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Doctoral Thesis

under the supervision of **Milton JARA** and **Wioletta RUSZEL**

Applications of Green Functions of Long-Range Diffusions in Probability

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Abstract

In this thesis, we study some probabilistic models which are all related to long-range diffusions. First, we will provide precise bounds for the speed for the convergence in law of a class of heavy-tailed random walks and detailed expansion of their Green functions in dimension 1. After that, we study the scaling limit of a class of random interfaces in the torus based on the divisible sandpile dynamics. Finally, we start to extend some analytical techniques for the study of stochastic partial differential equations in the context of non-local operators.

Resumo

Nesta tese, nós estudamos uma coleção de modelos probabilísticos relacionados a difusões de longo alcance. Primeiro, nós fornecemos cotas superiores precisas para a velocidade de convergência em distribuição para uma classe de passeios aleatórios de calda pesada e a expansão detalhadas de suas respectivas funções de Green em dimensão 1. Feito isso, nós provamos a convergência em distribuição de uma classe de interfaces aleatórias no toro baseada em modelos com dinâmica de pilhas de areia. Por fim, nós começamos a estender técnicas analíticas para o estudo de equações diferenciais parciais estocásticas para o contexto de operadores não locais.

Contents

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Introduction

Random interface models

This thesis is composed of a collection of papers in different contexts that try to illustrate the behaviour of random interface models once long-range behaviour is introduced. It is difficult to keep the selection of results under a single umbrella title, however, the thread that tries to connect these chapters is whether these interfaces defined in terms of some local diffusion will present an interesting change in behaviour when we redefine it in terms of a nonlocal diffusion instead.

We believe the term random interface can be broadly divided in two categories. In the first, by interface we mean the boundary of some random subset of $\mathbb{Z}^d$ or $\mathbb{R}^d$. The notion of boundary will be defined in terms of graphs in former case and in terms of the topology of $\mathbb{R}^d$ in the latter.

The second notion of boundary is the graph of a random function which could be seen as the boundary between its epigraph and the hypograph. Due to the 1-1 correspondence with its graph, we can see the function itself as a random interface. By allowing such point of view, we could also see random distributions (in the sense of Schwartz) as random interfaces, even though there is no well-defined notion of pointwise values (and therefore of graphs) of a distribution.

Although, most of the examples of the current thesis are related to the second class of random interfaces, we would like to give a quick example of both classes of interfaces.

First, consider the internal Diffusion Limited Agreggation (iDLA). In this model, introduced by Diaconis and Fulton, we start a Markov Chain in the space of finite subsets of $\mathbb{Z}^d$ with $\mathbf{P}[A(1) = \{0\}]$, then proceed with the following rule

$$\mathbf{P}\left[A(t+1) = A \cup \{x\} \mid A(t) = A\right] = \mathbb{P}_0\left[X_{\tau_{A^c}} = x\right],$$

where $\mathbb{P}_0$ is the law of the simple random walk starting at $X(0) = 0$ and $\tau_{A^c} := \inf\{t \geq 0 : X_t \notin A\}$. In [67], the authors showed that in event full probability, we have

$$B(0,(1-\varepsilon)n) \cap \mathbb{Z}^d \subset A(\lfloor \omega_d n^d \rfloor) \subset B(0,(1+\varepsilon)n) \cap \mathbb{Z}^d \text{ for all } n \text{ sufficiently large} \tag{1}$$

where $B(0,r)$ denotes the Euclidean ball of radius r and centre 0, ω_d denotes the volume of the unit ball of dimension d, and $\lfloor \cdot \rfloor$ denotes the floor function. This result characterises the interface $\partial A(t) := \{x \notin A(t) : d(x, A(t)) := 1\}$ up to its first order growth. Furthermore, the next order fluctuations are also understood, for $d = 2$ [60], they are $\mathcal{O}(\log n)$ and for $d \geq 3$, we have $\mathcal{O}(\sqrt{\log n})$ [61]. The equivalent results for $d = 1$, are usually considered trivial, as the geometry of $\mathbb{Z}$ renders the problem explicitly solvable in terms of the Gambler's ruin problem. See Fig 1 for a simulation of the iDLA.

For an example of the second type of random interface, we consider the discrete Gaussian Free Field. Given $\Lambda_n = [-\frac{n}{2}, \frac{n}{2})$, we can consider the Green function $G_n : \mathbb{Z}_n^d \times \mathbb{Z}_n^d \longrightarrow \mathbb{R}$ given by

$$G_n(x,y) := \mathbb{E}_x\left[\sum_{t=0}^{\tau_{(\Lambda_n)^c}-1} \mathbb{1}\left[X_t = y\right]\right]$$

where $\mathbb{E}_x$ is the expected value according to $\mathbb{P}_x$ and τ_{A^c} follows the same notation as the previous example. One can then define the discrete Gaussian Free Field (GFF) as the mean zero Gaussian vector $(\Xi^n(x))_{x\in\Lambda_n}$ such that

$$\operatorname{cov}\left(\Xi^n(x), \Xi^n(y)\right) := G_n(x,y).$$

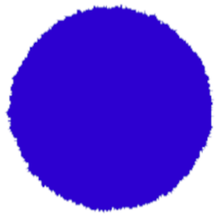

Fig. 1: A simulation of the iDLA after 20.000 particles are added.

We can see Ξ^n acting on functions $f \in C_c^\infty\left(\left[-\frac{1}{2},\frac{1}{2}\right)^d\right)$ as a distribution by taking

$$\Xi^n(f) := \frac{1}{n^{2d}} \sum_{x \in \Lambda_n} f\left(\frac{x}{n}\right) \Xi^n(x).$$

The discrete GFF does indeed converge as $n \longrightarrow \infty$, see [73, Proposition 12.2]. Only for $d = 1$, such distribution is achieved by a function in $L^1_{\text{loc}}\left(\left[-\frac{1}{2},\frac{1}{2}\right)^d\right)$, moreover, this function is the Brownian bridge. Again, this suggests that the one-dimensional case is trivial when compared to higher dimensions. Simulations of such fields can be found in Figure 3.1

In both examples, a good quantitative understanding of the underlying random walk becomes is necessary to complete the proofs. This understanding often takes the form of a good approximation for their associated Green functions.

Long-range and dimension increase

As exemplified by iDLA model, there are random interfaces whose behaviour is somewhat trivial in one dimension driven by a simple random walk (also called nearest-neighbour models), but also as an example in which long-range interactions (also referred as non-locality or heavy-tailed models) makes it difficult to even conjecture what to expect instead of (1).

Moreover, the equivalent of the continuous GFF defined in terms of fractional Laplacians (which are usually non-local) are also well understood and will be an important part of this thesis. Such fields are called fractional Gaussian Fields (fGF), see [73] for a comprehensive review on the subject. It is important to remark that such fractional fields depend on a parameter that reflects the power of the Laplacian. By tweaking this parameter for sufficiently low values, the limiting field is not a function even in $d = 1$. Moreover, we have that certain quantities of interest of specific fGF's behave precisely as their counterparts for the GFF in higher dimensions.

Indeed, long-range phenomena interaction can be seen as an increase of dimension of models. This is not a well-defined general result, but instead a useful rule of thumb. We will give two simple examples to try to illustrate how this happens.

The first example given in [95], is related to finite graphs. Let $G_n = (Z_n^2, E_n^2)$ be seen as a graph with periodic boundary, that is, for $x, y \in \mathbb{Z}_n^2$ we have that the bond $\{x, y\} \in E_n^2$ if $d_{\mathbb{Z}_n^2}(x, y) = 1$ where $d_{\mathbb{Z}_n^2}$ is the natural distance of the torus.

To derive our desired example, we select and remove all the edges that loop around the graph in parallel to the x-axis, whilst keeping the remaining edges untouched. Now, for each of the removed edges, we add a new edge connecting each vertex around the torus, but shifting its end point, so we have "diagonal" edges. We call the resulting graph G'_n, and we represent its construction graphically in Figure 2

Although this graph ceases to be translation invariant, it is still transitive. Moreover, one would expect that a probabilistic model defined in G'_n should display the same large-scale properties of models living in G_n. However, by starting from a fixed vertex in G'_n and always following the edge going to the right, we can see that G'_n is isomorphic to a graph $G''_n = (\mathbb{Z}_{n^2}, E''_{n^2})$ where $x, y \in \mathbb{Z}_{n^2}$ if $x - y \equiv \pm 1 \mod \mathbb{Z}_{n^2}$ or $x - y \equiv \pm n \mod \mathbb{Z}_{n^2}$. So a simple random walk in G''_n could be seen as a random walk with long jumps in $(\mathbb{Z}_{n^2}, E_{n^2})$. One would also expect that lattice models

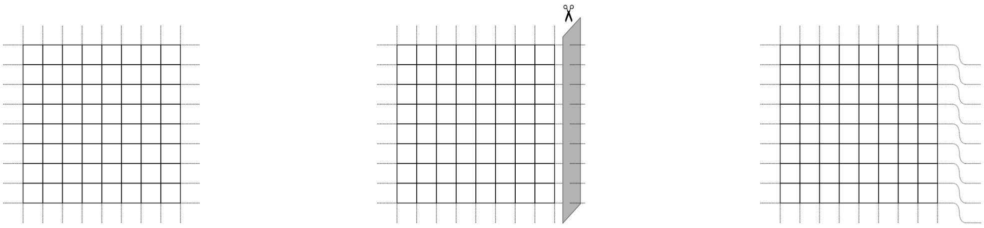

Fig. 2: Construction of the "twisted torus".

of statistical mechanics defined in $(\mathbb{Z}_n^2, E_n^2)$ should also present large scale behaviour similar to $(\mathbb{Z}_{n^2}, E_{n^2})$ with a corresponding long-range interaction. Notice that such constructions could be performed to reduce any d-dimensional torus to a lower dimension with extra edges.

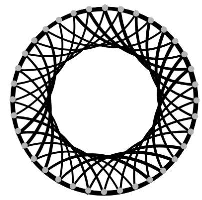

Fig. 3: The one-dimensional representation of G_6''

Another example of such dimension increase comes from analysis of PDE's. Consider the heat equation

$$\begin{cases} \partial_t u(x) - \Delta u(x) = 0, \text{ for } x \in \mathbb{R}^d, t > 0 \\ u(0,x) := u_0(x), \text{ for } x \in \mathbb{R}^d, \end{cases} \tag{2}$$

u_0 is a well-behaved function, we have that $u(t,x)$ can be computed as

$$u(t,x) := \mathbb{E}_x \left[u_0(B_t) \right], \tag{3}$$

where $\mathbb{E}_x$ denotes the law of the d-dimensional Brownian motion $\{B_t\}_{t\geq 0}$.

We also can characterise a similar solution for the half-Laplacian $-(-\Delta)^{1/2}$ (to be properly defined in the next chapter) which is a non-local operator, in the sense that if $f \equiv g$ in some domain $D \subset \mathbb{R}^d$, does not imply that $-(-\Delta)^{1/2} f$ is equal to $-(-\Delta)^{1/2} g$ in D. Such fractional powers of the Laplacian have many applications in physics, such as turbulent fluid motions [28, 40] or anomalous transport in fractured media [79]. For a general reference and more applications see also [37, 84].

We can then look at the equation

$$\begin{cases} \partial_t u(x) + (-\Delta)^{1/2} u(x) = 0, \text{ for } x \in \mathbb{R}^d, t > 0 \\ u(0,x) := u_0(x), \text{ for } x \in \mathbb{R}^d, \end{cases} \tag{4}$$

also admits a solution like (3), but with the Brownian motion substituted by a d-dimensional 1-stable Lévy Process, which is at times called Lévy flight, and has discontinuous trajectories due to long-jumps.

In order to connect the two, we look at the 2-dimensional elliptic equation

$$\begin{cases} -\Delta v(x) = 0, \text{ for } x \in \mathbb{R}^+ \times \mathbb{R} \\ v(x) := v_0(x_2), \text{ for } x = (0,x_2) \in \{0\} \times \mathbb{R}, \end{cases} \tag{5}$$

which also shares the same solution of (4) for $d = 1$ and $u(0, x) := v(0, x)$. Notice that the latter equation is entirely defined in terms of local operators.

Ising model as an example of why long-range is not a simple dimension increase

In the previous section, we tried to convince the reader that models with long-range interaction share similar behaviour with nearest-neighbour models in higher dimensions. However, as you look into deeper and more subtle properties of such models, this ceases to be the case. To illustrate this, we will use the Ising Model as an example.

As in the rest of this introduction, we will be somewhat vague, as the full description of the objects required to provide the correct statements goes beyond the scope of this subsection.

The Ising model in a finite set $\Lambda_n \subset \mathbb{Z}^d$ refers to the probability measure in $\{-1, +1\}^{\Lambda_n}$ satisfying

$$\mu^{\#}_{\beta,h,\Lambda_n} \propto e^{-\beta H^{\#}_{h,\Lambda_n}}$$

for some parameters $\beta > 0$, $h \in \mathbb{R}$ and $\# \in \{-1, +1\}^{\mathbb{Z}^d}$ and with Hamiltonian

$$H^{\#}_{h,\Lambda_n}(\sigma) = - \sum_{\substack{x \in \Lambda_n, y \in \mathbb{Z}^d \\ \|x-y\|_1 = 1}} p(x-y)\sigma_x \sigma^{\#}_y - \sum_{x \in \mathbb{Z}^d} h\sigma_x \tag{6}$$

where p is a symmetric probability kernel in $\mathbb{Z}^d$ and $\sigma^{\#} \in \{-1, +1\}^{\mathbb{Z}^d}$ is the configuration whose coordinates match with σ in Λ_n and with $\#$ in $\mathbb{Z}^d \setminus \Lambda_n$. It is not necessary in general that the factor p sums up to 1, but it is useful to keep the underlying random walk clear and could be achieved just by rescaling the parameters β and h.

Under the appropriate conditions of p and Λ_n, we have that the measure $\mu^{\#}_{\beta,h,\Lambda_n}$ converges a measure $\mu^{\#}_{\beta,h}$ as $n \longrightarrow \infty$. However, it might be the case that different choices of $\#$ lead to the same limit measure $\mu^{\#}_{\beta,h}$, when this fails to be the case, we say that the system undergoes *phase transition*, and one should expect that the existence depends on the parameters β and h as the first increases the effect of boundary effects and the second decreases it. For a good introduction to the Ising model, we recommend [41, Chapter 3]. One can prove that if $h = 0$ and p is the step distribution of the simple random walk, we do not have phase transition for any value of $\beta \in (0, \infty)$. However, under the same conditions but with $d \geq 2$, the model undergoes phase transition for all $\beta > \beta_c(d)$ and does not for $\beta < \beta_c(d)$. Such value $\beta_c(d)$ is the so-called critical temperature.

Again, if we introduce some long-range interaction in the one dimensional system, we can indeed cause the model to exhibit phase transition for β large enough. Indeed, let us focus on the so-called *Dyson model* which is the Ising model with $d = 1$ and $p(x) \propto \|x\|^{-2}$, this model was proven to display phase transition for $\beta > \beta_c(1)$, [42] "just like the 2-dimensional counterpart".

However, in the last two paragraphs do say anything about what happens at $\beta = \beta_c$. The nearest-neighbours Ising Model does not display phase transition at $\beta = \beta_c$ for any $d \geq 2$ [102, 3], this property is called *continuity of phase transition*. On the other hand, the Dyson model does indeed present phase transition at $\beta = \beta_c$ [2].

Moreover, another important quantitative behaviour measured in phase transition of statistical mechanics models is the so called *sharpness of phase transition*. This refers to the decay speed of the covariance $\mu^{\#}_{\beta,h}(\sigma_x\sigma_x) - \mu^{\#}_{\beta,h}(\sigma_x)\mu^{\#}_{\beta,h}(\sigma_x)$ as the distance between x and y diverges. Just like in the case of continuity, sharpness is a property that differentiate the phase transition of the Ising model in dimension $d \geq 2$ and the Dyson model, [59].

We can still see Ising models as random interfaces, that is given $\varepsilon > 0$, we can define the magnetisation field of as the distribution

$$\Phi_\varepsilon = a_\varepsilon \sum_{x \in \mathbb{Z}^d} \delta_{\varepsilon x} \sigma_x, \tag{7}$$

where $a_\varepsilon > 0$, $\delta_z = \delta(\cdot - z)$ is the delta function at z and $(\sigma_x)_{x \in \mathbb{Z}^d}$ is sampled according to $\mu^{\#}_{\beta,h}$. For the case $d = 2$, and $\beta = \beta_c$, and some appropriate choice of a_ε this field was proven to converge to a non-Gaussian limit, [20]. Moreover, in a dynamical setting, a similar result was proven in [78]. In that article, the author study the equivalent field to (7) for the Glauber dynamics of the Ising-Kac model, which is defined analogously with a change in the Hamiltonian (6) so that p takes ε in account. This time, the authors prove that the limit field is the so called *dynamical* Φ^4_2 *model*, which is the solution of a stochastic partial differential equation (SPDE).

A few extra difficulties of long-range systems

We would like to point out a few more differences of the study of long-range models, the first is in the discrete setting. When approximating the Brownian Motion via the Donsker's invariance principle, one tends to see the simple random walk as a natural candidate for such construction. Besides having a good geometry, the SRW is also useful because its law converges "fast" to the limit distribution. However, when looking for such analogue for processes in the domain of attraction of a α-stable random variable, it is not clear which random walk we should choose in order to obtain better quantitative bounds, this will be one of the central questions of Chapter 2.

Another difference can be seen in the continuous (in space) setting and comes from an analytical problem. The function $p_t^\alpha(x)$, defined as the heat semigroup of the fractional Laplacian $-(-\Delta)^{\alpha/2}$, has discontinuous time derivatives $\mathbb{R} \times \mathbb{R}^d \setminus \{0\} \times \mathbb{R}^d$. This presents a technical problem to construct the solution of a non-local SPDE and will be partially addressed in 4. Both the underlying differences in the nature of the phase transition and these technical problems make it less clear whether to expect the analogous result to the one in [78] should hold for any Dyson-Kac-like type of model.

Summary of the results

At the hears of each of the three main chapters of this thesis we will apply long-range there will be a different analytical tool to be studied in the context of long-range probabilistic models.

In Chapter 2, this tool will be the characteristic function (which can be seen as an inverse Fourier transform) which will be applied to random walks with heavy tails. There, we introduce a class (together with some examples of its elements) for which we can derive rates of convergence of the law of the heavy-tailed random walks to their scaling limits given by α-stable limits. Moreover, we derive a detailed expression for the potential kernel of the random walks in such class. This is based on the article [26]

In Chapter 3, we will use eigenvalues of the discrete fractional Laplacian in order to study the odometer function of the so-called divisible sandpile model (with long-range interaction). In practice this means computing the distribution of the maximum of the discrete forms of the fGF's and proving a central limit theorem type of behaviour for such discrete fields (even when we do not start with Gaussian random variables). This chapter follows the article [25]

Finally, in Chapter 4, we will turn to the study of Schauder estimates for the operator $-(-\Delta)^{1/2}$ in order to prove the local (in time) well posedness of a singular SPDE. We believe that a similar technique can be used to prove the so called *multi-scale Schauder estimates* in the context of regularity structures. This last chapter is based on the article [27].

Apart from Chapter 1, which goes over common notation for the remaining of the thesis, the chapters are all independent and can be read in any order. Finally, in Chapter 5 we discuss some possible directions of future research.

Chapter 1

Notation

In this chapter we will define some of the notation that will be used in multiple chapters of this thesis.

1.1 Miscellaneous

We start by fixing the notation of some numerical sets. For us, $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Moreover, $\mathbb{R}^+ := \{t \in \mathbb{R} : t > 0\}$ and $\mathbb{T} := [-\pi, \pi)$ is the torus, which is to be understood as $\mathbb{R}/(\bmod 2\pi)$. For $n \in \mathbb{N}$, we denote $\mathbb{Z}_n^d := [-\frac{n}{2}, \frac{n}{2}) \cap \mathbb{Z}^d$, and $\mathbb{T}_n^d := \frac{2\pi}{n}\mathbb{Z}_n^d$ the discrete torus.

For $x \in \mathbb{R}^d$, we will use $\|x\|$ to denote the ℓ^2 norm of x. We might also use $\|x\|_p$ to denote the ℓ^p norm of x. We for $r > 0$ will use $B(x, r)$ to denote the ball of centre x and radius r according to the ℓ^∞ norm, and we use $B_2(x, r)$ for its equivalent with the norm ℓ^2 norm. We will also denote $\lfloor \cdot \rfloor$ to be the floor function and $\lceil \cdot \rceil$ to be the ceiling function.

Given two functions f, g will use the notation $f \lesssim g$ if $f(\cdot) \leq Cg(\cdot)$ uniformly in the main variable, the constant may depend on the dimension d or some other variable, which should be clear from context of each chapter. If $f \lesssim g$ and $g \lesssim f$, we write $f \asymp g$. Finally, we use $\mathcal{O}(\cdot)$ and $o(\cdot)$ as the standard big-O or small-o notation respectively. We may use the abuse of notation $f(x) = \mathcal{O}\left(\|x\|^{\beta\pm}\right)$ to denote that $f(x) = \mathcal{O}\left(\|x\|^{\beta\pm\varepsilon}\right)$ for all $\varepsilon > 0$ small enough.

1.2 Function spaces

In this thesis we will use a few different function spaces, we will define then here for convenience.

Let $k \in \mathbb{N} \cup \{\infty\}$ and $\beta \in (0, 1)$ and $D \subseteq \mathbb{R}^d$ be a domain, we define $C^{k,\beta}(D)$ to be the set of functions that k derivatives and such that the derivatives of order k are β-Hölder continuous. We may abuse this notation to write $C^k(D) := C^{k,0}(D)$ the space of k-times differentiable function or $C^\beta(D) := C^{0,\beta}(D)$, which should be clear from context. We will use the following norm in $C^{k,0}(D)$

$$\|f\|_{C^k} := \sup_{\mathbf{m}:|\mathbf{m}| \leq k} \|D^{\mathbf{m}} f\|_{L^\infty(D)},$$

where $\mathbf{m}$ is taken over the multi-indexes $(\mathbb{N}_0)^d$ and $|\mathbf{m}| = \sum_{i=1}^d \mathbf{m}_i$, and $D^{\mathbf{m}} := \partial_{x_1}^{\mathbf{m}_1} \cdots \partial_{x_d}^{\mathbf{m}_d}$. Moreover, we define Hölder seminorm in $C^\beta(D)$:

$$[f]_{C^{0,\beta}(D)} := \sup_{\substack{x,x' \in D \\ x \neq x'}} \frac{|f(x) - f(x')|}{|x - x'|^\beta}.$$

Finally, we define the norm of the space $C^{k,\beta}(D)$ as

$$\|f\|_{C^{k,\beta}(D)} := \|f\|_{C^{k,0}(D)} + \sup_{\mathbf{m}:|\mathbf{m}|=k} [D^{\mathbf{m}} f]_{C^{0,\beta}(D)}.$$

We use $C_c^{k,\beta}(D)$ to denote function in $C^{k,\beta}(D)$ with a compact support. We denote by $C_b^{k,\beta}(D)$ the subspace of bounded functions in $C^{k,\alpha}$. Finally, by $C^{k,\beta}(\mathbb{T}^d)$, we denote the set of functions in

$C^{k,\alpha}(\mathbb{R}^d)$ which are 2π-periodic in each of its variables. We also use $f \in C^{k,\gamma-}(D)$ to denote that $f \in C^{k,\gamma-\varepsilon}(D)$ for all $\varepsilon > 0$ small enough.

Let $p \in [1,\infty)$, and $D \subseteq \mathbb{R}^d$ be a domain, we define

$$L^p(D) := \left\{ f : D \longrightarrow \mathbb{C} \text{ measurable } : \|f\|_{L^p(D)} := \left(\int_D |f(x)|^p dx \right)^{1/p} < \infty \right\}.$$

For $p = \infty$, we define

$$L^\infty(D) := \left\{ f : D \longrightarrow \mathbb{R} \text{ measurable } : \|f\|_{L^\infty(D)} := \operatorname{ess\,sup}_{x \in D} |f(x)| < \infty \right\}.$$

For $f, g \in L^2(D)$, we denote

$$\langle f, g \rangle := \int_D f(x)\overline{g(x)}dx$$

We will also use this same notation do describe to describe the dual pairing between a smooth function $g \in \mathcal{D}(D) := C_c^\infty(D)$ and a distribution $f \in \mathcal{D}'(D)$. We also use this same notation for the discrete case, that is, for $f, g \in \ell^2(\mathbb{T}_n^d)$, we denote

$$\langle f, g \rangle := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} f(x)\overline{g(x)}.$$

The meaning of $\langle \cdot, \cdot \rangle$ will always be clear from context. Although the definition of the $L^p(D)$ include functions in $\mathbb{C}$, we will mostly concentrate on $\mathbb{R}$-valued functions.

In Chapter 3 we will use Sobolev spaces in the torus and in Chapter 4 we will use negative Besov spaces. As such spaces will not be used in other chapters, we will delay their definitions until it is necessary.

1.3 Important operators

One of the most important operators we will be dealing with in this thesis is the Fractional Laplacian. Let $d \in \mathbb{N}$, $\alpha \in (0,2)$ and $f \in C_b^2(\mathbb{R}^d)$, we define

$$-(-\Delta)^{\alpha/2} f(x) := C_{d,\alpha} P.V. \int_{\mathbb{R}^d} \frac{f(x+y) + f(x-y) - 2f(x)}{2\|y\|^{d+\alpha}} dy, \tag{1.3.1}$$

where $P.V$ stands for principal value (in the Cauchy sense). The constant $c_{d,\alpha}$ is given by

$$C_{d,\alpha} := \frac{2^\alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{\pi^{d/2} |\Gamma(-\frac{\alpha}{2})|},$$

is chosen so that $(-\Delta)^{\alpha/2}(-\Delta)^{\beta/2} f = (-\Delta)^{(\alpha+\beta)/2} f$ for $\alpha + \beta < 2$.

We also consider its discrete counterpart. Let $f \in \ell^\infty(\mathbb{Z}^d)$, we define

$$-(-\Delta)_{\text{dis}}^{\alpha/2} f(x) := c_{d,\alpha} \sum_{y \in \mathbb{Z}^d \setminus \{0\}} \frac{f(x+y) + f(x-y) - 2f(x)}{2\|y\|^{d+\alpha}}. \tag{1.3.2}$$

Here, the constant $c_{d,\alpha}$ is the normalising constant

$$c_\alpha = c_{d,\alpha} := \left(\sum_{y \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{\|y\|^{d+\alpha}} \right)^{-1}.$$

Notice that we can see $-(-\Delta)^{\alpha/2}$ as an operator in $C^2(\mathbb{T}^d)$, as functions here have well-defined values in the whole $\mathbb{R}^d$. We will use the same notation for the two scenarios, but in case we want to emphasise the difference, we will use the notations $-(-\Delta)_{\mathbb{R}^d}^{\alpha/2}$ and $-(-\Delta)_{\mathbb{T}^d}^{\alpha/2}$.

We will also want to define $-(-\Delta)_{\text{dis}}^{\alpha/2}$ as an operator in $\ell^2(\mathbb{T}_n^d)$ by extending functions in $f \in \ell^2(\mathbb{T}_n^d)$ to the whole $\mathbb{Z}^d$ using the natural map

$$\tilde{f}(x) := f\left(\frac{nx^*}{2\pi}\right),$$

where $x^* \in \mathbb{T}_n^d$ is the unique value such that $x \equiv \frac{nx^*}{2\pi} \mod \mathbb{Z}_n^d$. We then define the operator

$$-(-\Delta)_n^{\alpha/2} f := -(-\Delta)_{\text{dis}}^{\alpha/2} \tilde{f} \tag{1.3.3}$$

and it will play an important role in Chapter 3.

Let us remark that for all $\alpha \in (0,2)$, $f \in C^\infty(\mathbb{T}^d)$ and all $x \in \bigcup_{n \geq 0} \mathbb{T}_n^d$ we have

$$\lim_{n \to \infty} n^\alpha (-\Delta)_n^{\alpha/2} f(x) = \frac{c_\alpha}{C_{d,\alpha}} (-\Delta)^{\alpha/2} f(x).$$

For $f \in L^p(\mathbb{R})$ and $p \in [1, \infty)$ and $x \in \mathbb{R}^d$, we denote

$$\mathcal{F}(f)(x) := \int_{\mathbb{R}^d} f(z) e^{-ix \cdot z} dz. \tag{1.3.4}$$

We chose this version due to its compatibility with characteristic functions.

For the periodic case, let $(e_k)_{\mathbb{Z}^d}$ be given by

$$e_k(z) := e^{ik \cdot z}, \tag{1.3.5}$$

notice that $(e_k)_{k \in \mathbb{Z}^d}$ is an orthogonal basis of $L^2(\mathbb{T}^d)$ but $\|e_k\|_{L^2(\mathbb{T}^d)} = (2\pi)^{d/2}$. For $f \in L^1(\mathbb{T}^d)$ and $k \in \mathbb{Z}^d$, we denote

$$\widehat{f}(k) = \mathcal{F}_{\mathbb{T}^d}(f)(k) := \int_{\mathbb{T}^d} f(z) e^{-iz \cdot k} dx \tag{1.3.6}$$

We will mostly stick to the first notation to avoid excessive notation, however, when we want to emphasise the difference between the Fourier transform in $\mathbb{R}^d$ and $\mathbb{T}^d$, we will use the second, this will mostly happen in Chapter 2. We extend such definitions to the set of tempered distributions in the usual way.

We also define the Fourier coefficients for the discrete torus for functions in $\ell^2(\mathbb{T}_n^d)$. Let $f \in C(\mathbb{T}^d)$, consider the projection of f into $\ell^2(\mathbb{T}_n^d)$ given by

$$\begin{aligned} \mathcal{P}_n : C(\mathbb{T}^d) &\longrightarrow \ell^2(\mathbb{T}_n^d) \\ f &\longmapsto \mathcal{P}_n(f) : x \in \mathbb{T}_n^d \mapsto f(x). \end{aligned} \tag{1.3.7}$$

Now, for $k \in \mathbb{Z}_n^d$, we define $e_k^n := \mathcal{P}^n(e_k)$. We have that $(e_k^n)_{k \in \mathbb{Z}_n^d}$ is an orthogonal basis of $\ell^2(\mathbb{T}_n^d)$ with the inner product $\langle \cdot, \cdot \rangle$. With this, we define the Fourier transform of a function $f \in \ell^2(\mathbb{T}_n^d)$ as

$$\widehat{f}(k) = \mathcal{F}_{\mathbb{T}_n^d}(f)(k) := \langle f, e_k^n \rangle. \tag{1.3.8}$$

Again, to avoid clumsy expressions, we will mostly use the first notation unless we want to emphasise that we are working on $\mathbb{T}_n^d$. Notice that for any fixed $f \in C(\mathbb{T}^d)$ and $k \in \mathbb{Z}^d$, we have $\mathcal{F}_n \circ \mathcal{P}_n(f)(k) \longrightarrow \mathcal{F}_{\mathbb{T}^d}(f)(k)$ as $n \longrightarrow \infty$.

An important observation is that, both at the discrete and the continuous level, the Fourier basis forms a basis of eigenvalues for the fractional Laplacians in the torus. This is particularly useful to compute the Green functions in the periodic setting.

1.4 Some stochastic processes of interest

Just as the fractional (discrete and continuous) Laplacians play a central role in this thesis, we will also study the stochastic processes generated by then.

Let $x, y \in \mathbb{Z}^d$, we define the probability kernel $p(x,y) = p(0, x-y)$ where

$$p_\alpha(0,x) := c_{d,\alpha} \frac{1}{\|x\|^{d+\alpha}} \mathbb{1}[x \neq 0]. \tag{1.4.1}$$

We will examine random walks with this probability transition multiple times in this thesis. In fact, this is the main example of class we study in Chapter 2 and will be used to drive the dynamics of the diffusion mechanism in Chapter 3.

Again, we will consider its embedding on the $\mathbb{T}_n^d$. To do so, let $x, y \in \mathbb{T}_n^d$, and let $p_\alpha^n(x,y) = p_\alpha^n(0, x-y)$ where

$$p_\alpha^n(0,x) := \sum_{x' \in \mathbb{Z}^d} p_\alpha\left(0, \frac{nx}{2\pi} + nx'\right). \tag{1.4.2}$$

For fixed $n \in \mathbb{N}$, we denote by $(X_{n,t}^\alpha)_{t \in \mathbb{N}_0}$ the random walk whose probability transitions are given by p_α^n.

We also consider the Lévy process starting at 0 and generated by $-(-\Delta)^{\alpha/2}$ which will be denoted by $(\bar{X}_t^\alpha)_{t \in \mathbb{R}^+}$. Let $\bar{p}_{\bar{X}_t^\alpha}(0,x)$ be the probability density of $\bar{X}_t^\alpha$ its probability density, which is given by

$$\bar{p}_{\bar{X}_t^\alpha}(0,x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t\|\theta\|^\alpha} e^{ix\cdot\theta} d\theta \tag{1.4.3}$$

we can also study the embedding of $(\bar{X}_t^\alpha)_{t \in \mathbb{R}^+}$ in $\mathbb{T}$, we will denote this process by $(\bar{W}_t^\alpha)_{t \in \mathbb{R}^+}$ whose probability density is given by

$$\bar{p}_{\bar{W}_t^\alpha}(0,x) := \sum_{x' \in \mathbb{Z}^d} \bar{p}_{\bar{X}_t^\alpha}(0, x + 2\pi x') \tag{1.4.4}$$

In Chapters 3 and 4 we need to deal with two types of randomness, one coming from the random walks/Lévy processes we just mentioned, the other coming from a white-noise (or its discrete equivalent). In order to separate the two, we will write probabilities/expectations according to the former as $\mathbf{P}_x$ or $\mathbf{E}_x$ where x denotes the starting point of the random walk/Lévy process. Probabilities and expectations taken according to the former will be denoted by $\mathbb{P}$ or $\mathbb{E}$.

1.5 Green functions and potential kernels

It is natural to care about Green functions when studying a particular pseudo-differential operator, in terms of the underlying stochastic process $(X_t)_{t \in T}$, this is simply

$$G_X(x,y) := \mathbf{E}_x\left[\sum_{t=0}^\infty \mathbb{1}_{[X_t = y]}\right] = \sum_{t=0}^\infty p_X^t(x,y) \tag{1.5.1}$$

if $T = \mathbb{N}_0$ where $p_X^t(0,x)$ is the t-th convolution of p_X with itself, not to be confused with p_α^n defined in (1.4.2). In the case $T = \mathbb{R}^+$, we would expect

$$G_X(x,y) := \mathbf{E}_x\left[\int_0^\infty \mathbb{1}_{[X_t = y]} dt\right] = \int_{\mathbb{R}} p_{X_t}(x,y) dt. \tag{1.5.2}$$

However, all the processes whose Green functions will be of interest in this thesis are actually recurrent: variations of $(X_t^\alpha)_{t \in \mathbb{N}_0}$ and $(\bar{X}_t^\alpha)_{t \in \mathbb{N}_0}$ in Chapter 2 for $d = 1$ and $\alpha \geq 1$, $(X_{n,t}^\alpha)_{t \in \mathbb{N}_0}$ in Chapter 3 and $(W_t^1)_{t \in \mathbb{R}^+}$ in Chapter 4.

Therefore, we need to add some form of renormalisation. In Chapter 2, we study potential kernels instead. That is, for a process $(X_t)_{t \in T}$, we define its potential kernel to be

$$a_X(0,x) := \sum_{t=0}^\infty \left(p_X^t(0,x) - p_X^t(0,0)\right) \tag{1.5.3}$$

if $T = \mathbb{N}_0$ or

$$a_X(0,x) := \int_{\mathbb{R}} \left(p_{X_t}(0,x) - p_{X_t}(0,0)\right) dt \tag{1.5.4}$$

if $T = \mathbb{R}^+$.

Therefore, in both cases $a_X(0,0) = 0$. For processes that live in the torus (either discrete or continuous) we will prefer to have a mean zero function. That is, for $x, y \in \mathbb{T}_n^d$, we will use the Green function

$$g_\alpha(x,y) = g_\alpha^n(x,y) := \frac{1}{n^d} \sum_{z \in \mathbb{T}_n^d} g_\alpha^z(x,y), \tag{1.5.5}$$

where

$$g_\alpha^z(x,y) := \mathbf{E}_x \left[\sum_{t=0}^{\tau_z - 1} \mathbb{1}_{[X_{n,t}^\alpha = y]} \right], \tag{1.5.6}$$

and $\tau_z := \inf\{t \geq 0 : X_t^n = z\}$. We then have

$$\left(-(-\Delta)_n^{\alpha/2} g_\alpha(x,\cdot) \right)(y) = -\left(\delta_{x,y} - \frac{1}{n^d} \right), \tag{1.5.7}$$

where $\delta_{x,y}$ is the Kronecker delta function. One can check that for any $x \in \mathbb{T}_n^d$ we have

$$\frac{1}{n^d} \sum_{y \in \mathbb{T}_n^d} g_\alpha(x,y) = 0,$$

which will be convenient in our computations.

In Chapter 4, just as we did for the discrete torus, we will also choose a Green function with the mean zero restriction, that is let G be the unique solution of

$$\begin{cases} -\Big((-\Delta)^{1/2} \quad G(x,\cdot) \Big)(y) = \delta(x-y), y \in \mathbb{T} \\ \qquad\qquad \int_{\mathbb{T}} G(x,y)dy = 0 \end{cases}$$

The function G can be calculated via its Fourier Transform. This results in

$$G(x,y) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{e_k(x)}{|k|}. \tag{1.5.8}$$

It is possible to evaluate this sum, this results in $G(x,y) = G(x-y)$, where

$$G(x) := -\frac{1}{\pi} \log\left(2 \Big| \sin\left(\frac{x}{2}\right) \Big| \right) \tag{1.5.9}$$

A straightforward computation on the Fourier space yields that for any function f in the domain of the operator $(-\Delta)^{1/2}$ and orthogonal to e_0 [that is, such that $\int_{\mathbb{T}} f(x)\,dx = 0$]

$$[(-\Delta)^{1/2} f] * G \;=\; f\,, \tag{1.5.10}$$

where $f * g$ stands for the convolution of two functions f, g in $L^2(\mathbb{T})$:

$$(f * g)(x) \;=\; \int_{\mathbb{T}} f(x-y)\, g(y)\, dy\,.$$

Part I

Random interfaces on equilibrium

Chapter 2

Local Central Theorems and potential kernel estimates

2.1 Introduction and overview of the results

When approximating the law of the Brownian motion, one usually appeals to Donsker's Theorem (see [64, Theorem 4.2]). Notice that, this construction allows us to choose the distribution of the i.i.d random variables that will be used. If we are interested in discrete approximations, the natural choice to take is the common law to be p_X the Rademacher distribution. Indeed, this choice leads is simple to compute, yet we have very good rates of convergence. When dealing with α-stable distributions with $\alpha \in (0,2)$ we cease to have a clear winner of the most practical choice of for the analogue approximation for its respective Lévy process. One of the ideas behind this chapter is to construct what we call *repaired* distributions, for which we will derive optimal rates of convergence.

Here, when discussing such rates, we are referring to local central limit theorems (LCLT) and potential kernel (or Green function) which are fundamental results in probability theory. They are important to study convergences of sequences of random variables for a variety of contexts in probability and statistical physics. Applications include mixing rates of Lorentz gases [82], asymptotic shapes in Internal Diffusion Limited Aggregation [67], scaling limit of the discrete Gaussian Free Field [19], convergence of discrete Gaussian multiplicative chaos [87] and bounds on size of the largest component for percolation on a box [83].

In this chapter, we study a class of i.i.d. heavy-tailed random variables $(X_i)_{i\in\mathbb{N}}$ with support on $\mathbb{Z}$ which are in the domain of attraction of a symmetric α-stable random variable $\bar{X}$ with index $\alpha \in (0,2)$ and satisfy a particular expansion of their characteristic function. We will prove a LCLT result providing sharp convergence rates for $p_X^n(\cdot)$, the law of $S_n := \sum_{i=1}^n X_i$, explicit asymptotic behaviour of its discrete potential kernel and additionally obtain a detailed expansion of the characteristic function for the step size of a long-range random walk in $\mathbb{Z}$.

There exists a vast literature providing different types of LCLT results (or local stable limit theorems) in the stable setting with explicit and implicit convergence rates, e.g. [11, 12, 13, 21, 47, 57, 77, 91, 96]. To our knowledge, the best explicit non-uniform convergence rate for 1d absolutely continuous X was proven in [36], where the author showed under some integrability conditions on the characteristic function that for any $\alpha \in (0,2)$:

$$|x|^\alpha \left|p_X^n(x) - p_{\bar{X}}^n(x)\right| \leq Cn^\gamma, \tag{2.1.1}$$

where $\bar{X}$ is the stable distribution of index α and $\gamma = 1 - \frac{2}{\alpha}$ if $\alpha \in [1,2)$ and $\gamma = 1 - \frac{1}{\alpha}$ if $\alpha \in (0,1)$. As for uniform bounds in x, one can use classical results of convergence of random variables (such as in [91, 96]) which imply that

$$n^{\frac{1}{\alpha}} \left|p_X^n(x) - p_{\bar{X}}^n(x)\right| = o(1). \tag{2.1.2}$$

In [13] the author studied LCLT and large deviation estimates for random variables in the Cauchy domain of attraction for mainly asymmetric 1d random walks using renewal theory. Renewal theory was also used in [21] to obtain large deviation results for Lévy walks and in [65, 75] in the dynamical systems setting. A different approach proving LCLT results was taken in a series of papers [62, 66, 74], where the authors use subadditivity of diverse metrics (Kolmogorov, Zoltarev or Mallows distance) to prove LCLT's for continuous heavy-tailed random variables.

Concerning discrete potential kernel or Green's function behaviour there has been some asymptotic estimates obtained in [4, 10, 13, 14, 101] and [100] in the continuum. In [101], the author proves that for $\alpha \in (0,2)$ the discrete potential kernel is asymptotic to $\|x\|^{d-\alpha}L(|x|)$ where $L(\cdot)$ is a slowly varying function, whereas [14] obtains similar asymptotic behaviour for processes on $\mathbb{Z}^d$ with index $\alpha = (\alpha_1, ..., \alpha_d)$ and $\alpha \in (0,2]^d$.

The uniform bound given in (2.1.2) is indeed sharp, as for each $\varepsilon \in (0,2)$ one can use examples from this chapter to construct sequences in which the term $o(1)$ in (2.1.2) is of order $\mathcal{O}(n^{-\varepsilon})$. Let us make a brief analogy to the LCLT rates in the classical domain of attraction of a Gaussian distribution. For convenience, we will stay in the symmetric distribution case. Under additional moment conditions, say $\mathbb{E}(|X|^3) < \infty$ or $\mathbb{E}(X^4) < \infty$, the speed of convergence in the LCLT can be improved from $\mathcal{O}(n^{-\frac{1}{2}})$, given in (2.1.2), to $\mathcal{O}(n^{-1})$ and $\mathcal{O}(n^{-\frac{3}{2}})$ respectively, see [68]. The Edgeworth expansion [35] tells us that these speeds are indeed optimal. In general, one can use cumulants of higher order to get an expansion of the characteristic function and to derive more information about the rate of convergence of such laws. Notice that this is not possible in the context of variables in the domain of attraction of an α-stable distribution, as moments, and therefore cumulants cease to exist. Therefore, we will need to derive the further expansions of the characteristic function analytically.

Let us state the main results from this chapter. Assume that the common characteristic function of the random variables X_i's satisfies the following expansion with respect to $\alpha \in (0,2)$ and regularity set $R_\alpha \subset (\alpha, 2+\alpha)$:

$$\phi_X(\theta) = 1 - \kappa_\alpha|\theta|^\alpha + \sum_{\beta \in R_\alpha} \kappa_\beta |\theta|^\beta + \mathcal{O}\Big(|\theta|^{2+\alpha}\Big) \tag{2.1.3}$$

as $|\theta| \longrightarrow 0$ with constants $\kappa_\alpha > 0, \kappa_\beta \in \mathbb{R}$. This class turns out to have nice properties, it is closed under e.g. addition and convex combinations. The concept of the regularity set R_α is similar to the *index set* A, which appears in the definition of *regularity structures* in [53].

We will show in Proposition 2.4.1 that the symmetric long-range random walk with transition probability $p(x,y) = c_\alpha|x-y|^{-(1+\alpha)}$ for $\alpha \in (0,2)$ falls into this class with $R_\alpha = \{2\}$ and determine the precise expansion of the characteristic function.

One of the main results, Theorem 2.3.2, yields sharp convergence rates:

$$\sup_{x\in\mathbb{Z}} \left|p_X^n(x) - p_{\bar{X}}^n(x)\right| \lesssim n^{-\frac{\beta_1+1-\alpha}{\alpha}}$$

where $\beta_1 = \min(J_\alpha^+)$ and $J_\alpha^+ \subset (\alpha, 2+\alpha)$ is a set which depends on the regularity set R_α. For a particular case where $R_\alpha \in \{\varnothing, \{2\}\}$ we prove in Theorem 2.3.1 that given a random variable Z symmetric, with finite support and variance $|\kappa_2|$ and $\bar{Z} \sim \mathcal{N}(0, |\kappa_2|)$. Then if

1. $\kappa_2 = 0$ we have that $\sup_{x\in\mathbb{Z}} |p_X^n(x) - p_{\bar{X}}^n(x)| \lesssim n^{-(1+\frac{1}{\alpha})}$
2. $\kappa_2 > 0$ we have that $\sup_{x\in\mathbb{Z}} |p_{X+Z}^n(x) - p_{\bar{X}}^n(x)| \lesssim n^{-(1+\frac{1}{\alpha})}$
3. $\kappa_2 < 0$ we have that $\sup_{x\in\mathbb{Z}} |p_X^n(x) - p_{\bar{X}+\bar{Z}}^n(x)| \lesssim n^{-(1+\frac{1}{\alpha})}$.

Note that depending on the sign of the constant κ_2 in the expansion we will modify either the original law $p_X^n(\cdot)$ or the limiting law $p_{\bar{X}}^n(\cdot)$ in such a way that the strong convergence rate $n^{-(1+\frac{1}{\alpha})}$ prevails. This modification introduces an error of order $\mathcal{O}(n^{-\frac{1}{\alpha}+(1-\frac{2}{\alpha})})$ which will vanish as $n \to \infty$.

The proofs involve a careful analysis of the laws $p_X^n(\cdot)$ and $p_{\bar{X}}^n(\cdot)$ in terms of their characteristic functions. The modification idea is natural and has shown to be very fruitful for example in [43] where the authors used it to obtain better convergence rates of a truncated Green's function in $\mathbb{Z}^2$. Furthermore, we provide explicit potential kernel bounds for $\alpha \in [1,2)$:

In Theorem 2.3.5 we will prove that there exist explicit constants C_α, C_0, C_δ such that for $|x| \to \infty$ and $\delta := \min(R_\alpha)$ we have

(i) If $\delta < 2\alpha - 1$, then there exists a constant C_δ such that

$$a_X(0,x) = C_\alpha |x|^{\alpha-1} + C_\delta |x|^{2\alpha-\delta-1} + \mathcal{O}(|x|^{2\alpha-\delta-1}),$$

(ii) if $\delta > 2\alpha - 1$, then there exists a constant C_0 such that

$$a_X(0,x) = C_\alpha |x|^{\alpha-1} + C_0 + o(1),$$

(iii) if $\delta = 2\alpha - 1$, then there exists a constant C_δ such that

$$a_X(0,x) = C_\alpha |x|^{\alpha-1} + C_\delta \log |x| + \mathcal{O}(1).$$

as $x \to \infty$.

In particular for $R_\alpha \in \{\varnothing, \{2\}\}$ we prove in Theorems 2.3.4 and 2.3.6 that there exist constants $C_0, C_\alpha, ..., C_{m_\alpha}$ such that:

(i) if $\alpha \in (1,2)$ and $\kappa_2 = 0$ then

$$a_X(0,x) = C_\alpha |x|^{\alpha-1} + C_0 + \mathcal{O}(|x|^{\frac{\alpha-2}{3}+\varepsilon}),$$

for any ε small enough

(ii) if $\alpha \in (1,2)$ and $\kappa_2 \neq 0$ then

$$a_X(0,x) = C_\alpha |x|^{\alpha-1} + \sum_{m=1}^{m_\alpha} C_m |x|^{\alpha-1-m(2-\alpha)} + C_0' \log |x| + \mathcal{O}(1)$$

(iii) if $\alpha = 1$ we have

$$a_X(0,x) = C_\alpha \log(|x|) + C_0 + o(1),$$

where the term $o(1)$ can be estimated if $\kappa_2 = 0$.

The proofs of the potential kernel bounds are original, and they exploit the asymptotics of the characteristic function together with Hölder continuity instead of using the LCLT as a starting point like in the classical case [68].

The novelty of this chapter includes to study the expansion of the characteristic function in terms of regularity sets, sharp convergence bounds in the LCLT and explicit asymptotic expansion with error bounds of the potential kernel and characteristic function of a random walk for a class of heavy-tailed random variables whose characteristic function satisfies (2.1.3) which did not exist in the literature yet.

Structure of the chapter

In Section 2.2, we provide the setting and introduce necessary definitions. In Section 2.3, we state our main Theorems. The next Section 2.4 deals with determining the expansion of the characteristic function for an explicit example of a long-range random walk and showing that it falls into the class we consider in this chapter. Section 2.5 contains all proofs regarding LCLT's and in Section 2.6 we demonstrate estimates on the discrete potential kernels. In Section 2.7, we present some final remarks on the possibility and limitations of generalising our techniques to the cases $\alpha < 1$ and/or $d \geq 2$, non-lattice and continuous time random walks. Some technical lemmata are postponed to the Appendix.

2.2 Definitions

We start by defining some notation specific to this section.:w

Given finite sets of positive real numbers $A, B \subset \mathbb{R}^+$, we define its sum by

$$A + B := \{a + b : a \in A, b \in B\}.$$

and

$$\text{span}(A) := \left\{ \sum_{a \in A} l_a a : l \in (N_0)^A \setminus \{0\} \right\} = \bigcup_{j \geq 1} \overbrace{(A + \cdots + A)}^{j \text{ times}}$$

Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables defined on some common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $p_X(\cdot)$ the probability distribution of X, with support in $\mathbb{Z}$ and assume that $p_X(-x) = p_X(x)$ for all $x \in \mathbb{Z}$. We write shorthand X instead of X_i when we refer to one single random variable. Call $S_t := \sum_{i=1}^t X_i$ its sum and abbreviate by $p_X^t(\cdot)$ the corresponding probability distribution. Denote by

$$\phi_X(\theta) := \mathbf{E}\Big[e^{i\theta \cdot X}\Big], \ \ \theta \in \mathbb{R}$$

its common characteristic function.

In the following let us define the class of random variables which we will consider in this chapter.

Definition 2.2.1. *Let $\alpha \in (0, 2]$ and let $R_\alpha \subset (\alpha, 2 + \alpha)$ be a finite set. We call the probability distribution $p_X(\cdot)$ of a symmetric random variable X with support in $\mathbb{Z}$ admissible of index α and regularity set R_α (or just admissible) if its corresponding characteristic function $\phi_X(\theta)$ admits the following expansion*

$$\phi_X(\theta) = 1 - \kappa_\alpha |\theta|^\alpha + \sum_{\beta \in R_\alpha} \kappa_\beta |\theta|^\beta + \mathcal{O}\Big(|\theta|^{2+\alpha}\Big) \tag{2.2.1}$$

as $|\theta| \longrightarrow 0$, for constants $\kappa_\alpha > 0$ and $\kappa_\beta \in \mathbb{R} \setminus \{0\}$, for all $\beta \in R_\alpha$.

It is important to recall that the constants $\kappa_\alpha, \kappa_\beta$, given in the definition above, depend on the law of $p_X(\cdot)$. However, to keep notation short, we omit to explicit this dependence.

Our *regularity set* R_α is a finite collection of powers of $|\theta|$ in the expansion of the characteristic function, up to orders which are strictly smaller than $2 + \alpha$.

In order to obtain sharp convergence rates of the LCLT, expansions up to an error term of order $\mathcal{O}(|\theta|^{2\alpha})$ are enough. In fact, for the LCLT this order of the error term is optimal. Regarding the potential kernel estimates choosing an error of order $\mathcal{O}(|\theta|^{2+\alpha})$ improves the expansion compared to choosing $\mathcal{O}(|\theta|^{2\alpha})$. For us, choosing $\mathcal{O}(|\theta|^{2+\alpha})$ is a natural choice since it appears in the expansion of the characteristic function for the distribution of the step size of a long-range random walk, see Section 2.4.

Furthermore, let

$$J_\alpha := \text{span}(R_\alpha^+) \cap (\alpha, 2 + \alpha), \tag{2.2.2}$$

where $R_\alpha^+ := R_\alpha \cup \{\alpha\}$. In a similar way, we define $J_\alpha^+ := J_\alpha \cup \{2 + \alpha\}$. Remark that if $R_\alpha = \emptyset$ we have that $J_\alpha = \alpha\mathbb{N} \cap (\alpha, 2 + \alpha)$ and in particular $\beta_1 = \min(J_\alpha^+) \leq 2\alpha$ for any admissible distribution.

Using the expansion given in (2.2.1) and the Taylor polynomial of $\log(1 + z)$ for $|z| < 1$, setting $z := \phi_X(\theta) - 1$, we get that $\phi_X(\cdot)$ can be written as

$$\phi_X(\theta) = e^{-\kappa_\alpha |\theta|^\alpha + r_X(\theta) + \mathcal{O}(|\theta|^{2+\alpha})}, \quad \text{as } |\theta| \longrightarrow 0, \tag{2.2.3}$$

where

$$r_X(\theta) = \sum_{j \in J_\alpha} \eta_j |\theta|^j,$$

and the coefficients η_j are combinations of coefficients coming from the expansion of the logarithm and the powers $|\theta|^\alpha$ resp. $|\theta|^\beta$. In particular, for $\alpha \in (1, 2)$ and $R_\alpha = \{2\}$, we have $r_X(\theta) = \kappa_2 |\theta|^2 - \frac{(\kappa_\alpha)^2}{2} |\theta|^{2\alpha}$.

The class of admissible probability distributions should be seen as a natural and well-behaved collection of probability distributions in the domain of attraction of an α-stable distribution. Indeed, in the classical central limit theorem case, one usually requires finite moments of order 3 or 4 to study LCLT's. This can be understood as a convenient way of making assumptions about characteristic functions of such variables. Once the term $|\theta|^\alpha$ for $\alpha \in (0,2)$ appears in the expansion of ϕ_X, $\lceil \alpha \rceil$ moments cease to exist. Hence, in the stable case we need to make assumptions directly in the terms of the expansion of the characteristic function instead of their moments.

Remark that symmetric random variables with support in $\mathbb{Z}$ and finite fourth moment have an admissible distribution of index $\alpha = 2$ and $R_\alpha = \varnothing$. Both LCLT and potential kernel estimates for such random variables are well understood, see [68]. For this reason, we will concentrate on the case $\alpha \in (0,2)$.

The class of admissible probability distributions is closed under natural operations. Let $p_{X_1}(\cdot)$ and $p_{X_2}(\cdot)$ be admissible distributions of independent random variables X_1 and X_2 of indexes $\alpha_1, \alpha_2 \in (0,2]$, $\alpha_1 \le \alpha_2$ and regularity sets R_{α_1}, R_{α_2} respectively. We have that their convolution equal to

$$p_X(x) := p_{X_1} * p_{X_2}(x)$$

is admissible of index α_1 and regularity set

$$R'_{\alpha_1} \subset (R_{\alpha_1} + R^+_{\alpha_2}) \cap (\alpha_1, 2 + \alpha_1).$$

Moreover convex combinations

$$p_{\tilde{X}}(x) := q \cdot p_{X_1}(x) + (1-q) p_{X_2}(x) \tag{2.2.4}$$

for $q \in (0,1)$ are admissible of index α_1 and regularity set

$$R^*_\alpha \subset \left(R_{\alpha_1} \cup R^+_{\alpha_2}\right) \cap (\alpha_1, 2 + \alpha_1).$$

We can only write the regularity sets as subsets since there might be cancellations due to the convolution or convex combinations.

Note that $\tilde{X} := UX_1 + (1-U)X_2$ where U is a Bernoulli r.v. with parameter q, independent of X_1 and X_2, has distribution $p_{\tilde{X}}(\cdot)$.

Our main example of an admissible distribution of index $\alpha \in (0,2)$ and $R_\alpha = \{2\}$ is p_α given by (1.4.1). We will discuss this example in Section 2.4. However, using similar ideas, one can show that the distribution given by

$$\tilde{p}_\alpha(x) = \tilde{p}_\alpha(-x) := \frac{1}{2|x|^\alpha} - \frac{1}{2(|x|+1)^\alpha}, \text{ for } x \in \mathbb{Z} \setminus \{0\}$$

is admissible of index α and regularity set $R_\alpha := \{2, 1+\alpha\}$.

An example of a distribution which is *not admissible* is $p_\alpha(\cdot)$, defined in (1.4.1) with $\alpha = 2$. In fact, in this case the characteristic function has the expansion

$$\phi_X(\theta) = 1 - \kappa_2 |\theta|^2 \log|\theta| + \mathcal{O}(|\theta|^2).$$

Let $p_{\bar{X}}(\cdot)$ denote the density of a symmetric *α-stable random variable* $\bar{X}$ of index $\alpha \in (0,2)$ and scale parameter $c = (\kappa_\alpha)^{1/\alpha}$. Its characteristic function is given by

$$\phi_{\bar{X}}(\theta) = e^{-\kappa_\alpha |\theta|^\alpha}. \tag{2.2.5}$$

Its t-th convolution will be abbreviated by $p^t_{\bar{X}}(\cdot)$. Notice that if $p_X(\cdot)$ is admissible, $t^{-\frac{1}{\alpha}} S_t$ converges to $\bar{X}$ in law. We will subdivide the class of admissible distributions in a subclass w.r.t. regularity sets $R_\alpha \in \{\varnothing, \{2\}\}$ and a subclass w.r.t. general R_α. The first subclass will be further subdivided in three classes which will have different asymptotic behaviour as $n \to \infty$.

Definition 2.2.2. *Let $p_X(\cdot)$ be admissible of index α with regularity set $R_\alpha \in \{\varnothing, \{2\}\}$. Then $p_X(\cdot)$ belongs to one of the following three classes:*

(i) repaired if $R_\alpha = \emptyset$

(ii) *locally repairable if* $R_\alpha = \{2\}$ *and* $\kappa_2 > 0$

(iii) *asymptotically repairable if* $R_\alpha = \{2\}$ *and* $\kappa_2 < 0$.

A *locally repairable* probability distribution $p_X(\cdot)$ can be *repaired* by convolving it with a simple discrete random variable with variance $2|\kappa_2|$ which plays the part of a *repairer*. Analogously, we can repair an *asymptotically repairable* probability distribution $p_X(\cdot)$. This repairing is not performed on $p_X(\cdot)$ itself. Instead, we repair its asymptotic distribution $p_{\bar{X}}(\cdot)$ by convolving $\bar{X}$ with a normal random variable with variance $2|\kappa_2|$. In both cases, the aim is to change either the original random variable X or its stable limit $\bar{X}$ in order to cancel the contribution from κ_2.

Definition 2.2.3. *Let* $p_X(\cdot)$ *be admissible of index* $\alpha \in (0,2)$ *with regularity set* $R_\alpha \in \{\varnothing, \{2\}\}$ *and let* κ_2 *be the constant defined in the expansion of* $\phi_X(\cdot)$.

(i) *If* $p_X(\cdot)$ *is locally repairable, we call the repairer an independent random variable* Z *with probability distribution given by*

$$p_Z(x) = \begin{cases} \frac{\kappa_2}{M^2}, & \text{if } |x| = M \\ 1 - \frac{2\kappa_2}{M^2}, & \text{if } x = 0 \\ 0, & \text{otherwise,} \end{cases} \tag{2.2.6}$$

where $M = \lceil \sqrt{2\kappa_2} \rceil \in \mathbb{N}$.

