xi^{(0)} = \xi \quad \text{and} \quad \xi^{(m)} = \xi^*, \\ d_H(\xi^{(i-1)}, \xi^{(i)}) = 1 \qquad \text{for } i \in [\![1, m]\!], \\ d_H(\xi^{(i)}, \xi^*) = m - i \qquad \text{for } i \in [\![1, m]\!]. \end{cases} \tag{4.20}$$

The choice we make for $\xi^{(0)},\dots,\xi^{(m)}$ is not relevant for the result but let us fix one for the sake of clarity. Let the sequences $(x_i)_{i=1}^m$ and $(y_i)_{i=1}^m$ be defined by

$$\begin{aligned} x_i &:= \min\left\{ x\in [\![1,N]\!] \ : \ \sum_{x=1}^N (\xi(x)-\xi^*(x))_+ = i \right\},\\ y_i &:= \min\left\{ y\in [\![1,N]\!] \ : \ \sum_{y=1}^N (\xi^*(y)-\xi(y))_+ = i \right\}. \end{aligned} \tag{4.21}$$

These sequences locate the discrepancies between ξ and ξ^*. Then we define $\xi^{(i)}$ inductively as being obtained from $\xi^{(i-1)}$ by moving the particle at x_i to y_i which is equivalent to setting

$$\xi^{(i)} = \xi\wedge\xi^* + \sum_{j=1}^i \mathbf{1}_{\{y_j\}} + \sum_{j'=i+1}^m \mathbf{1}_{\{x_{j'}\}}.$$

Finally, our path from ξ to ξ^* is defined by concatenating paths $\Gamma^{(i)}$, $i\in[\![1,m]\!]$, linking $\xi^{(i-1)}$ to $\xi^{(i)}$. We define $\Gamma^{(i)} = (\xi^{(i)}_0,\dots,\xi^{(i)}_{|x_i-y_i|})$ as a path of minimal length $|x_i-y_i|$ linking $\xi^{(i)}_0 := \xi^{(i-1)}$ to $\xi^{(i)}_{|x_i-y_i|}=\xi^{(i)}$. To define the intermediate steps, let us assume for notational simplicity (and without loss of generality) that $x_i<y_i$. Moreover, let $(z_j)_{j=1}^b$ be defined as the decreasing sequence such that

$$\xi^{(i-1)}|_{[\![x_i,y_i]\!]} = \mathbf{1}_{\{z_j\}_{j=1}^b}.$$

We then set $d_j := y_i - z_j$ if $j\in[\![1,b]\!]$ and $d_0:=0$, and define $(\xi^{(i)}_\ell)_{\ell=1}^{y_i-x_i}$ by setting if $d_{j-1}<\ell\le d_j$

$$\xi^{(i)}_\ell := \xi^{(i-1)}_\ell - \mathbf{1}_{\{z_j\}} + \mathbf{1}_{\{z_j+\ell-d_{j-1}\}}. \tag{4.22}$$

In other words, we move the particle at site z_j ($j\ge 1$) to site z_{j-1} (with $z_0=y_i$) starting from $j=1$ until $j=b$. We refer to Figure 4.2 for a graphical description.

LEMMA 4.1. *For the path collection* $(\Gamma_{\xi,\xi'})$ *constructed above, we have*

$$B := \max_e \frac{1}{2q(e)} \sum_{(\xi,\xi')\in\Omega_{Nk}\times\Omega_{N,k}\ :\ e\in\Gamma_{\xi,\xi'}} \pi^\omega_{N,k}(\xi)\pi^\omega_{N,k}(\xi')|\Gamma_{\xi,\xi'}| \le \alpha^{-1}N^2|\Omega_{N,k}|\left(\frac{1-\alpha}{\alpha}\right)^{N/2}. \tag{4.23}$$

Let us now conclude the proof of Proposition 2.2. By (4.18) and Lemma 4.1, we have

$$\mathrm{gap}^\omega_{N,k} \ge \alpha N^{-2}|\Omega_{N,k}|^{-1}\left(\frac{1-\alpha}{\alpha}\right)^{-N/2}. \tag{4.24}$$

Observe that

$$\max_{\xi,\xi'\in\Omega_{N,k}} \left(V^\omega(\xi)-V^\omega(\xi')\right) \le Nk\log\frac{1-\alpha}{\alpha},$$

and then

$$\min_{\xi\in\Omega_{N,k}} \pi^\omega_{N,k}(\xi) \ge |\Omega_{N,k}|^{-1}\left(\frac{1-\alpha}{\alpha}\right)^{-Nk}. \tag{4.25}$$

By (2.17), we have for $\varepsilon\in(0,1/2)$

$$t^{N,k,\omega}_{\mathrm{mix}}(\varepsilon) \le \alpha^{-1}N^2|\Omega_{N,k}|\left(\frac{1-\alpha}{\alpha}\right)^N\left(\log|\Omega_{N,k}| + Nk\log\frac{1-\alpha}{\alpha} - \log\varepsilon\right). \tag{4.26}$$

□

Proof of Lemma 4.1. A first observation is that by construction, our paths are of length smaller than N^2. Let e be an edge and (ξ,ξ') such that $e\in\Gamma_{\xi,\xi'}$. By symmetry and taking away the factor $1/2$, we can always assume that e belongs to the first part of the path linking ξ to ξ^*. After replacing $|\Gamma_{\xi,\xi'}|$ by the upper bound and summing over all ξ', we obtain that the quantity we want to bound is exactly

$$\frac{1}{2q(e)}\sum_{(\xi,\xi')\in\Omega_{N,k}\times\Omega_{N,k}\ :\ e\in\Gamma_{\xi,\xi'}}\pi^{\omega}_{N,k}(\xi)\pi^{\omega}_{N,k}(\xi')|\Gamma_{\xi,\xi'}|\le N^2\sum_{\xi\in\Omega_{N,k}\ :\ e\in\Gamma_{\xi,\xi^*}}\frac{\pi^{\omega}_{N,k}(\xi)}{q(e)}. \tag{4.27}$$

Now let $\chi_0(e,\xi)$ denote the first end of e which is visited by the path going from ξ to ξ^*. Now simply observing that $q(e)$ is at least α times the smallest probability $\pi^{\omega}_{N,k}$ of its two end points, we have

$$\frac{\pi^{\omega}_{N,k}(\xi)}{q(e)}\le\sup_{\xi'\in\Gamma_{\xi,\xi^*}}\alpha^{-1}e^{V(\xi')-V(\xi)}. \tag{4.28}$$

Hence using the bound in the sum in (4.27) we obtain that

$$\log B\le\log\alpha^{-1}N^2|\Omega_{N,k}|+\sup_{\substack{\xi\in\Omega_{N,k}\\ \xi'\in\Gamma_{\xi,\xi^*}}}V(\xi')-V(\xi). \tag{4.29}$$

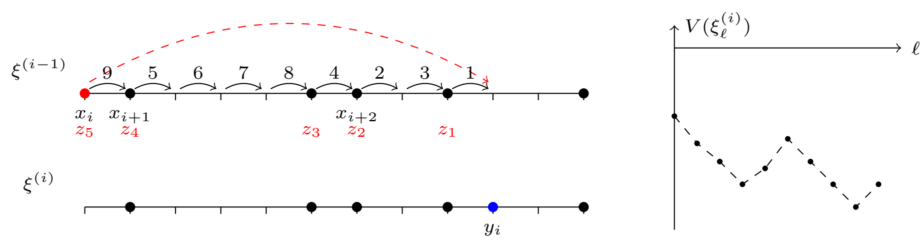
To conclude we only need to prove that for every $\xi\in\Omega_{N,k}$ and $\xi'\in\Gamma_{\xi,\xi^*}$ we have

$$V(\xi')-V(\xi)\le\frac{N}{2}\log\left(\frac{1-\alpha}{\alpha}\right). \tag{4.30}$$

This follows simply by inspection from the following observation which follows from our construction and our assumptions.

