Velocity averaging lemmas and applications

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To my father,
FERNANDO NARIYOSHI

YOU JUST CAN'T BEAT JESUS CHRIST

He was born to be known as everybody's brother
He is the Father's Son and Mary is His mother
He is a 'scuse my slanguage, well a compound country kinda guy
Ain't no way to get around it, you just can't beat Jesus Christ

I used to crank and drink until my back was to the floor
I'd take it to the limit, then I'd try to get some more
Yes, when it came to gamblin', well Lord God knows I'd roll them dice
Ain't no two ways about it, I have been saved by Jesus Christ

Even though I am a sinner He will always be my friend
Well He starts in the middle and He does not have an end
And when my soul was held for ransom, yea He is the one who paid the price
Ain't no reason to deny it, I owe it all to Jesus Christ

Billy Joe Shaver (1939–2020)

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Further, I desire to make the words of Tony Rice (1951–2020) my own: *"And to those I have trespassed against, I beg your forgiveness; to those who have trespassed against me, I love you."*

Last but certainly not least, I would like to thank the greatest man ever lived (and who is alive to this very day), my Lord and Savior, Jesus Christ. He is the one who has made us all number two.

J. F. Nariyoshi[1]

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Resumo

Equações parabólicas-hiperbólicas degeneradas são amplamente empregadas para modelar diversos importantes fenômenos naturais, como processos de sedimentação-consolidação e escoamentos multifásicos em meios porosos. Matematicamente, as soluções para tais equações são de difícil compreensão, pois essas exibem uma complicada mistura de comportamentos hiperbólicos e parabólicos. Uma contribuição fundamental para o entendimento de tais soluções foi a formulação cinética de P.-L. Lions *et al.*, que permite analisar essas equações microscopicamente. Como para retornar ao contexto macroscópico é necessário se tomar certas médias, pode-se deduzir várias propriedades não-triviais de tais soluções por meio dos chamados *"velocity averaging lemmas"*. Apesar de esta célebre técnica ser bem entendida em um âmbito puramente hiperbólico, ela ainda está possivelmente subdesenvolvida para equações parabólicas-hiperbólicas gerais.

Nesta tese de doutorado, introduzimos um método de se estabelecer velocity averaging lemmas para uma extensa classe de equações parabólicas-hiperbólicas. Subsequentemente, aplicamos tais lemas para provar novos resultados acerca de problemas não-lineares, a saber: princípios gerais de compacidade para soluções de entropia para equações parabólicas-hiperbólicas degeneradas determinísticas; a propriedade de traço forte para soluções de entropia para leis de conservação estocásticas; a boa colocação de um problema de Neumann não-linear para leis de conservação estocásticas e a regularidade de Sobolev das suas soluções. Finalmente, em capítulos complementares, elaboramos um método geral para se estudar problemas não-degenerados estocásticos e estudamos a suavidade de soluções para um problema parabólico-hiperbólico. A teoria desta tese desenvolve vários resultados bem-conhecidos, como alguns de P.-L. Lions *et al.*, E. Tadmor–T. Tao, A. Vasseur e outros.

Palavras-chave: Velocity averaging lemmas, equações parabólicas-hiperbólicas degeneradas, leis de conservação estocásticas.

Abstract

Degenerate parabolic-hyperbolic equations are widely employed to model several important natural phenomena, such as sedimentation-consolidation processes and multiphase flows in porous media. From a mathematical standpoint, the solutions to such equations are difficult to comprehend, for they display a complicated mixture of hyperbolic and parabolic behaviors. A fundamental contribution to understanding such solutions was the kinetic formulation introduced by P.-L. Lions *et al.*, which permits one to analyze these equations microscopically. Since it is required to take certain averages to return to the macroscopic context, one can deduce many nontrivial properties of such solutions via the so-called "velocity averaging lemmas". Even though this celebrated technique is well-understood in a purely hyperbolic framework, it is arguably still underdeveloped for general parabolic-hyperbolic equations.

In this Ph.D. thesis, we introduce a method for establishing velocity averaging lemmas for an extensive class of parabolic-hyperbolic equations. Subsequently, we apply such lemmas to prove new results on nonlinear problems, namely: some general compactness principles for entropy solutions to deterministic parabolic-hyperbolic equations; the strong trace property for entropy solutions to stochastic conservation laws; the well-posedness of a nonlinear Neumann problem for stochastic conservation laws, and the Sobolev regularity of its solutions. Finally, in complementary chapters, we elaborate a general method for studying stochastic nondegenerate problems, and we investigate the smoothness of solutions to a parabolic-hyperbolic problem. This thesis's theory develops many well-known theorems, including some due to P.-L. Lions *et al.*, E. Tadmor–T. Tao, and A. Vasseur, among others.

Keywords: Velocity averaging lemmas, degenerate parabolic-hyperbolic equations, stochastic conservation laws.

Contents

Chapter 1

Introduction

1.1 Motivation

This thesis is dedicated to the renowned technique of *velocity averaging*, and its profound consequences to the field of both deterministic and stochastic quasi-linear degenerate convection-diffusion equations.

In broad terms, a velocity averaging lemma, or just an averaging lemma, is a technical mathematical proposition regarding the regularity the so-called velocity averages

$$\int_{\mathbb{R}} f(t,x,v)\eta(v)\,dv, \tag{1.1}$$

where $\eta:\mathbb{R}\to\mathbb{R}$ is a weight function, $(t,x,v)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}$, and $f(t,x,v)$ is governed by a second-order multidimensional parabolic-hyperbolic equation of the general form

$$\frac{\partial f}{\partial t}(t,x,v)+\mathbf{a}(v)\cdot(\nabla_x f)(t,x,v)-\operatorname{div}_x\big(\mathbf{b}(v)(\nabla_x f)(t,x,v)\big)=\Lambda(t,x,v), \tag{1.2}$$

in which $\mathbf{a}(v)\in\mathbb{R}^N$ is a convection vector, $\mathbf{b}(v)\in\mathscr{L}(\mathbb{R}^N)$ is a nonnegative (but not necessarily uniformly positive) diffusion matrix, and $\Lambda(t,x,v)$ is a distribution that may contain measures, weak derivatives, stochastic noises *etc.* Although it is well known that a solution $f(t,x,v)$ to (1.2) may not exhibit any sort of smoothing effect, the startling observation is that the velocity averages (1.1)—which generally are the physically relevant quantities—may.

Thus, to illustrate the spirit and importance of the velocity averaging techniques, let us first recollect some aspects of the theory of entropy solutions to nonlinear degenerate problems.

A myriad of important natural phenomena, including—but not limited to—sedimentation-consolidation processes, the two- and three-phase flow in porous media, heat propagation by radiation in plasmas, and population dynamics, may be mathematically described by a degenerate parabolic–hyperbolic equation of the form

$$\frac{\partial u}{\partial t}+\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\mathbf{A}_j(u)-\sum_{j,k=1}^{N}\frac{\partial}{\partial x_j}\left(\mathbf{b}_{jk}(u)\frac{\partial u}{\partial x_k}\right)=\mathbf{S}(u), \tag{1.3}$$

where $N\geq 1$ is the spatial dimension, $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ represents the temporal-spatial variable, $u(t,x)\in\mathbb{R}$ describes the unknown field, $\mathbf{A}(u)=(\mathbf{A}_1(u),\ldots,\mathbf{A}_N(u))$ denotes a flux function, $\mathbf{b}(u)=(\mathbf{b}_{jk}(u))_{1\leq j,k\leq N}$, again, stands for a diffusion matrix, and $\mathbf{S}$ may be interpreted as a source term; see, *e.g.*, G. Chavent–J. Jaffre [21], G. Gagneux–M. Madaune-Tort [46], M. C. Bustos *et al.* [17], and J. L. Vazquez [111]. As a result, Equation (1.3) and its variants, also known as convection-diffusion equations, have been objects of great interest to scientists, engineers, and mathematicians throughout the years.

From a mathematical perspective, the fact that the diffusive matrix $\mathbf{b}_{jk}(u)$ may degenerate (*i.e.*, may vanish) at certain points poses crucial difficulties to the theoretical comprehension of (1.3). Indeed, by way of illustration, let us consider the extreme case in which $\mathbf{b}_{jk}(u) \equiv 0$ identically, so that Equation (1.3) is transformed into a hyperbolic conservation law. Then, the celebrated method of the characteristics shows that smooth solutions to (1.3) may develop shock discontinuities in finite time. Moreover, the classical Riemann problem demonstrates that weak solutions to (1.3) lack uniqueness properties in general (see S. ALINHAC [3] for details).

In virtue of its applications' importance, it becomes a challenging problem to determine a technical framework to such equations that is both mathematically and physically satisfactory. As it turns out, the adequate way of investigating (1.3) is by means of the notion of an entropy solution, which was firstly introduced by S. N. KRUZHKOV [74] in 1970 in the context of conservation laws, and only extended by J. CARRILLO [19] for parabolic-hyperbolic equations 29 years later. The fundamental feature of such solutions is that, for they can be formally obtained as limits of solutions to nondegenerate parabolic problems (in a procedure parallel to the inviscid limit in Fluid Dynamics), they hereby possess residual smoothness properties known as "entropy conditions". The extra assumption that the considered solutions are entropy solutions is by itself sufficient to ensure the well-posedness of the initial-value problem to (1.3) in many situations; see, *e.g.*, L. HÖRMANDER [64].

Therefore, one can see that a profound characteristic of (1.3) is the complex interplay between parabolic and hyperbolic behaviors. A fundamental milestone to decipher the complicated structure of its solutions was the kinetic formulation invented in 1994 by P.-L. LIONS–B. PERTHAME–E. TADMOR [82] for conservation laws, and then generalized by G.-Q. CHEN–B. PERTHAME [27] for general parabolic-hyperbolic equations in 2003. Essentially, they introduced a new variable $v \in \mathbb{R}$—traditionally called "velocity"—and a "change of variables" $u(t,x) \mapsto f(t,x,v)$, under which the entropy solutions $u(t,x)$ to (1.3) now observed a kinetic equation of the form

$$\frac{\partial f}{\partial t} + \sum_{j=1}^{N} \mathbf{a}_j(v) \frac{\partial f}{\partial x_j} - \sum_{j,k=1}^{N} \frac{\partial}{\partial x_j}\left(\mathbf{b}_{jk}(v) \frac{\partial f}{\partial x_k}\right) = \frac{\partial \mathbf{m}}{\partial v} + \mathbf{S}(v)\delta_{u(t,x)}(v), \tag{1.4}$$

where $\mathbf{a}(v) = (\mathbf{a}_1(v), \ldots, \mathbf{a}_N(v)) = \mathbf{A}'(v)$, and $\mathbf{m}(t,x,v)$ is a nonnegative measure sometimes known as the "entropy production measure". This formulation has an interesting connection with the kinetic theory of gases, and thus (1.4) may be thought of as the "microscopic" counterpart of the "macroscopic" Equation (1.3).

Although the kinetic formulation is mathematically equivalent to the entropy one when the considered solutions are bounded, there are certain advantages to analyzing (1.3) through (1.4). First of all, in spite of its right-hand side being somewhat singular, Equation (1.4) is linear in $f(t,x,v)$. Moreover, as G.-Q CHEN–B. PERTHAME [27] deftly demonstrated, the kinetic formulation provides a general and simplified approach to uniqueness theorems. Thirdly—and most importantly for us here—, P.-L. LIONS–B. PERTHAME–E. TADMOR [82] observed $u(t,x)$ may be reconstructed from $f(t,x,v)$ via the integral

$$u(t,x) = \int_{\mathbb{R}} f(t,x,v)\,dv.$$

Hence, for (1.4) is of the same type as (1.2), one can extract several nontrivial regularizing results for the original solutions $u(t,x)$ by means of velocity averaging lemmas, which, in some sense, quantify the aforementioned "residual smoothness" of the entropy solutions. So as to illustrate this point, applications of the velocity averaging lemmas allowed several authors to successfully establish

(a) the existence of entropy solutions employing the vanishing viscosity method (see, *e.g.*, R. BÜRGER–H. FRID–K. H. KARLSEN [15], H. FRID–Y. LI [42], B. GESS–M. HOFMANOVÁ [51]),

(b) the strong trace property (see, *e.g.*, A. VASSEUR [110], Y.-S. KWON–A. VASSEUR [75], H. FRID–Y. LI [42], and H. FRID *et al.* [43]), which is crucial to prove the uniqueness of solutions to many initial-boundary value problems (see the discussion in [110]),

(c) the existence of an asymptotic state (see, *e.g.*, G.-Q. Chen–H. Frid [23, 24], and G.-Q. Chen–B. Perthame [28]),

(d) the Sobolev regularity of entropy solutions (see, *e.g.*, P.-L. Lions–B. Perthame–E. Tadmor [82], T. Tao–E. Tadmor [107], B. Gess–M. Hofmanová [51], B. Gess–X. Lamy [52], B. Gess [50], and B. Gess–J. Sauer–E. Tadmor [53]),

among other propositions.

Velocity averaging lemmas possess a rich, albeit relatively short history, beginning with the original works of V. I. Agoshkov [2] and C. Bardos *et al.* [9] on transport equations. Later on, these averaging lemmas for transport equations were further delved into by F. Golse *et al.* [55], R. J. DiPerna–P.-L. Lions [36, 37] (with applications to the Boltzmann and Vlasov–Maxwell equations), R. J. DiPerna–P.-L. Lions–Y. Méyer [38] (with a general, noncritical source term in L^p), M. Bézard [11] and P.-L. Lions [81] (both of the latter studying optimal regularity in Sobolev spaces), B. Perthame–P. Souganidis [97] (with a general critical source term in L^p), R. DeVore–G. Petrova [34] (establishing optimal regularity in Besov spaces), L. Saint-Raymond [101] and F. Golse–L. Saint-Raymond [56, 57] (in an L^1–framework and with important consequences to the Navier–Stokes equations), P.-E. Jabin–H.-Y. Lin–E. Tadmor [69] (using commutator techniques), and D. Arsénio–N. Lerner [7] (employing an energy method), among many others.

The first applications to nonlinear conservation laws were given by P.-L. Lions–B. Perthame–E. Tadmor [82] with the introduction of the kinetic formulation. Their results were subsequently extended by the aforementioned work of B. Perthame–P. Souganidis [97], P.-E. Jabin–B. Perthame [70] (see also P.-E. Jabin–L. Vega [71, 72] for a similar theorems), M. Westdickenberg [112], and F. Berthelin–S. Junca [10], just to name a few.

Let us also point out that an L^2–theory of averaging lemmas for general partial differential operators was devised by P. Gérard [47, 48], P. Gérard–F. Golse [49], and M. Lazar–D. Mitrović [77, 78] using techniques of H-measures. Additionally, it is equally worth mentioning the applications of velocity averaging lemmas to numerical schemes by L. Desvillettes–S. Mischler [33], S. Mischler [85], F. Bouchut–L. Desvillettes [12], T. Horsin–S. Mischler–A. Vasseur [65], and N. Ayi–T. Goudon [8].

The vast majority of the aforesaid works was restricted to first-order equations, with notable exceptions being some statements in P.-L. Lions–B. Perthame–E. Tadmor [82] regarding hyperbolic-parabolic equations, the abstract theory of P. Gérard [47, 48] with F. Golse [49], and the parabolic averaging lemmas of M. Lazar–D. Mitrović [77, 78] (see also the very recent work of M. Erceg-M. Misur–D. Mitrovic [39]). As a matter of fact, the study of velocity averaging lemmas for convection-diffusion equations has a contrastingly much smaller body of literature and is largely influenced by the towering theory of E. Tadmor–T. Tao [107]. Their results delved into the Sobolev regularity of entropy solutions to such second-order equations, and they were based on dyadic partitions of the frequency space in terms of the Littlewood–Paley decomposition and the symbol of Equation (1.2),

$$\mathcal{L}(i\tau, i\kappa, v) \stackrel{\text{def}}{=} i(\tau + \mathbf{a}(v)\cdot\kappa) + \kappa\cdot\mathbf{b}(v)\kappa. \qquad \big[\,\tau\in\mathbb{R}, \kappa\in\mathbb{R}^N,\ \text{and } v\in\mathbb{R}.\big] \tag{1.5}$$

Consequently, in order to ensure the convergence of such expansions, it was necessary to impose uniform decay rates on the quantities

$$\omega(J;\delta) = \sup_{\sqrt{\tau^2+|\kappa|^2}\sim J} \operatorname{meas}\Big\{v\in\operatorname{supp}\eta;\, |\mathcal{L}(i\tau,i\kappa,v)|\le\delta\Big\}. \tag{1.6}$$

By carefully studying the L^r–norm of these parcels, one could then verify the $W^{s,r}$–regularity of the averages (1.1). This method was further expanded in a series of works by B. Gess–M. Hofmanová

[51] (with applications to stochastic quasilinear degenerate hyperbolic-parabolic equations), B. Gess–X. Lamy [52] (studying a conservation law with sources), B. Gess [50] and B. Gess–J. Sauer–E. Tadmor [53] (both of the latter establishing the optimal Sobolev regularity for the porous medium equation).

Despite the impressive power and elegance of such an approach, it is not without a few shortcomings. We enumerate some below.

- First of all, except for some elementary examples, the examination of the quantities (1.6) is somewhat laborious and so far have led only to partial results. For instance, concerning the simple parabolic-hyperbolic equation in $\mathbb{R}_t \times \mathbb{R}_x \times \mathbb{R}_y$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left\{\frac{1}{\ell+1}u^{\ell+1}\right\} - \frac{\partial^2}{\partial y^2}\left\{\frac{1}{n+1}|u|^n u\right\} = 0, \tag{1.7}$$

  where n and ℓ are positive integers, their theory has as yet been shown to be applicable under the restriction that $n \geq 2\ell$; see E. Tadmor–T. Tao [107]. (Equations resembling to (1.7) were encountered by L. Graetz [58, 59] and W. Nusselt [88] when investigating the phenomena of heat transfer in fluids).

- As the behavior of the symbol $\mathcal{L}(i\tau, i\kappa, v)$ is only treated obliquely via the quantities (1.6), it is not clear which class of tempered distributions $\Lambda(t, x, v)$ is admissible in the right-hand side of (1.2).

- Likewise, it is not clear if the Tadmor–Tao theory permits the diffusion matrix $\mathbf{b}(v)$ to degenerate on intervals, allowing Equation (1.2) to display a hyperbolic and a parabolic phase. This hypothesis is not of complete superficiality, as it appears naturally in applications to sedimentation-consolidation processes (see M. C. Bustos *et al.* [17]).

The purpose of this thesis is to present a novel approach to the theory of averaging lemmas that overcomes the difficulties previously listed. The most interesting features of our method include:

(i) The nondegeneracy conditions we consider are inspired by those introduced by P.-L. Lions–B. Perthame–E. Tadmor [82], once they are variants of

$$\text{“meas}\big\{v \in \operatorname{supp}\eta;\ \mathcal{L}(i\tau, i\kappa, v) = 0\big\} = 0 \text{ for all } (\tau, \kappa) \in \mathbb{R} \times \mathbb{R}^N \text{ with } \tau^2 + |\kappa|^2 = 1\text{”}. \tag{1.8}$$

As a consequence, they are of substantially easier verification.

(ii) The distributions $\Lambda(t, x, v)$ appearing in (1.2) are allowed to have the form $\mathscr{E}(-\Delta_v + 1)^{\ell/2} g$, where $\ell \geq 0$, $g \in L^q(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v)$ $(1 < q < \infty)$, and $\mathscr{E}$ is an elliptic operator that "tightly" dominates $\mathcal{L}\big(\frac{\partial}{\partial t}, \nabla_x, v\big)$. In particular, they can always involve full spatio-temporal derivatives of g, and they may contain second-order spatial derivatives of g if $\mathcal{L}\big(\frac{\partial}{\partial t}, \nabla_x, v\big)$ is parabolic for that particular velocity v, hence the "criticality" of our averaging lemmas. Accordingly, one gains an ample notion of the regularizing properties associated to the averaging process $f \mapsto \int_{\mathbb{R}} f\eta\, dv$.

(iii) The proofs are quite straightforward and transparent. Indeed, our arguments are based in the direct method of H. Frid *et al.* [43] (see also G.-Q. Chen–H. Frid [23], W. Neves [86], and the averaging lemma 2.1 in E. Tadmor–T. Tao), but they contain refinements in every aspect.

(iv) Our averaging lemmas are well-adapted to be used in several nonlinear problems of deterministic and stochastic nature. Furthermore, modifying our arguments conveniently, they may be employed to study the Sobolev regularity of entropy solutions in a plainer way as well.

This thesis's results were mainly motivated by the problem of proving the strong trace property for entropy solutions to stochastic parabolic-hyperbolic equations closely resembling (1.7). Such a problem was successfully solved with H. Frid, Y. Li, D. Marroquin, and Z. Zeng [45] via the techniques of this manuscript (see also the revised version of H. Frid–Y. Li [43]). This thesis, besides presenting new averaging lemmas inspired by the revered work of P.-L. Lions–B. Perthame–E. Tadmor [82], also investigates novel applications, as averaging lemmas are arguably only mathematical curiosities if devoid of nontrivial implementations. Even though we could have tackled more complex questions, our desire was to write an exposition emphasizing examples of problems that, to the best of our knowledge, cannot be analyzed without the fundamental contribution of the velocity averaging lemmas. Notwithstanding, one can rest assured that not only further applications but also developments on the velocity averaging technique are to be explored in the future.

1.2 Content and organization of the text

We have structured the present thesis as follows.

1.2.1 Chapter 2: Critical velocity averaging lemmas

In Chapter 2, we present the main results of this text: a novel set of "critical" velocity averaging lemmas in the style of P.-L. Lions–B. Perthame–E. Tadmor [82] for Equation (1.2). To be more specific, we focus our attention on partial differential equations having the general form

$$\frac{\partial f}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f - \mathrm{div}_x\big(\mathbf{b}(v)\nabla_x f\big) = \mathscr{E}(-\Delta_v + 1)^{\ell/2}[g] + \Phi\,\frac{dW}{dt}, \tag{1.9}$$

where f and $g \in L^q(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v)$ for some $1 < q < \infty$, $\ell \geq 0$, $\mathscr{E}$ is an elliptic operator that "tightly dominates" $\mathcal{L}(\frac{\partial}{\partial t}, \nabla_x, v)f = \frac{\partial f}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f - \mathrm{div}_x\big(\mathbf{b}(v)\nabla_x f\big)$, W is a cylindrical Wiener process, and $\Phi(t,x,v)$ are diffusion coefficients. These propositions are in "the style of P.-L. Lions–B. Perthame–E. Tadmor" in the sense that they concern the relative compactness of the velocity averages $\int_{\mathbb{R}} f\eta\,dv$, which turns out to be the sought-after property in many situations.

Moreover, we explore some local versions of such averaging lemmas, which, in practice, are the useful propositions. Their proofs, however, require some additional, careful analysis in virtue of the probabilistic nature of the equation, and the possible presence of the second-order term $\mathrm{div}_x(\mathbf{b}(v)\nabla_x f)$. Finally, numerous aspects of our method are minutely discussed; in particular, we compare the obtained results with several well-known theorems in the literature, including those of P.-L. Lions–B. Perthame–E. Tadmor [82], E. Tadmor–T. Tao [107], and B. Gess–M. Hofmanová [51].

1.2.2 Chapter 3: The relative compactness of entropy solutions to degenerate parabolic–hyperbolic equations

Evidently, by reading Chapter 2 and nothing else, one may fail to grasp the reason for being of the averaging lemmas. Thus, we dedicate Chapter 3 to expose how the velocity averaging technique can be employed to deduce the relative compactness of entropy solutions to the (deterministic) degenerate convection-diffusion equation

$$\frac{\partial u}{\partial t}(t,x) + \mathrm{div}_x\mathbf{A}\big(u(t,x)\big) - D^2_x : \mathbf{B}\big(u(t,x)\big) = 0, \tag{1.10}$$

where (t,x) lies in some open set $Q \subset \mathbb{R}_t \times \mathbb{R}^N_x$, $\mathbf{A} : \mathbb{R} \to \mathbb{R}^N$ is a continuously differentiable flux function, and $\mathbf{B}(v) \in \mathscr{L}(\mathbb{R}^N)$ is a continuously differentiable matrix such that $\mathbf{B}'(v) \geq 0$ everywhere. Of course, Equation (1.10) is exactly the same as (1.2) if one introduces the "monotone" matrices $\mathbf{B}(u) = \int_0^u \mathbf{b}(v)\,dv$, and extrapolates the so-called "Frobenius inner product"

$$\mathbf{T} : \mathbf{U} = \sum_{j,k=1}^{N} \mathbf{T}_{j,k}\mathbf{U}_{j,k} = \text{ trace of } (\mathbf{T}^\star \mathbf{U})$$

to the Hessian matrix $D_x^2 = (\frac{\partial^2}{\partial x_j \partial x_k})_{1 \leq j,k \leq N}$, in a fashion that

$$D_x^2 : \mathbf{B}(u) = \sum_{j,k=1}^{N} \frac{\partial^2}{\partial x_j \partial x_k} \mathbf{B}_{j,k}(u).$$

This restatement, sometimes called the "conservative form" of (1.2), is fairly convenient, for it does not require $\mathbf{b}(u)\nabla_x u$ to make any formal sense.

First, one needs to introduce the notions of an entropy solution and of a kinetic formulation. We have thus adapted the definitions of the influential work of G.-Q. Chen–B. Perthame [27] to a local setting, including that of a kinetic solution to (1.10), a concept extends the definition of entropy solution to a pure L^1–setting. (See also M. Bendahmane–K.-H. Karlsen [16] for the related and very similar notion of a renormalized solution). In possession of the kinetic formulation, we can then derive quite easily some general compactness principles for kinetic, and consequently, entropy solutions to (1.10) via the velocity averaging lemmas of Chapter 2.

Furthermore, in Chapter 3, we also consider some extensions and stability results proposed in P.-L. Lions–B. Perthame–E. Tadmor [82].

Although our compactness results partially improve on several results in the literature—such as the ones of E. Yu. Panov [92, 93, 95, 96], and M. Lazar–D. Mitrovic [78]—, the important takeaway of Chapter 3 is rather the robustness and simplicity of the velocity averaging technique.

Essentially, the procedure we employ is the following. Let $(u_\nu)_{\nu \in \mathscr{I}}$ be some set of entropy solutions to (1.10). According to the celebrated Morrey's and Rellich–Kondrachov theorems, given any open set $U \subset\subset \mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v$, the injection of $\mathfrak{M}(U)$, the space of the Radon measures supported on U, in the negative Sobolev space $W^{-\varepsilon, q_\varepsilon}(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v)$ is compact for all $0 < \varepsilon < 1$ and all $q_\varepsilon > 1$ sufficiently close to 1 (see Lemma 3.2). Thus, if the measures $\mathbf{m}(t, x, v)$ in (1.4) have some satisfactory a priori estimates, then one would have locally reduced such equation to

$$\frac{\partial f}{\partial t} + \mathbf{a}(v) \cdot \nabla_x f - \mathbf{b}(v) : D_x^2 f = (-\Delta_{t,x} + 1)^{1/2}(-\Delta_v + 1)^{-(1+\varepsilon)/2} g \tag{1.11}$$

where f belongs to a bounded set of $L^{q_\varepsilon}_{t,x,v}$, and g belongs to some compact set of $L^{q_\varepsilon}_{t,x,v}$. Hence, assuming some "nondegeneracy condition" on the coefficients $\mathbf{a}(v)$ and $\mathbf{b}(v)$, the averaging lemmas guarantee the relative compactness of the averages $\int_{\mathbb{R}} f\eta\,dv$. In many instances, such as when the set $(u_\nu)_{\nu \in \mathscr{I}}$ is bounded in $L^\infty(Q)$, this argument immediately yields the relative compactness of $(u_\nu)_{\nu \in \mathscr{I}}$ itself in $L^1_{\text{loc}}(Q)$.

In other words, in order to prove the relative compactness of entropy solutions to a degenerate convection-diffusion equation like (1.3), one needs to obtain a kinetic equation of the form (1.1), where the distributions Λ can be "tamed" by the operator $\mathcal{L}(\frac{\partial}{\partial t}, \nabla_x, v)$ in its best (*i.e.*, nondegenerate) regime. This philosophy is quite flexible and—as it will be hopefully illustrated in this thesis—can be applied to several nonlinear problems.

1.2.3 Chapter 4: Strong traces for solutions to multidimensional stochastic scalar conservation laws

In the remaining chapters, we turn to the study of the stochastic scalar conservation law

$$\frac{\partial u}{\partial t}(t, x) + \operatorname{div}(\mathbf{A}(u(t, x))) = \sum_{k=1}^{\infty} g_k(x, u(t, x))\,\frac{d\beta_k}{dt}(t), \tag{1.12}$$

where (t,x) belongs to some open set $Q \subset \mathbb{R}_t \times \mathbb{R}_x^N$, $\mathbf{A} : \mathbb{R}_u \to \mathbb{R}^N$ is a flux function, $g_k : \mathbb{R}_x^N \times \mathbb{R}_u \to \mathbb{R}$ are diffusion coefficients, and $(\beta_k)_{k \in \mathbb{N}}$ is a sequence of mutually independent Brownian motions.

The theory of entropy solutions to (1.12) is considerably more intricate than that of its deterministic counterpart, for the Itô's formula—the corresponding version of the chain rule for stochastic processes—requires the twice-differentiability of an entropy, excluding thence any usage of the traditional entropies of S.N. Kruzkov [74]

$$\eta(u;k) = |u-k|.$$

Fortunately, one can still deduce a kinetic formulation that allows one to elaborate an elegant well-posedness theory for equations of this form; see, *e.g.*, A. Debussche–J. Vovelle [31], A. Debussche–M. Hofmanová–J. Vovelle [30], and B. Gess–M. Hofmanová [51].

We thus begin the study of (1.12) by extending to the stochastic case the outstanding result of A. Vasseur [110] on strong traces of entropy solutions. Informally, the main theorem of Chapter 4 asserts that, even though an entropy solution to (1.12) is in general discontinuous, one may still define its values on surfaces as strong limits in L^1. Besides being a quite attractive proposition, this is very instrumental in proving the uniqueness of solutions to boundary-value problems involving (1.12).

The content of Chapter 4 is mostly contained in a previous joint work with H. Frid, D. Marroquin, Y. Li, and Z. Zeng [43]. Nonetheless, we have taken the opportunity to delve deeper into the details, simplify the arguments, and weaken some of the hypotheses. A difference between this thesis's theorem and the one of the aforementioned paper is that we can consider diffusion coefficients $g_k(x,u)$ that are not even continuous—let alone differentiable.

The technique of velocity averaging appears in a sudden yet decisive point of the proof. It is remarkable to point out that the averaging lemma used has to be "critical", as it must be applied to an equation like (1.2) with full derivatives on the right-hand side.