(ii) *If* $p_X(\cdot)$ *is asymptotically repairable, we call an asymptotic repairer a random variable* $\bar{Z}$ *such that* $\bar{Z} \sim \mathcal{N}(0, 2|\kappa_2|)$. $\bar{Z}$ *and* $\bar{X}$ *are independent and* $\bar{X}$ *be an r.v. with characteristic function given by* (2.2.5).

By construction, the characteristic function of a repairer Z satisfies the expansion

$$\phi_Z(\theta) = 1 - \kappa_2|\theta|^2 + \mathcal{O}(\theta^4), \qquad \text{as } |\theta| \longrightarrow 0.$$

It is easy to see that $p_{X+Z}(\cdot) = p_X * p_Z(\cdot)$ is in fact repaired. The asymptotic repairer $\bar{Z}$ is such that the characteristic function of $\bar{X} + \bar{Z}$ equal to

$$\phi_{\bar{X}+\bar{Z}}(\theta) = e^{-\kappa_\alpha|\theta|^\alpha - \kappa_2|\theta|^2}.$$

Note that in both cases we do not change the limiting distribution of $t^{-1/\alpha}S_t$. Indeed, this modification will introduce an error of order $\mathcal{O}(t^{1-\frac{3}{\alpha}})$ which vanishes as $n \to \infty$.

Let us remark that alternatively one could *repair* by taking a convex combination as in [43]. Different repairing methods might be more convenient depending on the context.

Finally, let us define the potential kernel for a random walk, whose transition probability $p_X(\cdot) := p_X(\cdot,\cdot)$ is admissible of index $\alpha \in [1,2)$ and regularity set R_α.

2.3 Results

2.3.1 Local central limit theorem

In this section we state our results regarding LCLT's for heavy-tailed i.i.d. random variables with admissible probability distribution. First for the subclass $R_\alpha \in \{\varnothing, \{2\}\}$ and then for general R_α.

Theorem 2.3.1. *Let* $\alpha \in (0,2)$ *and* $(X_i)_{i\in\mathbb{N}}$ *be a sequence of i.i.d. random variables with admissible law* $p_X(\cdot)$ *and* $R_\alpha \in \{\varnothing, \{2\}\}$. *Let furthermore* $p_{\bar{X}}(\cdot)$ *denote the law of the symmetric* α*-stable random variable with scale parameter* $(\kappa_\alpha)^{1/\alpha}$, $p_Z(\cdot)$ *the law of the repairer and* $p_{\bar{Z}}(\cdot)$ *the law of the asymptotic repairer. Then we have that,*

(i) *if* $p_X(\cdot)$ *is repaired,*

$$\sup_{x\in\mathbb{Z}} |p_X^t(x) - p_{\bar{X}}^t(x)| \lesssim t^{-(1+\frac{1}{\alpha})}.$$

(ii) *if* $p_X(\cdot)$ *is locally repairable,*

$$\sup_{x\in\mathbb{Z}} |p^t_{X+Z}(x) - p^t_{\bar{X}}(x)| \lesssim t^{-(1+\frac{1}{\alpha})}.$$

(iii) *if* $p_X(\cdot)$ *is asymptotically repairable,*

$$\sup_{x\in\mathbb{Z}} |p^t_X(x) - p^t_{\bar{X}+\bar{Z}}(x)| \lesssim t^{-(1+\frac{1}{\alpha})}.$$

The next theorem gives LCLT convergence rates for admissible distributions w.r.t. general R_α.

Theorem 2.3.2. *Let* $\alpha \in (0,2)$ *and* $(X_i)_{i\in\mathbb{N}}$ *be a sequence of i.i.d. random variables with common admissible law* $p_X(\cdot)$. *Let furthermore* $p_{\bar{X}}(\cdot)$ *denote the law of the symmetric* α*-stable random variable with scale parameter* $(\kappa_\alpha)^{1/\alpha}$. *Then, there exists a collection of constants* $\{C_j, j\in J_\alpha\}$ *s.t. for all* $x\in\mathbb{Z}$,

$$\left| p^t_X(x) - p^t_{\bar{X}}(x) - \sum_{j\in J_\alpha} C_j \frac{u_j\left(\frac{x}{t^{1/\alpha}}\right)}{t^{(1+j-\alpha)/\alpha}} \right| \lesssim t^{-\frac{3}{\alpha}}, \tag{2.3.1}$$

where

$$u_j(x) := \frac{1}{2\pi}\int_{\mathbb{R}} |\theta|^j e^{-\kappa_\alpha|\theta|^\alpha}\cos(\theta x)d\theta. \tag{2.3.2}$$

A careful analysis of the function $u_j(x)$ shows that

$$|u_j(x)| \lesssim \frac{1}{|x|^{\alpha+j+1}}. \tag{2.3.3}$$

Indeed, this bound is significantly weaker than its equivalent Theorem 2.3.7 in [68]. There, the integrands in (2.3.2) are given by $g_j(\theta) := \kappa_j\theta^j e^{-c|\theta|^2}$, and therefore $g_j(\cdot)$ are in Schwartz functions with rapidly decaying derivatives.

A simple triangular inequality leads us to the following corollary.

Corollary 2.3.3. *Under the conditions of Theorem 2.3.2, calling* $\beta_1 := \min(J^+_\alpha)$ *and* $\beta_2 := \min(J^+_\alpha \setminus \{\beta_1\})$, *we have that*

$$\left|p^t_X(x) - p^t_{\bar{X}}(x)\right| = o\left(\sum_{j\in J_\alpha} C_j \frac{u_j\left(\frac{x}{t^{1/\alpha}}\right)}{t^{(1+j-\alpha)/\alpha}}\right).$$

In particular we have that

$$\sup_{x\in\mathbb{Z}} \left|p^t_X(x) - p^t_{\bar{X}}(x)\right| \lesssim t^{-\frac{(\beta_1+1-\alpha)}{\alpha}}$$

and

$$\left|p^t_X(x) - p^t_{\bar{X}}(x)\right| \lesssim \left(t^{-\frac{(\beta_2+1-\alpha)}{\alpha}}\right) \vee \left(t^{\frac{2}{\alpha}}|x|^{-(\alpha+\beta_1+1)}\right).$$

Note that from Corollary 2.3.3 we can deduce that the rate of convergence is sharp. More precisely we have seen that the speed is of order $\mathcal{O}(n^{-\gamma})$ where $\gamma = \frac{\beta_1+1-\alpha}{\alpha}$. If $p_X(\cdot)$ is repaired, then $\beta_1 = \min\{2\alpha, 2+\alpha\} = 2\alpha \geq 2$ which leads to $\gamma = \frac{\alpha+1}{\alpha}$. For $\alpha \geq 1$ and $p_X(\cdot)$ is locally repairable we have that $\beta_1 = \min\{2, 2\alpha, 2+\alpha\} = 2$. Without repairing, the best uniform bound we can get is

$$\left|p^n_X(x) - p^n_{\bar{X}}(x)\right| \lesssim n^{1-3/\alpha},$$

which is much weaker than the bound in Theorem 2.3.1, especially for α close to 2. Theorem 2.3.1 states that repairing a probability distribution preserves the convergence rates. Note that for $\alpha < 1$, we have that $\beta_1 < 2$ so repairing will not provide better convergence bounds beyond the once in Corollary 2.3.3.

In Section 2.7, we discuss how one could potentially repair a distribution using heavy-tailed random variables instead of random variables with finite variance.

2.3.2 Potential kernel estimates for long-range random walks

The next Theorem 2.3.4 presents potential kernel estimates for long-range random walks with admissible law $p_X(\cdot)$. It exemplifies that repairing distributions provides good potential kernel expansions. This will be proven in Section 2.6. Note that the results in this Section hold for $\alpha \in [1, 2)$. For further considerations on $\alpha < 1$, we refer to Section 2.7.

We will first treat the case $\alpha \in (1, 2)$ and $\alpha = 1$ for the subclass described by $R_\alpha \in \{\varnothing, \{2\}\}$ separately. First we give bounds for repaired distributions when $\alpha \in (1, 2)$, where we have an expansion up to some vanishing error as $|x| \to \infty$. After that we compute all terms of the expansion for locally and asymptotically repairable distributions up to order $\mathcal{O}(1)$. Finally, we present the general admissible case, in which we obtain the first and second terms of the expansion which will depend on $\delta := \min(R_\alpha)$.

Theorem 2.3.4. *Let $\alpha \in (1, 2)$ and $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with common admissible distribution $p_X(\cdot)$ of index α and regularity set $R_\alpha \in \{\varnothing, \{2\}\}$.*

(i) Assume that $p_X(\cdot)$ is repaired, then there exist constants $C_0, C_\alpha \in \mathbb{R}$ such that

$$a_X(0, x) = C_\alpha |x|^{\alpha-1} + C_0 + \mathcal{O}(|x|^{\frac{\alpha-2}{3}+})$$

as $x \to \infty$, where

$$C_\alpha = \frac{1}{\pi\kappa_\alpha} \int_0^\infty \frac{\cos(\theta) - 1}{\theta^\alpha} d\theta$$

and

$$C_0 = -\frac{\pi^{1-\alpha}}{2\pi\kappa_\alpha(\alpha - 1)} + \frac{1}{\pi} \int_0^\pi \frac{\phi_X(\theta) - (1 - \kappa_\alpha\theta^\alpha)}{\kappa_\alpha\theta^\alpha(1 - \phi_X(\theta))} d\theta.$$

(ii) Assume that $p_X(\cdot)$ is locally or asymptotically repairable. Let $m_\alpha := \lceil \frac{\alpha-1}{2-\alpha} \rceil - 1$, then there exist constants $C_0', C_1, \ldots, C_{m_\alpha+1}$ such that

$$a_X(0, x) = C_\alpha |x|^{\alpha-1} + \sum_{m=1}^{m_\alpha} C_m |x|^{(\alpha-1)-m(2-\alpha)} + C_0' \log|x| + \mathcal{O}(1)$$

as $|x| \to \infty$, where for $1 \le m \le m_\alpha + 1$

$$C_m := \frac{\kappa_2^m}{\pi\kappa_\alpha^{m+1}} \int_0^\infty \theta^{m(2-\alpha)-\alpha}(\cos(\theta) - 1) d\theta,$$

and the sum is zero if $m_\alpha = 0$. Moreover,

$$C_0' := \begin{cases} 0, & \text{if } \frac{2}{2-\alpha} \notin \mathbb{N} \\ C_{m_\alpha+1}, & \text{if } \frac{2}{2-\alpha} \in \mathbb{N}. \end{cases}$$

Note that $m_\alpha \to \infty$ as $\alpha \to 2$, therefore, performing a repair (whenever possible) becomes more relevant for larger values of α. The following theorem treats the general admissible case.

Theorem 2.3.5. *Let $\alpha \in (1, 2)$ and $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with common admissible distribution $p_X(\cdot)$ of index α and regularity set R_α. Let $\delta := \min(R_\alpha)$ and*

$$C_\alpha = \frac{1}{\pi\kappa_\alpha} \int_0^\infty \frac{\cos(\theta) - 1}{\theta^\alpha} d\theta.$$

(i) If $\delta < 2\alpha - 1$, then there exists a constant C_δ such that

$$a_X(0, x) = C_\alpha |x|^{\alpha-1} + C_\delta |x|^{2\alpha-\delta-1} + \mathcal{O}(|x|^{2\alpha-\delta-1})$$

as $|x| \to \infty$, where

$$C_\delta = \frac{\kappa_\delta}{\pi\kappa_\alpha} \int_0^\infty \theta^{\delta-2\alpha}(\cos(\theta) - 1) d\theta.$$

(ii) If $\delta > 2\alpha - 1$, then there exists a constant C_0 such that

$$a_X(0,x) = C_\alpha |x|^{\alpha-1} + C_0 + o(1).$$

as $x \to \infty$, where

$$C_0 = -\frac{\pi^{1-\alpha}}{2\pi\kappa_\alpha(\alpha-1)} + \frac{1}{\pi}\int_0^\pi \frac{\phi_X(\theta) - (1-\kappa_\alpha\theta^\alpha)}{\kappa_\alpha\theta^\alpha(1-\phi_X(\theta))} d\theta.$$

(iii) If $\delta = 2\alpha - 1$, then there exists a constant C_δ such that

$$a_X(0,x) = C_\alpha |x|^{\alpha-1} + C_\delta \log|x| + \mathcal{O}(1).$$

as $x \to \infty$, where

$$C_\delta := \frac{\kappa_\delta}{\pi\kappa_\alpha}\int_0^\pi \frac{\cos(\theta)-1}{\theta} d\theta$$

Finally, we include the result for the potential kernel for $\alpha = 1$, when $R_\alpha \in \{\varnothing, \{2\}\}$.

Theorem 2.3.6. *Let $\alpha = 1$ and $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with common admissible law $p_X(\cdot)$ and $R_\alpha \in \{\varnothing, \{2\}\}$. Then*

$$a_X(0,x) = -\frac{1}{\pi\kappa_1}\log|x| + C_0 + o\left(1\right).$$

where

$$C_0 := \frac{\gamma + \log\pi}{\pi\kappa_1},$$

and γ is the Euler-Mascheroni constant. Additionally, if $p_X(\cdot)$ is repaired, we have that the term $o(1)$ is in fact $\mathcal{O}\left(|x|^{-\frac{1}{3}+}\right)$.

2.4 Example: 1d long-range random walk

In this section we will discuss a typical example of an admissible probability distribution with index $\alpha \in (0,2)$ and regularity set $R_\alpha = \{2\}$. This will be given by p_α defined in (1.4.1). Its characteristic function is equal to

$$\phi_\alpha(\theta) = c_\alpha \sum_{x\in\mathbb{Z}\setminus\{0\}} \frac{e^{ix\theta}}{|x|^{1+\alpha}}. \tag{2.4.1}$$

Proposition 2.4.1. *Let X denote the step size of the long-range random variable with probability distribution given by $p_\alpha(\cdot)$, $\alpha \in (0,2)$. The distribution $p_\alpha(\cdot)$ is admissible of index α and locally repairable, i.e. for $\alpha \neq 1$:*

$$\phi_\alpha(\theta) = 1 - \kappa_\alpha|\theta|^\alpha + \kappa_2|\theta|^2 + \mathcal{O}(|\theta|^{2+\alpha}) \text{ as } |\theta| \to 0$$

with coefficients κ_α, κ_2 given by

$$\kappa_\alpha = -2c_\alpha \cos\left(\frac{\pi\alpha}{2}\right)\Gamma(-\alpha)$$

and

$$\kappa_2 = 2c_\alpha\left(\frac{1}{2(2-\alpha)} - \frac{1}{4} - K_2\right)$$

where

$$K_2 = \frac{1-\alpha}{2}\Bigg(\left(\frac{2^{2-\alpha}-1}{2-\alpha} - \frac{3(2^{1-\alpha}-1)}{2(1-\alpha)}\right) + \frac{1}{2\Gamma(\alpha)}\sum_{m=1}^\infty (-1)^m(\zeta(m+\alpha)-1)\frac{m\Gamma(m+\alpha)}{\Gamma(m+2)(m+2)}\Bigg),$$

with $\zeta(\cdot)$ denoting the zeta function and $\Gamma(\cdot)$ the Gamma function. In the case $\alpha = 1$ we have that

$$\phi_1(\theta) = 1 - \frac{3}{\pi}|\theta| + \frac{3}{2\pi^2}|\theta|^2 + \mathcal{O}(|\theta|^3) \text{ as } |\theta| \to 0.$$

Proof. To prove this statement, we will use the Euler-Maclaurin formula [5], which states that for a given smooth function $f \in C^1(\mathbb{R})$, we have that

$$\sum_{x=1}^{M} f(x) - \int_1^M f(x)dx = \frac{f(1)+f(M)}{2} + R_\alpha^M, \tag{2.4.2}$$

where the remainder term R_α^M can be computed explicitly by

$$R_\alpha^M = \int_1^M f'(x)P_1(x)dx,$$

and $P_1(x) = (x - \lfloor x \rfloor) - \frac{1}{2}$. We will apply this formula to the function $f(x) = \frac{1-\cos(\theta x)}{|x|^{1+\alpha}}$. Without loss of generality, we assume that $\theta > 0$. The left-hand side of (2.4.2) becomes

$$\sum_{x=1}^{M} \frac{1-\cos(\theta x)}{x^{1+\alpha}} - \int_1^M \frac{1-\cos(\theta x)}{x^{1+\alpha}}dx.$$

Notice that, as we let M go to infinity, we get that the expression above converges to

$$\frac{1-\phi_\alpha(\theta)}{2c_\alpha} - \int_1^\infty \frac{1-\cos(\theta x)}{x^{1+\alpha}}dx, \tag{2.4.3}$$

where c_α was the normalising constant used in the definition of $p_\alpha(\cdot)$. By a change of variables $z = x\theta$ in the above integral, we get

$$\frac{1-\phi_\alpha(\theta)}{2c_\alpha} - \theta^\alpha \int_\theta^\infty \frac{1-\cos(z)}{z^{1+\alpha}}dz.$$

For $\alpha \in (0,2) \setminus \{1\}$, we can write

$$\int_0^\infty \frac{1-\cos(z)}{z^{1+\alpha}}dz = -\cos\Big(\frac{\pi\alpha}{2}\Big)\Gamma(-\alpha) > 0,$$

so, by writing

$$\begin{aligned}\theta^\alpha \int_\theta^\infty \frac{1-\cos(z)}{z^{1+\alpha}}dz &= \theta^\alpha \int_0^\infty \frac{1-\cos(z)}{z^{1+\alpha}}dz - \theta^\alpha \int_0^\theta \frac{1-\cos(z)}{z^{1+\alpha}}dz \\ &= \theta^\alpha\Big(-\cos\Big(\frac{\pi\alpha}{2}\Big)\Gamma(-\alpha)\Big) - \frac{1}{2(2-\alpha)}\theta^2 + \mathcal{O}(\theta^4),\end{aligned}$$

where in the last line we used a simple Taylor expansion of $\cos(\cdot)$.

Now we turn to the right-hand side of (2.4.2). Note that $f(M) \to 0$ as $M \to \infty$. Hence,

$$\begin{aligned}\lim_{M\to\infty} \frac{f(1)+f(M)}{2} + R_\alpha^M &= \frac{1}{2}(1-\cos(\theta)) + R_\alpha^\infty \\ &= \frac{1}{4}\theta^2 + \mathcal{O}(\theta^4) + R_\alpha^\infty,\end{aligned}$$

where

$$R_\alpha^\infty = \theta^{1+\alpha} \int_\theta^\infty \Big(\frac{z\sin z - (1+\alpha)(1-\cos(z))}{z^{2+\alpha}}\Big)P_1\Big(\frac{z}{\theta}\Big)dz. \tag{2.4.4}$$

We explore this integral in more detail in Lemma A.1.1, in which we prove that

$$R_\alpha^\infty = K_2\theta^2 + \mathcal{O}(\theta^{2+\alpha}),$$

where K_2 is a constant depending on α which is defined in (A.1.2). We will first focus on $\alpha > 1$ and express κ_2 as

$$\kappa_2 = 2c_\alpha\left(\frac{1}{2(2-\alpha)} - \frac{1}{4} - K_2\right).$$

To complete the proof that $\kappa_2 > 0$, we need to examine K_2. As $\alpha > 1$, for $m \geq 1$, we have $m + \alpha > 2$ and therefore

$$\begin{aligned}\zeta(m+\alpha) - 1 &= \frac{1}{2^{m+\alpha}} + \sum_{k\geq 3} \frac{3^{m+\alpha}}{3^{m+\alpha}} \frac{1}{k^{m+\alpha}} \\ &\leq \frac{1}{2^{m+\alpha}} + \frac{1}{3^{m+\alpha}} \sum_{k\geq 3} \left(\frac{3}{k}\right)^2 \\ &\leq \frac{1}{2^{m+\alpha}} \left(1 + 9\Big(\zeta(2) - \frac{5}{4}\Big)\right) \leq \frac{5}{2^{m+\alpha}},\end{aligned}$$

where $\zeta(z)$ is the zeta-function. Moreover, using Gautschi's inequality for the ratio of two Gamma functions, see e.g. [85], we can write

$$(m+2)^{\alpha-2} < \frac{\Gamma(m+\alpha)}{\Gamma(m+2)} < (m+1)^{\alpha-2} < m^{\alpha-2}.$$

The upper bound on K_2 will follow from the lower bound on $\frac{K_2}{1-\alpha}$. We remove all even summands m in the definition of K_2 and bound further

$$\begin{aligned}\frac{2K_2}{1-\alpha} &\geq \Bigg(\left(\frac{2^{2-\alpha}-1}{2-\alpha} - \frac{3(2^{1-\alpha}-1)}{2(1-\alpha)}\right) \\ &\quad - \frac{1}{2\Gamma(\alpha)} \sum_{m=0}^{\infty} (\zeta(2m+1+\alpha)-1) \frac{(2m+1)\Gamma(2m+1+\alpha)}{\Gamma(2m+3)(2m+3)}\Bigg) \\ &\geq \left(\left(\frac{2^{2-\alpha}-1}{2-\alpha} - \frac{3(2^{1-\alpha}-1)}{2(1-\alpha)}\right) - \frac{5}{2\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{(2m+2)^{\alpha-2}}{2^{2m+1+\alpha}}\right) \\ &\geq \left(\Big(\frac{2^{2-\alpha}-1}{2-\alpha} - \frac{3(2^{1-\alpha}-1)}{2(1-\alpha)}\Big) - \frac{5}{12\Gamma(\alpha)}\right).\end{aligned}$$

Call $u : (0,2) \to \mathbb{R}$ the map

$$t \mapsto \frac{1-t}{2}\left(\Big(\frac{2^{2-t}-1}{2-t} - \frac{3(2^{1-t}-1)}{2(1-t)}\Big) - \frac{5}{12\Gamma(t)}\right) \tag{2.4.5}$$

which is increasing for $t > 1$ and simple analysis shows that $u(t)$ is bounded from above by $\frac{1}{4}$. Now we collect all previous contributions to the constant κ_2 and show that the sum above cannot flip the sign. This concludes that

$$\kappa_2 = 2c_\alpha \left(\frac{1}{2(2-\alpha)} - \frac{1}{4} - K_2\right) > \frac{(\alpha-1)c_\alpha}{2-\alpha}$$

is positive for $\alpha > 1$.

For $\alpha < 1$, the strategy is similar, only this time, we proceed to get a function $u'(\cdot)$ similar to (2.4.5) but bounding $\frac{K_2}{2(1-\alpha)}$ from below (as $1-\alpha$ is now positive).

For the case $\alpha = 1$ the analysis becomes much simpler. This is because the first order term in (A.1.1) vanishes. Since $\alpha = 1$, the terms $\theta^{1+\alpha}$ and θ^2 collapse to the same term. The normalization constant is equal to $c_1 = \frac{1}{2\zeta(2)} = \frac{3}{\pi^2}$.

Again, using Euler-Maclaurin we get that, for $\theta > 0$

$$\frac{1-\phi_1(\theta)}{2c_1} - \int_1^\infty \frac{1-\cos(\theta x)}{x^2} dx = \frac{1-\cos(\theta)}{2} + R_1^\infty, \tag{2.4.6}$$

where the remainder term will be of order

$$R_1^\infty = \int_1^\infty \left(\frac{1-\cos(\theta\cdot)}{(\cdot)^2}\right)'(x) P_p(x) dx = \mathcal{O}(\theta^3).$$

Since

$$\int_0^\infty \frac{1-\cos(z)}{z^2}dz = \frac{\pi}{2},$$

we can write

$$\theta \int_\theta^\infty \frac{1-\cos(z)}{z^2}dz = \theta \int_0^\infty \frac{1-\cos(z)}{z^2}dz - \theta \int_0^\theta \frac{1-\cos(z)}{z^2}dz$$
$$= \frac{\pi}{2}\theta - \frac{1}{2}\theta^2 + \mathcal{O}(\theta^4)$$

where in the last line we used a simple Taylor expansion. Collecting all coefficients corresponding to the powers of θ we obtain the result.

□

2.5 Proofs of Local Central Limit Theorems

In this section we will prove Theorems 2.3.1 and 2.3.2.

Proof of Theorem 2.3.1. We will prove cases (i) and (iii) since case (ii) is a corollary of case (i).
Case (i): $p_X(\cdot)$ repaired
Consider $(X_i)_{i\in\mathbb{N}}$ resp. a sequence of symmetric i.i.d. α-stable random variables $(\bar{X}_i)_{i\in\mathbb{N}}$ with scale parameter $(\kappa_\alpha)^{1/\alpha}$ and laws $p_X(\cdot)$ resp. $p_{\bar{X}}(\cdot)$. Let $S_t = \sum_{i=1}^t X_i$ resp. $\bar{S}_t = \sum_{i=1}^t \bar{X}_i$ with probability distributions denoted by $p^t_X(\cdot)$ resp. $p^t_{\bar{X}}(\cdot)$. We want to compare the probability distributions $p^t_X(\cdot)$ and $p^t_{\bar{X}}(\cdot)$ using their representation in terms of its characteristic functions. More precisely we have that

$$p^t_X(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi^t_X(\theta)e^{-ix\theta}d\theta$$

resp.

$$p^t_{\bar{X}}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t\kappa_\alpha|\theta|^\alpha}e^{-i\theta\cdot x}d\theta.$$

Using a change of variable formula, we get

$$p^t_X(x) = \frac{1}{2\pi t^{1/\alpha}}\int_{-\pi t^{1/\alpha}}^{\pi t^{1/\alpha}} \phi^t_X\left(\frac{\theta}{t^{1/\alpha}}\right)e^{-ix\frac{\theta}{t^{1/\alpha}}}d\theta.$$

Given $\varepsilon > 0$, notice that $\sup_{\theta\in\mathbb{T}\setminus[-\varepsilon,\varepsilon]}|\phi_X(\theta)| < 1$, as X is supported in $\mathbb{Z}$, see [68, Lemma 2.3.2]. To get

$$p^t_X(x) = \frac{1}{2\pi t^{1/\alpha}}\int_{-\varepsilon t^{1/\alpha}}^{\varepsilon t^{1/\alpha}} \phi^t_X\left(\frac{\theta}{t^{1/\alpha}}\right)e^{-ix\frac{\theta}{t^{1/\alpha}}}d\theta + \mathcal{O}(e^{-ct})$$

for some positive constant $c > 0$. Analogously, we have that

$$p^t_{\bar{X}}(x) = \frac{1}{2\pi t^{1/\alpha}}\int_{-\infty}^{\infty} e^{-\kappa_\alpha|\theta|^\alpha}e^{-\frac{ix\theta}{t^{1/\alpha}}}d\theta$$
$$= \frac{1}{2\pi t^{1/\alpha}}\int_{-\varepsilon t^{1/\alpha}}^{\varepsilon t^{1/\alpha}} e^{-\kappa_\alpha|\theta|^\alpha}e^{-ix\frac{\theta}{t^{1/\alpha}}}d\theta$$
$$+ \frac{1}{2\pi t^{1/\alpha}}\int_{|\theta|>\varepsilon t^{1/\alpha}} e^{-\kappa_\alpha|\theta|^\alpha}e^{-ix\frac{\theta}{t^{1/\alpha}}}d\theta.$$

One can easily check that,

$$\int_{|\theta|>\varepsilon t^{1/\alpha}} e^{-\kappa_\alpha|\theta|^\alpha}e^{-\frac{ix\theta}{t^{1/\alpha}}}d\theta = \mathcal{O}(e^{-c't}),$$

for some constant $c' > 0$. Write $\phi^t_X\left(\frac{\theta}{t^{1/\alpha}}\right) = [1 + F_t(\theta)]e^{-\kappa_\alpha|\theta|^\alpha}$.

Hence, we can concentrate our efforts into bounding

$$\int_{-\varepsilon t^{1/\alpha}}^{\varepsilon t^{1/\alpha}} F_t(\theta) e^{-\kappa_\alpha |\theta|^\alpha} e^{-\frac{ix\theta}{t^{1/\alpha}}} d\theta. \tag{2.5.1}$$

Now we write $F_t(\theta) = e^{g_t(\theta)} - 1$ which is formally equal to

$$F_t(\theta) = \mathcal{O}\left(\frac{|\theta|^{2+\alpha}}{t^{2/\alpha}}\right) + \mathcal{O}\left(\left(\frac{|\theta|^{2+\alpha}}{t^{2/\alpha}}\right)^2\right)$$

and use that for $|\theta| < \varepsilon t^{1/\alpha}$ (possibly for smaller value of ε), we have for $k \in \{1, 2\}$

$$\left(\frac{|\theta|^{2+\alpha}}{t^{2/\alpha}}\right)^k = \mathcal{O}\left(\frac{|\theta|^{2\alpha}}{t}\right).$$

. With this, we get

$$\begin{aligned}
\left|p_X^t(x) - p_{\bar{X}}^t(x)\right| &= \Big|\frac{1}{2\pi t^{1/\alpha}} \int_{|\theta|<\varepsilon t^{1/\alpha}} e^{-\frac{ix\cdot\theta}{t^{1/\alpha}}} e^{-\kappa_\alpha|\theta|^\alpha} F_t(\theta) d\theta\Big| \\
&\lesssim \frac{1}{t^{1+1/\alpha}} \underbrace{\int_{|\theta|<\varepsilon t^{1/\alpha}} e^{-\kappa_\alpha|\theta|^\alpha} |\theta|^{2\alpha} d\theta}_{\mathcal{O}(1)}
\end{aligned}$$

and that the integral on the r.h.s. is bounded as $t \longrightarrow \infty$.
Case (iii): $p_X(\cdot)$ asymptotically repairable

We will prove the statement similarly, so we will only highlight the main differences. Write

$$\begin{aligned}
p_{\bar{X}+\bar{Z}}^t(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t\kappa_\alpha|\theta|^\alpha - t\kappa_2|\theta|^2} e^{-ix\theta} d\theta \\
&= \frac{1}{2\pi t^{1/\alpha}} \int_{-\infty}^{\infty} e^{-\kappa_\alpha|\theta|^\alpha - t^{(1-2/\alpha)}\kappa_2|\theta|^2} e^{-\frac{ix\theta}{t^{1/\alpha}}} d\theta
\end{aligned}$$

and write $\phi_X^t\left(\frac{\theta}{t^{1/\alpha}}\right) = [1 + F_t(\theta)] \exp\left(-\kappa_\alpha|\theta|^\alpha - t^{(1-2/\alpha)}\kappa_2|\theta|^2\right)$. Notice that $1 - \frac{2}{\alpha} < 0$.

One can easily check that,

$$\int_{|\theta|>\varepsilon t^{1/\alpha}} e^{-\kappa_\alpha|\theta|^\alpha - t^{1-\frac{2}{\alpha}}\kappa_2|\theta|^2} e^{-ix\frac{\theta}{t^{1/\alpha}}} d\theta = \mathcal{O}(e^{-ct}),$$

for some constant $c > 0$. The statement will follow once we bound

$$\int_{-\varepsilon t^{1/\alpha}}^{\varepsilon t^{1/\alpha}} F_t(\theta) e^{-\kappa_\alpha|\theta|^\alpha - t^{1-\frac{2}{\alpha}}\kappa_2|\theta|^2} e^{-ix\frac{\theta}{t^{1/\alpha}}} d\theta \lesssim t^{-1/\alpha}.$$

Analogously to before write $F_t(\theta) = e^{g_t(\theta)} - 1$ and note that for $|\theta| \le \varepsilon t^{1/\alpha}$, we have

$$|F_t(\theta)| \lesssim \frac{|\theta|^{2\alpha}}{t}.$$

This concludes the claim.

□

We proceed with the proof of Theorem 2.3.2.

Proof of Theorem 2.3.2. Using similar ideas as before in the proof of Theorem 2.3.1, assume again that $\theta > 0$, we write

$$p_X^t(x) = \frac{1}{2\pi t^{1/\alpha}} \int_{-\varepsilon t^{1/\alpha}}^{\varepsilon t^{1/\alpha}} [1 + F_t(\theta)] e^{-\kappa_\alpha|\theta|^\alpha} e^{-ix\theta t^{-\frac{1}{\alpha}}} d\theta + \mathcal{O}(e^{-ct^{1/\alpha}})$$

for some positive constant $c > 0$. We have that

$$F_t(\theta) = \sum_{j\in J_\alpha} C_j \frac{t}{t^{j/\alpha}}|\theta|^j + \mathcal{O}\left(\frac{|\theta|^{2+\alpha}}{t^{2/\alpha}}\right), \tag{2.5.2}$$

where we used the Taylor polynomial of

$$z \mapsto e^{\sum_{\beta\in R_\alpha} t^{1-\beta/\alpha}\kappa_\beta z^\beta}$$

truncated at level $\mathcal{O}(\frac{z^{2+\alpha}}{t^{2/\alpha}})$.

Define

$$u_j(x) := \frac{1}{2\pi}\int_{\mathbb{R}} |\theta|^j e^{-\kappa_\alpha|\theta|^\alpha}\cos(\theta x)d\theta,$$

hence we have that for $|\theta| < \varepsilon t^{1/\alpha}$

$$\begin{aligned}
&\left|p^t_X(x) - p^t_{\bar{X}}(x) - \sum_{j\in J_\alpha} C_j \frac{u_j\left(\frac{x}{t^{1/\alpha}}\right)}{t^{(1+j-\alpha)/\alpha}}\right| \\
&\lesssim \int_{-\varepsilon t^{1/\alpha}}^{\varepsilon t^{1/\alpha}} \frac{|\theta|^{2+\alpha}e^{-\kappa_\alpha|\theta|^\alpha}}{t^{3/\alpha}}d\theta + \sum_{j\in J_\alpha} C_j \int_{\mathbb{R}\setminus[-\varepsilon t^{1/\alpha},\varepsilon t^{1/\alpha}]} \frac{|\theta|^j e^{-\kappa_\alpha|\theta|^\alpha}}{t^{(1+j-\alpha)/\alpha}}d\theta, \\
&\lesssim t^{-3/\alpha} + \mathcal{O}\left(e^{-ct}\right)
\end{aligned}$$

for some $c > 0$ and t large enough.

□

2.6 Proofs of the Potential Kernel expansions

In this section we will develop potential kernel estimates stated in Theorems 2.3.4 and 2.3.6. The strategy will be to use detailed knowledge of the expansion $\phi_X(\cdot)$ and not the LCLT theorem as was done for the equivalent problem in the classical case in [68].

Proof of Theorem 2.3.4. **Case (i)** $p_X(\cdot)$ **repaired**
Let us evaluate the expression

$$\begin{aligned}
a_X(0,x) &= \sum_{t=0}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi^t_X(\theta)(\cos(\theta x) - 1)d\theta. \\
&= \lim_{T\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-\phi^t_X(\theta)}{1-\phi_X(\theta)}(\cos(\theta x) - 1)d\theta. \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1-\phi_X(\theta)}(\cos(\theta x) - 1)d\theta,
\end{aligned}$$

where the last identity holds due to the Dominated Convergence Theorem.

The idea is to compare $a_X(0,x)$ with the potential kernel $a_{\bar{X}}(\cdot,\cdot)$ of a symmetric stable process $(\bar{X}_t)_{t\in\mathbb{R}^+}$ with multiplicative constant κ_α whose characteristic function is given by $\phi_{\bar{X}_t}(\theta) = e^{-\kappa_\alpha t|\theta|^\alpha}$. This is more convenient since it can be explicitly computed. Using that $(t,\theta) \mapsto e^{-\kappa_\alpha t|\theta|^\alpha}(\cos(\theta x) - 1)$ is in $L^1(\mathbb{R}_+\times\mathbb{R})$, we can use Fubini to get

$$\begin{aligned}
a_{\bar{X}}(0,x) &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^\infty e^{-t\kappa_\alpha|\theta|^\alpha}dt(\cos(\theta x) - 1)d\theta \\
&= \left(\frac{1}{2\pi\kappa_\alpha}\int_{\mathbb{R}} \frac{1}{|\theta|^\alpha}(\cos(\theta) - 1)d\theta\right)|x|^{\alpha-1}
\end{aligned}$$

which gives the expression for the constant C_α. Now, we write

$$a_X(0,x) = a_{\bar{X}}(0,x) + \underbrace{\left(a_X(0,x) - a'_{\bar{X}}(0,x)\right)}_{A} + \underbrace{\left(a_{\bar{X}}(0,x) - a'_{\bar{X}}(0,x)\right)}_{B},$$

where

$$a'_{\bar{X}}(0,x) := \frac{1}{2\pi\kappa_\alpha}\int_{-\pi}^{\pi}\frac{1}{|\theta|^\alpha}(\cos(\theta x)-1)d\theta.$$

The remainder of the proof is divided into two parts: estimating the term in A by using Hölder continuity and then the term in B by using an interplay of Fourier transform in the torus $\mathbb{T}$ and in $\mathbb{R}$ plus a trick involving dyadic partitions of the unity.

We start by analysing the term

$$\begin{aligned} a_X(0,x) - a'_{\bar{X}}(0,x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{1-\phi_X(\theta)} - \frac{1}{\kappa_\alpha|\theta|^\alpha}\right)(\cos(\theta x)-1)d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_X(\theta)}{\kappa_\alpha|\theta|^\alpha(1-\phi_X(\theta))}(\cos(\theta x)-1)d\theta. \end{aligned}$$

Remember that h_X was defined as

$$h_X(\theta) := \phi_X(\theta) - (1-\kappa_\alpha|\theta|^\alpha) = \mathcal{O}(|\theta|^{2+\alpha}).$$

since $p_X(\cdot)$ is repaired.

It is important to notice that $h_X(\theta)$ is in $C^{1,\alpha-1-}(\mathbb{T})$ due to Lemma A.2.2 and the continuity of $\theta \mapsto 1-\kappa_\alpha|\theta|^\alpha$. Denote by $\tilde{h}_X(\theta) := \frac{h_X(\theta)}{\kappa_\alpha|\theta|^\alpha(1-\phi_X(\theta))}$ which is in $L^1(\mathbb{T})$, as $(1-\phi_X(\theta)) \neq 0$ for all $\theta \in \mathbb{T}\setminus\{0\}$ again due to the fact that X is supported in $\mathbb{Z}$.

Hence, we write for A

$$a_X(0,x) - a'_{\bar{X}}(0,x) = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{h}_X(\theta)d\theta + \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{h}_X(\theta)\cos(\theta x)d\theta}_{I(x)}.$$

The first integral in the r.h.s. is finite and does not depend on x. We will show that the second integral on the r.h.s. above is of order $\mathcal{O}(|x|^{\frac{\alpha-2}{3+\varepsilon}})$.

This estimate is based on the fact that such integrals are Fourier coefficients of a function in $C^{0,\frac{2-\alpha}{3+\varepsilon}}(\mathbb{T})$ for some $\varepsilon > 0$ small enough.

We write

$$f_1(\theta) := \frac{h_X(\theta)}{|\theta|^{2\alpha}}$$

and

$$f_2(\theta) := \frac{|\theta|^\alpha\left(\kappa_\alpha|\theta|^\alpha - h_X(\theta)\right)}{|\theta|^{2\alpha}} = \kappa_\alpha - \frac{h_X(\theta)}{|\theta|^\alpha}.$$

Now, we use Lemma A.2.1 to determine the degree of Hölder continuity of $f_1(\cdot)$ and $f_2(\cdot)$. For $f_1(\cdot)$ we can choose $\beta = \alpha-1-\varepsilon$ for any $\varepsilon \in (0,\alpha-1)$, $\beta_0 = 2+\alpha$ and $\beta_1 = 2\alpha$ to obtain that $f_1(\cdot)$ is Hölder continuous with $\alpha_1 = \frac{2-\alpha}{3+\varepsilon}$ for $\alpha > 1$. For $f_2(\cdot)$, we can choose $\beta = \alpha-1-\varepsilon$, $\beta_0 = 2+\alpha$ and $\beta_1 = \alpha$ which yields to an order $\alpha_2 = \frac{2}{3+\varepsilon}$. Since $f_2(\cdot)$ is bounded away from 0 we have that the reciprocal $1/f_2(\cdot)$ is Hölder continuous of order α_2 as well. Therefore, the product $f_1(\cdot)\cdot\frac{1}{f_2(\cdot)}$ is Hölder continuous of order $\alpha_1 \wedge \alpha_2 = \alpha_1$. This implies that $I(x) = \mathcal{O}(|x|^{-\alpha_1})$, see [48, Theorem 3.3.9].

For the second part of the proof, we estimate the term $B = a_{\bar{X}}(0,x) - a'_{\bar{X}}(0,x)$. To do so, let $\varphi \in C^\infty(\mathbb{R})$ be a symmetric cutoff function such that $\varphi \equiv 1$ in $\mathbb{R}\setminus[-\pi+\eta,\pi-\eta]$ for some arbitrarily small $\eta > 0$ and such that $\varphi \equiv 0$ in $[-\pi+2\eta,\pi-2\eta]$, we now have

$$\begin{aligned} 2\pi\kappa_\alpha\left[a_{\bar{X}}(0,x) - a'_{\bar{X}}(0,x)\right] &= \int_{\mathbb{R}\setminus\mathbb{T}}\frac{1}{|\theta|^\alpha}(\cos(\theta x)-1)d\theta \\ = -\underbrace{\int_{\mathbb{R}\setminus[-\pi,\pi]}\frac{1}{|\theta|^\alpha}d\theta}_{\frac{\pi^{1-\alpha}}{\alpha-1}} &+ \underbrace{\int_{\mathbb{R}}\varphi(\theta)\frac{1}{|\theta|^\alpha}\cos(\theta x)d\theta}_{J_1(x)} \\ &+ \underbrace{\int_{\mathbb{R}}\left[1_{[|\theta|>\pi]} - \varphi(\theta)\right]\frac{1}{|\theta|^\alpha}\cos(\theta x)d\theta}_{J_2(x)}. \end{aligned}$$

The constant $-\frac{\pi^{1-\alpha}}{2\pi\kappa_\alpha(\alpha-1)}$ gives the second contribution to C_0. We write $J_1(x) = \mathcal{F}\left(\frac{\varphi(\cdot)}{|\cdot|^\alpha}\right)(x)$,

In order to analyse $J_1(x)$ we need to use a dyadic partition of the unity to show that this term decays faster than any polynomial. Let ψ_{-1}, ψ_0 be two radial functions such that $\psi_{-1} \in C_c^\infty(B_\pi(0))$ and $\psi_0 \in C_c^\infty(B_{2\pi}(0) \setminus B_\pi(0))$. It satisfies

$$1 \equiv \psi_{-1}(\theta) + \sum_{j=0}^{\infty} \underbrace{\psi_0(2^{-j}\theta)}_{=:\psi_j(\theta)}. \tag{2.6.1}$$

Such functions exist by Proposition 2.10 in [7], it is an application of Littlewood-Payley theory.

Define

$$\mu(\theta) := \frac{\varphi(\theta)}{|\theta|^\alpha}\psi_{-1}(\theta) \qquad \text{and} \qquad \nu(\theta) := \frac{\varphi(\theta)}{|\theta|^\alpha}\psi_0(\theta) \equiv \frac{1}{|\theta|^\alpha}\psi_0(\theta),$$

where, in the identity, we used that $\varphi \equiv 1$ in the $\mathrm{supp}(\psi_0)$. We have that both $\mu, \nu \in C_c^\infty(\mathbb{R})$ and therefore their Fourier transforms decay faster than any polynomial, that is, for any $N > 1$, we have that

$$\mathcal{F}(\nu)(x), \mathcal{F}(\mu)(x) = \mathcal{O}(|x|^{-N}). \tag{2.6.2}$$

The fact that we can exchange the infinite sum with the Fourier transform is a result of the dominated convergence theorem.

Multiply both sides of (2.6.1) by $\varphi(\theta)/|\theta|^\alpha$, compute $\mathcal{F}$ and use the scaling property of the Fourier transform to get

$$J_1(x) = \mathcal{F}(\mu)(x) + \sum_{j=0}^{\infty} 2^{(1-\alpha)j}\mathcal{F}(\nu)(2^j x). \tag{2.6.3}$$

By using (2.6.2) and (2.6.3), we get that $J_1(x) = \mathcal{O}(|x|^{-N})$ for all $N \geq 1$. Finally, we estimate $J_2(x)$

$$\begin{aligned} J_2(x) &= \int_{-\pi}^{\pi} \left[1_{[|\theta|>\pi]} - \varphi(\theta)\right] \frac{1}{|\theta|^\alpha} \cos(\theta x) d\theta \\ &= -\int_{-\pi}^{\pi} \varphi(\theta) \frac{1}{|\theta|^\alpha} \cos(\theta x) d\theta \end{aligned}$$

where we used that $\varphi \equiv 1$ for $|x| > \pi$. We can write $J_2(x) = \mathcal{F}_{\mathbb{T}}(g)(x)$. Notice that g is $C^{0,1}(\mathbb{T})$, and therefore $J_2(x)$ decays as $\mathcal{O}(|x|^{-1})$ which is faster than $\mathcal{O}(|x|^{\frac{\alpha-2}{3+\varepsilon}})$ because $\alpha \in (1,2)$. This concludes the proof of the second part. Note that alternatively we could have interpreted the integral $a_{\bar{X}}(\cdot,\cdot) - a'_X(\cdot,\cdot)$ as a generalized hypergeometric function and study its series expansion which is more implicit. We preferred this more explicit way as it seems more feasible to generalise to higher dimensions.

Case (ii) $p_X(\cdot)$ locally or asymptotically repairable

Here we follow a similar idea as in case (ii). Write again

$$a_X(0,x) = \left(a_X(0,x) - a'_{\bar{X}}(0,x)\right) + \left(a_{\bar{X}}(0,x) - a'_{\bar{X}}(0,x)\right) + a'_{\bar{X}}(0,x).$$

The last two terms are exactly the same as in the proof of (i). However, the first term behaves differently due the presence of $\kappa_2|\theta|^2$. We have that

$$\frac{1}{(1-\phi_X(\theta))} - \frac{1}{\kappa_\alpha|\theta|^\alpha} = \frac{h_X(\theta)}{\kappa_\alpha|\theta|^\alpha(1-\phi_X(\theta))} = \mathcal{O}\left(|\theta|^{2-2\alpha}\right) \tag{2.6.4}$$

as $|\theta| \to 0$, which blows up slower than $\mathcal{O}(|\theta|^{-\alpha})$ for any $\alpha < 2$. The main idea is to perform a telescopic sum together with expression (2.6.4) until we get a function in $L^1(\mathbb{T})$, which will require exactly m_α iterations.

Note that, in this proof we are only interested in characterising the potential kernel up to a constant order, therefore, we will not need to use information on the degree of continuity of a remainder term as in previous proofs. Instead, we will compute the first m_α terms by hand and use that the remainder is in $L^1(\mathbb{T})$, for which an application of the Riemann-Lebesgue Lemma [48, Proposition 3.3.1] will be enough.

Let

$$a_X(0,x) - a'_{\bar{X}}(0,x) = \frac{1}{2\pi\kappa_\alpha}\int_{-\pi}^{\pi} \frac{h_X(\theta)}{|\theta|^\alpha(1-\phi_X(\theta))}\left(\cos(\theta x) - 1\right) d\theta.$$

For $\alpha < 3/2$ we have that $m_\alpha = 0$ and $\tilde{h}_X(\cdot) := \frac{h_X(\cdot)}{|\cdot|^\alpha(1-\phi_X(\cdot))}$ is in $L^1(\mathbb{T})$. Indeed,

$$a_X(0,x) - a'_{\bar{X}}(0,x) = \int_{-\pi}^{\pi} \tilde{h}_X(\theta)\cos(\theta x)\, d\theta - \int_{-\pi}^{\pi} \tilde{h}_X(\theta) d\theta.$$

The second term on the r.h.s. is a constant, whereas the first vanishes as $|x| \to \infty$ as before.

For the case $\alpha \in (\frac{3}{2}, \frac{5}{3})$ the proof is analogous to the proof of (i): we compare the integral to its counterpart with $1 - \phi_X(\theta)$ substituted by $\kappa_\alpha|\theta|^\alpha$ in the denominator. Here we have:

$$\begin{aligned} a_X(0,x) - a'_{\bar{X}}(0,x) := &\underbrace{\frac{\kappa_2}{2\pi(\kappa_\alpha)^2}\int_{-\pi}^{\pi} \frac{h_X(\theta)}{|\theta|^{2\alpha}}\left(\cos(\theta x) - 1\right) d\theta}_{I(x)} \\ &+ \underbrace{\frac{1}{2\pi\kappa_\alpha}\int_{-\pi}^{\pi} \left(\frac{h_X(\theta)}{|\theta|^\alpha(1-\phi_X(\theta))} - \frac{\kappa_2 h_X(\theta)}{\kappa_\alpha|\theta|^{2\alpha})}\right)\left(\cos(\theta x) - 1\right) d\theta}_{R_0(x)}. \end{aligned}$$

The last remainder term $R_0(x)$ is of order $\mathcal{O}(1)$ as $|x| \longrightarrow \infty$ for any $\alpha < 2$, again due to the fact that we can interpret it as the Fourier transform of a $L^1(\mathbb{T})$ function.

Since we assumed $\alpha > \frac{3}{2}$, $\theta \mapsto |\theta|^{2-2\alpha}\left(\cos(\theta x) - 1\right)$ is in $L^1(\mathbb{R})$ and therefore

$$\begin{aligned} I(x) &= |x|^{2\alpha-3}\frac{\kappa_2}{2\pi(\kappa_\alpha)^2}\int_{-\pi x}^{\pi x} |\theta|^{2-2\alpha}\left(\cos(\theta) - 1\right) d\theta \\ &\quad + \frac{\kappa_2}{2\pi\kappa_\alpha}\int_{-\pi}^{\pi} \frac{h_X(\theta) - |\theta|^2}{|\theta|^{2\alpha}}\left(\cos(\theta x) - 1\right) d\theta \\ &= \underbrace{|x|^{2\alpha-3}\frac{\kappa_2}{2\pi(\kappa_\alpha)^2}\int_{-\infty}^{\infty} |\theta|^{2-2\alpha}\left(\cos(\theta) - 1\right) d\theta}_{I_1(x)} \\ &\quad \underbrace{- |x|^{2\alpha-3}\frac{\kappa_2}{2\pi(\kappa_\alpha)^2}\int_{\mathbb{R}\setminus[-\pi x,\pi x]} |\theta|^{2-2\alpha}\left(\cos(\theta) - 1\right) d\theta}_{R_{1,1}(x)} \\ &\quad \underbrace{+ \frac{\kappa_2}{2\pi\kappa_\alpha}\int_{-\pi}^{\pi} \frac{h_X(\theta) - |\theta|^2}{|\theta|^{2\alpha}}\left(\cos(\theta x) - 1\right) d\theta}_{R_{1,2}(x)}. \end{aligned}$$

Both terms $R_{1,1}, R_{1,2} = \mathcal{O}(1)$ as $|x| \longrightarrow \infty$, since

$$|x|^{2\alpha-3}\left|\int_{\mathbb{R}\setminus[-\pi x,\pi x]} |\theta|^{2-2\alpha}\left(\cos(\theta) - 1\right) d\theta\right| = \mathcal{O}(1),$$

for any $\alpha < 2$. More generally, let $\alpha \in (1,2)$ and $2/(2-\alpha) \notin \mathbb{N}$, we write

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{h_X(\theta)}{|\theta|^\alpha(1-\phi_X(\theta))} (\cos(\theta x) - 1)\, d\theta \\
&= \sum_{m=1}^{m_\alpha} \underbrace{\int_{-\pi}^{\pi} \frac{\kappa_2^m}{\kappa_\alpha^m} \frac{(h_X(\theta))^m}{|\theta|^{(m+1)\alpha}} (\cos(\theta x) - 1)\, d\theta}_{I_m(x)} \\
&\quad + \underbrace{\int_{-\pi}^{\pi} \frac{\kappa_2^{m_\alpha+1}}{\kappa_\alpha^{m_\alpha+1}} \frac{(h_X(\theta))^{m_\alpha+1}}{|\theta|^{(m_\alpha+1)\cdot\alpha}(1-\phi_X(\theta))} (\cos(\theta x) - 1)\, d\theta}_{R(x)} \\
&= \sum_{m=1}^{m_\alpha} I_m(x) + R(x).
\end{aligned} \tag{2.6.5}$$

We chose $m_\alpha = \lceil \frac{\alpha-1}{2-\alpha} \rceil - 1$ as the minimal value of m such that

$$\frac{(h_X(\theta))^{m_\alpha+1}}{(1-\phi_X(\theta))|\theta|^{m_\alpha+1}} \in L^1(\mathbb{T}).$$

Analogously as before we argue that $R(x) = \mathcal{O}(1)$ as $|x| \longrightarrow \infty$.

Finally, for $m \leq m_\alpha$ we have

$$\frac{h_X^m(\theta)}{\kappa_\alpha^m |\theta|^{m\alpha}(1-\phi_X(\theta))} = \frac{\kappa_2^m}{\kappa_\alpha^{m+1}} |\theta|^{m(2-\alpha)-\alpha} + \mathcal{O}\left(|\theta|^{m(2-\alpha)-1}\right),$$

and as $\alpha < 2$, we have that $m(2-\alpha) - 1 > -1$, using a change of variable we get

$$\begin{aligned}
I_m(x) &= \frac{\kappa_2^m}{\kappa_\alpha^m} \int_{-\pi}^{\pi} |\theta|^{m(2-\alpha)-\alpha} (\cos(\theta x) - 1)\, d\theta + \mathcal{O}(1) \\
&= |x|^{(\alpha-1)-m(2-\alpha)} \frac{\kappa_2^m}{\kappa_\alpha^m} \int_{-\infty}^{\infty} |\theta|^{m(2-\alpha)-\alpha} (\cos(\theta) - 1)\, d\theta \\
&\quad - \frac{\kappa_2^m}{\kappa_\alpha^m} \int_{\mathbb{R}\setminus[-\pi|x|,\pi|x|]} |\theta|^{m(2-\alpha)-\alpha} (\cos(\theta x) - 1)\, d\theta + \mathcal{O}(1).
\end{aligned}$$

Where the first integral in the second line is finite because $m < m_\alpha$. Again, notice that the last integral is of order $\mathcal{O}(1)$ as $|x| \longrightarrow \infty$.

Finally, if $2/(2-\alpha) \in \mathbb{N}$, we have that

$$\frac{(h_X(\theta))^{m_\alpha+1}}{(1-\phi_X(\theta))|\theta|^{m_\alpha+1}} - \frac{\kappa_2^{m_\alpha+1}}{\kappa_\alpha^{m_\alpha+2}|\theta|} \in L^1(\mathbb{T}).$$

Now, we proceed like before, but also taking into account the contribution of the integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos(x\theta) - 1}{|\theta|} d\theta = \frac{1}{\pi} \int_0^{\pi|x|} \frac{\cos(\theta) - 1}{\theta} d\theta$$

and using Lemma A.1.2. This concludes the proof. □

Proof of Theorem 2.3.6. We will only prove the repaired case, as the other case is just an adaptation of the arguments of Theorem 2.3.4 case *(ii)* together with the considerations we will present here.

Instead of comparing $a_X(\cdot,\cdot)$ and $a_{\bar{X}}(\cdot,\cdot)$ and $a'_{\bar{X}}(\cdot,\cdot)$, we will only compare $a_X(\cdot,\cdot)$ and $a'_{\bar{X}}(\cdot,\cdot)$. That is, we have

$$a_X(0,x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1-\phi_X(\theta)} (\cos(\theta x) - 1) d\theta$$

and we define

$$a'_{\bar{X}}(0,x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{\kappa_1|\theta|}(\cos(\theta x) - 1)d\theta.$$

Write now

$$a_X(0,x) := a'_{\bar{X}}(0,x) + \left(a_X(0,x) - a'_{\bar{X}}(0,x)\right).$$

To evaluate the second term, we use a very similar approach to the one in the proof of Theorem 2.3.4. Using the second part of the statement of Lemma A.2.1, we get $\theta \mapsto \frac{1}{\kappa_1|\theta|} - \frac{1}{1-\phi_X(\theta)}$ is in $C^{0,1/3-}(\mathbb{T})$.

It remains to evaluate $a'_X(0,x)$. Note that

$$a'_{\bar{X}}(0,x) = \frac{1}{\pi\kappa_1}\int_0^{\pi|x|} \frac{\cos(\theta) - 1}{\theta}d\theta.$$

Again, using Lemma A.1.2, we conclude the result. □

2.7 Final remarks and generalisations

In this section we quickly discuss possible generalisations and limitations of our results and techniques.

LCLT for higher dimensions and the asymmetric case

The notion of an admissible distribution in higher dimensions is straightforward. Let $X_i \in \mathbb{Z}^d$, we do expect that the transition probability of a long-range random walk $p_\alpha(x) = c_{d,\alpha}\|x\|^{-(d+\alpha)}$ is admissible for any norm $\|\cdot\|$ in $\mathbb{Z}^d$.

Provided that such examples exist, we can generalise our LCLT results in Theorem 2.3.1, Theorem 2.3.2 and Corollary 2.3.3 immediately to d-dimensional walks. Let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. random variables on $\mathbb{Z}^d$ with admissible law of index $\alpha \in (0,2)$. Then

$$\sup_{x\in\mathbb{Z}^d} |p^t_X(x) - p^t_{\bar{X}}(x)| \lesssim t^{-\frac{\beta_1+d-\alpha}{\alpha}}.$$

and assuming that $p_X(\cdot)$ is repaired we obtain convergence rates of order $\mathcal{O}(t^{-(d+\alpha)/\alpha})$. The notions of repairer and asymptotic repairer can be trivially generalised to d-dimensions. We believe that an appropriate shift extends the results to the asymmetric case as well.

Continuous-time random walks

All the results presented here, could be easily extended to the continuous version of a random walk with admissible law. For the continuous time random walk, at each point, the walker waits a Poisson clock of rate 1 then makes a single step according to an admissible distribution. Both LCLT and potential kernel statements and proofs are essentially the same.

Further repairers

In this chapter we only studied repairers for probability distributions $p_X(\cdot)$ which are α-admissible with a regularity set $R_\alpha = \{2\}$. However, suppose that $p_X(\cdot)$ is an admissible distribution, let $\delta := \min(R_\alpha)$ and $\kappa_\delta > 0$, so we are in the locally repairable case. We could define a repairer Z as an admissible distribution $p_Z(\cdot)$ with index δ such that $\kappa_\delta = -\kappa'_\delta$, i.e. the coefficient corresponding to $|\theta|^\delta$ in the expansion of the characteristic function of X is equal to the negative value of the coefficient κ'_δ multiplying $|\theta|^\delta$ in the expansion of the characteristic function of Z. Then, $\min(R'_\alpha) > \delta$, where R'_α is the regularity set of $X + Z$. Hence, repairing would allow us to obtain precise estimates on its potential kernel beyond the constant order of the error. A similar idea could be used to improve the rates of convergence in the LCLT for distributions such that $\min(R_\alpha) < 2\alpha$, by performing multiple repairs to cancel each of the terms in $r_X(\theta)$.

Non-lattice walks/ Random variables in $\mathbb{R}$

We believe that a combination of the ideas of the present chapter and [96] would be enough to prove our results in the context of non-lattice walks and absolutely continuous random variables, possibly depending on a further integrability assumption over the characteristic function. However, we cannot say the same about potential kernel estimates. Here we are relying on the fact that smoothness implies decay of the Fourier coefficients on the torus. This relation is not simple in the full $\mathbb{R}^d$.