(i) In one step of Γ_{ξ,ξ^*}, V varies at most by $\log\left(\frac{1-\alpha}{\alpha}\right)$ in absolute value.
(ii) Along the sequence $(\xi^{(i)})_{i=1}^m$, $V(\xi^{(i)})$ is non-increasing.
(iii) Each concatenated path $\Gamma^{(i)}$ has a length smaller than N (hence each $\xi^{(i)}_\ell$ is within $N/2$ steps of either $\xi^{(i)}$ or $\xi^{(i-1)}$) so that we have

$$\max_{0\le\ell\le|x_i-y_i|}(V(\xi^{(i)}_\ell)-V(\xi))\le\max_{0\le\ell\le|x_i-y_i|}\left(V(\xi^{(i)}_\ell)-V(\xi^{(i)})\wedge V(\xi^{(i-1)})\right)\le\frac{N}{2}\log\frac{1-\alpha}{\alpha}.$$



(L) (R)

Figure 4. A bold circle represents a particle, and a particle at the same site for the configurations $\xi^{(i-1)}$ and $\xi^{(i)}$ is colored black. Otherwise, it is red or blue. (L) A graphical description of the movements of the particle at site x_i of $\xi^{(i-1)}$ to the empty site y_i and the numbers above the arrows are the relative order of the movements. (R) We draw the graph of $(\ell,V(\xi^{(i)}_\ell))_\ell$.

□

5. Lower bounds on the mixing time

Theorem 2.5 contains three separate lower bounds. The first one is a consequence of Proposition 2.1. In this section, we are going to prove the two remaining bounds which are restated below as Propositions 5.1 and 5.2 respectively. The proof of these propositions rely on the two mechanisms exposed in Subsection 2.5: The potential barrier created by rare fluctuations of V^ω (cf. Proposition 3.4) has the effect of trapping individual particles and slowing down the particle flow.

5.1. A lower bound from the position of the first particle.

PROPOSITION 5.1. *We have with high probability*

$$t_{\mathrm{mix}}^{N,k,\omega} \geq [N(\log N)^{-2}]^{\frac{1}{\lambda}} \tag{5.1}$$

PROOF. We let $y_1(\omega) > x_1(\omega)$ be such that $V(y_1(\omega)) - V(x_1(\omega))$ is maximized within $1 \leq x \leq y \leq N/4$ (the event that V^ω is non-increasing on $[\![1, N/4]\!]$ is unlikely, and then it can be ignored). We are going to prove that w.h.p.

$$t_{\mathrm{mix}}^{N,k,\omega} \geq \frac{1}{2e} e^{V(y_1)-V(x_1)} - 1. \tag{5.2}$$

As a consequence of Proposition 3.4 (applied to the segment $[\![1, N/4]\!]$), we have w.h.p.

$$V(y_1) - V(x_1) \geq \frac{1}{\lambda} \log N - \frac{2}{\lambda} \log\log N + \log 20,$$

so that (5.1) follows from (5.2). Recall the notation (3.1). Then considering that $\xi_{\min}$ should be the worst initial condition, we observe that

$$d_{N,k}^\omega(t) \geq \mathbf{P}\left[\bar{\sigma}_t^{\min}(1) \leq \frac{N}{4}\right] - \pi_{N,k}^\omega\left(\bar{\xi}(1) \leq N/4\right). \tag{5.3}$$

As a consequence of Lemma 2.3 we have w.h.p.

$$\pi_{N,k}^\omega(\bar{\xi}(1) \leq N/4) \leq 1/4.$$

To have an estimate on the mixing time, we must prove that $\mathbf{P}\left[\bar{\sigma}_t^{\min}(1) \leq \frac{N}{4}\right] \geq 1/2$. We define

$$\tau_{y_1} := \inf\left\{t \geq 0 : \bar{\sigma}_t^{\min}(1) = y_1\right\}.$$

We have

$$\mathbf{P}\left[\bar{\sigma}_t^{\min}(1) > \frac{N}{4}\right] \leq \mathbf{P}[\tau_{y_1} \leq t]. \tag{5.4}$$

We are going to show that

$$\mathbf{P}[\tau_{y_1} \leq t] \leq e(t+1)e^{V(x_1)-V(y_1)}, \tag{5.5}$$

which is sufficient for us to conclude that (5.2) holds. Using the graphical construction (with an enlargement of the probability space to sample the initial condition) we can couple $\sigma_t^{\min}$ with X_t^π a random walk on the interval $[\![1, y_1]\!]$ with transitions rates given by $q_{y_1}^\omega$ (cf. (1.1)) and starting with an initial distribution sampled from the equilibrium measure $\pi_{y_1,1}^\omega$, in such a way that

$$\forall t \leq \tau_{y_1}, \quad \bar{\sigma}_t^{\min}(1) \leq X_t^\pi.$$

Setting $\widetilde{\tau}_{y_1} := \inf\{t \geq 0 : X_t^\pi = y_1\}$, we then have

$$\mathbf{P}\left[\tau_{y_1} \leq t\right] \leq \mathbf{P}\left[\widetilde{\tau}_{y_1} \leq t\right]. \tag{5.6}$$

We define the occupation time

$$u(t) := \int_0^t \mathbf{1}_{\{y_1\}}(X_s^\pi)\mathrm{d}s.$$

We have

$$\mathbf{E}\left[u(t+1)\right] \geq \mathbf{P}[u(t+1) \geq 1] \geq \mathbf{P}[\widetilde{\tau}_{y_1} \leq t]\mathbf{P}\left[\forall s \in [0,1] : X^{\pi}_{\widetilde{\tau}_{y_1}+s} = y_1\right] \geq e^{-1}\mathbf{P}[\widetilde{\tau}_1 \leq t], \quad (5.7)$$

where in the last inequality we use the strong Markov property. As the process $(X^{\pi}_t)_{t\geq 0}$ is stationary,

$$\mathbf{E}[u(t+1)] = (t+1)\pi^{\omega}_{y_1,1}(y_1) \leq (t+1)e^{V(y_1)-V(x_1)},$$

which allows to conclude that

$$\mathbf{P}[\widetilde{\tau}_1 \leq t] \leq e(t+1)e^{V(y_1)-V(x_1)}. \quad (5.8)$$

□

5.2. A lower bound derived from flow consideration. Let us now derive the third bound which is necessary to complete the proof of Theorem 2.5.

PROPOSITION 5.2. *There exists a positive constant $c = c(\alpha, \mathbb{P})$ such that w.h.p. we have*

$$t^{N,k}_{\text{mix}} \geq ckN^{\frac{1}{2\lambda}}(\log N)^{-2\left(1+\frac{1}{\lambda}\right)}. \quad (5.9)$$

To prove the above result, we adopt the strategy developed in [Sch19, Proposition 4.2] by investigating the flow of particles through a *slow* segment of size of order $(\log N)$ where the drift of the random environment points to the left. This flow of particles is controlled via a comparison with a boundary driven exclusion process.