Let us also point out that the theorem of A. Vasseur [110] was equally and significantly improved in the works of Y.-S. Kwon–A. Vasseur [75], E. Yu. Panov [90, 91], W. Neves–E. Yu. Panov–J. Silva [87], and M. Erceg–D. Mitrovic [40] under different and quite fascinating (albeit deterministic) contexts. Additionally, let us mention that, again with H. Frid, D. Marroquin, Y. Li, and Z. Zeng [45], we have generalized this strong trace theorem to a degenerate parabolic-hyperbolic equation of the form

$$\frac{\partial u}{\partial t}(t,x,y) + \operatorname{div}_{x,y}(\mathbf{A}(u(t,x,y))) - D_y^2 : \mathbf{B}(u(t,x,y)) = \sum_{k=1}^{\infty} g_k(x,y,u(t,x,y))\,\frac{d\beta_k}{dt}(t), \tag{1.13}$$

where, this time, $(t,x,y) \in Q \subset \mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_y^M$ for some integers N and $M \geq 1$, $\mathbf{A} : \mathbb{R}_u \to \mathbb{R}^{N+M}$ is a flux function, $\mathbf{B} : \mathbb{R}_u \to \mathscr{L}(\mathbb{R}^M)$ is such that $\mathbf{B}'(u) \geq 0$ everywhere, $g_k : \mathbb{R}_x^N \times \mathbb{R}_y^M \times \mathbb{R}_u \to \mathbb{R}$ are diffusion coefficients, and $(\beta_k)_{k\in\mathbb{N}}$ is a sequence of mutually independent Brownian motions. The extension is not trivial, but it involves a mixture of the arguments of Chapter 4 with the ones in H. Frid–Y. Li [42].

1.2.4 Chapter 5: The zero-flux problem for stochastic conservation laws

In the last major chapter of this thesis, we study the so-called zero-flux problem for stochastic conservation laws

$$\begin{cases} \dfrac{\partial u}{\partial t} + \operatorname{div}_x(\mathbf{A}(u)) = \displaystyle\sum_{k=1}^{\infty} g_k(x,u)\,\frac{d\beta_k}{dt}(t) & \text{for } (t,x) \in Q, \\ \mathbf{A}(u)\cdot\nu = 0 & \text{for } (t,x) \in (0,T)\times\partial\mathscr{O}, \text{ and} \\ u(0,x) = u_0(x) & \text{for } x \in \mathscr{O}. \end{cases} \tag{1.14}$$

Here $T > 0$ is an arbitrary number, $N \geq 1$ is an integer, $\mathscr{O} \subset \mathbb{R}^N$ is a open set whose outward unit normal at a point $x \in \partial\mathscr{O}$ is $\nu(x)$, $Q = (0,T) \times \mathscr{O}$, $\mathbf{A} : \mathbb{R} \to \mathbb{R}^N$ is a flux function, $\beta_k(t)$ are mutually independent Brownian motions, and $g_k(x,u)$ are diffusion coefficients. Problems like this arise in many applications, such as the sedimentation of suspensions in closed vessels, and the dispersal of a single species of animals in a finite territory; see R. Bürger–H. Frid–K. H. Karlsen [15], and the references therein.

The goal of Chapter 5 is to establish a well-posedness result for Equation (1.14), which simultaneously extends the conclusions of R. Bürger–H. Frid–K. H. Karlsen [15] and enhances the theorem proven with H. Frid *et al.* [45] (we refer to both for the literature regarding this problem). Consequently, we have partitioned this chapter into three sections, in every single of which the velocity averaging technique plays a quite protagonist role.

Section 5.2: Uniqueness

We begin by showing that, for an appropriate notion of entropy solution, (1.14) has at most one solution—indeed, we establish the so-called comparison principle, which provides a fairly quantitative uniqueness statement. So as to prove such proposition, we employ the variant of Kruzkov's doubling of variables technique by A. Debussche–J. Vovelle [31]. Once the boundary condition in (1.14) is essentially a nonlinear Neumann condition, the boundary terms arising in the doubling of variables method cannot be approached as some entropy condition but have to be investigated via the strong trace theorem of the previous chapter.

This section is evidently deeply influenced by the work of A. Debussche–J. Vovelle [31], and it also was essentially in H. Frid *et al.* [43]. A novelty, however, is that, inspecting the arguments closely, we managed to significantly diminish the hypotheses on the diffusion coefficients $g_k(x,u)$. Now, the continuity assumptions on $g_k(x,u)$ are even weaker than that of A. Debussche–J. Vovelle [31], and such coefficients have some freedom to oscillate near the boundary.

Section 5.3: Existence

Subsequently, we turn to the proof of existence of entropy solutions to (1.14). As it is traditional in the field of the conservation laws, we firstly approximate (1.14) by the parabolic problem

$$\begin{cases} \dfrac{\partial u^{(\varepsilon)}}{\partial t} + \operatorname{div}_x \widetilde{\mathbf{A}}(u^{(\varepsilon)}) - \varepsilon \Delta_x u^{(\varepsilon)} = \displaystyle\sum_{k=1}^{\infty} g_k^{(\varepsilon)}(x, u^{(\varepsilon)}) \, \frac{d\beta_k}{dt} & \text{for } 0 < t < T \text{ and } x \in \mathscr{O}, \\ \widetilde{\mathbf{A}}(u^{(\varepsilon)}) \cdot \nu = \varepsilon \dfrac{\partial u^{(\varepsilon)}}{\partial \nu} & \text{for } 0 < t < T \text{ and } x \in \partial\mathscr{O}, \text{ and} \\ u(0,x) = u_0(x) & \text{for } t = 0 \text{ and } x \in \mathscr{O}, \end{cases} \tag{1.15}$$

where $\widetilde{\mathbf{A}}(u)$ and $g_k^{(\varepsilon)}(x,u)$ are suitable mollifications of the original coefficients $\mathbf{A}(u)$ and $g_k^{(\varepsilon)}(x,u)$. Assuming the existence of such approximate solutions for a moment, our desire is confirm some relative compactness of $u^{(\varepsilon)}$. In order to do so, we employ the kinetic formulation to write this parabolic equation into

$$\frac{\partial \mathbf{f}^{(\varepsilon)}}{\partial t} + \mathbf{a}(v) \cdot \nabla_x \mathbf{f}^{(\varepsilon)} = \frac{\partial \mathbf{q}^{(\varepsilon)}}{\partial v} + \varepsilon \Delta_x \mathbf{f}^{(\varepsilon)} + \sum_{k=1}^{\infty} g_k^{(\varepsilon)}(x,v) \delta_{u^{(\varepsilon)}}(v) \frac{d\beta_k}{dt},$$

where $\mathbf{q}^{(\varepsilon)}$ is some measure that can be uniformly bound in $0 < \varepsilon < 1$. Thus, the problem becomes how one can treat each and every stochastic source term $g_k^{(\varepsilon)}(x,v)\delta_{u^{(\varepsilon)}}(v)\frac{d\beta_k}{dt}$.

Basically, our method is the following. As it is well known, the stochastic integral $\int_0^t g_k^{(\varepsilon)}(x, u^{(\varepsilon)})$

$d\beta_k(t)$ may have some underlying Hölder continuity in t; since

$$g_k^{(\varepsilon)}(x, u^{(\varepsilon)}) \frac{d\beta_k}{dt} = \frac{\partial}{\partial t}\bigg(\int_0^t g_k^{(\varepsilon)}(x, u^{(\varepsilon)})\, d\beta_k(s) \bigg),$$

such term can thus be thought of as some derivative of order < 1 of an L^2–function, providing us some leeway to "naively intuit" how one can apply an averaging lemma. Unfortunately, one is hindered from directly proceed as such, in virtue of the natural lack of compactness in stochastic problems. Notwithstanding, by a scheme introduced by T. Yamada–S. Watanabe [113] and formalized by I. Gyöngy–N. Krylov [60], one may be able to indeed invoke velocity averaging lemmas provided that one has a sufficient number of "compactness" a priori estimates and the uniqueness of solutions. Luckly, we have both.

In the previous work with H. Frid *et al.* [43], the proof of the existence of solutions, while similar in spirit, depended on some uniform Sobolev space estimates given by the theory of B. Gess–M. Hofmanová [51]. This argument, however, was more complicated and required some more stringent nondegeneracy conditions. We were able to prove the very same theorem more directly and with more natural hypotheses, fully generalizing the result of R. Bürger–H. Frid–K. H. Karlsen [15] to the stochastic case. The method introduced in this chapter is also quite robust and may be applied to other initial-boundary value problems.

Section 5.4: Regularity

In the last section, we establish the Sobolev regularity of entropy solutions to (1.14) under some extra assumptions. The crux of the proof is a simplification and extension of the averaging lemma of B. Gess–M. Hofmanová [51] in the hyperbolic case. Two contributions of this section are: we can deduce the Sobolev regularity in the time variable; the regularization order is higher and indeed consistent with the theory of P.-L. Lions–B. Perthame–E. Tadmor [82].

1.2.5 Appendix A: The viscous approximation

In the first Appendix chapter, we delve into the approximated system (1.15). We solve this problem by constructing a general framework for studying nondegeneate equations, which mingles techniques of spectral theory, semigroup theory, and the theory of "intermediate spaces" of J.-L. Lions–E. Magenes [80]. This method will also be employed in H. Frid *et al.* [44, 45] to produce approximate solutions to different initial-boundary value problems involving stochastic convection-diffusion equations.

1.2.6 Appendix B: The Sobolev regularity of entropy solutions to a parabolic–hyperbolic equation

Finally, we revisit the problem of proving the Sobolev regularity for entropy solutions to (1.7). Through a "quadruple" Littlewood–Paley decomposition (which is indeed implicit in Chapter 2), we are able to lift the restriction of $n \geq 2\ell$ previously imposed by E. Tadmor–T. Tao [107]. The obtained result is again consistent with the theory of P.-L. Lions–B. Perthame–E. Tadmor [82] and E. Tadmor–T. Tao [107]; moreover, we are also able to consider a variation of (1.7) that could not be analyzed by the techniques of E. Tadmor–T. Tao [107].

Chapter 2

Critical velocity averaging lemmas

2.1 The main results

2.1.1 An illustrative example

Before properly stating our theorems, it is convenient to briefly look into a unidimensional model that not only explains our hypotheses but also portrays the general principles behind our theory.

Suppose that $N = 1$, and, for all $n \in \mathbb{N}$, the equation

$$\frac{\partial f_n}{\partial t} + v\frac{\partial f_n}{\partial x} - \frac{\partial}{\partial x}\left(\mathbf{b}(v)\frac{\partial f_n}{\partial x}\right) = (-\Delta_{t,x})^{1/2}\frac{\partial g_n}{\partial v} \tag{2.1}$$

is satisfied in $\mathscr{D}'(\mathbb{R}_t \times \mathbb{R}_x \times \mathbb{R}_v)$, where $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^2(\mathbb{R}_t \times \mathbb{R}_x \times \mathbb{R}_v)$, $(g_n)_{n\in\mathbb{N}}$ converges to zero in $L^2(\mathbb{R}_t \times \mathbb{R}_x \times \mathbb{R}_v)$, and $\mathbf{b} : \mathbb{R} \to \mathbb{R}$ is a smooth, nonnegative function. Our desire is to show that, given any weight function $\eta \in \mathscr{C}_c^\infty(\mathbb{R}_v)$, the averages $\int_{\mathbb{R}} f_n \eta\, dv$ are relatively compact in $L^2_{\mathrm{loc}}(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v)$.

Notice that one may assume that $f_n \rightharpoonup f$ weakly in $\sigma(L^2_{t,x,v}; L^2_{t,x,v})$; in this case, the weak limit $f(t, x, v)$ surely obeys the equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\left(\mathbf{b}(v)\frac{\partial f}{\partial x}\right) = 0.$$

Since $f \in L^2(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v)$, one may apply the classical techniques of Fourier analysis to deduce that $\int_{\mathbb{R}_t\times\mathbb{R}_x} |f(t, x, v)|^2 dxdt = 0$ for almost every $v \in \mathbb{R}$, hence $f \equiv 0$ in the $L^2_{t,x,v}$–sense. As a result, it is clear that, if $\int_{\mathbb{R}} f_n \eta\, dv$ is relatively compact, then it converges *a fortiori* to 0 in L^2_{loc}.

The traditional argument in the theory of the averaging lemmas is roughly as follows (see P.-L. Lions–B. Perthame–E. Tadmor [82]). If $\mathfrak{F}_{t,x}$ denotes the Fourier transform in (t, x), it can be seen that

$$\big(i(\tau + v\kappa) + \mathbf{b}(v)\kappa^2\big)(\mathfrak{F}_{t,x} f_n)(\tau, \kappa, v) = \sqrt{\tau^2 + |\kappa|^2}\frac{\partial}{\partial v}(\mathfrak{F}_{t,x} g_n)(\tau, \kappa, v).$$

This formula is very meaningful if $\mathcal{L}(i\tau, i\kappa, v) = i(\tau + v\kappa) + \mathbf{b}(v)\kappa^2$ is not too small, as one may then formally divide the equation by $\mathcal{L}(i\tau, i\kappa, v)$. In order to discern when $\mathcal{L}(i\tau, i\kappa, v)$ is acceptably far away from zero, let (τ', κ') denote the normalized frequency

$$(\tau', \kappa') = \frac{1}{\sqrt{\tau^2 + |\kappa|^2}}(\tau, \kappa) \tag{2.2}$$

for $(\tau, \kappa) \neq 0$, and introduce some $\psi \in \mathscr{C}^\infty(\mathbb{C}; \mathbb{R})$ such that $\psi(z) = 0$ for $|z| < 1/2$ and $\psi(z) = 1$ for $|z| > 1$. Then, for any $0 < \gamma$ and $\delta < 1$, one may decompose f_n as

$$f_n \overset{\text{def}}{=} f_n^{(1)} + f_n^{(2)} + f_n^{(3)},$$

where

$$\begin{cases} (\mathfrak{F}_{t,x} f_n^{(1)})(\tau,\kappa,v) \overset{\text{def}}{=} (1-\psi)\left(\dfrac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)(\mathfrak{F}_{t,x} f_n)(\tau,\kappa,v), \\ (\mathfrak{F}_{t,x} f_n^{(2)})(\tau,\kappa,v) \overset{\text{def}}{=} \psi\left(\dfrac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)(1-\psi)\left(\dfrac{\mathcal{L}(i\tau',i\kappa',v)}{\delta}\right)(\mathfrak{F}_{t,x} f_n)(\tau,\kappa,v), \text{ and} \\ (\mathfrak{F}_{t,x} f_n^{(3)})(\tau,\kappa,v) \overset{\text{def}}{=} \psi\left(\dfrac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\psi\left(\dfrac{\mathcal{L}(i\tau',i\kappa',v)}{\delta}\right)(\mathfrak{F}_{t,x} f_n)(\tau,\kappa,v). \end{cases}$$

One may interpret this division as follows. $f_n^{(1)}$ is formed by the low-frequencies of f_n, wherefore it is naturally well-behaved (recall, for instance, the Paley–Wiener theorem). On the other hand, $f_n^{(2)}$ is the part of f_n that is supported where $|\mathcal{L}(i\tau',i\kappa',v)|$ is small, and thus its average may be uniformly handled thanks to the nondegeneracy condition (1.8) (hence the necessity of such hypothesis). Observe that $\mathcal{L}(i\tau,i\kappa,v)$ verily satisfies (1.8), for its hyperbolic part $(\tau,\kappa,v)\mapsto i(\tau+v\kappa)$ certainly does.

At last, the remainder term, $f_n^{(3)}$, is the parcel of f_n located in the high frequencies such that $|\mathcal{L}(i\tau',i\kappa',v)|\geq \delta/2$. Therefore, it may be analyzed through the differential equation (2.1), in the sense that

$$(\mathfrak{F}_{t,x} f_n^{(3)}) = \psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\psi\left(\frac{\mathcal{L}(i\tau',i\kappa',v)}{\delta}\right)\frac{\sqrt{\tau^2+|\kappa|^2}}{\mathcal{L}(i\tau,i\kappa,v)}\frac{\partial}{\partial v}(\mathfrak{F}_{t,x} g_n). \tag{2.3}$$

As we argued, this is the sole element one should be preoccupied with, consequently we will only pay attention to it for now. Multiplying (2.3) by $\eta(v)$ and integrating in $v\in\mathbb{R}_v$ imply that

$$\begin{aligned} \mathfrak{F}_{t,x}\left(\int_{\mathbb{R}} f_n^{(3)}\eta\,dv\right) = \\ -\int_{\mathbb{R}}\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\frac{\partial}{\partial v}\left\{\eta(v)\psi\left(\frac{\mathcal{L}(i\tau',i\kappa',v)}{\delta}\right)\right\}\frac{\sqrt{\tau^2+|\kappa|^2}}{\mathcal{L}(i\tau,i\kappa,v)}(\mathfrak{F}_{t,x} g_n)\,dv \\ -\int_{\mathbb{R}}\eta(v)\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\psi\left(\frac{\mathcal{L}(i\tau',i\kappa',v)}{\delta}\right)\frac{\partial}{\partial v}\left\{\frac{\sqrt{\tau^2+|\kappa|^2}}{\mathcal{L}(i\tau,i\kappa,v)}\right\}(\mathfrak{F}_{t,x} g_n)\,dv. \end{aligned} \tag{2.4}$$

On the strength of the Plancherel theorem, the Cauchy–Schwarz inequality

$$\int_{\mathbb{R}^2_{t,x}}\left|\int_{\mathbb{R}_v}\Lambda(t,x,v)\phi(v)\,dv\right|^2 dxdt \leq \left(\int_{\mathbb{R}_v}|\phi(v)|^2\,dv\right)\left(\int_{\mathbb{R}^3_{t,x,v}}|\Lambda(t,x,v)|^2\,dvdxdt\right) \tag{2.5}$$

and the assumption that $g_n\to 0$ in $L^2_{t,x,v}$, it is not difficult to see after a moment of reflection that, so as to guarantee that $\int_{\mathbb{R}} f_n^{(3)}\eta\,dv$ also converges to 0, it suffices to establish that

$$\begin{cases} \sqrt{\tau^2+|\kappa|^2}\leq C|\mathcal{L}(i\tau,i\kappa,v)|, \text{ and} \\ |\mathcal{L}_v(i\tau,i\kappa,v)|\leq C|\mathcal{L}(i\tau,i\kappa,v)| \end{cases} \tag{2.6}$$

for $\mathcal{L}_v(i\tau,i\kappa,v)=\frac{\partial\mathcal{L}}{\partial v}(i\tau,i\kappa,v)$, $(\tau,\kappa)\in\mathscr{B}(v)=\{|\mathcal{L}(i\tau',i\kappa',v)|\geq\delta/2\}\cap\{\sqrt{\tau^2+|\kappa|^2}\geq\gamma/2\}$, and $v\in\operatorname{supp}\eta$.

Due to the restriction $(\tau,\kappa)\in\mathscr{B}(v)$, the first inequality (2.6) follows quite easily. Moreover, if $\mathbf{b}(v)\equiv 0$ for such v's (i.e., the equation is hyperbolic in the support of η), the second inequality is equally trivialized, for it then becomes a relation between two homogeneous functions of degree 0.

On the other hand, if $\mathbf{b}(v)\not\equiv 0$, the second desired estimate becomes much more delicate. For the sake of the argument, let us assume that $\mathbf{b}(v)=v^2$, so that $\mathcal{L}_v(i\tau,i\kappa,v)=i\kappa+2v\kappa^2$. Thus, choosing (τ',κ') such that τ' is very close to 1 (forcing $|\mathcal{L}(i\tau',i\kappa',v)|$ to be close to 1 as well), and

$v \neq 0$, one can infer that

$$\sup_{(\tau,\kappa)\in\mathscr{B}(v)} \left| \frac{\mathcal{L}_v(i\tau, i\kappa, v)}{\mathcal{L}(i\tau, i\kappa, v)} \right| \geq \frac{2}{|v|},$$

which becomes very singular—not even integrable—when v approaches the origin. As a corollary, (2.6) is not feasible if $0 \in \operatorname{supp} \eta$.

Nevertheless, this complicating velocity is a mere single point. Thus, one can truncate the weight function η near it, and indeed (2.6) would hold. The residual term, composed by the velocities neighboring 0, can be made uniformly small due to L^2–boundedness of f_n and (2.5). In this fashion, one can establish that $\int_{\mathbb{R}} f_n \eta \, dv \to 0$ in L^2_{loc}, as we wanted to show.

The issue above and its resolution indicate that solely employing the quantity $\mathcal{L}(i\tau', i\kappa', v)$ may not be adequate to measure the degeneracy of Equation (2.1) when $\mathbf{b}(v) \not\equiv 0$. In reality, the heart of the matter in the parabolic case is not that one should select the non-degenerate directions of $\mathcal{L}(i\tau, i\kappa, v)$, but that one should ensure that $\mathcal{L}(i\tau, i\kappa, v)$ behaves like the heat equation symbol $\mathcal{C}(\tau, \kappa) = i\tau + |\kappa|^2$. If this property is secured, not only can one bound $\mathcal{L}_v(i\tau, i\kappa, v)$, but also one may then control a stronger operator than $(-\Delta_{t,x})^{1/2}$: one may indeed substitute $(-\Delta_{t,x})^{1/2}$ for $(-\Delta_{t,x} + 1)^{1/2} - \Delta_x$, an elliptic operator that "tightly" dominates $\mathcal{C}(\tau, \kappa)$.

Furthermore, this toy model also suggests the following method for investigating (2.1) with a general $\mathbf{b}(v)$. One separates $\mathbb{R}_v$ into two subsets: the one where $\mathbf{b}(v) \equiv 0$ identically, and the one where $\mathbf{b}(v) > 0$. In the former, one can apply the simple argument of the hyperbolic case, whereas, in the latter, provided that one remains bounded away from $\{\mathbf{b}(v) = 0\}$, the argument for $\mathbf{b}(v) = v^2$ would hold fine. Then, assuming that the set where (2.1) mutates from a "hyperbolic" phase to a "parabolic" one—or vice versa—is "small", this agglutination would recover the complete average $\int_{\mathbb{R}} f_n \eta \, dv$, thence showing its convergence to 0 in L^2_{loc}. Theorems 2.1 and 2.2 of this thesis investigate this reasoning.

Notwithstanding, if $\mathbf{b}(v)$ does not degenerate in entire intervals but only in null sets (as, *e.g.*, $\mathbf{b}(v) = v^2$), a considerably better manner to evaluate the behavior of $\mathcal{L}(i\tau, i\kappa, v)$ would be to employ

$$\psi\left(\frac{\text{real part of } \mathcal{L}(i\tau, i\kappa, v)}{\delta |\kappa|^2} \right) = \psi\left(\frac{\mathbf{b}(v)}{\delta} \right), \tag{2.7}$$

as this function elegantly measures the diffuseness of $\mathcal{L}(i\tau, i\kappa, v)$. Notice that, when $\mathbf{b}(v) = v^2$, $\psi(\mathbf{b}(v)/\delta)$ only truncates the velocities near 0, exactly as we have argued before. This hypothesis on the set of degeneracy of $\mathbf{b}(v)$, which fundamentally says that Equation 2.1 possesses one unique regime (as opposed to the previous scenario), is considered in depth in Theorems 2.3 and 2.4.

One central matter we have not touched upon above is the extension from L^2 to a general L^p–space for $1 < p < \infty$. This is a quite dramatic paradigm shift, as the Plancherel theorem is unavailable, and thus the simple conditions (2.6) are no longer enough to prove that $\int_{\mathbb{R}} f_n^{(3)} \eta \, dv$ converges in L^p. Consequently, one is forced to apply multiplier theorems in order to analyze such averages; however, most L^p–multiplier theorems, such as the celebrated result of Mihlin–Hörmander, are not well-suited to examine functions like

$$\psi\left(\frac{\sqrt{\tau^2 + |\kappa|^2}}{\gamma} \right) \psi\left(\frac{\mathbf{b}(v)}{\delta} \right) \frac{\mathcal{L}_v(i\tau, i\kappa, v)}{\mathcal{L}(i\tau, i\kappa, v)} \tag{2.8}$$

in virtue of its lack of homogeneity for large $\sqrt{\tau^2 + |\kappa|^2}$. Fortunately, there exists a criterion that goes back to the original works of J. Marcinkiewicz that neatly facilitates the investigation of anisotropic multipliers such as (2.8). In this way, the principles we have just portrayed can be extended L^p, which is truly the case of interest in nonlinear problems.

2.1.2 The statement of the main results

With this philosophy in mind, let us determine some notations and hypotheses.

Inspired by the previous work of B. Gess–M. Hofmanová [51], we will also consider certain stochastic terms in the right-hand side of (1.2); even so, if one is interested in purely deterministic results, one only needs to let the Φ_n's appearing henceforth to be 0. In any event, our probabilistic framework is as follows. The triplet $(\Omega, \mathscr{F}, \mathbb{P})$ will stand for a probability space endowed with a complete, right-continuous filtration $(\mathscr{F}_t)_{t\geq 0}$. Furthermore, it will be assumed the existence of a sequence $(\beta_k(t))_{k\in\mathbb{N}}$ of mutually independent Brownian motions in $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, so that, if $\mathscr{H}$ is a separable Hilbert space with a hilbertian basis $(e_k)_{k\in\mathbb{N}}$, $W(t) = \sum_{k=1}^{\infty} \beta_k(t)e_k$ defines a cylindrical Wiener process. Recall that, if $\mathfrak{U}$ is another separable Hilbert space, $HS(\mathscr{H};\mathfrak{U})$ denotes the space of the Hilbert–Schmidt operators $T \in \mathscr{L}(\mathscr{H};\mathfrak{U})$.

Let $N \geq 1$ be an integer. The next definitions are central to the theory here exposed.

Definition 2.1. Let $\mathbf{b} : \mathbb{R} \to \mathscr{L}(\mathbb{R}^N)$ be a nonnegative matrix function.

1. $\mathbf{b}$ is said to have a *dichotomous range* if there exists a fixed linear subspace $M \subset \mathbb{R}^N$ such that, for every $v \in \mathbb{R}$, $R(\mathbf{b}(v))$, the range of $\mathbf{b}(v)$, is either M or $\{0\}$. The maximal subspace M for which such alternative holds is called the *effective range* of $\mathbf{b}$.

2. $\mathbf{b}$ is said to satisfy the *nontransiency condition* in a given measurable set $X \subset \mathbb{R}$ if, putting F to be the boundary of $\{v \in \mathbb{R}; \mathbf{b}(v) = 0\}$, $F \cap X$ is a null set with respect to the Lebesgue measure.

Remark 2.1. The nontransiency condition translates quantitatively the notion that the set of velocities in which (2.1) passes from a parabolic regime to a hyperbolic one, or *vice versa*, is small. On the other hand, the effective range hypothesis allows one to generalize the syllogism of Subsection 2.1.1 to multidimensional anisotropic equations.

Finally, recall that, given any linear subspace $M \subset \mathbb{R}^N$, the Laplacean operator restricted to M is defined as

$$\Delta_M \overset{\text{def}}{=} \operatorname{div}_x(P_M \nabla_x),$$

where P_M denotes the orthogonal projection onto M. Notice that, in terms of the Fourier transform, given any $\phi \in \mathscr{S}(\mathbb{R}^N_x)$,

$$\mathfrak{F}_x((-\Delta_M)\phi)(\kappa) = |P_M\kappa|^2(\mathfrak{F}_x\phi)(\kappa).$$

Likewise, recollect that, given any matrix $\mathbf{m} = (\mathbf{m}_{\mu,\nu})_{1\leq\mu,\nu\leq N} \in \mathscr{L}(\mathbb{R}^N)$, the differential operator $D_x^2 : \mathbf{m}$ is defined by

$$D_x^2 : \mathbf{m} \overset{\text{def}}{=} \sum_{\mu,\nu=1}^{N} \mathbf{m}_{\mu,\nu} \frac{\partial^2}{\partial x_\mu \partial x_\nu} = \operatorname{div}_x(\mathbf{m}\nabla_x).$$

With these conventions in mind, let us enunciate our first velocity averaging lemma.

Theorem 2.1 (The global "two-phase" averaging lemma)**.** *Let $\mathscr{J}$ be finite index set, and let be given exponents $1 < p, q_j < \infty$ ($j \in \mathscr{J}$), $1 \leq r \leq 2$ and $\ell \geq 0$. Assume that $\mathbf{a} \in \mathscr{C}^{k,\alpha}_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$ and $\mathbf{b} \in \mathscr{C}^{k,\alpha}_{\text{loc}}(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$, where the real numbers k and α are such that*

$$(k,\alpha) \in \begin{cases} \{0\} \times \{0\} & \text{if } \ell = 0, \\ \{\lfloor \ell \rfloor\} \times (\ell - \lfloor \ell \rfloor, 1] & \text{if } \ell > 0 \text{ is not an integer, and} \\ \{\ell - 1\} \times \{1\} & \text{if } \ell \geq 1 \text{ is an integer,} \end{cases} \tag{2.9}$$

and $\mathbf{b}(v)$ is nonnegative for all $v \in \mathbb{R}$ and has a dichotomous range. Let M be the effective range of $\mathbf{b}$.

Suppose that, for any integer $n \in \mathbb{N}$, the equation

$$\begin{aligned} \frac{\partial f_n}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f_n - \mathbf{b}(v) : D_x^2 f_n &= \sum_{j\in\mathscr{J}} (-\Delta_{t,x}+1)^{1/2}(-\Delta_v+1)^{\ell/2} g_{j,n} \\ &+ \sum_{j\in\mathscr{J}} (\Pi_j(v)\Delta_M)(-\Delta_v+1)^{\ell/2} h_{j,n} + (-\Delta_x+1)^{1/4}(-\Delta_v+1)^{\ell/2}\Phi_n \frac{dW}{dt} \end{aligned} \tag{2.10}$$

is almost surely obeyed in $\mathscr{D}'(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v)$*, where*

1. $(f_n)_{n\in\mathbb{N}}$ *is a bounded sequence in* $L^r(\Omega; L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v))$*,*
2. *for all* $j \in \mathscr{J}$*,* $(g_{j,n})_{n\in\mathbb{N}}$ *and* $(h_{j,n})_{n\in\mathbb{N}}$ *are relatively compact sequences in* $L^r(\Omega; L^{q_j}(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v))$*,*
3. *for all* $j \in \mathscr{J}$*,* $\Pi_j \in \mathscr{C}^{k,\alpha}_{\text{loc}}(\mathbb{R})$ *is such that* $\operatorname{supp} \Pi_j \subset \operatorname{supp} \mathbf{b}$*, and*
4. $(\Phi_n)_{n\in\mathbb{N}}$ *is a predictable and relatively compact sequence in* $L^2(\Omega \times [0,\infty)_t;\, HS(\mathscr{H};\, L^2(\mathbb{R}_x^N \times \mathbb{R}_v)))$*.*

Finally, let $\eta \in L^{p'}(\mathbb{R})$ *have compact support, and presume that the* nondegeneracy condition

$$\operatorname{meas}\big\{v \in \operatorname{supp}\eta; \tau + \mathbf{a}(v)\cdot\kappa = 0 \text{ and } \kappa\cdot\mathbf{b}(v)\kappa = 0\big\} = 0 \quad \text{for all } (\tau,\kappa) \in \mathbb{R}\times\mathbb{R}^N \text{ with } \tau^2 + |\kappa|^2 = 1 \tag{2.11}$$

holds, and that $\mathbf{b}(v)$ *satisfies the* nontransient condition *in* $\operatorname{supp}\eta$*.*

Then, with s *being the least number between* p*,* q_j *(*$j \in J$*), and* 2*, the sequence of averages* $\big(\varphi \int_{\mathbb{R}} f_n \eta\, dv\big)_{n\in\mathbb{N}}$ *is relatively compact in* $L^r(\Omega; L^s(\mathbb{R}_t \times \mathbb{R}_x^N))$ *for any* $\varphi \in (L^1 \cap L^\infty)(\mathbb{R}_t \times \mathbb{R}_x^N)$*.*

Some observations are in order.