Improvement of the error bounds in the Potential Kernel

Notice that the decay of the error term in the potential kernel remainder is equivalent to the degree of continuity of the function $\tilde{h}_X(\cdot)$ (defined in the proof of Theorem 2.3.4). This function contains the contribution of the regularity set and error term. In general, $h_X \in C^{1,\alpha-1-}(\mathbb{T})$ but we do not necessarily have $h_X \in C^{1,\alpha-1}(\mathbb{T})$. Under these assumptions, it falls upon the sharpness of Lemma A.2.1 (which we believe is close to optimal) to decide the maximum degree of continuity of $\tilde{h}_X(\cdot)$. There are two ways that one could try to obtain better bounds for a specific distribution. The first is by showing $\tilde{h}_X(\cdot)$ has a higher degree of continuity by hand for the specific examples. The second is by expanding the characteristic function up to a smaller error, which is computationally demanding.

Potential kernel estimates in higher dimensions and/or $\alpha < 1$, asymmetric case

Unfortunately, our results do not generalise for Green function estimates for $d \geq 2$ and $\alpha \in (0, 2)$ immediately without further assumptions on the degree of continuity of the remainder of the function $\tilde{h}_X(\cdot)$. We would need to guarantee that the remainder would decay faster than $\|x\|^{\alpha-d}$, which is the first order term.

The same argument applies to $\alpha < 1$ and $d = 1$, the degree of continuity of $\phi_X(\cdot)$ becomes too low to guarantee that its Fourier transform will decay faster than $|x|^{\alpha-1}$.

One could try to expand ideas from the proof of Theorem 1.4 in [43] to tackle the $d \geq 2$ and/or $\alpha < 1$ case. There the authors demonstrate a detailed expansion for the Green's function in $d = 2$, $\alpha \in (0, 2)$ for a truncated long-range random walk. Regarding adding asymmetry in the random walk, we expect that the potential kernel in this case can be written in terms of its continuous counterpart and an error term of order $\mathcal{O}(|x|^{-\alpha/3})$ for the repaired case and $\alpha > 1$.

Chapter 3

Odometer of long-range sandpiles

3.1 Introduction

The divisible sandpile model is the continuous fixed energy counterpart of the Abelian sandpile model, which was introduced in [8] as a discrete toy model displaying self-organised criticality. Self-organised critical models are characterised by a power-law behaviour of certain quantities such as two-point correlation functions without fine-tuning any external parameter. The divisible sandpile model was introduced in [71]. It gives insight into the behaviour of internal diffusion limited aggregation growth models on $\mathbb{Z}^d$ due to its similarity.

Consider a finite graph G (e.g. a discrete torus $(\mathbb{Z}/n\mathbb{Z})^d$) and initially assign randomly to each vertex a real number drawn from a given distribution. This real number plays the role of a *mass* in case the number is positive and a *hole* otherwise. At each time step, topple all vertices with mass strictly larger than 1 by keeping mass 1 and redistributing the excess to its neighbours. Two different redistribution types can be considered: either redistribution of mass happens to nearest neighbours (we will call the associated model nearest neighbour divisible sandpile) or to all neighbours according to their relative distance to the unstable vertex and depending on a parameter $\alpha > 0$ (long-range divisible sandpile). Under certain conditions (described in [70]), the sandpile configuration will stabilise, meaning that all heights will be equal to 1.

If we depict now the total amount of mass emitted from each vertex of the graph upon stabilisation (odometer), we can interpret the odometer function as a *random interface model* on the discrete graph G. Examples of interfaces in nature are hypersurfaces separating ice and water at 0^o C. A survey about random interface models can be found in [44] and about scaling limits of odometers of divisible sandpiles on the torus in [90].

For the nearest neighbour divisible sandpile, we get the following central limit type of behaviour. If the initial configuration satisfies a second moment and a certain independence condition, then the rescaled odometer converges to a bi-Laplacian field in some appropriate Sobolev space, see Theorem 2 in [33].

In this chapter, we study the divisible sandpile model, which is redistributing its excess mass to all the vertices of the d-dimensional torus upon each toppling. The amount of mass emitted from x and received by y depends on the distance $||x-y||^{-\alpha}$ (where $\|\cdot\|$ denotes the Euclidean norm) and is tuned by some parameter α, for $\alpha \in (0,\infty)$. A related problem was studied in [43] where the authors consider a divisible sandpile model on $\mathbb{Z}^d$ with a deterministic initial configuration, supported on a finite domain and redistributing the excess mass according to a truncated long-range random walk. They study the scaling limit of the odometer by exploring the connection of the limiting distribution to an obstacle problem for a truncated fractional Laplacian This connection was established for the nearest neighbour divisible sandpile model in Lemma 2.2 in [72].

The main results and novelty of the chapter include determining upper and lower bounds for the expected odometer on the discrete torus for an initial Gaussian configuration for all $\alpha \neq 2$, which is stated in Theorem 3.3.3, the scaling limit of the odometer function to a fractional Gaussian field $\mathrm{fGF}_\gamma(\mathbb{T}^d)$, $\gamma = \min\{\alpha, 2\}$ and $\alpha \in (0,\infty)$ on the continuous torus $\mathbb{T}^d$ in an appropriate Sobolev space depending on α in Theorem 3.3.4 and explicit asymptotics for the eigenvalues of discrete fractional Laplacians in the Lemmata 3.4.4,3.4.5 and 3.4.6 for all $\alpha > 0$. Note that the expected odometer is equal to the expected maximum of the discrete (massive) fractional Gaussian field on the discrete torus, when the initial configuration is Gaussian.

The structure of the proof of Theorem 3.3.3 is similar to the proof of Theorem 1.2 in [70] and for the scaling limit in Theorem 3.3.4 we rely on Theorem 2 in [33] proven for the nearest neighbour case. The crucial part of the proofs involves a careful analysis of the eigenvalues of the discrete fractional Laplacian for the different values of α.

In [31] the authors constructed fractional Gaussian fields $\mathrm{fGF}_\gamma(\mathbb{T}^d)$ with $\gamma \geq 2$ for correlated initial Gaussian configurations and nearest neighbour redistribution. Note that starting initially with correlated Gaussians can only produce fields which are in some sense *smoother* than the bi-Laplacian ($\gamma = 2$) and never of Gaussian Free Field (GFF) type ($\gamma = 1$) which is included in our results. The GFF is a very well known interface model which plays a crucial role in random field theory, lattice statistical physics, stochastic partial differential equations and quantum gravity theory in dimension $d = 2$.

Let us stress that for all $\alpha \geq 2$ our limiting field will be the bi-Laplacian field, also known as the membrane model, which is an important variation of the GFF. This model is becoming more studied over the past few years from a mathematical perspective, due to its own interest [18, 30] and its connections with uniform spanning trees [69, 97].

Let us give some heuristics for the change in behaviour according to α. For $\alpha \in (0, 2)$, the long-range random walk on the torus has a mixing time of order at most $n^\alpha \log(n)$ versus the usual $n^2 \log(n)$ of the simple random walk. The mixing time of the random walk is increasing in α, we expect the same to hold for the speed with which the sandpile configuration converges to its stable configuration. In this case, choosing small α implies that the sandpile configuration is close to stability after fewer toppling steps, hence the short-term behaviour of the dynamics dominates the odometer. Intuitively, each vertex x emits less mass upon stabilisation and its final odometer becomes less dependent on the odometer of vertices far away from x. As α increases, the long-time behaviour of the dynamics becomes more relevant for the odometer at each point x, smoothing the effects of the initial condition since toppling happens to close neighbours of x. For $\alpha > 2$, the central limit theorem guarantees that the long-term behaviour of the random walk (and therefore the sandpile dynamics) will behave similarly to the simple random walk. In other words, as the long-range random walk mixes faster, the odometer field becomes less regular and has a larger expectation.

Structure of the chapter

Section 3.2 provides all necessary definitions and notations. In particular, we define the long-range divisible sandpile model, abstract Wiener spaces and introduce notations for the Fourier analysis on the torus. The subsequent Section 3.3 contains our results regarding bounds for the expected odometer (expected maximum of the discrete fractional Gaussian field) and the scaling limit, including a few comments about generalisations. In Section 3.4 contains all the proofs, in particular asymptotics for the eigenvalues of the discrete fractional Laplacian. Finally, in Section 3.5, we go over possible generalisations the results in this chapter.

3.2 Notation and definitions

In this chapter we will use the following notation, given a domain $D \subset \mathbb{R}^d$ and a function $f \in L^\infty(D)$, we will use $\|f\|_D$ to denote $\|f\|_{L^\infty(D)}$.

The dynamics

Definition 3.2.1. *A divisible sandpile configuration s is a function $s : \mathbb{T}_n^d \to \mathbb{R}$.*

For $x \in \mathbb{T}_n^d$, if $s(x) \geq 0$, one should think of $s(x)$ as the quantity of mass at the site x. If $s(x) < 0$, it can be interpreted as the size of a hole in x. If a site x has mass $s(x) > 1$, we call it *unstable* and otherwise *stable*. We then evolve the sandpile according to the following dynamics: unstable vertices will topple by keeping mass 1 and distributing the excess over the other vertices proportionally according to the transition probabilities p_α^n at each discrete time step. Note that unstable sites in long-range divisible sandpile models distribute mass to *all* vertices (including itself) at every time step, contrary to nearest neighbour divisible sandpile models which distribute mass only to their nearest neighbours. One could generate a divisible sandpile on a graph from any random walk defined on it, where at each time step the mass is distributed proportional to the transition probabilities. We will elaborate more on this possibility in Section 3.5.

Let $s_t = (s_t(x))_{x\in\mathbb{T}_n^d}$ denote the sandpile configuration after $t \in \mathbb{N}$ discrete time steps (set $s_0 := s$ the initial configuration). The parallel toppling procedure is given by the following algorithm.

Algorithm 3.2.2 (Long-range divisible sandpile)**.** *Set $t = 1$, then run the following loop:*

1. *if* $\max_{x\in\mathbb{T}_n^d} s_{t-1}(x) \le 1$*, stop the algorithm;*
2. *for all* $x \in \mathbb{T}_n^d$*, set* $\mathrm{ex}_{t-1}(x) := (s_{t-1}(x) - 1)^+$ *to be the excess at the site x at time t;*
3. *set* $s_t(x) := s_{t-1}(x) - (-\Delta)_n^{\alpha/2}\,\mathrm{ex}_{t-1}(x)$*;*
4. *increase the value of t by 1 and go back to step 1.*

Definition 3.2.3. *For each $t \ge 0$ the odometer function is a map $u_t^\alpha : \mathbb{T}_n^d \to [0,\infty)$ defined as*

$$u_t^\alpha(x) := \sum_{i=0}^{t-1} \mathrm{ex}_i(x)$$

for all $x \in \mathbb{T}_n^d$. Using the fact that for each $x \in \mathbb{T}_n^d$, $u_t^\alpha(x)$ is non-decreasing in t, the limit $\lim_{t\to\infty} u_t^\alpha(x)$ is well-defined in $\mathbb{R}\cup\{\infty\}$, for all $x \in \mathbb{T}_n^d$. We will denote such a limit by $u_\infty^\alpha(x)$.

Analogously to Section 2 of [70], we have for every $x \in \mathbb{T}_n^d$ and $t > 0$:

$$s_t(x) = s_0(x) - (-\Delta)_n^{\alpha/2} u_t^\alpha(x). \tag{3.2.1}$$

From [70] we have the following dichotomy: either for all $x \in \mathbb{T}_n^d$ we have *stabilisation*, i.e. $u_\infty^\alpha(x) < \infty$ or *explosion*, i.e. for all $x \in \mathbb{T}_n^d : u_\infty^\alpha(x) = \infty$. We will see in Lemma 3.3.1 that given an initial configuration s_0, satisfying $\sum_{x\in\mathbb{T}_n^d} s_0(x) = n^d$, we have $u_\infty^\alpha(x) < \infty$ for all $x \in \mathbb{T}_n^d$ and $s_\infty \equiv 1$.

It is important to notice that the long-range divisible sandpile can be studied in terms of other toppling procedures as well, see Definition 2.1 in [70]. Moreover, the abelian property and least action principle, see Proposition 2.5 in [70], can be proved using essentially the same techniques used for the nearest neighbour divisible sandpile.

Define the initial configuration s_0 for $x \in \mathbb{T}_n^d$ by

$$s_0(x) := 1 + \sigma(x) - \frac{1}{n^d}\sum_{y\in\mathbb{T}_n^d} \sigma(y),$$

where $(\sigma(x))_{x\in\mathbb{T}_n^d}$ is a collection of i.i.d random variables with $\mathbb{E}[\sigma(x)] = 0$ and $\mathrm{var}[\sigma(x)] = 1$. s_0 chosen in this way guarantees that $\sum_{x\in\mathbb{T}_n^d} s_0(x) = n^d$. We will show in Proposition 3.3.2 the following equality in law

$$(u_\infty^\alpha(x))_{x\in\mathbb{T}_n^d} \overset{d}{=} \Big(\eta^\alpha(x) - \min_{z\in\mathbb{T}_n^d}(\eta^\alpha(z))\Big)_{x\in\mathbb{T}_n^d}, \tag{3.2.2}$$

where $(\eta^\alpha(x))_{x\in\mathbb{T}_n^d}$ are defined by

$$\eta^\alpha(x) := \sum_{y\in\mathbb{T}_n^d} g_\alpha(x,y)(s_0(y) - 1),$$

and g_α was defined in (1.5.5).

Note that the distribution of η^α is invariant by translations. Moreover, it has mean 0 and covariance given by

$$\mathbb{E}[\eta^\alpha(x)\eta^\alpha(y)] = \sum_{w\in\mathbb{T}_n^d} g_\alpha(x,w)g_\alpha(y,w). \tag{3.2.3}$$

We can see easily that the covariance solves the equation

$$\Big(-(-\Delta)_n^{\alpha/2}\Big)^2\Big[\mathbb{E}[\eta^\alpha(x)\eta^\alpha(y)]\Big] = \delta_x(y) - \frac{1}{n^d}.$$

Remark that when $(\sigma(x))_{x\in\mathbb{Z}_n^d}$ are i.i.d. Gaussians, $(\eta^\alpha(x))_{x\in\mathbb{Z}_n^d}$ can be interpreted as a massive discrete fractional Gaussian field on $\mathbb{Z}_n^d$.

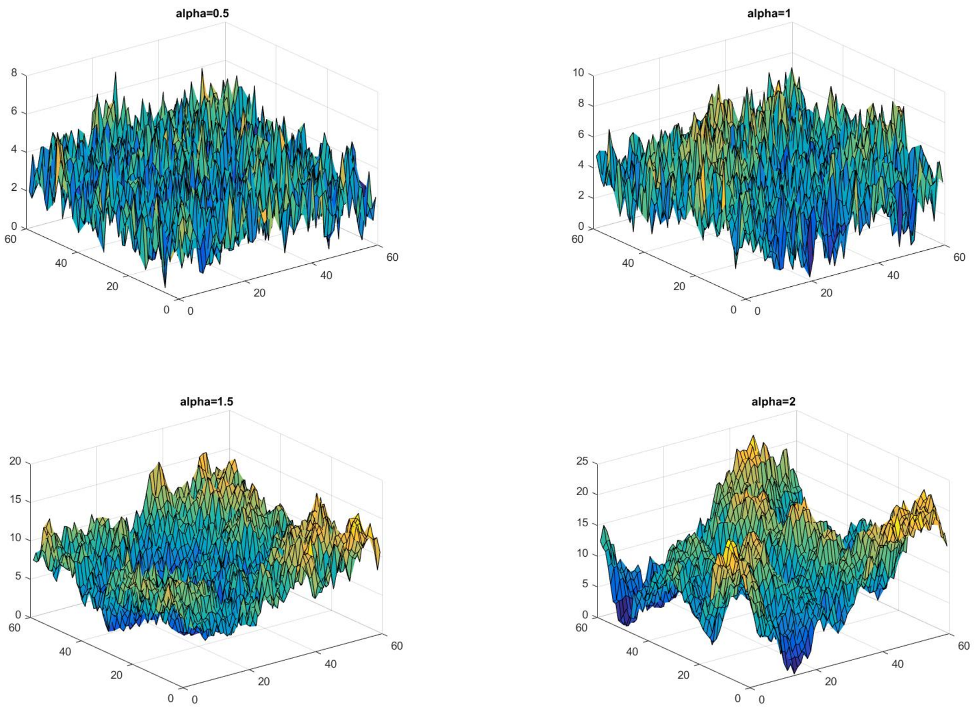

Fig. 3.1: Simulations of the odometer for different values of $\alpha \in [0.5, 2]$ in the discrete torus of length 60 and standard Gaussian initial random variables.

Remark 3.2.4. *One might feel tempted to write* $\Big((-\Delta)_n^{\alpha/2}\Big)^2 = (-\Delta)_n^\alpha$*, however this is not correct in the discrete case. Such property is valid in the continuous case because fractional Laplacians are fractional powers of each other. It fails in the discrete case as* $\mathbb{Z}^d$ *is not invariant by arbitrary rotations. The easiest way of seeing that, is to study the eigenvalues of* $(-\Delta)_n^{\alpha/2}$*. In case the property was valid, there should be a constant* $c = c(\alpha, d, n)$ *such that* $(\lambda_k^{\alpha,n})^2 = c\lambda_k^{2\alpha,n}$ *which is not true. For more discussion on the fractional powers of the discrete Laplacian, we refer to [29]. However, for* $\alpha, \beta \in (0, 2)$ *such that* $\alpha + \beta < 2$*, we have*

$$n^{\alpha+\beta}c_\alpha^{-1}c_\beta^{-1}(-\Delta)_n^{\beta/2}(-\Delta)_n^{\alpha/2}f(nx) \longrightarrow C_{d,\alpha+\beta}^{-1}(-\Delta)^{(\alpha+\beta)/2}f(x),$$

as $n \to \infty$*, so the powers of the fractional Laplacians are additive in the limit.*

Fourier analysis of the discrete Green function

In this section, we want to use the discrete Fourier transform (1.3.8) to describe the Green function g_α. Remember $(e_k)_{k\in\mathbb{Z}^d}$, the Fourier basis given in (1.3.5), notice that e_k^n (defined above

(1.3.8)) is the basis of eigenfunctions of $-(-\Delta)_n^{\alpha/2}$. Indeed, notice that

$$\begin{aligned}-(-\Delta)_n^{\alpha/2} e_k^n(x) &= \sum_{y\in\mathbb{T}_n^d} p_\alpha^n(x,y)(e_k^n(y)-e_k^n(x)) \\ &= e_k^n(x)\sum_{y\in\mathbb{T}_n^d} p_\alpha^n(x,y)\Big(e_k^n(y-x)-e_k^n(0)\Big) \\ &= e_k^n(x)\sum_{z\in\mathbb{T}_n^d} p_\alpha^n(0,z)\Big(e_k^n(z)-e_k^n(0)\Big) \\ &=: \lambda_k^{\alpha,n} e_k^n(x),\end{aligned}$$

where

$$\lambda_k^{\alpha,n} := -\frac{(2\pi)^{d+\alpha} c_\alpha}{n^{d+\alpha}} \sum_{z\in\frac{2\pi}{n}(\mathbb{Z}^d\setminus\{0\})} \frac{\sin^2\left(\frac{k\cdot z}{2}\right)}{\|z\|^{d+\alpha}}. \tag{3.2.4}$$

For a fixed $x\in\mathbb{T}_n^d$, denote by $g_{\alpha,x}:\mathbb{T}_n^d\longrightarrow\mathbb{R}$ the function $y\mapsto g_\alpha(x,y)$. Now, using that $-(-\Delta)^{\alpha/2}$ is a self-adjoint operator and that $\langle e_k^n, e_0^n\rangle = 0$, we get that for a fixed $x\in\mathbb{T}_n^d$ and $k\mathbb{Z}_n^d\setminus\{0\}$

$$\begin{aligned}\lambda_k^{\alpha,n}\widehat{g}_{\alpha,x}(k) &= \lambda_k^{\alpha,n}\langle g_{\alpha,x}, e_k^n\rangle \\ &= \langle g_{\alpha,x}, -(-\Delta)_n^{\alpha/2} e_k^n\rangle = \langle -(-\Delta)_n^{\alpha/2} g_{\alpha,x}, e_w^n\rangle \\ &= -\langle \delta_x, e_w^n(\cdot)\rangle = -\frac{1}{n^d} e_{-k}^n(x).\end{aligned} \tag{3.2.5}$$

Abstract Wiener Spaces and continuum fractional Laplacians

We need to define an abstract Wiener space (AWS) appropriately since the scaling limit will be a random distribution. Let us remark that we have to construct a different AWS than in [33], since we are dealing with general fractional Gaussian fields. Our presentation is based on Section 2 in [92] and Sections 6.1, 6.2 in [93].

An *abstract Wiener space* (AWS) is a triple (H, B, μ), where:

1. $(H, (\cdot,\cdot)_H)$ is a Hilbert space;
2. $(B, \|\cdot\|_B)$ is the Banach space completion of H with respect to the measurable norm $\|\cdot\|_B$ on H, equipped with the Borel σ-algebra $\mathcal{B}$ induced by $\|\cdot\|_B$; and
3. μ is the unique Borel probability measure on $(B,\mathcal{B})$ such that, if B^* denotes the dual space of B, then $\mu\circ\phi^{-1}\sim\mathcal{N}(0,\|\widetilde{\phi}\|_H^2)$ for all $\phi\in B^*$, where $\widetilde{\phi}$ is the unique element of H such that $\phi(h) = (\widetilde{\phi}, h)_H$ for all $h\in H$.

Note that, in order to construct a measurable norm $\|\cdot\|_B$ on H, it suffices to find a Hilbert-Schmidt operator T on H, and set $\|\cdot\|_B := \|T\cdot\|_H$.

Let us present the class of AWS which we will study and which is connected to the fractional powers of the Laplacian. Consider again $(e_k)_{k\in\mathbb{Z}^d}$ as the Fourier basis of $L^2(\mathbb{T}^d)$, we have $(e_k)_{k\in\mathbb{Z}^d}$ is a basis of eigenvectors of $-(-\Delta)^{\alpha/2}$, satisfying

$$(-\Delta)^{\alpha/2} e_k = \|k\|^\alpha e_k.$$

Also notice that

$$(-\Delta) e_k = \|k\|^2 e_k,$$

for the usual Laplacian. Hence, we can extend the definition (1.3.1) of the fractional Laplacian to $L^2(\mathbb{T}^d)$-functions in a very natural way, which also supports any power $a\in\mathbb{R}$ of $(-\Delta)$. Let $f\in L^2(\mathbb{T}^d)$ with Fourier expansion

$$f = \frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d}\widehat{f}(k) e_k,$$

and $a \in \mathbb{R}$. We define the operator $(-\Delta)^a$ as

$$(-\Delta)^a f(\cdot) = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \|k\|^{2a} \widehat{f}(k) e_k(\cdot), \tag{3.2.6}$$

whenever this sum is well-defined.

For all $a \in \mathbb{R}$, $(-\Delta)^a(f) = 0$ for all constant functions, hence we can study the operator $(-\Delta)^a$ acting only on functions $f \in L^2(\mathbb{T}^d)$ such that $\int_{\mathbb{T}^d} f(z)dz = 0$. With this in mind, let "$\sim$" be the equivalence relation on $C^\infty(\mathbb{T}^d)$ which identifies two functions differing by a constant. Let $H^a = H^a(\mathbb{T}^d)$ be the Hilbert space completion of $C^\infty(\mathbb{T}^d)/\sim$ under the norm

$$(f,\, g)_a := \frac{1}{(2\pi)^{2d}} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \|k\|^{4a} \widehat{f}(k) \overline{\widehat{g}(k)}. \tag{3.2.7}$$

Define the Hilbert space

$$\mathcal{H}_a := \Big\{ u \in L^2(\mathbb{T}^d) : \, (-\Delta)^a u \in L^2(\mathbb{T}^d) \Big\} / \sim$$

equipped with the norm

$$\|f\|^2_{\mathcal{H}_a(\mathbb{T}^d)} = \Big\langle (-\Delta)^a f,\, (-\Delta)^a f \Big\rangle. \tag{3.2.8}$$

In fact, $(-\Delta)^{-a}$ provides a Hilbert space isomorphism between $\mathcal{H}_a$ and H^a, which we identify when needed. For

$$b < a - \frac{d}{4} \tag{3.2.9}$$

one shows that $(-\Delta)^{b-a}$ is a Hilbert-Schmidt operator on H^a (cf. also [93, Proposition 5]). In our case, we will be setting $a := -\frac{\gamma}{2}$, where $\gamma := \min\{\alpha, 2\}$. Therefore, by (3.2.9), for any $-\varepsilon := b < 0$ which satisfies $\varepsilon > \frac{\gamma}{2} + \frac{d}{4}$, we have that $(H^{-\frac{\gamma}{2}}, \mathcal{H}_{-\varepsilon}, \mu_{-\varepsilon})$ is an AWS. The measure $\mu_{-\varepsilon}$ is the unique Gaussian law on $\mathcal{H}_{-\varepsilon}$ whose characteristic functional is equal to

$$\Phi(f) := \exp\left(-\frac{\|f\|^2_{-\frac{\gamma}{2}}}{2} \right).$$

The norm $\|\cdot\|_{-\gamma/2}$ is defined in (3.2.7) taking $a = -\gamma/2$. The field associated to Φ is called (continuous) fractional Gaussian Field with parameter γ, and it will be denoted by either Ξ^γ or $\mathrm{fGF}_\gamma(\mathbb{T}^d)$. It corresponds to the limiting field appearing in Theorem 3.3.4.

3.3 Results

3.3.1 Stabilisation and law of the odometer on $\mathbb{T}^d_n$

The following lemma is a simple result concerning stabilisation of a divisible sandpile model. The proof is analogous to the counterpart in the nearest neighbours case, which can be found in Lemma 7.1 in [70] and will be left for the reader. We consider $\alpha \in (0, \infty)$ and the toppling defined according to Algorithm 3.2.2.

Lemma 3.3.1. *Let $s_0 : \mathbb{T}^d_n \longrightarrow \mathbb{R}$ be any initial configuration satisfying $\sum_{x \in \mathbb{T}^d_n} s_0(x) = n^d$. Then s_0 stabilises to the all 1 configuration and its odometer u^α_∞ is the unique function satisfying*

$$s_0(x) - (-\Delta)^{\alpha/2}_n u^\alpha_\infty(x) = 1,$$

for all $x \in \mathbb{T}^d_n$ and $\min_{x \in \mathbb{T}^d_n} u^\alpha_\infty(x) = 0$.

Applying the above result, in an analogous manner as in Proposition 1.3 in [70], we get the following result.

Proposition 3.3.2. *Let* $(\sigma(x))_{x\in\mathbb{T}_n^d}$ *be i.i.d such that* $\mathbb{E}[\sigma(x)] = 0$ *and* $\mathrm{var}[\sigma(x)] = 1$*. Consider the long-range divisible sandpile with initial condition*

$$s_0(x) = 1 + \sigma(x) - \frac{1}{n^d} \sum_{y\in\mathbb{T}_n^d} \sigma(y).$$

Then s *stabilises to the all* 1 *configuration and the distribution of the odometer* $u_\infty^\alpha : \mathbb{T}_n^d \to \mathbb{R}$ *satisfies*

$$(u_\infty^\alpha(x))_{x\in\mathbb{T}_n^d} \stackrel{d}{=} \left(\eta^\alpha(x) - \min_{z\in\mathbb{T}_n^d} \eta^\alpha(z)\right)_{x\in\mathbb{T}_n^d},$$

where $\stackrel{d}{=}$ *means that the field share the same distribution with* η^α*, given by*

$$\begin{aligned}\eta^\alpha(x) &= \sum_{z\in\mathbb{T}_n^d} g_\alpha(x,z)\,(s(z)-1) \\ &= \sum_{z\in\mathbb{T}_n^d} g_\alpha(x,z) \left(\sigma(z) - \frac{1}{n^d}\sum_{y\in\mathbb{T}_n^d}\sigma(y)\right)\end{aligned}$$

with g_α *defined as in* (1.5.5) *and* $x \in \mathbb{T}_n^d$*. In particular,*

$$\mathbb{E}[\eta^\alpha(x)\eta^\alpha(y)] = \sum_{z\in\mathbb{T}_n^d} g_\alpha(x,z) g_\alpha(z,y).$$

3.3.2 The expected odometer on the finite torus

In this section we ask how the behaviour of the odometer is affected by the introduction of the long-range distribution on the finite grid $\mathbb{T}_n^d$ when $(\sigma(x))_{x\in\mathbb{T}_n^d}$ are i.i.d. standard Gaussians. We will prove here the equivalent version of Theorem 1.2 from [70].

Theorem 3.3.3. *Let* $\alpha \in \mathbb{R}_+\backslash\{2\}$*,* $d \geq 1$ *and* $(\sigma(x))_{x\in\mathbb{T}_n^d}$ *i.i.d standard normal random variables. Furthermore, let* s_0 *be the initial sandpile configuration given by*

$$s_0(x) = 1 + \sigma(x) - \frac{1}{n^d}\sum_{y\in\mathbb{T}_n^d} \sigma(y),\ x \in \mathbb{T}_n^d$$

and the redistribution rule defined by Algorithm 3.2.2. Then s *stabilises to the all* 1 *configuration and there exists a positive constant* $C = C(d,\alpha) > 0$*, such that the final odometer* u_∞^α *satisfies for all* $x \in \mathbb{T}_n^d$

$$\mathbb{E}[u_\infty^\alpha(x)] \asymp_{d,\alpha} \Phi_{d,\gamma}(n),$$

where $\gamma := \min\{\alpha, 2\}$ *and* $\Phi_{d,\gamma}$ *is given by*

$$\Phi_{d,\gamma}(n) := \begin{cases} n^{\gamma-\frac{d}{2}}, & \text{if } \gamma > \frac{d}{2} \\ \log n, & \text{if } \gamma = \frac{d}{2} \\ (\log n)^{\frac{1}{2}}, & \text{if } \gamma < \frac{d}{2}. \end{cases} \tag{3.3.1}$$

Let us make two remarks about this result. First, note that for $\alpha > 2$, comparing the result above with its counterpart Theorem 1.2 in [70], the asymptotic behaviour of the expected odometer is the same as for the nearest-neighbours divisible sandpile model. Secondly, for $\alpha = 2$ we expect that $\mathbb{E}[u_\infty^\alpha(x)]$ behaves like $\Phi_{d,2}(n)$ times some log factors that might depend on the dimension.

3.3.3 Scaling limit of the odometer

Theorem 3.3.4. *Let* $\alpha \in \mathbb{R}_+$*,* $d \geq 1$*, assume* $(\sigma(x))_{x\in\mathbb{T}_n^d}$ *is a collection of i.i.d. random variables with* $\mathrm{var}[\sigma(x)] = 1$ *for all* $x \in \mathbb{T}_n^d$*. Consider the long-range divisible sandpile in* $\mathbb{T}_n^d$ *with initial configuration*

$$s_0(x) = 1 + \sigma(x) - \frac{1}{n^d}\sum_{y\in\mathbb{T}_n^d}\sigma(y)$$

and redistribution defined by Algorithm 3.2.2. Define the field on $\mathbb{T}^d$ *by*

$$\Xi_n^\alpha(x) := \frac{a_\alpha(n)}{\tilde{c}^{(\alpha)}} \sum_{z\in\mathbb{T}_n^d} u_\infty^\alpha(z) \mathbb{1}_{B(z,\frac{\pi}{n})}(x), \; x \in \mathbb{T}^d, \tag{3.3.2}$$

where

$$\tilde{c}^{(\alpha)} := \begin{cases} \lim_{n\to\infty} \frac{-n^\alpha \lambda_w^{(\alpha,n)}}{\|w\|^\alpha} > 0, & \text{if } \alpha < 2; \\ \frac{c_2 \pi^{\frac{d+4}{2}}}{d\cdot\Gamma(d/2)} & \text{if } \alpha = 2; \\ \sum_{x\in\mathbb{Z}^d\setminus\{o\}} \frac{c_\alpha \pi^2 x_1^2}{\|x\|^{d+\alpha}} & \text{if } \alpha > 2 \end{cases}$$

and

$$a_\alpha(n) = \begin{cases} n^{\frac{d-2\alpha}{2}}, & \text{if } \alpha < 2; \\ n^{\frac{d-4}{2}} \left(\log(n)\right)^2, & \text{if } \alpha = 2; \\ n^{\frac{d-4}{2}}, & \text{if } \alpha > 2. \end{cases}$$

We identify Ξ_n^α *with the distribution acting on mean zero test functions* $f \in C^\infty(\mathbb{T}^d)$ *via the* $L^2(\mathbb{T}^d)$ *product. Then, we have that* Ξ_n^α *converges in law to a fractional Gaussian field with parameter* γ*, denoted by* Ξ^γ *or* $fGF_\gamma(\mathbb{T}^d)$*, with mean zero and covariance defined by*

$$\mathbb{E}\Big(\langle \Xi^\gamma, f\rangle, \langle \Xi^\gamma, g\rangle\Big) = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \|k\|^{-2\gamma} \widehat{f}(k)\overline{\widehat{g}(k)}, \tag{3.3.3}$$

where $\gamma := \min\{\alpha, 2\}$*. This convergence holds in* $\mathcal{H}_{-\varepsilon}$ *for* $\varepsilon > \max\{\frac{\gamma}{2} + \frac{d}{4}, \frac{d}{2}\}$*.*

Let us emphasize again two special cases included in the result above. $\gamma = 1$ corresponds to the GFF and $\gamma = 2$ to the bi-Laplacian model. Note further that it is enough to prove the theorem in the case $\mathbb{E}[\sigma(0)] = 0$. For random variables with non-zero mean write

$$s_0(x) = 1 + \sigma(x) - \frac{1}{n^d} \sum_{y\in\mathbb{T}_n^d} \sigma(y) = 1 + (\sigma(x) - \mathbb{E}[\sigma(0)]) - \frac{1}{n^d} \sum_{y\in\mathbb{T}_n^d} (\sigma(y) - \mathbb{E}[\sigma(0)])$$

which falls into the previous case. Let us discuss some further generalisations in the sequel.

3.4 Proofs

3.4.1 Estimates for the eigenvalues of discrete fractional Laplacians

The proofs of Theorem 3.3.3 resp. Theorem 3.3.4 follow similar ideas as the proofs of Theorem 1.2 in [70] resp. Theorem 2 in [33]. The main difference is exchanging the normalised graph Laplacian by the discrete fractional Laplacian given in (1.3.1). More specifically, we need a very sharp control over the eigenvalues associated to the discrete fractional Laplacian.

Note that for the nearest neighbour divisible sandpile model, one studies the normalised graph Laplacian $\Delta_n : \ell^2(\mathbb{T}_n^d) \to \ell^2(\mathbb{T}_n^d)$ given by

$$\Delta_n f(x) = \frac{1}{2d} \sum_{\substack{y\in\mathbb{T}_n^d \\ x\sim y}} (f(y) - f(x)),$$

where $x \sim y$ denotes nearest neighbours modulo $\mathbb{T}_n^d$. Remark that in [33] the authors consider the non-normalized Laplacian, but the factor $1/2d$ appears later in the definition of the discrete odometer u_t in Proposition 4. It is easy to see that, $(e_k)_{k\in\mathbb{Z}_n^d}$ as described in Section 3.2, are eigenvectors of Δ_n with respective eigenvalues given by

$$\lambda_k^n = -\frac{2}{d} \sum_{i=1}^d \sin^2\Big(\pi \frac{k_i}{n}\Big), \tag{3.4.1}$$

which, once properly rescaled, are close to $-\pi\|k\|^2$. However, as mentioned in Section 3.2 the eigenvalues $(\lambda_k^{\alpha,n})_{w\in\mathbb{Z}_n^d}$ of the discrete fractional Laplacian $-(-\Delta)_n^{\alpha/2}$ are given by (3.2.4).

A quick comparison between the eigenvalues (3.4.1) and (3.2.4) shows that we will need to proceed with some extra care to understand the asymptotic behaviour of $\lambda_w^{\alpha,n}$ in terms of n and w.

In fact, for $\alpha\in(0,2)$ one can easily show that

$$n^\alpha\lambda_k^{\alpha,n}\overset{n\to\infty}{\longrightarrow}-(2\pi)^{d+\alpha}c_\alpha\int_{\mathbb{R}^d}\frac{\sin^2(k\cdot z/2)}{\|z\|^{d+\alpha}}dz.$$

Let

$$\tilde{c}^{(\alpha)}:=(2\pi)^{d+\alpha}c_\alpha\int_{\mathbb{R}^d}\frac{\sin^2\left(\frac{z_1}{2}\right)}{\|z\|^{d+\alpha}}dz, \tag{3.4.2}$$

which stems from $\tilde{c}^{(\alpha)}=\lim_{n\to\infty}\frac{-n^\alpha\lambda_k^{\alpha,n}}{\|k\|^\alpha}$, then

$$\begin{aligned}\lambda_k^\alpha&:=-(2\pi)^{d+\alpha}c_\alpha\int_{\mathbb{R}^d}\frac{\sin^2\left(\frac{k\cdot z}{2}\right)}{\|z\|^{d+\alpha}}dz\\&=-(2\pi)^{d+\alpha}\|k\|^\alpha\int_{\mathbb{R}^d}\frac{\sin^2\left(\frac{k}{\|k\|}\cdot\frac{z}{2}\right)}{\|z\|^{d+\alpha}}dz\\&=-(2\pi)^{d+\alpha}c_\alpha\|k\|^\alpha\int_{\mathbb{R}^d}\frac{\sin^2\left(\frac{\pi z_1}{2}\right)}{\|z\|^{d+\alpha}}dz\\&=-\tilde{c}^{(\alpha)}\|k\|^\alpha.\end{aligned} \tag{3.4.3}$$

In the third equality we used a change of variables. The integral is finite, since for large values of z we can use that $\frac{\sin^2(\pi z_1)}{\|z\|^{d+\alpha}}\le\frac{1}{\|z\|^{d+\alpha}}$ and for small $\frac{\sin^2(\pi z_1)}{\|z\|^{d+\alpha}}\le\frac{\pi^2}{\|z\|^{d+\alpha-2}}$.

The best way to understand the asymptotic behaviour of $(n^\alpha\lambda_w^{\alpha,n})_n$ is to see it as a sequence the Riemann sums which converges to the integral $\lambda_w^{(\alpha,\infty)}$ as $n\to\infty$. In general, given a function $h\in C^2(\mathbb{R}^d)$ with sufficiently fast decay at infinity, it is easy to get the bound

$$\left|\frac{1}{n^d}\sum_{x\in\frac{1}{n}\mathbb{Z}^d}h(x)-\int_{\mathbb{R}^d}h(z)dz\right|\lesssim_h\frac{1}{n}\int_{\mathbb{R}^d}\|\nabla h(z)\|dz, \tag{3.4.4}$$

for some constant $c(h)>0$. Unfortunately, this bound is not good enough for us, as

$$h_k(z):=\frac{\sin^2\left(\frac{k\cdot z}{2}\right)}{\|z\|^{d+\alpha}} \tag{3.4.5}$$

and its derivatives have singularities at $z=0$.

The main technical result of this section is the following proposition, which presents the necessary bounds for the inverse of the eigenvalues in the case $\alpha\in(0,2)$. The equivalent of this proposition in the case $\alpha\ge2$ can be derived from the same techniques, but using Lemma 3.4.5 and 3.4.6 instead of Lemma 3.4.4.

Proposition 3.4.1. *Let $d\ge1$ and $\alpha\in(0,2)$ be fixed. For $n\ge1$ and $k\in\mathbb{Z}_n^d\setminus\{0\}$, we have*

$$\left|\frac{1}{n^\alpha\lambda_k^{\alpha,n}}-\frac{1}{\tilde{c}^{(\alpha)}\|k\|^\alpha}\right|\lesssim_{d,\alpha}\begin{cases}\frac{1}{n^{1-\alpha}\|k\|^{2\alpha-1}}, & \text{if }\alpha\in(0,1)\\\frac{1}{n}\log\left(\frac{n}{\|k\|}\right) & \text{if }\alpha=1\\\frac{1}{n^{2-\alpha}\|k\|^{2\alpha-2}}, & \text{if }\alpha\in(1,2),\end{cases} \tag{3.4.6}$$

where $\tilde{c}^{(\alpha)}$ was defined in (3.4.2).

The proof of the proposition is a consequence of Lemmas 3.4.2, 3.4.3 and 3.4.4 which we state and prove in the sequel.

Lemma 3.4.2. *Let $d \geq 1$ and $\alpha \in (0,2)$ be fixed. There exists a constant $C = C(d,\alpha) > 0$ such that, for all $n \geq 1$ and $k \in \mathbb{Z}_n^d \backslash \{0\}$, we have*

$$-\lambda_k^{\alpha,n} \asymp_{d,\alpha} \frac{\|k\|^\alpha}{n^\alpha}. \tag{3.4.7}$$

Proof. Note that

$$\begin{aligned}
-\lambda_k^{\alpha,n} &= -\frac{(2\pi)^{d+\alpha} c_\alpha}{n^{d+\alpha}} \sum_{y \in \frac{2\pi}{n}(\mathbb{Z}^d \backslash \{0\})} \frac{\sin^2\left(\frac{k \cdot y}{2}\right)}{\|y\|^{d+\alpha}} \\
&= -c_\alpha \left(\frac{2\pi\|k\|}{n}\right)^{d+\alpha} \sum_{y \in \frac{2\pi\|k\|}{n}(\mathbb{Z}^d \backslash \{0\})} \frac{\sin^2\left(\frac{k}{\|k\|} \cdot \frac{y}{2}\right)}{\|y\|^{d+\alpha}} \\
&= -c_\alpha \left(\frac{2\pi\|k\|}{n}\right)^{\alpha} \left(\left(\frac{2\pi\|k\|}{n}\right)^{d} \sum_{y \in \frac{2\pi\|k\|}{n}(\mathbb{Z}^d \backslash \{0\})} \frac{\sin^2\left(\frac{k}{\|k\|} \cdot \frac{y}{2}\right)}{\|y\|^{d+\alpha}} \right)
\end{aligned}$$

The term in the parenthesis is a Riemann sum, hence, we just need to prove that such a sum is uniformly bounded in n and k. Now, one proceeds by bounding the Riemann sum according to the upper and lower sum in the partition $\left\{B\left(\frac{2\pi\|k\|}{n} y, \frac{\pi\|k\|}{n}\right), y \in \mathbb{Z}^d\right\}$ and noticing that upper and lower sums are monotone according to the natural partition order. Therefore,

$$\begin{aligned}
\pi^d \sum_{y \in 2\pi(\mathbb{Z}^d \backslash \{0\})} \frac{\sin^2\left(\frac{k}{\|k\|} \cdot \frac{y_*(y)}{2}\right)}{\|y\|^{d+\alpha}} &\leq \left(\frac{2\pi\|k\|}{n}\right)^{d} \sum_{y \in \frac{2\pi\|k\|}{n}(\mathbb{Z}^d \backslash \{0\})} \frac{\sin^2\left(\frac{k}{\|k\|} \cdot \frac{y}{2}\right)}{\|y\|^{d+\alpha}} \\
&\leq \pi^d \sum_{y \in 2\pi(\mathbb{Z}^d \backslash \{0\})} \frac{\sin^2\left(\frac{k}{\|k\|} \cdot \frac{y^*(y)}{2}\right)}{\|y\|^{d+\alpha}}
\end{aligned}$$

where

$$y_*(y) = \operatorname*{argmin}_{z \in B(y,\pi)} \left\{ \frac{\sin^2(\pi z \cdot \frac{k}{\|k\|})}{\|z\|^{d+\alpha}} \right\} \text{ and } y^*(y) = \operatorname*{argmax}_{z \in B(y,\pi)} \left\{ \frac{\sin^2(\pi z \cdot \frac{k}{\|k\|})}{\|z\|^{d+\alpha}} \right\}.$$

Notice that, as $y \in 2\pi\mathbb{Z}^d$, we have that the ball $B(z,\pi)$ is bounded away from the origin. Finally, one can check that both of the sums are indeed finite and positive. ☐

The following lemma will be used to prove Lemma 3.4.4, it follows from basic calculus. Remember that we use $\|\cdot\|_D$ to denote the $L^\infty(D)$ of a function.

Lemma 3.4.3. *Let $d \geq 1$, $k \in \mathbb{Z}_n^d \backslash \{0\}$, $x \in \frac{2\pi\|k\|}{n}\left(\mathbb{Z}^d \backslash \{0\}\right)$ and $v \in \mathbb{R}^d$ such that $\|v\| = 1$. We have that*

$$h_v(z) = \frac{\sin^2\left(\frac{v \cdot z}{2}\right)}{\|z\|^{d+\alpha}}$$

satisfies

$$\|\nabla h_v(\cdot)\|_{B\left(x, \frac{\pi\|wk\|}{n}\right)} \lesssim_{d,\alpha} \min\left\{\frac{1}{\|x\|^{d+\alpha-1}}, \frac{1}{\|x\|^{d+\alpha}}\right\}.$$

The last ingredient for proving Proposition 3.4.1 is the following lemma.

Lemma 3.4.4. *For fixed $d \geq 1$ and $\alpha \in (0,2)$, for all $n \geq 1$ and $w \in \mathbb{Z}_n^d \backslash \{0\}$, we have*

$$|n^\alpha \lambda_k^{\alpha,n} - \tilde{c}^{(\alpha)}\|k\|^\alpha| \lesssim_{d,\alpha} \begin{cases} \frac{\|k\|}{n^{1-\alpha}}, & \text{if } \alpha \in (0,1) \\ \frac{\|k\|^2}{n} \log\left(\frac{n}{\|k\|}\right), & \text{if } \alpha = 1 \\ \frac{\|k\|^2}{n^{2-\alpha}}, & \text{if } \alpha \in (1,2), \end{cases} \tag{3.4.8}$$

where $\tilde{c}^{(\alpha)}$ is defined in (3.4.2).

Proof. We will study the rate of convergence of the Riemann sums of $h_k(z) = \frac{\sin^2(\pi z\cdot k)}{\|z\|^{d+\alpha}}$. The first step is to remove a neighbourhood around the origin.

$$\begin{aligned}
&|n^\alpha \lambda_k^{\alpha,n} - \tilde{c}^{(\alpha)}\|k\|^\alpha| \\
&\quad = (2\pi)^{d+\alpha} c_\alpha \Big| \frac{1}{n^d} \sum_{x\in \frac{2\pi}{n}(\mathbb{Z}^d\setminus\{0\})} h_k(x) - \int_{\mathbb{R}^d} h_k(z)dz \Big| \\
&\quad \lesssim \Big| \frac{1}{n^d} \sum_{x\in \frac{2\pi}{n}(\mathbb{Z}^d\setminus\{0\})} h_k(x) - \int_{\mathbb{R}^d\setminus B(0,\frac{\pi}{n})} h_k(z)dz \Big| + \int_{B(0,\frac{\pi}{n})} |h_k(z)|dz \\
&\quad \lesssim \underbrace{\|k\|^\alpha \Big| \frac{\|k\|^d}{n^d} \sum_{x\in \frac{2\pi\|k\|}{n}(\mathbb{Z}^d\setminus\{0\})} h_{\frac{k}{\|k\|}}(x) - \int_{\mathbb{R}^d\setminus B(0,\frac{\pi\|k\|}{n})} h_{\frac{k}{\|k\|}}(z)dz \Big|}_{I_n(k)} + \frac{\|k\|^2}{n^{2-\alpha}},
\end{aligned}$$

where in the second inequality, we used that $|h_k(z)| \lesssim \frac{\|k\|^2}{\|z\|^{d+\alpha-2}}$. Furthermore,

$$\begin{aligned}
I_n(k) &\lesssim \|k\|^\alpha \sum_{x\in \frac{2\pi\|k\|}{n}(\mathbb{Z}^d\setminus\{0\})} \int_{B(x,\frac{\pi\|k\|}{n})} \Big| h_{\frac{k}{\|k\|}}(x) - h_{\frac{k}{\|k\|}}(z) \Big| dz \\
&\lesssim \|k\|^\alpha \sum_{x\in \frac{\pi\|k\|}{n}\mathbb{Z}^d\setminus\{0\}} \int_{B(x,\frac{\pi\|k\|}{n})} \|z-x\| \int_0^1 \|\nabla h_{\frac{k}{\|k\|}}(tz+(1-t)x)\| dt dz. \qquad (3.4.9)
\end{aligned}$$

For points $z \in B(x, \frac{\pi\|k\|}{n})$, we can use the bound

$$\Big\|\nabla h_{\frac{k}{\|k\|}}(tz+(1-t)x)\Big\| \leq \|\nabla h_{\frac{k}{\|k\|}}\|_{B(x,\frac{\pi\|k\|}{n})}.$$

Hence,

$$(3.4.9) \lesssim \|k\|^\alpha \left(\frac{\|k\|}{n}\right)^{d+1} \sum_{x\in \frac{2\pi\|k\|}{n}(\mathbb{Z}^d\setminus\{0\})} \|\nabla h_{\frac{k}{\|k\|}}\|_{B(x,\frac{\pi\|k\|}{n})}$$

Using Lemma 3.4.3, we get that the above equation can be further bounded by

$$I_n(k) \lesssim \frac{\|k\|^{1+\alpha}}{n} \underbrace{\int_{\mathbb{R}^d\setminus B(0,\frac{\pi\|k\|}{n})} \min\Big\{\frac{1}{\|z\|^{d+\alpha-1}}, \frac{1}{\|z\|^{d+\alpha}}\Big\} dz}_{I'_n(k)}. \qquad (3.4.10)$$

To conclude the proof of the lemma, we bound the minimum in the integral above by $\frac{1}{\|\cdot\|^{d+\alpha-1}}$ for $\alpha > 1$ and by $\frac{1}{\|\cdot\|^{d+\alpha}}$ for $\alpha < 1$. Note that $\|k\| \lesssim_d n$ since $k \in \mathbb{Z}_n^d$, hence the dominant term for $\alpha < 1$ is $\frac{\|k\|}{n^{1-\alpha}}$. For $\alpha = 1$ we write

$$\begin{aligned}
I'_n(k) &= \int_{B(0,1)\setminus B(0,\frac{\pi\|k\|}{n})} \min\Big\{\frac{1}{\|z\|^d}, \frac{1}{\|z\|^{d+1}}\Big\} dz + \int_{\mathbb{R}^d\setminus B(0,1)} \min\Big\{\frac{1}{\|z\|^d}, \frac{1}{\|z\|^{d+1}}\Big\} dz \\
&\lesssim \int_{\frac{\pi\|k\|}{n}}^1 \frac{r^{d-1}}{r^d} dr + \int_1^\infty \frac{r^{d-1}}{r^{d+1}} dr \\
&\lesssim \log\left(\frac{n}{\|k\|}\right),
\end{aligned}$$

plugging this estimate in (3.4.10) concludes the proof. □

We finish the section with two lemmas that extend Lemma 3.4.4 to $\alpha \geq 2$. The equivalent statements for the other Lemmas in this section can also be adapted. We split the between the cases $\alpha = 2$ and $\alpha > 2$, as the proofs use different techniques.

Lemma 3.4.5. *For fixed $d \geq 1$ and $\alpha = 2$ we have for all $n \geq 1$ and $k \in \mathbb{Z}_n^d \setminus \{o\}$, for any $k \neq o$, we have*

$$\lambda_k^{(2,n)} = -\tilde{c}^{(2)} \frac{\|k\|^2 \log(n)}{n^2} + \mathcal{O}\left(\frac{\|k\|^2 \log(\|k\|)}{n^2}\right), \tag{3.4.11}$$

with

$$\tilde{c}^{(2)} := \frac{c_2 \pi^{\frac{d+4}{2}}}{d \cdot \Gamma(d/2)}.$$

The proof borrows ideas from Lemma 3.4.4. However, we need to keep track of some extra terms, due to the fact that h_k is not integrable around the origin. In the following, we sketch how to extend the proof.

This time, instead of comparing $n^2 \lambda_k^{(2,n)}$ with $\lambda_k^{(2,\infty)}$, we compare it to a second sequence $\bar{\lambda}_k^{(2,n)}$, defined as

$$-\bar{\lambda}_k^{(2,n)} := \|k\|^2 \int_{\frac{\|k\|}{2n} \leq \|z\| < \infty} \frac{\sin^2(\pi z \cdot v)}{\|z\|^{d+\alpha}} \mathrm{d}z.$$

One can prove that

$$(\log(n))^{-1} \cdot \bar{\lambda}_k^{(2,n)} \longrightarrow \tilde{c}^{(2)} \|k\|^2.$$

Indeed, for n sufficiently large, we can write

$$\begin{aligned} (\log(n))^{-1} \cdot \bar{\lambda}_k^{(2,n)} &= -\frac{\|k\|^2}{\log(n)} \int_{\frac{\|k\|}{2n} \leq \|z\| \leq 1} \frac{(\pi z \cdot v)^2}{\|z\|^{d+2}} \mathrm{d}z \\ &\quad - \frac{\|k\|^2}{\log(n)} \int_{\frac{\|k\|}{2n} \leq \|z\| \leq 1} \frac{\sin^2(\pi z \cdot v) - (\pi z \cdot v)^2}{\|z\|^{d+2}} \mathrm{d}z \\ &\quad - \frac{\|k\|^2}{\log(n)} \int_{1 \leq \|z\| < \infty} \frac{\sin^2(\pi z \cdot v)}{\|z\|^{d+2}} \mathrm{d}z \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Now, using invariance of the integral I_1 by orthonormal transformations and computing the integral explicitly via spherical coordinates, we get

$$I_1 = -\|k\|^2 \left(\pi^2 c_2 \int_{\mathbb{S}^{d-1}} x_1^2 \mathrm{d}\mu_{d-1}(x)\right) + \mathcal{O}\left(\frac{\|k\|^2 \log(\|k\|)}{\log(n)}\right),$$

moreover, we can evaluate the integral which is equal to $\frac{2\pi^{d/2}}{d \cdot \Gamma(d/2)}$ Using that $\sin^2(\pi z \cdot v) - (\pi z \cdot v)^2 = O(z^4)$ in the region of integration in I_2, we get that

$$|I_2| \lesssim \frac{\|k\|^2}{\log(n)}.$$

Finally, due to the integrability of h_v at infinity, we have again that

$$|I_3| \lesssim \frac{\|k\|^2}{\log(n)}.$$

One still needs to show that $(\log n)^{-1} |n^2 \lambda_k^{(2,n)} - \bar{\lambda}_k^{(2,n)}|$ is small, which is obtained by following the same strategy of Lemma 3.4.4 disregarding the region around the origin at the beginning of the argument. Moreover, we need to use the points of the grid $\frac{\pi\|k\|}{n}\left((\mathbb{Z}^d \setminus \{o\}) \cap [-1,1]^d\right)$ to also control the region $B(o, \pi\|k\|/n)$ with Euclidean norm larger than $\pi\|k\|/n$.

With this, we get

$$(\log(n))^{-1} \left| n^2 \lambda_k^{(2,n)} - \bar{\lambda}_k^{(2,n)} \right| \lesssim \frac{\|k\|^2}{n \log(n)} \int_{\|k\|/2n}^{1} r^{-2} \mathrm{d}r.$$

Keeping track of all these contributions, we get the desired bounds.

Lemma 3.4.6. *For fixed $d \geq 1$ and $\alpha \in (2, \infty)$ we have for all $n \geq 1$ and $k \in \mathbb{Z}_n^d \setminus \{o\}$, for any $k \neq o$, we have*

$$\lambda_k^{(\alpha,n)} = -\tilde{c}^{(\alpha)} \frac{\|k\|^2}{n^2} + \begin{cases} \mathcal{O}\left(\frac{\|k\|^4}{n^\alpha}\right), & \text{if } \alpha \in (2,4) \\ \mathcal{O}\left(\frac{\|k\|^4 \log n}{n^4} \log\left(\frac{\|k\|}{2n}\right)\right), & \text{if } \alpha = 4 \\ \mathcal{O}\left(\frac{\|k\|^4}{n^4}\right), & \text{if } \alpha \in (4,\infty), \end{cases} \tag{3.4.12}$$

with

$$\tilde{c}^{(\alpha)} := \pi^2 c_\alpha \sum_{x \in \mathbb{Z}^d \setminus \{o\}} \frac{x_1^2}{\|x\|^{d+\alpha}}.$$

Notice that the constant above is π^2 times the variance of any of the coordinates of the steps of the random walk, which recovers the clear probabilistic interpretation. We will also restrict ourselves to only sketch the proof, which is simpler than the case $\alpha \leq 2$ as it does not depend on rates of convergence of Riemann sums. Indeed, notice that using (3.2.4),

$$\begin{aligned} n^2 \lambda_k^{(\alpha,n)} &= -n^2 \cdot c_\alpha \sum_{\substack{x \in \mathbb{Z}^d \setminus \{o\} \\ \|x\| \leq n}} \frac{\sin^2(\pi x \cdot \frac{k}{n})}{\|x\|^{d+\alpha}} - n^2 \cdot c_\alpha \sum_{\substack{x \in \mathbb{Z}^d \setminus \{o\} \\ \|x\| \geq n}} \frac{\sin^2(\pi x \cdot \frac{k}{n})}{\|x\|^{d+\alpha}} \\ &= -c_\alpha \sum_{\substack{x \in \mathbb{Z}^d \setminus \{o\} \\ \|x\| \leq n}} \frac{(\pi x \cdot k)^2}{\|x\|^{d+\alpha}} + \mathcal{O}\left(\frac{\|k\|^4}{n^2} \int_1^n r^{3-\alpha} \mathrm{d}r\right) + \mathcal{O}\left(n^2 \int_n^\infty r^{-1-\alpha} \mathrm{d}r\right), \end{aligned}$$

where in the first sum, we used a Taylor expansion of the sine function. Now, we examine the first sum and get

$$\sum_{\substack{x \in \mathbb{Z}^d \setminus \{o\} \\ \|x\| \leq n}} \frac{(\pi x \cdot k)^2}{\|x\|^{d+\alpha}} = \|k\|^2 \sum_{x \in \mathbb{Z}^d \setminus \{o\}} \frac{\left(\pi x \cdot \frac{k}{\|k\|}\right)^2}{\|x\|^{d+\alpha}} + \mathcal{O}\left(\frac{\|k\|^2}{n^{\alpha-2}}\right).$$

Collecting all the terms, we get the desired error bounds. We still need to show that the first sum does not depend on $v = k/\|k\|$,

$$\sum_{x \in \mathbb{Z}^d \setminus \{o\}} \frac{(\pi x \cdot v)^2}{\|x\|^{d+\alpha}} = \pi^2 \sum_{i=1}^d v_i^2 \sum_{x \in \mathbb{Z}^d \setminus \{o\}} \frac{x_i^2}{\|x\|^{d+\alpha}} + 2\pi^2 \sum_{i \neq j} v_i v_j \sum_{x \in \mathbb{Z}^d \setminus \{o\}} \frac{x_i x_j}{\|x\|^{d+\alpha}}.$$

We have that $\sum_{x \in \mathbb{Z}^d \setminus \{o\}} \frac{x_i x_j}{\|x\|^{d+\alpha}} = 0$ and $\sum_{x \in \mathbb{Z}^d \setminus \{o\}} \frac{x_i^2}{\|x\|^{d+\alpha}}$ is finite and does not depend on the choice of i. Using that $\|v\| = 1$ we recover the constant.

3.4.2 Proof of Theorem 3.3.3

We will present the proof for $\alpha \in (0, 2)$, as the case $\alpha \in (2, \infty)$ uses the same techniques, with the exception that it relies on the Lemma 3.4.5, instead of Proposition 3.4.1. First note that using (3.2.2), we have for $x \in \mathbb{T}_n^d$,

$$\begin{aligned} \mathbb{E}[u_\infty^\alpha(x)] &= \mathbb{E}[\eta^\alpha(x)] - \mathbb{E}\left[\min_{z \in \mathbb{T}_n^d} \{\eta^\alpha(z)\}\right] \\ &= 0 + \mathbb{E}\left[\max_{z \in \mathbb{T}_n^d} \{\eta^\alpha(z)\}\right] \end{aligned}$$

since the field η^α is Gaussian and has 0 mean. Therefore, the expected odometer is equal to the expected value of the maximum of a Gaussian field. The key ingredients will be Dudley's bound [98, Proposition 1.2.1] and the majorising measure theorem [98, Theorem 2.1.1]. The idea is to study the mean of the extremes of a centred Gaussian field $(\eta^\alpha(x))_{x\in T}$ for some set of indexes through the metric on T defined by

$$\begin{aligned} d_\eta : T\times T &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto \mathbb{E}[(\eta^\alpha(x)-\eta^\alpha(y))^2]^{\frac{1}{2}}. \end{aligned} \tag{3.4.13}$$

Basically, good bounds for $d_\eta(x,y)$ will imply good bounds for $\mathbb{E}[\max_{x\in T}\{\eta^\alpha(x)\}]$. In the sequel we will prove upper and lower bounds for d_η for our case. Theorem 3.3.3 is a straightforward adaptation of the proofs of Propositions 8.3 and 8.8 made in [70].