In [Sch19] the *slow* segment is selected to be such that $\omega_x < 1/2$ for *every site*. It has the advantage of simplifying the computation since it allows for comparison with the homogeneous exclusion process for which computation has been performed in [BECE00]. Our approach brings an improvement by selecting the slow segment based on the potential function V^{ω}. The relevant quantity that limits the flow is the worst potential barrier that the particles have to overcome. Proposition 3.4 allows to identify the worst potential barrier in the system. We let $x_2(\omega) \leq y_2(\omega)$ be the smallest elements of $[\![N/2, 3N/4]\!]$ such that

$$V^{\omega}(y_2) - V^{\omega}(x_2) = \max_{N/2 \leq x \leq y \leq 3N/4} \left(V^{\omega}(y) - V^{\omega}(x)\right).$$

According to Proposition 3.4 we have w.h.p.

$$V(y_2) - V(x_2) \geq \frac{1}{\lambda}\left(\log N - 2\log\log N\right) \text{ and } y_2 - x_2 \leq q_N. \quad (5.10)$$

In order to illustrate how the mixing time can be controlled using the flow of particles, we start with a simple lemma. Let J_t denote the number of particles on the last portion of the segment,

$$J_t := \sum_{x \geq y_2+1} \sigma^{\min}_t(x). \quad (5.11)$$

LEMMA 5.3. *For any $\varepsilon > 0$, we have with high probability for every $t \geq 0$.*

$$d^{\omega}_{N,k}(t) \geq 1 - \frac{4\mathbf{E}[J_t]}{k} - \varepsilon. \quad (5.12)$$

PROOF. Setting $\mathcal{B} := \left\{\xi \in \Omega_{N,k} \ : \ \sum_{x\geq y_2+1}\xi(x) < k/4\right\}$, we have

$$d^{\omega}_{N,k}(t) \geq \|P^{\xi_{\min}}_t - \pi^{\omega}_{N,k}\|_{\text{TV}} \geq \mathbf{P}[\sigma^{\min}_t \in \mathcal{B}] - \pi^{\omega}_{N,k}(\mathcal{B}). \quad (5.13)$$

By Lemma 2.3, the second term is smaller than ε with high probability. Concerning the first term, we have by Markov's inequality

$$\mathbf{P}\left[\sigma^{\min}_t \in \mathcal{B}\right] = 1 - \mathbf{P}\left[J_t \geq k/4\right] \geq 1 - \frac{4\mathbf{E}[J_t]}{k}. \quad (5.14)$$

□

Now we can control $\mathbf{E}[J_t]$ by comparing our system with one in which the particles flow faster. We consider a process on a different state space

$$\widetilde{\Omega}_{x_2,y_2} := \{\xi : [\![x_2, y_2+1]\!] \to \mathbb{Z}_+ \ : \ \forall x \in [\![x_2, y_2]\!],\ \xi(x) \in \{0,1\}\}\,. \tag{5.15}$$

Under this new process the particles follow the exclusion dynamics in the bulk but new rules are added at the boundaries. If $\xi(x_2) = 0$ then a particle is added at site x_2 with rate one. At the other end of the segment particles can jump from site y_2 to site y_2+1 without respecting the exclusion rule (*i.e.*, the site y_2+1 is allowed to contain arbitrarily many particles) and particles at site y_2+1 remain there forever. We define the generator of the process to be (for $f : \widetilde{\Omega}_{x_2,y_2} \mapsto \mathbb{R}$)

$$\begin{aligned}\widetilde{\mathfrak{L}}^{\omega}_{x_2,y_2} f(\xi) := &\sum_{z=x_2}^{y_2-1} r^{\omega}(\xi, \xi^{z,z+1})\big[f(\xi^{z,z+1}) - f(\xi)\big] \\ &+ \omega_{x_2-1}\mathbf{1}_{\{\xi(x_2)=0\}}\big[f(\xi+\delta_{x_2}) - f(\xi)\big] + \omega_{y_2}\mathbf{1}_{\{\xi(y_2)=1\}}\big[f(\xi-\delta_{y_2}+\delta_{y_2+1}) - f(\xi)\big],\end{aligned} \tag{5.16}$$

where r^{ω} is defined in (2.8). We refer to Figure 5 for a graphical description. We let $(\widetilde{\sigma}^{\xi}_t)_{t\geq 0}$ denote the corresponding process starting from an initial condition $\xi \in \widetilde{\Omega}_{x_2,y_2}$.

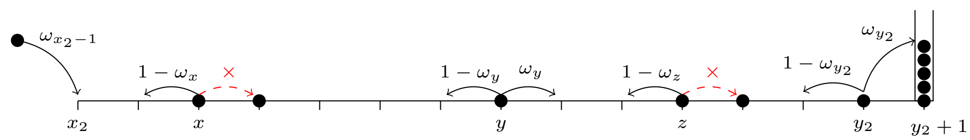


FIGURE 5. A graphical representation of the boundary driven process: a bold circle represents a particle, and the number above every arrow represents the jump rate while a red " $\times$ " represents an inadmissible jump. In addition, the site y_2+1 can accommodate infinite many particles and all particles at site y_2+1 stay put.

LEMMA 5.4. *Let* $\mathbf{0}$ *denote the configuration with all sites in* $[\![x_2, y_2+1]\!]$ *being empty, and then we have*

$$J_t \leq \widetilde{\sigma}^{\mathbf{0}}_t(y_2+1), \tag{5.17}$$

where J_t *is defined in* (5.11).

PROOF. The process $(\widetilde{\sigma}^{\mathbf{0}}_t)_{t\geq 0}$ can be constructed together with $(\sigma^{\min}_t)_{t\geq 0}$ on the same probability space using the graphical construction of Section 3.2 (with the obvious adaptation of the construction to fit the boundary condition for $(\widetilde{\sigma}^{\mathbf{0}}_t)_{t\geq 0}$ using the same clocks $(T^{(x)}_n)_{x,n\in\mathbb{N}}$ and auxiliary variables $(U^{(x)}_n)_{x,n\in\mathbb{N}}$ for the two processes. It can then be checked by inspection that for every $t \geq 0$

$$\forall x \in [\![x_2, y_2+1]\!] \quad \sum_{z=x}^{N} \sigma^{\min}_t(z) \leq \sum_{z=x}^{y_2+1} \widetilde{\sigma}^{\mathbf{0}}_t(z). \tag{5.18}$$

Since the above inequality is satisfied at $t = 0$, it is sufficient to check that it is conserved by any update of the two processes. The result then just corresponds to the case $x = y_2+1$. □

PROPOSITION 5.5. *There exists a constant* $C = C(\alpha, \mathbb{P})$ *such that for all* $t \geq 0$ *w.h.p. we have*

$$\mathbf{E}\left[\widetilde{\sigma}_t(y_2+1)\right] \leq tCN^{-\frac{1}{2\lambda}}(\log N)^{2(1+\frac{1}{\lambda})}. \tag{5.19}$$

With Proposition 5.5 whose proof is detailed in the next subsection, we are ready for the proof of Proposition 5.2.

Proof of Proposition 5.2. By Lemma 5.3 and Lemma 5.4, we have

$$d^{\omega}_{N,k}(t) \geq \frac{7}{8} - 4\frac{\mathbf{E}[\widetilde{\sigma}_t(y_2+1)]}{k}. \tag{5.20}$$

By Proposition 5.5, we take

$$t = \frac{1}{8C} k N^{\frac{1}{2\lambda}} (\log N)^{-2(1+\frac{1}{\lambda})}$$

in (5.20) to conclude the proof.

□

5.3. Proof of Proposition 5.5. Note that $\widetilde{\sigma}^{\mathbf{0}}_t(y_2+1)$ is a superadditive ergodic sequence. To see this we let ϑ_s denote the time shift operator on the graphical construction variables. Recalling (3.4) we set

$$\vartheta_s((T^{(x)}_i, U^{(x)}_i)_{x\in\mathbb{Z}, i\geq 1}) := \left(T^{(x)}_{i+i_0(x,s)} - s, U^{(x)}_{i+i_0(x,s)}\right)_{x\in\mathbb{Z}, i\geq 1}. \tag{5.21}$$

Now we observe that the graphical construction preserves the order $\preccurlyeq$ on $\widetilde{\Omega}_{x_2,y_2}$ defined by

$$\xi \preccurlyeq \xi' \quad \text{if} \quad \forall x \geq x_2, \quad \sum_{z=x}^{y_2+1} \xi(z) \leq \sum_{z=x}^{y_2+1} \xi'(z). \tag{5.22}$$

Hence comparing the dynamic in the interval $[s, s+t]$ with that starting from $\mathbf{0}$ at time s, we obtain

$$\widetilde{\sigma}^{\mathbf{0}}_{s+t}(y_2+1) \geq \widetilde{\sigma}^{\mathbf{0}}_s(y_2+1) + (\vartheta_s \circ \widetilde{\sigma})^{\mathbf{0}}_t(y_2+1). \tag{5.23}$$