Remark 2.2 (On the meaning of (2.10))*.* Conserving the assumptions of the first two paragraphs of Theorem 2.1, the differential equation (2.10) should be understood as follows: Almost surely, it holds that

$$\begin{aligned}
-\int_{\mathbb{R}_t}\int_{\mathbb{R}_x^N}\int_{\mathbb{R}_v} f_n\left(\frac{\partial\phi}{\partial t} + \mathbf{a}(v)\cdot\nabla_x\phi + \mathbf{b}(v) : D_x^2\phi\right) dv\,dx\,dt \\
= \sum_{j\in\mathscr{J}}\int_{\mathbb{R}_t}\int_{\mathbb{R}_x^N}\int_{\mathbb{R}_v}\Big((-\Delta_v+1)^{\ell/2}(-\Delta_{t,x}+1)^{1/2}\phi\Big)\, g_{j,n}\, dvdxdt \\
+ \sum_{j\in\mathscr{J}}\int_{\mathbb{R}_t}\int_{\mathbb{R}_x^N}\int_{\mathbb{R}_v}\Big((-\Delta_v+1)^{\ell/2}(\Pi_j(v)\Delta_M\phi)\Big)h_{j,n}\, dvdxdt \\
+ \int_0^\infty\int_{\mathbb{R}_x^N}\int_{\mathbb{R}_v}\Big((-\Delta_v+1)^{\ell/2}(-\Delta_x+1)^{1/4}\phi\Big)\Phi_n\, dvdxdW(t)
\end{aligned} \tag{2.12}$$

for all $\phi \in \mathscr{C}_c^\infty(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v)$ and $n \in \mathbb{N}$. Due to the Hölder regularity of the Π_j's and the compact support of ϕ, each and every term in (2.12) is almost surely well-defined—see, *e.g.*, Proposition 2.8. Clearly, this definition may be extended to the case in which, rather than in the entire space $\mathbb{R}_t \times \mathbb{R}_x^N$, one is only considering (t, x) lying in some smaller open set $Q \subset \mathbb{R}_t \times \mathbb{R}_x^N$.

Remark 2.3 (On the linear subspace M)*.* Certainly, one could have assumed without loss of generality that M had the form

$$M = \big\{x = (x_1, \dots, x_N) \in \mathbb{R}^N; x_\nu = 0 \text{ for } N' < \nu\big\},$$

where $N' = \dim M$ is a fixed integer. In this case, Δ_M would be simply

$$\Delta_M = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{N'}^2}.$$

Nevertheless, we have opted not to do so, as we reckon this would significantly clutter the notation. Anyhow, the linear subspace M is introduced in order to consider equations that are only diffusive in some variables (such as (1.7)).

Remark 2.4 (On the set $\mathscr{J}$, the functions $\Pi_j(v)$, etc)*.* Essentially, $\Pi_j(v)$'s are present in order that the deterministic source terms in (2.10) to carry full second–order derivatives in x during the "parabolic" phase of (2.10), ascertaining the criticality of Theorem 2.1. In accordance to our previous discussion, notice that, if $R(\mathbf{b}(v)) = M$, then $(-\Delta_{t,x}+1)^{1/2} + (-\Delta_M)$ is an elliptic that tightly dominates $\mathcal{L}(i\tau, i\kappa, v)$.

So as to be more consistent with this philosophy, the right-hand side of (2.10) could have also included terms of the form

$$\sum_{j\in\mathscr{J}} \Upsilon_j(v)(-\Delta_M)^{1/2}(-\Delta_v+1)^{\ell/2}\Psi_{j,n}, \tag{2.13}$$

where, for any $j \in \mathscr{J}$, $\Upsilon_j \in \mathscr{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R})$ with $\operatorname{supp}\Upsilon_j \subset \operatorname{supp}\mathbf{b}$, and $(\Psi_{j,n})_{n\in\mathbb{N}}$ is predictable and relatively compact in $L^2(\Omega\times[0,\infty); HS(\mathbb{R}^N_x\times\mathbb{R}_v))$. Indeed, it is well–known that solutions to stochastic differential equations involving the white noise possess one–half of the regularity one would expect from their deterministic counterparts (see, for instance, Lemma A.3). Nevertheless, we will omit such terms like (2.13) for simplicity's sake. For stochastic forcing terms involving derivatives in t, see Remark 2.11.

Let us mention that, in spite of the index set $\mathscr{J}$ commonly being a singleton with $\Pi_j \equiv 0$, it is important to let $\mathscr{J}$ be a general finite set so that (2.10) becomes "closed under localizations"—see the next theorem.

Even though the next averaging lemma is derivative of the former, its statement is better adapted to some applications. Again, let us first fix another notation. (Recall that $W^{z,p}$ stands for the usual Sobolev space of order z and exponent p.)

Let $1 \le p \le \infty$, $z \in \mathbb{R}$, $\mathscr{E}$ be an Euclidean space, and $\mathscr{U} \subset \mathscr{E}$ be an open set. $L^r(\Omega; W^{z,p}_{\mathrm{loc}}(\mathscr{U}))$ will represent the set of all mappings $f : \Omega \to W^{z,p}_{\mathrm{loc}}(\mathscr{U})$, such that $\theta f \in L^r(\Omega; W^{z,p}(\mathscr{U}))$ for any $\theta \in \mathscr{C}^\infty_c(\mathscr{U})$. This set clearly exemplifies the notion of a Fréchet space.

Theorem 2.2 (The local "two-phase" averaging lemma)**.** *Let $\mathscr{J}$ be finite index set, and let be given exponents $1 < p, q_j < \infty$ ($j \in \mathscr{J}$), $1 \le r \le 2$ and $\ell \ge 0$. Assume that $\mathbf{a} \in \mathscr{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N)$ and $\mathbf{b} \in \mathscr{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$, where the real numbers k and α satisfy the relation (2.9), and $\mathbf{b}(v)$ is nonnegative for all $v \in \mathbb{R}$ and has a dichotomous range. Moreover, let M be the effective range of $\mathbf{b}$, and let $Q \subset \mathbb{R}_t \times \mathbb{R}^N_x$ be an open set.*

Suppose that, for any $n \in \mathbb{N}$, the equation (2.10) is almost surely obeyed in $\mathscr{D}'(Q\times\mathbb{R}_v)$, where

1. *$(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^r(\Omega; L^p_{\mathrm{loc}}(Q\times\mathbb{R}_v))$ that is relatively compact in $L^r(\Omega; W^{-z_0,p}_{\mathrm{loc}}(Q\times\mathbb{R}_v))$ for some $z_0 > 0$,*
2. *for all $j \in \mathscr{J}$, $(g_{j,n})_{n\in\mathbb{N}}$ and $(h_{j,n})_{n\in\mathbb{N}}$ are relatively compact sequences in $L^r(\Omega; L^{q_j}(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v))$,*
3. *for all $j \in \mathscr{J}$, $\Pi_j \in \mathscr{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R})$ is such that $\operatorname{supp}\Pi_j \subset \operatorname{supp}\mathbf{b}$, and*
4. *$(\Phi_n)_{n\in\mathbb{N}}$ is a predictable and relatively sequence in $L^2(\Omega\times[0,\infty)_t;\, HS(\mathscr{H};\, L^2(\mathbb{R}^N_x\times\mathbb{R}_v)))$.*

Finally, let $\eta \in L^{p'}(\mathbb{R})$ have compact support, and presume that the nondegeneracy condition *(2.11) holds, and that $\mathbf{b}(v)$ satisfies the* nontransient condition *in $\operatorname{supp}\eta$.*

Then, the sequence of averages $(\int_{\mathbb{R}} f_n\eta\,dv)_{n\in\mathbb{N}}$ is relatively compact in $L^r(\Omega; L^s_{\mathrm{loc}}(Q))$, with s being the least number between p, q_j ($j \in \mathscr{J}$), and 2. In particular, if $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^r(\Omega; L^p(Q\times\operatorname{supp}\eta))$, and Q is of finite measure, the averages $\big(\int_{\mathbb{R}} f_n\eta\,dv\big)_{n\in\mathbb{N}}$ are relatively compact in $L^r(\Omega; L^z(Q))$ for any $1 \le z < p$.

Remark 2.5 (On the conditions on $(f_n)_{n\in\mathbb{N}}$)*.* In the probabilistic setting we are considering, it is pivotal to impose the relative compactness of (f_n) in a local negative Sobolev space, once this would not be a corollary of weak convergence arguments as it would have been in the deterministic case. Although such conditions do not hold in general, there exist certain procedures involving the Prohorov compactness theorem, the Skohorod representation theorem, and the Gyöngi–Krylov lemma that allow such hypotheses; see Chapter 5, and, for instance, A. Debussche–M. Hofmanová–J. Vovelle [30], H. Frid *et al.* [43], and the references therein.

We now turn to the averaging lemmas for equations displaying one specific behavior. We notice that, under such a circumstance, the statements of the results are quite facilitated.

Theorem 2.3 (The global "single-phase" averaging lemma)**.** *Let $\mathscr{J}$ be finite index set, and let be given exponents $1 < p, q_j < \infty$ ($j \in \mathscr{J}$), $1 \le r \le 2$ and $\ell \ge 0$. Assume that $\mathbf{a} \in \mathscr{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N)$ and $\mathbf{b} \in \mathscr{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$, where the real numbers k and α satisfy the relation (2.9). Furthermore, suppose that there exists a linear subspace $M \subset \mathbb{R}^N$, such that $R(\mathbf{b}(v)) \subset M$ and $\mathbf{b}(v)$ is nonnegative for all $v \in \mathbb{R}$.*

Assume that, for any $n \in \mathbb{N}$, the equation

$$\frac{\partial f_n}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f_n - \mathbf{b}(v) : D_x^2 f_n = \sum_{j\in\mathscr{J}} \big((-\Delta_{t,x}+1)^{1/2} - \Delta_M\big)(-\Delta_v+1)^{\ell/2} g_{j,n} + \big((-\Delta_x+1)^{1/4} + (-\Delta_M)^{1/2}\big)(-\Delta_v+1)^{\ell/2}\Phi_n \frac{dW}{dt} \tag{2.14}$$

is almost surely obeyed in $\mathscr{D}'(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v)$, where

1. *$(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^r(\Omega; L^p(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v))$,*
2. *for all $j \in \mathscr{J}$, $(g_{j,n})_{n\in\mathbb{N}}$ is a relatively compact sequence in $L^r(\Omega; L^{q_j}(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v))$, and*
3. *$(\Phi_n)_{n\in\mathbb{N}}$ is a predictable and relatively compact sequence in $L^2(\Omega\times[0,\infty)_t; HS(\mathscr{H}; L^2(\mathbb{R}_x^N\times\mathbb{R}_v)))$.*

Finally, let $\eta \in L^{p'}(\mathbb{R})$ have compact support, and presume that the nondegeneracy condition

$$\operatorname{meas}\big\{v\in\operatorname{supp}\eta; \tau + (P_{M^\perp}\mathbf{a})(v)\cdot\kappa = 0 \text{ and } \kappa\cdot\mathbf{b}(v)\kappa = 0\big\} = 0 \quad \text{for all } (\tau,\kappa)\in\mathbb{R}\times\mathbb{R}^N \text{ with } \tau^2+|\kappa|^2 = 1 \tag{2.15}$$

holds.

Then, with s being the least number between p, q_j ($j \in \mathscr{J}$), and 2, the sequence of averages $\big(\varphi\int_{\mathbb{R}} f_n\eta\,dv\big)_{n\in\mathbb{N}}$ is relatively compact in $L^r(\Omega; L^s(\mathbb{R}_t\times\mathbb{R}_x^N))$ for any $\varphi\in(L^1\cap L^\infty)(\mathbb{R}_t\times\mathbb{R}_x^N)$.

Remark 2.6 (On the nondegeneracy condition 2.15). In a nutshell, the nondegeneracy condition (2.15) forces that the "principal" symbol $(\tau,\kappa,v)\mapsto i(\tau + (P_{M^\perp}\mathbf{a})(v)\cdot\kappa) + \kappa\cdot\mathbf{b}(v)\kappa$ to obey the usual imposition (2.11), thus exempting any restriction on $(P_M\mathbf{a})(v)$ (the component of $\mathbf{a}(v)$ which acts on the "parabolic" variables). In accordance to the particular behavior of (2.14), the usage of the localizing functions Π_j could be dispensed.

Let us also state a local version of the previous theorem.

Theorem 2.4 (The local "single-phase" averaging lemma)**.** *Let $\mathscr{J}$ be finite index set, and let be given exponents $1 < p, q_j < \infty$ ($j \in \mathscr{J}$), $1 \le r \le 2$ and $\ell \ge 0$. Assume that $\mathbf{a} \in \mathscr{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N)$ and $\mathbf{b} \in \mathscr{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$, where the real numbers k and α satisfy the relation (2.9). Furthermore, suppose that there exists a linear subspace $M \subset \mathbb{R}^N$, such that $R(\mathbf{b}(v)) \subset M$ and $\mathbf{b}(v)$ is nonnegative for all $v \in \mathbb{R}$. Let $Q \subset \mathbb{R}_t\times\mathbb{R}_x^N$ be an open set.*

Assume that, for any $n \in \mathbb{N}$, Equation (2.14) is obeyed in $\mathscr{D}'(Q\times\mathbb{R}_v)$, where

1. *$(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^r(\Omega; L^p_{\mathrm{loc}}(Q\times\mathbb{R}_v))$ that is relatively compact in $L^r(\Omega; W^{-z_0,p}_{\mathrm{loc}}(Q\times\mathbb{R}_v))$ for some $z_0 > 0$,*
2. *for all $j \in \mathscr{J}$, $(g_{j,n})_{n\in\mathbb{N}}$ is a relatively compact sequence in $L^r(\Omega; L^{q_j}(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v))$, and*
3. *$(\Phi_n)_{n\in\mathbb{N}}$ is a predictable and relatively compact sequence in $L^2(\Omega\times[0,\infty)_t; HS(\mathscr{H}; L^2(\mathbb{R}_x^N\times\mathbb{R}_v)))$.*

Finally, let $\eta \in L^{p'}(\mathbb{R})$ *have compact support, and presume that the* nondegeneracy condition (2.15) *holds.*

Then, with s *being the least number between* p*,* q_j ($j \in \mathscr{J}$)*, and* 2*, the sequence of averages* $\left(\int_{\mathbb{R}} f_n \eta \, dv\right)_{n\in\mathbb{N}}$ *is relatively compact in* $L^r(\Omega; L^s_{\mathrm{loc}}(Q))$*. In particular, if* $(f_n)_{n\in\mathbb{N}}$ *is bounded in* $L^r(\Omega; L^p(Q \times \operatorname{supp} \eta))$*, and* Q *is of finite measure, then* $(\int_{\mathbb{R}} f_n \eta \, dv)_{n\in\mathbb{N}}$ *converges in* $L^r(\Omega; L^z(Q))$ *for any* $1 \leq z < p$*.*

Remark 2.7 (On the hypotheses on $\mathbf{b}(v)$). In the theory of flow in porous media, the matrix $\mathbf{b}(v)$ only degenerates in a single point. Therefore, $\mathbf{b}(v)$ evidently obeys the nontransiency condition, and both lines of theorem apply, even though Theorems 2.3 and 2.4 are likely preferable. On the other hand, in sedimentation-consolidation processes, $\mathbf{b}(v)$ has the isotropic form

$$\mathbf{b}(v) = \mathbf{q}(v) I_{\mathbb{R}^N}, \tag{2.16}$$

with $\mathbf{q} : \mathbb{R} \to \mathbb{R}$ satisfying $\mathbf{q}(v) > 0$ in some interval I, and $\mathbf{q}(v) = 0$ outside of I. Clearly again $\mathbf{b}(v)$ observes the nontransiency condition, and Theorems 2.1 and 2.2 are available.

On a more theoretical note, let us point out that, in contrast with Theorems 2.1 and 2.2, it is permissible that $R(\mathbf{b}(v)) \neq M$ everywhere. By way of illustration, if $N = 2$ and $M = \mathbb{R}^2$,

$$\mathbf{b}(v) = \begin{pmatrix} v^2 & v^3 \\ v^3 & v^4 \end{pmatrix}$$

satisfies the conditions of the last two theorems, in spite of $\dim R(\mathbf{b}(v)) < 2$ for all $v \in \mathbb{R}$.

2.1.3 Outline of the chapter

This segment of the manuscript is organized as follows. In Section 2, we will demonstrate Theorem 2.1. Subsequently, in Section 3, we will show how to reduce Theorem 2.2 to Theorem 2.1. In Section 4, we will concisely delineate the proof of both Theorems 2.3 and 2.4, once they are almost identical to the corresponding arguments of Theorems 2.1 and 2.2. Finally, in Section 2.5, we will discuss several details of the statement and proofs of such theorems; in particular, we will compare these results with theories of P.-L. Lions–B. Perthame–E. Tadmor [82] and of E. Tadmor–T. Tao [107].

2.2 Proof of Theorem 2.1

First of all, passing to a subsequence if necessary, we may assume that, for all $j \in \mathscr{J}$, $(g_{j,n})_{n\in\mathbb{N}}$ and $(h_{j,n})_{n\in\mathbb{N}}$ are convergent in $L^r(\Omega; L^{q_j}(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v))$, and that $(\Phi_n)_{n\in\mathbb{N}}$ is equally convergent in $L^2(\Omega \times [0,\infty); HS(\mathscr{H}; L^2(\mathbb{R}^N_x \times \mathbb{R}_v)))$. Accordingly, the conclusions of Theorem 2.1 will be accomplished once we verify that, for any $\varphi \in (L^1 \cap L^\infty)(\mathbb{R}_t \times \mathbb{R}^N_x)$, the averages $\varphi \int_{\mathbb{R}} f_n \eta \, dv$ define a convergent sequence in $L^r(\Omega; L^s(\mathbb{R}_t \times \mathbb{R}^N_x))$.

2.2.1 The decomposition of the average

In this subsection, we compartmentalize $\int_{\mathbb{R}} f_n \eta \, dv$ into components whose a priori estimates may be extracted from different hypotheses made in the statement of Theorem 2.1. In this fashion, the desired conclusion is established via a proper passage to the limit.

Let us define the differences

$$\mathfrak{f}_{m,n}(t,x,v) = f_m(t,x,v) - f_n(t,x,v). \tag{2.17}$$

Once (2.10) is linear, one may apply the theories of the elliptic operators and the Riesz transforms to verify that each $\mathfrak{f}_{m,n}$ obeys

$$\left(\frac{\partial}{\partial t} + \mathbf{a}(v)\cdot\nabla_x - \mathbf{b}(v) : D_x^2\right)\mathfrak{f}_{m,n} = \sum_{j\in\mathscr{J}} (-\Delta_{t,x}+1)^{1/2}\left[1 \pm \left(\frac{\partial^{\mathfrak{l}}}{\partial v^{\mathfrak{l}}}(-\Delta_v)^{\mathfrak{z}/2}\right)\right]\mathfrak{g}_{m,n}^{(j)}$$
$$+ \sum_{j\in\mathscr{J}} \Pi_j(v)\,(\Delta_M)\left[1 \pm \left(\frac{\partial^{\mathfrak{l}}}{\partial v^{\mathfrak{l}}}(-\Delta_v)^{\mathfrak{z}/2}\right)\right]\mathfrak{h}_{m,n}^{(j)}$$
$$+ (-\Delta_x+1)^{1/2}\left[1 \pm \left(\frac{\partial^{\mathfrak{l}}}{\partial v^{\mathfrak{l}}}(-\Delta_v)^{\mathfrak{z}/2}\right)\right]\left(\Psi_{m,n}\frac{dW}{dt}\right), \tag{2.18}$$

with the indices $\mathfrak{l}\in\mathbb{Z}$ and $0\le\mathfrak{z}<1$ being such that $\mathfrak{l}+\mathfrak{z}=\ell$, the sign $\pm$ being

$$\pm = \begin{cases} +, & \text{if } \mathfrak{l}\equiv 0 \bmod 4,\\ \text{arbitrary}, & \text{if } \mathfrak{l}\equiv 1 \bmod 4 \text{ or } 3 \bmod 4, \text{ and}\\ -, & \text{if } \mathfrak{l}\equiv 2 \bmod 4,\end{cases}$$

and, at last, each $(\mathfrak{g}_{m,n}^{(j)})_{m,n\in\mathbb{N}}$, $(\mathfrak{h}_{m,n}^{(j)})_{m,n\in\mathbb{N}}$ and $(\Psi_{m,n})_{m,n\in\mathbb{N}}$ satisfying for all $j\in\mathscr{J}$

$$\lim_{m,n\to\infty}\mathbb{E}\left(\int_{\mathbb{R}_t}\int_{\mathbb{R}_x^N}\int_{\mathbb{R}_v}|\mathfrak{g}_{m,n}^{(j)}(t,x,v)|^{q_j}\,dvdxdt\right)^{r/q_j} = 0, \tag{2.19}$$
$$\lim_{m,n\to\infty}\mathbb{E}\left(\int_{\mathbb{R}_t}\int_{\mathbb{R}_x^N}\int_{\mathbb{R}_v}|\mathfrak{h}_{m,n}^{(j)}(t,x,v)|^{q_j}\,dvdxdt\right)^{r/q_j} = 0, \text{ and} \tag{2.20}$$
$$\lim_{m,n\to\infty}\mathbb{E}\int_0^\infty \|\Psi_{m,n}(t)\|^2_{HS(\mathscr{H};L^2(\mathbb{R}_x^N\times\mathbb{R}_v))}\,dt = 0. \tag{2.21}$$

The mollification of the weigh function η.

Let us now introduce a certain smooth approximation of η that will allow us to handle the operator $\frac{\partial^{\mathfrak{l}}}{\partial v^{\mathfrak{l}}}(-\Delta_v)^{\mathfrak{z}/2}$ via integration by parts. This mollification, which we will symbolize by $\eta_{\delta,\gamma}$—as it will depend on two parameters γ and δ—, has a quite special support, whose role in our analysis can hardly be exaggerated.

Lemma 2.1. *Let $N\ge 1$ be an integer, $1<p<\infty$, $\eta\in L^{p'}(\mathbb{R})$ have compact support, and $\mathbf{b}:\mathbb{R}\to\mathscr{L}(\mathbb{R}^N)$ be nonnegative, continuous matrix function that has a dichotomous range and satisfies the nontransiency condition in* $\operatorname{supp}\eta$. *Let $\chi>0$ be given.*

For any $0<\delta$ and $\gamma<1$, there exist functions $\mathfrak{n}_\gamma$ and $\eta_{\delta,\gamma}$ in $L^{p'}(\mathbb{R})$ for which the following assertions hold.

(a) Regarding $\mathfrak{n}_\gamma$:

(a.i) $\mathfrak{n}_\gamma$ *in* $L^\infty(\mathbb{R})$ *with* $\|\mathfrak{n}_\gamma\|_{L^\infty(\mathbb{R}_v)}\le\gamma^{-\chi}$;

(a.ii) $\operatorname{supp}\mathfrak{n}_\gamma\subset\operatorname{supp}\eta$;

(a.iii) $\|\mathfrak{n}_\gamma-\eta\|_{L^{p'}(\mathbb{R})}\to 0$ *as* $\gamma\to 0_+$.

(b) Regarding $\eta_{\delta,\gamma}$:

(b.i) $\eta_{\delta,\gamma}\in\mathscr{C}_c^\infty(\mathbb{R})$ *and* $\|\eta_{\delta,\gamma}\|_{L^\infty(\mathbb{R})}\le\|\mathfrak{n}_\gamma\|_{L^\infty(\mathbb{R})}$;

(b.ii) $\operatorname{supp}\eta_{\delta,\gamma}\subset\operatorname{supp}\eta+(-\delta,\delta)$ *and is the disjoint union of two compact sets $K_h=K_h^{(\delta)}$ and $K_p=K_p^{(\delta)}$, such that*

$$\begin{cases}\mathbf{b}(v)\equiv 0 \ \textit{identically if } v\in K_h, \textit{ and}\\ \mathbf{b}(v)\ge\mathfrak{c}_\delta P_M \ \textit{whenever } v\in K_p,\end{cases} \tag{2.22}$$

where $\mathfrak{c}_\delta > 0$ depends only on δ, and M is the effective range of $\mathbf{b}$*;*

(b.iii) for any $0 < \gamma < 1$ fixed, $\|\eta_{\delta,\gamma} - \mathfrak{n}_\gamma\|_{L^{p'}(\mathbb{R})} \to 0$ as $\delta \to 0_+$.

Proof. In order to verify (a), it suffices to consider the truncations

$$\mathfrak{n}_\gamma(v) = \begin{cases} -\gamma^{-\chi} & \text{if } \eta(v) < -\gamma^{-\chi}, \\ \eta(v) & \text{if } |\eta(v)| \leq \gamma^{-\chi}, \text{ and} \\ \gamma^{-\chi} & \text{if } \eta(v) > \gamma^{-\chi}. \end{cases}$$

The construction of $\eta_{\delta,\gamma}$ is fairly more intricate. For this purpose, consider $(\varrho_\varepsilon)_{\varepsilon>0}$ to be standard mollifiers in the real line.

Were it not for the asserted decomposition of the support of $\eta_{\delta,\gamma}$, evidently we could have chosen this function to be $(\varrho_\delta \star \eta_\gamma)$. Indeed, if the boundary of $\{v \in \operatorname{supp}\eta; \mathbf{b}(v) = 0\}$ is empty, define $\eta_{\delta,\gamma}$ as such. Otherwise, so as to obtain this extra attribute, let us localize $(\varrho_\delta \star \eta_\gamma)$ by means of the next proposition of A. P. Calderón–A. Zygmund [18], whose proof may also be found in the classic book of E. M. Stein [103].

Proposition 2.1 (The existence of the "regularized distance")**.** *Let d be a positive integer, and $F \subset \mathbb{R}^d$ be a nonempty closed subset. There exists a continuous function $\mathfrak{d} : \mathbb{R}^d \to \mathbb{R}$ such that*

1. *$c_1 \operatorname{dist}(x, F) \leq \mathfrak{d}(x) \leq c_2 \operatorname{dist}(x, F)$ for all $x \in \mathbb{R}^d$,*
2. *$\mathfrak{d} \in \mathscr{C}^\infty(\mathbb{R}^d \setminus F)$, and, for all multi-indices $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_d)$,*

$$|(D^{\mathfrak{a}}\mathfrak{d})(x)| \leq B_{\mathfrak{a}} \operatorname{dist}(x, F)^{1-|\mathfrak{a}|} \text{ for all } x \in \mathbb{R}^d \setminus F,$$

where c_1, c_2, and $B_{\mathfrak{a}}$ are positive constants which do not depend on F.

We will employ this result as follows. Put $d = 1$, and let F be the boundary of $\{v \in \mathbb{R}; \mathbf{b}(v) = 0\}$. Once F is a closed set, there exists a function $\mathfrak{d}(v)$ with the properties listed above.

Given any $\varepsilon > 0$, define $H_\varepsilon : \mathbb{R} \to \mathbb{R}$ to be the regular approximations of the Heaviside function

$$H_\varepsilon(z) = \int_0^z \varrho_\varepsilon(w - 2\varepsilon)\, dw,$$

and introduce $\xi_\varepsilon(v) = H_\varepsilon(\mathfrak{d}(v))$. It is clear that $0 \leq \xi_\varepsilon(v) \leq 1$ everywhere, and that $\xi_\varepsilon(v) \to 1_{\mathbb{R}\setminus F}(v)$ pointwisely as $\varepsilon \to 0_+$. In addition, for $\operatorname{supp}\varrho_\varepsilon \subset (-\varepsilon, \varepsilon)$, $\xi_\varepsilon(v)$ actually vanishes if $\operatorname{dist}(v, F)$ is sufficiently small, hence $\xi_\varepsilon \in \mathscr{C}^\infty(\mathbb{R})$. Finally, because $F \cap \operatorname{supp}\eta$ is of measure zero (here is where the nontransiency condition is necessary),

$$\begin{aligned} \|\xi_\delta(\varrho_\delta \star \mathfrak{n}_\gamma) - \mathfrak{n}_\gamma\|_{L^{p'}(\mathbb{R})} &\leq \|\xi_\delta \mathfrak{n}_\gamma - \mathfrak{n}_\gamma\|_{L^{p'}(\mathbb{R})} + \|(\varrho_\delta \star \mathfrak{n}_\gamma) - \mathfrak{n}_\gamma\|_{L^{p'}(\mathbb{R})} \\ &\to 0 \text{ as } \delta \to 0_+. \end{aligned} \tag{2.23}$$

Let us therefore define $\eta_{\delta,\gamma}(v) = \xi_\delta(v)(\varrho_\delta \star \mathfrak{n}_\gamma)(v)$. Once now statements (b.i) and (b.iii) are easily verified for such $\eta_{\delta,\gamma}$, all that remains to finalize the proof of this lemma is property (b.ii).

To this end, perceive at first that $\operatorname{supp}\eta_{\delta,\gamma} \subset \operatorname{supp}\eta + (-\delta, \delta)$ is a basic result in the theory of convolution integrals. Per the properties of ξ_δ, the support of $\eta_{\delta,\gamma}$ is formed by the disjoint union of two closed sets, each of which, in virtue of the dichotomous range hypothesis, lies entirely in the interior of $\{v \in \mathbb{R}; \mathbf{b}(v) = 0\}$ or of $\{v \in \mathbb{R}; R(\mathbf{b}(v)) = M\}$. In case of the second alternative, being $\mathbf{b}(v)$ symmetric, $P_M\mathbf{b}(v)P_M$ can be seen as a linear isomorphism in M. Thus, the lower bound in (2.22) is derived from a simple continuity argument. □

The decomposition in the Fourier space.

Likewise, it is crucial that we introduce the next partitioning in the frequencies variables, which depends how degenerate is Equation (2.10) in that given region. So as to express such a division, let us define three Fourier symbols. Henceforth, $M \subset \mathbb{R}^N$ will denote the effective range of $\mathbf{b}(v)$. Furthermore, recall the definition of the symbol $\mathcal{L}(i\tau, i\kappa, v) = i(\tau + \mathbf{a}(v)\cdot\kappa) + \kappa\cdot\mathbf{b}(v)\kappa$ as given in (1.5).

Definition 2.2. The symbols $(R\mathcal{L})(i\tau, i\kappa, v)$, $\widetilde{\mathcal{L}}(i\tau, i\kappa, v)$ and $(\widetilde{R\mathcal{L}})(i\tau, i\kappa, v)$ ($\tau \in \mathbb{R}$, $\kappa \in \mathbb{R}^N$, and $v \in \mathbb{R}$) are defined as follows.