Proposition 3.4.7. *For fixed $d\geq 1$ and $\alpha\in(0,2)$, $n\in\mathbb{N}$ and all $x\in\mathbb{T}_n^d\backslash\{0\}$,*

$$\mathbb{E}[(\eta^\alpha(0)-\eta^\alpha(x))^2] \lesssim_{d,\alpha} \Psi_{d,\alpha}(n,\|x\|),$$

where

$$\Psi_{d,\alpha}(n,r) := \begin{cases} n^{2\alpha-d}r^2, & \text{if } \alpha\in(\frac{d}{2}+1,\infty) \\ \log(\frac{n}{r})n^2r^2, & \text{if } \alpha=\frac{d}{2}+1 \\ (nr)^{2\alpha-d}, & \text{if } \alpha\in(\frac{d}{2},\frac{d}{2}+1) \\ \log(1+nr), & \text{if } \alpha=\frac{d}{2} \\ 1, & \text{if } \alpha\in(0,\frac{d}{2}) \end{cases} \tag{3.4.14}$$

for $r>0$, we define $\Psi_{d,\alpha}(n,0):=0$ for any d,α,n.

Notice that the first two cases are only seen for $d=1$. We will split the proof in several parts. For any $x\in\mathbb{T}_n^d\backslash\{0\}$ we have

$$\begin{aligned} \mathbb{E}[(\eta^\alpha(0)-\eta^\alpha(x))^2] &= \sum_{w\in\mathbb{Z}_n^d}|g_\alpha(0,w)-g_\alpha(x,w)|^2 \\ &= n^d\sum_{w\in\mathbb{Z}_n^d}|\widehat{g_{\alpha,0}}(k)-\widehat{g_{\alpha,x}}(k)|^2 \\ &\overset{(3.2.5)}{=} \frac{4}{n^d}\sum_{k\in\mathbb{Z}_n^d\backslash\{0\}}\frac{\sin^2(\frac{k\cdot x}{2})}{(\lambda_k^{\alpha,n})^2} \\ &=: 4\cdot M_{n,d,\alpha}(x). \end{aligned}$$

One can relate the function $M_{n,d,\alpha}$ to

$$G_{n,d,\alpha,x}(y) := \sum_{k\in\mathbb{Z}_n^d\backslash\{0\}}\frac{\sin^2(\frac{k\cdot x}{2})}{(\lambda_k^{\alpha,n})^2}\mathbb{1}_{B(\frac{2\pi k}{n},\frac{\pi}{n})}(y)$$

by noticing that

$$M_{n,d,\alpha}(x) = \int_{\mathbb{R}^d} G_{n,d,\alpha,x}(y)dy. \tag{3.4.15}$$

We have the following property.

Lemma 3.4.8. *For fixed $d\geq 1$ and $\alpha\in(0,2)$, we have that*

$$G_{n,d,\alpha,x}(y) \asymp_{d,\alpha} H_{n,d,\alpha,x}(y),$$

for $x\in\mathbb{Z}_n^d\backslash\{0\}$, where

$$H_{n,d,\alpha,x}(y) := \sum_{k\in\mathbb{Z}_n^d\backslash\{0\}}\frac{\sin^2(\frac{k\cdot x}{2})}{\|y\|^{2\alpha}}\mathbb{1}_{B(\frac{2\pi k}{n},\frac{\pi}{n})}(y).$$

Proof. By the triangular inequality, we have $\|y\|\asymp\frac{\|k\|}{n}$, for $y\in B\Big(\frac{2\pi k}{n},\frac{\pi}{n}\Big)$, and $k\in\mathbb{Z}_n^d\backslash\{0\}$. Therefore, using Lemma 3.4.2, we can have that

$$\frac{\mathbb{1}_{B(\frac{k}{n},\frac{1}{2n})}(y)}{\|y\|^{2\alpha}} \lesssim \frac{\mathbb{1}_{B(\frac{\pi k}{n},\frac{\pi}{n})}(y)}{\Big(\frac{\|k\|}{n}\Big)^{2\alpha}} \lesssim \frac{\mathbb{1}_{B(\frac{\pi k}{n},\frac{\pi}{n})}(y)}{(\lambda_k^{\alpha,n})^2} \lesssim \frac{\mathbb{1}_{B(\frac{\pi k}{n},\frac{\pi}{n})}(y)}{\Big(\frac{\|k\|}{n}\Big)^{2\alpha}} \lesssim \frac{\mathbb{1}_{B(\frac{\pi k}{n},\frac{\pi}{n})}(y)}{\|y\|^{2\alpha}}.$$

Substituting these bounds in the definition of $G_{d,n,\alpha,x}$ concludes the proof. ☐

It follows from the previous lemma that

$$\mathbb{E}[(\eta^\alpha(0)-\eta^\alpha(x))^2] \asymp \int_{\mathbb{R}^d} H_{n,d,\alpha,x}(y)dy. \tag{3.4.16}$$

Note that the support of $H_{d,n,\alpha,x}(y)$ is contained in the annulus $B_2(0,2\pi\sqrt{d})\backslash B_2(0,\frac{\pi}{n})$ hence the above integral is well-defined. We have all the ingredients to prove Proposition 3.4.7 now.

Proof of Proposition 3.4.7. We split the integral

$$\int_{\mathbb{R}^d} H_{n,d,\alpha,x}(y)dy = \underbrace{\int_{\frac{\pi}{n}<\|y\|<\frac{2\pi\sqrt{d}}{n\|x\|}} H_{n,d,\alpha,x}(y)dy}_{:=I_1} + \underbrace{\int_{\frac{2\pi\sqrt{d}}{n\|x\|}<\|y\|<2\pi\sqrt{d}} H_{n,d,\alpha,x}(y)dy}_{:=I_2},$$

for $x\in\mathbb{T}_n^d\backslash\{0\}$. First let us look at I_1. Consider y such that $\frac{1}{2n}<\|y\|\le\frac{2\pi\sqrt{d}}{n\|x\|}$, we use the inequality $\sin^2(t)\le t^2$, the equivalence $\|y\|\asymp\frac{\|k\|}{n}$ and Cauchy-Schwarz to get

$$\begin{aligned} H_{n,d,\alpha,x}(y) &\lesssim \sum_{k\in\mathbb{Z}_n^d\backslash\{0\}} \frac{\|x\|^2\|k\|^2\frac{n^2}{n^2}}{\|y\|^{2\alpha}} \mathbb{1}_{B(\frac{2\pi k}{n},\frac{\pi}{n})}(y) \\ &\lesssim (n\|x\|)^2 \sum_{k\in\mathbb{Z}_n^d\backslash\{0\}} \frac{1}{\|y\|^{2\alpha-2}} \mathbb{1}_{B(\frac{2\pi k}{n},\frac{\pi}{2n})}(y). \end{aligned} \tag{3.4.17}$$

Therefore, we have

$$I_1 = \int_{\frac{\pi}{n}<\|y\|<\frac{2\pi\sqrt{d}}{n\|x\|}} H_{n,d,\alpha,x}(y)dy \lesssim (n\|x\|)^2 \int_{\frac{\pi}{n}}^{\frac{2\pi\sqrt{d}}{\|x\|}} r^{d+1-2\alpha}dr. \tag{3.4.18}$$

On the other hand, for y such that $\|y\|\in(\frac{n\sqrt{d}}{\|x\|},2\pi\sqrt{d})$, we just use the trivial bound $\sin^2(t)\le 1$. Therefore, the second integral can be bounded by

$$I_2 = \int_{\frac{2\pi\sqrt{d}}{n\|x\|}<\|y\|<2\pi\sqrt{d}} H_{n,d,\alpha,x}(y)dy \lesssim \int_{\frac{2\pi\sqrt{d}}{n\|x\|}}^{2\pi\sqrt{d}} r^{d-1-2\alpha}dr. \tag{3.4.19}$$

Computing the right-hand sides in both (3.4.18) and (3.4.19), one recovers the desired expression for $\Psi_{d,\alpha}(n,r)$. □

For $\alpha=2$, we can use expression (3.4.11) to get the right estimates. In fact, one has to estimate the rate of divergence of the Riemann sums of functions $\tilde{h}_w(x)=\frac{\sin^2(\pi w\cdot x)}{\|x\|^2\log(1/\|x\|)}$ for different dimensions involving log corrections.

For the lower bound we will distinguish different cases depending on α and d.

Lemma 3.4.9. *For $d=1$ and $\alpha\in(\frac{1}{2},2)$ we have that*

$$\mathbb{E}[(\eta^\alpha(0)-\eta^\alpha(x))^2] \gtrsim \Psi_{1,\alpha}(n,\|x\|),$$

for all $x\in\mathbb{T}_n^d\backslash\{0\}$.

Proof. We will use that $\sin(t)\ge\frac{2}{\pi}t$ for all $t\in(0,\frac{\pi}{2})$, then

$$\begin{aligned} \mathbb{E}[(\eta^\alpha(0)-\eta^\alpha(x))^2] &= M_{n,1,\alpha}(x) \\ &= \frac{1}{n}\sum_{k\in\mathbb{Z}_n\backslash\{0\}} \frac{\sin^2(\frac{kx}{2})}{(\lambda_k^{\alpha,n})^2} \\ &\overset{(3.4.7)}{\gtrsim} \frac{1}{n}\sum_{k\in\mathbb{Z}_n\backslash\{0\}} \frac{\sin^2(\frac{kx}{2})}{\frac{\|k\|^{2\alpha}}{n^{2\alpha}}} \\ &\gtrsim \frac{1}{n}\sum_{\substack{k\in\mathbb{Z}_n\backslash\{0\}\\ \|k\|<\frac{\pi}{\|x\|}}} \frac{\|x\|^2\|k\|^2}{\frac{\|k\|^{2\alpha}}{n^{2\alpha}}} = n^{2\alpha-1}\|x\|^2 \sum_{\substack{k\in\mathbb{Z}_n\backslash\{0\}\\ \|k\|<\frac{\pi}{\|x\|}}} \|k\|^{2-2\alpha}, \end{aligned} \tag{3.4.20}$$

then one recovers the right estimates by evaluating the sum, which will either be convergent ($\alpha > \frac{3}{2}$), diverges logarithmically ($\alpha = \frac{3}{2}$) or diverges polynomially ($\frac{1}{2} < \alpha < \frac{3}{2}$). □

Lemma 3.4.10. *For $d \in \{2,3\}$ and $\alpha \in (\frac{d}{2}, 2)$, we have*

$$\mathbb{E}[(\eta^\alpha(0) - \eta^\alpha(x))^2] \gtrsim \Psi_{d,\alpha}(n, \|x\|),$$

for $x \in \mathbb{T}_n^d \setminus \{0\}$.

Proof. Let $S_m \subset \mathbb{Z}^d = \mathbb{Z}^d \cap \partial B(0,m)$. For any $x \in \mathbb{R}^d$, we define $H_x = \{y \in \mathbb{Z}^d : |x \cdot y| \geq \frac{1}{\sqrt{d}}\|x\|\|y\|\}$. One can easily check that $|S_m \cap H_x| \gtrsim_d m^{d-1}$ for all $n \geq 1$ and all $x \in \mathbb{R}^d$. Let $a \in (1, \sqrt{d})$, if $\|k\| \leq \frac{2}{a\|x\|}$, then by Cauchy-Schwarz, we have $|x \cdot k|/2 \leq a^{-1}$. Now, we use the inequality $b|t| \leq |\sin(\pi \cdot t)| \leq |t|$ for all $|t| \leq a^{-1}$ with $b := a\sin(a^{-1})$. Hence, we get

$$\begin{aligned}
M_{n,d,\alpha}(x) &\geq \frac{1}{n^d} \sum_{m=1}^{\lfloor \frac{2}{a\|x\|} \rfloor \wedge \lfloor \frac{n}{4} \rfloor} \sum_{k \in S_m} \frac{\sin^2(\frac{k\cdot x}{2})}{(\lambda_k^{\alpha,n})^2} \\
&\gtrsim \frac{1}{n^d} \sum_{m=1}^{\lfloor \frac{2}{a\|x\|} \rfloor \wedge \lfloor \frac{n}{4} \rfloor} \sum_{k \in S_m \cap H_x} \frac{|k\cdot x|^2}{\frac{\|k\|^{2\alpha}}{n^{2\alpha}}} \\
&\gtrsim n^{2\alpha-d}\|x\|^2 \sum_{m=1}^{\lfloor \frac{2}{a\|x\|} \rfloor \wedge \lfloor \frac{n}{4} \rfloor} \sum_{k \in S_m} \|k\|^{2-2\alpha} \\
&\gtrsim n^{2\alpha-d}\|x\|^2 \sum_{m=1}^{\lfloor \frac{2}{a\|x\|} \rfloor \wedge \lfloor \frac{n}{4} \rfloor} m^{d-1}m^{2-2\alpha} \\
&\overset{(\alpha>d/2)}{\geq} n^{2\alpha-d}\|x\|^2 \left(\left\lfloor \frac{2}{a\|x\|} \right\rfloor \wedge \left\lfloor \frac{n}{4} \right\rfloor \right)^{d+2-2\alpha}.
\end{aligned} \tag{3.4.21}$$

As $\alpha \in (\frac{d}{2}, 2)$ and $a \in (0, \sqrt{d})$, the right-hand side of (3.4.21) is of order $\|x\|^{2\alpha-d} = \Psi_{d,\alpha}(n, \|x\|)$. □

For the case $d > 2\alpha$ and $x \neq 0$ we have to analyse the rate of convergence of the function $H_{n,d,\alpha,x}(y)$ to its almost everywhere pointwise limit, that is

$$H_{\infty,d,\alpha,x}(y) = \frac{\sin^2\left(\frac{y\cdot x}{2}\right)}{\|y\|^{2\alpha}} \mathbb{1}_{B(0,\frac{1}{2})\setminus\{0\}}(y).$$

In particular, for $d \geq 2$, it will be useful to express

$$\int_{r_1 \leq \|y\| \leq r_2} H_{\infty,d,\alpha,x}(y)dy \asymp \int_{r_1}^{r_2} \frac{v_d(r\|x\|)}{r^{2\alpha}} r^{d-1}dr, \tag{3.4.22}$$

where

$$v_d(t) := \int_{\mathbb{S}^{d-1}} \sin^2\left(\frac{ty_1}{2}\right) \mu_{d-1}(dy)$$

and μ_{d-1} is the surface measure on the sphere $\mathbb{S}^{d-1}$.

Lemma 3.4.11. *For $d \geq 2$, and for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $t \geq \varepsilon$, then $v_d(t) \geq \delta$.*

Proof. The case $d \geq 3$ is covered in Lemma 8.4, [70]. For $d = 2$, we need to prove that $\underline{\lim}_{t\to\infty} v_d(t) > 0$. By using [9, Corollary 4], we obtain

$$v_2(t) = c_2 \int_{-1}^{1} (1-y^2)^{-\frac{1}{2}} \sin^2\left(\frac{ty}{2}\right) dy \gtrsim \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin^2\left(\frac{ty}{2}\right) dy.$$

Now, the result follows by noticing that

$$\lim_{t\to\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin^2\left(\frac{ty}{2}\right) dy = \frac{1}{2}.$$

□

The proofs of the next two lemmas are equivalent to the proofs of Lemma 8.5 and 8.6 in [70].

Lemma 3.4.12. *Let $d \geq 1$. For all $\varepsilon > 0$, there exist $\delta, N > 0$ such that*

$$\left| H_{n,d,\alpha,x}(y) - \frac{\sin^2\left(\frac{y\cdot x}{2}\right)}{\|y\|^{2\alpha}} \right| \leq \frac{\varepsilon}{\|y\|^{2\alpha}}$$

for all $n \geq N$, $x \in \mathbb{T}_n^d\backslash\{0\}$ such that $\|x\| \leq \delta$ and for a.e y in the annulus $B_2(0, \frac{1}{4})\backslash B_2(0, \frac{\pi}{4n\|x\|})$.

Lemma 3.4.13. *Let $d \geq 2$ and $\alpha \in (0, 2)$ such that $\alpha \leq \frac{d}{2}$. There exist $\delta, N > 0$ such that*

$$\int_{\mathbb{R}^d} H_{n,d,\alpha,x}(y)dy \gtrsim_{d,\alpha} \int_{\frac{\pi}{4n\|x\|}}^{\frac{1}{4}} r^{d-2\alpha-1}dr$$

for all $n \geq N$, $x \in \mathbb{T}_n^d\backslash\{0\}$ satisfying $\|x\| \leq \delta$.

We are left with the case $d = 1$ and $\alpha \leq \frac{1}{2}$. Here we have to compute the lower bound of $\int_{\mathbb{R}^d} H_{n,d,\alpha,x}(y)dy$ directly as we cannot apply the same ideas as in the proofs of Lemma 3.4.9 or Lemma 3.4.11.

Lemma 3.4.14. *Let $d = 1$ and $\alpha \in (0, \frac{1}{2}]$. There exists $\delta, N > 0$ such that*

$$\int_{\mathbb{R}} H_{n,1,\alpha,x}(y)dy \gtrsim \Psi_{1,\alpha}(n, \|x\|)$$

for all $n \geq N$, $x \in \mathbb{T}_n\backslash\{0\}$ satisfying $\|x\| \leq \delta$.

Proof. Let $\varepsilon_1 > 0$ to be chosen later. By Lemma 3.4.12, we can find positive constants $\delta, N > 0$ such that

$$\left| H_{n,1,\alpha,x}(y) - \frac{\sin^2\left(\frac{y\cdot x}{2}\right)}{\|y\|^{2\alpha}} \right| \lesssim \frac{\varepsilon_1}{\|y\|^{2\alpha}}$$

for all $n \geq N$ and for all x such that $\|x\| \leq \delta$ and for a.e y in the annulus $B_2(0, \frac{1}{4})\backslash B_2(0, \frac{\pi}{4n\|x\|})$.

Therefore, for $n \geq N$ and x such that $\|x\| \leq \delta$, we have

$$\begin{aligned}
\int_{\mathbb{R}} H_{n,1,\alpha,x}(y)dy &\gtrsim \int_{\frac{\pi}{4n\|x\|} \leq \|y\| \leq \frac{1}{4}} H_{n,1,\alpha,x}(y)dy \\
&\gtrsim \int_{\frac{\pi}{4n\|x\|} \leq \|y\| \leq \frac{1}{4}} (H_{\infty,1,\alpha,x}(y) - \varepsilon_1 2\|y\|^{-2\alpha})dy \\
&\gtrsim \underbrace{\int_{\frac{\pi}{4n\|x\|} \leq r \leq \frac{1}{4}} r^{-2\alpha}\left(\sin^2\left(\frac{rx}{2}\right) - \varepsilon_1\right) dr}_{I_\alpha(x)} \\
&= \int_{0 \leq r \leq \frac{1}{4}} r^{-2\alpha} \sin^2\left(\frac{rx}{2}\right) dr - \varepsilon_1 \int_{0 \leq r \leq \frac{1}{4}} r^{-2\alpha}dr \\
&\quad - \int_{0 \leq r \leq \frac{\pi}{4n\|x\|}} r^{-2\alpha}\left(\sin^2\left(\frac{rx}{2}\right) - \varepsilon_1\right) dr
\end{aligned}$$

We first discuss the case $\alpha < \frac{1}{2}$. The last integral converges to 0 as $x \longrightarrow \infty$, the second is finite and the first is bounded below by a positive constant. Hence, if one chooses ε_1 small enough, one can show that the sum of the three integrals is bounded below by a positive constant (uniform in n and x). For $\alpha = \frac{1}{2}$, we have that

$$I_\alpha(x) \gtrsim \log(1 + \|x\|)$$

for $\varepsilon_1 > 0$ small enough. □

Proposition 3.4.15. *Let $d \geq 1$ and $\alpha \in (0, 2)$. There exist $\delta, N, C > 0$ such that*

$$\mathbb{E}[(\eta^\alpha(0) - \eta^\alpha(x))^2] \gtrsim \Psi_{d,\alpha}(n, \|x\|)$$

for all $n \geq N$, $x \in \mathbb{Z}_n^d \backslash \{0\}$ satisfying $\|x\| \leq \delta n$ and $\Psi_{d,\alpha}$ defined as in (3.4.14).

The proof of the proposition is a combination of Lemmas 3.4.9, 3.4.10, 3.4.13 and 3.4.14.

3.4.3 Proof of Theorem 3.3.4:

We will only present the proof for $\alpha \in (0, 2)$ as the general case follows similarly. It will be divided into two parts (analogously to the proof of Theorem 2 in [33]):

1. We first prove convergence of finite dimensional distributions to the field Ξ^α, that is the collection $\{\langle \Xi_n^\alpha, f \rangle\}_{f \in \mathcal{F}}$ converges to $\{\langle \Xi^\alpha, f \rangle\}_{f \in \mathcal{F}}$ for any finite collection $\mathcal{F}$ of test functions in the appropriate space.

2. Secondly, we prove tightness of the law of Ξ_n^α. We will take advantage of a classical result given by Theorem 3.4.22 which characterises compact embedding of Sobolev spaces.

The main difference between the proof of Theorem 3.3.4 and Theorem 2 in [33] is the asymptotics of the eigenvalues of $-(-\Delta)_n^{\alpha/2}$. In [33], the authors use Lemma 7 to bound the eigenvalues of the discrete Laplacian (up to the correct renormalisation) and with respect to its continuous counterpart. In particular, their lower-bound can be taken uniformly. However, in our case such bounds cannot be obtained in the same way. We rely on the asymptotic behaviour of the eigenvalues of $-(-\Delta)_n^{\alpha/2}$, as described throughout Subsection 3.4.1.

Moreover, once the comparison between the rescaled eigenvalues of the discrete fractional Laplacian and its continuous version is established, the rest of the proof follows easily for large values of α ($\alpha > 1$). However, for small values of α ($\alpha < 1$ and in particular $\alpha < 1/2$), the technical bounds necessary to make use of the dominated convergence theorem in the proof of finite-dimensional distributions has to be evaluated with more care. The rest of the proof follows similarly, with the analogous adaptations. However, we include its proof to keep the article more self-contained.

Note that, for all $m \geq 1$ and $\theta_1, \cdots, \theta_m \in \mathbb{R}$, $f^{(1)}, \cdots, f^{(m)} \in C^\infty(\mathbb{T}^d)$,

$$\langle \Xi_n^\alpha, \theta_1 f^{(1)} + \cdots + \theta_m f^{(m)} \rangle \stackrel{d}{=} \theta_1 \langle \Xi_n^\alpha, f^{(1)} \rangle + \cdots + \theta_m \langle \Xi_n^\alpha, f^{(m)} \rangle.$$

therefore, it will be enough to study the distribution of a single coordinate of the field, that is $\langle \Xi_n^\alpha, f \rangle$. By Proposition 3.3.2 the odometer can be represented as

$$u_\infty^\alpha(x) \stackrel{d}{=} \eta^\alpha(x) - \min_{z \in \mathbb{T}_n^d} \{\eta^\alpha(z)\}, \tag{3.4.23}$$

for each $x \in \mathbb{T}_n^d$, where

$$\begin{aligned} \eta^\alpha(x) &= \sum_{y \in \mathbb{T}_n^d} g_\alpha(x, y)(s(y) - 1) \\ &= \sum_{y \in \mathbb{T}_n^d} g_\alpha(x, y)\sigma(y) - \frac{1}{n^d} \sum_{y \in \mathbb{T}_n^d} g_\alpha(x, y) \sum_{z \in \mathbb{T}_n^d} \sigma(z) \\ &=: w_n(x) - \frac{1}{n^d} \sum_{y \in \mathbb{T}_n^d} g_\alpha(x, y) \sum_{z \in \mathbb{T}_n^d} \sigma(z). \end{aligned} \tag{3.4.24}$$

Given a function $h_n : \mathbb{Z}_n^d \longrightarrow \mathbb{R}$, one can define

$$\Xi_{h_n}^{\alpha}(x) := \tilde{c}^{(\alpha)} \sum_{z\in\mathbb{T}_n^d} n^{\frac{d-2\alpha}{2}} h_n(nz) \mathbb{1}_{B(z,\frac{1}{2n})}(x), \qquad x \in \mathbb{T}^d$$

and recall that we denoted by Ξ_n^α the field corresponding to $h_n = u_\infty^\alpha$ defined in (3.3.2) and $\tilde{c}^{(\alpha)}$ defined in (3.4.2). Then, for $f \in C^\infty(\mathbb{T}^d)$ such that $\int_{\mathbb{T}^d} f(z)dz = 0$, we have that

$$\langle \Xi_n^\alpha, f\rangle = \langle \Xi_{w_n}^\alpha, f\rangle,$$

since the last sum in (3.4.24) is invariant and does not depend on y. We prove convergence of all moments of $\langle \Xi_{w_n}^\alpha, f\rangle$ first for σ's for which all moments exist and then for the general case.

Convergence for weights with finite moments

In this section, we will prove the following theorem.

Theorem 3.4.16. *Assume that $(\sigma(x))_{x\in\mathbb{T}_n^d}$ is a collection of i.i.d random variables such that $\mathbb{E}[\sigma(x)] = 0$, $\mathbb{E}[\sigma^2(x)] = 1$ and $\mathbb{E}[|\sigma(x)|^k] < \infty$ for all $k \in \mathbb{N}$. Let $d \geq 1$ and u_∞^α the odometer for the long-range divisible sandpile in $\mathbb{T}_n^d$. Then the field Ξ_n^α defined in (3.3.2) converges weakly to Ξ^α as $n \to \infty$. The convergence holds in the same manner as in Theorem 3.3.4.*

First let us prove the following proposition.

Proposition 3.4.17. *Assume $\mathbb{E}[\sigma(x)] = 0$, $\mathbb{E}[\sigma^2(x)] = 1$ and that $\mathbb{E}[|\sigma(x)|^k] < \infty$ for all $k \in \mathbb{N}$ and $x \in \mathbb{Z}_n^d$. Then for all $m \geq 1$ and for all $f \in C^\infty(\mathbb{T}^d)$ with zero mean, the following limit holds:*

$$\lim_{n\to\infty} \mathbb{E}[(\Xi_{w_n}^\alpha, f)^m] = \begin{cases} (2m-1)!!\|f\|_{-\frac{\alpha}{2}}^m, & m \in 2\mathbb{N} \\ 0, & m \in 2\mathbb{N}+1. \end{cases} \tag{3.4.25}$$

Proof. For $f \in C^\infty(\mathbb{T}^d)$ define the map $T_n : \mathbb{T}^d \longrightarrow \mathbb{R}$ by

$$z \longmapsto \int_{B(z,\frac{\pi}{n})} f(y)dy. \tag{3.4.26}$$

Case $m = 2$: We have the equality

$$\begin{aligned}\mathbb{E}[w_n(y)w_n(y')] &= \sum_{x\in\mathbb{T}_n^d}\sum_{x'\in\mathbb{T}_n^d} g_\alpha(y,x)g_\alpha(x',y')\mathbb{E}[\sigma(x)\sigma(x')] \\ &= \sum_{x\in\mathbb{T}_n^d} g_\alpha(y,x)g_\alpha(x,y').\end{aligned}$$

This implies that

$$\mathbb{E}[\langle \Xi_{w_n}^\alpha, f\rangle^2] = (\tilde{c}^{(\alpha)})^2 n^{d-2\alpha} \sum_{x\in\mathbb{T}_n^d}\Big(\sum_{z\in\mathbb{T}_n^d} g_\alpha(x,z)T_n(z)\Big)^2.$$

We now use that, analogously to the proof of Proposition 4 in [33],

$$\begin{aligned}\sum_{x\in\mathbb{T}_n^d} g_\alpha(x,y)g_\alpha(x,y') &= n^d \widehat{g}_{\alpha,y}(0)\widehat{g}_{\alpha,y'}(0) + n^d \sum_{k\in\mathbb{Z}_n^d\setminus\{0\}} \widehat{g}_{\alpha,y}(k)\widehat{g}_{\alpha,y'}(k) \\ &= n^d L^2 + C_n^\alpha(y,y'),\end{aligned} \tag{3.4.27}$$

where $L = \widehat{g}_{\alpha,\cdot}(0)$ is constant. The term n^dL^2 can be dealt with by defining a common Gaussian random variable, independent of the rest of the field, with mean zero and variance n^dL^2. This common random variable will not matter as we are restricting ourselves to mean zero functions. The second part of (3.4.27) can be written as

$$C_n^\alpha(x,y) := \frac{1}{n^d}\sum_{k\in\mathbb{Z}_n^d\setminus\{0\}} \frac{\exp(ik\cdot(y-x))}{(\lambda_k^{\alpha,n})^2}. \tag{3.4.28}$$

Hence,

$$\begin{aligned}\mathbb{E}[(\Xi^\alpha_{w_n}, f)^2] &= (\tilde{c}^{(\alpha)})^2 n^{d-2\alpha} \sum_{z,z'\in\mathbb{T}^d_n} C^\alpha_n(z,z') T_n(z) T_n(z') \\ &= (\tilde{c}^{(\alpha)})^2 n^{d-2\alpha} \sum_{z,z'\in\mathbb{T}^d_n} C^\alpha_n(z,z') \int_{B(z,\frac{\pi}{n})} f(x)dx \int_{B(z',\frac{\pi}{n})} f(x')dx'.\end{aligned}$$

Our strategy will be to divide the above sum in three parts:

$$\begin{aligned}\mathbb{E}[\langle\Xi^\alpha_{w_n}, f\rangle^2] &= (\tilde{c}^{(\alpha)})^2 n^{-d-2\alpha} \sum_{z,z'\in\mathbb{T}^d_n} C^\alpha_n(z,z') f(z) f(z') \\ &+ (\tilde{c}^{(\alpha)})^2 n^{-d-2\alpha} \sum_{z,z'\in\mathbb{T}^d_n} C^\alpha_n(z,z') K_n(f)(z) K_n(f)(z') \\ &+ 2(\tilde{c}^{(\alpha)})^2 n^{-d-2\alpha} \sum_{z,z'\in\mathbb{T}^d_n} C^\alpha_n(z,z') f(z) K_n(f)(z'),\end{aligned}$$

where K_n is defined as

$$K_n(f)(z) := n^d \Big[\int_{B(z,\frac{\pi}{n})} (f(x) - f(z))dx \Big]. \tag{3.4.29}$$

Using Propositions 3.4.18 and 3.4.19 and Cauchy-Schwarz inequality we will prove that

$$\lim_{n\to\infty} \mathbb{E}[(\Xi^\alpha_{w_n}, f)^2] = \|f\|^2_{-\frac{\alpha}{2}},$$

concluding the proof for the case $m = 2$.

Proposition 3.4.18. *For any $f \in C^\infty(\mathbb{T}^d)$ with $\int_{\mathbb{T}^d} f(x)dx = 0$, we have that*

$$\lim_{n\to\infty} (\tilde{c}^{(\alpha)})^2 n^{-d-2\alpha} \sum_{z,z'\in\mathbb{T}^d_n} f(z) f(z') C^\alpha_n(z,z') = \|f\|^2_{-\frac{\alpha}{2}}.$$

Proposition 3.4.19. *For any $f \in C^\infty(\mathbb{T}^d)$ with $\int_{\mathbb{T}^d} f(x)dx = 0$,*

$$\lim_{n\to\infty} (\tilde{c}^{(\alpha)})^2 n^{-d-2\alpha} \sum_{z,z'\in\mathbb{T}^d_n} C^\alpha_n(z,z') K_n(f)(z) K_n(f)(z') = 0.$$

We will first prove Proposition 3.4.19, it is an easy consequence of the following lemma.

Lemma 3.4.20. *For each $f \in C^\infty(\mathbb{T}^d)$ $\sup_{z\in\mathbb{T}^d} |K_n(f)(z)| \lesssim_f \frac{1}{n}$.*

Proof. Using the mean value inequality, we have that there exists $c_{x,z} \in (0,1)$ such that

$$\begin{aligned}|K_n(f)(z)| &\le n^d \int_{B(z,\frac{\pi}{n})} |f(x) - f(z)|dx \\ &\le n^d \int_{B\ (z,\frac{\pi}{n})} \|\nabla f(c_{x,z}x + (1-c_{x,z})z)\| \|z - x\| dx \\ &\lesssim_d \frac{n^d}{n} \int_{B\ (z,\frac{1}{2n})} \|\nabla f(c_{x,z}x + (1-c_{x,z})z)\| dx \\ &\lesssim \frac{1}{n} \|\nabla f(\cdot)\|_{\mathbb{T}^d}.\end{aligned}$$

The lemma follows from the fact that $\|\nabla f(\cdot)\|_{\mathbb{T}^d} < \infty$. ☐

Let $K_n'(f)(z) = K_n(f)(\frac{z}{n})$ and write

$$\begin{aligned}
&(\tilde{c}^{(\alpha)})^2 n^{-2d} \sum_{z,z'\in\mathbb{T}_n^d} n^{d-2\alpha} C_n^\alpha(z,z') K_n(f)(z) K_n(f)(z') \\
&= (\tilde{c}^{(\alpha)})^2 n^{-2d} \sum_{z,z'\in\mathbb{T}_n^d} \sum_{k\in\mathbb{Z}_n^d\setminus\{0\}} \frac{\exp(ik\cdot(z-z'))}{(n^\alpha \lambda_k^{\alpha,n})^2} K_n(f)(z) K_n(f)(z') \\
&\overset{\text{Lemma 3.4.2}}{\lesssim} \sum_{k\in\mathbb{Z}_n^d\setminus\{0\}} |\widehat{K_n(f)}(k)|^2,
\end{aligned}$$

where, we used that $\alpha > 0$ and $\|k\| \geq 1$. Notice that

$$\begin{aligned}
\sum_{k\in\mathbb{Z}_n^d\setminus\{0\}} |\widehat{K_n(f)}(k)|^2 &\leq \sum_{k\in\mathbb{Z}_n^d} |\widehat{K_n(f)}(k)|^2 \\
&\leq n^{-d} \sum_{k\in\mathbb{Z}_n^d} |K_n(f)(k)|^2 \\
&\leq \|K_n(f)\|_{\mathbb{T}^d}^2 \lesssim \frac{1}{n^2}.
\end{aligned}$$

This completes the proof of Proposition 3.4.19.

Proof of Proposition 3.4.18. To prove Proposition 3.4.18 we will rely on information about the speed of convergence of the eigenvalues $\lambda_k^{\alpha,n}$, proven in Proposition 3.4.1 in Subsection 3.4.1. Notice that

$$\begin{aligned}
&\lim_{n\to\infty} n^{-d-2\alpha} (\tilde{c}^{(\alpha)})^2 \sum_{z,z'\in\mathbb{T}_n^d} f(z) f(z') C_n^\alpha(z,z') \\
&= \lim_{n\to\infty} n^{-2d} (\tilde{c}^{(\alpha)})^2 \sum_{z,z'\in\mathbb{T}_n^d} f(z) f(z') \sum_{k\in\mathbb{Z}_n^d\setminus\{0\}} \frac{\exp(ik\cdot(z-z'))}{(n^\alpha \lambda_k^{\alpha,n})^2} \\
&\overset{\text{Proposition 3.4.1}}{=} \lim_{n\to\infty} n^{-2d} (\tilde{c}^{(\alpha)})^2 \sum_{z,z'\in\mathbb{T}_n^d} f(z) f(z') \Bigg(\sum_{k\in\mathbb{Z}_n^d\setminus\{0\}} \frac{\exp(ik\cdot(z-z'))}{(\tilde{c}^{(\alpha)})^2 \|k\|^{2\alpha}} \\
&\quad + 2\exp(ik\cdot(z-z')) \Big(\frac{1}{n^\alpha \lambda_k^{\alpha,n}} - \frac{1}{\tilde{c}^{(\alpha)} \|k\|^\alpha} \Big) \frac{1}{\tilde{c}^{(\alpha)} \|k\|^\alpha} \\
&\quad + \exp(ik\cdot(z-z')) \Big(\frac{1}{n^\alpha \lambda_k^{\alpha,n}} - \frac{1}{\tilde{c}^{(\alpha)} \|k\|^\alpha} \Big)^2 \Bigg) \\
&= I + II + III. \qquad (3.4.30)
\end{aligned}$$

However, we will show that the last two summands are irrelevant. Remember the operator $\mathcal{P}_n : C(\mathbb{T}^d) \longrightarrow \ell^2(\mathbb{T}_n^d)$ was defined in (1.3.7). First we will prove that the third term is irrelevant. Case $\alpha \in (1,2)$: We have that

$$\begin{aligned}
&\left| (\tilde{c}^{(\alpha)})^2 n^{-2d} \sum_{z,z'\in\mathbb{T}_n^d} f(z) f(z') \sum_{k\in\mathbb{Z}_n^d\setminus\{0\}} \exp\Big(ik\cdot(z-z')\Big) \Big(\frac{1}{n^\alpha \lambda_k^{\alpha,n}} - \frac{1}{\tilde{c}^{(\alpha)} \|k\|^\alpha} \Big)^2 \right| \\
&\overset{(3.4.6)}{\lesssim} \frac{1}{n^{4-2\alpha}} \sum_{k\in\mathbb{Z}_n^d\setminus\{0\}} \frac{|\widehat{\mathcal{P}_n f}(k)|^2}{\|k\|^{4\alpha-4}} \\
&\overset{\|k\|\geq 1}{\lesssim} \frac{1}{n^{4-2\alpha}} \sum_{k\in\mathbb{Z}_n^d} |\widehat{\mathcal{P}_n f}(k)|^2 = \frac{1}{n^{4-2\alpha}} \frac{1}{n^d} \sum_{z\in\mathbb{T}_n^d} |f(z)|^2
\end{aligned}$$

where in the last equality we used Parseval's identity. As $\frac{1}{n^d}\sum_{z\in\mathbb{T}_n^d}|f(z)|^2 \longrightarrow \int_{\mathbb{T}^d}|f(z)|^2dz$, the last term in the above expression vanishes as $n\longrightarrow\infty$.

<u>*Case $\alpha\in(\frac{1}{2},1)$:*</u> The proof follows analogously to the previous one, we look at the third term in the brackets of expression (3.4.30) to get

$$\Big|(\tilde{c}^{(\alpha)})^2 n^{-2d}\sum_{z,z'\in\mathbb{T}_n^d} f(z)f(z')\sum_{k\in\mathbb{Z}_n^d\setminus\{0\}}\exp\Big(ik\cdot(z-z')\Big)\Big(\frac{1}{n^\alpha\lambda_k^{\alpha,n}}-\frac{1}{\tilde{c}^{(\alpha)}\|k\|^\alpha}\Big)^2\Big|$$
$$\overset{(3.4.6)}{\lesssim}\frac{1}{n^{2-2\alpha}}\sum_{k\in\mathbb{Z}_n^d\setminus\{0\}}\frac{|\widehat{\mathcal{P}_n f}(k)|^2}{\|k\|^{4\alpha-2}}\lesssim\frac{1}{n^{2-2\alpha}}\frac{1}{n^d}\sum_{z\in\mathbb{T}_n^d}|f(z)|^2\longrightarrow 0,$$

as $n\to\infty$ using the same reasoning as before.

<u>*Case $\alpha\in(0,\frac{1}{2}]$:*</u> In this case we write the third term of (3.4.30) as

$$\Big|(\tilde{c}^{(\alpha)})^2 n^{-2d}\sum_{z,z'\in\mathbb{T}_n^d} f(z)f(z')\sum_{k\in\mathbb{Z}_n^d\setminus\{0\}}\exp\Big(ik\cdot(z-z')\Big)\Big(\frac{1}{n^\alpha\lambda_k^{\alpha,n}}-\frac{1}{\tilde{c}^{(\alpha)}\|k\|^\alpha}\Big)^2\Big|$$
$$\overset{(3.4.6)}{\lesssim}\frac{1}{n^{2-2\alpha}}\sum_{k\in\mathbb{Z}_n^d\setminus\{0\}}\frac{|\widehat{\mathcal{P}_n f}(k)|^2}{\|k\|^{4\alpha-2}}\le c\frac{1}{n^{2\alpha}}\frac{1}{n^d}\sum_{z\in\mathbb{T}_n^d}|f(z)|^2\longrightarrow 0,$$

where, in the last inequality, we used that for $k\in\mathbb{Z}_n^d$, $\|k\|^{2-4\alpha}\lesssim n^{2-4\alpha}$ together with Parseval's identity.

<u>*Case $\alpha=1$:*</u> We compute

$$III\lesssim\frac{1}{n^2}\sum_{k\in\mathbb{Z}_n^d\setminus\{0\}}|\widehat{\mathcal{P}_n f}(w)|^2\log^2\left(\frac{n}{\|k\|}\right)\lesssim\frac{\log^2(n)}{n^2}\frac{1}{n^d}\sum_{z\in\mathbb{T}_n^d}|f(z)|^2\longrightarrow 0,$$

as $n\to\infty$. This proves that the third term of the summand inside the brackets in (3.4.30) vanishes as $n\to\infty$. To prove that the second term II in (3.4.30) vanishes, one can proceed similarly, distinguishing the cases $\alpha\in(1,2)$, $\alpha\in(1/3,1)$, and $\alpha\in(0,1/3)$ and then considering the special cases $\alpha=\frac{1}{3}$ and $\alpha=1$. In fact, for $\alpha=\frac{1}{3}$ we have

$$II\lesssim\frac{1}{n^{2/3}}\sum_{k\in\mathbb{Z}_n^d\setminus\{0\}}|\widehat{\mathcal{P}_n f}(k)|^2\lesssim\frac{1}{n^{2/3}}\frac{1}{n^d}\sum_{z\in\mathbb{T}_n^d}|f(z)|^2\longrightarrow 0,$$

For $\alpha=1$, we now use

$$II\lesssim\frac{1}{n}\sum_{k\in\mathbb{Z}_n^d\setminus\{0\}}\frac{|\widehat{\mathcal{P}_n f}(k)|^2}{\|k\|}\log\left(\frac{n}{\|k\|}\right)\overset{\|k\|\ge 1}{\lesssim}\frac{\log(n)}{n}\frac{1}{n^d}\sum_{z\in\mathbb{T}_n^d}|f(z)|^2\longrightarrow 0.$$

It remains to prove that for all $\alpha\in(0,2)$, we have

$$\lim_{n\to\infty}(\tilde{c}^{(\alpha)})^2 n^{-2d}\sum_{z,z'\in\mathbb{T}_n^d}f(z)f(z')\sum_{w\in\mathbb{Z}_n^d\setminus\{0\}}\frac{\exp(ik\cdot(z-z'))}{\|k\|^{2\alpha}}=\|f\|^2_{-\frac{\alpha}{2}}. \tag{3.4.31}$$

We will distinguish different cases, depending on dimension d and α, for which $\sum_{x\in\mathbb{Z}^d\setminus\{0\}}\|x\|^{-2\alpha}$ is convergent or not.

<u>*Case $d<2\alpha$*:</u> In this simple case we have that

$$n^{-2d}\sum_{z,z'\in\mathbb{T}_n^d}f(z)f(z')\sum_{k\in\mathbb{Z}_n^d\setminus\{0\}}\frac{\exp(ik\cdot(z-z'))}{\|k\|^{2\alpha}}$$
$$=\sum_{k\in\mathbb{Z}^d}\frac{\mathbb{1}_{k\in\mathbb{Z}_n^d\setminus\{0\}}}{\|k\|^{2\alpha}}\frac{\sum_{z\in\mathbb{T}_n^d}f(z)\exp(ik\cdot z)}{n^d}\frac{\sum_{z'\in\mathbb{T}_n^d}f(z')\exp(-ik\cdot z')}{n^d},$$

applying the dominated convergence theorem (notice the uniform bound as f is bounded on the torus $\mathbb{T}^d$), we get (3.4.31).
Case $d \geq 2\alpha$: Here we need to make use of mollifiers. Let $\varrho \in C^\infty(\mathbb{R}^d)$ a positive function in the Schwartz space with support in $[-\frac{1}{2}, \frac{1}{2})^d$ and satisfying $\int_{\mathbb{R}^d} \varrho(x)dx = 1$, let $\varrho_\varepsilon(x) := \frac{1}{\varepsilon^d}\varrho\left(\frac{x}{\varepsilon}\right)$, for $\varepsilon > 0$. As ϱ is in the Schwartz class, [48, Proposition 2.2.11] guarantees that $\widehat{\varrho_\varepsilon}$ is also in this same class, hence for any $m \in \{0, 1, 2, \dots\}$, we have

$$\left|\widehat{\varrho_\kappa}(k)\right| \lesssim_{m,\varepsilon} \frac{1}{(1 + \|k\|)^m}. \tag{3.4.32}$$

We will prove in the following that the convergence in (3.4.31) is equivalent to the convergence of

$$\lim_{\varepsilon \to 0^+} \lim_{n \to \infty} (\tilde{c}^{(\alpha)})^2 n^{-2d} \sum_{z, z' \in \mathbb{T}_n^d} f(z)f(z') \sum_{k \in \mathbb{Z}_n^d \setminus \{0\}} \widehat{\varrho_\varepsilon}(k) \frac{\exp(ik \cdot (z - z'))}{\|k\|^{2\alpha}} = \|f\|^2_{-\frac{\alpha}{2}}. \tag{3.4.33}$$

To do so, we will show that

$$\lim_{\varepsilon \to 0^+} \overline{\lim_{n \to \infty}} \left| n^{-2d} \sum_{z, z' \in \mathbb{T}_n^d} f(z)f(z') \sum_{k \in \mathbb{Z}_n^d \setminus \{0\}} \left(1 - \widehat{\varrho_\varepsilon}(k)\right) \frac{\exp(ik \cdot (z - z'))}{\|k\|^{2\alpha}} \right| = 0. \tag{3.4.34}$$

Since $\int_{\mathbb{R}^d} \varrho_\kappa(x)dx = 1$ and ϱ is positive, we have that

$$|\widehat{\varrho_\varepsilon}(k) - 1| \leq \int_{\mathbb{R}^d} \varrho_\varepsilon(y)|e^{ik \cdot y} - 1|dy.$$

Moreover from $|\exp(ix) - 1|^2 = 4\sin^2(x/2)$ and $|\sin(x)| \leq |x|$ we obtain

$$|\widehat{\varrho_\varepsilon}(k) - 1| \lesssim \varepsilon\|k\| \int_{\mathbb{R}^d} \|y\|\varrho(y)dy \lesssim \varepsilon\|k\|. \tag{3.4.35}$$

Therefore,

$$\begin{aligned} &\left| n^{-2d} \sum_{k \in \mathbb{Z}_n^d \setminus \{0\}} \frac{\widehat{\varrho_\varepsilon}(k) - 1}{\|k\|^{2\alpha}} \sum_{z, z' \in \mathbb{T}_n^d} f(z)f(z') \exp\left(ik \cdot (z - z')\right) \right| \\ &\lesssim \varepsilon \sum_{k \in \mathbb{Z}_n^d \setminus \{0\}} \|k\|^{1-2\alpha} |\widehat{\mathcal{P}_n f}(k)|^2. \end{aligned}$$

For $\alpha \geq \frac{1}{2}$, as $\|k\| \geq 1$, we have

$$\sum_{k \in \mathbb{Z}_n^d \setminus \{0\}} \|k\|^{1-2\alpha} |\widehat{\mathcal{P}_n f}(k)|^2 \leq \sum_{k \in \mathbb{Z}_n^d \setminus \{0\}} |\widehat{\mathcal{P}_n f}(k)|^2 \leq \frac{1}{n^d} \sum_{z \in \mathbb{T}_n^d} |f(z)|^2$$

where we used Parseval's identity, as before. Hence, we have

$$\begin{aligned} &\overline{\lim_{n \to \infty}} \left| n^{-2d} \sum_{z, z' \in \mathbb{T}_n^d} f(z)f(z') \sum_{k \in \mathbb{Z}_n^d \setminus \{0\}} \left(1 - \widehat{\varrho_\varepsilon}(k)\right) \frac{\exp(ik \cdot (z - z'))}{\|k\|^{2\alpha}} \right| \\ &\lesssim \varepsilon \|f(\cdot)\|^2_{\mathbb{T}^d}, \end{aligned}$$

which proves (3.4.34) letting ε go to 0. For the case $\alpha < \frac{1}{2}$, we use the bound

$$|\widehat{\varrho_\varepsilon}(k) - 1| \lesssim \min\{\varepsilon\|k\|, 1\}, \tag{3.4.36}$$

So we can repeat the approach

$$\begin{aligned}
&\left| n^{-2d} \sum_{k\in\mathbb{Z}_n^d\setminus\{0\}} \frac{\widehat{\varrho_\varepsilon}(k)-1}{\|k\|^{2\alpha}} \sum_{z,z'\in\mathbb{T}_n^d} f(z)f(z')\exp\Big(ik\cdot(z-z')\Big)\right| \\
&\quad\lesssim \sum_{k\in\mathbb{Z}_n^d\setminus\{0\}} \min\{\kappa\|k\|^{1-2\alpha}, \|k\|^{-2\alpha}\}|\widehat{\mathcal{P}_n f}(k)|^2 \\
&\quad\le \sum_{\substack{k\in\mathbb{Z}_n^d\setminus\{0\}\\ \|k\|\le\frac{1}{\kappa}}} \kappa \underbrace{\|k\|^{1-2\alpha}}_{\le\kappa^{2\alpha-1}} |\widehat{\mathcal{P}_n f}(k)|^2 + \sum_{\substack{k\in\mathbb{Z}_n^d\setminus\{0\}\\ \|k\|\ge\frac{1}{\kappa}}} \underbrace{\|k\|^{-2\alpha}}_{\le\kappa^{2\alpha}} |\widehat{\mathcal{P}_n f}(k)|^2 \\
&\quad\le \kappa^{2\alpha} \sum_{k\in\mathbb{Z}_n^d\setminus\{0\}} |\widehat{\mathcal{P}_n f}(k)|^2,
\end{aligned}$$

where in the last inequality we used that $\alpha < \frac{1}{2}$ and recover (3.4.34). The proof will be complete once we show (3.4.33). We will apply the dominated convergence theorem twice. First note that, as $\widehat{\varrho_\varepsilon}$ decays fast at infinity,

$$\lim_{n\longrightarrow\infty} n^{-d} \sum_{z\in\mathbb{T}_n^d} f(z)\exp(ik\cdot z) = \widehat{f}(k),$$

and

$$\begin{aligned}
&\lim_{n\to\infty} (\tilde{c}^{(\alpha)})^2 n^{-2d} \sum_{z,z'\in\mathbb{T}_n^d} f(z)f(z') \sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} \widehat{\varrho_\varepsilon}(k)\frac{\exp(ik\cdot(z-z'))}{\|k\|^{2\alpha}} \\
&\quad= (\tilde{c}^{(\alpha)})^2 \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \widehat{\varrho_\varepsilon}(k)\frac{|\widehat{f}(k)|^2}{\|k\|^{2\alpha}}.
\end{aligned}$$

As $\|\widehat{\varrho_\varepsilon}\|_\infty \le 1$, we get the desired equation (3.4.31). That concludes the proof of Proposition 3.4.18. □

Case $m \ge 3$**.** We still need to prove Proposition 3.4.17 for higher order moments, however this will be a much easier result as we can now rely on Propositions 3.4.18 and 3.4.19. We will also need this auxiliary Lemma 12 from [33].

Lemma 3.4.21. *Let $f \in C^\infty(\mathbb{T}^d)$ with mean zero, T_n specified in* (3.4.26) *and $\mathcal{T}_n := \mathcal{P}_n T_n$. Then*

$$n^d \sum_{z\in\mathbb{Z}_n^d} |\widehat{\mathcal{T}_n}(z)| \lesssim_{d,f} 1.$$

For $m \in \{1, 2, \dots\}$, define $\mathcal{P}(m)$ the set of partitions of $\{1, 2, \dots, m\}$. Moreover, denote by Π the elements of a partition $P \in \mathcal{P}(m)$. We will denote $|\Pi|$ the number of elements in Π. Call $\mathcal{P}_2(m) \subset \mathcal{P}(m)$ the pair partitions, that is, partitions $P \in \mathcal{P}(m)$ such that for all $\Pi \in P$, $|\Pi| = 2$. We obtain

$$\begin{aligned}
\mathbb{E}[(\Xi^\alpha_{w_n}, f)^m] &= \Big(\tilde{c}^{(\alpha)} n^{\frac{d-2\alpha}{2}}\Big)^m \sum_{z_1,\dots,z_n\in\mathbb{T}_n^d} \mathbb{E}\left[\prod_{j=1}^m w_n(nz_j)\right] \prod_{j=1}^m T_n(z_j) \\
&= \Big(\tilde{c}^{(\alpha)} n^{\frac{d-2\alpha}{2}}\Big)^m \sum_{P\in\mathcal{P}(m)} \prod_{\Pi\in P} \mathbb{E}\Big[\sigma^{|\Pi|}(0)\Big] \sum_{x\in\mathbb{T}_n^d} \left(\sum_{z_j\in\mathbb{T}_n^d : j\in\Pi} \prod_{j\in\Pi} g_\alpha(x,z_j)T_n(z_j)\right) \\
&= \sum_{P\in\mathcal{P}(m)} \prod_{\Pi\in P} \Big(\tilde{c}^{(\alpha)} n^{\frac{d-2\alpha}{2}}\Big)^{|\Pi|} \mathbb{E}\Big[\sigma^{|\Pi|}(0)\Big] \sum_{x\in\mathbb{Z}_n^d} \left(\sum_{z\in\mathbb{T}_n^d} g_\alpha(x,z)T_n(z)\right)^{|\Pi|} \qquad (3.4.37)
\end{aligned}$$

For a fixed P, let us consider in the product $\Pi \in P$ any term corresponding to a block Π with $|\Pi| = 1$, this will give no contribution to the sum as σ have mean zero. Now consider $\Pi \in P$ with $j := |\Pi| > 2$. We have that

$$\left(\tilde{c}^{(\alpha)} n^{\frac{d-2\alpha}{2}}\right)^j \mathbb{E}[\sigma^j(0)] \sum_{x\in\mathbb{T}_n^d} \left(\sum_{z\in\mathbb{T}_n^d} g_\alpha(x,z) T_n(z)\right)^j$$
$$= \left(\tilde{c}^{(\alpha)} n^{\frac{d-2\alpha}{2}}\right)^j \mathbb{E}[\sigma^j(0)] \sum_{x\in\mathbb{Z}_n^d} \Big(\sum_{z\in\mathbb{T}_n^d} g_\alpha(x,z) \mathcal{T}_n(z)\Big)^j.$$

Now we apply Parseval's identity and get

$$\left(\tilde{c}^{(\alpha)} n^{\frac{d-2\alpha}{2}}\right)^j \mathbb{E}[\sigma^j(0)] \sum_{x\in\mathbb{T}_n^d} \left(n^d \sum_{k\in\mathbb{Z}_n^d} \widehat{g_{\alpha,x}}(k) \widehat{\mathcal{T}_n}(k)\right)^j$$
$$\overset{(3.2.5)}{=} \left(\tilde{c}^{(\alpha)} n^{\frac{d-2\alpha}{2}}\right)^k \mathbb{E}[\sigma^j(0)] \sum_{x\in\mathbb{T}_n^d} \left(\sum_{z\in\mathbb{Z}_n^d\setminus\{0\}} \frac{e^n_{-k}(x)}{-\lambda_k^{\alpha,n}} \widehat{\mathcal{T}_n}(z)\right)^j. \tag{3.4.38}$$

We used that $\widehat{\mathcal{T}_n}(0) = 0$. Now, we evoke Lemma 3.4.2 and $\|k\| \geq 1$ to obtain that $-\lambda_k^{\alpha,n} \geq cn^{-\alpha}$ for all $k \in \mathbb{Z}_n^d$. Therefore, the above expression is bounded from above by

$$\left(\tilde{c}^{(\alpha)} n^{\frac{d-2\alpha}{2}}\right)^j \mathbb{E}[\sigma^j(0)] \sum_{x\in\mathbb{T}_n^d} \left(\sum_{z\in\mathbb{T}_n^d} g_\alpha(x,z) T_n(z)\right)^j$$
$$\lesssim n^{\frac{dj}{2}+d} \mathbb{E}[\sigma^j(0)] \left(\sum_{z\in\mathbb{T}_n^d} \left|\widehat{\mathcal{T}_n}(z)\right|\right)^j. \tag{3.4.39}$$

As the moments of σ are finite, we can use Lemma 3.4.21 to bound the term in parenthesis above. Hence, each block of cardinality $j > 2$ has order at most $n^{\frac{jd}{2}-(j-1)d} = o(1)$. Therefore, the only terms of (3.4.37) that contribute as $n \longrightarrow \infty$ are the ones with $j = 2$, only the pair partitions. Since $\mathcal{P}_2(2m+1) = \emptyset$, the odd moments will vanish. Therefore,

$$\mathbb{E}\Big[\langle \Xi_{w_n}^\alpha, f\rangle^{2m}\Big] = \sum_{P\in\mathcal{P}_2(2m)} \left(\left(\tilde{c}^{(\alpha)}\right)^2 n^{d-2\alpha} \sum_{x\in\mathbb{T}_n^d} \Big(\sum_{z\in\mathbb{T}_n^d} g_\alpha(x,z)\mathcal{T}_n(z)\Big)^2\right)^m + o(1).$$

Note that $|\mathcal{P}_2(2m)| = (2m-1)!!$ and that the bracket term above converges to $\|f\|^2_{-\frac{\alpha}{2}}$. This concludes the proof of Proposition 3.4.17 and with it, the proof of the convergence of distribution in finite dimensions in the essentially bounded case. □

Tightness: For proving tightness we will need the following result which is proven in Theorem 5.8 in [89].