Since the shift operator ϑ_s on (T, U) is ergodic, we obtain from Kingman's subbadditive ergodic Theorem [Kin73] (continuous time version) that

$$\mathbf{E}\left[\widetilde{\sigma}^{\mathbf{0}}_t(y_2+1)\right] \leq t\left[\lim_{s\to\infty} \frac{1}{s}\widetilde{\sigma}^{\mathbf{0}}_s(y_2+1)\right]. \tag{5.24}$$

Letting $\mathcal{N}_s := \sum_{x=x_2}^{y_2} \widetilde{\sigma}^{\mathbf{0}}_s(x)$ denote the number of mobile particles in the system (particles at site y_2+1 which have stopped moving are not counted), we have

$$\widetilde{\sigma}^{\mathbf{0}}_t(y_2+1) = \sum_{s\in(0,t]} \mathbf{1}_{\{\mathcal{N}_s < \mathcal{N}_{s_-}\}}. \tag{5.25}$$

Letting $(\mathcal{T}_n)_{n\geq 1}$ denote the sequence of time at which $\mathcal{N}_t < \mathcal{N}_{t_-}$ (in increasing order), we have

$$\lim_{s\to\infty} \frac{1}{s}\widetilde{\sigma}^{\mathbf{0}}_s(y_2+1) = \lim_{n\to\infty} \frac{n}{\mathcal{T}_n}. \tag{5.26}$$

Similarly to (5.23), using preservation of order and the fact that

$$\sigma^{\mathbf{0}}_s \preccurlyeq [\sigma^{\mathbf{0}}_s(y_2+1) + y_2 - x_2 + 1]\mathbf{1}_{\{y_2+1\}},$$

we have for every $s > 0$

$$\widetilde{\sigma}^{\mathbf{0}}_{s+t}(y_2+1) \leq \widetilde{\sigma}^{\mathbf{0}}_s(y_2+1) + (\vartheta_s \circ \widetilde{\sigma})^{\mathbf{0}}_t(y_2+1) + y_2 - x_2 + 1. \tag{5.27}$$

Now as a consequence of (5.27)

$$\mathcal{T}_{l+y_2-x_2+2} \geq \mathcal{T}_l + \vartheta_{\mathcal{T}_l} \circ \mathcal{T}_1. \tag{5.28}$$

Since $\mathcal{T}_l$ is a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$ (recall (3.5)), by the strong Markov property $\vartheta_{\mathcal{T}_l} \circ \mathcal{T}_1$ is independent of $\mathcal{T}_1$ and has the same distribution. Iterating the process we obtain that

$$\mathcal{T}_{(r-1)(y_2-x_2+2)+1} \geq \mathcal{T}^{(1)}_1 + \cdots + \mathcal{T}^{(r)}_1 \tag{5.29}$$

where $(\mathcal{T}_1^{(a)})_{a=1}^r$ is a sequence of IID copies of $\mathcal{T}_1$. This yields that

$$\liminf_{n\to\infty} \frac{\mathcal{T}_n}{n} \geq \frac{1}{y_2 - x_2 + 2} \mathbf{E}\left[\mathcal{T}_1\right]. \tag{5.30}$$

Finally let us compare $(\widetilde{\sigma}_t^{\mathbf{0}})_{t\geq 0}$ with $(\widetilde{\sigma}_t')_{t\geq 0}$ starting from another initial condition. Now we specify the initial condition. Let us first choose the number of particle by setting

$$\begin{aligned} &\Lambda(\omega) := \{x \in [\![x_2, y_2]\!] \ : \ V(x) \leq [V(y_2) + V(x_2)]/2\}, \\ &k'(\omega) := \#\Lambda(\omega). \end{aligned} \tag{5.31}$$

We let $(\widetilde{\sigma}_t')_{t\geq 0}$ be the dynamic with generator (5.16) and initial configuration $\widetilde{\sigma}_0'$ is obtained by setting $\widetilde{\sigma}_0'(y_2+1) = 0$ and sampling $\pi^{\omega}_{[x_2,y_2],k'}$ (the invariant probability measure for the exclusion process on the segment $[\![x_2, y_2]\!]$ with k' particles) to set the values of $(\widetilde{\sigma}_0'(x))_{x\in[\![x_2,y_2]\!]}$. Using monotonicity again we have

$$\mathcal{T}_1 \geq \inf\{t \geq 0 : \widetilde{\sigma}_t'(y_2+1) = 1\} \geq \inf\{t \geq 0 : \widetilde{\sigma}_t'(x_2) = 0 \ \text{ or } \ \widetilde{\sigma}_t'(y_2) = 1\} =: \mathcal{T}'. \tag{5.32}$$

Now let us observe that until time $\mathcal{T}'$, the process $(\widetilde{\sigma}_t')_{t\geq 0}$ (or rather, its restriction to $[\![x_2, y_2]\!]$) coincides with the exclusion process on the segment $[\![x_2, y_2]\!]$ with k' particles. Using this we can prove the following (the proof is postponed to the end of the section).

Lemma 5.6. *We have*

$$\mathbf{E}\left[\mathcal{T}'\right] \geq \frac{1}{16e^2(y_2 - x_2)} e^{\frac{V(y_2) - V(x_2)}{2}}. \tag{5.33}$$

Let us now conclude the proof of Proposition 5.5. Combing (5.24), (5.26), (5.30) and (5.32), we have

$$\mathbf{E}\left[\widetilde{\sigma}_t^{\mathbf{0}}(y_2+1)\right] \leq t\left[\lim_{s\to\infty} \frac{1}{s}\widetilde{\sigma}_s^{\mathbf{0}}(y_2+1)\right] \leq \frac{t(y_2 - x_2 + 2)}{\mathbf{E}\left[\mathcal{T}_1\right]} \leq \frac{t(y_2 - x_2 + 2)}{\mathbf{E}\left[\mathcal{T}'\right]}. \tag{5.34}$$

Using Lemma 5.6, we obtain

$$\mathbf{E}\left[\widetilde{\sigma}_t^{\mathbf{0}}(y_2+1)\right] \leq t16e^2(y_2 - x_2 + 2)^2 e^{-\frac{V(y_2) - V(x_2)}{2}}. \tag{5.35}$$

By (5.10), we have w.h.p.

$$\mathbf{E}\left[\widetilde{\sigma}_t^{\mathbf{0}}(y_2+1)\right] \leq t16e^2(q_N + 2)^2 N^{-\frac{1}{2\lambda}} (\log N)^{\frac{1}{\lambda}}. \tag{5.36}$$

□

Proof. Proof of Lemma 5.6 With a small abuse of notation, in this proof $(\widetilde{\sigma}_t')_{t\geq 0}$ denotes the exclusion process on the segment $[\![x_2, y_2]\!]$ with k' particles starting from stationarity. Since $\mathbf{E}\left[\mathcal{T}'\right] \geq t\mathbf{P}\left[\mathcal{T}' > t\right]$, our goal is to provide a lower bound on $\mathbf{P}\left[\mathcal{T}' > t\right]$. We define

$$\begin{aligned} \mathcal{B}_1 &:= \left\{\xi \in \Omega_{[\![x_2,y_2]\!],k'} : \xi(x_2) = 0\right\}, \\ \mathcal{B}_2 &:= \left\{\xi \in \Omega_{[\![x_2,y_2]\!],k'} : \xi(y_2) = 1\right\}. \end{aligned} \tag{5.37}$$

Using the strong Markov property at $\mathcal{T}'$ and the fact that jumping rates for particles are bounded from above by one at every site, we have

$$\mathbf{P}\left[\forall t \in [\mathcal{T}', \mathcal{T}'+1], \ \widetilde{\sigma}_t' \in \mathcal{B}_1 \cup \mathcal{B}_2\right] \geq e^{-2}.$$

Using independence as in (5.7), we have

$$\mathbf{P}\left[\mathcal{T}' \leq t\right] \leq e^2(t+1)\pi^{\omega}_{[x_2,y_2],k'}\left(\mathcal{B}_1 \cup \mathcal{B}_2\right). \tag{5.38}$$