1. By $(R\mathcal{L})(i\tau, i\kappa, v)$, it will be understood the so-called *restricted symbol*:

$$(R\mathcal{L})(i\tau, i\kappa, v) \overset{\text{def}}{=} i\big(\tau + (P_{M^\perp}\mathbf{a})(v)\cdot\kappa\big) = \mathcal{L}\big(i\tau, iP_{M^\perp}\kappa, v\big). \tag{2.24}$$

2. By $\widetilde{\mathcal{L}}(i\tau, i\kappa, v)$, it will be understood the so-called *normalized symbol*:

$$\widetilde{\mathcal{L}}(i\tau, i\kappa, v) \overset{\text{def}}{=} \mathcal{L}\left(\frac{i\tau}{\sqrt{\tau^2+|\kappa|^2}}, \frac{i\kappa}{\sqrt{\tau^2+|\kappa|^2}}, v\right). \tag{2.25}$$

3. By $(\widetilde{R\mathcal{L}})(i\tau, i\kappa, v)$, it will be understood the so-called *restricted normalized symbol*:

$$(\widetilde{R\mathcal{L}})(i\tau, i\kappa, v) \overset{\text{def}}{=} (R\mathcal{L})\left(\frac{i\tau}{\sqrt{\tau^2+|P_{M^\perp}\kappa|^2}}, \frac{i(P_{M^\perp}\kappa)}{\sqrt{\tau^2+|P_{M^\perp}\kappa|^2}}, v\right). \tag{2.26}$$

Choose two functions λ and $\psi \in \mathscr{C}^\infty(\mathbb{C};\mathbb{R})$ such that

1. $\lambda(z) = 1$ for $|z| < \frac{1}{2}$,
2. $0 \le \lambda(z) \le 1$ for $\frac{1}{2} \le |z| \le 1$,
3. $\lambda(z) = 0$ for $|z| > 1$, and
4. $\lambda(z) + \psi(z) = 1$ everywhere.

For any $0 < \delta$ and $\gamma < 1$, which will be fixed for now—but will be let go to 0 eventually—, let us then write

$$\mathfrak{f}_{m,n}(t,x,v) = \sum_{\nu=1}^{4} \mathfrak{f}_{m,n}^{(\nu)}(t,x,v),$$

where, with $\mathfrak{F}_{t,x}$ denoting the Fourier transform in (t,x),

$$\begin{cases} \mathfrak{f}_{m,n}^{(1)} = \mathfrak{F}_{t,x}^{-1}\left[\lambda\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)(\mathfrak{F}_{t,x}\mathfrak{f}_{m,n})\right], \\ \mathfrak{f}_{m,n}^{(2)} = \mathfrak{F}_{t,x}^{-1}\left[\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\lambda\left(\frac{\widetilde{\mathcal{L}}(i\tau,i\kappa,v)}{\delta}\right)(\mathfrak{F}_{t,x}\mathfrak{f}_{m,n})\right], \\ \mathfrak{f}_{m,n}^{(3)} = \mathfrak{F}_{t,x}^{-1}\left[\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\psi\left(\frac{\widetilde{\mathcal{L}}(i\tau,i\kappa,v)}{\delta}\right)\right. \\ \qquad\qquad \left.\lambda\left(\frac{(\widetilde{R\mathcal{L}})(i\tau,i\kappa,v)}{\delta}\right)(\mathfrak{F}_{t,x}\mathfrak{f}_{m,n})\right], \text{ and} \\ \mathfrak{f}_{m,n}^{(4)} = \mathfrak{F}_{t,x}^{-1}\left[\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\psi\left(\frac{\widetilde{\mathcal{L}}(i\tau,i\kappa,v)}{\delta}\right)\right. \\ \qquad\qquad \left.\psi\left(\frac{(\widetilde{R\mathcal{L}})(i\tau,i\kappa,v)}{\delta}\right)(\mathfrak{F}_{t,x}\mathfrak{f}_{m,n})\right]. \end{cases} \tag{2.27}$$

Even though neither $\widetilde{\mathcal{L}}(i\tau, i\kappa, v)$ nor $(\widetilde{R\mathcal{L}})(i\tau, i\kappa, v)$ are defined in the entire space $\mathbb{R}_\tau \times \mathbb{R}_\kappa^N \times \mathbb{R}_v$, this does not pose a problem, as their domain is of total measure nonetheless. Recall that it is admissible to take the spatio-temporal Fourier transform of $\mathfrak{f}_{m,n}$, as it almost surely lies in $L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v)$ and, consequently, defines almost surely a tempered distribution. The tacit affirmation that each $\mathfrak{f}_{m,n}^{(\nu)}$ is indeed a function will be justified afterwards.

Conclusion.

All things considered, we thus establish the decomposition

$$\begin{aligned}\int_{\mathbb{R}} \eta \mathfrak{f}_{m,n}\, dv &= \int_{\mathbb{R}} \mathfrak{f}_{m,n}(\eta - \eta_{\delta,\gamma})\, dv + \int_{\mathbb{R}} \mathfrak{f}_{m,n}^{(1)} \eta_{\delta,\gamma}\, dv \\ &\qquad + \int_{\mathbb{R}} \mathfrak{f}_{m,n}^{(3)} \eta_{\delta,\gamma}\, dv + \int_{\mathbb{R}} \mathfrak{f}_{m,n}^{(4)} \eta_{\delta,\gamma}\, dv \\ &\overset{\text{def}}{=} \mathfrak{v}_{m,n}^{(0)} + \mathfrak{v}_{m,n}^{(1)} + \mathfrak{v}_{m,n}^{(2)} + \mathfrak{v}_{m,n}^{(3)} + \mathfrak{v}_{m,n}^{(4)}. \end{aligned} \tag{2.28}$$

As a consequence, the definition of $\mathfrak{f}_{m,n}$ (2.17) yields

$$\varphi\left(\int_{\mathbb{R}} f_m \eta\, dv - \int_{\mathbb{R}} f_n \eta\, dv \right) = \sum_{\nu=0}^{4} \varphi \mathfrak{v}_{m,n}^{(\nu)}, \tag{2.29}$$

in such a manner that our main objection is reduced to the extraction of a priori estimates in $L_\omega^r L_{t,x}^s$ for each $\varphi \mathfrak{v}_{m,n}^{(j)}$ as m and $n \to \infty$.

2.2.2 The analysis of $\mathfrak{v}_{m,n}^{(0)}$.

Proposition 2.2. *There exists a constant* $C = C\left(\|\varphi\|_{L_{t,x}^1 \cap L_{t,x}^\infty}, \sup_{\nu\in\mathbb{N}} \|f_\nu\|_{L_\omega^r L_{t,x,v}^p} \right)$ *such that, for all* m *and* $n \in \mathbb{N}$,

$$\mathbb{E}\|\varphi \mathfrak{v}_{m,n}^{(0)}\|_{L^s(\mathbb{R}_t \times \mathbb{R}_x^N)}^r \leq C \|\eta_{\delta,\gamma} - \eta\|_{L^{p'}(\mathbb{R})}^r. \tag{2.30}$$

By virtue of Lemma 2.1, this is an interesting estimate as δ and γ separately tend to 0_+. Before we demonstrate this bound for $\mathfrak{v}_{m,n}^{(0)}$, let us state the following elementary yet fairly useful estimate, whose proof is an immediate corollary to Hölder's inequality.

Lemma 2.2. *For any exponent* $1 \leq \mathfrak{s} \leq \infty$, $\phi \in L^{\mathfrak{s}'}(\mathbb{R}_v)$ *and* $\Lambda \in L^{\mathfrak{s}}(\mathbb{R}_t \times \mathbb{R}_x^N \times \operatorname{supp}\phi)$,

$$\left\| \int_{\mathbb{R}} \phi(v) \Lambda(\,\cdot\,,\,\cdot\,,v)\, dv \right\|_{L^{\mathfrak{s}}(\mathbb{R}_t \times \mathbb{R}_x^N)} \leq \|\phi\|_{L^{\mathfrak{s}'}(\mathbb{R}_v)} \|\Lambda\|_{L^{\mathfrak{s}}(\mathbb{R}_t \times \mathbb{R}_x^N \times \operatorname{supp}\phi)}.$$

In particular, if $\Lambda \in L^{\mathfrak{s}}(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v)$,

$$\left\| \int_{\mathbb{R}} \phi(v) \Lambda(\,\cdot\,,\,\cdot\,,v)\, dv \right\|_{L^{\mathfrak{s}}(\mathbb{R}_t \times \mathbb{R}_x^N)} \leq \|\phi\|_{L^{\mathfrak{s}'}(\mathbb{R}_v)} \|\Lambda\|_{L^{\mathfrak{s}}(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v)}. \tag{2.31}$$

Proof of Proposition 2.2. Applying (2.31) to the definition of $\mathfrak{v}_{m,n}^{(0)}$, we deduce that

$$\begin{aligned}\mathbb{E}\|\mathfrak{v}_{m,n}^{(0)}\|_{L^p(\mathbb{R}_t \times \mathbb{R}_x^N)}^r &\leq \|\eta_{\delta,\gamma} - \eta\|_{L^{p'}(\mathbb{R}_v)}^r \mathbb{E}\|f_m - f_n\|_{L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v)}^r \\ &\leq 2^r \left(\sup_{\nu\in\mathbb{N}} \|f_\nu\|_{L_\omega^r L_{t,x,v}^p}^r \right) \|\eta_{\delta,\gamma} - \eta\|_{L^{p'}(\mathbb{R}_v)}^r;\end{aligned}$$

i.e.,

$$\mathbb{E}\|\varphi \mathfrak{v}_{m,n}^{(0)}\|_{L^s(\mathbb{R}_t \times \mathbb{R}_x^N)}^r \leq 2^r \|\varphi\|_{L_{t,x}^1 \cap L_{t,x}^\infty}^r \left(\sup_{\nu\in\mathbb{N}} \|f_\nu\|_{L_\omega^r L_{t,x,v}^p}^r \right) \|\eta_{\delta,\gamma} - \eta\|_{L^{p'}(\mathbb{R})}^r,$$

which establishes (2.30). □

2.2.3 The analysis of $\mathfrak{v}^{(1)}_{m,n}$.

Proposition 2.3. *Let $\phi \in \mathscr{C}^\infty_c(\mathbb{C};\mathbb{C})$, and $\varepsilon > 0$. There exists a function $\mathfrak{K} \in \cap_{\nu=0}^\infty W^{\nu,1}(\mathbb{R}_t \times \mathbb{R}^N_x)$ such that, for any $\Lambda \in \mathscr{S}(\mathbb{R}_t \times \mathbb{R}^N_x)$,*

$$\mathfrak{F}^{-1}_{t,x}\left[\phi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\varepsilon}\right)(\mathfrak{F}_{t,x}\Lambda)\right] = \varepsilon^{N+1}(\mathfrak{K}(\varepsilon\,\cdot\,,\varepsilon\,\cdot\,)\star_{t,x}\Lambda).$$

Moreover, for any integer $\nu \geq 0$,

$$\|\mathfrak{K}\|_{W^{\nu,1}(\mathbb{R}_t\times\mathbb{R}^N_x)} \leq C(\nu, \operatorname{supp}\phi, \|\phi\|_{\mathscr{C}^{N+1}}).$$

Proof. Put $\mathfrak{G}(\tau,\kappa) = \phi(\sqrt{\tau^2+|\kappa|^2})$, and let $P(\tau,\kappa)$ be an arbitrary complex polynomial function. It is not hard to see that $P\mathfrak{G} \in W^{N+1,1}(\mathbb{R}_\tau \times \mathbb{R}^N_\kappa)$, and, for every multi-index $\mathfrak{a} = (\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_N)$ of length $N+1$, one has that

$$|D^{\mathfrak{a}}(P\mathfrak{G})(\tau,\kappa)| \leq C_{A,P}\|\phi\|_{\mathscr{C}^{N+1}}\frac{1_{(0,A)}(\sqrt{\tau^2+|\kappa|^2})}{(\tau^2+|\kappa|^2)^{\frac{N}{2}}},$$

where $A > 0$ is any real number for which $\phi(z) = 0$ if $|z| > A$. Thus, $P\mathfrak{G} \in W^{N+1,\mathfrak{s}}(\mathbb{R}_\tau\times\mathbb{R}^N_\kappa)$ for any $1 \leq \mathfrak{s} < \frac{N+1}{N}$. As a result, the Haussdorf–Young inequality mingled with the Riemann–Lebesgue lemma asserts that $\mathfrak{K} = \mathfrak{F}^{-1}_{t,x}\mathfrak{G}$ satisfies the pointwise estimate

$$|(D^{\mathfrak{b}}\mathfrak{K})(t,x)| \leq \frac{H_{\mathfrak{b}}(t,x)}{(1+\sqrt{t^2+|x|^2})^{N+1}} \text{ for all } (t,x) \in \mathbb{R}_t \times \mathbb{R}^N_x,$$

where $\mathfrak{b} = (\mathfrak{b}_0, \mathfrak{b}_1, \ldots, \mathfrak{b}_N)$ is any multi-index, and $H_{\mathfrak{b}} \in L^{\mathfrak{t}}(\mathbb{R}_t \times \mathbb{R}^N_x)$ for $N+1 < \mathfrak{t} \leq \infty$ with $\|H_{\mathfrak{b}}\|_{L^{\mathfrak{t}}_{t,x}} \leq C(\mathfrak{b}, \mathfrak{t}, \|\phi\|_{\mathscr{C}^{N+1}}, \operatorname{supp}\phi)$. The desired conclusion now follows from the Hölder's inequality and the Fourier analysis operational rules. □

Remark 2.8. The argument above would have also been greatly simplified, had one assumed that ϕ is constant near the origin (as, for instance, λ is); indeed, in this case $\mathfrak{G} \in \mathscr{C}^\infty_c(\mathbb{R}_\tau \times \mathbb{R}^N_\kappa)$, hence $\mathfrak{K} \in \mathscr{S}(\mathbb{R}_t \times \mathbb{R}^N_x)$. Despite this, we have opted for this proof, seeing that this result will be summoned in the next subsection.

Proposition 2.4. *There exist a constant $C = C(\|\varphi\|_{L^p_{t,x}}, \|\eta\|_{L^{p'}_v}, \sup_{\nu\in\mathbb{N}}\|f_\nu\|_{L^r_\omega L^p_{t,x,v}})$ and an exponent $\mathfrak{q} > 0$, such that, for all $0 < \gamma < 1$, and m and $n \in \mathbb{N}$,*

$$\mathbb{E}\|\varphi\mathfrak{v}^{(1)}_{m,n}\|^r_{L^s(\mathbb{R}_t\times\mathbb{R}^N_x)} \leq C\gamma^{\mathfrak{q}}. \tag{2.32}$$

Proof. According to Proposition 2.3,

$$\begin{aligned}
\mathfrak{v}^{(1)}_{m,n}(t,x) &= \gamma^{N+1}\left(\int_{\mathbb{R}_v}(\mathfrak{K}(\gamma\,\cdot\,,\gamma\,\cdot\,)\star_{t,x}\mathfrak{f}_{m,n})(\,\cdot\,,\,\cdot\,,v)\eta_{\delta,\gamma}(v)\,dv\right)(t,x)\\
&= \gamma^{N+1}\left(\mathfrak{K}(\gamma\,\cdot\,,\gamma\,\cdot\,)\star\int_{\mathbb{R}_v}\mathfrak{f}_{m,n}(\,\cdot\,,\,\cdot\,,v)\eta_{\delta,\gamma}\,dv\right)(t,x),
\end{aligned}$$

Thus, applying the Young's inequality for convolutions and the trivial estimate (2.31), we see that, for almost any $\omega \in \Omega$,

$$\begin{aligned}
\|\mathfrak{v}^{(1)}_{m,n}\|_{\mathscr{C}_0(\mathbb{R}_t\times\mathbb{R}^N_x)} &\leq \gamma^{\frac{N+1}{p}}\|\mathfrak{K}\|_{L^{p'}_{t,x}}\left\|\int_{\mathbb{R}_v}\mathfrak{f}_{m,n}(\,\cdot\,,\,\cdot\,,v)\eta_{\delta,\gamma}\,dv\right\|_{L^p_{t,x}}\\
&\leq \gamma^{\frac{N+1}{p}}\|\mathfrak{K}\|_{L^{p'}_{t,x}}\|\eta_{\delta,\gamma}\|_{L^{p'}_v}\|\mathfrak{f}_{m,n}\|_{L^p_{t,x,v}}
\end{aligned}$$

(notice that the Sobolev inequality implies that $W^{N+1,1}_{t,x} \subset L^1_{t,x} \cap L^\infty_{t,x}$). The asserted bound with $\mathfrak{q} = r\frac{N+1}{p}$ now follows from a joint application of the Hölder's inequality and Lemma 2.1. □

Remark 2.9. Were $(f_n)_{n\in\mathbb{N}}$ also bounded in $L^r_\omega L^\varsigma_{t,x,v}$ for some $1 \le \varsigma < p$, the Young's inequality for convolutions could have been invoked to refine (2.32) into

$$\mathbb{E}\|\varphi\mathfrak{v}^{(1)}_{m,n}\|^r_{L^p(\mathbb{R}_t\times\mathbb{R}^N_x)} \le C\gamma^{r(N+1)\left(\frac{1}{\varsigma}-\frac{1}{p}\right)}\|\eta_{\delta,\gamma}\|^r_{L^{\varsigma'}_v}\mathbb{E}\|\mathfrak{f}_{m,n}\|^r_{L^\varsigma_{t,x,v}}.$$

Thus, estimating $\|\eta_{\delta,\gamma}\|^r_{L^{\varsigma'}_v} \le C\|\eta_{\delta,\gamma}\|^{r/\varsigma'}_{L^\infty_v} \le C\gamma^{-r\chi/\varsigma'}$, we see that, provided that $\chi = \chi(p,\varsigma)$ is chosen sufficiently small,

$$\mathbb{E}\|\varphi\mathfrak{v}^{(1)}_{m,n}\|^r_{L^p(\mathbb{R}_t\times\mathbb{R}^N_x)} \le C\gamma^{\mathfrak{q}}$$

for all m and $n \in \mathbb{N}$, and $\gamma > 0$, with $C = C(\|\varphi\|_{L^\infty_{t,x}}, \sup_{\nu\in\mathbb{N}}\|f_\nu\|_{L^r_\omega L^\varsigma_{t,x,v}})$, and $\mathfrak{q} = \mathfrak{q}(p,\varsigma) > 0$.

2.2.4 The analysis of $\mathfrak{v}^{(2)}_{m,n}$.

Let us recall some results arising from the E. Tadmor–T. Tao theory [107].

Definition 2.3.

1. A Fourier multiplier $m(\tau,\kappa)$ on $\mathbb{R}_\tau \times \mathbb{R}^N_\kappa$ is said to satisfy the *truncation property* if, for any $\phi \in \mathscr{C}^\infty_c(\mathbb{C};\mathbb{C})$, $\varepsilon > 0$, and $1 < \mathfrak{s} < \infty$, the formula
$$\Lambda \in \mathscr{S}(\mathbb{R}_t \times \mathbb{R}^N_x) \mapsto \mathfrak{F}^{-1}_{t,x}\left[\phi\left(\frac{m(\tau,\kappa)}{\varepsilon}\right)(\mathfrak{F}_{t,x}\Lambda)\right] \tag{2.33}$$
defines a bounded linear operator in $L^\mathfrak{s}(\mathbb{R}_t \times \mathbb{R}^N_x)$ whose norm may depend on $\mathfrak{s}$, and on the support and $\mathscr{C}^\nu$–norm of ϕ for some nonnegative integer ν, but not on $\varepsilon > 0$. In other words, there exists some integer $\nu \ge 0$ and some constant $C = C(\mathfrak{s}, \operatorname{supp}\phi, \|\phi\|_{\mathscr{C}^\nu})$ such that
$$\left\|\mathfrak{F}^{-1}_{t,x}\left[\phi\left(\frac{m(\tau,\kappa)}{\varepsilon}\right)(\mathfrak{F}_{t,x}\Lambda)\right]\right\|_{L^\mathfrak{s}(\mathbb{R}_t\times\mathbb{R}^N_x)} \le C\|\Lambda\|_{L^\mathfrak{s}(\mathbb{R}_t\times\mathbb{R}^N_x)} \tag{2.34}$$
for all $\Lambda \in \mathscr{S}(\mathbb{R}_t \times \mathbb{R}^N_x)$ and $\varepsilon > 0$.
2. Let $m(\tau,\kappa,v)$ be a Fourier multiplier on $\mathbb{R}_\tau \times \mathbb{R}^N_\kappa$ depending on a parameter $v \in \mathbb{R}_v$. $m(\tau,\kappa,v)$ is said to satisfy the *truncation property uniformly in* v if, given any compact subset $K \subset \mathbb{R}_v$, the symbol $(\tau,\kappa) \mapsto m(\tau,\kappa,v)$ satisfies the truncation property, and the integer $\nu \ge 0$ and the constant C appearing in (2.34) may be uniformly chosen for $v \in K$.

Let us also remember the following Fourier multiplier theorem, which can be seen as a corollary of the so-called Marcinkiewicz multiplier theorem (see E. M. Stein [103]) and whose statement we adapt from F. Zimmermann [114]. Other demonstrations and further improvements may also be found in P.I. Lizorkin [83], R. Haller–H. Heck–A. Noll [61], P.C. Kunstmann–L. Weiss [76], and T. P. Hytönen [66, 67], and the references therein. (Recollect that, for any $w \in \mathbb{R}^d_y$, the differential operator $\frac{\partial}{\partial w}$ is defined as $w \cdot \nabla_y$. Further, recall that the Fourier transform is well-behaved under linear changes of coordinates).

Theorem 2.5. *Let d be a positive integer, and $m \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Assume that there exists an orthonormal basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$ such that, for any multi-index $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_d)$ observing $\mathfrak{a} \le 1 = (1, \ldots, 1)$, one has that*
$$\frac{\partial^{\mathfrak{a}_1+\cdots+\mathfrak{a}_d} m}{\partial e_1^{\mathfrak{a}_1}\cdots\partial e_d^{\mathfrak{a}_d}} \in L^1_{\mathrm{loc}}(\mathbb{R}^d),$$
and
$$\sum_{\mathfrak{a}\le 1}\operatorname*{ess\,sup}_{y\in\mathbb{R}^d}\left|(y\cdot e_1)^{\mathfrak{a}_1}\cdots(y\cdot e_d)^{\mathfrak{a}_d}\frac{\partial^{\mathfrak{a}_1+\cdots+\mathfrak{a}_d} m}{\partial e_1^{\mathfrak{a}_1}\cdots\partial e_d^{\mathfrak{a}_d}}(y)\right| = B < \infty. \tag{2.35}$$

Then, for any $1 < \mathfrak{s} < \infty$, m *is an* $L^{\mathfrak{s}}(\mathbb{R}^d)$*–multiplier, and there exists a constant* $C = C_{\mathfrak{s},d} > 0$ *such that*

$$\left\| \mathfrak{F}_y^{-1}\left[m(\,\cdot\,)(\mathfrak{F}_y f)(\,\cdot\,) \right] \right\|_{L^{\mathfrak{s}}(\mathbb{R}^d_y)} \leq CB \|f\|_{L^{\mathfrak{s}}(\mathbb{R}^d_y)} \text{ for all } f \in \mathscr{S}(\mathbb{R}^d_y). \tag{2.36}$$

Let us now show that the symbols employed in the decomposition (2.27) indeed have the truncation property.

Proposition 2.5. *The following statements hold.*

1. *The symbol* $(\tau, \kappa) \in \mathbb{R}_\tau \times \mathbb{R}^N_\kappa \mapsto \sqrt{\tau^2 + |\kappa|^2}$ *satisfies the truncation property.*
2. *The normalized symbol* $\widetilde{\mathcal{L}}(i\tau, i\kappa, v)$ *observes the truncation property uniformly in* v.
3. *Likewise, the normalized restricted symbol* $(\widetilde{R\mathcal{L}})(i\tau, i\kappa, v)$ *enjoys the truncation property uniformly in* v.

Proof. First of all, statement (1) is an obvious conclusion flowing from Proposition 2.3. On the other hand, the second assertion's verification is trivialized after the constatation of the following two facts.

Claim #1: The symbols m_h and $m_p : \big(\mathbb{R}_\tau \times \mathbb{R}^N_\kappa \setminus \{0\}\big) \times \mathbb{R}_v \to \mathbb{R}$ given by

$$\begin{cases} m_h(\tau, \kappa, v) = \dfrac{\tau}{\sqrt{\tau^2 + |\kappa|^2}} + \mathbf{a}(v) \cdot \dfrac{\kappa}{\sqrt{\tau^2 + |\kappa|^2}}, \text{ and} \\ m_p(\tau, \kappa, v) = \dfrac{\kappa \cdot \mathbf{b}(v)\kappa}{\tau^2 + |\kappa|^2} \end{cases}$$

satisfy the truncation property uniformly in $v \in \mathbb{R}$. (Indeed, this follows directly from Theorem 2.5. So as to facilitate such an inspection, notice that one may assume without loss of generality that

$$\begin{cases} m_h(\tau, \kappa, v) = \sqrt{1 + |\mathbf{a}(v)|^2} \dfrac{\tau}{\sqrt{\tau^2 + |\kappa|^2}}, \text{ and} \\ m_p(\tau, \kappa, v) = \dfrac{\lambda_1(v)\kappa_1^2 + \cdots + \lambda_N(v)\kappa_N^2}{\tau^2 + |\kappa|^2}, \end{cases}$$

where $0 \leq \lambda_1(v) \leq \cdots \leq \lambda_N(v) = \|\mathbf{b}(v)\|_{\mathscr{L}(\mathbb{R}^N)}$. Then, one can inspect that

$$\begin{aligned} &\operatorname*{ess\,sup}_{(\tau,\kappa)\times\mathbb{R}\times\mathbb{R}^N} \left| \big(\tau^{\mathfrak{a}_0}\kappa_1^{\mathfrak{a}_1} \cdots \kappa_N^{\mathfrak{a}_N}\big) \frac{\partial^{\mathfrak{a}_0 + \mathfrak{a}_1 + \cdots + \mathfrak{a}_N}}{\partial\tau^{\mathfrak{a}_0}\partial\kappa_1^{\mathfrak{a}_1} \cdots \partial\kappa_N^{\mathfrak{a}_N}} \left[\phi\left(\frac{m(\tau,\kappa,v)}{\varepsilon} \right) \right] \right| \\ &\qquad \leq C_{\mathfrak{a}} \left\{ \sup_{x \in \mathbb{R}} |\phi(x)| + \sup_{x\in\mathbb{R}} \left| x \frac{d\phi}{dx}(x) \right| + \cdots + \sup_{x\in\mathbb{R}} \left| x^{\mathfrak{a}_0 + \cdots + \mathfrak{a}_N} \frac{d^{\mathfrak{a}_0 + \cdots + \mathfrak{a}_N}\phi}{dx^{\mathfrak{a}_0 + \cdots + \mathfrak{a}_N}}(x) \right| \right\} \end{aligned}$$

where $C_{\mathfrak{a}}$ does not depend on $\varepsilon > 0$, m is either m_h or m_p, $\phi \in \mathscr{C}^\infty_c(\mathbb{R})$ is arbitrary, and $\mathfrak{a} = (\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_N)$ is any multi-index ≤ 1. This evidently yields the desired conclusion).

Claim #2: If m_1 and $m_2 : \big(\mathbb{R}_\tau \times \mathbb{R}^N_\kappa \setminus \{0\}\big) \times \mathbb{R}_v \to \mathbb{R}$ are two real-valued multipliers satisfying the truncation property uniformly on v, then so does the complex-valued multiplier $m(\tau, \kappa, v) = m_1(\tau, \kappa, v) + im_2(\tau, \kappa, v)$. (The proof of this statement utilizes Fourier series and may be found in E. Tadmor–T. Tao [107]).

This couple of claims shows assertion (2), leaving us to inspect the statement (3). Comprehending $(\widetilde{R\mathcal{L}})(i\tau, i\kappa, v)$ as a multiplier in $\mathbb{R}_\tau \times M^\perp$, the demonstration that this symbol possesses the truncation property uniformly in v becomes—aside from minor technicalities—parallel to the analysis already described; thus we will omit it. The proof is now complete. □

Remark 2.10. Observe that the statement (1) could have been proven via Theorem 2.5 (or the Mihlin–Hörmander theorem). Nevertheless, the presented reasoning, besides being certainly more elementary, shows that the endpoints $\mathfrak{s} = 1$ and $\mathfrak{s} = \infty$ in Definition 2.3 are valid for the particular symbol $(\tau, \kappa) \mapsto \sqrt{\tau^2 + |\kappa|^2}$.

What is more, let us point out that Claim #1 answers positively a question posed in Tadmor–Tao [107]; see also R.J. DiPerna–P.-L. Lions–Y. Meyer [38].

Lemma 2.3. *There exist constants $C = C_p$ and $\mathfrak{p} = \mathfrak{p}_p > 0$, both independent of $0 < \delta$ and $\gamma < 1$, such that, almost surely, and for all m and $n \in \mathbb{N}$,*

$$\|\mathfrak{v}^{(2)}_{m,n}\|_{L^p(\mathbb{R}_t\times\mathbb{R}^N_x)} \leq C\|\eta_{\delta,\gamma}\|_{L^\infty(\mathbb{R})}\|\mathfrak{f}_{m,n}\|_{L^p(\mathbb{R}_t\times\mathbb{R}^N_x\times\mathbb{R}_v)} \left(\sup_{\tau^2+|\kappa|^2=1} \operatorname{meas}\Big\{v \in \operatorname{supp}\eta_{\delta,\gamma};\, |\mathcal{L}(i\tau, i\kappa, v)| \leq \delta\Big\}\right)^{\mathfrak{p}}. \tag{2.37}$$

As a result, for all m and $n \in \mathbb{N}$,

$$\mathbb{E}\|\varphi\mathfrak{v}^{(2)}_{m,n}\|^r_{L^s(\mathbb{R}_t\times\mathbb{R}^N_x)} \leq C\|\eta_{\delta,\gamma}\|^r_{L^\infty(\mathbb{R})}\left(\sup_{\tau^2+|\kappa|^2=1} \operatorname{meas}\Big\{v \in \operatorname{supp}\eta_{\delta,\gamma};\, |\mathcal{L}(i\tau, i\kappa, v)| \leq \delta\Big\}\right)^{r\mathfrak{p}}, \tag{2.38}$$

where $C = C\big(\|\varphi\|_{L^1_{t,x}\cap L^\infty_{t,x}}, \sup_{\nu\in\mathbb{N}}\|f_\nu\|_{L^r_\omega L^p_{t,x,v}}\big)$ is independent of $0 < \delta$ and $\gamma < 1$.

Proof. The result will follow from the investigation of the norm of the linear transformation

$$(T_{\delta,\gamma}f)(t,x) = \mathfrak{F}^{-1}_{t,x}\left[\int_{\mathbb{R}_v} \eta_{\delta,\gamma}(v)\lambda\left(\frac{\widetilde{\mathcal{L}}(i\tau, i\kappa, v)}{\delta}\right)\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)(\mathfrak{F}_{t,x}f)\,dv\right](t,x).$$

According the previous proposition—once that $\psi(\sqrt{\tau^2+|\kappa|^2}/\gamma) = 1 - \lambda(\sqrt{\tau^2+|\kappa|^2}/\gamma)$—, the trivial estimate (2.31) asserts that $T_{\delta,\gamma} : L^{\mathfrak{s}}_{t,x,v} \to L^{\mathfrak{s}}_{t,x}$ is continuous for any $1 < \mathfrak{s} < \infty$, and

$$\|T_{\delta,\gamma}\|_{\mathscr{L}(L^{\mathfrak{s}}_{t,x,v};L^{\mathfrak{s}}_{t,x})} \leq C_{\mathfrak{s}}\|\eta_{\delta,\gamma}\|_{L^\infty(\mathbb{R})} \tag{2.39}$$

for some $C_{\mathfrak{s}}$ which is independent of $0 < \delta$ and $\gamma < 1$.