Theorem 3.4.22 (Rellich's theorem)**.** *If $\beta_1 < \beta_2$ the inclusion operator $H^{\beta_2}(\mathbb{T}^d) \hookrightarrow H^{\beta_1}(\mathbb{T}^d)$ is a compact linear operator. In particular for any radius $R > 0$, the closed ball $\overline{B_{\mathcal{H}_{-\frac{\varepsilon}{2}}}(0, R)}$ is compact in $\mathcal{H}_{-\varepsilon}$.*

Choose $-\varepsilon < -\frac{d}{2}$. Observe that

$$\|\Xi_{w_n}^\alpha\|^2_{L^2(\mathbb{T}^d)} = (\tilde{c}^{(\alpha)})^2 n^{d-2\alpha} \sum_{x,\,y\in\mathbb{Z}_n^d} g_\alpha(x,y)\sigma(x) \sum_{x',\,y'\in\mathbb{Z}_n^d} g_\alpha(x',y')\sigma(x')$$

is a.s. finite, as, for any fixed n, it is a finite combination of essentially bounded random variables. Therefore, $\Xi_{w_n}^\alpha \in L^2(\mathbb{T}^d) \subset \mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$ a.s. Due to Rellich's Theorem, it is enough to show that, for all $\delta > 0$, there exists a constant $R = R(\delta) > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{P}\Big(\|\Xi_{w_n}^\alpha\|_{\mathcal{H}_{-\frac{\varepsilon}{2}}} \geq R\Big) \leq \delta.$$

However, one can use Markov's inequality to show that it is enough to get

$$\sup_{n\in\mathbb{N}}\mathbb{E}\Big[\|\Xi^{\alpha}_{w_n}\|^2_{\mathcal{H}_{-\frac{\varepsilon}{2}}}\Big]\lesssim 1.$$

Since $\Xi^{\alpha}_{w_n}\in L^2(\mathbb{T}^d)$, we get a representation

$$\Xi^{\alpha}_{w_n}(z)=\sum_{k\in\mathbb{Z}^d}\widehat{\Xi^{\alpha}_{w_n}}(k)\phi_k(z)$$

in terms of eigenfunctions, we use the notation $\widehat{\Xi^{\alpha}_{w_n}}(k):=\langle\Xi^{\alpha}_{w_n},\,\phi_k\rangle$. Thus, we can express

$$\|\Xi^{\alpha}_{w_n}\|^2_{\mathcal{H}_{-\frac{\varepsilon}{2}}}=\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\|k\|^{-2\varepsilon}\Big|\widehat{\Xi^{\alpha}_{w_n}}(k)\Big|^2.$$

Note that

$$\widehat{\Xi^{\alpha}_{w_n}}(k)=\int_{\mathbb{T}^d}\Xi^{\alpha}_{w_n}(z)\phi_\nu(z)dz=\tilde{c}^{(\alpha)}\sum_{x\in\mathbb{T}^d_n}n^{\frac{d-2\alpha}{2}}w_n(x)\int_{B(x,\,\frac{\pi}{n})}e_k(z)dz.$$

This gives

$$\begin{aligned}&\mathbb{E}\Big[\|\Xi^{\alpha}_{w_n}\|^2_{\mathcal{H}_{-\frac{\varepsilon}{2}}}\Big]\\&=\left(\tilde{c}^{(\alpha)}\right)^2\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\sum_{x,y\in\mathbb{T}^d_n}\|k\|^{-2\varepsilon}n^{d-2\alpha}\mathbb{E}\Big[w_n(x)w_n(y)\Big]\int_{B(x,\,\frac{\pi}{n})}e_k(z)dz\int_{B(y,\,\frac{\pi}{n})}\overline{e_k(z)}dz\\&\overset{(3.4.27)}{=}\left(\tilde{c}^{(\alpha)}\right)^2\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\sum_{x,\,y\in\mathbb{T}^d_n}\frac{n^{d-2\alpha}}{\|k\|^{2\varepsilon}}\Big(n^dL^2+C^{\alpha}_n(x,\,y)\Big)\int_{B(x,\,\frac{\pi}{n})}e_k(z)dz\int_{B(y,\,\frac{\pi}{n})}\overline{e_k(z)}dz.\end{aligned}\quad(3.4.40)$$

Since $\int_{\mathbb{T}^d}\phi_\nu(z)dz=0$, the previous expression reduces to

$$\left(\tilde{c}^{(\alpha)}\right)^2\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\sum_{x,\,y\in\mathbb{T}^d_n}\|k\|^{-2\varepsilon}n^{d-2\alpha}C^{\alpha}_n(x,\,y)\int_{B(x,\frac{\pi}{n})}e_k(z)dz\int_{B(y,\,\frac{\pi}{n})}\overline{e_k}(z)dz.$$

Define $F_{n,k}:\mathbb{T}^d_n\to\mathbb{C}$ as the function $F_{n,k}(x):=\int_{B(x,\frac{\pi}{n})}e_k(z)dz$. Since $e_k\in L^2(\mathbb{T}^d)$, by Cauchy-Schwarz inequality we get $F_{n,k}\in L^1(\mathbb{T}^d_n)$. Now, we claim that

Lemma 3.4.23. *We have that*

$$\sup_{k\in\mathbb{Z}^d}\sup_{n\in\mathbb{N}}\sum_{x,\,y\in\mathbb{T}^d_n}n^{d-2\alpha}C^{\alpha}_n(x,\,y)F_{n,k}(x)\overline{F_{n,k}(y)}\lesssim 1.\quad(3.4.41)$$

Supposing that the lemma above is valid, we have that

$$\begin{aligned}\mathbb{E}\Big[\|\Xi^{\alpha}_{w_n}\|^2_{\mathcal{H}_{-\frac{\varepsilon}{2}}}\Big]&=\left(\tilde{c}^{(\alpha)}\right)^2\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\|k\|^{-2\varepsilon}\sum_{x,\,y\in\mathbb{T}^d_n}n^{d-2\alpha}C^{\alpha}_n(x,\,y)F_{n,k}(x)\overline{F_{n,k}(y)}\\&\lesssim\sum_{k\geq 1}k^{d-1-2\varepsilon}\lesssim 1.\end{aligned}$$

It remains to prove the Lemma 3.4.23.

Proof of Lemma 3.4.23. Again, we will rely on the bounds of Proposition 3.4.1, we will also use that

$$\sum_{x,\,y\in\mathbb{T}^d_n}\exp\Big(ik\cdot(x-y)\Big)F_{n,k}(x)\overline{F_{n,k}(y)}=\Big|\widehat{F_{n,k}}(w)\Big|^2n^{2d}\geq 0.$$

Now, we analyse

$$\begin{aligned}
&\sum_{x,\, y\in\mathbb{T}_n^d} n^{d-2\alpha} C_n^\alpha(x,\, y) F_{n,k}(x)\overline{F_{n,k}(y)} \\
&= \sum_{x,\, y\in\mathbb{T}_n^d} \sum_{k'\in\mathbb{Z}_n^d\setminus\{0\}} \frac{\exp\Big(ik'(x-y)\Big)}{(n^\alpha \lambda_{k'}^{\alpha,n})^2} F_{n,k}(x)\overline{F_{n,k}(y)} \\
&\lesssim \sum_{x,\, y\in\mathbb{T}_n^d} \sum_{k'\in\mathbb{Z}_n^d\setminus\{0\}} \frac{\exp\Big(ik'\cdot(x-y)\Big)}{\|k\|^{2\alpha}} F_{n,k}(x)\overline{F_{n,k}(y)}. \qquad (3.4.42)
\end{aligned}$$

Again, consider mollifiers ϱ_ε as before. We rewrite the right-hand side of (3.4.42) as

$$\begin{aligned}
&\sum_{x,\, y\in\mathbb{T}_n^d} \sum_{k'\in\mathbb{Z}_n^d\setminus\{0\}} \widehat{\varrho_\varepsilon}(k') \frac{\exp\Big(ik'\cdot(x-y)\Big)}{\|k'\|^{2\alpha}} F_{n,k}(x)\overline{F_{n,k}(y)} \\
&+ \sum_{x,\, y\in\mathbb{T}_n^d} \sum_{k'\in\mathbb{Z}_n^d\setminus\{0\}} \Big(1-\widehat{\varrho_\varepsilon}(k')\Big) \frac{\exp\Big(ik'\cdot(x-y)\Big)}{\|k'\|^{2\alpha}} F_{n,k}(x)\overline{F_{n,k}(y)}. \qquad (3.4.43)
\end{aligned}$$

In the sequel we will bound the two summands independently, starting with the second. Consider $G_{n,\nu}:\mathbb{Z}^d \longrightarrow \mathbb{C}$, given by $G_{n,k} := \mathcal{P}_n F_{n,k}$. We have

$$\begin{aligned}
&\sum_{x,\, y\in\mathbb{T}_n^d} \sum_{k'\in\mathbb{Z}_n^d\setminus\{0\}} \Big(1-\widehat{\varrho_\varepsilon}(k')\Big) \frac{\exp\Big(ik'\cdot(x-y)\Big)}{\|k'\|^{2\alpha}} F_{n,k}(x)\overline{F_{n,k}(y)} \\
&= \sum_{k'\in\mathbb{Z}_n^d\setminus\{0\}} \left(\frac{1-\widehat{\varrho_\varepsilon}(k')}{\|k'\|^{2\alpha}}\right) \sum_{x,\, y\in\mathbb{Z}_n^d} \exp\Big(ik'(x-y)\Big) F_{n,k}(x)\overline{F_{n,k}(y)} \\
&= \sum_{k'\in\mathbb{Z}_n^d\setminus\{0\}} \left(\frac{1-\widehat{\varrho_\varepsilon}(k')}{\|k'\|^{2\alpha}}\right) \widehat{G_{n,k}}(k')\overline{\widehat{G_{n,k}}(k')} \\
&\overset{(3.4.36)}{\lesssim} \varepsilon^\delta n^{2d} \sum_{k'\in\mathbb{Z}_n^d} |\widehat{G_{n,k}}(k')|^2,
\end{aligned}$$

where, $\delta := \min\{1, 2\alpha\}$, as done before. In the last inequality, we also used that $|\widehat{G_{n,k}}(0)|^2 \geq 0$. Since $|F_{n,k}(x)| \leq n^{-d}$ and due to Parseval's identity we get

$$\begin{aligned}
\sum_{k'\in\mathbb{Z}_n^d} |\widehat{G_{n,\nu}}(w)|^2 &= n^{-d} \sum_{x\in\mathbb{T}_n^d} |F_{n,k}(x)|^2 \\
&\leq n^{-2d} \sum_{x\in\mathbb{T}_n^d} \int_{B(x,\frac{\pi}{n})} |e_k(z)| dz = n^{-2d} \|e_k\|_{L^1(\mathbb{T}^d)} \lesssim n^{-2d}. \qquad (3.4.44)
\end{aligned}$$

Therefore,

$$\sum_{x,\, y\in\mathbb{T}_n^d} \sum_{k'\in\mathbb{Z}_n^d\setminus\{0\}} \Big(1-\widehat{\varrho_\varepsilon}(k')\Big) \frac{\exp\Big(ik'(x-y)\Big)}{\|k'\|^{2\alpha}} F_{n,k}(x)\overline{F_{n,k}(y)} \lesssim \varepsilon^\delta. \qquad (3.4.45)$$

We can then concentrate on bounding the first term of (3.4.43).

$$\begin{aligned}
&\sum_{x,\,y\in\mathbb{T}_n^d}\sum_{k'\in\mathbb{Z}_n^d\setminus\{0\}}\widehat{\varrho_\varepsilon}(k')\frac{\exp\Big(ik'\cdot(x-y)\Big)}{\|k'\|^{2\alpha}}F_{n,k}(x)\overline{F_{n,k}(y)}\\
&=\sum_{x,\,y\in\mathbb{T}_n^d}\sum_{k'\in\mathbb{Z}^d\setminus\{0\}}\widehat{\varrho_\varepsilon}(k')\frac{\exp\Big(ik'\cdot(x-y)\Big)}{\|k'\|^{2\alpha}}F_{n,k}(x)\overline{F_{n,k}(y)}\\
&\quad-\sum_{x,\,y\in\mathbb{T}_n^d}\sum_{\substack{k'\in\mathbb{Z}^d\\ \|k'\|_\infty>\frac{n}{2}}}\widehat{\varrho_\varepsilon}(k')\frac{\exp\Big(ik'\cdot(x-y)\Big)}{\|k'\|^{2\alpha}}F_{n,k}(x)\overline{F_{n,k}(y)}. \qquad (3.4.46)
\end{aligned}$$

Again using the fast decay of $\widehat{\varrho_\varepsilon}$ as in (3.4.32) we get

$$\begin{aligned}
&\sum_{x,\,y\in\mathbb{T}_n^d}\sum_{\substack{k'\in\mathbb{Z}^d\\ \|k'\|_\infty>\frac{n}{2}}}\widehat{\varrho_\varepsilon}(k')\frac{\exp\Big(ik'\cdot(x-y)\Big)}{\|k'\|^{2\alpha}}F_{n,k}(x)\overline{F_{n,k}(y)}\\
&\lesssim\frac{1}{n^{2\alpha}}\sum_{\substack{k'\in\mathbb{Z}^d\\ \|k'\|_\infty>\frac{n}{2}}}\widehat{\varrho_\varepsilon}(k')\sum_{x,\,y\in\mathbb{T}_n^d}F_{n,k}(x)\overline{F_{n,k}(y)}\\
&\le\sum_{\substack{k'\in\mathbb{Z}^d\\ \|k'\|_\infty>\frac{n}{2}}}\widehat{\varrho_\varepsilon}(k')\left|\sum_{x\in\mathbb{T}_n^d}F_{n,k}(x)\right|^2\lesssim\sum_{\substack{k'\in\mathbb{Z}^d\\ \|k'\|_\infty>\frac{n}{2}}}\frac{\|e_k\|^2_{L^1(\mathbb{T}^d)}}{(1+\|k'\|)^m}\lesssim 1, \qquad (3.4.47)
\end{aligned}$$

where in the last inequality we used that we can choose m as large as necessary.

Analogously,

$$\begin{aligned}
&\sum_{x,\,y\in\mathbb{T}_n^d}\sum_{k'\in\mathbb{Z}^d\setminus\{0\}}\widehat{\varrho_\varepsilon}(k')\frac{\exp\Big(ik'(x-y)\Big)}{\|k'\|^{2\alpha}}F_{n,k}(x)\overline{F_{n,k}(y)}\\
&\qquad\overset{(3.4.32)}{\lesssim}\sum_{x,\,y\in\mathbb{T}_n^d}\sum_{k'\in\mathbb{Z}^d\setminus\{0\}}\frac{1}{(1+\|k'\|)^m}|F_{n,k}(x)\overline{F_{n,k}(y)}|\\
&\qquad\lesssim\sum_{k'\in\mathbb{Z}^d\setminus\{0\}}\frac{\|e_k\|^2_{L^1(\mathbb{T}^d)}}{(1+\|k'\|)^m}\lesssim 1. \qquad (3.4.48)
\end{aligned}$$

By plugging (3.4.43), (3.4.45),(3.4.46), (3.4.47) and (3.4.48) in (3.4.42), we conclude the proof of the lemma, and hence of the Theorem 3.3.4. □

Truncation method

In the first part of the argument, we had to restrict ourselves to essentially bounded weights. We will now show how to reconstruct the general case. We will need to fix an arbitrarily large (but finite) constant $\mathcal{R}>0$. Set

$$\begin{aligned}
w_n^{<\mathcal{R}}(x)&:=\sum_{y\in\mathbb{Z}_n^d}g_\alpha(x,\,y)\sigma(y)1_{\{|\sigma(y)|<\mathcal{R}\}},\\
w_n^{\geq\mathcal{R}}(x)&:=\sum_{y\in\mathbb{Z}_n^d}g_\alpha(x,\,y)\sigma(y)1_{\{|\sigma(y)|\geq\mathcal{R}\}}.
\end{aligned}$$

Clearly we have that $w_n(\cdot)=w_n^{<\mathcal{R}}(\cdot)+w_n^{\geq\mathcal{R}}(\cdot)$. To prove our result, we will use the following theorem from Theorem 4.2 from [16].

Theorem 3.4.24. *Let S be a metric space with metric ϱ. Suppose that $(X_{n,u}, X_n)$ are elements of $S \times S$. If*

$$\lim_{u\to\infty} \overline{\lim_{n\to\infty}} \mathbb{P}\Big(\varrho(X_{n,u}, X_n) \geq \tau\Big) = 0$$

for all $\tau > 0$, and $X_{n,u} \longrightarrow_n Z_u \longrightarrow_u X$, where "$\longrightarrow_x$" indicates convergence in law as $x \longrightarrow \infty$, then $X_n \longrightarrow_n X$.

Therefore, we need to prove two statements.

(S1) $\lim_{\mathcal{R}\to\infty} \overline{\lim_{n\to\infty}} \mathbb{P}\Big(\Big\|\Xi^\alpha_{w_n} - \Xi^\alpha_{w_n^{\leq \mathcal{R}}}\Big\|_{\mathcal{H}_{-\varepsilon}} \geq \tau\Big) = 0$ for all $\tau > 0$.

(S2) For a constant $v_{\mathcal{R}} > 0$, we have $\Xi^\alpha_{w_n^{\leq \mathcal{R}}} \longrightarrow_n \sqrt{v_{\mathcal{R}}}\Xi^\alpha \longrightarrow_{\mathcal{R}} \Xi^\alpha$ in the topology of $\mathcal{H}_{-\varepsilon}$.

It follows that $\Xi^\alpha_{w_n}$ converges to $\Xi^{(\alpha)}$ in law in the topology of $\mathcal{H}_{-\varepsilon}$.

Since the proof of (S1) and (S2) does not present any extra technical difficulties, and therefore the argument is almost unchanged when compared to the proof in Section 5.2 in [33], we will leave it to the reader.

3.5 Final remarks and generalisations

Sandpiles based on general α-admissible random walks

Note that the redistribution of the mass, specified in Algorithm 3.2.2, depends on $(-\Delta)^{\alpha/2}_n$ which is defined w.r.t. the long-range random walk with transition probabilities p^n_α given in (1.4.1). The fact that one obtains fractional Gaussian fields with parameter γ as scaling limits of the odometer should not depend on the particular law p^n_α but rather on its asymptotic properties. We expect the following generalisation to hold. Let $(X_t)_{t\geq 0}$ be a random walk with transition probabilities given by $p(x,y) = p(\|x-y\|)$ be in the domain of attraction of a α-stable random variable for $\alpha \in (0,2]$. Define its periodisation by

$$p^n_X(x) = \sum_{z\in\mathbb{Z}^d\setminus\{0\}} p_X\left(\frac{nx}{2\pi} + z\right).$$

Consider the divisible sandpile model on $\mathbb{Z}^d_n$ where the mass is distributed according to p_n. Denote its final odometer by $u^{(p)}_\infty$ and the formal field on $\mathbb{T}^d$ by

$$\Xi^{(p)}_n(x) := a^{(p)}_n \sum_{z\in\mathbb{T}^d_n} u^{(p)}_\infty(z)\mathbb{1}_{B(z,\frac{\pi}{n})}(x),\ x\in\mathbb{T}^d$$

where $a^{(p)}_n = n^{\frac{d-2\alpha}{2}}$. We believe that that $\Xi^{(p)}$ converges in law to a fractional Gaussian field $\mathrm{fGF}_\alpha(\mathbb{T}^d)$ with parameter α.

Subcritical vs supercritical sandpiles

We showed that if the initial configuration s_0 for the long-range divisible sandpile model is chosen in such a way that $\sum_{x\in\mathbb{Z}^d_n} s_0(x) = n^d$, then the odometer u^α_∞ is finite a.s. and $s_\infty \equiv 1$. Consider now

$$s'(x) = 1 + c_0 + \sigma(x) - \frac{1}{n^d}\sum_{y\in\mathbb{T}^d_n}\sigma(y)$$

for some $c_0 \in \mathbb{R}$. If $c_0 > 0$, then clearly $u^\alpha_\infty \equiv \infty$ for every realisation. However, if we define

$$\tilde{u}^\alpha_t(x) := u^\alpha_t(x) - \frac{1}{n^d}\sum_{y\in\mathbb{T}^d_n} u^\alpha_t(y),$$

we still have $s_t(x) = s'(x) - (-\Delta)^{\alpha/2}_n \tilde{u}^\alpha_t(x)$ for all $x \in \mathbb{T}^d_n$. In this case we can prove that, for all $x \in \mathbb{T}^d_n$, $s_t(x) \to 1 + c_0$ and $\tilde{u}^\alpha_\infty(x) \in [0,\infty)$. The scaling limit of the field $\tilde{u}^\alpha_\infty$ is the same as of u^α_∞. However, for $c_0 < 0$ it is less clear what happens since we do not know how the configurations s' and s'_∞ correlate.

Potential kernel of $(W_t^\alpha)_{t\in\mathbb{R}^+}$ in the torus

Finally let us remark that the asymptotics of potential kernels can be used to recover the kernel of the fractional Laplacian for $\alpha \in (0,2)$ for dimension $d > 2\alpha$. The proof is analogous to the one of Theorem 3 in [33], hence we leave it to the reader.

Part II

Dynamical random interfaces

Chapter 4

Wellposedness of a non-local SPDE

4.1 Introduction

In this article, we study the local wellposedness of the following formal SPDE

$$\partial_t \mathfrak{u} = -(-\Delta)^{1/2}\mathfrak{u} - \sinh(\gamma\mathfrak{u}) + \xi \quad \text{in } \mathbb{R}_+ \times \mathbb{T} \tag{4.1.1}$$

in which $\mathbb{T}$ is the one-dimensional torus, ξ is the space-time white-noise, $(-\Delta)^{1/2}$ is the half-Laplacian operator, $\sinh(\gamma\mathfrak{u})$ is taken in a Gaussian Multiplicative Chaos (GMC) sense and $\gamma \in \mathbb{R}$ belongs to a small interval around the origin. This equation is a long-range counterpart to the equation

$$\partial_t \mathfrak{u} = \Delta\mathfrak{u} - \sinh(\gamma\mathfrak{u}) + \xi \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^2$$

which appears in the context of Liouville quantum gravity [45] and is related to the cosh-interaction, and in Quantum Field Theory [56] when sinh is replaced by sin. We will refer to it as sinh-Gordon equation.

The lack of regularity of the white-noise ξ prevents the existence of function-valued solutions. In consequence, the meaning of the non-linear terms of the equation is not clear. Several approaches to circumvent this problem were proposed. The first one, the so-called Da Prato-Debussche perturbative method [34], provides local existence in time of a certain class of equations. More recently, Gubinelli and co-authors [50, 51] introduced an approach to study singular SPDEs based on techniques from paradifferential calculus and controlled rough paths, and Hairer [52] proposed a theory for studying a large class, so-called *subcritical*, of non-linear SPDE's by using *regularity structures*.

The equation (4.1.1) studied in this chapter falls short of the scope of the theory of regularity structures for two reasons. The first one, discussed here in the regime γ small, is the non-locality of the operator $(-\Delta)^{1/2}$ responsible for the lack of smoothness of the semigroup. In consequence, to apply the theory of regularity structures, one needs new methods to prove the wellposedness of the operator (denoted by $\mathcal{K}_\gamma$ in [52]) which works as the abstract counterpart of the integration against the kernel for regularity structures. This question has been addressed in [15] in the context of polynomial non-linearity. It is also not clear if the lack of regularity of the semigroup is not an obstacle to estimate the remainder of the Taylor expansions appearing in the BPHZ renormalisation presented in [22].

The second problem is related to the lack of regularity of the exponential non-linearity. In [22], the authors require the noise (or the non-linearity) to have finite cumulants of all orders. However, as the classical theory indicates, the GMC does not have finite moments of order larger than $4\pi/\gamma^2$. Notice that in [54], by changing the perspective from renormalisation of regularity structures to renormalisation of graphs, the author does not need to refer to cumulants with regard to *negative renormalisation.* However, one would require *positive renormalisation* in order to construct a local solution of (4.1.1) when leaving the Da Prato-Debussche regime.

Consider, for example, the second of these problems in relation to the sine-Gordon equation [56, 23]

$$\partial_t \mathfrak{u} = \Delta \mathfrak{u} + \sin(\gamma \mathfrak{u}) + \xi \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^2 , \tag{4.1.2}$$

Although this equation also comes from a GMC type of process (by seeing it as the real part of $\exp(i\beta\mathfrak{u})$), the boundedness of the sin function implies the finiteness of all moments. Moreover, the moments only grow linearly. This makes the equations (4.1.1), (4.1.2) very different from the perspective of regularity structures.

With regard to the second problem, an exponential non-linearity has been examined before in the Da Prato-Debussche regime by Garban [45] for the equation

$$\partial_t \mathfrak{u} = \Delta \mathfrak{u} + \exp(\gamma \mathfrak{u}) + \xi \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^2 \tag{4.1.3}$$

in dimension 2. There is no difference between proving local wellposedness between the non-linearities $\sinh(\gamma\mathfrak{u})$ and $\exp(\gamma\mathfrak{u})$ when restricted to small values of γ. However, the symmetries of the latter make it more likely to have global wellposedness and for it to appear as a scaling limit of some class of particle systems. Recently, [80] have also studied this equation in the context of stochastic quantization on 2-dimensional manifolds.

One of the main contributions of this chapter, presented in Section 4.4, is a Schauder estimate for the half-Laplacian on the Torus in the context of negative Besov spaces. The equivalent result has already been proven for Hölder spaces (or equivalently, positive Besov spaces) in [38]. The strategy that we will implement follows the idea used in [52], that is, given a fundamental solution K of an equation, we try to decompose it as $K = R + \sum_{n\geq 0} K_n$ where R and K_n are C^∞ smooth functions and the support of K_n's are contained in balls of exponentially small radii. However, the nonlocality of out operator $(-\Delta)^{1/2}$ implies that such decomposition cannot be achieved, instead, we have that the derivatives of K_n's blow up at the line $\{t = 0\}$. Therefore, we need apply such idea once more and decompose each K_n further into smooth functions whose support is contained in a set exponentially close to the line $\{t = 0\}$.

It is important to mention that other strategies could also be chosen as a starting point. For instance, in [58, 81, 86], the authors use an equivalent norm for the negative Besov spaces based on the semigroup which is chosen to keep the scaling relations of the original problem. This type of approach was initially found on the paper [81], in which the authors explore a quasilinear equation; however, they restricted themselves to the setting of periodic-in-time solutions of a (local) operator. Later, [86], managed to remove the periodicity in the time coordinate but still restricted to local operators. Both papers rely on classical Schauder type estimates for their respective linear equations with "frozen coefficients", which is precisely the step we are trying to prove. On the other hand, in [58], the authors do work with nonlocal type equations. But again, they restrict themselves to the periodic-in-time setting, this allows them to approach the time-coordinate via Fourier Transform methods that do not immediately transfer to the initial-value problem. Hence, our Schauder type result is no simple consequence of their methods.

The same strategy presented here can be used to prove Schauder estimates in Besov space for any $\alpha < 0$. This result could be extended to other fractional powers of the Laplacian, provided that one is able to derive the necessary bounds for the right-side derivatives of the semigroup at $t = 0$. Finally, we believe that the strategy presented here could also be used to prove wellposedness of the operator $\mathcal{K}_\gamma$, mentioned above, in the context of regularity structures. This would be an important (although technical) step in the in order to use regularity structures for nonlocal equations.

Structure of chapter

The chapter is organised as follows: In section 4.2, we will introduce the notation required and state our results. In Section 4.3, we present the main properties of the fractional Laplacian needed in the article. In Section 4.4, we prove Theorem 4.2.4. In Section 4.5, we examine the properties of the log-correlated Gaussian random field $\mathfrak{v}_\varepsilon$, introduced in (4.2.6). In Section 4.6, we prove Theorem 4.2.1 and, in Section 4.7, Theorem 4.2.2.

4.2 Notation and Results

We state in this section the main results of the article. We start introducing some spaces of continuous functions. We will work with the one-dimensional torus $\mathbb{T} = [-\pi, \pi)$. Elements of $\mathbb{R}\times\mathbb{T}$ are represented by the letter $z = (t, x)$. Denote by d the distance on $\mathbb{R} \times \mathbb{T}$ given by

$$d(z, z') \;=\; d\big((t, x)\,,\,(t', x')\big) \;=\; |t - t'| \;+\; d_{\mathbb{T}}(x, x')\;,$$

where $d_{\mathbb{T}}(\cdot, \cdot)$ stands for the standard metric in $\mathbb{T}$. We will often use the notation $\|z\|$ to denote $d(0, z)$ even though it is not a norm.

In this chapter, to start with, we need to update the definition of Hölder space to reflect the distance d defined above. Denote by $C^\beta = C^\beta(\mathbb{R} \times \mathbb{T})$, $\beta \in (0, 1)$, the set of continuous functions $f : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ such that for all S, $T \in \mathbb{R}$, $S < T$,

$$\|f\|_{C^\beta([S,T]\times\mathbb{T})} \;:=\; \sup_{z,z'\in[S,T]\times\mathbb{T}} \frac{|f(z) - f(z')|}{d(z, z')^\beta} \;<\; \infty\,. \tag{4.2.1}$$

Let $C^\beta_+ = C^\beta_+(\mathbb{R} \times \mathbb{T})$ be the subset of functions f in $C^\beta(\mathbb{R} \times \mathbb{T})$ such that $f(t, x) = 0$ for all $t \le 0$. The space $C^\beta_+([0, T] \times \mathbb{T})$ is defined analogously. We will denote by $C^m_c(\mathbb{R} \times \mathbb{T})$ represents the subset of functions in $C^m(\mathbb{R} \times \mathbb{T})$ which have compact support. Notice we are using the quotient topology in $\mathbb{T}$ in order to define such compact sets.

Besov Spaces

Let $\mathfrak{C}^m(\mathbb{R} \times \mathbb{T})$, $m \in \mathbb{N}_0$, be the dual of $C^m_c(\mathbb{R} \times \mathbb{T})$. Elements of $\mathfrak{C}^m(\mathbb{R} \times \mathbb{T})$ are denoted by X, and by the gothic characters $\mathfrak{u}$, $\mathfrak{v}$. We represent by $X(f)$ or $\langle X\,,\, f\rangle$ the value at $f \in C^m_c(\mathbb{R} \times \mathbb{T})$ of the bounded linear functional X.

For $z \in \mathbb{R} \times \mathbb{T}$, denote by $\mathbb{B}(z, a)$ the open ball with regards the distance d in $\mathbb{R} \times \mathbb{T}$ of radius $a > 0$ centered at z. Let B_m, $m \in \mathbb{N}_0$, be the set of all functions g in $C^m_c(\mathbb{R} \times \mathbb{T})$ whose support is contained in the ball $\mathbb{B}(0, \pi/2)$ and such that $\|g\|_{C^m(\mathbb{R}\times\mathbb{T})} \le 1$.

For $0 < \delta \le 1$, $z \in \mathbb{R} \times \mathbb{T}$, and a continuous function $g\colon \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ whose support is contained in $\mathbb{B}(0, \pi/2)$, denote by $S^\delta_z g$ the function defined by

$$(S^\delta_z g)\,(w) \;:=\; \begin{cases} \delta^{-2}\, g\big(\,[w - z]/\delta\,\big)\,, & w \in \mathbb{B}(z, \delta)\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{4.2.2}$$

where, for $z = (t, x)$, we define $[z] := (t, x')$ such that $x' \equiv x \mod 2\pi$ and $x' \in [-\pi, \pi)$.

Fix $\alpha < 0$ and set $m = -\lfloor\alpha\rfloor$. For $S < T$, and an element X in $\mathfrak{C}^m(\mathbb{R} \times \mathbb{T})$, let $\|\cdot\|_{C^\alpha([S,T]\times\mathbb{T})}$ be the semi-norm defined by

$$\|X\|_{C^\alpha([S,T]\times\mathbb{T})} \;:=\; \sup_{\delta\in(0,1]}\; \sup_{z\in[S,T]\times\mathbb{T}}\; \sup_{g\in B_m} \frac{1}{\delta^\alpha}\,\big|\,\langle X\,,\, S^\delta_z g\rangle\,\big|\,. \tag{4.2.3}$$

Denote by $C^\alpha = C^\alpha(\mathbb{R}\times\mathbb{T})$ the subspace of $\mathfrak{C}^m(\mathbb{R}\times\mathbb{T})$ of all elements X such that $\|X\|_{C^\alpha([S,T]\times\mathbb{T})} < \infty$ for all $S < T$.

White Noise

Let ξ be a white-noise on $\mathbb{R} \times \mathbb{T}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Expectation with respect to $\mathbb{P}$ is represented by $\mathbb{E}$. Hence, ξ is a centered Gaussian field on $\mathbb{R} \times \mathbb{T}$ whose covariance is formally given by $\mathbb{E}[\,\xi(t, x)\, \xi(t', x')\,] = \delta(x - x')\, \delta(t - t')$, where δ is the Dirac distribution.

An elementary computation shows that for every $N \ge 1$, there exists a finite constant C_N such that

$$\mathbb{E}\left[\bigg(\int (S^\delta_z f)(w)\, \xi(w)\, dw\bigg)^{2N}\right] \;\lesssim_{f,N}\; \delta^{-2N}$$

for all $z \in \mathbb{R}\times\mathbb{T}$, $0 < \delta \le 1$ and continuous function $f : \mathbb{R}\times\mathbb{T} \to \mathbb{R}$ whose absolute value is bounded by 1 and support is contained in $[-1, 1] \times \mathbb{T}$, By [24, Theorem 2.7], for every $\alpha < -1$, there is a version of ξ which belongs to $C^\alpha(\mathbb{R}\times\mathbb{T})$. More precisely, there exists a centered Gaussian field $\tilde\xi$ such that $\langle\tilde\xi\,,\,f\rangle = \langle\xi\,,\,f\rangle$ almost surely for all continuous function f with bounded support, and $\mathbb{E}[\,\|\,\tilde\xi\,\|_{C^\alpha([-T,T]\times\mathbb{T})}\,] < \infty$ for all $T>0$.

Denote by $\varrho:\mathbb{R}^2\to\mathbb{R}_+$ a non-negative, symmetric, smooth mollifier whose support is contained in $(-\pi/2\sqrt{2},\pi/2\sqrt{2})^2$, and which integrates to 1:

$$\varrho(z)\;\ge\;0\,,\quad \varrho(-z)\;=\;\varrho(z)\,,\quad \int_{\mathbb{R}^2}\varrho(z)\,dz\;=\;1\,. \tag{4.2.4}$$

As the support in contained in $(-\pi/2\sqrt{2},\pi/2\sqrt{2})^2$, we also consider ϱ as a mollifier acting on $\mathbb{R}\times\mathbb{T}$ and in this context, we have that the support of ϱ is contained in $\mathbb{B}(0,\pi/2)$. For $\varepsilon>0$, let

$$\varrho_\varepsilon\;=\;S_0^\varepsilon\varrho\,,\quad \xi_\varepsilon\;=\;\varrho_\varepsilon\,*\,\xi\,. \tag{4.2.5}$$

By [1, Theorem 1.4.2], for every $\varepsilon>0$, almost surely, the mollified field ξ_ε is smooth in the sense that it has derivatives of all orders.

Fractional Laplacian

Remember the definition of Fractional Laplacian given in (1.3.1). In this chapter, we will concentrate in the case $\alpha=1$. The operator $-(-\Delta)^{1/2}$ corresponds to the generator of the Cauchy process. Some properties of this operator and its semigroup are reviewed in Section 4.3.

Let $(P_t:t\ge 0)$ be the associated semigroup, which acts on continuous functions, and let $p(t,x)$ be its density, so that $(P_tf)(x)=\int_{\mathbb{T}}p(t,y-x)\,f(y)\,dy$ for all continuous function $f:\mathbb{T}\to\mathbb{R}$, $t\ge 0$. We present in (4.3.2) and explicit formula for $p(t,x)$.

Denote by $\mathcal{H}:\mathbb{R}\to[0,1]$ a smooth function such that $\mathcal{H}(t)=1$ for $t\le 2\pi$ and $\mathcal{H}(t)=0$ for $t\ge 4\pi$. Let $q:\mathbb{R}\times\mathbb{T}\to\mathbb{R}_+$ be given by

$$q(t,x)\;=\;p(t,x)\,\mathcal{H}(t)\,.$$

Clearly, q coincides with p on $(-\infty,2\pi]\times\mathbb{T}$, and it has support contained in $[0,4\pi]\times\mathbb{T}$.

Let $\mathfrak{v}_\varepsilon := q*\xi_\varepsilon$, $\varepsilon\ge 0$, be the centered Gaussian random field on $\mathbb{R}\times\mathbb{T}$ defined by:

$$\mathfrak{v}_\varepsilon(t,x)\;:=\;\int_{\mathbb{R}}ds\int_{\mathbb{T}}dy\;q(t-s,x-y)\,\xi_\varepsilon(s,y)\,. \tag{4.2.6}$$

Here, $\mathfrak{v}_0$, also denoted by $\mathfrak{v}$, is the Gaussian random field given by the previous formula with $\xi_0=\xi$.

Denote by Q_ε, $\varepsilon\ge 0$, the covariances of the fields $\mathfrak{v}_\varepsilon$: For z, z' in $\mathbb{R}\times\mathbb{T}$,

$$Q_\varepsilon(z,z')\;=\;\mathbb{E}\big[\,\mathfrak{v}_\varepsilon(z)\,\mathfrak{v}_\varepsilon(z')\,\big]\,.$$

A change of variables yields that $Q_\varepsilon(z,z')=Q_\varepsilon(0,z'-z)$. Denote this later quantity by $Q_\varepsilon(z'-z)$. According to Proposition 4.5.1, $\{\mathfrak{v}(z):z\in\mathbb{R}\times\mathbb{T}\}$ is a log-correlated Gaussian field. More precisely, there exists a continuous, bounded function $R:\mathbb{R}\times\mathbb{T}\to\mathbb{R}$, such that

$$Q(z)\;=\;\frac{1}{\pi}\,\log^+\frac{\pi}{2\,\|z\|}\;+\;R(z)\,,\quad z\;\in\;\mathbb{R}\times\mathbb{T}\,, \tag{4.2.7}$$

where $Q(z) =: Q_0(z)$ and $\log^+t=\max\{\log t,0\}$.

Gaussian Multiplicative Chaos

Let $X_{\gamma,\varepsilon}$, $\varepsilon>0$, $\gamma\in\mathbb{R}$, be the random field defined by

$$\langle X_{\gamma,\varepsilon}\,,\,f\rangle\;:=\;\int f(z)\,e^{\gamma\,\mathfrak{v}_\varepsilon(z)\,-\,(\gamma^2/2)\,E[\mathfrak{v}_\varepsilon(0)^2]}\,dz\,,\quad f\in C_c^\infty(\mathbb{R}\times\mathbb{T})\,. \tag{4.2.8}$$

It follows from (4.2.7), as stated in Lemma 4.5.6, that there exists a finite constant $C(\varrho)$ such that

$$E[\mathfrak{v}_\varepsilon(0)^2]\;=\;\frac{1}{\pi}\,\log\frac{1}{\varepsilon}\;+\;C(\varrho)\;+\;R(\varrho,\varepsilon)\,,$$

where $R(\varrho,\varepsilon)$ represents a remainder whose absolute value is bounded by $C_0(\varrho)\varepsilon^2$. Hence,

$$\langle X_{\gamma,\varepsilon}\,,\, f\rangle \;=\; [1 \,+\, o(1)] \int f(z)\, A(\varrho)\, \varepsilon^{\gamma^2/2\pi}\, e^{\gamma\, \mathfrak{v}_\varepsilon(z)}\, dz\,,$$

where $A(\varrho) \,=\, \exp\{-\,(\gamma^2/2)\, C(\rho)\,\}$.

Theorem 4.2.1. *Fix $0<\gamma^2 \,< 2\sqrt{2\pi}-\sqrt{6\pi}$, and $\alpha<\alpha_\gamma := \gamma^2/4\pi - 2\gamma/\sqrt{2\pi}$. Then, as $\varepsilon\to 0$, $X_{\gamma,\varepsilon}$ converges in probability in C^α to a random field, denoted by X_γ. The limit does not depend on the mollifier ϱ. Moreover, for each $p\in\mathbb{N}$, $1\le p<8\pi/\gamma^2$, then*

$$\mathbb{E}\,\big[\,|\,\langle X_\gamma\,,\, S^\delta_z f\rangle\,|^p\,\big] \;\lesssim_{p,\gamma}\; \|f\|^p_\infty\, \delta^{-p(p-1)(\gamma^2/4\pi)}$$

for every δ in $(0,1)$, $z\in\mathbb{R}\times\mathbb{T}$ and continuous function $f:\mathbb{R}\times\mathbb{T}\to\mathbb{R}$ whose support is contained in $\mathbb{B}(0,1/4)$

As $\mathfrak{v}$ is a log-correlated Gaussian field, X_γ is the so-called Gaussian multiplicative chaos (GMC), introduced by Kahane in [63].

A long-range Sinh-Gordon Equation

Due to the lack of regularity necessary to define the objects needed in equation (4.1.1) pointwise we need to provide another notion of solution. Given the mollifier ϱ used in the definition (4.2.6), we say that $\mathfrak{u}^\varrho$ is a solution of (4.1.1) if $\mathfrak{u}^\varrho$ is the limit in probability (in an appropriate Besov space) of the pointwise solutions of the renormalised and regularised equations given by (4.2.9). If the limit does not depend on the specific choice of the mollifier ϱ, we say that the solution is $\mathfrak{u}$ is unique.

Fix u_0 in $C^{\beta_0}(\mathbb{T})$ for some $\beta_0>0$, $\gamma\in\mathbb{R}$, and denote by $\mathfrak{u}_\varepsilon$, $0<\varepsilon<1$, the solution of

$$\begin{cases}\partial_t\mathfrak{u}_\varepsilon \;=\; -\,(-\Delta)^{1/2}\mathfrak{u}_\varepsilon \;-\; A(\varrho)\;\varepsilon^{\gamma^2/2\pi}\sinh(\gamma\mathfrak{u}_\varepsilon) \,+\, \xi_\varepsilon \\ \mathfrak{u}_\varepsilon(0) \;=\; u_0 \,+\, \mathfrak{v}_\varepsilon(0)\,,\end{cases} \tag{4.2.9}$$

where $\mathfrak{v}_\varepsilon(0)$ is given by (4.2.6).

Let $\alpha_\gamma := \gamma^2/4\pi - 2\gamma/\sqrt{2\pi}$, by definition, $\alpha_\gamma\in(-1/2,0)$ as long as $0<\gamma^2<2\sqrt{2\pi}-\sqrt{6\pi}$. Fix $\alpha\in(-1/2,\alpha_\gamma)$ and choose κ small enough for $0<2\kappa<1+2\alpha$. Let $\beta=\alpha+1-2\kappa$. Note that $0<\beta<1$ and $\alpha+\beta>0$.

Theorem 4.2.2. *Fix $0<\gamma^2<2\sqrt{2\pi}-\sqrt{6\pi}$, $\alpha\in(-1/2,\alpha_\gamma)$ and u_0 in $C^\beta(\mathbb{T})$. There exists an almost surely, strictly positive random variable τ, $\mathbb{P}[\tau>0]=1$, with the following property.*

For each $0<\varepsilon<1$, there exists a unique solution in $C^\beta([0,\tau]\times\mathbb{T})$ of the equation (4.2.9), denoted by $\mathfrak{u}_\varepsilon$. As $\varepsilon\to 0$, the sequence $\mathfrak{u}_\varepsilon$ converges in probability in $C^{-\kappa}([0,\tau]\times\mathbb{T})$ to a random field $\mathfrak{u}$ which does not depend on the mollifier ϱ.

The proof of Theorem 4.2.2 follow the approach proposed by [56] in the context of sine-Gordon equations, and [45] for dynamical Liouville equation. It relies on a Schauder estimate.

Remark 4.2.3. *By extending to $\mathbb{R}\times\mathbb{T}$ the theory of Gaussian multiplicative chaos, along the lines of [88], one can extend the validity of Theorems 4.2.1 and 4.2.2 to a larger range of γ. We leave this for a future work in which regularity structures will be used to extend the range up to criticality.*

A Schauder estimate for the fractional Laplacian

Let $q_z:\mathbb{R}\times\mathbb{T}\to\mathbb{R}_+$, $z=(t,x)\in(0,\infty)\times\mathbb{R}$, be given by $q_z(w)=q(z-w)$.

Theorem 4.2.4. *Fix $-1<\alpha<0$, $0<\kappa<1+\alpha$. Then,*

$$\big|\,X(q_z)\,-\,X(q_{z'})\,\big| \;\lesssim_{\kappa_0}\; \big\|\,z\,-\,z'\,\big\|^{1+\alpha-\kappa}\,\big\|\,X\,\big\|_{C^\alpha([S-4\pi,T+3\pi]\times\mathbb{T})}\,.$$

for all $X\in C^\alpha(\mathbb{R}\times\mathbb{T})$, $S<T$, $z=(t,x)$, $z'=(t',x')\in[S,T]\times\mathbb{T}$ such that $\|z-z'\|\;\le\pi/2$.

This result is one of the main novelties of this article. One of the major difficulties of the proof lies on the fact that the transition density $p(t,x)$ of the fractional Laplacian does not belong to C^1 due to the long jumps. One needs, in particular, to provide a meaning to $X(q_z)$, approximating q_z by smooth functions.

Sketch of the proof. Following Da Prato and Debussche [34], we expand the solution $\mathfrak{u}_\varepsilon(t,x)$ around the solution of the linear equation

$$\begin{cases} \partial_t \mathfrak{f}_\varepsilon \;=\; -(-\Delta)^{1/2}\, \mathfrak{f}_\varepsilon \;+\; \xi_\varepsilon \;, \quad (t,x) \;\in\; (0,\infty)\times \mathbb{T} \;, \\ \mathfrak{f}_\varepsilon(0,x) \;=\; f(x) \;, \quad x\in\mathbb{T} \;. \end{cases} \tag{4.2.10}$$

The solution of (4.2.10) can be represented in terms of the semigroup $(P_t : t \geq 0)$ of the Cauchy process as

$$\mathfrak{f}_\varepsilon(t) \;=\; \int_0^t P_{t-s}\, \xi_\varepsilon(s)\, ds \;+\; P_t f \;.$$

Recall the definition of the Gaussian field $\mathfrak{v}_\varepsilon$ introduced on (4.2.6). Comparing $\mathfrak{v}_\varepsilon$ to $\mathfrak{f}_\varepsilon$ and choosing an appropriate initial condition f yields that

$$\partial_t \mathfrak{v}_\varepsilon \;+\; (-\Delta)^{1/2}\, \mathfrak{v}_\varepsilon \;-\; \xi_\varepsilon \;=\; R_\varepsilon \;,$$

where R_ε is a smooth function with nice asymptotic properties.

By writing the solution $\mathfrak{u}_\varepsilon$ of equation (4.2.9) as $\mathfrak{v}_\varepsilon + \mathfrak{w}_\varepsilon$ yields that $\mathfrak{w}_\varepsilon$ solves the equation

$$\begin{cases} \partial_t \mathfrak{w}_\varepsilon \;=\; -(-\Delta)^{1/2}\mathfrak{w}_\varepsilon \;-\; \frac{1}{2}\, X_{\gamma,\varepsilon}\, e^{\gamma \mathfrak{w}_\varepsilon} \;+\; \frac{1}{2}\, X_{-\gamma,\varepsilon}\, e^{-\gamma \mathfrak{w}_\varepsilon} \;-\; R_\varepsilon \;, \\ \mathfrak{w}_\varepsilon(0) \;=\; u_0 \;, \end{cases} \tag{4.2.11}$$

where $X_{\gamma,\varepsilon}$ has been introduced in (4.2.8).

It is not difficult to show that the sequence of random fields $\mathfrak{v}_\varepsilon$ converges as $\varepsilon\to 0$. The proof of the convergence of $\mathfrak{u}_\varepsilon$ is thus reduced to the one of $\mathfrak{w}_\varepsilon$.

The proof of local existence and uniqueness of solutions to (4.2.11) is divided in two steps.We first show that the sequence $X_{\gamma,\varepsilon}$ converges in probability in C^α to a random field, represented by $X_{\gamma,0}$. This is the content of Theorem 4.2.1. Then, writing the solution of (4.2.11) as

$$\begin{aligned} \mathfrak{w}_\varepsilon(t) \;&=\; -\frac{1}{2}\int_0^t P_{t-s}\Big\{ e^{\gamma \mathfrak{w}_\varepsilon(s)}\, X_{\gamma,\varepsilon}(s) \;-\; e^{-\gamma \mathfrak{w}_\varepsilon(s)}\, X_{\gamma,\varepsilon}(s)\Big\}\, ds \\ &\quad - \int_0^t P_{t-s} R_\varepsilon(s)\, ds \;+\; P_t u_0 \;, \end{aligned}$$

we prove the existence and uniqueness of a fixed point for this equation, including the case $\varepsilon=0$, in an appropriate Besov space. Moreover, we show that $\mathfrak{w}_\varepsilon$ converges to $\mathfrak{w}_0$ as $\varepsilon\to$ in some Hölder space. The Schauder estimate is one of main tools here.

4.3 The Cauchy process

Remember we calculated the Green function G given in (1.5.9), which will serve as the inverse operator of $-(-\Delta)^{1/2} = -(-\Delta)^{1/2}_{\mathbb{T}}$.

The semigroup $p(t,x)$ associated to the fractional Laplacian $(-\Delta)^{1/2}$ can be computed explicitly. Denote by $\boldsymbol{p}(t,x) := p_{\bar{X}^1_t}(x)$ defined in (1.4.3) the solution on $\mathbb{R}$ of the differential equation

$$\begin{cases} \partial_t \boldsymbol{p} \;=\; -(-\Delta)^{1/2}_{\mathbb{R}}\, \boldsymbol{p} \\ p(0,x) \;=\; \delta_0(x) \;, \end{cases}$$

In the case $\alpha = 1$, we can evaluate the Fourier inverse formula to get

$$\boldsymbol{p}(t,x) \;=\; \frac{1}{\pi}\,\frac{t}{x^2+t^2} \;, \quad x\,\in\,\mathbb{R} \;,\;\; t\,>\,0 \;, \tag{4.3.1}$$

and $\boldsymbol{p}(t,x) = 0$ if $t < 0$.

In order to simplify the notation, let $p(t,x) := p_{\bar{W}^1_t}(x)$ defined in (1.4.4) be the projection of the transition probability $\boldsymbol{p}$ on the torus:

$$p(t,x) \;=\; \boldsymbol{p}(t,x) \;+\; p_\star(t,x) \;:=\; \boldsymbol{p}(t,x) \;+\; \sum_{k\neq 0} \boldsymbol{p}(t,x+2\pi x')\;, \quad x\in\mathbb{T}\;, \tag{4.3.2}$$

where the last sum is performed over all integers $k\in\mathbb{Z}$ different from 0. An elementary computation shows that the function p is smooth in its domain of definition $(0,\infty)\times\mathbb{T}$.

Although p is smooth as a function defined on the torus, this is not the case of $p_\star$. However, if we assume that $p_\star$ is defined on $(0,\infty)\times(-\pi-\kappa,\pi)$ for some $0<\kappa<\pi/2$, it is not difficult to show that this function is smooth on $(0,\infty)\times(-\pi-\kappa,\pi)$ and that it is uniformly bounded, as well as its derivatives: For all j, $k\geq 0$, we have

$$\sup_{z\in(0,\infty)\times(-\pi-\kappa,\pi)} \big|\, \partial^j_t\, \partial^k_x\, p_\star(z)\,\big| \;\lesssim_{j,k}\; 1\;.$$

Let $\mathcal{A}$ be the annulus on $\mathbb{R}^2$ given by $\mathcal{A} = \{(t,x)\in\mathbb{R}^2 : 1/2 < t^2+x^2 < 2\}$, and set $\mathcal{A}_n = \{(t,x)\in\mathbb{R}^2 : (2^n t, 2^n x)\in\mathcal{A}\}$, $\mathcal{A}^+_n = \{(t,x)\in(0,\infty)\times\mathbb{R} : (t,x)\in\mathcal{A}_n\}$. It follows from the previous estimates on p and elementary computations that for all $j\geq 0$, $k\geq 0$ we have that for all $n\geq 2$,

$$\sup_{z\in\mathcal{A}^+_n} \big|\, \partial^j_t\, \partial^k_x\, p(z)\,\big| \;\lesssim_{j,k}\; 2^{(1+j+k)n}\;. \tag{4.3.3}$$

It is also not difficult to show from (4.3.2) that

$$\big|\, \partial_t p\,(z)\,\big| \;\lesssim\; \frac{1}{\|z\|}\;, \quad \big|\, \partial_x p\,(z)\,\big| \;\lesssim\; \frac{1}{\|z\|}\;, \quad p\,(z) \;\lesssim\; \Big\{ 1 \,+\, \frac{1}{\|z\|} \Big\} \tag{4.3.4}$$

for all $z=(t,x)$ such that $t>0$.

Recall from Section 4.2 that we denote by $(P_t : t\geq 0)$ the semigroup associated to the generator $-(-\Delta)^{1/2}_{\mathbb{R}}$: P_t acts on continuous functions $f:\mathbb{T}\to\mathbb{R}$ as $(P_t f)(x) = \int_{\mathbb{T}} p(t,y-x)\, f(y)\, dy$, where p is the transition density introduced in (4.3.2).

Denote by $(\bar{X}^1_t : t\geq 0)$ the Cauchy process. This is the Markov process on $\mathbb{R}$ which starts from the origin and whose semigroup is given by $\boldsymbol{p}$, introduced in (4.3.1).

Lemma 4.3.1. *Fix $\beta\in(0,1)$, for all $T>0$, $u\in C^\beta(\mathbb{T})$,*

$$\|\, P_t\, u\,\|_{C^\beta([0,T]\times\mathbb{T})} \lesssim_\beta \;\|\, u\,\|_{C^\beta(\mathbb{T})}\;.$$

Proof. Fix $u\in C^\beta(\mathbb{T})$, $T>0$, x, $y\in\mathbb{T}$ and $0\leq s<t\leq T$. Remember that $\bar{W}^1_t$ denotes the projection of $\bar{X}^1_t$ in $\mathbb{T}$, we then have

$$\big|\,(P_t u)(x) \,-\, (P_s u)(y)\,\big| \;=\; \Big|\mathbb{E}\,\big[\,u(x+\bar{W}^1_t) \,-\, u(y+\bar{W}^1_s)\,\big]\,\Big|\;,$$

where $\mathbb{E}$ represents the expectation with respect to the Cauchy process $\bar{X}^1_t$. Recall that we represent by $|\cdot|$ the distance on the torus, although is not a norm. The previous expression is bounded by

$$\|u\|_{C^\beta(\mathbb{T})}\,\mathbb{E}\,\big[\,|\,x+\mathbb{Z}_t - y - \mathbb{Z}_s\,|^\beta\,\big] \;\leq\; \|u\|_{C^\beta(\mathbb{T})} \Big\{\,|\,x-y\,|^\beta \;+\; \mathbb{E}\,\big[\,|\,\bar{W}^1_t \,-\, \bar{W}^1_s\,|^\beta\,\big]\,\Big\}$$

because $(a+b)^\theta \leq a^\theta + b^\theta$ for a, $b>0$, $0<\theta<1$. Since $|\,\bar{W}^1_t - \bar{W}^1_s\,| \,\leq\, |\,\bar{X}^1_t - \bar{X}^1_s\,|$, the increments are stationary and the process is self-similar,

$$\mathbb{E}\,\big[\,|\,\bar{W}^1_t \,-\, \bar{W}^1_s\,|^\beta\,\big] \;\leq\; \mathbb{E}\,\big[\,|\,\bar{X}^1_t \,-\, \bar{X}^1_s\,|^\beta\,\big] \;=\; \mathbb{E}\,\big[\,|\,\bar{X}^1_{t-s}\,|^\beta\,\big] \;=\; \mathbb{E}\,\big[\,|\,(t-s)\bar{X}^1_1\,|^\beta\,\big]\;.$$

The right-hand side of the penultimate formula is thus bounded above by

$$\|u\|_{C^\beta(\mathbb{T})} \Big\{\,|\,x-y\,|^\beta \;+\; C_0\,(t-s)^\beta\,\Big\}\;,$$

where $C_0 = \mathbb{E}\,[\,|\,\bar{X}^1_1\,|^\beta\,]$. This completes the proof of the lemma. □

4.4 A Schauder estimate

We prove in this section a Schauder estimate for the kernel $p(t,x)$ of the fractional Laplacian on the torus. We follow the approach based on the homogeneity of the kernel under scaling, in the sense that $\boldsymbol{p}(t/\delta, x/\delta) = \delta\, \boldsymbol{p}(t,x)$ for all $(t,x) \in \mathbb{R}^2 \setminus \{0\}$, $\delta > 0$, cf. [94, 52]. However, on the torus, the kernel is not homogeneous, and, more importantly, due to the non-locality of the generator, the transition density $p(t,x)$ is not C^1 at $t = 0$. In particular, it does not belong to the domain of the distributions in C^α, and a plethora of arguments and bounds are needed to define and bound the main quantities such as $X(p)$.

Denote by $\mathcal{H} : \mathbb{R} \to [0,1]$ a smooth function such that $\mathcal{H}(t) = 1$ for $t \le 2\pi$ and $\mathcal{H}(t) = 0$ for $t \ge 4\pi$. Let $q : \mathbb{R} \times \mathbb{T} \to \mathbb{R}_+$ be given by

$$q(t,x) \;=\; p(t,x)\, \mathcal{H}(t)\;. \tag{4.4.1}$$

Clearly, q coincides with p on $(-\infty, 2\pi] \times \mathbb{T}$, it has support contained in $[0, 4\pi] \times \mathbb{T}$, it belongs to $C^2(\Omega_{0\bullet})$ and for every $t_0 > 0$, $m \ge 1$, we have

$$\| q \|_{C^m(\Omega_{t_0 \bullet})} \;\lesssim_m\; \| p \|_{C^m(\Omega_{t_0\bullet})}\;.$$

Here and below, for $s < t$, $\Omega_{s,t} = (s,t) \times \mathbb{T}$, $\Omega_{\bullet t} = (-\infty, t) \times \mathbb{T}$, $\Omega_{t\bullet} = (t, \infty) \times \mathbb{T}$.

Let $q_z : \mathbb{R} \times \mathbb{T} \to \mathbb{R}_+$, $z = (t,x) \in \Omega_{0\bullet}$, be given by $q_z(w) = q(z - w)$. The main result of this section reads as follows.

Theorem 4.4.1. *Fix $-1 < \alpha < 0$, $0 < \kappa < 1 + \alpha$. Then,*

$$\big|\, X(q_z) \,-\, X(q_{z'})\,\big| \;\lesssim_\kappa\; \|\, z \,-\, z'\, \|^{1+\alpha-\kappa}\, \|\, X\,\|_{C^\alpha([S-4\pi, T+3\pi]\times\mathbb{T})}\;.$$

for all $X \in C^\alpha(\mathbb{R}\times\mathbb{T})$, $S < T$, $z = (t,x)$, $z' = (t', x') \in [S,T] \times \mathbb{T}$ such that $\|z - z'\| \;\le \pi/2$.

Corollary 4.4.2. *Fix $-1 < \alpha < 0$, $0 < \kappa < 1 + \alpha$. Then,*

$$\|u\|_{C^{1+\alpha-\kappa}([S,T]\times\mathbb{T})} \;\lesssim_\kappa\; \|\, X\,\|_{C^\alpha([S-4\pi, T+3\pi]\times\mathbb{T})}\;.$$

for all $X \in C^\alpha(\mathbb{R}\times\mathbb{T})$, $S < T$, where $u = u : \mathbb{R}\times\mathbb{T} \to \mathbb{R}$ is given by $u(z) = X(q_z)$.

Part of the proof of Theorem 4.4.1 consists in giving a meaning to $X(q_z)$ since, as pointed our earlier, q_z does not belong to the domain of a distribution in C^α. We start with a simple estimate on C^α.

Lemma 4.4.3. *Fix $\alpha < 0$ and let $m = -\lfloor \alpha \rfloor$. For all $a < b$, $S < T$, $0 < \delta \le 1$, $z \in \Omega_{S,T}$, and function g in $C^m(\mathbb{R}\times\mathbb{T})$ whose support is contained in $\Omega_{a,b}$,*

$$\big|\, \langle X \,,\, S^\delta_z g\rangle\,\big| \;\lesssim\; (1 + b - a)\, \delta^\alpha\, \|g\|_{C^m(\mathbb{R}\times\mathbb{T})}\, \|X\|_{C^\alpha([S+a-\pi/2, T+b+\pi/2]\times\mathbb{T})}\;.$$

Proof. It follows from the definition of the seminorms $\|X\|_{C^\alpha([S,T]\times\mathbb{T})}$, introduced in (4.2.3), that for all functions g in $C^m(\mathbb{R}\times\mathbb{T})$ whose support is contained in $\mathbb{B}(0, \pi/2)$, every $0 < \delta \le 1$, $z \in [S,T] \times \mathbb{T}$.

$$\big|\, \langle X \,,\, S^\delta_z g\rangle\,\big| \;\le\; \delta^\alpha\, \|g\|_{C^m(\mathbb{R}\times\mathbb{T})}\, \|X\|_{C^\alpha([S,T]\times\mathbb{T})}\;. \tag{4.4.2}$$

For each $p \in \mathbb{N}$, there exists a function φ in $C^p(\mathbb{R}^2)$ whose support is contained in $\mathbb{B}(0, \pi/2)$ and such that

$$\sum_{j\in\mathbb{Z}} \sum_{k=0}^{7} \varphi_{j,k}(t,x) \;=\; 1 \quad \text{for all} \;\; (t,x) \;\in\; \mathbb{R}\times\mathbb{T}\;,$$

where $\varphi_{j,k}(t,x) = \varphi(\,t - (\pi j/4)\,,\, x - (\pi k/4)\,)$.