We now head to provide an upper bound on $\pi^{\omega}_{[x_2,y_2],k'}(\mathcal{B}_1)$. Recalling the definition of Λ in (5.31), we observe that when $\xi \in \mathcal{B}_1$, since $x_2 \in \Lambda$ and there are k' particles, there must be a particle in $\Lambda^{\complement} := [\![x_2, y_2]\!] \setminus \Lambda$. Let $\mathtt{R}(\xi)$ be the position of the rightmost such particle

$$\mathtt{R}(\xi) := \sup\left\{z \in \Lambda^{\complement} : \ \xi(z) = 1\right\},$$

and set for $z \in \Lambda^{\complement}$

$$\mathcal{B}_{1,z} := \{\xi \in \mathcal{B}_1 : \mathtt{R}(\xi) = z\}.$$

By moving the particle from site z to site x_2 as in (3.28), we obtain

$$\pi^{\omega}_{[x_2,y_2],k'}\left(\mathcal{B}_{1,z}\right) = \sum_{\xi \in \mathcal{B}_{1,z}} \pi^{\omega}_{[x_2,y_2],k'}(\xi^{x_2,z}) e^{-V(z)+V(x_2)} \le e^{-V(z)+V(x_2)} \le e^{-\frac{V(y_2)-V(x_2)}{2}},$$

and then

$$\pi^{\omega}_{[x_2,y_2],k'}(\mathcal{B}_1) = \sum_{z \in \Lambda^{\complement}} \pi^{\omega}_{[x_2,y_2],k'}(\mathcal{B}_{1,z}) \le (y_2 - x_2) e^{-\frac{V(y_2)-V(x_2)}{2}}. \tag{5.39}$$

Similarly, we can obtain

$$\pi^{\omega}_{[x_2,y_2],k'}(\mathcal{B}_2) \le (y_2 - x_2) e^{-\frac{V(y_2)-V(x_2)}{2}}. \tag{5.40}$$

Combining (5.39) with (5.40), in (5.38) we take

$$t = \frac{1}{4e^2(y_2 - x_2)} e^{\frac{V(y_2)-V(x_2)}{2}} - 1$$

to obtain

$$\mathbf{E}\left[\mathcal{T}'\right] \ge \frac{1}{2}\left(\frac{1}{4e^2(y_2-x_2)} e^{\frac{V(y_2)-V(x_2)}{2}} - 1\right) \ge \frac{1}{16e^2(y_2-x_2)} e^{\frac{V(y_2)-V(x_2)}{2}}. \tag{5.41}$$

□

6. Upper bound on the mixing time

This section is dedicated to the proof of Theorem 2.4. First in Section 6.1 we are going to reduce the problem to the estimation of the hitting time of $\xi_{\max}$. Afterwards using Proposition 3.2 and Proposition 3.3 we are going to provide estimate of this hitting time using a modified censored dynamics. First in Section 6.2 we treat the case of $k \le q_N$ which is a bit simpler and treat the more general case $q_N < k \le N/2$ in Section 6.4.

6.1. Deducing the mixing time from the hitting time of the maximal configuration. Let us first show that the study of the mixing time can be reduced to that of the probability of hitting the configuration $\xi_{\max}$ starting from the other extremal configuration $\xi_{\min}$.

Proposition 6.1. *We have for every $t > 0$ and $n \in \mathbb{N}$*

$$d^{\omega}_{N,k}(nt) \le \left(1 - P_t(\xi_{\min}, \xi_{\max})\right)^n. \tag{6.1}$$

Proof. We have (see for instance [LP17, Lemma 4.10])

$$d^{\omega}_{N,k}(t) \le \bar{d}^{\omega}_{N,k}(t) := \max_{\xi,\xi'} \|P^{\xi}_t - P^{\xi'}_t\|_{\mathrm{TV}} \le \max_{\xi,\xi'} \mathbf{P}\left[\sigma^{\xi}_t \ne \sigma^{\xi'}_t\right] \tag{6.2}$$

Using the monotonicity under the graphical construction (cf. Proposition 3.1) for all $\xi \in \Omega_{N,k}$ and $t \ge 0$ we have

$$\sigma^{\min}_t \le \sigma^{\xi}_t \le \sigma^{\max}_t,$$

where $\sigma^{\min}$ and $\sigma^{\max}$ are starting from the extremal conditions $\xi_{\min}$ and $\xi_{\max}$ in (3.3). As a consequence for arbitrary ξ and ξ' with $\tau' := \inf\{t \geq 0 \ : \sigma_t^{\xi} = \sigma_t^{\xi'}\}$, we have

$$\forall t \geq \tau', \quad \sigma_t^{\xi} = \sigma_t^{\xi'}. \tag{6.3}$$

On the other hand we have

$$\tau' \geq \tau := \inf\left\{t \geq 0 : \sigma_t^{\min} = \xi_{\max}\right\}. \tag{6.4}$$

Therefore (6.2) implies that

$$d^{\omega}_{N,k}(t) \leq \mathbf{P}\big(\tau > t\big). \tag{6.5}$$

Using again the Markov property and the monotonicity in Proposition 3.1, we have for any positive integer n

$$\mathbf{P}\left(\tau > nt\right) \leq \mathbf{P}\left(\sigma^{\min}_{it} \neq \xi_{\max}, \forall i \in [\![1, n]\!]\right) \leq \mathbf{P}\left(\sigma^{\min}_{t} \neq \xi_{\max}\right)^n. \tag{6.6}$$

□

6.2. The case $k_N \leq q_N$. Before stating the main result of this section, let us present a strategy to bound $P_t(\xi_{\min}, \xi_{\max})$ from below. We present in the process a few key technical lemmas whose proof is presented in the next subsection. We consider environment within the following event

$$\mathcal{A}_N := \left\{\omega : \max_{\substack{1 \leq x \leq y \leq N \\ y - x \geq q_N}} (V(y) - V(x)) \leq -3 \log N\right\}. \tag{6.7}$$

Note that by Proposition 3.4, this is an high probability event. The event $\mathcal{A}_N$ ensures that on segments of length $4q_N$, at equilibrium the particles concentrate on the right half of the segment with high probability. It also ensures that with high probability the last site is occupied by a particle.

Lemma 6.2. *If $\omega \in \mathcal{A}_N$, then we have for any $x \in [\![0, N - 4q_N]\!]$ and any $k \leq q_N$,*

$$\begin{aligned} \pi^{\omega}_{[x+1, x+4q_N], k}\left[\bar{\xi}(1) \leq x + 2q_N\right] &\leq 2q_N^2 N^{-3}, \\ \pi^{\omega}_{[x+1, x+4q_N], q_N}\left[\xi(x + 4q_N) = 0\right] &\leq 3q_N N^{-3}. \end{aligned} \tag{6.8}$$

Our second technical lemma is a direct consequence of Proposition 2.2. It allows to bound the mixing time of the system for each of the intervals of length $4q_N$ in a quantitative way. We define

$$T = T_N := 80\alpha^{-1} q_N^4 \binom{4q_N}{q_N} \left(\frac{1-\alpha}{\alpha}\right)^{2q_N} \log\left(\frac{1-\alpha}{\alpha}\right).$$

The following result is obtained by taking $\varepsilon = N^{-3}$ in Proposition 2.2.

Lemma 6.3. *Under the assumption* (2.21) *we have for all $k \leq q_N$, all ω and all $x \in [\![0, N - 4q_N]\!]$*

$$d^{\omega}_{[x+1, x+4q_N], k}(T) \leq N^{-3}. \tag{6.9}$$

We are going to use the censoring inequality to guide all the particles to the right with the following plan. We are going to design our censoring such that on the time interval $[iT, (i+1)T)$, with $i \in \mathbb{Z}_+$ satisfying $2(i+2)q_N < N$, our k particles perform the exclusion process restricted in the interval on the interval $[\![2iq_N + 1, 2(i+2)q_N]\!]$ (of length $4q_N$). Hence at each such time step, particles take a time T to shift towards the right of an amount $2q_N$. After the whole $\lceil N/(2q_N)\rceil - 1$ steps have been performed, all particles are in $[\![N - 4q_N + 1, N]\!]$. Once this is done we conclude using censoring again by showing that the dynamics in $[\![N - 4q_N + 1, N]\!]$ with less than q_N particles hits $\xi_{\max}$ after time T with a positive probability. For this last step we need the following result which states roughly that $\xi_{\max}$ has a positive weight with high probability under the equilibrium measure.