Let us consider initially the case $p = 2$. In this scenario, we may sharpen the trivial estimate (2.31) by means of the Plancherel identity, in order to obtain

$$\begin{aligned}\|T_{\delta,\gamma}f\|^2_{L^2(\mathbb{R}_t\times\mathbb{R}^N_x)} &\leq \int_{\mathbb{R}_\tau}\int_{\mathbb{R}^N_\kappa}\left(\int_{\{w\in\mathbb{R};|\widetilde{\mathcal{L}}(i\tau,i\kappa,w)|\leq\delta\}} |\eta_{\delta,\gamma}(w)|^2\,dw\right) \\ &\qquad \int_{\mathbb{R}_v}\left|\lambda\left(\frac{\widetilde{\mathcal{L}}(i\tau, i\kappa, v)}{\delta}\right)\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)(\mathfrak{F}_{t,x}f)(\tau,\kappa,v)\right|^2 dv d\kappa d\tau \\ &\leq \|\eta_{\delta,\gamma}\|^2_{L^\infty(\mathbb{R})}\|f\|^2_{L^2(\mathbb{R}_t\times\mathbb{R}^N_x\times\mathbb{R}_v)} \\ &\qquad \left(\sup_{\tau^2+|\kappa|^2=1} \operatorname{meas}\Big\{v \in \operatorname{supp}\eta_{\delta,\gamma};\, |\mathcal{L}(i\tau, i\kappa, v)| \leq \delta\Big\}\right).\end{aligned} \tag{2.40}$$

In other words,

$$\|T_{\delta,\gamma}\|_{\mathscr{L}(L^2_{t,x,v};L^2_{t,x})} \leq \|\eta_{\delta,\gamma}\|_{L^\infty(\mathbb{R})}\left(\sup_{\tau^2+|\kappa|^2=1} \operatorname{meas}\Big\{v \in \operatorname{supp}\eta_{\delta,\gamma};\, |\mathcal{L}(i\tau, i\kappa, v)| \leq \delta\Big\}\right)^{1/2}. \tag{2.41}$$

This proves (2.37) if $p = 2$. For a general exponent $1 < p < \infty$, one can interpolate (2.41) with (2.39) via the Riesz–Thorin theorem with exponents, say, $\mathfrak{s} = \frac{1+p}{2}$ if $1 < p < 2$, and $\mathfrak{s} = 2p$ if $2 < p < \infty$. □

Before we close this subsection, let us state and prove the following topological fact that guarantees the utility of the estimate in (2.38).

Lemma 2.4. *It holds that*

$$\sup_{\tau^2+|\kappa|^2=1} \operatorname{meas}\Big\{v \in \operatorname{supp}\eta_{\delta,\gamma}; |\mathcal{L}(i\tau, i\kappa, v)| \leq \delta\Big\} \to 0 \text{ as } \delta \to 0_+. \tag{2.42}$$

Proof. Assume, by absurd, that the conclusion (2.42) is false, and denote by $\mathbb{S}^N$ the sphere in $\mathbb{R} \times \mathbb{R}^N$. Under such an assumption, there would exist some $\vartheta > 0$, $\delta_n \to 0_+$, and $(\tau_n, \kappa_n) \in \mathbb{S}^N$ such that

$$\operatorname{meas}\Big\{v \in \operatorname{supp}\eta_{\delta_n,\gamma}; |\mathcal{L}(i\tau_n, i\kappa_n, v)| \leq \delta_n\Big\} \geq \vartheta \text{ for all } n \in \mathbb{N}. \tag{2.43}$$

Passing to a subsequence if necessary, we may assume that $(\tau_n, \kappa_n) \to (\tau_\infty, \kappa_\infty) \in \mathbb{S}^N$. In light of the uniform continuity of $\mathcal{L}(i\,\cdot, i\,\cdot, \cdot)$ over compact sets of $\mathbb{S}^N \times \mathbb{R}_v$, and of the assertion (b.i) in Lemma 2.1, (2.43) implies that

$$\operatorname{meas}\Big\{v \in \operatorname{supp}\eta + (-\delta_n, \delta_n); |\mathcal{L}(i\tau_\infty, i\kappa_\infty, v)| \leq \delta_n + \varepsilon_n\Big\} \geq \vartheta \tag{2.44}$$

for all $n \in \mathbb{N}$ and some $\varepsilon_n \to 0_+$. Notwithstanding, amalgamating the Lebesgue dominated convergence theorem and the nondegeneracy condition (2.11),

$$\lim_{n\to\infty} \operatorname{meas}\Big\{v \in \operatorname{supp}\eta + (-\delta_n, \delta_n); |\mathcal{L}(i\tau_\infty, i\kappa_\infty, v)| \leq \delta_n + \varepsilon_n\Big\} = 0,$$

which is a blatant contradiction of (2.44). Once the absurd hypothesis cannot hold, the desired limit (2.42) is thus established. □

2.2.5 The analysis of $\mathfrak{v}^{(3)}_{m,n}$.

Let us reinterpret the results of the previous subsection to the context of $\mathfrak{v}^{(3)}_{m,n}$.

Lemma 2.5. *The following statements hold.*

1. *There exists an exponent $\mathfrak{r} = \mathfrak{r}_p > 0$ independent of $0 < \delta$ and $\gamma < 1$, such that, for all m and $n \in \mathbb{N}$,*

$$\begin{aligned}\mathbb{E}\|\varphi\mathfrak{v}^{(3)}_{m,n}\|^r_{L^s(\mathbb{R}_t\times\mathbb{R}^N_x)} &\leq C\|\eta_{\delta,\gamma}\|^r_{L^\infty(\mathbb{R})} \\ &\left(\sup_{\substack{(\tau,\kappa)\in\mathbb{R}\times M^\perp \\ \tau^2+|\kappa|^2=1}} \operatorname{meas}\Big\{v \in \operatorname{supp}\eta_{\delta,\gamma}; |\mathcal{L}(i\tau, i\kappa, v)| \leq \delta\Big\}\right)^{r\mathfrak{r}},\end{aligned} \tag{2.45}$$

where $C = C\left(\|\varphi\|_{L^s_{t,x}\cap L^\infty_{t,x}}, \sup_{n\in\mathbb{N}}\|f_n\|_{L^r_\omega L^p_{t,x,v}}\right)$.

2. *It holds that*

$$\sup_{\substack{(\tau,\kappa)\in\mathbb{R}\times M^\perp \\ \tau^2+|\kappa|^2=1}} \operatorname{meas}\Big\{v \in \operatorname{supp}\eta_{\delta,\gamma}; \mathcal{L}(i\tau, i\kappa, v)| \leq \delta\Big\} \to 0 \text{ as } \delta \to 0_+. \tag{2.46}$$

Proof. Observe that $(R\mathcal{L})(i\tau, i\kappa, v)$ and $\widetilde{(R\mathcal{L})}(i\tau, i\kappa, v)$ can be seen as, respectively, $\mathcal{L}(i\tau, i\kappa, v)$ and $\widetilde{\mathcal{L}}(i\tau, i\kappa, v)$ restricted to $(\tau, \kappa, v) \in \mathbb{R} \times (M^\perp \setminus \{0\}) \times \mathbb{R}_v$—hence their name. For it was already shown in Proposition 2.5 that $\widetilde{(R\mathcal{L})}(i\tau, i\kappa, v)$ satisfies the truncation property uniformly in v, the derivation of the first statement becomes now indistinguishable from the proof of Lemma 2.3.

Finally, choosing $\kappa \in M^\perp$ in the nondegeneracy condition (2.11), we deduce that

$$\operatorname{meas}\Big\{v \in \operatorname{supp}\eta; \mathcal{L}(i\tau, i\kappa, v) = 0\Big\} = 0 \ \forall(\tau, \kappa) \in \mathbb{R} \times M^\perp \text{ such that } \tau^2 + |\kappa|^2 = 1.$$

Therefore, reprising the argument behind Lemma 2.4, (2.46) follows. □

2.2.6 The analysis of $\mathfrak{v}^{(4)}_{m,n}$.

Initial manipulations.

It is not difficult to see

$$(\tau,\kappa,v)\mapsto \psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\psi\left(\frac{\widetilde{\mathcal{L}}(i\tau,i\kappa,v)}{\delta}\right)\frac{1}{\mathcal{L}(i\tau,i\kappa,v)}$$

is a well-defined function in $(\mathscr{C}^{k,\alpha}_{\text{loc}}\cap L^\infty)(\mathbb{R}_t\times\mathbb{R}^N_x\times\mathbb{R}_v)$ provided we understand it to be 0 where $\mathcal{L}(i\tau,i\kappa,v)=0$. Accordingly, if we apply the Fourier transform to (2.18) and recall the definition $\mathfrak{f}^{(4)}_{m,n}$ as expressed in (2.27), we thus are able to justify the formula

$$\begin{aligned}(\mathfrak{F}_{t,x}\mathfrak{f}^{(4)}_{m,n}) = {}&\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\psi\left(\frac{\widetilde{\mathcal{L}}(i\tau,i\kappa,v)}{\delta}\right)\psi\left(\frac{(\widetilde{R\mathcal{L}})(i\tau,i\kappa,v)}{\delta}\right)\\ &\left[\sum_{j\in\mathscr{J}}\frac{\left(\tau^2+|\kappa|^2+1\right)^{1/2}}{\mathcal{L}(i\tau,i\kappa,v)}\left(1\pm\frac{\partial^{\mathfrak{l}}}{\partial v^{\mathfrak{l}}}(-\Delta_v)^{\mathfrak{z}/2}\right)(\mathfrak{F}_{t,x}\mathfrak{g}^{(j)}_{m,n})\right.\\ &\quad-\sum_{j\in\mathscr{J}}\frac{\Pi_j(v)|P_M\kappa|^2}{\mathcal{L}(i\tau,i\kappa,v)}\left(1\pm\frac{\partial^{\mathfrak{l}}}{\partial v^{\mathfrak{l}}}(-\Delta_v)^{\mathfrak{z}/2}\right)(\mathfrak{F}_{t,x}\mathfrak{h}^{(j)}_{m,n})\\ &\quad\left.+\frac{\left(|\kappa|^2+1\right)^{1/4}}{\mathcal{L}(i\tau,i\kappa,v)}\left(1\pm\frac{\partial^{\mathfrak{l}}}{\partial v^{\mathfrak{l}}}(-\Delta_v)^{\mathfrak{z}/2}\right)\mathfrak{F}_{t,x}\left(\Psi_{m,n}\frac{dW}{dt}\right)\right].\end{aligned}$$

Additionally, taking advantage that $\psi(\sqrt{\tau^2+|\kappa|^2}/\gamma)$ cancels near the origin, we may substitute the term $(\tau^2+|\kappa|^2+1)^{1/2}$ with $\sqrt{\tau^2+|\kappa|^2}$ by modifying $\mathfrak{g}_{m,n}$. Therefore, this alteration yields the subdivision

$$\mathfrak{v}^{(3)}_{m,n}=\sum_{j\in\mathscr{J}}(I)^{(j)}_{m,n}+\sum_{j\in\mathscr{J}}(II)^{(j)}_{m,n}+(III)_{m,n}, \tag{2.47}$$

where these parcels are given by

$$\begin{aligned}(I)^{(j)}_{m,n}={}&\mathfrak{F}^{-1}_{t,x}\left\{\int_{\mathbb{R}}\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)(\mathfrak{F}_{t,x}\widetilde{\mathfrak{g}}^{(j)}_{m,n})\left(1\pm(-1)^{\mathfrak{l}}\frac{\partial^{\mathfrak{l}}}{\partial v^{\mathfrak{l}}}(-\Delta_v)^{\mathfrak{z}/2}\right)\right.\\ &\left.\left[\eta_{\delta,\gamma}(v)\psi\left(\frac{\widetilde{\mathcal{L}}(i\tau,i\kappa,v)}{\delta}\right)\psi\left(\frac{(\widetilde{R\mathcal{L}})(i\tau,i\kappa,v)}{\delta}\right)\frac{\sqrt{\tau^2+|\kappa|^2}}{\mathcal{L}(i\tau,i\kappa,v)}\right]dv\right\},\\ (II)^{(j)}_{m,n}={}&\mathfrak{F}^{-1}_{t,x}\left\{\int_{\mathbb{R}}\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)(\mathfrak{F}_{t,x}\mathfrak{h}^{(j)}_{m,n})\left(1\pm(-1)^{\mathfrak{l}}\frac{\partial^{\mathfrak{l}}}{\partial v^{\mathfrak{l}}}(-\Delta_v)^{\mathfrak{z}/2}\right)\right.\\ &\left.\left[\eta_{\delta,\gamma}(v)\psi\left(\frac{\widetilde{\mathcal{L}}(i\tau,i\kappa,v)}{\delta}\right)\psi\left(\frac{(\widetilde{R\mathcal{L}})(i\tau,i\kappa,v)}{\delta}\right)\frac{-\Pi_j(v)|P_M\kappa|^2}{\mathcal{L}(i\tau,i\kappa,v)}\right]dv\right\},\text{ and}\\ (III)_{m,n}={}&\mathfrak{F}^{-1}_{t,x}\left\{\int_{\mathbb{R}}\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\mathfrak{F}_{t,x}\left(\Psi_{m,n}\frac{dW}{dt}\right)\left(1\pm(-1)^{\mathfrak{l}}\frac{\partial^{\mathfrak{l}}}{\partial v^{\mathfrak{l}}}(-\Delta_v)^{\mathfrak{z}/2}\right)\right.\\ &\left.\left[\eta_{\delta,\gamma}(v)\psi\left(\frac{\widetilde{\mathcal{L}}(i\tau,i\kappa,v)}{\delta}\right)\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\delta}\right)\frac{\left(|\kappa|^2+1\right)^{1/4}}{\mathcal{L}(i\tau,i\kappa,v)}\right]dv\right\},\end{aligned}$$

and, for any $j\in\mathscr{J}$, $(\widetilde{\mathfrak{g}}^{(j)}_{m,n})$ still satisfies

$$\lim_{m,n\to\infty}\mathbb{E}\left(\int_{\mathbb{R}_t}\int_{\mathbb{R}^N_x}\int_{\mathbb{R}_v}|\widetilde{\mathfrak{g}}^{(j)}_{m,n}(t,x,v)|^{q_j}\,dvdxdt\right)^{r/q_j}=0. \tag{2.48}$$

Even though each $\widetilde{\mathfrak{g}}_{m,n}^{(j)}$ depends on $\gamma > 0$, this will not be of substance for now. Let us inspect each term $(I)_{m,n}^{(j)}$, $(II)_{m,n}^{(j)}$ and $(III)_{m,n}$ separately.

The analysis of $(I)_{m,n}^{(j)}$.

Lemma 2.6. *There exists a constant, independent of m and $n \in \mathbb{N}$, such that, for all $j \in \mathscr{J}$ and almost surely,*

$$\|(I)_{m,n}^{(j)}\|_{L^{q_j}(\mathbb{R}_t\times\mathbb{R}_x^N)} \leq C\|\widetilde{\mathfrak{g}}_{m,n}^{(j)}\|_{L^{q_j}(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v)}. \tag{2.49}$$

Consequently,

$$\lim_{m,n\to\infty} \mathbb{E}\Big\|\varphi \sum_{j\in\mathscr{J}} (I)_{m,n}^{(j)}\Big\|_{L^s(\mathbb{R}_t\times\mathbb{R}_x^N)}^r = 0. \tag{2.50}$$

Proof. Step #1: In order to fix ideas, let us assume firstly that $\mathfrak{z} = 0$, i.e., $\mathfrak{l} = \ell$, so that, if $\ell \geq 1$, $\mathcal{L}(i\tau, i\kappa, v)$ is of class $\mathscr{C}_{\text{loc}}^{\ell-1,1}$ with respect to the velocity variable v (notice that $\mathcal{L}(i\tau, i\kappa, v)$ is polynomial in τ and κ, and hence infinitely differentiable in these arguments). The crux of our reasoning is based on the construction of $\eta_{\delta,\gamma}$—more specifically on assertion (b.ii) of Lemma 2.1—; thus, let us engage the same notations of this proposition here as well. Since the integrand defining $(I)_{m,n}^{(j)}$ is supported for $v \in \operatorname{supp}\eta_{\delta,\gamma}$, we may bifurcate our attention between the alternatives that $v \in K_h$ or $v \in K_p$.

Step #1.1: Let us first investigate the case $v \in K_h$, in which, because $\mathbf{b}(v) = 0$, Equation (2.18) has a hyperbolic character. Letting (τ', κ') be the normalized frequency as defined in (2.2), it holds that

$$\begin{cases} \widetilde{\mathcal{L}}(i\tau, i\kappa, v) = \mathcal{L}(i\tau', i\kappa', v), \text{and} \\ \mathcal{L}(i\tau, i\kappa, v) = \sqrt{\tau^2 + |\kappa|^2}\mathcal{L}(i\tau', i\kappa', v). \end{cases}$$

Observe that the last relations above remains true if one substitutes v with another $w \in \mathbb{R}$, provided that $|v - w| < \operatorname{dist}(K_h, K_p)$.

As a result, putting $\widetilde{\psi}(z) = \frac{1}{z}\psi(z)$ (which is, by all means, a regular function), each integrand of $(I)_{m,n}^{(j)}$ is transformed into

$$\begin{aligned} &\mathfrak{F}_{t,x}^{-1}\Bigg\{\frac{1}{\delta}\psi\bigg(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\bigg)(\mathfrak{F}_{t,x}\widetilde{\mathfrak{g}}_{m,n}^{(j)})\bigg(1 \pm (-1)^\ell\frac{\partial^\ell}{\partial v^\ell}\bigg) \\ &\qquad\bigg[\eta_{\delta,\gamma}(v)\widetilde{\psi}\bigg(\frac{\mathcal{L}(i\tau', i\kappa', v)}{\delta}\bigg)\psi\bigg(\frac{(\widetilde{R\mathcal{L}})(i\tau', i\kappa', v)}{\delta}\bigg)\bigg]\Bigg\} \\ &\qquad\qquad = \sum_{\nu=0}^{\ell}\eta_{\delta,\gamma}^{(\nu)}(v)\mathfrak{F}_{t,x}^{-1}\Bigg\{\psi\bigg(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\bigg)m_\nu(\tau,\kappa,\xi)(\mathfrak{F}_{t,x}\widetilde{\mathfrak{g}}_{m,n}^{(j)})\Bigg\}, \end{aligned} \tag{2.51}$$

with each $m_\nu(\tau, \kappa, v)$ being given by

$$\begin{cases} m_0(\tau,\kappa,v) = \dfrac{1}{\delta}\bigg(1 \pm (-1)^\ell\dfrac{\partial^\ell}{\partial v^\ell}\bigg)\bigg[\widetilde{\psi}\bigg(\dfrac{\mathcal{L}(i\tau', i\kappa', v)}{\delta}\bigg)\psi\bigg(\dfrac{(\widetilde{R\mathcal{L}})(i\tau, i\kappa, v)}{\delta}\bigg)\bigg], \text{ and} \\ m_\nu(\tau,\kappa,v) = \pm\dfrac{(-1)^\ell}{\delta}\dbinom{\ell}{\nu}\bigg(\dfrac{\partial^{\ell-\nu}}{\partial v^{\ell-\nu}}\bigg)\bigg[\widetilde{\psi}\bigg(\dfrac{\mathcal{L}(i\tau', i\kappa', v)}{\delta}\bigg)\psi\bigg(\dfrac{(\widetilde{R\mathcal{L}})(i\tau, i\kappa, v)}{\delta}\bigg)\bigg] \end{cases}$$

for $\nu = 1, \ldots, \ell$. On the grounds of Theorem 2.5, all of these symbols m_ν are $L^{q_j}(\mathbb{R}_t \times \mathbb{R}_x^N)$–multipliers for every $j \in \mathscr{J}$ and their norms are bounded in $v \in K_h$; in other words, for $v \in K_h$, (2.51) implies

$$\Bigg\| \mathfrak{F}_{t,x}^{-1} \Bigg\{ \frac{1}{\delta} \psi\Bigg(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma} \Bigg) (\mathfrak{F}_{t,x} \widetilde{\mathfrak{g}}_{m,n}) \Bigg(1 \pm (-1)^\ell \frac{\partial^\ell}{\partial v^\ell} \Bigg) \Bigg[\eta_{\delta,\gamma}(v) \widetilde{\psi}\Bigg(\frac{\mathcal{L}(i\tau', i\kappa', v)}{\delta} \Bigg) \psi\Bigg(\frac{(\widetilde{R\mathcal{L}})(i\tau', i\kappa', v)}{\delta} \Bigg) \Bigg] \Bigg\} \Bigg\|_{L^{q_j}(\mathbb{R}_t \times \mathbb{R}_x^N)} \leq C_j \Bigg(\sum_{\nu=0}^{\ell} |\eta_{\delta,\gamma}^{(\nu)}(v)| \Bigg) \|\widetilde{\mathfrak{g}}_{m,n}^{(j)}(\,\cdot\,,\,\cdot\,,v)\|_{L^{q_j}(\mathbb{R}_t \times \mathbb{R}_x^N)} \text{ almost surely,} \tag{2.52}$$

for all $j \in \mathscr{J}$, where C_j does not depend on $v \in K_h$, and on m and $n \in \mathbb{N}$.

Step #1.2: The last estimate is enough to control the integral $(I)_{m,n}^{(j)}$ when v ranges over K_h. Let us now investigate the other dichotomic option: let $v \in K_p$ be given. Even though now there is no simplification in the integrand of $(I)_{m,n}^{(j)}$, we may still perform the necessary differentiations, arriving at the formula

$$(I)_{m,n}^{(j)} = \mathfrak{F}_{t,x}^{-1} \Bigg[\pm (-1)^\ell \int_{\mathbb{R}} \eta_{\delta,\gamma}^{(\ell)}(v) \psi\Bigg(\frac{\widetilde{\mathcal{L}}(i\tau, i\kappa, v)}{\delta} \Bigg) \psi\Bigg(\frac{(\widetilde{R\mathcal{L}})(i\tau, i\kappa, v)}{\delta} \Bigg) \psi\Bigg(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma} \Bigg) \frac{\sqrt{\tau^2+|\kappa|^2}}{\mathcal{L}(i\tau, i\kappa, v)} (\mathfrak{F}_{t,x} \widetilde{\mathfrak{g}}_{m,n}^{(j)})\, dv \Bigg] + [\text{ similar terms }]. \tag{2.53}$$

Although the omitted parcels could have been explicitly expressed via Leibniz's and Faà di Bruno's rules, all portions can be handled analogously. Consequently, we will concentrate on the sole portion above.

We are thus led to examine the Fourier operator

$$f \mapsto \mathfrak{F}_{t,x}^{-1} \Bigg[\psi\Bigg(\frac{\widetilde{\mathcal{L}}(i\tau, i\kappa, v)}{\delta} \Bigg) \psi\Bigg(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma} \Bigg) \psi\Bigg(\frac{(\widetilde{R\mathcal{L}})(i\tau, i\kappa, v)}{\delta} \Bigg) \frac{\sqrt{\tau^2+|\kappa|^2}}{\mathcal{L}(i\tau, i\kappa, \xi)} (\mathfrak{F}_{t,x} f) \Bigg]. \tag{2.54}$$

In order to verify that such an expression defines an $L_{t,x}^{q_j}$–multiplier, let us first establish a simple bound that will be stated as a lemma, since it will again be instrumental later on as well.

Lemma 2.7. *There exists a constant $C = C(\delta, \gamma) > 0$, such that*

$$\left| \frac{\sqrt{\tau^2+|\kappa|^2} + |P_M \kappa|^2}{\mathcal{L}(i\tau, i\kappa, v)} \right| \leq C \tag{2.55}$$

for all $v \in K_p$, and (τ, κ, v) in the support of $\psi\Big(\frac{\widetilde{\mathcal{L}}(i\tau, i\kappa, v)}{\delta} \Big) \psi\Big(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma} \Big) \psi\Big(\frac{(\widetilde{R\mathcal{L}})(i\tau, i\kappa, v)}{\delta} \Big)$.

Proof. Fix $v \in K_p$. If (τ, κ) is such that $\psi((\widetilde{R\mathcal{L}})(i\tau, i\kappa, v)/\delta) \neq 0$, then

$$\begin{aligned} |\tau + \mathbf{a}(v) \cdot \kappa| &= |(\tau + \mathbf{a}(v) \cdot P_{M^\perp} \kappa) + (\mathbf{a}(v) \cdot P_M \kappa)| \\ &\geq \frac{\delta}{2} \sqrt{\tau^2 + |P_{M^\perp} \kappa|^2} - \Big(\sup_{v \in K_p} |\mathbf{a}(v)| \Big) |P_M \kappa| \\ &\geq \frac{\delta}{2} \sqrt{\tau^2 + |P_{M^\perp} \kappa|^2} - \frac{\mathfrak{c}_\delta}{2} |P_M \kappa|^2 - A, \end{aligned}$$

where A depends only on $\mathfrak{c}_\delta$ and K_p, thus solely on δ and γ (recall we are employing the notations of Lemma 2.1). Hence, from the trivial inequality $\frac{1}{\sqrt{2}}(|a| + |b|) \leq \sqrt{a^2 + b^2}$, and the fact that

$\kappa \cdot \mathbf{b}(v)\kappa \geq \mathfrak{c}_\delta |P_M \kappa|^2$ for $v \in K_p$,

$$|\mathcal{L}(i\tau, i\kappa, v)| \geq \frac{1}{\sqrt{2}} \left(\frac{\delta}{2} \sqrt{\tau^2 + |P_{M^\perp}\kappa|^2} + \frac{\mathfrak{c}_\delta}{2} |P_M \kappa|^2 - A \right),$$

concluding the existence of constants B and $R > 0$, depending only on δ and γ, such that

$$|\mathcal{L}(i\tau, i\kappa, v)| \geq B \Big(\sqrt{\tau^2 + |\kappa|^2} + |P_M \kappa|^2 \Big)$$

if $\sqrt{\tau^2 + |\kappa|^2} \geq R$ and $\psi((\widetilde{R\mathcal{L}})(i\tau, i\kappa, v)/\delta) \neq 0$. This shows (2.55) for $\sqrt{\tau^2 + |\kappa|^2}$ sufficiently large.

On the other land, because $\{\frac{\gamma}{2} \leq \sqrt{\tau^2 + |\kappa|^2} \leq R\} \times K_p$ is compact, the continuity of

$$(\tau, \kappa, v) \mapsto \psi\left(\frac{\widetilde{\mathcal{L}}(i\tau, i\kappa, v)}{\delta} \right) \frac{\sqrt{\tau^2 + |\kappa|^2} + |P_M \kappa|^2}{\mathcal{L}(i\tau, i\kappa, v)}$$

in this region proves (2.55) for $\sqrt{\tau^2 + |\kappa|^2}$ of "intermediate size". Finally, for $\sqrt{\tau^2 + |\kappa|^2} < \frac{\gamma}{2}$, $\psi(\sqrt{\tau^2 + |\kappa|^2}/\gamma) = 0$, and, therefore, the desired bound is immediate in this region as well. The lemma is hereby demonstrated. □

As a consequence, in order to apply Theorem 2.5, we can argue just as in Proposition 2.5: choose an orthonormal basis $e_0, e_1, \ldots, e_N$ in $\mathbb{R} \times \mathbb{R}^N$ such that $e_0 = (1, 0)$, and, for $1 \leq \nu \leq N$, $e_\nu = (0, \phi_\nu)$, with ϕ_ν belonging either to M or $M^\perp$. In these coordinates, it is not troublesome to verify the estimate (2.35) uniformly for $v \in K_p$. Hence, according to Theorem 2.5 and the bound (2.36), (2.54) indeed defines an $L^{q_j}_{t,x}$–multiplier whose norm is bounded for $v \in K_p$.

Therefore, reprising this reasoning, and agglutinating all parcels, the $L^{q_j}_{t,x}$–norm of the left-side of (2.53) can be estimated by

$$\leq C_j \left(\sum_{\nu=0}^{\ell} |\eta^{(\nu)}_{\delta,\gamma}(v)| \right) \|\widetilde{\mathfrak{g}}^{(j)}_{m,n}(\,\cdot\,, \cdot\,, v)\|_{L^{q_j}(\mathbb{R}_t \times \mathbb{R}^N_x)} \text{ almost surely,} \tag{2.56}$$

where $C_j > 0$ is uniform for $v \in K_p$, and m and $n \in \mathbb{N}$.

Step #1: (Conclusion). Once (2.56) is exactly the same estimate as (2.52), it is valid for all $v \in \operatorname{supp} \eta_{\delta,\gamma} = K_h \cup K_p$. Consequently, integrating in v, invoking the trivial estimate (2.31), and taking the expected value, we deduce (2.49). Lastly, (2.50) is a direct byproduct of (2.48).

Step #2: Assume now the fractional case $0 < \mathfrak{z} < 1$. Then, Equation (2.53) reads

$$\begin{aligned} (I)_{m,n} = \mathfrak{F}^{-1}_{t,x} \bigg[\pm (-1)^{\mathfrak{l}} \int_{\mathbb{R}} (-\Delta_v)^{\mathfrak{z}/2} \bigg\{ \eta^{(\mathfrak{l})}_{\delta,\gamma}(v) \psi\left(\frac{\widetilde{\mathcal{L}}(i\tau, i\kappa, v)}{\delta} \right) \psi\left(\frac{(\widetilde{R\mathcal{L}})(i\tau, i\kappa, v)}{\delta} \right) \\ \psi\left(\frac{\sqrt{\tau^2 + |\kappa|^2}}{\gamma} \right) \frac{\sqrt{\tau^2 + |\kappa|^2}}{\mathcal{L}(i\tau, i\kappa, v)} \bigg\} (\mathfrak{F}_{t,x} \widetilde{\mathfrak{g}}^{(j)}_{m,n})\, dv \bigg] + [\text{ similar terms }]. \end{aligned} \tag{2.57}$$

Once more, let us exclusively focus on the leading term.

First of all, recall that the operator $(-\Delta_v)^{\mathfrak{z}/2}$ can be defined for all sufficiently smooth functions $\phi : \mathbb{R} \to \mathbb{R}$ as

$$\Big((-\Delta_v)^{\mathfrak{z}/2} \phi \Big)(v) = c_{\mathfrak{z}} \int_{\mathbb{R}} \frac{\phi(v) - \phi(w)}{|v - w|^{1+\mathfrak{z}}}\, dw; \tag{2.58}$$

(see, *e.g.*, P.R. Stinga [105]). While the numerical constant $c_{\mathfrak{z}}$ is given by

$$c_{\mathfrak{z}} = \frac{2^{\mathfrak{z}}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1+\mathfrak{z}}{2}\right)}{|\Gamma\left(-\frac{\mathfrak{z}}{2}\right)|},$$

its precise value will not be needed. In contrast to the first step, we observe that it is not possible to detach $\eta_{\delta,\gamma}(v)$ from the other factors, and so the inequality (2.31) is no longer of applicability here. Moreover, due to the nonlocality of the fractional Laplacian, v now varies through the entire real line rather than on the compact $\operatorname{supp}\eta_{\delta,\gamma} = K_h \cup K_p$. As a result, we have no other choice than to show that $(-\Delta_v)^{\mathfrak{z}/2}m$, with $m(\tau,\kappa,v)$ given by

$$m(\tau,\kappa,v) = \eta^{(\mathfrak{l})}_{\delta,\gamma}(v)\psi\left(\frac{\widetilde{\mathcal{L}}(i\tau,i\kappa,v)}{\delta}\right)\psi\left(\frac{\widetilde{(R\mathcal{L})}(i\tau,i\kappa,v)}{\delta}\right)\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\frac{\sqrt{\tau^2+|\kappa|^2}}{\mathcal{L}(i\tau,i\kappa,v)}, \tag{2.59}$$

is an $L^{q_j}_{t,x}$–multiplier with a well-behaved norm as $|v|\to\infty$.