Fix $p \ge m$ and write g as $\sum_{j\in\mathbb{Z}} \sum_{0\le k\le 7} g_{j,k}$, where $g_{j,k} \;=\; g\, \varphi_{j,k}$. Since the support of g is contained in $[a,b] \times \mathbb{T}$, in the previous sum there are at most $B_0(1 + b - a)$ terms which do not vanish, for some finite constant B_0. Moreover, for $0 < \delta \le 1$, $z \in [S,T] \times \mathbb{T}$,

$$\big|\, \langle X \,,\, S^\delta_z g\rangle\,\big| \;\le\; \sum_{j,k} \big|\, \langle X \,,\, S^\delta_z g_{j,k}\rangle\,\big|\;.$$

where the sum is performed over the non-vanishing terms. We may write $g_{j,k}$ as $S^1_{z_{j,k}} S^1_{-z_{j,k}} g_{j,k}$, where $z_{j,k} = (\pi j/4\,,\,\pi k/4)$. Let $F_{j,k} = S^1_{-z_{j,k}} g_{j,k}$, and note that the support of $F_{j,k}$ is contained in $\mathbb{B}(0,\pi/2)$. Since $S^\delta_z S^1_{z_{j,k}} F_{j,k} = S^\delta_{z+\delta\, z_{j,k}} F_{j,k}$, the right-hand side of the previous displayed equation is equal to

$$\sum_{j,k} \big|\, \langle X\,,\, S^\delta_{z+\delta\, z_{j,k}} F_{j,k}\rangle \,\big|\,.$$

By (4.4.2), this sum is less than or equal to

$$\delta^\alpha \sum_{j,k} \|F_{j,k}\|_{C^m(\mathbb{R}\times\mathbb{T})}\, \|X\|_{C^\alpha([S+a-\pi/2,T+b+\pi/2]\times\mathbb{T})}$$

because $z+\delta\, z_{j,k} \in [S+a-\pi/2, T+b+\pi/2]\times\mathbb{T}$ for all (j,k) for which $F_{j,k}$ does not vanish. Since $\|F_{j,k}\|_{C^m(\mathbb{R}\times\mathbb{T})} \lesssim_\varphi \|g\|_{C^m(\mathbb{R}\times\mathbb{T})}$, to complete the proof of the lemma, it remains to recall that are at most $B_0(1+b-a)$ non-vanishing terms in the sum. □

As mentioned in the introduction of this section, the kernel $p(t,x)$ does not belong to $C^1(\mathbb{R}\times\mathbb{T})$. In particular, if P_z, $z\in(0,\infty)\times\mathbb{T}$, stands for the functions defined by $P_z(w) = p(z-w)$, $X(P_z)$ is not defined for distributions X in $C^\alpha(\mathbb{R}\times\mathbb{T})$, $-1<\alpha<0$. The next lemmata provide sufficient conditions which permit defining $X(P_z)$ as a limit.

Let Ω be an open set of $\mathbb{R}\times\mathbb{T}$ and let $f:\Omega\to\mathbb{R}$ be a continuously differentiable function. We denote by $\|\,f\,\|_{C^1(\Omega)}$ the norm defined by

$$\|\,f\,\|_{C^1(\Omega)} \;=\; \sum_{j,k} \|\,\partial^j_x\,\partial^k_t\, f\,\|_{L^\infty(\Omega)}\;,$$

where the sum is carried out over all j, k in $\mathbb{N}_0$ such that $j+k\le 1$.

Let $\varphi:\mathbb{R}_+\to[0,1]$ be the germ of a dyadic partition of the unity: φ is a smooth function such that

$$\varphi(r) \;=\; 0 \;\text{ if }\; r\not\in(\pi/8\,,\,\pi/2)\;, \qquad \sum_{n\in\mathbb{Z}} \varphi(2^n r) \;=\; 1 \;\text{ for }\; r>0\;. \tag{4.4.3}$$

We refer to [7, Proposition 2.10] for the existence of φ. Let $\varphi_n(r) = \varphi(2^n r)$. Note that the supports of φ_n and φ_m are disjoints whenever $|n-m|\ge 2$.

Let $\psi:\mathbb{R}\to[0,1]$ be a symmetric, smooth function whose support is contained in $(-\pi\,,\,\pi)$ and such that

$$\sum_{k\in\mathbb{Z}} \psi(x\,-\,\pi k) \;=\; 1\;, \quad x\;\in\;\mathbb{R}\;. \tag{4.4.4}$$

Let $\psi_n(x) = \psi(2^n x)$, $n\ge 1$. Consider ψ, ψ_n as defined on $\mathbb{R}\times\mathbb{T}$ and depending only on the second coordinate. We abuse of notation below and denote by $k/2^n$ the element $(0,k/2^n)$ of $\mathbb{R}\times\mathbb{T}$. Note that

$$1 \;=\; \sum_{k=-2^n+1}^{2^n} \psi(2^n x\,-\,\pi k) \;=\; \sum_{k=-2^n+1}^{2^n} \psi_n(x\,-\,\pi k 2^{-n}) \tag{4.4.5}$$

for all $x\in\mathbb{T}=[-\pi,\pi)$.

Lemma 4.4.4. *Fix $-1<\alpha<0$ and a continuously differentiable function $f:(0,\infty)\times\mathbb{T}\to\mathbb{R}$. Assume that there exists $T_1<\infty$ such that $f(t,x)=0$ for $t\ge T_1$ and that $\|\,f\,\|_{C^1(\Omega_{0\bullet})}<\infty$. Let $f_n(t,x) = f(t,x)\,\varphi_n(t)$, $n\in\mathbb{N}_0$. Then,*

$$\big|\,X(\,f_n\,)\,\big| \;\lesssim\; 2^{-n(1+\alpha)}\|\,f\,\|_{C^1(\Omega_{0\bullet})}\,\big\|X\big\|_{C^\alpha([-\pi/2,\pi]\times\mathbb{T})}$$

for all X in C^α, $n\in\mathbb{N}_0$.

Proof. For each $n\ge 0$, the function f_n belongs to $C^1_c(\mathbb{R}\times\mathbb{T})$ and its support is contained in $[\pi 2^{-(n+3)}, \pi 2^{-(n+1)}]\times\mathbb{T}$. In particular, $X(f_n)$ is well-defined.

Recall the definition of ψ introduced in (4.4.4). By (4.4.5),

$$f_n(s,y) \;=\; \sum_{k=-2^n+1}^{2^n} f(s,y)\,\varphi_n(s)\,\psi_n(x\,-\,\pi k 2^{-n})\,.$$

Let $H_n : \mathbb{R}\times\mathbb{T}\to\mathbb{R}$ be given by

$$H_n(s,y) \;=\; 2^{-2n}\, f\big(2^{-n}(s,y) \;+\; (0,\pi k 2^{-n})\big)\,\varphi(s)\,\psi(y)\;,$$

so that $(S^{2^{-n}}_{(0,\pi k2^{-n})}H_n)(s,y) = f(s,y)\varphi_n(s)\,\psi_n(x\,-\,\pi k 2^{-n})$. In particular,

$$X(f_n) \;=\; \sum_{k=-2^n+1}^{2^n} X\big(\,S^{2^{-n}}_{(0,\pi k2^{-n})}H_n\,\big)\;. \tag{4.4.6}$$

The function H_n belongs to $C^1_c(\mathbb{R}\times\mathbb{T})$, it has support contained in $[\pi/8,\pi/2]\times\mathbb{T}$, and $\|H_n\|_{C^1}\le 2^{-2n}\|f\|_{C^1(\Omega_{0\bullet})}$. Therefore, by Lemma 4.4.3,

$$\big|\,X(\,f_n\,)\,\big| \;\lesssim\; 2^n\,2^{-2n}\,2^{-n\alpha}\|\,f\,\|_{C^1(\Omega_{0\bullet})}\,\big\|X\big\|_{C^\alpha([-\pi/2,\pi]\times\mathbb{T})}.$$

The factor 2^n comes from the number of terms in the sum over k. □

Remark 4.4.5. *One could be tempted to define H_n as $H_n(s,y) = 2^{-2n}f(2^{-n}(s,y))\,\varphi(s)$. But this function is not periodic. This is the reason for introducing ψ_n.*

Let $f:(0,\infty)\times\mathbb{T}\to\mathbb{R}$ be a function which satisfies the assumptions of Lemma 4.4.4 and whose support is contained in $[0,T_1]\times\mathbb{T}$. Set

$$\Upsilon(s) \;=\; \sum_{n\ge 0}\varphi_n(s)\;,$$

and write f as $f = f^{(0)} + f^{(1)}$, where $f^{(0)}(t,x) = f(t,x)\,\Upsilon(t)$, $f^{(1)}(t,x) = f(t,x)\,[1-\Upsilon(t)]$. In view of Lemma 4.4.4, we may define $X(f^{(0)})$ as $\sum_{n\ge0}X(f_n)$. On the other hand, $f^{(1)}$ belongs to $C^1_c(\mathbb{R}\times\mathbb{T})$ and $X(f^{(1)})$ is well-defined. Moreover, since the support of f is contained in $[0,T_1]\times\mathbb{T}$, by Lemma 4.4.3 with $\delta=1$ and $z=0$,

$$\big|\,X(f^{(1)})\,\big| \;\lesssim\; (1+T_1)\,\|\,f\,\|_{C^1(\Omega_{0\bullet})}\,\big\|X\big\|_{C^\alpha([-\pi/2,T_1+\pi/2]\times\mathbb{T})}\;. \tag{4.4.7}$$

We summarize these observations in the next result.

Corollary 4.4.6. *Fix $-1<\alpha<0$. Let $f:(0,\infty)\times\mathbb{T}\to\mathbb{R}$ be a function which satisfies the assumptions of Lemma 4.4.4. Assume that the support of f is contained in $[0,T_1]\times\mathbb{T}$. Define $X(f)$ as*

$$X(f) \;=\; \sum_{n\ge0}X(f_n) \;+\; X(f^{(1)})\;.$$

Then,

$$|X(f)| \;\lesssim\; (1+T_1)\,\|\,f\,\|_{C^1(\Omega_{0\bullet})}\,\big\|X\big\|_{C^\alpha([-\pi/2,T_1+\pi/2]\times\mathbb{T})}\;.$$

Proof. This result follows from (4.4.7) and from Lemma 4.4.4 which asserts that

$$\begin{aligned}\sum_{n\ge0}\big|\,X(\,f_n\,)\,\big| \;&\lesssim\; \sum_{n\ge0}2^{-n(1+\alpha)}\,\|\,f\,\|_{C^1(\Omega_{0\bullet})}\,\big\|X\big\|_{C^\alpha([-\pi/2,T_1+\pi/2]\times\mathbb{T})}\\ &\lesssim\; \|\,f\,\|_{C^1(\Omega_{0\bullet})}\,\big\|X\big\|_{C^\alpha([-\pi/2,T_1+\pi/2]\times\mathbb{T})}\end{aligned}$$

where we used the fact that $\alpha>-1$. □

An elementary computation yields that for all z, z' in $\mathbb{R}\times\mathbb{T}$, δ, δ' in $(0,1]$ and continuous functions f,

$$S^\delta_z\,S^{\delta'}_{z'}\,f \;=\; S^{\delta\,\delta'}_{z+\delta z'}\,f\;. \tag{4.4.8}$$

Corollary 4.4.7. *Fix $-1<\alpha<0$. Let $f:(0,\infty)\times\mathbb{T}\to\mathbb{R}$ be a function which satisfies the hypotheses of Lemma 4.4.4. Assume that the support of f is contained in $[0,T_1]\times\mathbb{T}$. Then,*

$$\big|\,X\big(\,S^\delta_z f\,\big)\,\big| \;\lesssim\; (1+T_1)\,\delta^\alpha\,\|\,f\,\|_{C^1(\Omega_{0\bullet})}\,\big\|X\big\|_{C^\alpha([S-\pi/2,T+T_1+\pi/2]\times\mathbb{T})}$$

for all $S<T$, X in C^α, $z\in[S,T]\times\mathbb{T}$, $0<\delta\le1$.

Proof. Recall the decomposition of f as $\sum_{n\geq 0} f_n + f^{(1)}$ introduced in Corollary 4.4.6. Since $f^{(1)}$ belongs to $C^1_c(\mathbb{R}\times\mathbb{T})$, by Lemma 4.4.3,

$$\big|\, X\big(\, S^\delta_z f^{(1)}\,\big)\,\big| \;\lesssim\; (1+T_1)\,\delta^\alpha\,\|\, f\,\|_{C^1(\Omega_{0\bullet})}\,\big\|X\big\|_{C^\alpha([S-\pi/2,T+T_1+\pi/2]\times\mathbb{T})}$$

We turn to $X(\, S^\delta_z f_n\,)$. By (4.4.6) and (4.4.8),

$$X(\, S^\delta_z f_n\,) \;=\; \sum_{k=-2^n+1}^{2^n} X\big(\, S^\delta_z\, S^{2^{-n}}_{(0,k2^{-(n+1)})} H_n\,\big) \;=\; \sum_{k=-2^n+1}^{2^n} X\big(\, S^{\delta\, 2^{-n}}_{z+\delta(0,k2^{-(n+1)})} H_n\,\big)\;.$$

The function H_n belongs to $C^1_c(\mathbb{R}\times\mathbb{T})$, its support is contained in $[\pi/8,\pi/2]\times\mathbb{T}$, and $\|H_n\|_{C^1}\leq 2^{-2n}\|f\|_{C^1(\Omega_{0\bullet})}$. Thus, by Lemma 4.4.3,

$$\big|\, X(\, S^\delta_z f_n\,)\,\big| \;\lesssim\; \sum_{k=-2^n+1}^{2^n} \delta^\alpha\, 2^{-n\alpha}\, 2^{-2n}\,\|\, f\,\|_{C^1(\Omega_{0\bullet})}\,\big\|X\big\|_{C^\alpha([S-\pi/2,T+\pi]\times\mathbb{T})}.$$

As $\alpha > -1$, summing over n we get that

$$\sum_{n\geq 0} \big|\, X(\, S^\delta_z f_n\,)\,\big| \;\lesssim\; \delta^\alpha\,\|\, f\,\|_{C^1(\Omega_{0\bullet})}\,\big\|X\big\|_{C^\alpha([S-\pi/2,T+\pi]\times\mathbb{T})}\;,$$

which completes the proof of the corollary. □

Remark 4.4.8. *The set $\{(t,x) : t=0\}$ plays no role in the proof of the previous results. A similar statement holds for functions f which are smooth on the set $\Omega_{\bullet t_0}\cup\Omega_{t_0\bullet}$, $t_0\in\mathbb{R}$. The result also applies to functions which are smooth on sets of the form $\Omega_{\bullet t_0}\cup\Omega_{t_0,t_1}\cup\Omega_{t_1,\bullet}$. For example, in the next lemma, for a function f given by $f(w)=g(w-z)-g(w-z')$, where g, fulfills the assumptions of Lemma 4.4.4.*

Lemma 4.4.9. *Fix a function f satisfying the assumptions of Lemma 4.4.4, and assume that its support is contained in $[0,T_1]\times\mathbb{T}$. Let $g_z:\mathbb{R}\times\mathbb{T}\to\mathbb{R}$ be the function given by $g_z(w)=f(w-z)-f(w)$, where $z=(t_0,x_0)\in(0,\infty)\times\mathbb{T}$ is such that $\|z\|\leq\pi/2$. Then,*

$$\big|\, X\big(\, g_z\,\big)\,\big| \;\lesssim\; (1+T_1)\,\|z\|^{1+\alpha}\,\|\, f\,\|_{C^2(\Omega_{0\bullet})}\,\big\|X\big\|_{C^\alpha([-19\pi/8,19\pi/8+T_1/2)]\times\mathbb{T})}$$

for all X in C^α.

Note that on the right-hand side we have the norm of f in $C^2(\Omega_{0\bullet})$. This is not a misprint. It comes from the fact that we estimate the L^∞ norm of $(\partial_t f)(w-z)-(\partial_t f)(w)]$ by $\|z\|\,\{\,\|\,\partial^2_t f\,\|_{L^\infty(\Omega_{0\bullet})}+\|\,\partial^2_{t,x} f\,\|_{L^\infty(\Omega_{0\bullet})}\,\}$.

Proof of Lemma 4.4.9. Note that g_z is a continuously-differentiable function on $\Omega_{0,t_0}\cup\Omega_{t_0\bullet}$ which vanishes on $\Omega_{\bullet 0}$. It might be discontinuous at $t=0$ and $t=t_0$. Using the dyadic partition of the unity, we estimate separately $X(g_z)$ in the regions $\Omega_{kr,(k+1)r}$, $0\leq k\leq 2$ and $\Omega_{t_0\bullet}$, where $r=t_0/3$.

We start with the first region, $\Omega_{0,r}$. The argument is similar to the one presented in the proof of Lemma 4.4.4. Let $n_1\in\mathbb{Z}$ such that $\pi 2^{-n_1+1}\leq t_0<\pi 2^{-n_1+2}$, and denote by $A(s,y)$ the function given by

$$A(s,y) \;=\; \sum_{n\geq n_1} \varphi_n(s)\, g_z(s,y)\;.$$

Note that $A(s,y)=0$ if $s\geq t_0/4$.

Recall the definition of the function ψ introduced in (4.4.4). By (4.4.5), $X(A)$ can be written as

$$\sum_{n\geq n_1} X\big(\,\varphi_n\, g_z\,\big) \;=\; \sum_{n\geq n_1}\sum_{k=-2^n+1}^{2^n} X\big(\,\psi_n(y-\pi k 2^{-n})\,\varphi_n(s)\, g_z(s,y)\,\big)\;.$$

Let $H_n : \mathbb{R}\times\mathbb{T}\to\mathbb{R}$ be given by $H_n(s,y) = 2^{-2n} g_z(2^{-n}(s,y)+(0,\pi k/2^n))\,\varphi(s)\,\psi(y)$ so that $(S^{2^{-n}}_{(0,\pi k/2^n)} H_n)(s,y) = \psi_n(y-\pi k 2^{-n})\,\varphi_n(s)\,g_z(s,y)$. In particular, the previous sum is equal to

$$\sum_{n\ge n_1}\sum_{k=-2^n+1}^{2^n} X\big(\,S^{2^{-n}}_{(0,\pi k/2^n)} H_n\,\big)\ .$$

The function H_n belongs to $C^1_c(\mathbb{R}\times\mathbb{T})$, it has support contained in $[\pi/8,\pi/2]\times\mathbb{T}$, and $\|H_n\|_{C^1}\le 2^{-2n}\|f\|_{C^1(\Omega_{0\bullet})}$. Therefore, by Lemma 4.4.3, as $\alpha>-1$,

$$\begin{aligned}\big|\,X(\,A\,)\,\big| \;&\lesssim\; \sum_{n\ge n_1} 2^{-n(1+\alpha)}\|\,f\,\|_{C^1(\Omega_{0\bullet})}\,\big\|X\big\|_{C^\alpha([-\pi/2,\pi]\times\mathbb{T})}\\ &\lesssim\; 2^{-n_1(1+\alpha)}\|\,f\,\|_{C^1(\Omega_{0\bullet})}\,\big\|X\big\|_{C^\alpha([-\pi/2,\pi]\times\mathbb{T})}.\end{aligned}$$

By definition of n_1, $2^{-n_1(1+\alpha)}\le t_0^{1+\alpha}\le\|z\|^{1+\alpha}$. This completes the proof of the first estimate.

We turn to the second one, $\Omega_{t_0/3,2t_0/3}$, squeezed between the first and the third regions. Let $\Upsilon_1(s)=\sum_{n\ge n_1}\varphi_n(s)$, $\Upsilon_2(s)=\sum_{n\ge n_1}\varphi_n(t_0-s)$. We need to estimate $\tilde g_z = g_z[1-\Upsilon_1-\Upsilon_2] = -f\,[1-\Upsilon_1-\Upsilon_2]$.

At the beginning of the proof, we pointed out that the support of Υ_1 is contained in $[0,t_0/4]$. On the other hand, since the supports of φ_n and φ_m are disjoints whenever $|n-m|\ge 2$, $\Upsilon_1(s)=1$ for $0<s\le t_0/32$. A similar result holds for Υ_2, so that the support of $\tilde g_z$ is contained in $[a,b]\times\mathbb{T}$, where $a=t_0/32$, $b=(31/32)t_0$. In this set the function f is C^1, which implies that $\tilde g_z$ belongs to $C^1_c(\mathbb{R}\times\mathbb{T})$.

Recall that $\pi 2^{-n_1+1}\le t_0<\pi 2^{-n_1+2}$ and write $\tilde g_z$ as

$$\begin{aligned}\tilde g_z(s,y) \;&=\; \sum_{k=-2^{n_1}+1}^{2^{n_1}} \psi_{n_1}(y-\pi k 2^{-n_1})\;\tilde g_z(s,y)\\ &:=\; \sum_{k=-2^{n_1}+1}^{2^{n_1}} (S^{2^{-n_1}}_{(t_0/2,\pi k/2^{n_1})}\tilde H_z)(s,y)\ .\end{aligned}$$

From the last identity we get that $\tilde H_z(t,x) = 2^{-2n_1}\psi(x)\tilde g_z(t_0/2+t/2^{n_1},\pi k/2^{n_1}+\pi x/2^{n_1})$. The support of $\tilde H_z$ is thus contained in $[-15\pi/8,15\pi/8]\times\mathbb{T}$ and $\|\tilde H_z\|_{C^1_c(\mathbb{R}\times\mathbb{T})}\le 2^{-2n_1}\|f\|_{C^1(\Omega_{0\bullet})}$. In this later estimate, observe that the time derivative of Υ_1 is of order t_0^{-1}. These bounds and Lemma 4.4.3 yield that

$$\begin{aligned}\big|\,X(\,\tilde g_z\,)\,\big| \;&\le\; \sum_{k=-2^{n_1}+1}^{2^{n_1}} \big|\,X(\,S^{2^{-n_1}}_{(t_0/2,\pi k/2^{n_1})}\tilde H_z\,)\,\big|\\ &\lesssim\; 2^{-n_1(1+\alpha)}\|\,f\,\|_{C^1(\Omega_{0\bullet})}\,\big\|X\big\|_{C^\alpha([-19\pi/8,19\pi/8+(t_0/2)]\times\mathbb{T})}\ .\end{aligned}$$

This provides a bound for the second region since $2^{-n_1}\le t_0\le\|z\|$.

We estimate $X(\,g_z\,)$ in the third region, $\Omega_{2t_0/3,t_0}$, as in the first one. It remains to consider the set $\Omega_{t_0\bullet}$. The result follows from Corollary 4.4.6, Remark 4.4.8 and the fact that the L^∞ norm of $g_z(w)=f(w-z)-f(w)$ is bounded by $\{\,\|\partial_t f\|_{L^\infty(\Omega_{0\bullet})}+\|\partial_x f\|_{L^\infty(\Omega_{0\bullet})}\,\}\,\|z\|$. A similar inequality holds for the L^∞ norm of the first derivatives of g_z. This requires f to be in C^2 and provides an estimate of $\|g_z\|_{C^1(\Omega_{t_0\bullet})}$ in terms of $\|z\|\,\|f\|_{C^2(\Omega_{0\bullet})}$. □

We turn to the proof of the Schauder estimate. Recall the definition of the kernel q introduced at the beginning of this section. The function q is smooth in $(0,\infty)\times\mathbb{T}$, it diverges at the origin and is not C^1 at $t=0$. For $n\ge 0$, let $q_n:\mathbb{R}_+\times\mathbb{T}\to\mathbb{R}_+$ be given by $q_n(z)=\phi_n(z)\,q(z)$, where $\phi_n(z)=\varphi_n(|z|)$.

Let $q_{n,z}:\mathbb{R}\times\mathbb{T}\to\mathbb{R}$, $z\in\mathbb{R}\times\mathbb{T}$, $n\in\mathbb{N}_0$, be given by

$$q_{n,z}(w) \;=\; q_n(z-w) \;=\; \phi_n(z-w)\,q(z-w)\ .$$

The function $q_{n,z}$ fulfills the assumptions of Lemma 4.4.4. Hence, by Corollary 4.4.6, we may define $X(q_{n,z})$. The next lemma provides a bound for this quantity.

Lemma 4.4.10. *For all $X \in C^\alpha$, $S < T$, $z = (t,x)$ in $[S,T] \times \mathbb{T}$ and $n \ge 0$,*

$$|\, X(q_{n,z}) \,| \;\lesssim\; 2^{-(1+\alpha)n} \, \|\, X \,\|_{C^\alpha([S-\pi, T+\pi]\times\mathbb{T})} \,.$$

Proof. Let $\delta = 2^{-n} \le 1$, and $g : \mathbb{R}\times\mathbb{T} \to \mathbb{R}$ be given by $g(w) = q_n(-\delta w)$. Since ϕ is symmetric and $\delta = 2^{-n}$, $g(w) = \phi(w)\, q(-\delta w)$. In particular, the support of g is contained in $\mathbb{B}(0,\pi/2)$. As q and p coincide on $\mathbb{B}(0,\pi/2)$, $g(w) = \phi(w)\, p(-\delta w)$. Hence, $g(s,y) = 0$ for $s \ge 0$, g satisfies the assumptions of Lemma 4.4.4, and, by (4.3.3), we have

$$\|g\|_{C^1((-\infty,0)\times\mathbb{T})} \;\lesssim\; \delta^{-1} \,.$$

On the other hand, an elementary computation yields that $\delta^2\, (S^\delta_z g)(w) = q_{n,z}(w)$. Hence, by Corollary 4.4.7 and Remark 4.4.8, there exists a finite constant C_0 such that

$$\begin{aligned} \big|\, X\big(\, q_{n,z}\,\big)\,\big| \;&=\; \delta^2 \,\big|\, X\big(\, S^\delta_z g\,\big)\,\big| \\ &\lesssim\; \delta^{2+\alpha}\, \|g\|_{C^1((-\infty,0)\times\mathbb{T})}\, \|X\|_{C^\alpha([S-\pi,T+\pi]\times\mathbb{T})} \\ &\lesssim\; 2^{-(1+\alpha)n}\, \|X\|_{C^\alpha([S-\pi,T+\pi]\times\mathbb{T})} \end{aligned}$$

for all $n \ge 0$, $S < T$, $z = (t,x)$ in $[S,T]\times\mathbb{T}$. This completes the proof of the lemma. □

Let

$$\Phi \;=\; \sum_{k\ge 0} \phi_k \;, \quad \Psi \;=\; \sum_{k<0} \phi_k \;\; \text{so that} \;\; \Phi \,+\, \Psi \;=\; 1 \,,$$
$$Q \;=\; \Phi\, q \;, \quad R \;=\; \Psi\, q \,.$$

Since the supports of φ_n and φ_m are disjoints whenever $|n-m| \ge 2$, $\Psi(w) = 0$ for $w \in \mathbb{B}(0,\pi/2)$. Let $Q_z : \mathbb{R}\times\mathbb{T} \to \mathbb{R}_+$ be given by $Q_z(w) = Q(z-w) = \Phi(z-w)\, q(z-w)$. The previous lemma permits defining $X(Q_z)$ as

$$X(Q_z) \;=\; \sum_{k\ge 0} X(q_{k,z}) \,. \tag{4.4.9}$$

Lemma 4.4.11. *Fix $-1 < \alpha < 0$, $0 < \kappa < 1+\alpha$. Then,*

$$\big|\, X(Q_z) \,-\, X(Q_{z'})\,\big| \;\lesssim\; \big\|\, z \,-\, z' \,\big\|^{1+\alpha-\kappa} \, \|\, X \,\|_{C^\alpha([S-4\pi/3-1, T+4\pi/3+1]\times\mathbb{T})}$$

for all $X \in C^\alpha(\mathbb{R}\times\mathbb{T})$, $S < T$, $z = (t,x)$, $z' = (t',x') \in [S,T]\times\mathbb{T}$ such that $\|z - z'\| \;\le \pi/2$.

Proof. In view of (4.4.9), we have to estimate $|\, X(q_{n,z}) \,-\, X(q_{n,z'})\,\big|$, $n \ge 0$. Let $n_1 \in \mathbb{Z}$ such that $\pi 2^{-(n_1)} < \|z - z'\| \le \pi 2^{-n_1+1}$. We first bound this expression for n large, and then we examine the case of n small.

Note that for $n \ge 0$, $q_{n,z} = p_{n,z}$, where $p_{n,z}(w) = \phi_n(z-w)\, p(z-w)$, because, in this range of n, $q(z-w) = p(z-w)$ if $\phi_n(z-w) \ne 0$. Moreover, by (4.3.3), we have

$$\begin{aligned} \sup_{w\in\mathbb{R}\times\mathbb{T}} \big|\, p_{n,z}(w) - p_{n,z'}(w)\,\big| \;&\lesssim\; 2^{2n}\, \|z' - z\| \;, \\ \sup_{w\in\mathbb{R}\times\mathbb{T}} \big|\, Dp_{n,z}(w) - Dp_{n,z'}(w)\,\big| \;&\lesssim\; 2^{3n}\, \|z' - z\| \end{aligned} \tag{4.4.10}$$

for all z, z' in $\mathbb{R}\times\mathbb{T}$, $n \in \mathbb{N}_0$. In this formula, D stands for either ∂_t or ∂_x. As $q_{n,z} = p_{n,z}$, these bounds hold for $q_{n,z}$ and we keep working with q.

Fix $n \ge n_1$. By Lemma 4.4.10, since t, $t' \in [S,T]$, we have

$$\begin{aligned} \big|\, X(q_{n,z}) \,-\, X(q_{n,z'})\,\big| \;&\le\; \big|\, X(q_{n,z})\,\big| \,+\, \big|\, X(q_{n,z'})\,\big| \\ &\lesssim\; 2^{-(1+\alpha)n}\, \|\, X \,\|_{C^\alpha([S-\pi,T+\pi]\times\mathbb{T})} \,. \end{aligned} \tag{4.4.11}$$

We turn to small n's. Assume that $0 \le n < n_1$. We claim that

$$\big|\, X(q_{n,z}) \,-\, X(q_{n,z'})\,\big| \;\lesssim\; \|z' - z\|^{1+\alpha}\, \|X\|_{C^\alpha([-4\pi/3+1, t+4\pi+1]\times\mathbb{T})} \,. \tag{4.4.12}$$

Let $\delta = (1/2)(\|z'-z\|/\pi + 2^{-n})$. Note that $\delta \le 1$ because $\|z'-z\| \le \pi/2$ and $n \ge 0$. Let $g : \mathbb{R}\times\mathbb{T}\to\mathbb{R}$ be given by $g(w) = \delta^2[q_{n,z'-z}(\delta w) - q_{n,0}(\delta w)]$.

Assume, without loss of generality, that $t' > t$, and set $\Omega = [(-\infty, 0)\cup(0,a)]\times\mathbb{T}$, where $a = (t'-t)/\delta > 0$. The function g is smooth on Ω, it vanishes on $\Omega_{a\bullet}$ and its time-derivative is discontinuous at $s = 0$ and $s = a$, where $w = (s,y)$. Moreover, by (4.4.10),

$$\|g\|_{L^\infty(\mathbb{R}\times\mathbb{T})} \lesssim \delta^2\, 2^{2n}\, \|z'-z\| \;, \quad \|Dg\|_{L^\infty(\mathbb{R}\times\mathbb{T})} \lesssim \delta^3\, 2^{3n}\, \|z'-z\| \;,$$

where, as above, D stands either for ∂_t or for ∂_x. By definition of n_1, since $n < n_1$, $2^n\,\|z'-z\| \le 2^{n_1}\,\|z'-z\| \le 1$ so that $\delta 2^n \le 1$. In particular, $\|g\|_{C^1(\Omega)} \lesssim \|z'-z\|$.

On the other hand, by definition of φ, introduced in (4.4.3), the support of g is contained in $\mathbb{B}(\delta^{-1}(z'-z)\,,\,\delta^{-1}2^{-(n+1)}) \cup \mathbb{B}(0\,,\,\delta^{-1}2^{-(n+1)})$. Since $\delta^{-1}\|z'-z\| \le 4\pi/3$ and $\delta^{-1}2^{-(n+1)} \le 1$, the latter set is contained in $\mathbb{B}(0\,,\,4\pi/3+1)$.

An elementary computation yields that $(S^\delta_z g)(w) = q_{n,z'}(w) - q_{n,z}(w)$, so that

$$\big|\, X(q_{n,z}) - X(q_{n,z'})\,\big| \;=\; \big|\, X(S^\delta_z g)\,\big| \,.$$

Since the support of g is contained in $\mathbb{B}(0, 4\pi/3+1)$, X belongs to $C^\alpha(\mathbb{R}\times\mathbb{T})$ and z to $[S,T]\times\mathbb{T}$, by Corollary 4.4.7 and Remark 4.4.8,

$$\big|\, X(S^\delta_z g)\,\big| \;\lesssim\; \delta^\alpha\, \|g\|_{C^1(\Omega)}\, \|X\|_{C^\alpha([t-4\pi/3-1, t+4\pi/3+1]\times\mathbb{T})}.$$

As $\|g\|_{C^1(\Omega)} \lesssim \|z'-z\|$ and $\delta \ge (1/2\pi)\|z'-z\|$, the previous expression is less than or equal to (up to a constant)

$$\|z'-z\|^{1+\alpha}\, \|X\|_{C^\alpha([S-4\pi/3-1, T+4\pi/3+1]\times\mathbb{T})} \;.$$

This proves (4.4.12).

We are now in a position to prove the lemma. By definition,

$$\big|\, X(Q_z) - X(Q_{z'})\,\big| \;\le\; \sum_{n\ge 0} \big|\, X(q_{n,z}) - X(q_{n,z'})\,\big| \,. \tag{4.4.13}$$

By (4.4.11) and (4.4.12), the right-hand side of this expression is bounded above by

$$\|z'-z\|^{1+\alpha}\, \|X\|_{C^\alpha([S-4\pi/3-1, T+4\pi/3+1]\times\mathbb{T})}\, n_1 \;+\; C_0\, \big\|\, X\,\big\|_{C^\alpha([S-\pi, T+\pi]\times\mathbb{T})} \sum_{n\ge n_1} 2^{-(1+\alpha)n} \;.$$

As $2^{n_1} \le \|z'-z\|^{-1}$, we may estimate $n_1 \lesssim \log\|z'-z\|^{-1}$. Since $0 < \kappa < 1+\alpha$ and the map $t \mapsto t^\kappa[1+\log t^{-1}]$ is bounded in the interval $[0,1]$, the previous expression is less than or equal to

$$\lesssim_\kappa \|z'-z\|^{1+\alpha-\kappa}\, \|X\|_{C^\alpha([S-4\pi/3-1, T+4\pi/3+1]\times\mathbb{T})} \;+\; \|z'-z\|^{1+\alpha}\, \big\|\, X\,\big\|_{C^\alpha([S-\pi, T+\pi]\times\mathbb{T})} \;.$$

for some finite constant $C_0(\kappa)$. We used here that $2^{-n_1} \le 2\|z'-z\|$ to bound the second term. This completes the proof of the lemma. □

Recall the definition of R given just above (4.4.9). The function R fulfills the hypotheses of Lemma 4.4.4 and its support is contained in $[0, 4\pi]\times\mathbb{T}$.

Let $R_z : \mathbb{R}\times\mathbb{T}\to\mathbb{R}_+$ be given by $R_z(w) = R(z-w) = \Psi(z-w)\, q(z-w)$. The next lemma provides estimates for $X(R_z)$.

Lemma 4.4.12. *Fix $-1 < \alpha < 0$. Then,*

$$\big|\, X(R_z) - X(R_{z'})\,\big| \;\lesssim\; \big\|\, z - z'\,\big\|^{1+\alpha}\, \big\|\, X\,\big\|_{C^\alpha([S-51\pi/8, T+19\pi/8]\times\mathbb{T})}$$

for all $X \in C^\alpha(\mathbb{R}\times\mathbb{T})$, $S < T$, $z = (t,x)$, $z' = (t',x') \in [S,T]\times\mathbb{T}$ such that $\|z - z'\| \le 2\pi$.

Proof. Fix $z = (t,x)$, $z' = (t',x')$ such that $\|z'-z\| \le 2\pi$. Without loss of generality, assume that $t < t'$ and let $f(w) = R_{z'}(w)$. The function f satisfies the assumptions of Lemma 4.4.4, with the plane $\{(s,y) : s = t'\}$ replacing $\{(s,y) : s = 0\}$. Its support in contained in $[t'-4\pi, t']$. Clearly, $R_z(w) - R_{z'}(w) = f(w-\hat z) - f(w)$, where $\hat z = z - z'$. By Lemma 4.4.9 applied to the function f,

$$\big|\, X(R_z) - X(R_{z'})\,\big| \;\lesssim\; \|\hat z\|^{1+\alpha}\, \|\, f\,\|_{C^2(\Omega_{\bullet t'})}\, \big\|X\big\|_{C^\alpha([t'-4\pi-19\pi/8, t'+19\pi/8]\times\mathbb{T})} \;.$$

By definition of f and R, and since $\Psi(w) = 0$ for $w \in \mathbb{B}(0, \pi/4)$, we have that

$$\| f \|_{C^2(\Omega_{\bullet t'})} \;=\; \| R \|_{C^2(\Omega_{0\bullet})} \;\lesssim\; \| p \|_{C^2(\Omega_\star)} \;,$$

where $\Omega_\star = \Omega_{0,4\pi} \setminus \mathbb{B}(0, \pi/4)$. To complete the proof, it remains to recall that $z' \in [S, T] \times \mathbb{T}$. □

Remark 4.4.13. *Note that the proofs of Lemmata 4.4.4, 4.4.9, 4.4.10 and 4.4.11 permit to extend the domain of definition of a distribution X in C^α to functions which do not belong to $C^1(\mathbb{R}\times\mathbb{T})$. For instance to functions of the type $g(t, x) = f(t, x)\mathbb{1}_{(S,T)}(t)$, where f belongs to $C^1(\mathbb{R} \times \mathbb{T})$ and $\mathbb{1}_A$ is the indicator of the set A. This property is further exploited below in the definition of the distribution X^+.*

Proof of Theorem 4.4.1. The proof is a consequence of Lemmata 4.4.11 and 4.4.12. □

Remark 4.4.14. *The proof of Lemma 4.4.11 can be extended to $\alpha < -1$, but this result will not be needed here.*

4.4.1 The distribution X^+

Let f be a function in $C^1_c(\mathbb{R}\times\mathbb{T})$ and denote by $\mathbb{1}_A$, $A \subset \mathbb{R}\times\mathbb{T}$, the indicator function of the set A. Lemma 4.4.4 permits also to define $X(f\mathbb{1}_{\mathbb{R}_+\times\mathbb{T}})$ as a sum. We denote this quantity by $X^+(f)$:

$$X^+(f) \;:=\; X\big(\, f\, \mathbb{1}_{\mathbb{R}_+\times\mathbb{T}}\,\big)\;.$$

Clearly, $X^+(f) = 0$ for all f in $C^1_c(\mathbb{R} \times \mathbb{T})$ whose support is contained in $(-\infty, 0] \times \mathbb{T}$.

Fix $-1 < \alpha < 0$, $0 < \kappa < 1 + \alpha$. We claim that there exists a that

$$\big|\, X^+(q_{(t,x)})\,\big| \;\lesssim_\kappa\; t^{1+\alpha-\kappa}\, \|\, X\,\|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})} \tag{4.4.14}$$

for all $X \in C^\alpha(\mathbb{R} \times \mathbb{T})$, $0 \le t \le \pi/2$.

To prove this claim, fix $z = (t, x)$ and write $X^+(q_{(t,x)}) = X\big(\, q_{(t,x)}\, \mathbb{1}_{\mathbb{R}_+\times\mathbb{T}}\,\big)$ as

$$X\big(\, q_{(t,x)} \;-\; q_{(0,x)}\,\big) \;+\; X\big(\, q_{(0,x)} \;-\; q_{(t,x)}\,[1 - \mathbb{1}_{\mathbb{R}_+\times\mathbb{T}}]\,\big)\;. \tag{4.4.15}$$

By Theorem 4.4.1 with $S = 0$, $T = \pi$, the absolute value of the first term is bounded by $t^{1+\alpha-\kappa}\, \|\, X\,\|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})}$ up to some constant that depends only on κ. On the other hand, since the support of $q_{(0,x)}$ is contained in $(-\infty, 0] \times \mathbb{T}$, the function appearing in the second term can be written as $[\, q_{(0,x)} \;-\; q_{(t,x)}\,]\,[1 - \mathbb{1}_{\mathbb{R}_+\times\mathbb{T}}]$. In Lemmata 4.4.11 and 4.4.12, we estimated $X(q_{(0,x)} \;-\; q_{(t,x)})$ in each region separately. In particular, it follows from these results that the absolute value of the second term in (4.4.15) is bounded by $t^{1+\alpha-\kappa}\, \|\, X\,\|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})}$ up to some positive constant that depends only on κ. This proves (4.4.14).

For similar reasons, the proofs of Lemmata 4.4.11 and 4.4.12 yield that for fixed $-1 < \alpha < 0$, $0 < \kappa < 1 + \alpha$, we have that

$$\big|\, X^+(q_z) \;-\; X^+(q_{z'})\,\big| \;\lesssim_\kappa\; \|z - z'\|^{1+\alpha-\kappa}\, \|\, X\,\|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})} \tag{4.4.16}$$

for all $X \in C^\alpha(\mathbb{R} \times \mathbb{T})$, z, $z' \in [0, \pi] \times \mathbb{T}$ such that $\|z - z'\| \;\le \pi/2$.

Corollary 4.4.15. *Fix $-1 < \alpha < 0$, $0 < 2\kappa < 1 + \alpha$. Then,*

$$\|u\|_{C^{1+\alpha-2\kappa}([0,T]\times\mathbb{T})} \;\le\; T^\kappa\, \|\, X\,\|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})}\;.$$

for all $X \in C^\alpha(\mathbb{R} \times \mathbb{T})$, $0 < T \le \pi/2$, where $u : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ is given by $u(z) = X^+(q_z)$.

Proof. Fix $X \in C^\alpha(\mathbb{R} \times \mathbb{T})$, $0 < T \le \pi/2$, $z = (t, x)$, $z' = (t', x') \in [0, T] \times \mathbb{T}$. Suppose first that $|x - x'| \le T$. In this case, $\|\, z \;-\; z'\,\|^\kappa \le (2T)^\kappa$. Hence, by (4.4.16),

$$\big|\, X^+(q_z) \;-\; X^+(q_{z'})\,\big| \;\lesssim_\kappa\; \big\|\, z \;-\; z'\,\big\|^{1+\alpha-2\kappa}\, T^\kappa \|\, X\,\|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})}\;.$$

In contrast, if $|x - x'| > T$, $t^{1+\alpha-\kappa} \le T^{1+\alpha-2\kappa}\, T^\kappa \le |x - x'|^{1+\alpha-2\kappa}\, T^\kappa \le \|z - z'\|^{1+\alpha-2\kappa}\, T^\kappa$. A similar inequality holds with t' in place of t. Hence, by (4.4.14),

$$\begin{aligned}\big|\, X^+(q_z) \;-\; X^+(q_{z'})\,\big| \;&\le\; \big|\, X^+(q_z)\,\big| \;+\; \big|\, X^+(q_{z'})\,\big| \\ &\lesssim_\kappa \big\|\, z \;-\; z'\,\big\|^{1+\alpha-2\kappa}\, T^\kappa \|\, X\,\|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})}\;.\end{aligned}$$

The lemma follows from the two previous estimates. □

4.5 A log-correlated Gaussian random field

We introduce in this section a Gaussian random field closely related to the linear SPDE

$$\begin{cases} \partial_t \mathfrak{f} \;=\; -(-\Delta)^{1/2}\,\mathfrak{f} + \xi\;, \\ \mathfrak{f}(0,x) \;=\; f(x)\;, \end{cases} \tag{4.5.1}$$

where ξ represents the space-time white noise and $f : \mathbb{T} \to \mathbb{R}$ a continuous function. Denote by $\mathfrak{f}_\varepsilon$ the solution of the previous equation with ξ replaced by its regularized version ξ_ε introduced in (4.2.5).

The solution of these equations can be expressed in terms of the semigroup $p(t,x)$ introduced in (4.3.2):

$$\mathfrak{f}_\varepsilon(t,x) \;=\; \int_0^t \int_{\mathbb{T}} p(t-s, x-y)\,\xi_\varepsilon(s,y) dy ds \;+\; \int_{\mathbb{T}} p(t,x-y)\, f(y)\; dy \tag{4.5.2}$$

because $p(t,\cdot)$ is symmetric. As $p(t,x) = 0$ for $t < 0$, in the first term we may change the interval of integration from $[0,t]$ to $[0,\infty)$. Moreover, if we set the initial condition f to be $f(x) = \int_{-\infty}^0 \int_{\mathbb{T}} p(-s,x-y)\xi_\varepsilon(s,y)\, dy ds$, as p is a semigroup,

$$\mathfrak{f}_\varepsilon(t,x) \;=\; \int_{-\infty}^{\infty} \int_{\mathbb{T}} p(t-s,x-y)\,\xi_\varepsilon(s,y) dy ds\;.$$

We replace, in the previous convolution, p by a kernel q with bounded support to avoid problems of integrability at infinity. Recall the definition of the function $q : \mathbb{R}\times\mathbb{T} \to \mathbb{R}_+$ introduced in (4.4.1). Note that

$$q(t,x) \;=\; p(t,x) \text{ for } t \le 2\pi \text{ and that } q(s,\cdot) \text{ is symmetric} \tag{4.5.3}$$

for all $s \in \mathbb{R}$, $q(s,x) = q(s,-x)$, because so is $p(s,\cdot)$.

Let $\mathfrak{v} := q * \xi$, $\mathfrak{v}_\varepsilon := q * \xi_\varepsilon$ be the centered Gaussian random fields on $\mathbb{R}\times\mathbb{T}$ defined by:

$$\begin{aligned} \mathfrak{v}(t,x) \;&:=\; \int_{\mathbb{R}} \int_{\mathbb{T}} q(t-s,x-y)\,\xi(s,y) dy ds\;, \\ \mathfrak{v}_\varepsilon(t,x) \;&:=\; \int_{\mathbb{R}} \int_{\mathbb{T}} q(t-s,x-y)\,\xi_\varepsilon(s,y) dy ds\;. \end{aligned} \tag{4.5.4}$$

Let $r = q - p$, and note that r is smooth and vanishes in the time-interval $(-\infty, 2\pi]$. As q coincides with p on the time-interval $(-\infty, 2\pi]$ and vanishes outside the interval $[0,4\pi]$, for $0 < t < 2\pi$, the field $\mathfrak{v}_\varepsilon(t)$ can be rewritten as

$$\mathfrak{v}_\varepsilon(t) \;=\; \int_{-4\pi}^{2\pi} r(t-s)\,\xi_\varepsilon(s)\; ds \;+\; \mathfrak{g}_\varepsilon(t) \tag{4.5.5}$$

where

$$\mathfrak{g}_\varepsilon(t) \;=\; \int_0^t p(t-s)\,\xi_\varepsilon(s)\; ds \;+\; p(t)\int_{-4\pi}^0 p(-s)\,\xi_\varepsilon(s)\; ds\;.$$

We used in the last step the fact that p is a semigroup so that $p(t-s) = p(t)p(-s)$ for all $s < 0 < t$.

Denote by $\mathfrak{G}_\varepsilon(t)$ the first term on the right-hand side of (4.5.5). Since r is a smooth function, by [1, Theorem 1.4.2], almost surely, the field $\mathfrak{G}_\varepsilon$ is C^∞ in the set $\mathbb{R}\times\mathbb{T}$. On the other hand, by (4.5.2), the second term on the right-hand side of (4.5.5), represented by $\mathfrak{g}_\varepsilon$, solves (4.5.1) with initial condition $\mathfrak{g}_\varepsilon(0) = \int_{[-4\pi,0]} p(-s)\,\xi_\varepsilon(s)\; ds$. Therefore, for $0 < t < 2\pi$,

$$\partial_t \mathfrak{v}_\varepsilon \;+\; (-\Delta)^{1/2}\,\mathfrak{v}_\varepsilon \;-\; \xi_\varepsilon \;=\; \partial_t \mathfrak{G}_\varepsilon \;+\; (-\Delta)^{1/2}\,\mathfrak{G}_\varepsilon\;, \tag{4.5.6}$$

A similar conclusion holds with $\mathfrak{v}_\varepsilon$, ξ_ε, $\mathfrak{G}_\varepsilon$ replaced by $\mathfrak{v}$, ξ, $\mathfrak{G} = \int_{[-4\pi,2\pi]} r(t-s)\,\xi(s)\; ds$, respectively.

4.5.1 The correlations of the fields $\mathfrak{v}_\varepsilon$, $\mathfrak{v}$

Denote by Q, Q_ε the covariances of the fields $\mathfrak{v}$, $\mathfrak{v}_\varepsilon$, respectively: For z, z' in $\mathbb{R}\times\mathbb{T}$,

$$Q(z,z') \;=\; \mathbb{E}\big[\,\mathfrak{v}(z)\,\mathfrak{v}(z')\,\big]\;, \quad Q_\varepsilon(z,z') \;=\; \mathbb{E}\big[\,\mathfrak{v}_\varepsilon(z)\,\mathfrak{v}_\varepsilon(z')\,\big]\;.$$

A change of variables yields that $Q_\varepsilon(z,z') = Q_\varepsilon(0,z'-z)$. Denote this later quantity by $Q_\varepsilon(z'-z)$ and define $Q(\cdot)$ similarly.

The main result of this section reads as follows. Recall that $\log^+ t = \log t$ if $t \ge 1$ and $\log^+ t = 0$ if $0 < t \le 1$.

Proposition 4.5.1. *There exist a continuous, bounded function $R : \mathbb{R}\times\mathbb{T}\to\mathbb{R}$, such that*

$$Q(z) \;=\; \frac{1}{2\pi}\log^+\frac{\pi}{2\,\|z\|} \;+\; R(z)\;.$$

The proof of Proposition 4.5.1 relies on some lemmata. An elementary computation yields that

$$\begin{aligned} Q(z) \;&=\; \int_{\mathbb{R}\times\mathbb{T}} q(-w)\,q(z-w)\;dw \;=\; \int_{\mathbb{R}\times\mathbb{T}} q(w)\,q(z+w)\;dw\;, \\ Q_\varepsilon(z) \;&=\; \int_{\mathbb{R}\times\mathbb{T}} (q*\varrho_\varepsilon)(w)\,(q*\varrho_\varepsilon)(z+w)\;dw\;. \end{aligned} \tag{4.5.7}$$

A change of variables shows that $Q(-z) = Q(z)$. On the other hand, as $q(t,-x) = q(t,x)$, it is not difficult to see that $Q(t,-x) = Q(t,x)$ for all $(t,x)\in\mathbb{R}\times\mathbb{T}$. Moreover, as the support of q is contained in $[0,4\pi]\times\mathbb{T}$, and the support of Q is contained in $[-4\pi,4\pi]\times\mathbb{T}$.

Let $\mathcal{P} = \{(0,x) : x\in\mathbb{T}\}$. It is not difficult to show that Q is smooth in $(\mathbb{R}\times\mathbb{T})\setminus\{(0,0)\}$, that $(-\Delta)^{1/2}Q$ is well-defined in the set $(\mathbb{R}\times\mathbb{T})\setminus\mathcal{P}$, and that

$$[(-\Delta)^{1/2}Q](z) \;=\; \int_{\mathbb{R}\times\mathbb{T}} q(w)\;[(-\Delta)^{1/2}q]\,(w+z)\;dw\;, \quad z\;\in\;(\mathbb{R}\times\mathbb{T})\setminus\mathcal{P}\;. \tag{4.5.8}$$

Finally, by the definition (1.3.1) of the operator $(-\Delta)^{1/2}$, as $Q(-z) = Q(z)$, a change of variables yields that $[(-\Delta)^{1/2}Q](-z) = [(-\Delta)^{1/2}Q](z)$. We summarize these properties in the next formula:

$$\begin{gathered} Q(-z) \;=\; Q(z)\;,\;\; Q(t,-x) \;=\; Q(t,x)\;, \\ [(-\Delta)^{1/2}Q]\,(-z) = [(-\Delta)^{1/2}Q]\,(z)\;, \quad z\;\in\;(\mathbb{R}\times\mathbb{T})\setminus\mathcal{P}\;. \end{gathered} \tag{4.5.9}$$

These identities extend to Q_ε. Moreover,

$$Q_\varepsilon(z) \;=\; (Q*\bar\varrho_\varepsilon)(z)\;, \tag{4.5.10}$$

where $\bar\varrho_\varepsilon$ is the mollifier given by $\bar\varrho_\varepsilon := \varrho_\varepsilon * (\mathcal{T}\varrho_\varepsilon)$, and $\mathcal{T}$ is the operator defined by $(\mathcal{T}f)(z) := f(-z)$. We define $\bar\varrho$ similarly.

Lemma 4.5.2. *There exists a continuous, bounded function $R_0 : \mathbb{R}\times\mathbb{T}\to\mathbb{R}$ such that*

$$[(-\Delta)^{1/2}Q](z) \;=\; (1/2)\,\big[\,q(-z) \,+\, q(z)\,\big] \;+\; R_0(z)\;, \quad z\;\in\;(\mathbb{R}\times\mathbb{T})\setminus\mathcal{P}\;.$$

Proof. By the properties (4.5.3) of the kernel q,

$$\begin{cases} \partial_t q \;+\; (-\Delta)^{1/2}\,q \;=\; R\;, & t>0\;, \\ q(0,\cdot) \;=\; \delta_0(\cdot)\;, & t\;=\;0\;, \\ q(t,\cdot) \;=\; 0\;, & t\;<\;0\;, \end{cases} \tag{4.5.11}$$

where δ_0 is the Dirac distribution concentrated at $x = 0$ and $R : \mathbb{R}\times\mathbb{T}\to\mathbb{R}$, given by $R = [\,\partial_t + (-\Delta)^{1/2}\,](q-p)$, is a smooth function with compact support. Note that $R(s,y) = 0$ if $s\le 0$.

Fix $z = (t,x)\in\mathbb{R}\times\mathbb{T}$ and assume that $t<0$. By (4.5.8),

$$[(-\Delta)^{1/2}Q](z) \;=\; \int_{\mathbb{R}\times\mathbb{T}} q(w)\;[(-\Delta)^{1/2}q]\,(w+z)\;dw\;.$$

Since $q(s,y)$ vanishes for $s<0$, we may restrict the integration to the time-interval $(-t,+\infty)$. By (4.5.11), on $(0,+\infty)\times\mathbb{T}$, $[(-\Delta)^{1/2}q](s,y) = R(s,y)-(\partial_s q)(s,y)$. The previous integral is thus equal to

$$\int_{-t}^{\infty}\int_{\mathbb{T}} q(w)\, R(w+z)dyds \;-\; \int_{-t}^{\infty}\int_{\mathbb{T}} q(w)\, (\partial_s q)\, (w+z)dyds\;,$$

where $w=(s,y)$ As $R(s,y)$ vanishes for $s\le 0$, in the first integral we may integrate over $\mathbb{R}\times\mathbb{T}$. After an integration by parts, as $q(0,\cdot)=\delta_0(\cdot)$, the second integral becomes

$$q(-z) \;+\; \int_{-t}^{\infty}\int_{\mathbb{T}} (\partial_s q)\,(w)\; q\,(w+z)dyds\;.$$

Using again the first identity in (4.5.11), we can write this expression as

$$q(-z) \;+\; \int_{(-t,\infty)\times\mathbb{T}} R\,(w)\; q\,(w+z)\; dw \;-\; \int_{(-t,\infty)\times\mathbb{T}} [(-\Delta)^{1/2}q]\,(w)\; q\,(w+z)\; dw\;.$$

Since $(-\Delta)^{1/2}$ is a symmetric operator, $q(s,y)$ is smooth away from $\mathcal{P}$ and vanishes for $s<0$, the last integral is equal to

$$\begin{aligned}&-\int_{(-t,\infty)\times\mathbb{T}} q\,(w)\;[(-\Delta)^{1/2}q]\,(w+z)\; dw\\ &\quad=\;-\int_{\mathbb{R}\times\mathbb{T}} q\,(w)\;[(-\Delta)^{1/2}q]\,(w+z)\; dw \;=\; -\,[(-\Delta)^{1/2}Q]\,(z)\;.\end{aligned}$$

Putting together the previous terms yields that for $z=(t,x)$ with $t<0$,

$$2\,[(-\Delta)^{1/2}Q](z) \;=\; q(-z) \;+\; \int_{\mathbb{R}\times\mathbb{T}} q\,(w)\,\big[\,R\,(w+z)\;+\;R\,(w-z)\,\big]\; dw\;.$$

Since $q(z)=0$, we may add $q(z)$ to the right-hand side to complete the proof of the lemma in the case $t<0$ with

$$R_0(z) \;=\; \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} q\,(w)\,\big[\,R\,(w+z)\;+\;R\,(w-z)\,\big]\; dw\;.$$

The proof in the case $t>0$ is analogous.

The function R_0 is continuous because R is uniformly continuous and q is integrable. It is bounded because R is uniformly bounded and q is integrable. □

Recall from (1.5.9) that we denote by G the Green function associated to $(-\Delta)^{1/2}$. Let $q_s(z) = (1/2)(q(-z)+q(z))$.

Lemma 4.5.3. *There exists a continuous, bounded function $R_1:\mathbb{R}\times\mathbb{T}\to\mathbb{R}$ such that for all $(t,x)\in\mathbb{R}\times\mathbb{T}$, $(t,x)\neq(0,0)$,*

$$Q(t,x) \;=\; [\,q_s(t,\cdot)*G\,](x) \;+\; R_1(t,x)\;.$$

Proof. Recall from (4.5.7) the expression of the function Q. An elementary computation yields that for each $t\in\mathbb{R}$,

$$\mathcal{Q}(t) \;:=\; \int_{\mathbb{T}} Q(t,x)\, dx \;=\; \int_{\mathbb{R}} \mathfrak{q}(s)\, \mathfrak{q}(t+s)\, ds\;,$$

where $\mathfrak{q}(t) = \int_{\mathbb{T}} q(t,y)\, dy$. By definition of $q(s,y)$, the function $\mathfrak{q}(t)$ is bounded, integrable and discontinuous only at the origin. Moreover, it vanishes outside a compact set, and it is equal to 0, resp. 1, for $s<0$, resp. $0\le s\le 2\pi$. In particular, $\mathcal{Q}$ is bounded and continuous.

Let $\bar{Q}(t,x) = Q(t,x)-\mathcal{Q}(t)$. Clearly, for all $(t,x)\in(\mathbb{R}\times\mathbb{T})\setminus\mathcal{P}$, $[(-\Delta)^{1/2}\bar{Q}(t,\cdot)](x) = [(-\Delta)^{1/2}Q(t,\cdot)](x)$. Hence, by the previous lemma,

$$[(-\Delta)^{1/2}\bar{Q}(t,\cdot)]\,(x) \;=\; q_s(t,x) \;+\; R_0(t,x)\;, \quad (t,x)\;\in\;(\mathbb{R}\times\mathbb{T})\setminus\mathcal{P}\;.$$

Since, for all $t' \in \mathbb{R}$, $\int_{\mathbb{T}} \bar{Q}(t', y)\, dy = 0$, by (1.5.10), taking the convolution with respect to G on both sides of the previous equation yields that

$$Q(t,x) \;=\; [\, q_s(t,\cdot) * G\,](x) \;+\; R_1(t,x) \quad (t,x) \;\in\; (\mathbb{R}\times\mathbb{T})\setminus\mathcal{P}\;,$$

with $R_1(t,x) = [R_0(t,\cdot) * G](x) + Q(t)$.