LEMMA 6.4. *We have*

$$\lim_{\varepsilon\to 0} \inf_{\substack{N\geq 1\\ k\in[\![1,N/2]\!]}} \mathbb{P}\left[\pi^{\omega}_{N,k}(\xi_{\max})>\varepsilon\right]=1. \tag{6.10}$$

In particular if $\mathcal{B}_{N,k} := \left\{\omega : \pi^{\omega}_{[N-4q_N+1,N],k}(\xi_{\max}) \geq 2q_N^{-1}\right\}$, *we have*

$$\lim_{N\to\infty} \inf_{k\in[\![1,q_N]\!]} \mathbb{P}\left[\mathcal{B}_{N,k}\right]=1. \tag{6.11}$$

PROPOSITION 6.5. *If* $k\leq q_N$, *if* $\omega\in\mathcal{A}_N\cap\mathcal{B}_{N,k}$ *and setting* $t_0 := T\left(\left\lceil \frac{N}{2q_N}\right\rceil -1\right)$, *we have*

$$P_{t_0}(\xi_{\min},\xi_{\max}) \geq \frac{3}{2q_N}. \tag{6.12}$$

In particular the inequality holds with high probability.

The last part of the statement is of course a direct consequence of the first part combined with (6.11) and of Proposition 3.4 (which ensures that $\mathcal{A}_N$ and $\mathcal{B}_{N,k}$ are high probability events). Before proving a proof of Proposition 6.5 using the strategy exposed above, let us use it to conclude the proof of the upper bound on the mixing time.

PROOF OF THEOREM 2.4 WHEN $k\leq q_N$. By Proposition 6.1 and Proposition 6.5, we have

$$d^{\omega}_{N,k}(2q_N t_0) \leq (1-P_{t_0}(\xi_{\min},\xi_{\max}))^{2q_N} \leq \left(1-\frac{1}{q_N}\right)^{2q_N} \leq \frac{1}{4}, \tag{6.13}$$

which allows us to conclude the proof for the case $k\leq q_N$ with the inequality

$$\binom{4q_N}{q_N} \leq \left(\frac{4^4}{3^3}\right)^{q_N}.$$

□

Now we move to prove Proposition 6.5 using the censoring inequality (Proposition 3.2). More precisely, we define for $i\in[\![0,\lceil N/(2q_N)\rceil-3]\!]$

$$\mathcal{C}_i := \Big\{\{i2q_N, i2q_N+1\},\{(i+2)2q_N,(i+2)2q_N+1\}\Big\} \tag{6.14}$$

and set

$$\mathcal{C}_{\lceil N/(2q_N)\rceil-2} := \{N-4q_N, N-4q_N+1\}. \tag{6.15}$$

We define a censoring scheme by setting

$$\mathcal{C}(t) := \mathcal{C}_i \quad \text{ for } t\in[iT,(i+1)T), i\in[\![0,\lceil N/(2q_N)\rceil-2]\!], \tag{6.16}$$

and $\mathcal{C}(t)=\emptyset$ for $t\geq \lceil N/(2q_N)\rceil-1$. Let us write

$$A_{\text{fin}} := \{\xi\in\Omega_{N,k} \ : \ \forall x\in[\![0,N-4q_N]\!], \xi(x)=0\}. \tag{6.17}$$

Recalling the notation of Section 3.3, we let $(\sigma^{\min,\mathcal{C}}_t)_{t\geq 0}$ denote the corresponding censored dynamics with initial condition $\xi_{\min}$.

LEMMA 6.6. *If* $\omega\in\mathcal{A}_N$, *we have*

$$\mathbf{P}\left[\sigma^{\min,\mathcal{C}}_{(\lceil N/2q_N\rceil-2)T}\in A_{\text{fin}}\right] \geq 1-N^{-1}. \tag{6.18}$$

Proof. For $i \in [\![0, \lceil N/2q_N \rceil - 2]\!]$, we define

$$A_i := \left\{ \xi \in \Omega_{N,k} : 2iq_N < \bar{\xi}(1) \le \bar{\xi}(k) \le 2(i+2)q_N \right\}.$$

Now we prove by induction that for all $i \in [\![0, \lceil N/2q_N \rceil - 3]\!]$

$$\mathbf{P}\left[\sigma^{\min,\mathcal{C}}_{iT} \in A_i\right] \ge 1 - i\frac{4q_N^2}{N^3}. \tag{6.19}$$

From the definitions of $\mathcal{C}$ and $\xi_{\min}$, the inequality in (6.19) holds for $i = 0$. Assuming that (6.19) holds for i, then k particles perform the simple exclusion process restricted in the interval $[\![2iq_N + 1, 2(i+2)q_N]\!]$. By Lemma 6.2 and Lemma 6.3 with $x = 2iq_N$, we have

$$\begin{aligned} &\mathbf{P}\left[\sigma^{\min,\mathcal{C}}_{(i+1)T} \in A_{i+1}\right] \\ &\quad \ge \mathbf{P}\left[\sigma^{\min,\mathcal{C}}_{iT} \in A_i\right] - \left(\pi^{\omega}_{[2iq_N+1,2(i+2)q_N],k}\left(\bar{\xi}(1) \le 2(i+1)q_N\right) + d^{\omega}_{[2iq_N+1,2(i+2)q_N],k}(T)\right) \\ &\quad \ge 1 - i\frac{4q_N^2}{N^3} - \frac{4q_N^2}{N^3}. \end{aligned} \tag{6.20}$$

This concludes the induction and the case $i = \lceil N/2q_N \rceil - 3$ in (6.20) to concludes the proof of the lemma since

$$2(\lceil N/2q_N \rceil - 2)q_N \ge N - 4q_N.$$

□

Proof of Proposition 6.5. Using Proposition 3.2, it is sufficient to bound the corresponding probability for the censored dynamics, that is, $P^{\mathcal{C}}_{(\lceil N/2q_N \rceil - 1)T}(\xi_{\min}, \xi_{\max})$. If

$$\sigma^{\min,\mathcal{C}}_{(\lceil N/2q_N \rceil - 2)T} \in A_{\text{fin}},$$

then the restriction to the segment $[\![N - 4q_N + 1, N]\!]$ of the dynamics corresponds to an exclusion process with k particles on a segment of length $4q_N$. Let $\pi_{[N-4q_N+1,N],k}$ and $d_{[N-4q_N+1,N],k}(t)$ denote respectively the equilibrium measure and the distance to equilibrium for this dynamics, and then we have

$$\begin{aligned} P^{\mathcal{C}}_{(\lceil N/2q_N \rceil - 1)T}(\xi_{\min}, \xi_{\max}) &\ge \mathbf{P}[\sigma^{\xi_{\min},\mathcal{C}}_{(\lceil N/2q_N \rceil - 2)T} \in A_{\text{fin}}](\pi_{[N-4q_N+1,N],k}(\xi_{\max}) - d_{[N-4q_N+1,N],k}(T)) \\ &\ge (1 - N^{-1})(2q_N^{-1} - N^{-3}) \ge \frac{3}{2q_N} \end{aligned} \tag{6.21}$$

where we have used the definition of $\mathcal{B}_{N,k}$ (recall (6.11)) and Lemma 6.3 with $x = N - 4q_N$.