Proposition 2.6. *For any $v\in\mathbb{R}_v$ and $j\in\mathscr{J}$, $(\tau,\kappa)\mapsto\big((-\Delta_v)^{\mathfrak{z}/2}m\big)(\tau,\kappa,v)$ is an $L^{q_j}(\mathbb{R}_t\times\mathbb{R}^N)$–multiplier. Moreover, if T_v is the associated linear transformation*

$$(T_v\phi)(t,x) = \mathfrak{F}^{-1}_{t,x}\Big\{\big((-\Delta_v)^{\mathfrak{z}/2}m\big)(\,\cdot\,,\,\cdot\,,v)\mathfrak{F}_{t,x}\phi(\,\cdot\,,\,\cdot\,)\Big\}(t,x),$$

then, there exists a constant $C_j>0$ such that

$$\|T_v\|_{\mathscr{L}(L^{q_j}_{t,x})} \le \frac{C_j}{(1+|v|)^{1+\mathfrak{z}}} \tag{2.60}$$

for all $v\in\mathbb{R}$.

Proof. *Step #1*: Let us first show that the $\big((-\Delta)^{\mathfrak{z}/2}m\big)(\tau,\kappa,v)$ is an $L^q(\mathbb{R}_t\times\mathbb{R}^N)$–Fourier multiplier for any $v\in\mathbb{R}$ and $1<q<\infty$.

Write

$$\begin{aligned}\Big((-\Delta_v)^{\mathfrak{z}/2}m\Big)(\tau,\kappa,v) &= c_{\mathfrak{z}}\int_{|w|<\varepsilon_0}\frac{m(\tau,\kappa,v)-m(\tau,\kappa,v+w)}{|w|^{1+\mathfrak{z}}}\,dw\\ &\quad + c_{\mathfrak{z}}\int_{|w|>\varepsilon_0}\frac{m(\tau,\kappa,v)-m(\tau,\kappa,v+w)}{|w|^{1+\mathfrak{z}}}\,dv,\end{aligned} \tag{2.61}$$

where ε_0 is the least number between $\operatorname{dist}(K_h,K_p)$ and, say, 1. Evidently, as $m(\tau,\kappa,v)$ has compact support in v and is an $L^q_{t,x}$–multiplier whose norm is globally bounded in $v\in\mathbb{R}_v$, the second integral above poses no difficulty: it represents an $L^q_{t,x}$–multiplier as well.

On the other hand, for any fixed $(\tau,\kappa)\in\mathbb{R}\times\big(\mathbb{R}^N\setminus M\big)$ and any multi-index $\mathfrak{a}$ in $\mathbb{R}\times\mathbb{R}^N$, the function $v\in\mathbb{R}\mapsto(D^{\mathfrak{a}}m)(\tau,\kappa,v)$ lies in the Hölder class $\mathscr{C}^\alpha_c(\mathbb{R}_v)$. Thus, once that $\alpha>\mathfrak{z}$, the singular integral in (2.61) not only converges absolutely for any $v\in\mathbb{R}$, but also may be freely differentiated in (τ,κ) under the integral sign.

Dividing between the cases $v\in K_h$ (in which m is homogeneous of degree 0), and $v\in K_p$ (for which one may justifiably employ Lemma 2.7), one can apply Theorem 2.5 to once more show that $\big((-\Delta_v)^{\mathfrak{z}/2}m\big)(\tau,\kappa,v)$ is an $L^q_{t,x}$–multiplier.

Step #2: Let us now confirm the decay estimate (2.60). Evidently, as a corollary of the previous argument, $\|T_v\|_{\mathscr{L}(L^q_{t,x})}$ is bounded as long as $v\in\mathbb{R}$ also remains bounded.

That said, let $L>0$ be any number for which $\operatorname{supp}\eta_{\delta,\gamma}\subset(-L,L)$, so that

$$\big((-\Delta_v)^{\mathfrak{z}/2}m\big)(\tau,\kappa,v) = -c_{\mathfrak{z}}\int_{-L}^{L}\frac{m(\tau,\kappa,w)}{|v-w|^{1+\mathfrak{z}}}\,dw$$

whenever $|v|>2L$. From this formula, it is easily seen the existence of some constant $C_q>0$ such that

$$\|T_v\|_{\mathscr{L}(L^q_{t,x})} \le \frac{C_q}{|v|^{1+\mathfrak{z}}} \text{ for } |v|>2L.$$

The amalgamation of the former two paragraphs' statements yields (2.60), proving hereby the proposition. ∎

Consequently, in virtue of the last lemma and the Minkowski's and Hölder's inequalities,

$$\begin{aligned}\Big\| \int_{\mathbb{R}} \mathfrak{F}_{t,x}^{-1}\Big\{\Big((-\Delta_v)^{\mathfrak{z}/2} m\Big)(\,\cdot\,,\,\cdot\,,v)(\mathfrak{F}_{t,x}\widetilde{\mathfrak{g}}_{m,n}^{(j)})(\,\cdot\,,\,\cdot\,,v)\,dv\Big\}\Big\|_{L^{q_j}(\mathbb{R}_t\times\mathbb{R}_x^N)} \\ \leq C_j \int_{\mathbb{R}} \frac{\|\widetilde{\mathfrak{g}}_{m,n}^{(j)}(\,\cdot\,,\,\cdot\,,v)\|_{L^{q_j}(\mathbb{R}_t\times\mathbb{R}_x^N)}}{(1+|v|)^{1+\mathfrak{z}}}\,dv \\ \leq C_j \|\widetilde{\mathfrak{g}}_{m,n}^{(j)}\|_{L^{q_j}(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v)} \text{ almost surely,}\end{aligned}$$

where C_j is independent of m and n. Returning to (2.57) and repeating this investigation to each and every element defining $(I)_{m,n}^{(j)}$, we once more conclude the estimate (2.49), and hence (2.50) per (2.48). The lemma is proven. □

The analysis of $(II)_{m,n}^{(j)}$.

The investigation of $(II)_{m,n}^{(j)}$ is virtually identical to the one of $(I)_{m,n}^{(j)}$; thus, the details will be omitted. In spite of this, let us only indicate that, once $\Pi_j(v) = 0$ whenever $v \in K_h$, one needs to investigate the alternative $v \in K_p$.

Lemma 2.8. *There exists a constant C, independent of m and $n \in \mathbb{N}$, such that, almost surely and for all $j \in \mathscr{J}$,*

$$\|(II)_{m,n}^{(j)}\|_{L^{q_j}(\mathbb{R}_t\times\mathbb{R}_x^N)} \leq C\|\mathfrak{h}_{m,n}^{(j)}\|_{L^{q_j}(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v)}.$$

Consequently,

$$\lim_{m,n\to\infty} \mathbb{E}\Big\|\varphi \sum_{j\in\mathscr{J}} (II)_{m,n}^{(j)}\Big\|_{L^s(\mathbb{R}_t\times\mathbb{R}_x^N)}^r = 0. \tag{2.62}$$

The analysis of $(III)_{m,n}$.

Lemma 2.9. *There exists a constant C, independent of m and $n \in \mathbb{N}$, such that*

$$\mathbb{E}\|(III)\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x^N)}^2 \leq C\mathbb{E}\int_0^\infty \|\Psi_{m,n}(t)\|_{HS(\mathscr{H},L^2(\mathbb{R}_x^N\times\mathbb{R}_v))}^2\,dt. \tag{2.63}$$

Consequently,

$$\lim_{m,n\to\infty} \mathbb{E}\|\varphi\,(III)\|_{L^s(\mathbb{R}_t\times\mathbb{R}_x^N)}^r = 0. \tag{2.64}$$

Before presenting the proof of this lemma, let us explicate and explore the meaning of the expression $\Psi_{m,n}\frac{dW}{dt}$ and its Fourier transform. As (2.12) suggests, $\Psi_{m,n}\frac{dW}{dt}$ is defined as the linear functional

$$\phi \in \mathscr{S}(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v) \mapsto \int_0^\infty \int_{\mathbb{R}_x^N}\int_{\mathbb{R}_v} \phi(t,x,v)\Psi_{m,n}(t,x,v)\,dvdxdW(t). \tag{2.65}$$

Proposition 2.7. $\Psi_{m,n}\frac{dW}{dt}$*, given by formula (2.65), is almost surely a tempered distribution in $\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_x^N$; more precisely, letting $\int_0^t \Psi\,dW = 0$ for $t<0$, then it holds the "intuitive" relation*

$$\Psi_{m,n}\frac{dW}{dt} = \frac{\partial}{\partial t}\Big(\int_0^t \Psi_{m,n}\,dW\Big) \text{ almost surely in } \mathscr{S}'(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v). \tag{2.66}$$

Furthermore, its spatio-temporal Fourier transform $\mathfrak{F}_{t,x}(\Psi_{m,n}\frac{dW}{dt})$ is, almost surely, formally given by

$$\mathfrak{F}_{t,x}\Big(\Psi_{m,n}\frac{dW}{dt}\Big)(\tau,\kappa,v) = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-it\tau}(\mathfrak{F}_x\Psi_{m,n})(t,\kappa,v)\,dW(t); \tag{2.67}$$

that is, for any $\phi \in \mathscr{S}(\mathbb{R}_\tau \times \mathbb{R}^N_\kappa \times \mathbb{R}_v)$ and almost surely,

$$\left\langle \mathfrak{F}_{t,x}\left(\Psi_{m,n}\frac{dW}{dt}\right), \phi\right\rangle_{\mathscr{S}',\mathscr{S}} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}_\tau}\int_0^\infty\int_{\mathbb{R}^N_\kappa}\int_{\mathbb{R}_v} e^{-it\tau}(\mathfrak{F}_x\Psi_{m,n})(\tau,\kappa,v)\phi(\tau,\kappa,v)\,dvd\kappa dW(t)d\tau. \tag{2.68}$$

Proof. Pick $\phi \in \mathscr{S}(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v)$. For the Burkholder inequality asserts that

$$\mathbb{E}\sup_{t>0}\left\|\int_0^t \Psi_{m,n}(t')\,dW(t')\right\|^2_{L^2(\mathbb{R}^N_x\times\mathbb{R}_v)} \leq C\mathbb{E}\int_0^\infty \left\|\Psi_{m,n}(t')\right\|^2_{HS(\mathscr{H},L^2(\mathbb{R}^N_x\times\mathbb{R}_v))}\,dt' < \infty, \tag{2.69}$$

one may combine the stochastic Fubini theorem (see, *e.g.*, G. Da Prato–J. Zabczyk [29]) with the usual formula $\phi(t,x,v) = -\int_t^\infty \frac{\partial\phi}{\partial t'}(t',x,v)\,dt'$ to translate the right-hand side of (2.65) into

$$-\int_0^\infty\left(\int_{\mathbb{R}^N_x}\int_{\mathbb{R}_v}\frac{\partial\phi}{\partial t}(t',x,v)\left[\int_0^{t'}\Psi_{m,n}(t,x,v)\,dW(t)\right]dvdx\right)dt'.$$

This establishes (2.66). Thus, thanks to (2.69) again, it is not difficult to argue from this that indeed $\Psi_{m,n}\frac{dW}{dt}$ defines almost surely a tempered distribution.

Let us now establish (2.67). Via the stochastic Fubini theorem once more, it may be shown that

$$\begin{aligned}\left\langle \Psi_{m,n}\frac{dW}{dt}, \mathfrak{F}_{t,x}\phi\right\rangle &= \int_0^\infty\int_{\mathbb{R}^N_x}\int_{\mathbb{R}_v}(\mathfrak{F}_{t,x}\phi)\Psi_{m,n}\,dvdxdW(t)\\ &= \int_0^\infty\int_{\mathbb{R}^N_x}\int_{\mathbb{R}_v}(\mathfrak{F}_t\phi)(\mathfrak{F}_x\Psi_{m,n})\,dvd\kappa dW(t)\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}_\tau}\left(\int_{\mathbb{R}^N_x}\int_{\mathbb{R}_v}\left[\int_0^\infty e^{-it\tau}\big(\mathfrak{F}_x\Psi_{m,n}\big)\,dW(t)\right]\phi\,dvdx\right)d\tau,\end{aligned}$$

hence (2.68). □

Proof of Lemma 2.9. On the strength of the previous proposition, we deduce that

$$(III)_{m,n} = \mathfrak{F}^{-1}_{t,x}\left\{\int_{\mathbb{R}} m(\tau,\kappa,v)\left(\int_0^\infty e^{-it\tau}(\mathfrak{F}_x\Psi_{m,n})(t,\kappa,v)\,dW(t)\right)dv\right\}, \tag{2.70}$$

where $m : \mathbb{R}_\tau \times \mathbb{R}^N_\kappa \times \mathbb{R}_v \to \mathbb{C}$ is given by

$$m(\tau,\kappa,v) = \psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\left(1 \pm (-1)^{\mathfrak{l}}\frac{\partial^{\mathfrak{l}}}{\partial v^{\mathfrak{l}}}(-\Delta_v)^{\mathfrak{z}/2}\right)\left[\eta_{\delta,\gamma}(v)\psi\left(\frac{\widetilde{\mathcal{L}}(i\tau,i\kappa,v)}{\delta}\right)\psi\left(\frac{\widetilde{R\mathcal{L}}(i\tau,i\kappa,v)}{\delta}\right)\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\frac{\big(|\kappa|^2+1\big)^{1/4}}{\mathcal{L}(i\tau,i\kappa,v)}\right] \tag{2.71}$$

(a formal fashion to prove (2.70) can be found in B. Gess–M. Hofmanová [51]). Notice that, mingling the bound (2.55) of Lemma 2.7 and reasoning of Proposition 2.6, it is not difficult to corroborate the existence of a constant $C = C_{\delta,\gamma} > 0$ such that

$$|m(\tau,\kappa,v)| \leq C\frac{1_{(\frac{\gamma}{2},\infty)}(\sqrt{\tau^2+|\kappa|^2})}{(1+|v|)^{1+\mathfrak{z}}}\frac{\sqrt{1+|\kappa|}}{\sqrt{\tau^2+|\kappa|^2}} \tag{2.72}$$

for all $(\tau,\kappa,v) \in \big(\mathbb{R}\times\mathbb{R}^N\setminus\{0\}\times M\big)\times\mathbb{R}$. Hence, a joint application of the Plancherel formula, the Cauchy–Schwarz inequality, (2.72), the Fubini theorem, and the Itô isometry to (2.70) yields

$$\begin{aligned}
&\mathbb{E}\|(III)_{m,n}\|^2_{L^2(\mathbb{R}_t\times\mathbb{R}^N_x)}\\
&\quad= \mathbb{E}\int_{\mathbb{R}_\tau}\int_{\mathbb{R}^N_\kappa}\left|\int_{\mathbb{R}} m(\tau,\kappa,v)\left(\int_0^\infty e^{-it\tau}(\mathfrak{F}_x\Psi_{m,n})(t,\kappa,v)\,dW(t)\right)dv\right|^2 d\kappa d\tau\\
&\quad\le \mathbb{E}\int_{\mathbb{R}_\tau}\int_{\mathbb{R}^N_\kappa}\left(\int_{\mathbb{R}}|m(\tau,\kappa,w)|^2\,dw\right)\\
&\qquad\qquad\left(\int_{\mathbb{R}}\left|\int_0^\infty e^{-it\tau}(\mathfrak{F}_x\Psi_{m,n})(t,\kappa,v)\,dW(t)\right|^2 dv\right)d\kappa d\tau\\
&\quad\le C\mathbb{E}\int_0^\infty\int_{\sqrt{\tau^2+|\kappa|^2}\ge\frac{\gamma}{2}}\int_{\mathbb{R}_v}\frac{1+|\kappa|}{\tau^2+|\kappa|^2}\|(\mathfrak{F}_x\Psi_{m,n})(t,\kappa,v)\|^2_{HS(\mathscr{H};\mathbb{R})}\,dvd\kappa d\tau dt,
\end{aligned}\tag{2.73}$$

where we introduced the notation

$$\|(\mathfrak{F}_x\Psi_{m,n})(t,\kappa,v)\|^2_{HS(\mathscr{H};\mathbb{R})} = \text{trace of }\Big\{(\mathfrak{F}_x\Psi_{m,n})(t,\kappa,v)^\star(\mathfrak{F}_x\Psi_{m,n})(t,\kappa,v)\Big\},$$

which, by assumption, lies in $L^1_\omega L^1_{t,\kappa,v}$. Integrating (2.73) firstly in the τ-variable, we obtain that

$$\begin{aligned}
&\mathbb{E}\|(III)_{m,n}\|^2_{L^2(\mathbb{R}_t\times\mathbb{R}^N_x)}\\
&\qquad\le C\mathbb{E}\int_0^\infty\int_{\mathbb{R}^N_\kappa}\int_{\mathbb{R}_v}(1+|\kappa|)\zeta_\gamma(\kappa)\|(\mathfrak{F}_x\Psi_{m,n})(t,\kappa,v)\|^2_{HS(\mathscr{H};\mathbb{R})}\,dvd\kappa d\tau dt,
\end{aligned}$$

with the function $\zeta_\gamma:\mathbb{R}^N_\kappa\to\mathbb{R}$ being defined as

$$\zeta_\gamma(\kappa)\stackrel{\text{def}}{=}\int_{\sqrt{\tau^2+|\kappa|^2}\ge\frac{\gamma}{2}}\frac{1}{\tau^2+|\kappa|^2}\,d\tau=\begin{cases}\dfrac{4}{\gamma} & \text{for } |\kappa|=0,\\[2ex] \dfrac{\pi-2\arctan\sqrt{\frac{\gamma^2}{4|\kappa|^2}-1}}{|\kappa|} & \text{for } 0<|\kappa|<\dfrac{\gamma}{2}, \text{ and}\\[2ex] \dfrac{\pi}{|\kappa|} & \text{for } |\kappa|\ge\dfrac{\gamma}{2}.\end{cases}$$

Due to the boundedness of $(1+|\kappa|)\zeta_\gamma(\kappa)$, (2.63) is thus verified. Finally, (2.64) follows from (2.21). □

Remark 2.11. Reviewing Equations (2.70)–(2.73), it is clear that we could have included a term of the form

$$(-\Delta_{t,x}+1)^{\sigma/2}(-\Delta_v+1)^{\ell/2}\left\{\Theta_n\frac{dW}{dt}\right\}$$

in (2.10) where $0\le\sigma<1/2$ (naturally, we are tacitly imagining that $(\Theta_n)_{n\in\mathbb{N}}$ is a predictable and relatively compact sequence in $L^2(\Omega\times[0,\infty)_t;\,HS(\mathscr{H};\,L^2(\mathbb{R}^N_x\times\mathbb{R}_v))))$. This shows that, in the stochastic case, one can still expect some regularization in the time variable t of order $1/2-$, which is undoubtely a very fascinating information.

However, we have decided not to add such terms, as they do not seem to be well-behaved under localizations. Despite that, one should keep this fact in mind when investigating, for instance, the Sobolev regularity of averages of solutions to this type of equation.

The conclusion of the analysis of $\mathfrak{v}^{(4)}_{m,n}$.

Recalling the decomposition (2.47), the limits (2.50), (2.62), and (2.64) affirm the next proposition.

Lemma 2.10. *It holds the limit*

$$\lim_{m,n\to\infty} \mathbb{E}\|\varphi \mathfrak{v}^{(4)}_{m,n}\|^r_{L^s(\mathbb{R}_t\times\mathbb{R}^N_x)} = 0. \tag{2.74}$$

2.2.7 The conclusion of the proof of Theorem 2.1.

Returning to (2.29), the merger of the estimates (2.30), (2.32), (2.38), (2.45) and (2.74) results in

$$\begin{aligned}\limsup_{m,n\to\infty} \mathbb{E}\left\|\varphi\int_{\mathbb{R}_v}(f_m-f_n)\eta\,dv\right\|^r_{L^s(\mathbb{R}_t\times\mathbb{R}^N_x)} &\le C\|\eta_{\delta,\gamma}-\eta\|^r_{L^{p'}(\mathbb{R})} + C\gamma^{\mathfrak{q}} \\ &+ C\|\eta_{\delta,\gamma}\|^r_{L^\infty(\mathbb{R})}\left[\left(\sup_{\tau^2+|\kappa|^2=1}\operatorname{meas}\Big\{v\in\operatorname{supp}\eta_{\delta,\gamma};\,|\mathcal{L}(i\tau,i\kappa,v)|\le\delta\Big\}\right)^{r\mathfrak{p}}\right. \\ &\left.+\left(\sup_{\substack{(\tau,\kappa)\in\mathbb{R}\times M^\perp\\ \tau^2+|\kappa|^2=1}}\operatorname{meas}\Big\{v\in\operatorname{supp}\eta_{\delta,\gamma};\,|\mathcal{L}(i\tau,i\kappa,v)|\le\delta\Big\}\right)^{r\mathfrak{r}}\right],\end{aligned}$$

where the positive constants C, $\mathfrak{p}$, $\mathfrak{q}$, and $\mathfrak{r}$ do not depend on the integers m and n, nor on $0<\delta$ and $\gamma<1$. Letting first $\delta\to 0_+$, Lemmas 2.1, 2.4 and 2.5 imply that

$$\limsup_{m,n\to\infty} \mathbb{E}\left\|\varphi\int_{\mathbb{R}_v}(f_m-f_n)\eta\,dv\right\|^r_{L^s(\mathbb{R}_t\times\mathbb{R}^N_x)} \le C\|\mathfrak{n}_\gamma-\eta\|^r_{L^{p'}(\mathbb{R})} + C\gamma^{\mathfrak{q}}.$$

Finally, passing $\gamma\to 0_+$ and applying Lemma 2.1 one last time, we conclude

$$\lim_{m,n\to\infty} \mathbb{E}\left\|\varphi\int_{\mathbb{R}_v}(f_m-f_n)\eta\,dv\right\|^r_{L^s(\mathbb{R}_t\times\mathbb{R}^N_x)} = 0.$$

Therefore, the sequence of the averages $(\varphi\int_{\mathbb{R}_v} f_n\eta\,dv)$ is Cauchy on the Banach space $L^r(\Omega;L^s(\mathbb{R}_t\times\mathbb{R}^N_x))$. Theorem 2.1 is hereby demonstrated. □

2.3 Proof of Theorem 2.2

We will reduce Theorem 2.2 to a corollary of Theorem 2.1.

Let us commence by observing that, should $(f_n)_{n\in\mathbb{N}}$ be relatively compact in $L^r(\Omega;W^{-z_0,p}_{\mathrm{loc}}(Q\times\mathbb{R}_v))$ for some $z_0>0$, then the same assertion is valid for all $z_0>0$, as an interpolation argument readily shows. As a result, given any $\varphi\in\mathscr{C}^\infty_c(Q)$ and $\zeta\in\mathscr{C}^\infty_c(\mathbb{R}_v)$,

$$(-\Delta_{t,x}+1)^{-1/4}(-\Delta_v+1)^{-\ell/2}(\varphi\zeta f_n) \text{ is relatively compact in } L^r(\Omega;L^p(\mathbb{R}_t\times\mathbb{R}^N_x\times\mathbb{R}_v)). \tag{2.75}$$

With this is mind, let $\theta\in\mathscr{C}^\infty_c(Q)$ be arbitrary, and consider also some $\vartheta\in\mathscr{C}^\infty_c(\mathbb{R}_v)$, such that

$$\begin{cases}0\le\vartheta\le 1 \text{ everywhere, and}\\ \vartheta\equiv 1 \text{ in } \operatorname{supp}\eta+(-1,1).\end{cases}$$

Put $\widetilde{f}_n(t,x,v)=\theta(t,x)\vartheta(v)^2 f_n(t,x,v)$. Hence, conserving the notation $\mathcal{L}(i\tau,i\kappa,v)=i(\tau+\mathbf{a}(v)\cdot$

$\kappa) + \kappa\cdot\mathbf{b}(v)\kappa$, each $\widetilde{f}_n$ obeys the equation

$$\begin{aligned}\frac{\partial\widetilde{f}_n}{\partial t} + \mathbf{a}(v)\cdot\nabla_x\widetilde{f}_n - \mathbf{b}(v):D_x^2\widetilde{f}_n = f_n\mathcal{L}\Big(\frac{\partial}{\partial t},\nabla_x,v\Big)(\theta\vartheta^2) + 2\mathrm{div}_x(f_n\mathbf{b})\cdot\nabla_x(\theta\vartheta^2)\\ + \sum_{j\in\mathscr{J}}\theta\vartheta^2(-\Delta_{t,x}+1)^{1/2}(-\Delta_v+1)^{\ell/2}g_{j,n}\\ + \sum_{j\in\mathscr{J}}\theta\vartheta^2(\Pi_j(v)\Delta_M)(-\Delta_v+1)^{\ell/2}h_{j,n}\\ + \theta\vartheta^2(-\Delta_{t,x}+1)^{1/4}(-\Delta_v+1)^{\ell/2}\Phi_n\frac{dW}{dt}\end{aligned} \tag{2.76}$$

almost surely in the sense of the distributions in $\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v$.

Lemma 2.11. *Equation* (2.76) *may be written as*

$$\begin{aligned}\frac{\partial\widetilde{f}_n}{\partial t} + \mathbf{a}(v)\cdot\nabla_x\widetilde{f}_n - \mathbf{b}(v):D_x^2\widetilde{f}_n = \sum_{j\in\widetilde{\mathscr{J}}}(-\Delta_{t,x}+1)^{1/2}(-\Delta_v+1)^{\ell/2}\widetilde{g}_{j,n}\\ - \sum_{j\in\widetilde{\mathscr{J}}}(\widetilde{\Pi}_j(v)\Delta_M)(-\Delta_v+1)^{\ell/2}\widetilde{h}_{j,n} + (-\Delta_x+1)^{1/4}(-\Delta_v+1)^{\ell/2}\widetilde{\Phi}_n\frac{dW}{dt},\end{aligned} \tag{2.77}$$

where $\widetilde{\mathscr{J}}$ *is a finite index set such that, for any* $j\in\widetilde{\mathscr{J}}$,

1. $s\le q_j<\infty$,
2. $(\widetilde{g}_{j,n})_{n\in\mathbb{N}}$ *and* $(\widetilde{h}_{j,n})_{n\in\mathbb{N}}$ *are relatively compact sequences in* $L^r(\Omega;L^{q_j}(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v))$,
3. $\widetilde{\Pi}_j\in\mathscr{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R})$ *is such that* $\mathrm{supp}\,\widetilde{\Pi}_j\subset\mathrm{supp}\,\mathbf{b}$, *and*
4. $(\widetilde{\Phi}_n)_{n\in\mathbb{N}}$ *is a predictable and relatively compact sequence in* $L^2(\Omega\times[0,\infty)_t;\,HS(\mathscr{H};\,L^2(\mathbb{R}_x^N\times\mathbb{R}_v)))$.

In order to rewrite each term in (2.76) to our liking, let us state and prove the next proposition.

Proposition 2.8. *Let d be a positive integer, $\mathscr{U}\subset\mathbb{R}^d$ be a nonempty open set, $1<p<\infty$ be an exponent, $\ell\ge 0$, and $(k,\alpha)\in\mathbb{Z}\times[0,1]$ satisfy the relation* (2.9).

Then, for any Λ belonging to the Sobolev space $W^{-\ell,p}(\mathscr{U})$ and $\phi\in\mathscr{C}_c^{k,\alpha}(\mathscr{U})$, the distribution $\phi\Lambda$ lies in $W^{-\ell,p}(\mathbb{R}^d)$. Moreover, there exists a constant $C=C(d)$ such that

$$\|\phi\Lambda\|_{W^{-\ell,p}(\mathbb{R}^d)}\le C\|\phi\|_{\mathscr{C}^{k,\alpha}(\mathscr{U})}\|\Lambda\|_{W^{-\ell,p}(\mathscr{U})}.$$

Proof. On account of the definition of multiplication of distributions by regular functions and the duality relation $W^{-\ell,p}(\mathbb{R}^d)=W^{\ell,p'}(\mathbb{R}^d)^\star$, it suffices to show that there exists a constant $C=C(d)$ such that

$$\|\phi u\|_{W^{\ell,p'}(\mathbb{R}^d)}\le C\,\|\phi\|_{\mathscr{C}^{k,\alpha}(\mathbb{R}^d)}\,\|u\|_{W^{\ell,p'}(\mathscr{U})},$$

for every $u\in W^{\ell,p'}(\mathbb{R}^d)$.

If ℓ is an integer, then this inequality is derived directly from the Leibniz's rule. In the case that ℓ is not an integer, recall, since $\phi u\in W_0^{\ell,p'}(\mathscr{U})$, its $W^{\ell,p'}$–norm is equivalent to

$$\|\phi u\|_{L^{p'}(\mathbb{R}^d)} + \sum_{j=1}^d\left\|(-\Delta_y)^{\frac{\ell-\lfloor\ell\rfloor}{2}}\frac{\partial^{\lfloor\ell\rfloor}}{\partial y_j^{\lfloor\ell\rfloor}}(\phi u)\right\|_{L^{p'}(\mathbb{R}^d)}.$$

Therefore, in virtue of the L. Grafakos–S. Oh's Kato–Ponce inequality [54]

$$\begin{aligned}\|(-\Delta)^{\mathfrak{s}/2}[FG]\|_{L^{p'}(\mathbb{R}^d)} &\leq C\|(-\Delta)^{\mathfrak{s}/2}F\|_{L^{p'}(\mathbb{R}^d)}\|G\|_{L^\infty(\mathbb{R}^d)} \\ &\quad + C\|(-\Delta)^{\mathfrak{s}/2}G\|_{L^\infty(\mathbb{R}^d)}\|F\|_{L^{p'}(\mathbb{R}^d)},\end{aligned} \tag{2.78}$$

which is valid for any $0 < \mathfrak{s} < 1$, and F and $G \in \mathscr{S}(\mathbb{R}^d)$, the desired conclusion now follows. □

Proof of Lemma 2.11. Let us write (2.76) as

$$\left(\frac{\partial}{\partial t} + \mathbf{a}(v)\cdot\nabla_x - \mathbf{b}(v) : D_x^2\right)\widetilde{f}_n = (I)_n + \sum_{j\in\mathscr{J}}(II)_n^{(j)} + \sum_{j\in\mathscr{J}}(III)_n^{(j)} + (IV)_n, \tag{2.79}$$

where we are denoting

$$\begin{cases}(I)_n = f_n\mathcal{L}\Big(\dfrac{\partial}{\partial t},\nabla_x, v\Big)(\theta\vartheta^2) + 2\mathrm{div}_x(f_n\mathbf{b})\cdot\nabla_x(\theta\vartheta^2) & \\ (II)_n^{(j)} = \theta\vartheta^2(-\Delta_{t,x}+1)^{1/2}(-\Delta_v+1)^{\ell/2}g_{j,n} & [j\in\mathscr{J}\,] \\ (III)_n^{(j)} = \theta\vartheta^2(\Pi_j(v)\Delta_M)(-\Delta_v+1)^{\ell/2}h_{j,n} & [j\in\mathscr{J}\,] \\ (IV)_n = (-\Delta_x+1)^{1/4}(-\Delta_v+1)^{\ell/2}\Phi_n\dfrac{dW}{dt}. & \end{cases} \tag{2.80}$$

Step #1: First of all, let us inspect $(I)_n$.