Since R_0 is bounded and continuous, and G is integrable, $[R_0(t,\cdot) * G](x)$ is bounded and continuous. As we already showed that $Q(t)$ is bounded and continuous, the proof is complete for $(t,x) \in (\mathbb{R}\times\mathbb{T})\setminus\mathcal{P}$. Since all terms are continuous on $(\mathbb{R}\times\mathbb{T})\setminus\{(0,0)\}$, this identity can be extended to $(t,x) \neq (0,0)$. □

We are now in a position to prove Proposition 4.5.1.

Proof of Proposition 4.5.1. In view of Lemma 4.5.3, we need to show that

$$R_3(t,x) \;:=\; [\, q_s(t,\cdot) * G\,](x) \;-\; \frac{1}{2\pi}\log^+ \frac{\pi}{2\,\|z\|}$$

is a continuous, bounded function.

Recall that $q_s(t,x) = (1/2)\,[\, q(t,x) + q(-t,-x)\,]$. Since $q(s,y) = 0$ for $s < 0$ and since $q(s,\cdot)$ is symmetric, $q_s(t,x) = (1/2)\, q(|t|,x)$ for $t \neq 0$. By the explicit expression (1.5.9) of the Green function,

$$R_3(t,x) \;=\; -\frac{1}{2\pi}\int_{\mathbb{T}} q(|t|,y)\, \log\Big\{ 2\,\big|\, \sin(|x-y|)\,\big| \Big\}\, dy \;-\; \frac{1}{2\pi}\log^+ \frac{\pi}{2\,\|z\|}\;.$$

By definition of q, for any $\delta > 0$, R_3 is bounded and continuous on $\mathbb{B}(0,\delta)^c$ because the function $\log t$ is integrable on the interval $(0,1)$.

We turn to the behavior of R_3 on $\mathbb{B}(0,\delta)$. Fix $0 < \delta < \pi/16$. On the set $\mathbb{B}(0,\delta)$, $\log^+(\pi/(\,2\,\|z\|\,)) = \log(\pi/(\,2\,\|z\|\,))$, and the right-hand side of the previous equation can be written as

$$-\,q(|t|)\,\frac{1}{2\,\pi}\,\log\,2 \;-\; \frac{\pi}{4\,\pi}\int_{\mathbb{T}} q(|t|,y)\,\log\Big\{\,\big|\,\sin([x-y]/2)\,\big|\,\Big\}\,dy \;-\; \frac{1}{2\pi}\log\frac{\pi}{2\,\|z\|}$$

where $t \mapsto q(|t|)$ is the continuous and bounded function defined in the previous proof as $q(t) := \int_{\mathbb{T}} q(t,y)\, dy$.

On the set $\{(s,y) : |s| \le \delta\}$, $q(s,y) = p(s,y)$. We may, therefore, replace q by p in the previous integral. At this point, it remains to show that the function R_4 given by

$$R_4(t,x) \;=\; \int_{\mathbb{T}} p(|t|,y)\,\log\,\big|\,\sin([x-y]/2)\,\big|\,dy \;-\; \log\|z\|$$

is bounded and continuous in $\mathbb{B}(0,\delta)$.

We prove the boundedness, the continuity being similar. Assume that $t > 0$. Let $F : \mathbb{R}\to\mathbb{R}$ be the one-periodic function given by $F(x) = \log|\,\sin(x/2)\,|$. Note that F is symmetric: $F(-x) = F(x)$. We claim that

$$\Big|\int_{\mathbb{T}} p(t,y)\,F(x-y)\,dy \;-\; \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log\,|\,ty\,|}{1+[y+(x/t)]^2}\,dy\Big| \;\lesssim_\delta\; 1\,. \tag{4.5.12}$$

The proof of (4.5.12) is divided in several steps. Recall the definition (4.3.1) of $\boldsymbol{p}(t,x)$. By (4.3.2) and a change of variables, the first term in (4.5.12) is equal to

$$\sum_{k\in\mathbb{Z}}\int_{2\pi k-\pi}^{2\pi k+\pi}\boldsymbol{p}(t,y)\,F(x-y)\,dy \;=\; \int_{\mathbb{R}}\boldsymbol{p}(t,y)\,F(x-y)\,dy$$

because F is periodic.

Fix $4\delta < a < \pi/4$. We claim that

$$\int_{[x-a,x+a]^c}\boldsymbol{p}(t,y)\,|\,F(y-x)\,|\,dy \;\lesssim_{\delta,a}\; 1. \tag{4.5.13}$$

Let $A = \cup_{k\in\mathbb{Z}}[2\pi k - a, 2\pi k + a]$. The function F is uniformly bounded on the complement of A, i.e, $|\,F(y)\,| \lesssim_a 1$ for all $y \in A^c$. Let $x + A = \{x + y : y \in A\}$. Since $|\,F(y-x)\,| \lesssim 1$ for all $y \in [x+A]^c$, and $\boldsymbol{p}(t,\cdot)$ is a probability density,

$$\int_{[x+A]^c} \boldsymbol{p}(t,y)\,|\,F(y-x)\,|\;dy \;\lesssim\; 1\;.$$

On the other hand, by the explicit form of the density $\boldsymbol{p}(t,y)$, and a change of variable,

$$\sum_{k\ge 1}\int_{2\pi k+x-a}^{2\pi k+x+a} \boldsymbol{p}(t,y)\,|\,F(y-x)\,|\;dy \;\le\; \frac{1}{\pi}\sum_{k\ge 1}\frac{t}{t^2+[2\pi k+x-a]^2}\int_{-a}^{a}|\,F(y)\,|\;dy\;.$$

Since F is integrable in a neighborhood of the origin, and since $|x| + a \le \pi/2$, the previous sum is bounded (up to a constant) by $t\sum_{k\ge1}[2\pi k - (\pi/2)]^{-2} \lesssim t$. A similar bound can be derived for the sum $k \le -1$. This proves (4.5.13).

By the explicit form of $\boldsymbol{p}(t,\cdot)$ and a change of variables,

$$\int_{x-a}^{x+a} \boldsymbol{p}(t,y)\,F(y-x)\;dy \;=\; \frac{1}{\pi}\int_{-a/t}^{a/t}\frac{F(ty)}{1+[y+(x/t)]^2}\;dy\;.$$

We have that $|\,\sin(y/2)\,/y| \;\lesssim_a\; 1$ for all y in the interval $[-a,a]$. Hence, since $\boldsymbol{p}(t,\cdot)$ is a probability density,

$$\Big|\int_{-a/t}^{a/t}\frac{F(ty)}{1+[y+(x/t)]^2}\;dy \;-\; \int_{-a/t}^{a/t}\frac{\log\,|\,ty\,|}{1+[y+(x/t)]^2}\;dy\Big| \;\lesssim\; 1\;.$$

We claim that

$$\int_{a/t}^{\infty}\frac{|\,\log\,(\,ty\,)\,|}{1+[y+(x/t)]^2}\;\;dy \;\lesssim\; 1\;, \tag{4.5.14}$$

with a similar bound if the domain of integration is replaced by $(-\infty,-a/t]$. By a change of variables, this integral is bounded (up to multiplicative constant) by

$$\int_{(a+x)/t}^{4\pi/t}\frac{1}{y^2}\;dy \;+\; \int_{4\pi/t}^{\infty}\frac{\log t+\log y}{y^2}\;dy \;\lesssim\; \{t\,\log t + t^{1/2}\}$$

because $\log y \le C_0 y^{1/2}$ for $y \ge 1$. This proves (4.5.14). Putting together all previous estimates yields (4.5.12).

It remains to show that the absolute value of

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log\,|\,ty\,|}{1+[y+(x/t)]^2}\;dy \;-\; \log\|z\|$$

is bounded on $\mathbb{B}(0,\delta)$. Since $\|z\| = |x| + t = t(1 + |x|/t)$ and $\boldsymbol{p}(t,\cdot)$ is a probability density, this difference can be written as

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log\,(t\,|y|\,)}{1+[y+\eta]^2}\,dy \;-\; \log t\,(1+\eta) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log|y|}{1+[y+\eta]^2}\,dy \;-\; \log(1+\eta)\;,$$

where $\eta = |x|/t$.

Fix $K \ge 1$. If $\eta \le K$, it is easy to show that both terms are bounded separately by a constant which depends on K. Assume that $\eta > K$. We may replace $\log(1+\eta)$ by $\log\eta$, paying a constant. After this replacement, by a change of variables the previous difference becomes

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\eta}{1+\eta^2[y+1]^2}\;\log|y|\;dy\;.$$

Fix $0 < c_1 < 1/2$, $c_2 > 2$. On the interval $|y| \le c_1$, the function $\log|y|$ is integrable and the ratio is bounded by C_0/η. On the interval $|y| \in [c_1,c_2]$, the function $\log|y|$ is bounded and the ratio is a probability density. Finally, on the interval $|y| \in [c_2,\infty)$, as $c_2 \ge 2$, the ratio is bounded by $4/\eta y^2$, and $\log|y|/y^2$ is integrable in this interval. This completes the proof of the lemma. □

We conclude this section with some consequences of Proposition 4.5.1. The next two results are Lemmas 3.1 and 3.7 in [56]. Recall the definition of the mollifier $\bar\varrho_\varepsilon$, $0<\varepsilon<1$, introduced just before Lemma 4.5.2, and that the support of ϱ is contained in $\mathbb{B}(0,\pi/2)$.

Lemma 4.5.4. *For every $r>0$, we have*

$$Q_\varepsilon(z) + \frac{1}{2\pi}\log\big(\,\|z\|+\varepsilon\,\big)\;\lesssim_r\;1$$

for all $0<\varepsilon<1$, $\|z\|\le r$.

Proof. Fix $r>0$ and $z\in\mathbb{R}\times\mathbb{T}$ such that $\|z\|\le r$. By Proposition 4.5.1 and by (4.5.10), there exists a finite constant C_0, whose value may change from line to line, such that

$$Q_\varepsilon(z)\;\le\;\frac{1}{2\pi}\int_{D_\varepsilon}\log\frac{\pi}{2\,\|z-\varepsilon w\|}\,\bar\varrho(w)\,dw\;+\;C_0\,, \tag{4.5.15}$$

where the integral is performed over the set $D_\varepsilon=\{w:\|z-\varepsilon w\|\le\pi/2\}$.

Let $A=2\pi$ and assume first that $\|z\|\le A\varepsilon$. By extracting the factor $2\varepsilon/\pi$ from the logarithm, we may bound the right-hand side of the previous equation by

$$\begin{aligned}&\frac{1}{2\pi}\log\frac{1}{\varepsilon}\;+\;\frac{1}{2\pi}\int_{D_\varepsilon}\log\frac{1}{\|z/\varepsilon-w\|}\,\bar\varrho(w)\,dw\;+\;C_0\\&\quad\le\;\frac{1}{2\pi}\log\frac{1}{\varepsilon}\;+\;\frac{1}{2\pi}\int_{D'_\varepsilon}\log\frac{1}{\|w\|}\,\bar\varrho(z/\varepsilon-w)\,dw\;+\;C_0\,,\end{aligned}$$

where we performed a change of variables and $D'_\varepsilon=\{w:\|w\|\le\pi/2\varepsilon\}$. By hypothesis, the support of ϱ is contained in $\mathbb{B}(0,\pi/2)$. Hence, by definition, the support of $\bar\varrho$ is contained in $\mathbb{B}(0,\pi)$, and we may restrict the previous integral to points w such that $\|w\|\le A+\pi=3\pi$ because $\|z\|\le A\varepsilon$. The previous expression is thus bounded by

$$\frac{1}{2\pi}\log\frac{1}{\varepsilon}\;+\;\frac{\|\bar\varrho\|_\infty}{2\pi}\int_{\|w\|\le3\pi}\Big|\log\|w\|\Big|\,dw\;+\;C_0\;\le\;\frac{1}{2\pi}\log\frac{1}{\varepsilon}\;+\;C_0\,.$$

To complete the argument note that $\varepsilon^{-1}\le(A+1)/(\|z\|+\varepsilon)$ on the set where $\|z\|\le A\varepsilon$. Hence, the previous expression is less than or equal to

$$\frac{1}{2\pi}\log\frac{1}{\|z\|+\varepsilon}\;+\;C_0\,.$$

We turn to the case $A\varepsilon<\|z\|\le r$. In this case, for w in the support of $\bar\varrho$, $\|z-\varepsilon w\|\ge\|z\|-\varepsilon\|w\|\ge\|z\|-\pi\varepsilon\ge\|z\|/2$ because $\|z\|>2\pi\varepsilon$. Therefore, the expression on the right-hand side of (4.5.15) is bounded above by

$$\frac{1}{2\pi}\log\frac{1}{\|z\|}\int_{D_\varepsilon}\bar\varrho(w)\,dw\;+\;C_0\,,$$

where we extracted the factor 2 from the logarithm. Considering separately the cases $\|z\|<1$ and $1\le\|z\|\le r$, we can bound the previous expression by

$$\frac{1}{2\pi}\log\frac{1}{\|z\|}\;+\;C_0(r)$$

for some finite constant $C_0(r)$ depending on r. As $\|z\|>A\varepsilon$, this term is less than or equal to

$$\frac{1}{2\pi}\log\frac{A+1}{A\,(\|z\|+\varepsilon)}\;+\;C_0(r)\;\le\;\frac{1}{2\pi}\log\frac{1}{\|z\|+\varepsilon}\;+\;C_0(r)\,,$$

absorbing the A's in the constant $C_0(r)$. This completes the proof of the lemma. □

Lemma 4.5.5. *For each $r > 0$, we have*

$$\big| Q(z) - Q_\varepsilon(z) \big| \lesssim_r \frac{\varepsilon}{\|z\|} \cdot$$

for all $\|z\| \le r$, $0 < \varepsilon \le 1$.

Proof. Fix $r > 0$. By (4.3.4) and the definition of q, we have that

$$\big| \partial_t q\,(z) \big| \lesssim \frac{1}{\|z\|} \,, \quad \big| \partial_x q\,(z) \big| \lesssim \frac{1}{\|z\|} \,, \quad q\,(z) \lesssim \Big\{ 1 + \frac{1}{\|z\|} \Big\} \tag{4.5.16}$$

for all $z = (t, x)$ such that $t \neq 0$.

By the definition (4.5.7) of Q and by (4.5.10),

$$\big| Q_\varepsilon(z) - Q(z) \big| \le \int \bar\varrho(w) \int q(u) \, \big| q(u + z - \varepsilon w) - q(u + z) \big| du dw \,.$$

By (4.5.16), this expression is bounded above (up to multiplicative constant) by

$$\int_0^\varepsilon \int \bar\varrho(w) \int_0^1 \Big\{ 1 + \frac{1}{\|u\|} \Big\} \frac{1}{\|\, u + z - tw \,\|} \, \mathbb{1}_A(u) dt dw du \,, \tag{4.5.17}$$

where $A = [0, 2\pi] \times \mathbb{T}$. Fix $z' \neq 0$. Decompose the set A in four pieces: $A_1 = \mathbb{B}(0, \|z'\|/2)$, $A_2 = \mathbb{B}(z', \|z'\|/2)$, $A_3 = \mathbb{B}(0, 4\|z'\|) \setminus (A_1 \cup A_2)$ and $A_4 = A \setminus \mathbb{B}(0, 4\|z'\|)$. Estimating the integral below in each of these sets, we show that

$$\int \frac{1}{\|\, u + z' \,\|} \, \mathbb{1}_A(u) du \lesssim \int \frac{1}{\|u\|} \frac{1}{\|\, u + z' \,\|} \, \mathbb{1}_A(u) du \lesssim \Big\{ 1 + \log \frac{1}{\|z'\,\|} \Big\} \,.$$

The region A_4 is responsible for the log factor. Hence, (4.5.17) is bounded above (up to multiplicative constant) by

$$\varepsilon + \int_0^\varepsilon dt \int dw \; \bar\varrho(w) \, \log \frac{1}{\|z - tw\,\|} \,. \tag{4.5.18}$$

Assume that $\|z\| \ge 2\pi\varepsilon$, where, recall, the ball of radius π contains the support of $\bar\varrho$. In this case $\|\, z - tw \,\| \ge \|\, z \,\| - 2\,\varepsilon s_1 \ge (1/2) \|\, z \,\|$. The previous expression is thus bounded above by

$$\varepsilon \Big\{ 1 + \log \frac{1}{\|\, z \,\|} \Big\} \lesssim_r \frac{\varepsilon}{\|\, z \,\|}$$

because $\|z\| \le r$.

Assume, now, that $\|z\| \le 2\pi\varepsilon$ and consider the second integral in (4.5.18). We first examine the integral in the interval $0 \le t \le \|z\|/2\pi$ (note that $\|z\|/2\pi \le \varepsilon$). In this case, as the support of $\bar\varrho$ is contained in $\mathbb{B}(0, \pi)$, $\|\, z - tw \,\| \ge \|\, z \,\|/2$. Hence,

$$\int_0^{\|z\|/2\pi} dt \int dw \; \bar\varrho(w) \, \log \frac{1}{\|z - tw\,\|} \le \frac{\|z\|}{2\pi} \log \frac{2}{\|z\|} \lesssim 1$$

because $\|z\| \le 2\pi\varepsilon \le 2\pi$.

We turn to the integral on the interval $\|z\|/2\pi \le t \le \varepsilon$. Rewriting $\|z - tw\,\|$ as $t \,\|(z/t) - w\,\|$ and changing variables as $w' = (z/t) - w$, the corresponding integral becomes

$$\int_{\|z\|/2\pi}^\varepsilon dt \int dw \; \bar\varrho((z/t) - w) \Big\{ \log \frac{1}{t} + \log \frac{1}{\|w\,\|} \Big\} \,.$$

As $\log t^{-1}$ is integrable in the interval $[0, 1]$ and since $\|z/t\| \le 2\pi$, the previous expression is bounded by

$$1 + \int_{\|z\|/2\pi}^\varepsilon dt \int_{\mathbb{B}(0, 3\pi)} dw \, \log \frac{1}{\|\, w \,\|} \lesssim 1 \,.$$

In conclusion, if $\|z\| \le 2\pi\varepsilon$, the sum in (4.5.18) is bounded above by $1 \lesssim \varepsilon/\|z\|$. This completes the proof of the lemma. □

The proof of the next lemma follows from a straightforward computation based on the formula for Q presented in Proposition 4.5.1, and on the fact that ϱ has compact support.

Lemma 4.5.6. *For all* $0 < \varepsilon \le 1$,

$$Q_\varepsilon(0) \;=\; \frac{1}{2\pi}\,\log\frac{1}{\varepsilon} \;+\; \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{T}} \bar\varrho(w)\,\log\frac{\pi}{2\,\|w\|}\,dw \;+\; \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{T}} \bar\varrho(w)\,R(\varepsilon\, w)\,dw\;,$$

where R is the function appearing in the statement of Proposition 4.5.1.

We conclude this section with some result whose proofs are similar to the previous ones. For $0<\varepsilon'$, $\varepsilon\le 1$, let $Q_{\varepsilon,\varepsilon'} : \mathbb{R}\times\mathbb{T}\to\mathbb{R}$ be given by

$$Q_{\varepsilon,\varepsilon'}(w) \;=\; \mathbb{E}[\,\mathfrak{v}_\varepsilon(0)\,\mathfrak{v}_{\varepsilon'}(w)\,]\;.$$

It follows from the proofs of Lemmata 4.5.4 and 4.5.5 that for every $r>0$, we have

$$\begin{aligned} Q_{\varepsilon,\varepsilon'}(z) + \frac{1}{2\pi}\,\log\big(\,\|z\|+\varepsilon'\,\big) \;&\lesssim_r\; 1\\ \big|\,Q(z) \,-\, Q_{\varepsilon,\varepsilon'}(z)\,\big| \;&\lesssim_r\; \frac{\varepsilon}{\|z\|} \end{aligned} \tag{4.5.19}$$

for all $0<\varepsilon'\le\varepsilon<1$, $\|z\|\le r$.

Let $\vartheta : \mathbb{R}^2\to\mathbb{R}_+$ be a mollifier satisfying the conditions (4.2.4). For $0<\varepsilon\le 1$, let $Q^{\varrho,\vartheta}_\varepsilon$: $\mathbb{R}\times\mathbb{T}\to\mathbb{R}$ be given by

$$Q^{\varrho,\vartheta}_\varepsilon(w) \;=\; \mathbb{E}[\,\mathfrak{v}_\varepsilon(0)\,\tilde{\mathfrak{v}}_\varepsilon(w)\,]\;,$$

where $\tilde{\mathfrak{v}}_\varepsilon(w) := q*\tilde\xi_\varepsilon$, $\tilde\xi_\varepsilon = \vartheta_\varepsilon * \xi$, $\vartheta_\varepsilon = S^\varepsilon_0\vartheta$. By the proofs of Lemmata 4.5.4 and 4.5.5, for every $r>0$,

$$\begin{aligned} Q^{\varrho,\vartheta}_\varepsilon(z) + \frac{1}{2\pi}\,\log\big(\,\|z\|+\varepsilon\,\big) \;&\lesssim_r\; 1\\ \big|\,Q(z) \,-\, Q^{\varrho,\vartheta}_\varepsilon(z)\,\big| \;&\lesssim_r\; \frac{\varepsilon}{\|z\|} \end{aligned} \tag{4.5.20}$$

for all $0<\varepsilon<1$, $\|z\|\le r$.

4.6 Gaussian multiplicative chaos

Recall the definition of the Gaussian random field $\mathfrak{v}$, $\mathfrak{v}_\varepsilon$, $\varepsilon>0$, introduced in (4.5.4). Let $X_{\gamma,\varepsilon}$, $\gamma\in\mathbb{R}$, be the random field defined by

$$X_{\gamma,\varepsilon}(f) \;=\; \langle X_{\gamma,\varepsilon}\,,\, f\rangle \;:=\; \int f(z)\,e^{\gamma\,\mathfrak{v}_\varepsilon(z)\,-\,(\gamma^2/2)\,E[\mathfrak{v}_\varepsilon(z)^2]}\,dz\;,$$

for $f\in C^\infty_c(\mathbb{R}\times\mathbb{T})$.

By Lemma 4.5.6 and since $\int_{\mathbb{R}\times\mathbb{T}}\bar\varrho(w)\,dw \;=\; 1$, there exists a finite constant $C(\varrho)$ such that

$$E[\mathfrak{v}_\varepsilon(0)^2] \;=\; \frac{1}{2\pi}\,\log\frac{1}{\varepsilon} \;+\; C(\varrho) \;+\; R(\varrho,\varepsilon)\;,$$

where $R(\varrho,\varepsilon) = (\bar\varrho_\varepsilon * R)(0)$, which converges to $R(0)$ where R is the function given by Lemma 4.5.6. Hence, for a smooth function f such that $\operatorname{supp} f\subset[S,T]\times\mathbb{T}$, we have

$$\langle X_{\gamma,\varepsilon}\,,\, f\rangle \;=\; [1\,+\,\omega_{R,\delta}(\varepsilon)]\int f(z)\,A(\varrho)\,\varepsilon^{\gamma^2/4\pi}\,e^{\gamma\,\mathfrak{v}_\varepsilon(z)}\,dz\;,$$

where $A(\varrho) \;=\; \exp\{-\,(\gamma^2/2)\,(C(\rho){+}R(0))\,\}$ and $\omega_{R,\delta}(\varepsilon)$ is the modulus of continuity of the function R restricted to the set $[-\delta,\delta]\times\mathbb{T}$ for some positive δ, which vanishes as $\varepsilon\longrightarrow 0^+$.

The main result of this section, Theorem 4.6.1 below, states that, for certain values of γ, the sequence of random fields $X_{\gamma,\varepsilon}$ converge in probability, as $\varepsilon\to 0$, in C^α to a random field X_γ and that the limit does not depend on the mollifier ϱ chosen.

Theorem 4.6.1. *Fix* $0 < \gamma^2 < 2\sqrt{2\pi}$, $\alpha < \alpha_\gamma := \gamma^2/4\pi - 2\gamma/\sqrt{2\pi}$. *Then, as* $\varepsilon \to 0$, $X_{\gamma,\varepsilon}$ *converges in probability in* C^α *to a random field, denoted by* X_γ. *The limit does not depend on the mollifier* ϱ. *Moreover, for each* $p \in \mathbb{N}$, $1 \le p < 8\pi/\gamma^2$,

$$\mathbb{E}\left[\, |\, \langle X_\gamma \,,\, S_z^\delta f\rangle \,|^p \,\right] \lesssim_{p,\gamma} \|f\|_\infty^p \, \delta^{-p(p-1)(\gamma^2/4\pi)}$$

for every δ *in* $(0,1)$, $z \in \mathbb{R}\times\mathbb{T}$ *and continuous function* $f : \mathbb{R}\times\mathbb{T} \to \mathbb{R}$ *whose support is contained in* $\mathbb{B}(0,\pi/2)$

Recall from (4.5.7) that we represent by Q the covariances of the Gaussian field $\{\mathfrak{v}(z) : z \in \mathbb{R}\times\mathbb{T}\}$. The proof relies essentially on the fact that the field is log-correlated: According to Proposition 4.5.1,

$$Q(z) \;=\; \frac{1}{2\pi}\log^+\frac{\pi}{2\,\|z\|} \;+\; R(z)\,, \tag{4.6.1}$$

where R is a bounded, continuous function.

The proof of the next result is similar to the one of [45, Proposition A1].

Lemma 4.6.2. *Fix* $0 < \gamma^2 < 4\pi$. *For each* $p \in \mathbb{R}$, $1 \le p < 8\pi/\gamma^2$, *we have that*

$$\mathbb{E}\left[\, |\, \langle X_{\gamma,\varepsilon} \,,\, S_z^\delta f\rangle \,|^p \,\right] \lesssim_{p,\gamma} \|f\|_\infty^p \, \delta^{-p(p-1)(\gamma^2/4\pi)}$$

for all $0 < \varepsilon \le 1$, δ *in* $(0,1)$, $z \in \mathbb{R}\times\mathbb{T}$ *and continuous function* $f : \mathbb{R}\times\mathbb{T} \to \mathbb{R}$ *whose support is contained in* $\mathbb{B}(0,\pi/2)$,

Proof. The idea of this proof is simply to compare our Gaussian field, which is periodic in space, with another Gaussian field which is "flat" in space.

Fix $z = (t,x) \in \mathbb{R}\times\mathbb{T}$, let $\tilde{\mathfrak{v}}^z$, be the restriction to the set $B_1(z,\pi/2)$ of the periodic extension (in space) of the field $\mathfrak{v}$. Notice that for $w, w' \in B_1(z,\pi/8)$ we have that $\|z_1 - z_2\| = \|w_1 - w_2\|_{\mathbb{R}^2}$, where for $i \in \{1,2\}$ we have $w_i = [z_i]$, where $[\cdot]$ was defined just after (4.2.2).

In particular, the covariance kernel of $\tilde{\mathfrak{v}}^z$ is given by

$$\tilde{Q}^z(w-z) = \frac{1}{2\pi}\log^+\left(\frac{\pi}{2\|w-z\|_{\mathbb{R}^2}}\right) + R(w-z)$$

with R bounded as $\|w\|_{\mathbb{R}^2} \to 0$. Moreover, the fact that $\tilde{\mathcal{Q}}^z$ the positive-definite derives from that $\mathcal{Q}$ has the same property. We can then define $\tilde{X}^z_{\gamma,\varepsilon}$ by substituting $\mathfrak{v}$ by $\tilde{\mathfrak{v}}^z$ in the definition of $X_{\gamma,\varepsilon}$. Notice that for f continuous with support contained in $\mathbb{B}(0,\pi/2)$ we have

$$\langle X_{\gamma,\varepsilon}, S_z^\delta f\rangle = \langle \tilde{X}^z_{\gamma,\varepsilon}, S_z^\delta f\rangle$$

Therefore, we are under the conditions of [88, Proposition 3.7], which implies that

$$\langle \tilde{X}^z_{\gamma,\varepsilon}, S_z^\delta \mathbb{1}_{\mathbb{B}(0,\pi/2)}\rangle \lesssim_\gamma \delta^{-p(p-1)(\gamma^2/4\pi)}$$

Now, using that f is bounded and $\tilde{X}^z_{\gamma,\varepsilon}$ is positive we get the result. Notice that the invariance of $\mathfrak{v}$ by translations guarantees that the constant obtained is independent of z.

□

The proofs of the next two lemmas are similar to the one of [56, Theorem 3.2].

Lemma 4.6.3. *Fix* $0 < \gamma^2 < 4\pi$ *and let* $a = \gamma^2/2\pi < 2$. *Then*

$$\mathbb{E}\left[\, |\, \langle X_{\gamma,\varepsilon} \,-\, X_{\gamma,\varepsilon'} \,,\, S_z^\delta f\rangle \,|^2 \,\right] \lesssim_\gamma \|f\|_\infty^2 \, \varepsilon^{2\kappa}\, \delta^{-a-2\kappa}$$

for all ε, ε', δ *in* $(0,1)$, $0 < 2\kappa < 1 \wedge (2-a)$, $z \in \mathbb{R}\times\mathbb{T}$ *and continuous function* $f : \mathbb{R}\times\mathbb{T} \to \mathbb{R}$ *whose support is contained in* $\mathbb{B}(0,\pi/2)$, *Proof.* Fix ε, ε', δ in $(0,1)$ and assume, without loss of generality that $\varepsilon' \le \varepsilon$. By definition of $X_{\gamma,\varepsilon}$, the left-hand side of the previous formula is equal to

$$\int_{\mathbb{R}\times\mathbb{T}}\int_{\mathbb{R}\times\mathbb{T}} (S^\delta_z f)(z_1)\,(S^\delta_z f)(z_2)\,R_{\gamma,\varepsilon,\varepsilon'}(z_2-z_1)dz_2dz_1\ ,$$

where

$$R_{\gamma,\varepsilon,\varepsilon'}(w)\ =\ e^{\gamma^2 Q_\varepsilon(w)}\ -\ 2\,e^{\gamma^2 Q_{\varepsilon,\varepsilon'}(w)}\ +\ e^{\gamma^2 Q_{\varepsilon'}(w)}\ ,$$

and $Q_{\varepsilon,\varepsilon'}(w)$ has been introduced at the end of Section 4.5. By definition of $S^\delta_z f$ and since the support of f is contained in $\mathbb{B}(0,\pi/2)$, the absolute value of the last integral is bounded above (up to multiplicative constant) by

$$\frac{\|f\|_\infty^2}{\delta^2}\int_{\mathbb{B}(0,\pi\delta)}\big|\,R_{\gamma,\varepsilon,\varepsilon'}(w)\,\big|\,dw. \tag{4.6.2}$$

Suppose first that $\varepsilon > \pi\delta$. By Lemma 4.5.4, $\exp\{\gamma^2 Q_\varepsilon(w)\}\lesssim_\gamma \|w\|^{-a}$ for all $\|w\|\le 1$. Here, recall, $a=\gamma^2/2\pi<2$. By (4.5.19), a similar bound holds for $Q_{\varepsilon,\varepsilon'}(w)$ and $Q_{\varepsilon'}(w)$ in place of $Q_\varepsilon(w)$. Hence, (4.6.2) is less than or equal than a constant times

$$\frac{\|f\|_\infty^2}{\delta^2}\int_{\mathbb{B}(0,\delta/2)}\frac{1}{\|w\|^a}\,dw\ \lesssim_\gamma\ \|f\|_\infty^2\,\delta^{-a}\ \le\ \|f\|_\infty^2\,\varepsilon^{2\kappa}\,\delta^{-a-2\kappa}$$

for all $\kappa>0$ because $\delta/2\le\varepsilon$.

We turn to the case $\varepsilon\le\pi\delta$. We first consider the integral appearing in (4.6.2) on the set $\mathbb{B}(0,\varepsilon)$. By the bounds on Q_ε, $Q_{\varepsilon'}$, $Q_{\varepsilon,\varepsilon'}$ presented above,

$$\frac{1}{\delta^2}\int_{\mathbb{B}(0,\varepsilon)}\big|\,R_{\gamma,\varepsilon,\varepsilon'}(w)\,\big|\,dw\ \lesssim_\gamma\ \frac{\varepsilon^{2-a}}{\delta^2}\lesssim\varepsilon^{2\kappa}\,\delta^{-a-2\kappa} \tag{4.6.3}$$

provided $2\kappa<2-a$ because $\varepsilon\le\pi\delta$.

We next consider the integral (4.6.2) on $\mathbb{B}(0,\pi\delta)\setminus\mathbb{B}(0,\varepsilon)$. Note that

$$\begin{aligned}&\Big|\,e^{\gamma^2 Q_\varepsilon(w)}\ -\ e^{\gamma^2 Q_{\varepsilon,\varepsilon'}(w)}\,\Big|\\ &\quad\le\ e^{\gamma^2 Q(w)}\Big\{\Big|\,e^{\gamma^2[Q_\varepsilon(w)-Q(w)]}\ -\ 1\,\Big|\ +\ \Big|\,e^{\gamma^2[Q_{\varepsilon,\varepsilon'}(w)-Q(w)]}\ -\ 1\,\Big|\Big\}\,.\end{aligned}$$

By Proposition 4.5.1, $\exp\{\gamma^2 Q(w)\}\lesssim\|w\|^{-a}$ for all $\|w\|\le\pi$ (consider, separately, the cases $\|w\|\le\pi/2$ and $\pi/2<\|w\|\le\pi$). Hence, by Lemma 4.5.5 and (4.5.19), as $\varepsilon\le\|w\|$,

$$\Big|\,e^{\gamma^2 Q_\varepsilon(w)}\ -\ e^{\gamma^2 Q_{\varepsilon,\varepsilon'}(w)}\,\Big|\ \lesssim_\gamma\ \frac{\varepsilon}{\|w\|^{1+a}}\ .$$

A similar bound holds for $Q_{\varepsilon'}$ instead of Q_ε (because $\varepsilon'\le\varepsilon$). Thus,

$$\frac{1}{\delta^2}\int_{\mathbb{B}(0,\pi\delta)\setminus\mathbb{B}(0,\varepsilon)}\big|\,R_{\gamma,\varepsilon,\varepsilon'}(w)\,\big|\,dw\ \lesssim_\gamma\ \frac{1}{\delta^2}\int_{\mathbb{B}(0,\delta/2)\setminus\mathbb{B}(0,\varepsilon)}\frac{\varepsilon}{\|w\|^{1+a}}\,dw\ .$$

This expression is bounded above by a constant times

$$\frac{1}{\delta^2}\begin{cases}\varepsilon\,\delta^{1-a} & \text{if } a<1\ ,\\ \varepsilon\,\log(\delta/\varepsilon) & \text{if } a=1\ ,\\ \varepsilon^{2-a} & \text{if } 1<a<2\ .\end{cases}$$

It is easy to check that these expressions are bounded by $\varepsilon^{2\kappa}\,\delta^{-a-2\kappa}$ in all three cases provided $2\kappa<1\wedge(2-a)$. In conclusion, under the previous assumption on κ,

$$\frac{1}{\delta^2}\int_{\mathbb{B}(0,\delta/2)\setminus\mathbb{B}(0,\varepsilon)}\big|\,R_{\gamma,\varepsilon,\varepsilon'}(w)\,\big|\,dw\ \lesssim_\gamma\ \varepsilon^{2\kappa}\,\delta^{-a-2\kappa}\ .$$

This estimate together with (4.6.3) completes the proof of the lemma. □

Recall that ϑ is another mollifier and recall the definition of the Gaussian random field $\{\tilde{\mathfrak{v}}_\varepsilon(w) : w \in \mathbb{R}\times\mathbb{T}\}$ introduced above (4.5.20). Let $\tilde{X}_{\gamma,\varepsilon}$, $\gamma\in\mathbb{R}$, be the random field defined by

$$\tilde{X}_{\gamma,\varepsilon}(f) \;=\; \langle \tilde{X}_{\gamma,\varepsilon}\,,\, f\rangle \;:=\; \int f(z)\, e^{\gamma\,\tilde{\mathfrak{v}}_\varepsilon(z) \,-\, (\gamma^2/2)\,E[\tilde{\mathfrak{v}}_\varepsilon(z)^2]}\, dz\;,$$

for $f \in C^\infty_c(\mathbb{R}\times\mathbb{T})$. The proof of the previous result and the estimates (4.5.20) yield the next result.

Lemma 4.6.4. *Fix $0<\gamma^2<4\pi$ and let $a=\gamma^2/2\pi<2$. Then,*

$$\mathbb{E}\big[\,|\,\langle X_{\gamma,\varepsilon} \,-\, \tilde{X}_{\gamma,\varepsilon}\,,\, S^\delta_z f\rangle\,|^2\,\big] \;\lesssim_\gamma\; \|f\|^2_\infty\, \varepsilon^{2\kappa}\,\delta^{-a-2\kappa}$$

for all ε, δ in $(0,1)$, $0<2\kappa<1\wedge(2-a)$, $z\in\mathbb{R}\times\mathbb{T}$ and continuous function $f:\mathbb{R}\times\mathbb{T}\to\mathbb{R}$ whose support is contained in $\mathbb{B}(0,1/4)$,

Recall from Section 4.2 that we denote by $\mathfrak{C}^n(\mathbb{R}\times\mathbb{T})$, $n\in\mathbb{N}$ the dual of $C^n_c(\mathbb{R}\times\mathbb{T})$. Consider the random fields $X_{\gamma,\varepsilon}$ as elements of $\mathfrak{C}^n(\mathbb{R}\times\mathbb{T})$ for some $n\in\mathbb{N}$. Next result follows from Lemmata 4.6.2, 4.6.3 and 4.6.4.

Corollary 4.6.5. *For every $0<\gamma^2<4\pi$, as $\varepsilon\to 0$, the sequence of random fields $X_{\gamma,\varepsilon}$ converges in L^2 to a random field represented by X_γ. The limit does not depend on the mollifier ϱ. Moreover, for each $1\le p<8\pi/\gamma^2$, we have that*

$$\mathbb{E}\big[\,|\,\langle X_\gamma\,,\, S^\delta_z f\rangle\,|^p\,\big] \;\lesssim_{p,\gamma}\; \|f\|^p_\infty\,\delta^{-p(p-1)(\gamma^2/4\pi)}$$

for every δ in $(0,1)$, $z\in\mathbb{R}\times\mathbb{T}$ and continuous function $f:\mathbb{R}\times\mathbb{T}\to\mathbb{R}$ whose support is contained in $\mathbb{B}(0,\pi/2)$

Remark 4.6.6. *The limit field X_γ is called a Gaussian multiplicative chaos (GMC). It has been introduced by Kahane [63]. We refer to the reviews [88, 87, 39] for properties of these fields.*

Lemma 4.6.7. *Fix $0<\gamma^2<4\pi$, $0<\nu<1$ and $2\le p<8\pi/\gamma^2$. Let $p_\nu = p-\nu(p-2)$. Then,*

$$\mathbb{P}\big[\,|\,\langle X_{\gamma,\varepsilon} \,-\, X_\gamma\,,\, S^\delta_z f\rangle\,| \,>\, \eta\,\big] \;\lesssim_{p,\gamma}\; \frac{\|f\|^{p_\nu}_\infty}{\eta^{p_\nu}}\,(\varepsilon/\delta)^{2\kappa\nu}\,\delta^{-(\gamma^2/4\pi)p(p-1)}$$

for all $\eta>0$, $0<\varepsilon\le 1$, $0<\delta\le 1$, $0<2\kappa<1\wedge 2a$, $z\in\mathbb{R}\times\mathbb{T}$ and continuous function $f:\mathbb{R}\times\mathbb{T}\to\mathbb{R}$ whose support is contained in $\mathbb{B}(0,\pi/2)$.

Proof. Fix $\eta>0$, $0<\varepsilon\le 1$, $0<\delta\le 1$, $z\in\mathbb{R}\times\mathbb{T}$, and a continuous function $f:\mathbb{R}\times\mathbb{T}\to\mathbb{R}$ whose support is contained in $\mathbb{B}(0,1/4)$.

Recall that $a=\gamma^2/2\pi$. By Lemma 4.6.3, after sending $\varepsilon'\to 0$, and using Markov inequality, we have that

$$\mathbb{P}\big[\,|\,\langle X_{\gamma,\varepsilon} \,-\, X_\gamma\,,\, S^\delta_z f\rangle\,| \,>\, \eta\,\big] \;\lesssim_\gamma\; \frac{1}{\eta^2}\,\|f\|^2_\infty\,(\varepsilon/\delta)^{2\kappa}\,\delta^{-a}$$

for all $0<2\kappa<1\wedge 2a$.

By Lemma 4.6.2, Corollary 4.6.5 and Markov Inequality, we have

$$\mathbb{P}\big[\,|\,\langle X_{\gamma,\varepsilon} \,-\, X_\gamma\,,\, S^\delta_z f\rangle\,| \,>\, \eta\,\big] \;\lesssim_{p,\gamma}\; \frac{1}{\eta^p}\,\|f\|^p_\infty\,\delta^{-p(p-1)a/2}\;.$$

We have used above that $(a+b)^p\le 2^p(a^p+b^p)$ and that $a=\gamma^2/2\pi$.

Fix $0<\nu<1$. Take the first inequality to the power ν, the second one to the power $1-\nu$ and multiply them to get that

$$\mathbb{P}\big[\,|\,\langle X_{\gamma,\varepsilon} \,-\, X_\gamma\,,\, S^\delta_z f\rangle\,| \,>\, \eta\,\big] \;\lesssim_{p,\gamma}\; \frac{1}{\eta^{p_\nu}}\,\|f\|^{p_\nu}_\infty\,(\varepsilon/\delta)^{2\kappa\nu}\,\delta^{-p(p-1)a/2}\,\delta^{a\nu[p(p-1)-2]/2}\;,$$

where $p_\nu = 2\nu+(1-\nu)p = p-\nu(p-2)$. This completes the proof of the lemma because $p\ge 2$. $\square$

4.6.1 Convergence in C^α

We prove in Theorem 4.6.1 below that the sequence of random fields $X_{\gamma,\varepsilon}$ converges in $C^\alpha([-T,T]\times \mathbb{T})$ for all $T>0$.

We start introducing an orthonormal basis of $L^2(\mathbb{R}\times\mathbb{T})$. We refer to [76, Chapter 3], [99, Chapter 1] and [52, Section 3] for a proof of all lemmata made below. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be the scaling function of a multiresolution of $\mathbb{R}$, the "father wavelet". This is a function in $L^2(\mathbb{R})$ such that

(i) $\displaystyle\int_{\mathbb{R}} \varphi(x)\,\varphi(x+2\pi k)\,dx \;=\; \delta_{0,k}$ for every $k\in\mathbb{Z}$;

(ii) There exist constants $(a_k : k\in\mathbb{Z})$ auch that $\varphi(x)=\displaystyle\sum_{k\in\mathbb{Z}} a_k\,\varphi(2x-2\pi k)$.

For every $r\in\mathbb{N}$, there exists a compactly supported function φ in $C^r(\mathbb{R})$ satisfying (i) and (ii). Moreover, in (ii), $a_k=0$ for all but a finite number of integers k.

For l, $n\in\mathbb{Z}$, let $\varphi^n_l(x) \;=\; 2^{n/2}\,\varphi(2^n x-2\pi l)$,

$$\psi(x) \;=\; \sum_{k\in\mathbb{Z}}(-1)^k\,a_{1-k}\,\varphi(2x-2\pi k)\;,\quad \psi^n_l(x) \;=\; 2^{n/2}\,\psi(2^n x-2\pi l)\;.$$

For each integer $0\le m\le r$,

$$\int_{\mathbb{R}}\psi(x)\,x^m\,dx \;=\; 0\;, \tag{4.6.4}$$

and, for every $n\in\mathbb{Z}$, the set

$$\big\{\varphi^n_p : p\in\mathbb{Z}\big\}\,\cup\,\big\{\psi^m_p : m\ge n\,,\, p\in\mathbb{Z}\big\} \tag{4.6.5}$$

forms an orthonormal basis of $L^2(\mathbb{R})$.

A multiresolution analysis is also available on the torus $\mathbb{T}$. Fix L sufficiently large for φ^L_0, ψ^L_0 to have a support contained in $(-\pi,\pi)$. Let $P_j=\{m\in\mathbb{Z} : 0\le 2\pi m<2^j\}$.

For $j\ge L$, $m\in P_j$, let

$$\varphi^j_{\tau,m}(x) \;=\; \sum_{\ell\in\mathbb{Z}}\varphi^j_m(x-2\pi\ell)\;,\quad \psi^j_{\tau,m}(x) \;=\; \sum_{\ell\in\mathbb{Z}}\psi^j_m(x-2\pi\ell)\;. \tag{4.6.6}$$

The functions $\varphi^j_{\tau,m}$, $\psi^j_{\tau,m}$ are periodic, with period 2π. Let $\varphi^{\tau,j}_m$, $\psi^{\tau,j}_m:\mathbb{T}\to\mathbb{R}$ be the functions defined by

$$\varphi^{\tau,j}_m(x) \;=\; \varphi^j_{\tau,m}(x)\;,\quad \psi^{\tau,j}_m(x) \;=\; \psi^j_{\tau,m}(x)\;,\quad x\in\mathbb{T} \;=\; [-\pi,\pi)\;.$$

Since the support of φ^L_0, ψ^L_0 are contained in $(-\pi,\pi)$, for each fixed $x\in\mathbb{R}$, $j\ge L$ and $m\in P_j$, in the sums (4.6.6) there is only one $\ell\in\mathbb{Z}$ such that $\varphi^j_m(x-2\pi\ell)\neq 0$.

Extend the operator S^δ_z introduced in (4.2.2) to functions defined on $\mathbb{R}$: For $0<\delta\le 1$, $y\in\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$, let $(S^\delta_y g)(x)=\delta^{-1}g(\delta^{-1}(x-y))$. By definition,

$$\varphi^{\tau,j}_m \;=\; 2^{-j/2}\,S^{2^{-j}}_{2\pi m/2^j}\varphi\;,\quad \psi^{\tau,j}_m \;=\; 2^{-j/2}\,S^{2^{-j}}_{2\pi m/2^j}\psi\;. \tag{4.6.7}$$

There is a slight abuse of notation in this identify, as φ, ψ are functions defined on $\mathbb{R}$. For $x\in\mathbb{T}$, $2^{-j}\,(S^{2^{-j}}_{2\pi m/2^j}\varphi)(x)$ has to be understood as $\varphi([2^j(x-(2\pi m/2^j))])$.

The set $\{\varphi^{\tau,L}_m : m\in P_L\}\cup\{\psi^{\tau,n}_m : n\ge L\,,\, m\in P_n\}$ forms an orthonormal basis of $L^2(\mathbb{T})$. Clearly, tensor products provide an orthonormal basis of $L^2(\mathbb{R}\times\mathbb{T})$, but we proceed differently to have products of functions equally scaled. Let $\mathcal{B}=\{\phi^{0,L}_{l,k}\,,\phi^{\iota,n}_{l,m} : \iota\in\{1,2,3\}\,,\, l\in\mathbb{Z}\,,\, k\in P_L\,,\, n\ge L\,,\, m\in P_n\}$, where

$$\phi^{0,L}_{l,k}(t,x) \;=\; \varphi^L_l(t)\,\varphi^{\tau,L}_k(x)\;,\quad \phi^{1,n}_{l,m}(t,x) \;=\; \varphi^n_l(t)\,\psi^{\tau,n}_m(x)\;,$$
$$\phi^{2,n}_{l,m}(t,x) \;=\; \psi^n_l(t)\,\varphi^{\tau,n}_m(x)\;,\quad \phi^{3,n}_{l,m}(t,x) \;=\; \psi^n_l(t)\,\psi^{\tau,n}_m(x)\;.$$

It is not difficult to show that this family is orthogonal. It follows from property (ii) and (4.6.5) that it generates $L^2(\mathbb{R}\times\mathbb{T})$. Moreover, in view of (4.6.7), the elements of this basis can be represented in terms of the operator S^δ_z:

$$\phi^{0,L}_{l,k} \;=\; 2^{-L}\, S^{2^{-L}}_{(2\pi l/2^L, 2\pi k/2^L)}\, \Phi_0\;, \quad \phi^{\iota,n}_{l,m} \;=\; 2^{-n}\, S^{2^{-n}}_{(2\pi l/2^n, 2\pi k/2^n)}\, \Phi_\iota\;, \tag{4.6.8}$$

with the same convention as in (4.6.7), and where $\Phi_0(t,x) = \phi(t)\,\phi(x)$,

$$\Phi_1(t,x) = \varphi(t)\,\psi(x)\;, \quad \Phi_2(t,x) = \psi(t)\,\varphi(x)\;, \quad \Phi_3(t,x) = \psi(t)\,\psi(x)\;.$$

Let X be an element in the dual of $C^r_0(\mathbb{R}\times\mathbb{T})$ for some $r>0$. Fix $T_1>0$, and let $A_\iota = A_\iota(T_1)$, $A = A(T_1)$ be given by

$$\begin{aligned} A_\iota \;&=\; \sup_{n\ge L}\max_{m\in P_n}\max_{l} 2^{n\alpha}\,\big|\,\langle X\,,\, S^{2^{-n}}_{(2\pi l/2^n, 2\pi m/2^n)}\Phi_\iota\rangle\,\big|\;, \quad 1\,\le\,\iota\,\le\,3\;, \\ A_0 \;&=\; \max_{m\in P_L}\max_{l} 2^{L\alpha}\,\big|\,\langle X\,,\, S^{2^{-L}}_{(2\pi l/2^L, 2\pi m/2^L)}\Phi_0\rangle\,\big|\;, \quad A \;=\; \max_{0\le\iota\le 3} A_\iota\;, \end{aligned} \tag{4.6.9}$$

where the maximum in the first line is carried over all $l\in\mathbb{Z}$ such that $|2\pi l/2^n|\le T_1+1$. In the second line, it is carried over all $l\in\mathbb{Z}$ such that $|2\pi l/2^L|\le T_1+1$.

Lemma 4.6.8. *Let $\alpha<0$, X be an element in the dual of $C^r_0(\mathbb{R}\times\mathbb{T})$, for some $r\in\mathbb{N}$, $r>-\alpha$. Fix $T_1>0$. Then, there exists a constant C_0 such that*

$$\big|\,\langle X\,,\, S^\delta_z h\rangle\,\big| \;\lesssim\; A\,\delta^\alpha$$

for all $z\in[-T_1,T_1]\times\mathbb{T}$, $\delta\in(0,1]$ and function h in $C^r_0(\mathbb{R}\times\mathbb{T})$ whose support is contained in $\mathbb{B}(0,\pi/2)$ and such that $\|h\|_{C^r}\le 1$, In particular, if $A<\infty$, $X\in C^\alpha([-T_1,T_1]\times\mathbb{T})$ and $\|X\|_{C^\alpha([-T_1,T_1]\times\mathbb{T})}\lesssim A$.

Proof. Fix $z\in[-T_1,T_1]\times\mathbb{T}$, $\delta\in(0,1]$ and a function h in $C^r_0(\mathbb{R}\times\mathbb{T})$ whose support is contained in $\mathbb{B}(0,1/4)$ and such that $\|h\|_{C^r}\le 1$.

As $S^\delta_z h$ belongs to $L^2(\mathbb{R}\times\mathbb{T})$ and $\phi^{0,L}_{l,k}$, $\phi^{\iota,n}_{l,m}$ to $C^r_0(\mathbb{R}\times\mathbb{T})$,

$$\begin{aligned} \langle X\,,\, S^\delta_z h\rangle \;&=\; \sum_{\iota=1}^{3}\sum_{n=L}^{\infty}\sum_{l\in\mathbb{Z}}\sum_{m\in P_n}\langle X\,,\,\phi^{\iota,n}_{l,m}\rangle\,\langle\phi^{\iota,n}_{l,m}\,,\,S^\delta_z h\rangle \\ &\quad+\;\sum_{l\in\mathbb{Z}}\sum_{k\in P_L}\langle X\,,\,\phi^{0,L}_{l,k}\rangle\,\langle\phi^{0,L}_{l,k}\,,\,S^\delta_z h\rangle\;. \end{aligned} \tag{4.6.10}$$

Clearly, $\langle\phi^{\iota,n}_{l,m}\,,\,S^\delta_z h\rangle = 0$ if the supports of $\phi^{\iota,n}_{l,m}$ and $S^\delta_z h$ are disjoint. Hence, in view of (4.6.8), (4.6.9), the absolute value of the first sum is bounded by

$$A\sum_{\iota=1}^{3}\sum_{n=L}^{\infty}\sum_{l\in\mathbb{Z}}\sum_{m\in P_n} 2^{-n(1+\alpha)}\,\big|\,\langle\phi^{\iota,n}_{l,m}\,,\,S^\delta_z h\rangle\,\big|\;. \tag{4.6.11}$$

Let n_0 be the integer such that $2^{-n_0}\le\delta<2^{-n_0+1}$. Since h belongs to $C^r_0(\mathbb{R}\times\mathbb{T})$ and $\|h\|_{C^r}\le 1$, by a Taylor expansion, (4.6.4) and Schwarz inequality, for $\iota\in\{1,2,3\}$, $l\in\mathbb{Z}$, $n\ge n_0$, $m\in P_n$,

$$\big|\,\langle\phi^{\iota,n}_{l,m}\,,\,S^\delta_z h\rangle\,\big| \;\lesssim\; 2^{-(n-n_0)(1+r)}\,2^{n_0}\;.$$

Here and below, one only uses the fact that the support of h is contained in $\mathbb{B}(0,\pi/2)$ and that $\|h\|_{C^r}\le 1$.

For a fixed δ, there are less than $(3\delta)^2 2^{2n}\le C_0 2^{2(n-n_0)}$ pairs (l,m) for which the supports of $\phi^{\iota,n}_{l,m}$ and $S^\delta_z h$ are not disjoints. Hence, the sum of the terms $n\ge n_0$ in (4.6.11) is bounded by a constant times

$$\sum_{n=n_0}^{\infty} 2^{-n(1+\alpha)}\,2^{2(n-n_0)}\,2^{-(n-n_0)(1+r)}\,2^{n_0} \;\lesssim\; \sum_{n=n_0}^{\infty} 2^{-(n-n_0)(r+\alpha)}\,2^{-n_0\alpha}\;.$$

As $r > -\alpha$, this expression is bounded by $2^{-n_0\alpha} \lesssim \delta^\alpha$.

We turn to the terms $n \le n_0$. Estimating $\phi^{\iota,n}_{l,m}$ by its L^∞ norm yields that

$$\big| \langle \phi^{\iota,n}_{l,m} \,,\, S^\delta_z h\rangle \big| \;\lesssim\; 2^n \;.$$

The number of pairs for which $\langle \phi^{\iota,n}_{l,m} \,,\, S^\delta_z h\rangle$ does not vanish is bounded by a constant times $\delta^2 2^{2n}$. Hence, the contribution of the terms $n \le n_0$ to the sum in (4.6.11) is bounded by a constant times

$$\sum_{n=L}^{n_0} 2^{-n(1+\alpha)}\, 2^n\, \delta^2\, 2^{2n} \;=\; \sum_{n=L}^{n_0} 2^{(2-\alpha)n}\, \delta^2 \;\lesssim\; 2^{(2-\alpha)n_0}\, \delta^2 \;\lesssim\; \delta^\alpha \;.$$

It remains to estimate the second sum in (4.6.10). We may proceed as for the terms $n \le n_0$ to conclude that the absolute value of the second term on the right-hand side of (4.6.10) is bounded above by a constant times

$$A\, 2^{(2-\alpha)L}\, \delta^2 \;=\; A\, \delta^2 \;\lesssim\; A\, \delta^\alpha \;.$$

This completes the proof of the lemma. □

We turn to the proof of Theorem 4.6.1. We showed in Lemma 4.6.4 that the sequence of random fields $X_{\gamma,\varepsilon}$ converges in L^2 and that the limit does not depend on the mollifier ϱ. We also derived the bounds claimed in the statement of the theorem. It remains to show that the convergence also takes place in C^α.

Proof of Theorem 4.6.1. We will show that

$$\lim_{\varepsilon\to 0} \mathbb{P}\big[\, \|X_{\gamma,\varepsilon} - X_\gamma\|_{C^{\alpha'}([-T_1,T_1]\times\mathbb{T})} \,>\, \eta \,\big] \;=\; 0 \;.$$

Let $A_{\iota,\varepsilon} = A(T_1,\iota,\varepsilon,\gamma)$, $0 \le \iota \le 3$, be given by equation (4.6.9) with X replaced by $X_{\gamma,\varepsilon} - X_\gamma$. By Lemma 4.6.8, it is enough to show that for all $\eta > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}\big[\, A_{\iota,\varepsilon} \,>\, \eta \,\big] \;=\; 0 \;.$$

Denote by $B^L_{0,\varepsilon,\eta}$, $B^n_{\iota,\varepsilon,\eta}$, $1 \le \iota \le 3$, $n \ge L$, the events defined by

$$B^L_{0,\varepsilon,\eta} \;:=\; \Big\{ \max_{m\in P_L} \max_{l} 2^{L\alpha'} \,\big|\, \langle X_{\gamma,\varepsilon} - X_\gamma \,,\, S^{2^{-L}}_{(2\pi l/2^L,\, 2\pi m/2^L)} \Phi_0 \rangle \,\big| \,>\, \eta \Big\} \;,$$

$$B^n_{\iota,\varepsilon,\eta} \;:=\; \Big\{ \max_{m\in P_n} \max_{l} 2^{n\alpha'} \,\big|\, \langle X_{\gamma,\varepsilon} - X_\gamma \,,\, S^{2^{-n}}_{(2\pi l/2^n,\, 2\pi m/2^n)} \Phi_\iota \rangle \,\big| \,>\, \eta \Big\} \;,$$

where the maximum over l is carried over the same set appearing in (4.6.9).