□

6.3. Proof of auxiliary lemmas.

Proof of Lemma 6.2. To provide an upper bound on $\pi^{\omega}_{[x+1,x+4q_N],k}\left[\bar{\xi}(1) \le x + 2q_N\right]$, for $\xi \in \Omega_{[x+1,x+4q_N],k}$ we define its rightmost empty site to be

$$\bar{R}(\xi) := \sup\left\{y \in [\![x+1, x+4q_N]\!] :\ \xi(y) = 0\right\}. \tag{6.22}$$

As in (3.28), we have

$$\pi^{\omega}_{[x+1,x+4q_N],k}\left[\bar{\xi}(1) = z, \bar{R}(\xi) = y\right] \le e^{V^{\omega}(y) - V^{\omega}(z)} \le N^{-3} \tag{6.23}$$

where we have used $y - z \geq q_N$ and $\omega \in \mathcal{A}_N$. Then we have

$$\pi^{\omega}_{[x+1,x+4q_N],k}\left[\bar{\xi}(1) \leq x + 2q_N\right] = \sum_{\substack{z \in [\![x+1,x+2q_N]\!] \\ y \in [\![x+4q_N-k+2,x+4q_N]\!]}} \pi_{[x+1,x+4q_N],k}\left[\bar{\xi}(1) = z, \bar{R}(\xi) = y\right] \leq 2q_N^2 N^{-3}. \quad (6.24)$$

We now move to deal with $\pi^{\omega}_{[x+1,x+4q_N],q_N}[\xi(x+4q_N)=0]$. For $\xi \in \Omega_{[x+1,x+4q_N],q_N}$, we define its leftmost particle to be

$$\bar{L}(\xi) := \inf\left\{y \in [\![x+1, x+4q_N]\!] :\ \xi(y) = 1\right\}.$$

As in (3.28), we have

$$\pi^{\omega}_{[x+1,x+4q_N],q_N}\left[\xi(x+4q_N) = 0; \bar{L}(\xi) = y\right] \leq e^{V^{\omega}(x+4q_N) - V^{\omega}(y)} \leq N^{-3} \quad (6.25)$$

where we have used $y \leq x + 3q_N$ and $\omega \in \mathcal{A}_N$. Then

$$\pi^{\omega}_{[x+1,x+4q_N],q_N}\left[\xi(x+4q_N) = 0\right] = \sum_{y \in [\![x+1,x+3q_N]\!]} \pi^{\omega}_{[x+1,x+4q_N],q_N}\left[\xi(x+4q_N) = 0; \bar{L}(\xi) = y\right] \leq 3q_N N^{-3}. \quad (6.26)$$

□

Proof of Lemma 6.4. Recall the event $\mathcal{A}_r$ in (2.25). Observe that

$$\max_{\xi \in \mathcal{A}_r}\left(V^{\omega}(\xi_{\max}) - V^{\omega}(\xi)\right) \leq 2r^2 \log\frac{1-\alpha}{\alpha}, \quad (6.27)$$

and then we have

$$\frac{\pi^{\omega}_{N,k}(\xi_{\max})}{\pi^{\omega}_{N,k}(\mathcal{A}_r)} \geq |\mathcal{A}_r|^{-1} \exp\left(-\max_{\xi \in \mathcal{A}_r}\left(V^{\omega}(\xi_{\max}) - V^{\omega}(\xi)\right)\right) \geq 2^{-2r} e^{-2r^2 \log\frac{1-\alpha}{\alpha}}. \quad (6.28)$$

For given $\varepsilon > 0$ sufficiently small, we take

$$r(\varepsilon) := \left(\frac{-\log 2\varepsilon}{\log\frac{2(1-\alpha)}{\alpha}}\right)^{1/2} \quad (6.29)$$

so that the rightmost hand-side of (6.28) is larger than 2ε. Moreover, by (3.30) we know that

$$\lim_{r\to\infty} \inf_{\substack{N \geq 1 \\ k \in [\![1,N/2]\!]}} \mathbb{P}\left[\pi^{\omega}_{N,k}(\mathcal{A}_r) \geq 1 - 2(1 - e^{\frac{\mathbb{E}[\log \rho_1]}{2}})^{-2} e^{\frac{\mathbb{E}[\log \rho_1] r}{2}}\right] = 1. \quad (6.30)$$

Since when r is sufficiently large we have

$$1 - 2(1 - e^{\frac{\mathbb{E}[\log \rho_1]}{2}})^{-2} e^{\frac{\mathbb{E}[\log \rho_1] r}{2}} \geq \frac{1}{2},$$

then by (6.30) with r chosen as in (6.29) we obtain

$$\lim_{\varepsilon\to 0} \inf_{\substack{N \geq 1 \\ k \in [\![1,N/2]\!]}} \mathbb{P}\left[\pi^{\omega}_{N,k}(\xi_{\max}) \geq \varepsilon\right] = 1. \quad (6.31)$$

□

6.4. The case $k_N \geq q_N$. To treat the case of a larger number of particles, the small problem there is with the strategy of the previous subsection is that it does not allow to channel all the k particles to the right at the same time. What we do instead is that we use the process to transport one particle to the right, and then use Proposition 3.3 to be able to move all other particles to the left and iterate the process. We largely recycle the strategy used in the previous section. In the final step as in (6.21), we need to deal with the leftmost q_N particles performing the exclusion process restricted in the interval $[\![N-k-3q_N+1, N-k+q_N]\!]$, and then define

$$\mathcal{B}'_{N,k} = \left\{\omega : \pi^{\omega}_{[N-k-3q_N+1,N-k+q_N],q_N}(\xi'_{\max}) \geq 2q_N^{-1}\right\}$$

where $\xi'_{\max} := \mathbf{1}_{\{N-k+1 \leq x \leq N-k+q_N\}}$. By Lemma 6.4 we have

$$\lim_{N\to\infty} \inf_{k\in[\![q_N+1,N/2]\!]} \mathbb{P}\left[\mathcal{B}'_{N,k}\right] = 1. \tag{6.32}$$

Proposition 6.7. *If $k > q_N$ and $\omega \in \mathcal{A}_N \cap \mathcal{B}'_{N,k}$, setting $t_1 := \left(\left\lceil \frac{N-k+q_N}{2q_N}\right\rceil - 1\right)(k-q_N+1)T$ we have*

$$P_{t_1}(\xi_{\min}, \xi_{\max}) \geq \frac{1}{q_N}. \tag{6.33}$$

Proof of Theorem 2.4 when $k > q_N$. By Proposition 6.1 and Proposition 6.7, we have

$$d^{\omega}_{N,k}(2q_N t_1) \leq (1 - P_{t_1}(\xi_{\min}, \xi_{\max}))^{2q_N} \leq \left(1 - \frac{1}{q_N}\right)^{2q_N} \leq \frac{1}{4}, \tag{6.34}$$

which allows us to conclude the proof for the case $k > q_N$ with the inequality

$$\binom{4q_N}{q_N} \leq \left(\frac{4^4}{3^3}\right)^{q_N}.$$

□

The remaining of the subsection is devoted to the proof of Proposition 6.7. This time we need to combine our censoring scheme with displacements of particles to the left (using Proposition 3.3). Our plan is to first move (one by one) the rightmost $k - q_N$ particles to the segment $[\![N-k+q_N+1, N]\!]$ and use censoring to block the these $k - q_N$ particles afterwards. We are then left with the problem of moving the remaining q_N particles, and this can be treated as in Proposition 6.5.

Let us explain our plan to move the the rightmost $k - q_N$ particles one by one with censoring and displacement. We proceed by induction (each step is going to leave aside an event of small probability, and our technical estimates are such that the sum over all steps of these probabilities will remain small). We set $r = \lceil (N-k+q_N)/2q_N \rceil - 1$, and define for $j \in [\![0, k-q_N]\!]$, $i \in [\![0, \lceil (N-k+q_N)/2q_N \rceil - 3]\!]$, $a_{i,j} := k - q_N - j + 2q_N i$,

$$\begin{aligned} \mathcal{C}_{i,j} &:= \Big\{\{a_{i,j}, a_{i,j}+1\}, \{a_{i,j}+4q_N, a_{i,j}+4q_N+1\}, \{N-j, N-j+1\}\Big\}, \\ \mathcal{C}^*_j &= \Big\{\{N-4q_N-j, N-4q_N-j+1\}, \{N-j, N-j+1\}\Big\}. \end{aligned} \tag{6.35}$$

We define the censoring scheme $\mathcal{C}$ by setting

$$\begin{cases} \mathcal{C}(t) = \mathcal{C}_{i,j} & \text{if } t \in [(i+rj)T, (i+rj+1)T), \\ \mathcal{C}(t) = \mathcal{C}^*_j & \text{if } t \in [(r(j+1)-1)T, r(j+1)T), \\ \mathcal{C}(t) = \emptyset & \text{if } t \geq r(k-q_N+1)T. \end{cases} \tag{6.36}$$

The censored dynamic $(\sigma^{\mathcal{C},\min}_t)$ moves the first particle to the right in a time rT. Indeed, the same mechanism used in the proof of Proposition 6.5 moves (w.h.p) the last q_N particles in the segment $[\![N-4q_N+1,N]\!]$ by time $(r-1)T$. Then we mix the q_N particles within the segment $[\![N-4q_N+1,N]\!]$ and Lemma 6.2 ensures that after an additional time T, the last site N is occupied by a particle.