It is clear from (2.75) that, applying Proposition 2.8 to the v–variable,

$$f_n\mathcal{L}\Big(\frac{\partial}{\partial t},\nabla_x, v\Big)(\theta\vartheta^2) = (-\Delta_{t,x}+1)^{1/2}(-\Delta_v+1)^{\ell/2}Y_n(t,x,v), \tag{2.81}$$

where $(Y_n)_{n\in\mathbb{N}}$ is relatively compact in $L^r_\omega L^p_{t,x,v}$. On the other hand,

$$2\mathrm{div}_x(f_n\mathbf{b})\cdot\nabla_x(\theta\vartheta^2) = -2f_n\vartheta\mathbf{b} : D^2(\vartheta\theta) + 2\mathrm{div}_x\big(f_n\mathbf{b}\nabla_x(\theta\vartheta^2)\big), \tag{2.82}$$

and, repeating the very same argument of (2.81),

$$-2f_n\vartheta\mathbf{b} : D^2(\vartheta\theta) = (-\Delta_{t,x}+1)^{1/2}(-\Delta_v+1)^{\ell/2}Y_n' \tag{2.83}$$

with $(Y_n')_{n\in\mathbb{N}}$ being also relatively compact in $L^r_\omega L^p_{t,x,v}$.

To facilitate the investigation of the complementary parcel, notice that we can assume without loss of generality that

$$M = \{x = (x_1,\dots,x_N)\in\mathbb{R}^N; x_\nu = 0 \text{ if } N' < \nu \leq N\}$$

for some $0 \leq N' \leq N$. Thence, the second part in (2.82) has the form

$$2\mathrm{div}_x\big(f_n\mathbf{b}\nabla_x(\theta\vartheta^2)\big) = (\vartheta\mathbf{b}) : P_M\nabla_x\otimes(-\Delta_{t,x}+1)^{1/4}(-\Delta_v+1)^{\ell/2}V_n,$$

where $(V_n)_{n\in\mathbb{N}} = \big([V_n^{(1)},\dots,V_n^{(N)}]\big)_{n\in\mathbb{N}}$ is relatively compact now in $L^r_\omega(L^p_{t,x,v})^N$. According to Theorem 2.5, for any $1 \leq \nu \leq N'$,

$$\left(\frac{\partial}{\partial x_\nu}\right)(-\Delta_{t,x}+1)^{1/4}((-\Delta_{t,x}+1)^{1/2} - \Delta_M)^{-1}$$

defines a bounded operator in $L^p_{t,x}$; for this reason, writing $\mathbf{b}$ matricially as $\mathbf{b} = (\mathbf{b}_{\mu,\nu})_{1\leq\mu,\nu\leq N}$,

$$2\mathrm{div}_x\big(\mathbf{b}f_n\nabla_x(\theta\vartheta^2)\big) = \sum_{\mu,\nu=1}^{N'}(\vartheta\mathbf{b}_{\mu,\nu})((-\Delta_{t,x}+1)^{1/2} - \Delta_M)(-\Delta_v+1)^{\ell/2}Y''_{\mu,\nu,n}, \tag{2.84}$$

with each $(Y''_n)_{\mu,\nu,n}$ being relatively compact in $L^p_{t,x,v}$.

Returning to the representation formulas (2.81)—(2.84), we conclude that

$$(I)_n = \sum_{\mu,\nu=1}^{N'}(\vartheta\mathbf{b}_{\mu,\nu})(-\Delta_M)(-\Delta_v+1)^{\ell/2}K'_{n,\mu,\nu} + (-\Delta_{t,x}+1)^{1/2}(-\Delta_v+1)^{\ell/2}K''_n,$$

where each and every $K'_{n,\mu,\nu}$ and K''_n is relatively compact in $L^r_\omega L^p_{t,x,v}$, as we wanted to show.

Step #2: In an analogous fashion, all the other terms $(II)^{(j)}_n$, $(III)^{(j)}_n$, and $(IV)^{(j)}_n$ may be handled. Let us only point out a difference appearing in the analysis of $(III)^{(j)}_n$, in which we write

$$\begin{aligned}(II)^{(j)}_n &= \theta\vartheta^2\Pi_j\Delta_M(-\Delta_v+1)^{\ell/2}h_{j,n}\\ &= (\vartheta^2\Pi_j)\big(\Delta_M(-\Delta_v+1)^{\ell/2}(\theta h_{j,n})\big)\\ &\quad - 2\Big[(P_M\nabla_x)(\theta\Pi_j\vartheta^2)\Big]\cdot(P_M\nabla_x)(-\Delta_v+1)^{\ell/2}h_{j,n}\\ &\quad - \Big[(\Delta_M)(\theta\Pi_j\vartheta^2)\Big](-\Delta_v+1)^{\ell/2}h_{j,n}(t,x,v).\end{aligned}$$

Evidently, the first term has the form $\vartheta^2\Pi_j(\Delta_M)(-\Delta_v+1)^{\ell/2}H_{j,n}$, where $H_{j,n}$ is relatively compact in $L^r_\omega L^{q_j}_{t,x,v}$. Moreover, according to Proposition 2.8, the last two parts are equal to $(-\Delta_{t,x}+1)^{1/2}(-\Delta_v+1)^{\ell/2}H'_{j,n}$, with $H'_{j,n}$ being again relatively compact in $L^r_\omega L^{q_j}_{t,x,v}$. The lemma is hereby proven. □

Since trivially $(\widetilde{f}_n)$ is bounded in $L^r_\omega L^p_{t,x,v}$ and $\theta\int_{\mathbb{R}}\eta f_n\,dv = \int_{\mathbb{R}}\eta\widetilde{f}_n\,dv$, the relative compactness of the averages now in $L^r_\omega L^s_{t,x}$ is guaranteed by Theorem 2.1. The final assertion in the statement of Theorem 2.2 is a consequence of Proposition 2.2. □

2.4 Proof of Theorems 2.3 and 2.4

We will only briefly depict the proof of Theorem 2.3, for the remaining details are indistinguishable from the ones found in Theorems 2.1 and 2.2—as a matter of fact, the verification of Theorem 2.4 is sensibly more unproblematic than that of Theorem 2.2.

First of all, we may assume that $M \neq \{0\}$, otherwise the conclusions can be derived from Theorems 2.1 and 2.2. Furthermore, we may suppose, passing to a subsequence if necessary, to assume again that all $(g_{j,n})_{n\in\mathbb{N}}$ and $(\Phi_n)_{n\in\mathbb{N}}$ are convergent in their respective spaces. Whereas we will still define $\mathfrak{n}_\gamma$ as in Lemma 2.1, we will now simply put $\eta_{\delta,\gamma} = (\varrho_\delta \star \mathfrak{n}_\gamma)$, where (ϱ_ε) is a mollifier in the real line. Define also the Fourier multiplier

$$(\widetilde{R\mathcal{E}})(i\kappa, v) = \frac{(P_M\kappa)\cdot\mathbf{b}(v)(P_M\kappa)}{(P_M\kappa\cdot P_M\kappa)} = \text{“the restricted normalized elliptic symbol”},$$

which can be shown to satisfy the truncation property (recall that $M^\perp \subset N(\mathbf{b}(v))$ for all $v\in\mathbb{R}$).

Thus, if $\mathfrak{f}_{m,n} = f_m - f_n$, and $0 < \delta$ and $\gamma < 1$ once more, introduce the Fourier decomposition

$$\begin{aligned}
\mathfrak{f}^{(1)}_{m,n} &= \mathfrak{F}^{-1}_{t,x}\left[\lambda\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)(\mathfrak{F}_{t,x}\mathfrak{f}_{m,n})\right],\\
\mathfrak{f}^{(2)}_{m,n} &= \mathfrak{F}^{-1}_{t,x}\left[\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\lambda\left(\frac{(\widetilde{R\mathcal{E}})(i\kappa,v)}{\delta}\right)(\mathfrak{F}_{t,x}\mathfrak{f}_{m,n})\right],\\
\mathfrak{f}^{(3)}_{m,n} &= \mathfrak{F}^{-1}_{t,x}\left[\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\psi\left(\frac{(\widetilde{R\mathcal{E}})(i\kappa,v)}{\delta}\right)\lambda\left(\frac{(\widetilde{R\mathcal{L}})(i\tau,i\kappa,v)}{\delta}\right)(\mathfrak{F}_{t,x}\mathfrak{f}_{m,n})\right], \text{ and}\\
\mathfrak{f}^{(4)}_{m,n} &= \mathfrak{F}^{-1}_{t,x}\left[\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\psi\left(\frac{(\widetilde{R\mathcal{E}})(i\kappa,v)}{\delta}\right)\psi\left(\frac{(\widetilde{R\mathcal{L}})(i\tau,i\kappa,v)}{\delta}\right)(\mathfrak{F}_{t,x}\mathfrak{f}_{m,n})\right],
\end{aligned}$$

where $\psi(z)$, $\lambda(z)$, $\widetilde{\mathcal{L}}(i\tau,i\kappa,v)$ and $(\widetilde{R\mathcal{L}})(i\tau,i\kappa,v)$ are also as before. Finally, write

$$\int_{\mathbb{R}} \mathfrak{f}_{m,n}\eta\,dv = \int_{\mathbb{R}} \mathfrak{f}_{m,n}(\eta-\eta_{\delta,\gamma})\,dv + \sum_{\nu=1}^{4}\int_{\mathbb{R}} \mathfrak{f}^{(\nu)}_{m,n}\eta_{\delta,\gamma}\,dv \stackrel{\text{def}}{=} \sum_{\nu=0}^{4}\mathfrak{v}^{(\nu)}_{m,n}.$$

Let $\varphi \in (L^1\cap L^\infty)(\mathbb{R}_t\times\mathbb{R}^N_x)$ be given. Reprising the manipulations performed in the proof of Theorem 2.1, for any $0\le\nu\le 3$, $\varphi\mathfrak{v}^{(\nu)}_{m,n}$ have all an uniformly "small" $L^r_\omega L^s_{t,x,v}$–norm as δ and γ tend to 0 in a regulated manner. On the other hand, once the estimate in (2.55) now reads as follows.

Lemma 2.12. *There exists a constant $C_{\delta,\gamma}$ such that*

$$\left|\frac{\sqrt{\tau^2+|\kappa|^2+|P_M\kappa|^2}}{\mathcal{L}(i\tau,i\kappa,v)}\right| \le C_{\delta,\gamma} \tag{2.85}$$

for all $(\tau,\kappa,v)\in \operatorname{supp}\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\delta}\right)\psi\left(\frac{(\widetilde{R\mathcal{E}})(i\tau,i\kappa,v)}{\delta}\right)\psi\left(\frac{(\widetilde{R\mathcal{L}})(i\tau,i\kappa,v)}{\delta}\right)$ and $v\in\operatorname{supp}\eta+(-1,1)$.

Proof. Firstly, let us demonstrate that $\mathcal{L}(i\tau,i\kappa,v)$ cannot vanish in the support of the expression in the left-hand side of (2.85). Put

$$\begin{cases}
X = \mathbb{R}\times\mathbb{R}^N\setminus(\{0\}\times M\cup\mathbb{R}\times M^\perp),\\
V = \operatorname{supp}\eta+(-1,1),\\
\theta_\delta(\tau,\kappa,v) = \psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\delta}\right)\psi\left(\frac{(\widetilde{R\mathcal{E}})(i\tau,i\kappa,v)}{\delta}\right)\psi\left(\frac{(\widetilde{R\mathcal{L}})(i\tau,i\kappa,v)}{\delta}\right), \text{ and}\\
B = \sup\limits_{\substack{v\in V \text{ and } \kappa\in M\\ \text{with } |\kappa|=1}} |\mathbf{a}(v)\cdot\kappa|.
\end{cases}$$

Then, if $(\tau,\kappa,v)\in(X\times V)\cap\operatorname{supp}\theta$ lies inside the cone $\left\{\sqrt{\tau^2+|P_{M^\perp}\kappa|^2}\ge 4\frac{B+1}{\delta}|P_M\kappa|\right\}$,

$$\begin{aligned}
|\tau+\mathbf{a}(v)\cdot\kappa| &\ge \delta\sqrt{\tau^2+|P_{M^\perp}\kappa|^2} - B_\delta|P_M\kappa|\\
&\ge \frac{\delta}{2}\sqrt{\tau^2+|P_{M^\perp}\kappa|^2} + (B_\delta+2)|P_M\kappa|\\
&\ge c_{1,\delta}\sqrt{\tau^2+|\kappa|^2}.
\end{aligned} \tag{2.86}$$

On the other hand, if $(\tau,\kappa,v)\in(X\times V)\cap\operatorname{supp}\theta$ but with $\sqrt{\tau^2+|P_{M^\perp}\kappa|^2} < 4\frac{B+1}{\delta}|P_M\kappa|$, then $|P_M\kappa| > c_{2,\delta}\sqrt{\tau^2+|\kappa|^2}$ for some $c_{2,\delta}>0$, and

$$(P_M\kappa)\cdot\mathbf{b}(v)(P_M\kappa) \ge \delta|P_M\kappa|^2 \ge c_{2,\delta}(\tau^2+|\kappa|^2). \tag{2.87}$$

Thus, mingling both (2.86) and (2.87) with the fact that *a fortiori* $\sqrt{\tau^2+|\kappa|^2}\ge\gamma$ for $(\tau,\kappa,v)\in$

supp θ, one concludes that

$$|\mathcal{L}(i\tau, i\kappa, v)| \geq c_{\delta,\gamma} \; \forall(\tau,\kappa,v) \in (X \times V) \cap \operatorname{supp}\theta. \tag{2.88}$$

This shows that the bound (2.85) is at least well-defined; it remains only to prove its validity. Reprising the previous reasoning, it is not difficult to verify that

$$|\mathcal{L}(i\tau, i\kappa, v)| \geq c\Big(\delta\sqrt{\tau^2 + |P_{M^\perp}\kappa|^2} - B|P_M\kappa| + \delta|P_M\kappa|^2\Big)$$

for all $(\tau,\kappa,v) \in (X \times \operatorname{supp}\eta) \cap \operatorname{supp}\theta$. Thus, there exists some $R_\delta > 0$ such that (2.85) holds true provided that $(\tau,\kappa,v) \in X \times \eta + (-1,1)$ and $\sqrt{\tau^2 + |\kappa|^2} \geq R_\delta$. On the other hand, if $\sqrt{\tau^2+|\kappa|^2} \leq R_\delta$, (2.85) is a direct consequence of (2.88). The proposition is hereby demonstrated. □

Hence, it is clear that

$$\lim_{m,n\to\infty} \mathbb{E}\|\varphi\mathfrak{v}^{(4)}_{m,n}\|^r_{L^s(\mathbb{R}_t\times\mathbb{R}^N_x)} = 0,$$

in spite of $\mathbf{b}(v)$ possibly not having total rank in M and the right-hand side of (2.14) being relatively more singular. Based on these observations, Theorem 2.1 follows. □

2.5 Last remarks

Remark 2.12 (On the spatially periodic case)*.* It is not difficult to see that our results may be translated from $\mathbb{R}^N_x$ to $\mathbb{T}^N_x$, the N-dimensional torus, if one employs the so-called De Leeuw's theorem; see, *e.g.*, E. M. Stein–G. Weiss [104], theorem 3.8 in chapter VII.

Remark 2.13 (On the convection vector function $\mathbf{a}(v)$)*.* Evidently, nothing prevents one from extending this manuscript's results for velocity variables v lying in some multidimensional Euclidean space $\mathbb{R}^K_v$.

Likewise, the left-hand side of Equations (2.10) and (2.14) could perfectly had been

$$\mathbf{a}_0(v)\frac{\partial f_n}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f_n - \mathbf{b}(v) : D^2_x f_n,$$

for some temporal convection function $\mathbf{a}_0 \in \mathscr{C}^{k,\alpha}_{\text{loc}}$. In this case, minor alterations in the statements and proofs must be made, as induced ripples from the symbol now being $(i\tau, i\kappa, v) \mapsto i(\mathbf{a}_0(v)\tau + \mathbf{a}(v)\cdot\kappa) + \kappa\cdot\mathbf{b}(v)\kappa$.

Remark 2.14 (On the nonnegativeness of $\mathbf{b}(v)$)*.* A moment of reflection reveals that the hypothesis that $\mathbf{b}(v) \geq 0$ was not strictly necessary, but one could have exchanged it with the following "sign" condition: *"for all $v \in \mathbb{R}$, either $\mathbf{b}(v) \geq 0$, or $\mathbf{b}(v) \leq 0$"*.

Remark 2.15 (On the exponents p, q_j and r)*.* Should the stochastic terms $(\Phi_n)_{n\in\mathbb{N}}$ be absent in our averaging lemmas—i.e., we are in a deterministic setting—, not only the range $1 \leq r < \infty$ is allowed, but one also can choose s to be least number between q_j and p. This represents a slight improvement on the exponent conditions of P.-L. Lions–B. Perthame–E. Tadmor [82], which assumed card. $\mathscr{J} = 1$, $p = q_j$ and $1 < p \leq 2$.

Remark 2.16 (On the exponents p, q and r, part II)*.* In a nutshell, the role of the function $\varphi \in L^1_{t,x} \cap L^\infty_{t,x}$ in Theorems 2.1 and 2.3 was to convert all the L^p–, L^{q_j}– and L^2–estimates into L^s–ones. Therefore, as Remark 2.9 indicates, φ is immaterial if such exponents are identical and one possesses an additional a priori estimate.

Corollary 2.1. *In the context of Theorems 2.1 and 2.3, assume in addition that*

1. *there exists some $1 \leq \varsigma < p$ such that $(f_n)_{n\in\mathbb{N}}$ is also bounded in $L^r(\Omega; L^\varsigma(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v))$, and*

2. *$p = q_j$ for all $j \in \mathscr{J}$.*

Then, if either $p = 2$, *or* $\Phi_n \equiv 0$, *the sequence of averages* $(\int_{\mathbb{R}_v} f_n \eta \, dv)_{n\in\mathbb{N}}$ *is relatively compact in* $L^r(\Omega; L^p(\mathbb{R}_t \times \mathbb{R}^N_x))$.

Although this assumption that $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^r_\omega L^\varsigma_{t,x,v}$ is commonly not found in the literature, in the applications to kinetic equations, the boundedness in $L^r_\omega L^p_{t,x,v}$ is equivalent to one in $L^r_\omega L^1_{t,x,v}$, wherefore it is not of extraordinary character.

Remark 2.17 (On the exponents p, q and r, part III). In the same spirit of the last two remarks, notice that essentially the low-frequency truncations $\lambda(\sqrt{\tau^2 + |\kappa|^2/\gamma})$ are introduced so that one could to replace the operators $(-\Delta_{t,x}+1)$ with its homogeneous counter-part $-\Delta_{t,x}$. Nevertheless, it is clear that, if $\mathbf{b}(v) \equiv 0$ and, in Equation (2.10), $(-\Delta_{t,x}+1)$ and $(-\Delta_x+1)$ are substituted respectively by $-\Delta_{t,x}$ and $-\Delta_x$, then these truncations may be discarded. One can thus deduce the next global averaging lemma, which recuperates a relative compactness result of B. Perthame–P.E. Souganidis [97].

Proposition 2.9 (The "global" hyperbolic averaging lemma). *Given exponents* $1 < p < \infty$, $1 \le r \le 2$ *and* $\ell \ge 0$, *let* $\mathbf{a} \in \mathscr{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^N)$, *where the real numbers* k *and* α *satisfy the relation* (2.9).

Assume that, for any integer $n \in \mathbb{N}$, *the equation*

$$\frac{\partial f_n}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f_n = (-\Delta_{t,x})^{1/2}(-\Delta_v+1)^{\ell/2} g_n + (-\Delta_x)^{1/4}(-\Delta_v+1)^{\ell/2}\Phi_n \frac{dW}{dt}$$

is almost surely obeyed in $\mathscr{D}'(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v)$, *where*

1. $(f_n)_{n\in\mathbb{N}}$ *is a bounded sequence in* $L^r\big(\Omega; L^p(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v)\big)$,
2. $(g_n)_{n\in\mathbb{N}}$ *is a convergent sequence in* $L^r(\Omega; L^p(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v))$, *and*
3. $(\Phi_n)_{n\in\mathbb{N}}$ *is a predictable and convergent sequence in* $L^2(\Omega\times[0,\infty)_t; HS(\mathscr{H}; L^2(\mathbb{R}^N_x \times \mathbb{R}_v)))$.

Finally, let $\eta \in L^{p'}(\mathbb{R})$ *have compact support, and presume that the* nondegeneracy condition

$$\operatorname{meas}\big\{v \in \operatorname{supp}\eta; \tau + \mathbf{a}(v)\cdot\kappa = 0\big\} = 0 \textit{ for all } (\tau,\kappa) \in \mathbb{R}\times\mathbb{R}^N \textit{ with } \tau^2 + |\kappa|^2 = 1$$

holds.

Then, if either $p = 2$, *or* $\Phi_n \equiv 0$, *the sequence of averages* $(\int_{\mathbb{R}_v} f_n\eta\, dv)_{n\in\mathbb{N}}$ *is relatively compact in* $L^r(\Omega; L^p(\mathbb{R}_t \times \mathbb{R}^N_x))$.

Remark 2.18 (Equations with discontinuous coefficients). In certain models, one considers $\mathbf{b}(v)$ having the isotropic form (2.16), where $\mathbf{q}(v) = 0$ for v belonging to some interval I, and $\mathbf{q}(v) = \mathbf{q}_c > 0$ for $v \notin I$, making thus (2.10) strongly degenerate; see, *e.g.*, R. Bürger–S. Evje–K. H. Karlsen [14] and R. Bürger–K. H. Karlsen [16]. Despite possessing now discontinuous coefficients, our theory may still apply to Equation (2.10) if one performs the following adjustment.

Assume that, in any of the averaging lemmas we have studied here, all hypotheses are preserved, but one weakens the requirement on $\mathcal{L}(i\tau, i\kappa, v)$ to $\mathbf{a} \in (\mathscr{C}^{k,\alpha}_{\mathrm{loc}} \cap L^{p'}_{\mathrm{loc}})(\mathbb{R}\setminus G; \mathbb{R}^N)$ and $\mathbf{b} \in (\mathscr{C}^{k,\alpha}_{\mathrm{loc}} \cap L^{p'}_{\mathrm{loc}})(\mathbb{R}\setminus G; \mathscr{L}(\mathbb{R}^N))$, where $G \subset \mathbb{R}$ is a closed set of zero Lebesgue measure. (The condition that $\mathbf{a}(v)$ and $\mathbf{b}(v)$ belong to $L^{p'}_{\mathrm{loc}}$ is only made so as to Equations (2.10) and (2.14) to make sense).

Following the proof of Lemma 2.1, one may construct a family of functions $(\Xi_\varepsilon)_{0<\varepsilon<1}$, such that

1. for all $0 < \varepsilon < 1$, $\Xi_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}_v)$,
2. $0 \le \Xi_\varepsilon(v) \le 1$ for all $0 < \varepsilon < 1$ and $v \in \mathbb{R}$,
3. for all $0 < \varepsilon < 1$, there exists some $c_\varepsilon > 0$ such that $\Xi_\varepsilon(v) = 0$ if $\operatorname{dist}(v, G) < c_\varepsilon$, and
4. $\Xi_\varepsilon(v) \to 1_{\mathbb{R}\setminus G}(v)$ for all $v \in \mathbb{R}$ as $\varepsilon \to 0_+$.

Repeating our techniques, it is not difficult to verify that $(\int_{\mathbb{R}} f_n \Xi_\varepsilon \eta \, dv)$ is relatively compact in $L^r(\Omega; L^s_{\mathrm{loc}}(Q))$ for any $0 < \varepsilon < 1$ (here Q may be $\mathbb{R}_t \times \mathbb{R}^N_x$). Therefore, in virtue of Proposition 2.2, one derives that the original velocity averages $(\int_{\mathbb{R}} f_n \eta \, dv)_{n\in\mathbb{N}}$ are indeed relatively compact in $L^r(\Omega; L^s_{\mathrm{loc}}(Q))$.

Notice that in the preceding argument it is not necessary to suppose that $\mathbf{a}(v)$ and $\mathbf{b}(v)$ lie in, respectively, $L^\infty_{\mathrm{loc}}(\mathbb{R}_v; \mathbb{R}^N)$ and $L^\infty_{\mathrm{loc}}(\mathbb{R}_v; \mathscr{L}(\mathbb{R}^N))$. Generally, f_n has uniformly bounded $L^r_\omega L^p_{t,x,v}$–norms for all $1 \le p \le \infty$, permitting one to consider $\mathbf{a} \in L^1_{\mathrm{loc}}(\mathbb{R} \setminus G; \mathbb{R}^N)$ and $\mathbf{b} \in L^1_{\mathrm{loc}}(\mathbb{R}; \mathscr{L}(\mathbb{R}^N))$

Remark 2.19 (Comparison with the work of P.L. Lions, B. Perthame and E. Tadmor). Following the previous Remark 2.15, let us continue juxtaposing our results with the classical averaging lemmas of P.-L. Lions–B. Perthame–E. Tadmor [82].

Regarding the differences between our theory and theirs, let us mention this minor one: when ℓ was not an integer, they permitted the indices $(k, \alpha) = (\lfloor \ell \rfloor, \ell - \lfloor \ell \rfloor)$. Alas, this assumption could not be made in our arguments. Indeed, as we have seen, the operator $(-\Delta_v + 1)^{\ell/2}$ acts (via "integrations by parts") on the symbol $\mathcal{L}(i\tau, i\kappa, v)$, forcing it to be Hölder–regular enough in order to $(-\Delta_v + 1)^{\ell/2}\mathcal{L}(i\tau, i\kappa, v)$ to make sense. As a consequence, except when ℓ is an integer—which permits $(-\Delta_v + 1)^{\ell/2}$ to be transformed into a regular derivative—, $\mathbf{a}(v)$ and $\mathbf{b}(v)$ need to have the sort of smoothness "leeway" we have imposed in (2.9); see, *e.g.*, P. R. Stinga [105]. To illustrate this point, notice that the function $\mathbf{b}(v) = |v|^{3/2} I_{\mathbb{R}^N}$ belongs to the Hölder class $\mathscr{C}^{1,1/2}_{\mathrm{loc}}(\mathbb{R}; \mathscr{L}(\mathbb{R}^N))$, but not to, say, $H^{3/2}_{\mathrm{loc}}(\mathbb{R}; \mathscr{L}(\mathbb{R}^N))$.

Despite this, we should point out that, in most applications, ℓ can be chosen as any number > 1, hence the negligibility of this inconvenience.

Therefore, having these observations in mind, we conclude that Theorem 2.2 may be understood as an extension of the hyperbolic compactness result of Lions–Perthame–Tadmor if $\mathbf{b}(v) \equiv 0$. The case $\mathbf{b}(v) \not\equiv 0$ is, however, distinct, for their theorem was stated for general diffusion matrices. Nevertheless, besides requiring $\mathbf{b}(v)$ to be smooth, they do not seem to allow a derivative of order higher than one in the forcing terms, which is instrumental for localization procedures—see the proof of Theorem 2.2.

Curiously enough, there is one particular instance in which we can treat general diffusion matrices, even though this case is of no pertinence to the theory of entropy solutions (see, however, Remark 2.23).

Proposition 2.10. *Let exponents $1 < p, q < \infty$, and $1 \le r \le 2$ be given. Let also $\mathbf{a} \in \mathscr{C}(\mathbb{R}; \mathbb{R}^N)$ and $\mathbf{b} \in \mathscr{C}(\mathbb{R}; \mathscr{L}(\mathbb{R}^N))$, with $\mathbf{b}(v)$ being nonnegative for all $v \in \mathbb{R}$.*

Assume that, for any $n \in \mathbb{N}$, the equation

$$\frac{\partial f_n}{\partial t} + \mathbf{a}(v) \cdot \nabla_x f_n - \mathbf{b}(v) : D^2_x f_n = (-\Delta_{t,x} + 1)^{1/2} g_n + (-\Delta_x + 1)^{1/4} \Phi_n \frac{dW}{dt} \tag{2.89}$$

is almost surely obeyed in $\mathscr{D}'(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v)$, where

1. *$(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^r(\Omega; L^p(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v))$,*
2. *$(g_n)_{n\in\mathbb{N}}$ is relatively compact in $L^r(\Omega; L^q(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v))$, and*
3. *$(\Phi_n)_{n\in\mathbb{N}}$ is a predictable and convergent sequence in $L^2(\Omega \times [0,\infty)_t;\, HS(\mathscr{H};\, L^2(\mathbb{R}^N_x \times \mathbb{R}_v)))$.*

Finally, let $\eta \in L^{p'}(\mathbb{R})$ have compact support, and presume that the nondegeneracy condition *(2.11) holds.*

Then, with s being the least number between p, q, and 2, for any $\varphi \in (L^1 \cap L^\infty)(\mathbb{R}_t \times \mathbb{R}^N_x)$, the sequence of averages $\left(\varphi \int_{\mathbb{R}} f_n \eta \, dv\right)_{n\in\mathbb{N}}$ converges in $L^r(\Omega; L^s(\mathbb{R}_t \times \mathbb{R}^N_x))$.

Sketch of the proof. Let us keep the notations of the proof of Theorems 2.3 and 2.4. If $\eta_{\delta,\gamma}$ is the same as then, define now the decomposition

$$\begin{cases} \mathfrak{f}^{(1)}_{m,n} = \mathfrak{F}^{-1}_{t,x}\left[\lambda\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)(\mathfrak{F}_{t,x}\mathfrak{f}_{m,n})\right], \\ \mathfrak{f}^{(2)}_{m,n} = \mathfrak{F}^{-1}_{t,x}\left[\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\lambda\left(\frac{\mathcal{L}(i\tau,i\kappa,v)}{\delta\sqrt{\tau^2+|\kappa|^2}}\right)(\mathfrak{F}_{t,x}\mathfrak{f}_{m,n})\right], \text{ and} \\ \mathfrak{f}^{(3)}_{m,n} = \mathfrak{F}^{-1}_{t,x}\left[\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\psi\left(\frac{\mathcal{L}(i\tau,i\kappa,v)}{\delta\sqrt{\tau^2+|\kappa|^2}}\right)(\mathfrak{F}_{t,x}\mathfrak{f}_{m,n})\right], \end{cases}$$

and write

$$\int_{\mathbb{R}} \mathfrak{f}_{m,n}\eta\, dv = \int_{\mathbb{R}} \mathfrak{f}_{m,n}(\eta-\eta_{\delta,\gamma})\, dv + \sum_{\nu=1}^{3}\int_{\mathbb{R}} \mathfrak{f}^{(\nu)}_{m,n}\eta_{\delta,\gamma}\, dv \stackrel{\text{def}}{=} \sum_{\nu=0}^{3} \mathfrak{v}^{(\nu)}_{m,n}.$$

The only term which needs some explanation is evidently $\mathfrak{f}^{(2)}_{m,n}$. Based on our techniques, it is not hard to see that $\mathcal{L}(i\tau,i\kappa,v)/\sqrt{\tau^2+|\kappa|^2}$ satisfies the truncation property uniformly on v. Moreover, it is not hard to see that

$$\mathbb{E}\|\mathfrak{v}^{(2)}_{m,n}\|^r_{L^p(\mathbb{R}_t\times\mathbb{R}^N_x)} \le C\|\eta_{\delta,\gamma}\|^r_{L^\infty(\mathbb{R})}\mathbb{E}\|\mathfrak{f}_{m,n}\|^r_{L^p(\mathbb{R}_t\times\mathbb{R}^N_x\times\mathbb{R}_v)} \left(\sup_{\tau^2+|\kappa|^2=1} \operatorname{meas}\left\{v\in\operatorname{supp}\eta_{\delta,\gamma};\, |\mathcal{L}(i\tau,i\kappa,v)|\le\frac{2\delta}{\gamma}\right\}\right)^{r\mathfrak{p}},$$

where C and $\mathfrak{p}>0$ are independent of m and n. Since we pass δ to zero prior to applying the same limit to γ, the factor $2\delta/\gamma$ brings no hindrances.