Clearly, for $1 \le \iota \le 3$,

$$\mathbb{P}\big[\, A_{\iota,\varepsilon} \,>\, \eta \,\big] \;\le\; \mathbb{P}\Big[\, \bigcup_{n\ge L} B^n_{\iota,\varepsilon,\eta} \,\Big] \;\le\; \sum_{n\ge L} \mathbb{P}\big[\, B^n_{\iota,\varepsilon,\eta} \,\big] \;.$$

Now, fix $n \ge L$ and let p, ν, κ satisfy the conditions of Lemma 4.6.7, we have

$$\begin{aligned}
\mathbb{P}\big[\, B^n_{\iota,\varepsilon,\eta} \,\big] \;&\le\; \sum_{m\in P_n} \sum_{l} \mathbb{P}\Big[\, 2^{n\alpha'} \big|\, \langle X_\gamma - X_{\gamma,\varepsilon} \,,\, S^{2^{-n}}_{(2\pi l/2^n,\, 2\pi m/2^n)} \Phi_\iota \rangle \,\big| \,>\, \eta \,\Big] \\
&\lesssim_{p,\gamma}\; \frac{1}{\eta^{p_\nu}}\, \varepsilon^{2\kappa} \sum_{m\in P_n} \sum_{p} 2^{n\big(p - \nu(p-2) + \kappa\nu + \frac{\gamma^2}{4\pi} p(p-1)\big)} \;,
\end{aligned}$$

Summing over $n \ge L$ we get that

$$\mathbb{P}\big[\, A_{\iota,\varepsilon} \,>\, \eta \,\big] \;\lesssim_{p,\gamma,T_1}\; \frac{\varepsilon^{2\kappa}}{\eta^{p_\nu}} \sum_{n\ge L} 2^{n\big(p - \nu(p-2) + \kappa\nu + \frac{\gamma^2}{4\pi} p(p-1) + 2\big)]} \tag{4.6.12}$$

where the factor 2^{2n} appeared to take care of the volume. Notice that, if we choose γ, p, ν, κ such that

$$M = M(\alpha', \gamma, p, \nu, \kappa) := p - \nu(p-2) + \kappa\nu + \frac{\gamma^2}{4\pi}p(p-1) < 0$$

Then we have that the

$$\mathbb{P}\big[\, A_{\iota,\varepsilon} > \eta \,\big] \lesssim_{\alpha',\gamma,p,T_1,\nu,\kappa} \varepsilon^{\kappa},$$

which would be able to complete the proof of the theorem. Noticing that $M < 0$ if and only if

$$\alpha' < -\frac{\frac{\gamma^2}{4\pi}p(p-1) + 2 + \kappa\nu}{p - \nu(p-2)} = -\frac{\gamma^2}{4\pi}(p-1) - \frac{2}{p} + R_{\gamma,p,\nu,\kappa},$$

where $R_{\gamma,p,\nu,\kappa}$ vanishes as $\nu \to 0^+$. Hence, we can focus on optimising the function

$$N(p,\gamma) := -\frac{\gamma^2}{4\pi}(p-1) - \frac{2}{p}.$$

A simple computation shows that for a fixed $\gamma < 2\sqrt{2\pi}$, N achieves its maximum in the interval at $p = 2\sqrt{2\pi}$. Substituting such value in N, we get that for any $\alpha' < \alpha_\gamma := \gamma^2/4\pi - 2\gamma/\sqrt{2\pi}$ by taking ν sufficiently small, we have that $M < 0$ which completes the proof. □

Remark 4.6.9. *As mentioned in [45] , there is an heuristic argument based on the thick points (see [87], Section 4) for why one does not expect convergence in any space C^α for $\alpha > \frac{\gamma^2}{4\pi} - 2\sqrt{2\pi}\gamma$. This is based in* (4.17) *in [87], which holds at least for the class of star scale invariant random measures. In this case, by defining the random variable*

$$C(x,\delta) = \frac{X_\gamma(B(z,\delta))}{\delta^2 e^{\gamma \mathfrak{v}_\delta(z) - \frac{\gamma^2}{2}\frac{\log(1/\delta)}{2\pi}}}$$

we have that, there exists a constant $C_x > 0$ such that for all δ,

$$C_x^{-1} \le \mathbb{E}[C(x,\delta)] \le C_x.$$

Therefore, in order to have $\langle X_\gamma, S_z^\delta 1_{B(0,1/4)} \rangle \overset{\delta\to 0}{\sim} \delta^\alpha$ we need

$$\mathfrak{v}_\delta(z) \overset{\delta\to 0}{\sim} -\Big(\frac{\gamma}{4\pi} - \frac{\alpha}{\gamma}\Big)\log\delta. \tag{4.6.13}$$

By denoting $b := (\frac{\gamma}{4\pi} - \frac{\alpha}{\gamma})$ and $\mathcal{H}_b$ the set of points z satisfying (4.6.13)*. For any $b \in \mathbb{R}$, we have an explicit formula for the Haussdorf dimension of $\mathcal{H}_b$*

$$\dim \mathcal{H}_b = \max\Big\{2 - \pi b^2, 0\Big\}.$$

Notice that $2 - \pi b^2 = 0$ is achievied exactly at $\alpha = \frac{\gamma^2}{4\pi} - \sqrt{\frac{2}{\pi}}\gamma$. And for $\alpha > \frac{\gamma^2}{4\pi} - 2\sqrt{2\pi}\gamma$, one has that H_b is empty a.s see [32].

4.7 Proof of Theorem 4.2.2

We prove in this section Theorem 4.2.2. To avoid an additional term, we prove Theorem 4.2.2 for the equation (4.2.9) with the hyperbolic sinus replaced by the exponential. The arguments presented below apply without modifications to the original equation.

Recall the definition of the Gaussian random fields $\mathfrak{v}_\varepsilon$, $\mathfrak{v}$ introduced in (4.5.4). Fix $\beta_0 > 0$, u_0 in $C^{\beta_0}(\mathbb{T})$, $\gamma \in \mathbb{R}$, and denote by $\mathfrak{u}_\varepsilon$, $0 < \varepsilon < 1$, the solution of

$$\begin{cases} \partial_t \mathfrak{u}_\varepsilon \;=\; -(-\Delta)^{1/2}\mathfrak{u}_\varepsilon \;-\; A(\varrho)\,\varepsilon^{\gamma^2/4\pi}\,e^{\gamma\mathfrak{u}_\varepsilon} \;+\; \xi_\varepsilon \\ \mathfrak{u}_\varepsilon(0) \;=\; u_0 \;+\; \mathfrak{v}_\varepsilon(0)\;. \end{cases} \tag{4.7.1}$$

Let $\mathfrak{w}_\varepsilon = \mathfrak{u}_\varepsilon - \mathfrak{v}_\varepsilon$. An elementary computation yields that

$$\begin{cases}\partial_t \mathfrak{w}_\varepsilon \;=\; -(-\Delta)^{1/2}\mathfrak{w}_\varepsilon \;-\; X_{\gamma,\varepsilon}\, e^{\gamma \mathfrak{w}_\varepsilon} \;-\; R_\varepsilon \;, \\ \mathfrak{w}_\varepsilon(0) \;=\; u_0 \;, \end{cases} \tag{4.7.2}$$

where $X_{\gamma,\varepsilon}$ is the random field $A(\varrho)\, \varepsilon^{\gamma^2/4\pi} e^{\gamma \mathfrak{v}_\varepsilon}$ examined in the previous section, and R_ε the one given by

$$R_\varepsilon \;=\; \partial_t \mathfrak{v}_\varepsilon \;+\; (-\Delta)^{1/2}\mathfrak{v}_\varepsilon \;-\; \xi_\varepsilon \;. \tag{4.7.3}$$

We denote by $\mathfrak{w}$ the solution of the same equation with $X_{\gamma,\varepsilon}$, R_ε replaced by X_γ, $R = \partial_t \mathfrak{v} \;+\; (-\Delta)^{1/2}\mathfrak{v} \;-\; \xi$, respectively.

The solutions $\mathfrak{w}_\varepsilon$ can be represented as

$$\mathfrak{w}_\varepsilon(t) \;=\; -\int_0^t P_{t-s}\Big\{ X_{\gamma,\varepsilon}\, e^{\gamma \mathfrak{w}_\varepsilon(S)} \;+\; R_\varepsilon(s)\Big\}\, ds \;+\; P_t u_0 \;, \tag{4.7.4}$$

where $(P_t : t \ge 0)$ represents the semigroup of the generator $-(-\Delta)^{1/2} = -(-\Delta)^{1/2}_{\mathbb{T}}$.

Theorem 4.7.3 below, a fixed point theorem, establishes that this equation, for $0 \le \varepsilon \le 1$, has a unique solution in $C^{\beta_0}([0,T]\times\mathbb{R})$ for T sufficiently small.

We first recall Theorem 2.52 in [7], which permits to define the product of a distributions function and a distribution provided they are not too irregular.

Proposition 4.7.1. *Fix α_0, $\beta_0 \in \mathbb{R}$ such that $\alpha_0 + \beta_0 > 0$, and $S < T$. Then, there exists a bilinear form $B : C^{\alpha_0}([S,T]\times\mathbb{T}) \times C^{\beta_0}([S,T]\times\mathbb{T}) \to C^{\alpha_0\wedge\beta_0}([S,T]\times\mathbb{T})$ such that $B(f,g) = f\,g$ if f and g belong to $C^\infty([S,T]\times\mathbb{T})$. Moreover,*

$$\|B(f,g)\|_{C^{\alpha_0\wedge\beta_0}([S,T]\times\mathbb{T})} \;\lesssim_{S,T,\alpha_0,\beta_0}\; \|f\|_{C^{\alpha_0}([S,T]\times\mathbb{T})}\, \|g\|_{C^{\beta_0}([S,T]\times\mathbb{T})}$$

for all $f \in C^{\alpha_0}(\mathbb{R}\times\mathbb{T})$, $g \in C^{\beta_0}(\mathbb{R}\times\mathbb{T})$.

Remark 4.7.2. *We apply below this proposition to a distribution X in C^α and to a function $\mathfrak{w}$ in C^β. This explains the hypothesis below that $\alpha > -1/2$ which yields that $\alpha + \beta > 0$.*

Recall the definition of the function q introduced in (4.4.1). Fix $\alpha_0 < 0$, $\beta_0 > 0$, such that $\alpha_0 + \beta_0 > 0$, R in $C^1(\mathbb{R}\times\mathbb{T})$, u in $C^{\beta_0}(\mathbb{T})$, X in $C^{\alpha_0}(\mathbb{R}\times\mathbb{T})$, $\gamma \in \mathbb{R}$ and $0 < T_1 < \pi/2$. For $\mathfrak{w}$ in $C^{\beta_0}([0,T_1]\times\mathbb{T})$ such that $\mathfrak{w}(0,\cdot) = u(\cdot)$, let $\Psi_{T_1,\gamma,X,R,u}(\mathfrak{w}) = \Psi(\mathfrak{w})$ be given by

$$\Psi(\mathfrak{w})(t) \;:=\; \int_0^t q_{t-s}\big\{\, X(s)\, e^{\gamma \mathfrak{w}(s)} \,\big\}\, ds \;+\; \int_0^t P_{t-s}\, R(s)\, ds \;+\; P_t u \;, \quad 0 \le t \le T_1 \;.$$

Note that $\Psi(\mathfrak{w})(0,\cdot) = u(\cdot)$. Sometimes we write $\Psi(\mathfrak{w})$ as $\Psi_{T_1}(\mathfrak{w})$ to stress its dependence on T_1.

Denote by $\mathfrak{x}$ the first term on the right-hand side. It has to be understood as follows. Extend the definition of $\mathfrak{w}$ to $\mathbb{R}\times\mathbb{T}$ by setting $\mathfrak{w}(t,x) = \mathfrak{w}(T_1,x)$ for $t \ge T_1$ and $\mathfrak{w}(t,x) = \mathfrak{u}(x)$ for $t \le 0$. Denote the extended function by $\tilde{\mathfrak{w}}$. It is clear that $\tilde{\mathfrak{w}}$ belongs to $C^{\beta_0}(\mathbb{R}\times\mathbb{T})$ and that

$$\|\,\tilde{\mathfrak{w}}\,\|_{L^\infty([S,T]\times\mathbb{T})} \;=\; \|\,\mathfrak{w}\,\|_{L^\infty([S',T']\times\mathbb{T})} \;, \quad \|\,\tilde{\mathfrak{w}}\,\|_{C^{\beta_0}([S,T]\times\mathbb{T})} \;=\; \|\,\mathfrak{w}\,\|_{C^{\beta_0}([S',T']\times\mathbb{T})} \tag{4.7.5}$$

for all $S < T\wedge T_1$, $T > 0$, where $S' = S \vee 0$, $T' = T\wedge T_1$. We may also replace $\mathfrak{w}$ by $\tilde{\mathfrak{w}}$ in the formula for Ψ because they coincide on $[0,T_1]\times\mathbb{T}$.

By Lemma A.3.2 below, $\exp\{\gamma\tilde{\mathfrak{w}}\}$ belongs to $C^{\beta_0}(\mathbb{R}\times\mathbb{T})$. Hence, by Proposition 4.7.1, $X_{\mathfrak{w}} := X\,\exp\{\gamma\tilde{\mathfrak{w}}\}$ belongs to $C^{\alpha_0}(\mathbb{R}\times\mathbb{T})$.

As q vanishes for $(-\infty,0)\times\mathbb{T}$, we may include in the domain of integration the time-interval $[t,\infty)$. As X belongs to $C^{\alpha_0}(\mathbb{R}\times\mathbb{T})$,

$$\int_0^\infty q_{t-s}\, X_{\mathfrak{w}}(s)\, ds \;=\; X_{\mathfrak{w}}\big(\, q_z\, \mathbb{1}_{\mathbb{R}_+\times\mathbb{T}}\,\big) \;=\; X^+_{\mathfrak{w}}\big(\, q_z\,\big) \;,$$

where, recall, $\mathbb{1}_A$ represents the indicator function of the set A, q_z has been introduced just before the statement of Theorem 4.4.1, and the distribution $X^+_{\mathfrak{w}}$ at the end of Section 4.4.

From now on, α, β and κ are fixed. We first pick $-1/2 < \alpha < 0$ and then choose κ small enough for $0 < 2\kappa < 1 + 2\alpha$. Let $\beta = \alpha + 1 - 2\kappa$. Note that $0 < \beta < 1$ and $\alpha + \beta > 0$: On the one hand, $\beta > \beta + \alpha = 1 + 2\alpha - 2\kappa > 0$. On the other, $\beta = 1 + \alpha - 2\kappa < 1$. In Theorem 4.7.4, we further require $\alpha < \alpha_\gamma$.

In the remaining of this section we will be a bit more careful and name the constants used in upper bounds, as each of them depend on different parameters. All constants below may depend on α, β, κ without any reference. In contrast, any dependence on other variables will be explicitly mentioned.

Theorem 4.7.3. *For any $\gamma \in \mathbb{R}$, R in $C^1(\mathbb{R}\times\mathbb{T})$, X in $C^\alpha(\mathbb{R}\times\mathbb{T})$ for $\alpha > -1/2$, and u in $C^\beta(\mathbb{T})$ with $0 < \beta < 1 + \alpha$ and $\alpha + \beta > 0$. There exists $0 < \tau < \pi/2$ such that the equation*

$$\Psi_T(\mathfrak{w}) \;=\; \mathfrak{w} \tag{4.7.6}$$

has a unique solution in $C^\beta([0,T]\times\mathbb{T})$ for all $0 < T \le \tau$.

Notice that for $\gamma \in [0, 2\sqrt{2\pi} - \sqrt{6\pi})$ we have that $\alpha_\gamma > -1/2$.

Theorem 4.7.4. *Fix $0 < \gamma^2 < 2\sqrt{2\pi} - \sqrt{6\pi}$, $\alpha \in (-1/2, \alpha_\gamma)$ and u in $C^\beta(\mathbb{T})$. There exists a strictly positive random variable τ, $\mathbb{P}[\tau > 0] = 1$, satisfying the next statement.*

Denote by $\mathfrak{w}_\varepsilon$, $0 \le \varepsilon \le 1$, the solution of the fixed point problem (4.7.6) in $C^\beta([0,\tau]\times\mathbb{T})$, with R_ε given by (4.7.3) and $X_{\gamma,\varepsilon} = A(\varrho)\,\varepsilon^{\gamma^2/4\pi} e^{\gamma\mathfrak{v}_\varepsilon}$. Then, $\mathfrak{w}_\varepsilon$ converges in probability to $\mathfrak{w}_0 = \mathfrak{w}$, as $\varepsilon \to 0$, in $C^\beta([0,\tau]\times\mathbb{T})$.

Proof of Theorem 4.2.2. Let τ be the a. s. strictly positive random time given by Theorem 4.7.4. The solution $\mathfrak{u}_\varepsilon$ of (4.2.9) can be represented as $\mathfrak{v}_\varepsilon + \mathfrak{w}_\varepsilon$. According to Theorem 4.7.4, $\mathfrak{w}_\varepsilon$ converges in probability to $\mathfrak{w}$, as $\varepsilon \to 0$, in $C^\beta([0,\tau]\times\mathbb{T})$. On the other hand, it is not difficult to show that $\mathfrak{v}_\varepsilon$ converges in probability to $\mathfrak{v}$, as $\varepsilon \to 0$, in $C^{-\kappa}([0,\tau]\times\mathbb{T})$. Since neither $\mathfrak{w}$ nor $\mathfrak{v}$ depend on the mollifier ϱ, the theorem is proved. □

Proposition 4.7.5. *Fix $0 < T_1 < \pi/2$. For all $\mathfrak{w} \in C^\beta([0,T_1]\times\mathbb{T})$, the function $\Psi_{T_1}(\mathfrak{w})$ belongs to $C^\beta([0,T_1]\times\mathbb{T})$. Moreover, there exist finite constants $A_1 = A_1(\|u\|_{C^\beta(\mathbb{T})})$, $A_2 = A_2(\gamma, \|u\|_{L^\infty(\mathbb{T})})$, $A_3 = A_3(\gamma)$ such that*

$$\begin{aligned}\|\Psi(\mathfrak{w})\|_{C^\beta([0,T_1]\times\mathbb{T})} \;&\le\; A_1\left(1 \,+\, \|R\|_{C^1([0,\pi/2]\times\mathbb{T})}\right) \\ &\quad +\; A_2\, T_1^\kappa\, \|X\|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})} \exp\Big\{ A_3\, \|\mathfrak{w}\|_{C^\beta([0,T_1]\times\mathbb{T})} \Big\}\,.\end{aligned}$$

Proof. We examine separately each term appearing in the definition of $\Psi(\mathfrak{w})$. We start with $P_t u$. By Lemma 4.3.1, $P_t\, u$ belongs to $C^\beta(\mathbb{R}\times\mathbb{T})$, and there exists a finite constant C_0, depending only on β, such that

$$\|P_t\, u\|_{C^\beta([0,T_1]\times\mathbb{T})} \lesssim_\beta \|u\|_{C^\beta(\mathbb{T})}\,.$$

We turn to the term involving $R(s)$. Let $\mathfrak{r}(t) = \int_0^t P_{t-s} R(s)\, ds$. Let $M_0 = \|(\partial_x R)\|_{L^\infty([0,2\pi]\times\mathbb{T})}$,

$$M_1 \;=\; M_0\, 2\pi \left\{ 1 \,+\, E\big[\,|Z_1|^\beta\,\big] \right\} \,+\, 2\pi^{1-\beta}\, \|R\|_{L^\infty([0,2\pi]\times\mathbb{T})}\,.$$

A computation, similar to the one presented in the proof of Lemma 4.3.1, yields that

$$\big|\,\mathfrak{r}(t,x) \,-\, \mathfrak{r}(t',x')\,\big| \;\le\; M_1 \left\{ |x'-x|^\beta \,+\, |t'-t|^\beta \right\}$$

for all t, $t' \in [0,T_1]$, x, $x' \in \mathbb{T}$. We used here the fact that $T_1 \le 2\pi$. This proves that $\mathfrak{r}$ belongs to $C^\beta([0,T_1]\times\mathbb{T})$ and that $\|\mathfrak{r}\|_{C^\beta([0,T_1]\times\mathbb{T})} \le M_1$.

Finally, let $\mathfrak{x}(z) = \int_0^t q_{t-s}[\,X(s)\, e^{\gamma\mathfrak{w}(s)}\,]\, ds = X^+_{\mathfrak{w}}(q_z)$. Since $X_{\mathfrak{w}}$ belongs to $C^\alpha(\mathbb{R}\times\mathbb{T})$, as $\beta = 1 + \alpha - 2\kappa$, by Corollary 4.4.15, $\mathfrak{x}$, belongs to $C^\beta([0,T_1]\times\mathbb{T})$ and there exists a finite constant M_3, whose value may change from line to line, such that

$$\|\mathfrak{x}\|_{C^\beta([0,T_1]\times\mathbb{T})} \;\le\; M_3\, T_1^\kappa\, \|X_{\mathfrak{w}}\|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})}\,.$$

By Proposition 4.7.1 and (A.3.2), $\|X_{\mathfrak{w}}\|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})}$ is less than or equal to

$$M_3\,\|X\|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})}\,\exp\Big\{\,|\gamma|\,\|\,\tilde{\mathfrak{w}}\,\|_{L^\infty([-4\pi,4\pi]\times\mathbb{T})}\Big\}\,|\gamma|\,\|\,\tilde{\mathfrak{w}}\,\|_{C^\beta([-4\pi,4\pi]\times\mathbb{T})}\;.$$

By definition of $\tilde{\mathfrak{w}}$ and (4.7.5), we may replace $\tilde{\mathfrak{w}}$ by $\mathfrak{w}$ and the interval $[-4\pi,4\pi]$ by $[0,T_1]$.

Let $M_4 = M_3\,\|X\|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})}$, use the bound $a\le e^a$, $a>0$, and apply the inequality (A.3.1) below to bound the previous expression by

$$M_5\,\exp\Big\{\,|\gamma|\,(1+2\pi^\beta)\,\|\,\mathfrak{w}\,\|_{C^\beta([0,T_1]\times\mathbb{T})}\Big\}\;,$$

where $M_5 = M_4\,\exp\{\,|\gamma|\,\|\,u\,\|_{L^\infty(\mathbb{T})}\,\}$.

To complete the proof of the proposition, it remains to recollect the previous estimates. □

Next result asserts that the function $\Psi_{T,\gamma,X,R,u}$ depends continuously on the parameters X and R.

Lemma 4.7.6. *Fix $0<T_1<\pi/2$, $\gamma\in\mathbb{R}$, $u\in C^\beta(\mathbb{T})$. There exists a finite constant $A_4 = A_4(\gamma,u)$ such that*

$$\begin{aligned}\big\|\,\Psi_{T_1,\gamma,X,R,u}(\mathfrak{w})\;-\;\Psi_{T_1,\gamma,X',R',u}(\mathfrak{w})\,\big\|_{C^\beta([0,T_1]\times\mathbb{T})}\;&\le\;A_4\,\big\|\,R\;-\;R'\,\big\|_{C^1([0,2\pi]\times\mathbb{T})}\\ +\;A_4\,T_1^\kappa\,\big\|\,X\;-\;X'\,\big\|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})}&\,\exp\Big\{\,A_4\,\|\,\mathfrak{w}\,\|_{C^\beta([0,T_1]\times\mathbb{T})}\Big\}\end{aligned}$$

for all X, X' in $C^\alpha(\mathbb{R}\times\mathbb{T})$, R, R' in $C^1(\mathbb{R}\times\mathbb{T})$, and $\mathfrak{w}$ in $C^\beta([0,T_1]\times\mathbb{T})$.

Proof. We estimate the difference term by term. We start with the one involving R. For a function U in $C^1(\mathbb{R}\times\mathbb{T})$, let $\mathfrak{x}_U(t) = \int_0^t P_{t-s}U(s)\,ds$. The proof of Proposition 4.7.5 yields that there exists a finite constant M_0 such that $\|\mathfrak{x}_R - \mathfrak{x}_{R'}\|_{C^\beta([0,T]\times\mathbb{T})}\;\le\;M_0\,\big\|\,R\;-\;R'\,\big\|_{C^1([0,2\pi]\times\mathbb{T})}$.

For Y in $C^\alpha(\mathbb{R}\times\mathbb{T})$, let $\mathfrak{x}_Y(z) = \int_0^\infty q_{t-s}[\,Y(s)\,e^{\gamma\mathfrak{w}(s)}\,]\,ds = Y^+_{\mathfrak{w}}(q_z)$. By the proof of Proposition 4.7.5, there exists a constant $M_1 = M_1(\gamma,u)$ such that $\|\,\mathfrak{x}_X - \mathfrak{x}_{X'}\,\|_{C^\beta([0,T_1]\times\mathbb{T})}$ is bounded above by

$$M_1\,T_1^\kappa\,\big\|\,X\;-\;X'\,\big\|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})}\,\exp\Big\{\,M_1\,\|\,\mathfrak{w}\,\|_{C^\beta([0,T_1]\times\mathbb{T})}\Big\}\;.$$

The assertion of the lemma follows from the two previous estimates. □

The next result asserts that Ψ is a contraction provided the time-interval is small enough. It follows from the third part of the proof of Proposition 4.7.5 and from Lemma A.3.3 below.

Lemma 4.7.7. *Fix $0<T_1<\pi/2$, $\gamma\in\mathbb{R}$, $u\in C^\beta(\mathbb{T})$. There exists a finite constant $A_5 = A_5(\gamma,u)$ such that*

$$\begin{aligned}\big\|\,\Psi(\mathfrak{w}_1)\;-\;\Psi(\mathfrak{w}_2)\,\big\|_{C^\beta([0,T_1]\times\mathbb{T})}\;&\le\;A_5\,T_1^\kappa\,\|\,X\,\|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})}\;\times\\ \times\;\exp A_5\,\big\{\,\|\,\mathfrak{w}_1\,\|_{C^\beta([0,T_1]\times\mathbb{T})}+\|\,\mathfrak{w}_2\,\|_{C^\beta([0,T_1]\times\mathbb{T})}\big\}&\,\big\|\,\mathfrak{w}_1\;-\;\mathfrak{w}_2\,\big\|_{C^\beta([0,T_1]\times\mathbb{T})}\end{aligned}$$

for all X in $C^\alpha(\mathbb{R}\times\mathbb{T})$, R in $C^1(\mathbb{R}\times\mathbb{T})$, and $\mathfrak{w}_1$, $\mathfrak{w}_2$ in $C^\beta([0,T_1]\times\mathbb{T})$ such that $\mathfrak{w}_k(0,\cdot) = u(\cdot)$, $k=1,\,2$.

Let $B_{K_1} = B(\gamma,u,K_1)$, and $\tau_{K_1,K_2,B} = \tau(\gamma,u,K_1,K_2,B)$, K_1, K_2, $B>0$ be given by

$$B_{K_1}\;=\;1\;+\;A_1\,\big(\,1\;+\;K_1\,\big)\;,\quad \tau_{K_1,K_2,B}\;=\;\tau_1\wedge\tau_2\;,$$

where A_1, A_2, A_3, A_5 are the constants appearing in the statement of Proposition 4.7.5 and Lemma 4.7.7 and

$$\tau_1^{-\kappa}\;=\;A_2\,K_2\,e^{A_3\,B}\;,\quad \tau_2^{-\kappa}\;=\;2\,A_5\,K_2\,e^{2\,A_5\,B}\;.$$

Next result is a straightforward consequence of Proposition 4.7.5 and Lemma 4.7.7. It asserts that Ψ is a contraction from the ball of radius B in the $C^\beta([0,T]\times\mathbb{T})$-topology to itself provided $T\le\tau$.

Lemma 4.7.8. *Fix $\gamma \in \mathbb{R}$, $u \in C^\beta(\mathbb{T})$, $K_1 > 0$, $K_2 > 0$ and $B \ge B_{K_1}$. Then, $\big\| \Psi(\mathfrak{w}) \big\|_{C^\beta([0,T]\times\mathbb{T})} \le B$ if $\big\| \mathfrak{w} \big\|_{C^\beta([0,T]\times\mathbb{T})} \le B$ and*

$$\big\| \Psi(\mathfrak{w}_1) \,-\, \Psi(\mathfrak{w}_2) \big\|_{C^\beta([0,T]\times\mathbb{T})} \;\le\; (1/2) \big\| \mathfrak{w}_1 \,-\, \mathfrak{w}_2 \big\|_{C^\beta([0,T]\times\mathbb{T})}$$

for all $0 < T \le \tau_{K_1,K_2,B}$, $R \in C^1(\mathbb{R}\times\mathbb{T})$ such that $\|R\|_{C^1([0,2\pi]\times\mathbb{T})} \le K_1$, $X \in C^\alpha(\mathbb{R}\times\mathbb{T})$ such that $\| X_\gamma \|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})} \le K_2$, $\mathfrak{w}_1$, $\mathfrak{w}_2$ in $C^\beta([0,T]\times\mathbb{T})$ such that $\mathfrak{w}_k(0,\cdot) = u(\cdot)$, $\big\| \mathfrak{w}_k \big\|_{C^\beta([0,T]\times\mathbb{T})} \le B$, $k = 1, 2$.

Proof of Theorem 4.7.3. The result follows from the previous lemma and a fixed point theorem in Banach spaces. □

Proof of Theorem 4.7.4. Fix $0 < \gamma < 2\sqrt{2\pi} - \sqrt{6\pi}$ and $\alpha \in (-1/2, \alpha_\gamma)$. By Theorem 4.6.1, $X_{\gamma,\varepsilon}$ converges in probability to X_γ in C^α, so that $\| X_\gamma \|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})}$ is almost surely finite. On the other hand, by Proposition A.3.5 below, $\|R\|_{C^1([0,2\pi]\times\mathbb{T})}$ is almost surely finite and $\| R - R_\varepsilon \|_{C^1([0,2\pi]\times\mathbb{T})}$ converges to 0 in probability.

Fix $0 < \zeta \le 1$, $\eta > 0$. It follows from the previous observations that there exists $K > 0$ and $\varepsilon_0 > 0$ such that

$$\begin{aligned} &\mathbb{P}\big[\, \|R\|_{C^1([0,2\pi]\times\mathbb{T})} > K \,\big] \;\le\; \eta\,, \quad \mathbb{P}\big[\, \| X_\gamma \|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})} > K \,\big] \;\le\; \eta\,, \\ &\mathbb{P}\big[\, \|R_\varepsilon \,-\, R\|_{C^1([0,2\pi]\times\mathbb{T})} > \zeta \,\big] \;\le\; \eta\,, \quad \mathbb{P}\big[\, \| X_{\gamma,\varepsilon} \,-\, X_\gamma \|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})} > \zeta \,\big] \;\le\; \eta\,, \end{aligned}$$

for all $0 \le \varepsilon < \varepsilon_0$. Note that we included $\varepsilon = 0$.

Denote by $\Omega_{K,\zeta}$ the union of the four sets appearing in the previous displayed formula. Let $B = B_{K+1} = 1 + A_1(2+K)$, $\tau = \tau_{K+1,K+1,B}$. On the set $\Omega^c_{K,\zeta}$, by Theorem 4.7.3, $\| \mathfrak{w}_\varepsilon \|_{C^\beta([0,\tau]\times\mathbb{T})} \le B$ for all $0 \le \varepsilon \le \varepsilon_0$.

We claim that on the set $\Omega^c_{K,\zeta}$

$$\| \mathfrak{w} \,-\, \mathfrak{w}_\varepsilon \|_{C^\beta([0,\tau]\times\mathbb{T})} \;\le\; A\, \zeta\, e^{AB} \tag{4.7.7}$$

for some constant $A = A(\gamma, u)$.

To prove this claim, let $\Psi = \Psi_{X,R,u}$, $\Psi_\varepsilon = \Psi_{X_\varepsilon,R_\varepsilon,u}$. Since $\mathfrak{w}$, $\mathfrak{w}_\varepsilon$ are solutions of the fixed point problem stated in Theorem 4.7.3,

$$\begin{aligned} \| \mathfrak{w} \,-\, \mathfrak{w}_\varepsilon \|_{C^\beta([0,\tau]\times\mathbb{T})} \;&=\; \| \Psi(\mathfrak{w}) \,-\, \Psi_\varepsilon(\mathfrak{w}_\varepsilon) \|_{C^\beta([0,\tau]\times\mathbb{T})} \\ &\le\; \| \Psi(\mathfrak{w}) \,-\, \Psi_\varepsilon(\mathfrak{w}) \|_{C^\beta([0,\tau]\times\mathbb{T})} \;+\; \| \Psi_\varepsilon(\mathfrak{w}) \,-\, \Psi_\varepsilon(\mathfrak{w}_\varepsilon) \|_{C^\beta([0,\tau]\times\mathbb{T})}\,. \end{aligned}$$

By Lemma 4.7.6, on the set $\Omega^c_{K,\zeta}$, the first term is bounded above by

$$A \Big\{ \big\| R \,-\, R_\varepsilon \big\|_{C^1([0,2\pi]\times\mathbb{T})} \;+\; \big\| X \,-\, X_\varepsilon \big\|_{C^\alpha([-4\pi,4\pi]\times\mathbb{T})}\, e^{A\,B} \Big\} \;\le\; A\, \zeta\, e^{AB}$$

for some constant $A = A(\gamma, u)$. By Lemma 4.7.8 with $K_1 = K_2 = K + 1$, $B = B_{K+1}$, on the set $\Omega^c_{K,\zeta}$, the second term is bounded by $(1/2) \| \mathfrak{w}_1 \,-\, \mathfrak{w}_2 \big\|_{C^\beta([0,\tau]\times\mathbb{T})}$. This proves (4.7.7).

Hence, there exists a finite constant $A = A(\gamma, u)$ with the following property. For all $\zeta > 0$, $\eta > 0$, there exists $\varepsilon_0 > 0$ such that for all $0 \le \varepsilon < \varepsilon_0$

$$\mathbb{P}\Big[\, \| \mathfrak{w} \,-\, \mathfrak{w}_\varepsilon \|_{C^\beta([0,\tau]\times\mathbb{T})} \;>\; A\, \zeta\, e^{AB} \,\Big] \;\le\; 4\,\eta\,.$$

This proves the theorem. □

Part III

Conclusion, Appendix and Bibliography

Chapter 5

Where to go from here

In this final chapter we discuss briefly some possible future directions for the methods used here

Green functions restricted to a domain

Observe that one of the main applications of the usage of the potential kernel in the setting of random walks is to be able to evaluate the function

$$G_A^X(x,y) := \mathbb{E}_x\left[\sum_{t=0}^{\tau_{A^c}} \mathbb{1}_{[X_t=y]}\right].$$

For $\alpha \in (1,2)$ we can still use the optional stopping time theorem in order to prove that

$$G_A^X(x,y) = -a_X(x,y) + \sum_{z\notin A} \mathbb{P}_x[X_{\tau_{A_c}=z}]a_X(z,y)$$

In the case of the simple random walk, the analysis of the function $H_A^X(x,z) := P_x[X_{\tau_{A_c}=z}]$ is simpler as its support is bounded. In order to approximate by its continuous counterpart, which can be explicitly calculated, see [17, Theorem A]. However, these computations come from an integral problem where one of the parameters is the potential kernel of the Lévy process, which is also explicit. At this point it is not clear that our expansion of a_X around $a_{\bar{X}}$ given in Theorem 2.3.5 would translate in a good expansion of $H_A^X(x,z)$ around its continuous counterpart $H_A^{\bar{X}}(x,z)$. If we manage to do so, we could translate it into a good understanding of G_A^X as well and apply this to the study of many models in probability.

Homogenization of Gaussian Fields

In Chapter 3, we proved the convergence of discrete random fields defined in terms of a diffusion given by a random walk. Similarly, we could define a non-homogeneous model for which each site redistributes differently. In this case, we can even choose these local laws for the redistribution at random, and provided some reasonable conditions on the joint law, we can prove that we still have the same scaling limit, up to a multiplicative constant. This project is currently in progress, and using techniques based on [6, 46] we seem to be able to prove it both in the continuous (in the spirit of the Generalised Gaussian Free Fields present in [49]) and the discrete setting.

Schauder estimates and BPHZ renormalisation for fractional Laplacians

Finally, we believe that the strategy that we used to prove Schauder estimates in Chapter 4 can be extended to the context of regularity structures in any dimension and for any $\alpha \in \mathbb{Q}\cap(0,2)$.

There, we would need to prove the existence of the abstract convolution with the kernel breaking it again in smoother parts with support bounded away from the $\{t = 0\}$ hyperplane.

A more delicate question is lies in further tools developed after the original article [52]. More specifically, when we try to renormalise fields constructed as nonlinear maps of singular fields, we need to evaluate certain integrals. Such integrals involve multiple convolutions of the heat kernel with itself and are usually easier to understand via a graphical representation. The theoretical technology for manipulating such objects is an interesting topic by itself and has inspired a series of papers [55, 22, 54]. Although expected, it is not clear whether the lack of smoothness of long-range kernels could somehow break the current results for such representations. Again, it would be interesting to explore such ideas in the future.

Appendix A

Appendix

A.1 Evaluation of some special integrals

Lemma A.1.1. *For $\alpha \in (0,2) \setminus \{1\}$, we have that R_α^∞ defined in* (2.4.4) *satisfies*

$$R_\alpha^\infty(\theta) = K_2|\theta|^2 + \mathcal{O}(|\theta|^{2+\alpha}) \tag{A.1.1}$$

where

$$K_2 = \frac{1-\alpha}{2}\Bigg(\left(\frac{2^{2-\alpha}-1}{2-\alpha} - \frac{3(2^{1-\alpha}-1)}{2(1-\alpha)}\right) + \frac{1}{2\Gamma(\alpha)}\sum_{m=1}^{\infty}(-1)^m(\zeta(m+\alpha)-1)\frac{m\Gamma(m+\alpha)}{\Gamma(m+2)(m+2)}\Bigg). \tag{A.1.2}$$

Proof. Recall that $\theta > 0$ and

$$R_\alpha^\infty = \theta^{1+\alpha}\int_\theta^\infty \Big(\frac{z\sin(z)-(1+\alpha)(1-\cos(z))}{z^{2+\alpha}}\Big)P_1\Big(\frac{z}{\theta}\Big)dz$$

and $P_1(x) = (x - \lfloor x \rfloor) - \frac{1}{2}$. Note that this integral is finite. Indeed, one can prove this by observing that $|P(z)| \le \frac{1}{2}$. We shall now divide the integral in R_α^∞ in two parts, one going from θ to 1 and the other 1 to ∞, as we will use different techniques to bound them.

$$R_\alpha^\infty = \underbrace{\theta^{1+\alpha}\int_\theta^1 \frac{z\sin(z)-(1+\alpha)(1-\cos(z))}{z^{2+\alpha}}P_1\Big(\frac{z}{\theta}\Big)dz}_{I_1} + \underbrace{\theta^{1+\alpha}\int_1^\infty \frac{z\sin(z)-(1+\alpha)(1-\cos(z))}{z^{2+\alpha}}P_1\Big(\frac{z}{\theta}\Big)dz}_{I_2}.$$

We start by analysing I_2 and proving that $I_2 = \mathcal{O}(|\theta|^{2+\alpha})$,

$$I_2 = \theta^{1+\alpha}\int_1^\infty \frac{z\sin(z)-(1+\alpha)(1-\cos(z))}{z^{2+\alpha}}P_1\Big(\frac{z}{\theta}\Big)dz.$$

For convenience, we assume that $\theta^{-1} \in \mathbb{N}$. To treat the general case we need to compare the expressions between for θ^{-1} and $\lfloor \theta^{-1} \rfloor$.

In this case, we can write the integral above as

$$I_2 = \theta^{1+\alpha}\sum_{k=1/\theta}^{\infty}\int_{k\theta}^{(k+1)\theta} g(z)\left(\frac{z}{\theta} - k - \frac{1}{2}\right)dz,$$

where $g(z) := \frac{z\sin(z)-(1+\alpha)(1-\cos(z))}{z^{2+\alpha}}$. Now, we will use that $\int_{k\theta}^{(k+1)\theta} P_1\left(\frac{z}{\theta}\right) dz = 0$ and sum and subtract the term $g(k\theta)$ in each term of the summands. Hence,

$$\begin{aligned}|I_2| &= |\theta|^{1+\alpha}\left|\sum_{k=1/\theta}^{\infty}\int_{k\theta}^{(k+1)\theta}(g(z)-g(k\theta))\left(\frac{z}{\theta}-k-\frac{1}{2}\right)dz,\right| \\ &\le |\theta|^{1+\alpha}\sum_{k=1/\theta}^{\infty}\sup_{y\in[k\theta,(k+1)\theta]}|g'(y)|\int_{k\theta}^{(k+1)\theta}|z-k\theta|\left|\frac{z}{\theta}-k-\frac{1}{2}\right|dz, \\ &\le \frac{1}{4}|\theta|^{3+\alpha}\sum_{k=1/\theta}^{\infty}\sup_{y\in[k\theta,(k+1)\theta]}|g'(y)|,\end{aligned}$$

where we used in the second inequality both a change of variables and that $|z-k\theta|\le\theta$.

For $z>0$, we have

$$g'(z) = \frac{\cos(z)}{z^{1+\alpha}} - 2(1+\alpha)\frac{\sin(z)}{z^{2+\alpha}} + (1+\alpha)(2+\alpha)\frac{1-\cos(z)}{z^{3+\alpha}}$$

and therefore

$$|g'(z)| \lesssim_\alpha \frac{1}{z^{1+\alpha}}$$

which implies

$$\sup_{[k\theta,(k+1)\theta]}|g'(z)| \lesssim_\alpha \frac{1}{(\theta k)^{1+\alpha}}.$$

We can now use this in the estimate of $|I_2|$ to get

$$|I_2| \lesssim \theta^2\sum_{k=1/\theta}^{\infty}\frac{1}{k^{1+\alpha}} \lesssim |\theta|^{2+\alpha}$$

and $I_2 = \mathcal{O}(|\theta|^{2+\alpha})$

Now, for I_1, we use Taylor expansion of the function $h(z) = z\sin z - (1+\alpha)(1-\cos z)$ to get

$$h(z) = \frac{1-\alpha}{2}z^2 - \frac{3-\alpha}{24}z^4 + r(z),$$

where $r(z) = \mathcal{O}(z^6)$. We get

$$\begin{aligned}I_1 &= \theta^{1+\alpha}\frac{1-\alpha}{2}\int_\theta^1\frac{1}{z^\alpha}P_1\Big(\frac{z}{\theta}\Big)dz - \theta^{1+\alpha}\frac{3-\alpha}{24}\int_\theta^1 z^{2-\alpha}P_1\Big(\frac{z}{\theta}\Big)dz \\ &\quad + \theta^{1+\alpha}\int_\theta^1\frac{r(z)}{z^{2+\alpha}}P_1\Big(\frac{z}{\theta}\Big)dz \\ &= \frac{1-\alpha}{2}I_{1,1} - \frac{3-\alpha}{24}I_{1,2} + I_{1,3}\end{aligned}$$

Again we examine each of the terms separately. We start with the last one. For this, notice that $r(\cdot)$ is a $C^\infty([-1,1])$ function, as it is the difference of two such functions. Moreover, we know that $\tilde r(z) := \left|\frac{r(z)}{z^{2+\alpha}}\right|$ and therefore, applying Lemma A.2.1 we have that $\tilde r(\cdot)$ is in $C^{0,\frac{4-\alpha}{6}-}([-1,1])$. Now we can proceed like we did for I_2 to get that $I_{1,3}$ is of order $\mathcal{O}(\theta^{2+\alpha})$.

The first integral $I_{1,1}$ can be written as, again assuming that $\theta^{-1}\in\mathbb{N}$,

$$\begin{aligned}I_{1,1} &= \theta^{1+\alpha}\sum_{k=1}^{\lfloor\frac{1}{\theta}\rfloor-1}\int_{k\theta}^{(k+1)\theta}\frac{1}{z^\alpha}\Big(\frac{z}{\theta}-k-\frac{1}{2}\Big)dz \\ &= \theta^2\sum_{k=1}^{\lfloor\frac{1}{\theta}\rfloor-1}k^{2-\alpha}\left[\frac{\left(1+\frac{1}{k}\right)^{2-\alpha}-1}{2-\alpha} - \Big(1+\frac{1}{2k}\Big)\frac{\left(1+\frac{1}{k}\right)^{1-\alpha}-1}{1-\alpha}\right].\end{aligned}$$

We now split between $k = 1$ and $k > 1$.

$$I_{1,1} = \theta^2\Big(\frac{2^{2-\alpha}-1}{2-\alpha} - \frac{3(2^{1-\alpha}-1)}{2(1-\alpha)}\Big) + \theta^2 \sum_{k=2}^{\lfloor \frac{1}{\theta}\rfloor -1} k^{2-\alpha}\left[\frac{\Big(1+\frac{1}{k}\Big)^{2-\alpha}-1}{2-\alpha} - \Big(1+\frac{1}{2k}\Big)\frac{\Big(1+\frac{1}{k}\Big)^{1-\alpha}-1}{1-\alpha}\right]$$

Use now the full Taylor series of both $(1+x)^{2-\alpha}$ and $(1+x)^{1-\alpha}$ where we are taking $x = \frac{1}{k} \in (0,1)$ to explore the cancellations. We then get that the last sum is equal to

$$\theta^2 \sum_{k=2}^{\lfloor \frac{1}{\theta}\rfloor -1} \sum_{j=3}^{\infty} k^{2-\alpha-j}\frac{(j-2)\Gamma(1-\alpha)}{2j!\Gamma(-\alpha-j+3)}. \tag{A.1.3}$$

Therefore using the reflection formula for the Gamma function and a change of variables $m = j-2$, we get

$$(\text{A.1.3}) = \frac{\theta^2}{2\Gamma(\alpha)} \sum_{k=2}^{\lfloor \frac{1}{\theta}\rfloor -1} \sum_{m=1}^{\infty} (-1)^m k^{-\alpha-m}\frac{m\Gamma(m+\alpha)}{(m+2)!}.$$

Now, using Euler-Maclaurin again, one can easily prove that for $\alpha \in (0,2)$ and $m \geq 1$

$$\left|\sum_{k=2}^{\lfloor \frac{1}{\theta}\rfloor -1} k^{-\alpha-m} - (\zeta(m+\alpha)-1) + \frac{\theta^{m+\alpha-1}}{m+\alpha-1}\right| \lesssim \theta^{m+\alpha}. \tag{A.1.4}$$

Therefore there exist an explicit constant K_2

$$I_{1,1} = K_2|\theta|^2 + K_{1+\alpha}|\theta|^{1+\alpha} + \mathcal{O}(|\theta|^{2+\alpha}).$$

Finally, we can show in an analogous way that $I_{1,2} = \mathcal{O}(|\theta|^4)$. For the case $\alpha = 1$ we proceed similarly. We need to evaluate

$$R_1^\infty = \theta^2 \int_\theta^\infty \Big(\frac{z\sin(z) - 2(1-\cos(z))}{z^3}\Big) P_1\Big(\frac{z}{\theta}\Big) dz,$$

Using similar ideas as before and the fact that $z\sin(z) - 2(1-\cos(z)) = \mathcal{O}(z^4)$ when $|z| \to 0$ instead of the order $\mathcal{O}(z^2)$ that we got for the case $\alpha \in (1,2)$ we conclude the proof. □

Lemma A.1.2. *Let $z \in [1,\infty)$ define*

$$\text{cin}(z) := \int_0^z \frac{1-\cos(t)}{t} dt.$$

We have that

$$\text{cin}(z) = \log z + \gamma + \mathcal{O}(z^{-1}).$$

as $z \longrightarrow \infty$ where γ is the Euler-Mascheroni constant

Proof. By defining

$$\text{ci}(z) := -\int_z^\infty \frac{\cos(t)}{t} dt$$

the linearity of the integral implies that

$$\text{cin}(z) = \log z - \text{ci}(z) + \int_1^\infty \frac{\cos t}{t} dt + \int_0^1 \frac{1-\cos t}{t} dt.$$

The exact value of the sum of the two integrals is not relevant for us, but it is known to be γ. Therefore,

$$\text{cin}(z) = -\text{ci}(z) + \log z + \gamma.$$

finally, it is easy to show that $\text{ci} = \mathcal{O}(z^{-1})$ as $z \to \infty$. □

A.2 Continuity estimates

Lemma A.2.1. *Let $f \in C^{1,\beta}(I)$ for a closed interval I containing the origin. Additionally, suppose that*

$$f(x) = \mathcal{O}\left(|x|^{\beta_0}\right) \ \text{ as } |x| \longrightarrow 0$$

for some $\beta_0 \geq 1 + \beta$. Let $1 < \beta_1 < \beta_0$ and define the function

$$h(x) := \frac{f(x)}{|x|^{\beta_1}}.$$

Then we have that the function h is in $C^{0,\bar{\beta}}(I)$ where $\bar{\beta} = \frac{\beta_0 - \beta_1}{\beta_0 - \beta}$. If instead, we have that $f \in C^{0,\beta}(I)$ for some $\beta \in (0,1)$, and $1/2 \leq \beta_1 < \beta_0 = 1$, we get that $h \in C^{0,\bar{\beta}}(I)$ with $\bar{\beta} := \beta(1 - \beta_1)$.

Proof. We will prove the first claim, the second can be proved analogously. Let $x, y \in I$ and assume, without loss of generality, that $|x| < |y|$,

$$\begin{aligned}\left|\frac{f(x)}{|x|^{\beta_1}} - \frac{f(y)}{|y|^{\beta_1}} \pm \frac{f(x)}{|y|^{\beta_1}}\right| &= \left|\frac{f(x)}{|x|^{\beta_1}}\left(\frac{|y|^{\beta_1} - |x|^{\beta_1}}{|y|^{\beta_1}}\right) + \frac{f(x) - f(y)}{|y|^{\beta_1}}\right| \\ &\lesssim |x|^{\beta_0 - \beta_1}\frac{\left||y|^{\beta_1} - |x|^{\beta_1}\right|}{|y|^{\beta_1}} + \frac{|f(x) - f(y)|}{|y|^{\beta_1}}\end{aligned}$$

Now, we use that for $A, B > C > 0$ real numbers and $\delta \in [0,1]$, we have $C \leq A^{\delta}B^{1-\delta}$. Regarding the first term on the right-hand side, notice that

$$\left||y|^{\beta_1} - |x|^{\beta_1}\right| \lesssim \min\{|y|^{\beta_1}, |y|^{\beta_1 - 1}|x - y|\}$$

so choosing $A = |y|^{\beta_1}, B = |y|^{\beta_1 - 1}|x - y|$ and $\delta = \beta_0 - \beta_1$ we can easily see that

$$|x|^{\beta_0 - \beta_1}\frac{\left||y|^{\beta_1} - |x|^{\beta_1}\right|}{|y|^{\beta_1}} \lesssim |x - y|^{\delta} \leq |x - y|^{\bar{\beta}}.$$

To bound the second term, remark that $|f'(z)| \lesssim |y|^{\beta}$ for all $|z| \leq |y|$ since $f' \in C^{0,\beta}(I)$ and $f'(0) = 0$, so

$$|f(x) - f(y)| \lesssim \min\{|y|^{\beta_0}, |y|^{\beta}|x - y|\}$$

and again choosing $A = |y|^{\beta_0}, B = |y|^{\beta}|x - y|$ and $\delta = \bar{\beta}$ the claim follows. □

Lemma A.2.2. *If $p_X(\cdot)$ is admissible of index $\alpha \in (1,2)$, then $\phi_X(\cdot)$ is in $C^{1,\alpha-1-}(\mathbb{T})$. If $p_X(\cdot)$ is admissible of index 1, then ϕ_X is $C^{0,1-}(\mathbb{T})$.*

Proof. Notice that $p_X(\cdot)$ being admissible implies that it is in the basin of attraction of a α-stable distribution. Therefore, given $\beta \geq 0$ we have $\mathbb{E}_X[|X|^{\beta}] < \infty$ for $\beta \in (0, \alpha)$ and $p_X(x) \lesssim |x|^{-\alpha+}$. Now, we just write that $p_X(\cdot)$ as the Fourier transform of ϕ_X

$$\mathcal{F}_{\mathbb{T}}(\phi_X)(-x) = p_X(x).$$

Then use the classic relations between continuity and decay of Fourier coefficients, see [48, Proposition 3.3.12]. □

A.3 Some technical Lemmata for Section 4.7

Recall from (4.2.1) that we denote by $C^b_+(\mathbb{R}\times\mathbb{T})$, $0<b<1$, the elements f of C^b such $f(t)=0$ for $t\le 0$. Here we follow the same policy about constants specified in Section 4.7, that is: All constants below may depend on α,β,κ without any reference but not in any other variables.

Lemma A.3.1. *Fix $0<b<1$. For all f in $C^b_+(\mathbb{R}\times\mathbb{T})$ and $T>0$,*

$$\| f \|_{L^\infty([0,T]\times\mathbb{T})} \;\le\; T^b \, \| f \|_{C^b([0,T]\times\mathbb{T})}\;.$$

Proof. Fix $T>0$ and $(t,x)\in[0,T]\times\mathbb{T}$. Since $f(0)=0$, $|\, f(t,x)\,| \;=\; |\, f(t,x)-f(0,x)\,|$. Hence, by definition of the norm $\|\cdot\|_{C^b([0,T]\times\mathbb{T})}$, $|\, f(t,x)\,|$ is bounded by $T^b\,\| f\|_{C^b([0,T]\times\mathbb{T})}$, which proves the lemma. □

Fix $0<b<1$. It follows from the previous lemma that for all f in $C^b(\mathbb{R}\times\mathbb{T})$ and $T>0$,

$$\| f \|_{L^\infty([0,T]\times\mathbb{T})} \;\le\; \| f(0,\cdot) \|_{L^\infty(\mathbb{T})} \;+\; T^b \, \| f \|_{C^b([0,T]\times\mathbb{T})}\;, \tag{A.3.1}$$

where $f(0,\cdot)$ represents the restriction of the function f to $\{0\}\times\mathbb{T}$.

Lemma A.3.2. *Fix $\gamma\in\mathbb{R}$, $0<b<1$ and an element f in $C^b(\mathbb{R}\times\mathbb{T})$. Then, $\exp\{\gamma f\}$ belongs to $C^b(\mathbb{R}\times\mathbb{T})$ and for all $T>0$, $-T\le T_1<T_2\le T$,*

$$\| e^{\gamma f} \|_{C^b([T_1,T_2]\times\mathbb{T})} \;\le\; C_3(T)\, |\gamma|\, \| f \|_{C^b([T_1,T_2]\times\mathbb{T})}\;, \tag{A.3.2}$$

where $C_3(T)=\exp\{\,|\gamma|\,\| f\|_{L^\infty([-T,T]\times\mathbb{T})}\,\}$. Moreover, if f belongs to $C^b_+(\mathbb{R}\times\mathbb{T})$,

$$\| e^{\gamma f} \|_{C^b([0,T]\times\mathbb{T})} \;\le\; \exp\Big\{ (1+T^b)\,|\gamma|\, \| f \|_{C^b([0,T]\times\mathbb{T})} \Big\}\;. \tag{A.3.3}$$

Proof. The first claim follows from the bound $|e^y-e^x|\le \max\{e^{|x|},e^{|y|}\}\,|y-x|$, $x,\,y\in\mathbb{R}$. Now, suppose that f belongs to $C^b_+(\mathbb{R}\times\mathbb{T})$. By Lemma A.3.1, $\| f\|_{L^\infty([0,T]\times\mathbb{T})}\le \| f\|_{C^b([0,T]\times\mathbb{T})}T^b$. To complete the proof of the second lemma, it remains to recall that $a\le e^a$ for $a\ge 0$. □

Lemma A.3.3. *Fix $\gamma\in\mathbb{R}$, $0<b<1$ and f, g in $C^b(\mathbb{R}\times\mathbb{T})$. Assume that $f(0,x)=g(0,x)=J(x)$. Then, for all $T>0$,*

$$\| e^{\gamma f} \,-\, e^{\gamma g} \|_{C^b([0,T]\times\mathbb{T})} \;\le\; A_0\, e^{4\gamma A_1}\, \| f \,-\, g \|_{C^b([0,T]\times\mathbb{T})}\;,$$

where $A_0 \;=\; (1+2T^b)\,\gamma$ and

$$A_1 = \| J \|_{L^\infty(\mathbb{T})} \;+\; T^b \Big\{ \| f \|_{C^b([0,T]\times\mathbb{T})} \;+\; \| g \|_{C^b([0,T]\times\mathbb{T})} \Big\}\;.$$

Proof. Fix $T>0$, $z,\,w\in[0,T]\times\mathbb{T}$. Let $h=f-g$, and write $[\exp\{\gamma\, f(z)\} \,-\, \exp\{\gamma\, g(z)\}\,] \,-\, [\exp\{\gamma\, f(w)\} \,-\, \exp\{\gamma\, g(w)\}\,]$ as

$$\frac{e^{\gamma\, f(z)} \,-\, e^{\gamma\, g(z)}}{f(z)\,-\,g(z)}\,\big[\, h(z)-h(w)\,\big] \;+\; h(w)\Big(e^{\gamma\, g(z)} \,-\, e^{\gamma\, g(w)}\Big)\frac{e^{\gamma h(z)}-1}{h(z)}$$
$$+\; h(w)\, e^{\gamma\, g(w)}\Big(\frac{e^{\gamma h(z)}-1}{h(z)} \;-\; \frac{e^{\gamma h(w)}-1}{h(w)}\Big)\;.$$

Denote by f_∞, g_∞, h_∞ the $L^\infty([0,T]\times\mathbb{T})$ norms of f, g and h, respectively, and by f_b, g_b, h_b, the $C^b([0,T]\times\mathbb{T})$ norms of these three functions. Clearly, h belongs to $C^b_+([0,T]\times\mathbb{T})$. Hence, by Lemma A.3.1, $h_\infty\le T^b h_b$, while, by (A.3.1), $f_\infty\le J_\infty+T^b f_b$, with a similar inequality for g. Here, J_∞ stands for the $L^\infty(\mathbb{T})$ norm of J.

Consider separately the three terms of the previous displayed equation. It is not difficult to show that the first one is bounded by $\gamma\,\exp\{\gamma\,[g_\infty\,+\,f_\infty]\}\,h_b\,\|z\,-\,z'\|^b$, and the second one by $\gamma^2\,\exp\{\gamma\,[g_\infty\,+\,h_\infty]\}\,h_\infty\,g_b\,\|z\,-\,z'\|^b$. Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(\theta)=(e^\theta-1)/\theta$. Since $f'(\theta)\le e^{2|\theta|}$, the third term is bounded by $\gamma^2\,\exp\{\gamma\,[g_\infty+2h_\infty]\}\,h_b\,h_\infty\|z-z'\|^b$.

To complete the proof of the lemma it remains to add the bounds and to recall the estimates of f_∞, g_∞, h_∞ in terms of J_∞, f_b, g_b and h_b. □

Let f be a function in $C^1(\mathbb{R}_+ \times \mathbb{T})$. Denote by $\tilde{f} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ the function which is 2π-periodic in space and which coincides with f on $\mathbb{R}_+ \times [-\pi, \pi)$.

Lemma A.3.4. *For all $T > 0$, $0 \le s \le T$, x, $y \in \mathbb{R}$,*

$$\big| \tilde{f}(s,x) \,-\, \tilde{f}(s,y) \big| \;\le\; M_0 \, \big| x \,-\, y \big|^{\beta} \,, \tag{A.3.4}$$

where

$$M_0 \;=\; M_0(f,T) \;:=\; \big\| \, (\partial_x f) \, \big\|_{L^\infty([0,T]\times\mathbb{T})} \,.$$

Proof. As $\tilde{f}$ is 2π-periodic, we may replace y by y' such that $|y' - x| \le 1$. Then, use that f is uniformly Lipschitz on $[0,T] \times \mathbb{T}$, and finally that $|y' - x| \le |y' - x|^\beta \le |y - x|^\beta$ because $|y' - x| \le 1$, $\beta < 1$. □

We conclude this section proving that the sequence of random fields R_ε introduced in (4.7.2) converges in probability to $R = R_0$.

Proposition A.3.5. *We have that $\mathbb{P}\big[\, \|R\|_{C^1([0,2\pi]\times\mathbb{T})} \,<\, \infty \big] \,=\, 1$. Moreover, for every $\eta > 0$,*

$$\lim_{\varepsilon\to 0} \mathbb{P}\big[\, \|\, R \,-\, R_\varepsilon \,\|_{C^1([0,2\pi]\times\mathbb{T})} \,>\, \eta \,\big] \;=\; 0 \,.$$

Proof. By (4.5.6),

$$R_\varepsilon \;=\; \partial_t \mathfrak{G}_\varepsilon \,+\, (-\Delta)^{1/2}\, \mathfrak{G}_\varepsilon \;=\; \int_{-4\pi}^{2\pi} h(t-s)\, \xi_\varepsilon(s)\, ds$$

where $h = [\, \partial_t \,+\, (-\Delta)^{1/2} \,]\, r$ and $r = q - p$ is a smooth function. A similar identity holds with R_ε, ξ_ε replaced by R, ξ, respectively.

By [1, Proposition 1.3.3], $\sup_{z\in[0,2\pi]\times\mathbb{T}} R(z)$ has finite expectation as well as $-\inf_{z\in[0,2\pi]\times\mathbb{T}} R(z)$. The same bound holds for $\partial_x R$, $\partial_t R$. This proves that $\|R\|_{C^1([0,2\pi]\times\mathbb{T})}$ is almost-surely finite.

As r is smooth, the same theorem guarantees that there exists a finite constant C_0 such that $\mathbb{E}\big[\, \sup_{z\in[0,2\pi]\times\mathbb{T}} \{\, R_\varepsilon(z) - R(z) \,\} \,\big] \,<\, C_0\, \varepsilon$ for all $0 < \varepsilon \le 1$. The same result holds for $R(z) - R_\varepsilon(z)$ and for the first partial derivatives. It follows from these estimates that $\|\, R \,-\, R_\varepsilon \,\|_{C^1([0,2\pi]\times\mathbb{T})}$ converges to 0 in probability as $\varepsilon \to 0$. □

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