We then proceed by induction to show that for $j \le k-q_N$ all the sites in the segment $[\![N-j+1,N]\!]$ are occupied by particles by time rjT. Our censoring is designed so that after time rjT the number of particles in the j rightmost sites does not change.

In order to facilitate the induction (this is not strictly necessary though) at each time of the form $rjT =: s_j$ we move all the leftmost $N-j$ particles to the left on the segment $[\![1,N-j]\!]$, so that the beginning of each induction step looks the same. We define thus Q_j by setting

$$Q_j(\xi,\xi_j^*)=1, \quad Q_j(\xi,\xi')=0 \text{ if } \xi'\neq\xi_j^* \tag{6.37}$$

where the function $\xi\to\xi_j^*$ is defined by (recall (3.1))

$$\bar\xi_j^*(\ell)=\begin{cases}\ell & \text{if } l\le k-j,\\ \bar\xi(\ell) & \text{if } \ell>k-j.\end{cases} \tag{6.38}$$

Since $\xi_j^*\le\xi$, Q_j satisfies (3.9). We let $(\widetilde\sigma_t)_{t\ge0}$ denote the composed censored dynamics (recall (3.10)) corresponding to $\mathcal{C}$, $(s_j)_{j=1}^{k-q_N}$ and $(Q_j)_{j=1}^{k-q_N}$ and starting from $\xi_{\min}$. We set

$$\xi_j^0 := \mathbf{1}_{[\![1,k-j]\!]}+\mathbf{1}_{[\![N-j+1,N]\!]}.$$

The following lemma formalizes in a quantitative manner the induction described above.

Lemma 6.8. *For all $j\in[\![0,k-q_N]\!]$, we have*

$$\mathbf{P}\left[\widetilde\sigma_{rjT}=\xi_j^0\right]\ge 1-4jq_NN^{-2}. \tag{6.39}$$

Proof. The statement is trivial for $j=0$. For the induction step it is sufficient to prove that

$$\mathbf{P}\left[\widetilde\sigma_{r(j+1)T}=\xi_{j+1}^0 \mid \widetilde\sigma_{rjT}=\xi_j^0\right]\ge 1-4q_NN^{-2}. \tag{6.40}$$

With our choice for $\mathcal{C}$, the j particles in the interval $[\![N-j+1,N]\!]$ do not move between time instants rjT and $r(j+1)T$, it is therefore sufficient to control $\mathbf{P}\left[\widetilde\sigma_{r(j+1)T}(N-j)=1 \mid \widetilde\sigma_{rjT}=\xi_j^0\right]$. Let us define

$$B_j := \left\{\xi\in\Omega_{N,k} : \sum_{N-j-4q_N+1}^{N-j}\xi(x)=q_N\right\} \tag{6.41}$$

We can repeat the proof of Lemma 6.6 to obtain that

$$\mathbf{P}\left[\widetilde\sigma_{rjT+(r-1)T}\in B_j \mid \widetilde\sigma_{rjT}=\xi_j^*\right]\ge 1-(r-1)\frac{4q_N^2}{N^3}. \tag{6.42}$$

Now in the time interval $[rjT+(r-1)T, r(j+1)T)$, the censoring makes the restriction of the dynamics to the segment $[\![N-j-4q_N+1,N-j]\!]$ an exclusion process with q_N particles. Hence using Lemma 6.3 and the second estimate in Lemma 6.2 we have for any $\chi\in B_j$

$$\mathbf{P}\left[\widetilde\sigma_{r(j+1)T}(N-j)=1 \mid \widetilde\sigma_{rjT+(r-1)T}=\chi\right]\ge 1-N^{-3}(1+3q_N^2). \tag{6.43}$$

Combining (6.42) and (6.43), we obtain

$$\mathbf{P}\left[\widetilde\sigma_{r(j+1)T}=\xi_j^0\right]\ge \mathbf{P}\left[\widetilde\sigma_{r(j+1)T}=\xi_j^0\right]-r\frac{4q_N^2}{N^3}\ge 1-4(j+1)q_NN^{-2}. \tag{6.44}$$

□

Proof of Proposition 6.7. Taking $j = k - q_N$ in Lemma 6.8, from now on we assume that the event $\{\widetilde{\sigma}_{(k-q_N)rT} = \xi^0_{k-q_N}\}$ holds. Then the rightmost $k - q_N$ particles are frozen in the rightmost $k - q_N$ sites for $t \geq (k - q_N)rT$, and at $t = (k - q_N)rT$ the leftmost q_N particles are in the leftmost q_N sites. Thus we can repeat the proof in Proposition 6.5 to obtain

$$\mathbf{P}\left[\widetilde{\sigma}_{r(k-q_N+1)T} = \xi_{\max}\right] \geq \frac{3}{2} q_N^{-1} \left(1 - (k - q_N)\frac{4q_N}{N^2}\right) \geq \frac{1}{q_N} \tag{6.45}$$

where we have used $\omega \in \mathcal{B}'_{N,k}$. We conclude the proof by Proposition 3.2 and Proposition 3.3. $\square$

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Resumo: Esta tese estuda o tempo de mistura em variação total para a dinâmica do banho de calor de dois modelos de interface e um sistema de partículas.
O primeiro modelo de interface que consideramos é o modelo de fixação de polímero interagindo com uma linha defeituosa impenetrável e, na fase repulsiva, mostramos que a distância de variação total ao equilíbrio cai abruptamente de um para zero.
O outro modelo de interface é uma variante do modelo de fixação de polímero que também está sujeito a outra força externa puxando a interface para longe da linha defeituosa. Identificamos a fase localizada/deslocalizada para a estática, e para a dinâmica identificamos a fase de mistura lenta/rápida onde o tempo de mistura cresce polinomialmente/superpolinomialmente.
Finalmente, estudamos o processo de exclusão simples assimétrico em um ambiente aleatório onde as taxas de salto das partículas são obtidas independentemente de uma lei comum. Supondo que o ambiente aleatório seja transiente para a direita, provamos que com alta probabilidade o tempo de mistura do processo cresce polinomialmente.

Palavras-chave: Cadeia de markov, Dinâmica do banho de calor, Tempo de mistura, Modelos de interface, processo de exclusão, ambiente aleatório.

Abstract: This thesis studies the total variation mixing times for the heat-bath dynamics of two interface models and a particle system.
The first interface model we consider is the polymer pinning model interacting with an impenetrable defected line, and in the repulsive phase we show that the total variation distance to equilibrium abruptly drops from one to zero.
The other interface model is a variant of the polymer pinning model which is also subjected to another external force pulling the interface away from the defected line. We identify the localized/delocalized phase for the statics, and for the dynamics we identify the slow/rapidly mixing phase where the mixing time grows polynomially/superpolynomially.
Finally, we study the asymmetric simple exclusion process in a random environment where the jump rates of particles are independently sampled from a common law. Assuming that the random environment is transient to the right, we prove that the mixing time of the process grows polynomially with high probability.

Key words: Markov chain, Heat-bath dynamics, Mixing times, Interface models, Exclusion process, random environment.

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