Furthermore, it is not hard to see that

$$\left|\psi\left(\frac{\mathcal{L}(i\tau,i\kappa,v)}{\delta\sqrt{\tau^2+|\kappa|^2}}\right)\psi\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{\gamma}\right)\frac{\sqrt{\tau^2+|\kappa|^2}}{\mathcal{L}(i\tau,i\kappa,v)}\right| \le \frac{2}{\delta} \text{ for all } (\tau,\kappa,v)\in\left(\mathbb{R}\times\mathbb{R}^N\setminus\{0\}\right)\times\mathbb{R}_v.$$

Hence, the proposition may be demonstrated following the same lines of the proof of Theorem 2.1. □

Let us mention that, were $\mathbf{a}(v)$ and $\mathbf{b}(v)$ locally Lipschitz, we could also have added some term of the form $(-\Delta_v+1)^{1/2}h_n$ into (2.89), where, evidently, $(h_n)_{n\in\mathbb{N}}$ is relatively compact in, say, $L^r(\Omega;L^q(\mathbb{R}_t\times\mathbb{R}^N_x\times\mathbb{R}_v))$. Notwithstanding, let us stress that the argument above is not valid for Equation (2.10) if $\ell>0$, as we have discoursed in Subsection 2.1.1.

Remark 2.20 (Comparison with the work of M. Lazar and D. Mitrovic [78]). Using an extension of the celebrated technique of H-measures, M. Lazar–D. Mitrovic [78] invented a very intriguing general theory for averaging lemmas to parabolic–hyperbolic equations suchlike ours; see also N. Antonic–M. Lazar [5, 6], E. Yu. Panov [93, 95], and M. Lazar–D. Mitrovic [77, 79]. It is worth mentioning that they could handle scenarios where the coefficients $\mathbf{a}$ and $\mathbf{b}$ depend discontinuously in x and v, which is impossible by our method.

An interesting instance that they consider and that can be easily comparable to Theorem 2.3 is as follows.

Theorem 2.6. *Let $2\le s<\infty$ be a real number, and let $N\ge 1$ and $\ell\ge 0$ be integers. Assume that $\mathbf{a}\in L^2(\mathbb{R}_v;\mathbb{R}^N)$ and $\mathbf{b}\in L^2(\mathbb{R}_v;\mathscr{L}(\mathbb{R}^N))$, with, for some fixed subspace $M\subset\mathbb{R}^N$, $R(\mathbf{b}(v))\subset M$ for all $v\in\mathbb{R}$.*

Suppose that, for any integer $n \in \mathbb{N}$, the equation

$$\frac{\partial f_n}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f_n - \mathbf{b}(v) : D_x^2 f_n = \big((-\Delta_{t,x}+1)^{1/2} - \Delta_M\big)\frac{\partial^\ell g_n}{\partial v^\ell} \tag{2.90}$$

is obeyed in $\mathscr{D}'(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v)$, where $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in in $L^2_{\mathrm{loc}}(\mathbb{R}_v; L^s_{\mathrm{loc}}(\mathbb{R}_t\times\mathbb{R}_x^N))$, and $(g_n)_{n\in\mathbb{N}}$ is a relatively compact sequence in $L^2(\mathbb{R}_v; L^{s'}(\mathbb{R}_t\times\mathbb{R}_x^N))$.

Furthermore, the following nondegeneracy condition is valid:

$$\operatorname{meas}\Big\{v\in\mathbb{R};\ i(\tau + (P_{M^\perp}\mathbf{a})(v)\cdot\kappa) + \kappa\cdot\mathbf{b}(v)\kappa = 0\Big\} = 0$$
$$\forall(\tau,\kappa)\in\mathbb{R}\times\mathbb{R}^N \text{ with } \tau^2 + |P_{M^\perp}\kappa|^2 + |P_M\kappa|^4 = 1.$$

Then, for all $\eta\in L^2(\mathbb{R})$ with compact support, the averages $(\int_{\mathbb{R}} f_n\eta\,dv)_{n\in\mathbb{N}}$ are relatively compact in $L^2_{\mathrm{loc}}(\mathbb{R}_t\times\mathbb{R}_x^N)$.

The most important points to be made are the following.

(i) Even though their results are critical in the same way ours are, the theory presented here can be applied to stochastic problems.

(ii) Their result still requires a certain L^2–property in the v-variable, which is generally a quite strong hypothesis. Nevertheless, for some specific applications in the study of entropy solutions, this issue can be circumvented provided one performs some clever remarks (see the section 5 in [78], G.-Q. Chen–H. Frid [23], W. Neves [86], and H. Frid *et al.* [43]).

(iii) They consider some very rough coefficients (see, however, Remark 2.18). Consequently, this gives Theorem 2.6 some flexibility to be locally used in some situations similar to ours. Still, we note that the right-hand of (2.90) can solely involve pure derivatives in v, which is not as propitious for localization procedures as the source terms in Equations (2.10) and (2.14) (see, withal, M. Lazar–D. Mitrovic [77]).

Remark 2.21 (Comparison with the work of M. Erceg, M. Misur, and D. Mitrovic [39]). While this manuscript was being written, a fascinating paper by M. Erceg–M. Misur–D. Mitrovic [39] has emerged. The main result of their work contains the next theorem, which is commensurable to our theory.

Theorem 2.7. *Let $N\geq 1$, and $\mathscr{O}\subset\subset\mathbb{R}_t\times\mathbb{R}_x^N$, and $\mathscr{V}\subset\subset\mathbb{R}_v$ be open sets. Assume that, for some $2<q<\infty$, $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^q(\mathscr{O}\times\mathscr{V})$, and each f_n solves the equation*

$$\frac{\partial f_n}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f_n - \mathbf{b}(v) : D_x^2 f_n = \frac{\partial g_n}{\partial v} + \operatorname{div}_{t,x} H_n \tag{2.91}$$

in $\mathscr{D}'(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v)$, where

1. *$\mathbf{a}\in L^p(\mathscr{V};\mathbb{R}^N)$ for some $p>q'$,*
2. *$\mathbf{b}\in\mathscr{C}^{0,1}(\mathscr{V};\mathscr{L}(\mathbb{R}^N))$ has the form $\mathbf{b}(v)=\sigma(v)^\star\sigma(v)$ for some $\sigma\in\mathscr{C}^{0,1}(\mathscr{V};\mathscr{L}(\mathbb{R}^N))$,*
3. *$(g_n)_{n\in\mathbb{N}}$ is relatively compact in $L^r_{\mathrm{loc}}(\mathbb{R}_v; W^{-1/2,r}_{\mathrm{loc}}(\mathbb{R}_t\times\mathbb{R}_x^N))$ for some $1<r<\infty$,*
4. *$(H_n)_{n\in\mathbb{N}}$ is relatively compact in $L^s_{\mathrm{loc}}(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v;\mathbb{R}\times\mathbb{R}^N)$ for some $1<s<\infty$, and*
5. *the following nondegeneracy condition is valid:*

$$\operatorname{meas}\Big\{v\in\mathscr{V};\ i(\tau+\mathbf{a}(v)\cdot\kappa) + \kappa\cdot\mathbf{b}(v)\kappa = 0\Big\} = 0$$
$$\forall(\tau,\kappa)\in\mathbb{R}\times\mathbb{R}^N \text{ with } \tau^2+|\kappa|^2 = 1. \tag{2.92}$$

Then, for all $\eta \in \mathscr{C}_c(\mathscr{V})$, the sequence of the averages $(\int_{\mathbb{R}} f_n \eta \, dv)$ is relatively compact in $L^1(\mathbb{R}^N)$.

Of course, the novelty here is the nondegeneracy condition (2.92), which is weaker and more general than ours (see, however, Remark 2.23). The particular structure of $\mathbf{b}(v)$ allows them to handle the distributions $\frac{\partial g_n}{\partial v}$ in a brilliant fashion; as a matter of fact, we believe that their analysis may perfectly be incorporated to the argument of Proposition 2.10, so as to our result also fit similar source terms. Still, in spite of Assumption (2.92), our theory has some advantages. Besides the previous observation that ours is of probabilistic nature, and stable under spatio-temporal "localizations", let us briefly mention the next two aspects.

(i) Their result is not critical. Unfortunately, the particular equation (2.91) is not suitable to the applications we have in mind due to the particular form of the source term $\frac{\partial g_n}{\partial v}$, which can only contain one derivative in v and one-half one in (t, x).

(ii) The diffusion matrix possessing the structure $\mathbf{b}(v) = \sigma(v)^\star \sigma(v)$ for $\sigma \in \mathscr{C}^{0,1}$ excludes several well-known diffusion matrices (see however Remark 2.18).

Remark 2.22 (In comparison with the work of E. Tadmor and T. Tao, and of B. Gess and M. Hofmanová). Even though the averaging lemmas of E. Tadmor–T. Tao [107] and B. Gess–M. Hofmanová [51] deal with the Sobolev regularity of the averages and hence are of a different kind than ours, in several situations, this type of result is used in the same context: to corroborate the existence of kinetic solutions to nonlinear degenerate convection–diffusion equations. It is thus interesting to contrast our theory with theirs.

Well-understood, the crux of our argument is the regularizing effects of the Fourier quotient $\frac{1}{\mathcal{L}(i\tau, i\kappa, v)}$. In contrast, as we have commented in the Introduction, theirs was founded on dyadic decompositions and some uniform rates on the quantities $\omega(J; \delta)$ expressed in (1.6). Hence, their method treats both the degree of $\mathcal{L}(i\tau, i\kappa, v)$ and its behavior (parabolic or hyperbolic) quite indirectly and more abstractly. Even though this leads to a theorem enunciated in broader terms, not only are their conditions much more arduous to be verified but also all concrete examples provided by both works are also valid in our setting.

A particular and impressive attribute of work of B. Gess–M. Hofmanová [51] is that, under some conditions on $\mathbf{a}(v)$ and $\mathbf{b}(v)$, they could let the weight function η not possess compact support, which seems to be a quite unprecedented assumption in the theory of the velocity averaging lemmas. Furthermore, they did not assume any Hölder regularity on $\mathbf{a}(v)$ and $\mathbf{b}(v)$ (nevertheless, one usually employs some Hölder regularity in order to investigate (1.6)).

Anyhow, it remains an intriguing conjecture to verify if the nontransient condition is somehow implicit in their hypotheses, or, conversely, if it is essential at all.

A more tangible fashion to pose this conjecture is as follows. Like in Subsection 2.1.1, put $N = 1$, let $\mathbf{b} \in \mathscr{C}_c^\infty(0, 1)$ be a nonnegative function vanishing exactly in a Cantor set of positive measure in $[0, 1]$, and define $\mathcal{L} : \mathbb{R}_\tau \times \mathbb{R}_\kappa \times \mathbb{R}_v \to \mathbb{C}$ by $\mathcal{L}(i\tau, i\kappa, v) = i(\tau + v\kappa) + \mathbf{b}(v)\kappa^2$. Evidently, $\mathbf{b}(v)$ does not obey the nontransient condition in $[0, 1]$, consequently our theorem does apply to this particular symbol. Do, however, the hypotheses of Tadmor–Tao or Gess–Hofmanová apply? (Notice, since $\mathbf{b}(v)$ vanishes at infinite order in this Cantor set, it is not clear how to reproduce the analysis featured in section 4.2 of [107]; neither seems their condition (2.20) easily verifiable). If not, can an averaging lemma like Theorem 2.1 still be proven to this symbol?

Remark 2.23 (On the nondegeneracy condition, and the real analytic case). The core of this final remark is the following observation.

Proposition 2.11. *Let $N \geq 1$ be an integer, I be an open interval, and $\mathbf{a} \in \mathscr{C}(I; \mathbb{R}^N)$ and $\mathbf{b} \in \mathscr{C}(I; \mathscr{L}(\mathbb{R}^N))$. Assume also that $\mathbf{b}(v)$ is nonnegative and real analytic in I.*

If the general nondegeneracy condition

$$\text{“meas}\Big\{ v \in I;\ \tau + \mathbf{a}(v) \cdot \kappa = 0 \ \textit{and}\ \kappa \cdot \mathbf{b}(v)\kappa = 0 \Big\} = 0 \qquad \forall (\tau, \kappa) \in \mathbb{R} \times \mathbb{R}^N \ \textit{with}\ \tau^2 + |\kappa|^2 = 1\text{”} \tag{2.93}$$

holds, then there exists some subspace $M \subset \mathbb{R}^N$ such that $R(\mathbf{b}(v)) \subset M$ for all $v \in I$, and

$$\operatorname{meas}\Big\{v \in I;\ \tau + (P_{M^\perp}\mathbf{a})(v)\cdot\kappa = 0 \text{ and } \kappa\cdot\mathbf{b}(v)\kappa = 0\Big\} = 0 \quad \forall(\tau,\kappa) \in \mathbb{R}\times\mathbb{R}^N \text{ with } \tau^2 + |\kappa|^2 = 1. \tag{2.94}$$

Proof. Logically, one has the alternative:

- Either for all $\kappa \in \mathbb{R}^N$ with $|\kappa| = 1$, $\operatorname{meas}\{v \in I; \kappa\cdot\mathbf{b}(v)\kappa = 0\} = 0$,
- or there exists some $\kappa_1 \in \mathbb{R}^N$ with $|\kappa_1| = 1$ such that $\operatorname{meas}\{v \in I; \kappa_1\cdot\mathbf{b}(v)\kappa_1 = 0\} > 0$.

In the first case, it suffices to take $M = \mathbb{R}^N$, and the desired conclusion would follow; in the latter, however, the analyticity of $\mathbf{b}(v)$ implies that $\kappa_1\cdot\mathbf{b}(v)\kappa_1 \equiv 0$ for $v \in I$. Since $\mathbf{b}(v) \geq 0$, such an identity would be the same as $|\mathbf{b}(v)^{1/2}\kappa_1|^2 \equiv 0$, which evidently forces κ_1 to be in the null space of all $\mathbf{b}(v)$'s.

We may continue such a procedure:

- Either for all $\kappa \in \mathbb{R}^N$ such that $|\kappa| = 1$ and $\kappa \perp \kappa_1$, $\operatorname{meas}\{v \in I; \kappa\cdot\mathbf{b}(v)\kappa = 0\} = 0$,
- or there exists some $\kappa_2 \in \mathbb{R}^N$ such that $|\kappa_2| = 1$, $\kappa_2 \perp \kappa_1$, and $\operatorname{meas}\{v \in I; \kappa_1\cdot\mathbf{b}(v)\kappa_1 = 0\} > 0$.

Again, in the former hypothesis, one may take $M = \kappa_1^\perp$ in (2.94), but, in the latter, κ_1 and κ_2 are two orthogonal elements lying in the null space of each and every $\mathbf{b}(v)$. Reprising this reasoning at most $N-2$ more times, one would obtain the desired conclusion. □

Consequently, one can see that the nondegeneracy condition (2.93) implies the ones we have considered in this manuscript if $\mathbf{b}(v)$ is analytic.

So as to weaken considerably the smoothness condition on $\mathbf{b}(v)$, we can apply the ideas of Remark 2.18, and consider $\mathbf{b}(v)$'s for which there exists some closed set $G \subset \mathbb{R}$ of measure 0 such that $\mathbf{b}(v)$ is analytic outside of G. For $\mathbb{R}\setminus G$ is a countable union of open intervals, the previous argument would hold, and we would see that (2.93) locally implies (2.94), which is, evidently, (2.15).

Accordingly, our nondegeneracy conditions end up containing the classical nondegeneracy condition of P.-L. Lions–B. Perthame–E. Tadmor [82] in numerous interesting and important cases. Let us summarize this discussion in the next two theorems.

Theorem 2.8 (The global analytical averaging lemma). *Let exponents $1 < p, q < \infty$, $1 \leq r \leq 2$ and $\ell \geq 0$ be given. Assume that $\mathbf{a} \in \mathscr{C}^{k,\alpha}_{\text{loc}} \cap L^{p'}(\mathbb{R}\setminus G; \mathbb{R}^N)$ and $\mathbf{b} \in \mathscr{C}^{\infty}_{\text{loc}} \cap L^{p'}(\mathbb{R}\setminus G; \mathscr{L}(\mathbb{R}^N))$, where the real numbers k and α satisfy the relation (2.9), and $G \subset \mathbb{R}$ is a closed set of measure zero. Furthermore, suppose that $\mathbf{b}(v)$ is nonnegative for all $v \in \mathbb{R}$, and $\mathbf{b}(v)$ is real analytic outside G.*

Assume that, for any $n \in \mathbb{N}$, the equation

$$\frac{\partial f_n}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f_n - \mathbf{b}(v) : D_x^2 f_n = (-\Delta_{t,x}+1)^{1/2}(-\Delta_v+1)^{\ell/2} g_n + (-\Delta_x+1)^{1/4}(-\Delta_v+1)^{\ell/2}\Phi_n\frac{dW}{dt} \tag{2.95}$$

is almost surely obeyed in $\mathscr{D}'(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v)$, where

1. *$(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^r(\Omega; L^p(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v))$,*
2. *$(g_n)_{n\in\mathbb{N}}$ is a relatively compact sequence in $L^r(\Omega; L^q(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v))$, and*
3. *$(\Phi_n)_{n\in\mathbb{N}}$ is a predictable and relatively compact sequence in $L^2(\Omega\times[0,\infty)_t; HS(\mathscr{H}; L^2(\mathbb{R}_x^N\times\mathbb{R}_v)))$.*

(If each and every f_n has finite $L^1(\Omega; L^\infty_{\mathrm{loc}}(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v))$–"norms", one can assume that $\mathbf{a} \in L^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^N)$ and $\mathbf{b} \in L^1_{\mathrm{loc}}(\mathbb{R}; \mathscr{L}(\mathbb{R}^N))$).

Finally, let $\eta \in L^{p'}(\mathbb{R})$ have compact support, and presume that the following nondegeneracy condition *holds:*

$$\operatorname{meas}\Big\{v \in \operatorname{supp}\eta;\ \tau + \mathbf{a}(v)\cdot\kappa = 0 \text{ and } \kappa \cdot \mathbf{b}(v)\kappa = 0\Big\} = 0 \qquad \forall(\tau,\kappa) \in \mathbb{R}\times\mathbb{R}^N \text{ with } \tau^2 + |\kappa|^2 = 1. \tag{2.96}$$

Then, with s being the least number between p, q, and 2, the sequence of averages $\big(\varphi \int_{\mathbb{R}} f_n \eta\, dv\big)_{n\in\mathbb{N}}$ is relatively compact in $L^r(\Omega; L^s(\mathbb{R}_t \times \mathbb{R}^N_x))$ for any $\varphi \in (L^1 \cap L^\infty)(\mathbb{R}_t \times \mathbb{R}^N_x)$.

Theorem 2.9 (The local analytical averaging lemma). *Let exponents $1 < p, q < \infty$, $1 \le r \le 2$ and $\ell \ge 0$ be given. Assume that $\mathbf{a} \in \mathscr{C}^{k,\alpha}_{\mathrm{loc}} \cap L^{p'}(\mathbb{R}\setminus G; \mathbb{R}^N)$ and $\mathbf{b} \in \mathscr{C}^\infty_{\mathrm{loc}} \cap L^{p'}(\mathbb{R}\setminus G; \mathscr{L}(\mathbb{R}^N))$, where the real numbers k and α satisfy the relation (2.9), and $G \subset \mathbb{R}$ is a closed set of measure zero. Furthermore, suppose that $\mathbf{b}(v)$ is nonnegative for all $v \in \mathbb{R}$, and $\mathbf{b}(v)$ is real analytic outside G. Let $Q \subset \mathbb{R}_t \times \mathbb{R}^N_x$ be an open set.*

Assume that, for any $n \in \mathbb{N}$, Equation (2.95) is obeyed in $\mathscr{D}'(Q \times \mathbb{R}_v)$, where

1. *$(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^r(\Omega; L^p_{\mathrm{loc}}(Q \times \mathbb{R}_v))$ that is relatively compact in $L^r(\Omega; W^{-z_0,p}_{\mathrm{loc}}(Q \times \mathbb{R}_v))$ for some $z_0 > 0$,*
2. *$(g_n)_{n\in\mathbb{N}}$ is a relatively compact sequence in $L^r(\Omega; L^q(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v))$, and*
3. *$(\Phi_n)_{n\in\mathbb{N}}$ is a predictable and relatively compact sequence in $L^2(\Omega \times [0,\infty)_t; HS(\mathscr{H}; L^2(\mathbb{R}^N_x \times \mathbb{R}_v)))$.*

(If each and every f_n has finite $L^1(\Omega; L^\infty_{\mathrm{loc}}(Q \times \mathbb{R}_v))$–"norms", one can assume that $\mathbf{a} \in L^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^N)$ and $\mathbf{b} \in L^1_{\mathrm{loc}}(\mathbb{R}; \mathscr{L}(\mathbb{R}^N))$).

Finally, let $\eta \in L^{p'}(\mathbb{R})$ have compact support, and presume that the nondegeneracy condition *(2.96) holds.*

Then, with s being the least number between p, q, and 2, the sequence of averages $\big(\int_{\mathbb{R}} f_n \eta\, dv\big)_{n\in\mathbb{N}}$ is relatively compact in $L^r(\Omega; L^s_{\mathrm{loc}}(Q))$. In particular, if $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^r(\Omega; L^p(Q \times \operatorname{supp}\eta))$, and Q is of finite measure, then $(\int_{\mathbb{R}} f_n \eta\, dv)_{n\in\mathbb{N}}$ converges in $L^r(\Omega; L^z(Q))$ for any $1 \le z < p$.

Chapter 3

The relative compactness of entropy solutions to degenerate parabolic–hyperbolic equations

3.1 The definition of entropy solution and the main result

Let us now present how one may employ the previous chapter's averaging lemmas to straightforwardly derive the relative compactness of entropy solutions to diffusion–convection equations.

Let $N \geq 1$ be an integer, and $Q \subset \mathbb{R}_t \times \mathbb{R}_x^N$ be an open set, and consider the quasilinear partial differential equation

$$\frac{\partial u}{\partial t}(t, x) + \mathrm{div}_x \mathbf{A}(u(t, x)) - D_x^2 : \mathbf{B}(u(t, x)) = 0, \tag{3.1}$$

where (t, x) lies in some open set $Q \subset \mathbb{R}_t \times \mathbb{R}_x^N$, $\mathbf{A} : \mathbb{R} \to \mathbb{R}^N$ is a continuously differentiable flux function, and $\mathbf{B}(v) \in \mathscr{L}(\mathbb{R}^N)$ is a continuously differentiable matrix such that $\mathbf{B}'(v) \geq 0$ everywhere. Throughout this chapter, put $\mathbf{A}'(v) = \mathbf{a}(v)$ and $\mathbf{B}'(v) = \mathbf{b}(v)$.

Based on the celebrated work of G.-Q. Chen–B. Perthame [27], let us first state what we mean by an entropy solution to (3.1).

Definition 3.1. Let $u \in L^\infty_{\mathrm{loc}}(Q)$. One says that u is an *entropy solution* to (3.1) if the following conditions hold.

1. (Regularity). If $\sigma(v) = \mathbf{b}(v)^{1/2}$, and $\beta(v) = \int_0^v \sigma(w)\, dw$, then $\mathrm{div}_x(\beta(u)) \in L^2_{\mathrm{loc}}(Q; \mathbb{R}^N)$.

2. (Chain rule). For any nonnegative function $\psi \in \mathscr{C}(\mathbb{R}_v)$, put $\beta^\psi(v) = \int_0^v \psi(w)^{1/2}\sigma(w)\, dw$, and $\mathfrak{n}^\psi(t, x) = \psi(u(t, x))|\mathrm{div}_x \sigma(u(t, x))|^2$. Then,

$$\begin{cases} \mathrm{div}_x(\beta^\psi(u)) = \psi(u)^{1/2}\mathrm{div}_x \beta(u) \in L^2_{\mathrm{loc}}(Q; \mathbb{R}^N), \text{ and} \\ \mathfrak{n}^\psi(t, x) = |\mathrm{div}_x\, \beta^\psi(u(t, x))|^2. \end{cases}$$

3. (The entropy condition). There exists a nonnegative measure $\mathfrak{m}(t, x, v)$ supported on $Q \times \mathbb{R}_v$, such that, for any function $\eta \in \mathscr{C}^2(\mathbb{R})$, one has that

$$\frac{\partial}{\partial t}\eta(u) + \mathrm{div}_x \mathbf{A}^\eta(u) - D_x^2 : \mathbf{B}^\eta(u) = -(\mathfrak{m}^{\eta''} + \mathfrak{n}^{\eta''}) \text{ in } \mathscr{D}'(Q), \tag{3.2}$$

where

$$\begin{cases} \mathbf{A}^\eta(v) = \displaystyle\int_0^v \eta'(\xi)\mathbf{a}(\xi)\,d\xi, \\ \mathbf{B}^\eta(v) = \displaystyle\int_0^v \eta'(\xi)\mathbf{b}(\xi)\,d\xi, \text{ and} \\ \mathfrak{m}^{\eta''}(t,x) = \displaystyle\int_{\mathbb{R}_\xi} \eta''(\xi)\,\mathfrak{m}(t,x,d\xi). \end{cases} \tag{3.3}$$

Remark 3.1 (The formal derivation of the entropy condition)*.* Let us motivate the definition above.

Assume that u is a smooth solution to (3.1). Then, according to the usual chain rule, we may multiply (3.1) by $\eta'(u)\varphi$, where $\eta \in \mathscr{C}^2(\mathbb{R})$ with $\eta'' \geq 0$ and $\varphi \in \mathscr{C}_c^\infty(Q)$, and deduce that

$$\begin{aligned} 0 &= \int_Q \eta'(u)\varphi\bigg(\frac{\partial u}{\partial t} + \mathbf{a}(u)\cdot\nabla_x u - \nabla_x\cdot\big(\mathbf{b}(u)\nabla_x u\big)\bigg)\,dxdt \\ &= \int_Q \varphi\bigg(\frac{\partial \eta(u)}{\partial t} + \mathrm{div}_x\mathbf{A}^\eta(u) - \eta'(u)\mathrm{div}_x\big(\mathbf{b}(u)\nabla_x u\big)\bigg)\,dxdt \\ &= -\int_Q \bigg(\eta(u)\frac{\partial \varphi}{\partial t} + \mathbf{A}^\eta(u)\cdot\nabla_x\varphi - \big(\varphi\eta''(u)\nabla_x u + \eta'(u)\nabla\varphi\big)\cdot\big(\mathbf{b}(u)\nabla_x u\big)\bigg)\,dxdt \\ &= -\int_Q \bigg(\eta(u)\frac{\partial \varphi}{\partial t} + \mathbf{A}^\eta(u)\cdot\nabla_x\varphi - |\eta''(u)^{1/2}\sigma(u)\nabla_x u|^2\varphi - \eta'(u)\mathbf{b}(u)\nabla_x u\cdot\nabla_x\varphi\bigg)\,dxdt \\ &= -\int_Q \bigg(\eta(u)\frac{\partial \varphi}{\partial t} + \mathbf{A}^\eta(u)\cdot\nabla_x\varphi - |\mathrm{div}\beta^{\eta''}(u)|^2\varphi + \mathbf{B}^\eta(u) : D_x^2\varphi\bigg)\,dxdt \end{aligned} \tag{3.4}$$

Clearly, this leads to (3.1) without $\mathfrak{m}^{\eta''}$. Alas, due to the fact that $\mathbf{b}(v)$ may degenerate, generally one cannot construct regular enough solutions to justify all the previous calculations; in fact, if $\mathbf{B}(v) \equiv 0$, (3.1) becomes a first-order quasilinear equation; hence, the well-known method of the characteristics shows that classical solutions may develop singularities in finite time.

Notwithstanding, in most applications, one firstly approximates $\mathbf{B}(v)$ by adding a viscous part, *i.e.*, one introduces

$$\mathbf{B}_\nu(v) = \mathbf{B}(v) + \nu v I_{\mathbb{R}^N},$$

where $\nu > 0$, thus making $(\mathbf{B}_\nu)'(v) = \mathbf{b}_\nu(v)$ "uniformly elliptic". This regularization in reality has its basis in Physics—specifically, Fluid Dynamics—, as the inclusion of such additional viscosity allows the equation to present "internal frictional forces", which is more reasonable from the point of view of mathematical modeling. From a theoretical perspective, the fact that $\mathbf{b}_\nu(v) \geq \nu I_{\mathbb{R}^N}$ permits one—among other propositions—to obtain L^2–estimates for $\nabla_x u$.

Thus, for the sake of argument, let us assume that one may construct a family of weak solutions $u_\nu \in L^\infty(Q)$ with $\nabla_x u_\nu \in L^2(Q)$ to (3.1) with this "better" matrix $\mathbf{B}_\nu(v)$ replacing $\mathbf{B}(v)$. Observe that these regularity assumptions suffice to carry out the manipulations in (3.4). Furthermore, let us presume that the next a priori estimates hold:

1. $\sup_{0<\nu<1} \|u_\nu\|_{L^\infty(Q)} < \infty$, and
2. $\sup_{0<\nu<1} \int_Q |\mathrm{div}_x\,\beta(u_\nu(t,x))|^2\,dxdt < \infty$.

Equally, neither of these bounds are of extraordinary character, as in practice they may be deduced via the classical comparison principles and energy methods for parabolic equations. At last, let us suppose that one can also ensure that, for some sequence $\nu_n \to 0_+$, u_{ν_n} converges almost everywhere to some $u \in L^\infty(Q)$ (which is of course a substantially more sensitive hypothesis). Then, for $|\mathrm{div}_x\beta(u_\nu)|^2 \leq |\mathrm{div}_x\beta_\nu(u_\nu)|^2$ (where $\beta_\nu(v) = \int_0^v \mathbf{b}_\nu(w)^{1/2}\,dw$), classical weak convergence methods assert that

$$\frac{\partial}{\partial t}\eta(u) + \mathrm{div}_x\mathbf{A}^\eta(u) - D_x^2 : \mathbf{B}^\eta(u) \leq -\mathfrak{n}^{\eta''} \text{ in } \mathscr{D}'(Q).$$

Therefore, one may apply the Riesz representation theorem in order to deduce that

$$\varphi(t,x)\eta''(v) \in \mathscr{C}_c^\infty(Q\times\mathbb{R}) \mapsto \int_Q \left(\eta(u)\frac{\partial\varphi}{\partial t} + \mathbf{A}^\eta(u)\cdot\nabla_x\varphi + \mathbf{B}^\eta(u) : D_x^2\varphi - \mathfrak{n}^{\eta''}\varphi\right) dtdx$$

defines a nonnegative σ–finite measure in $Q\times\mathbb{R}_v$. This explains the presence of the "hyperbolic dissipation measure" $\mathfrak{m}$ in (3.1). (Appropriately, $\mathfrak{n}$ is usually called the "parabolic dissipation measure".)

The reasoning just delineated displays the main driving force behind Definition 3.1. Incidentally, the chain rule in it may also be corroborated via this viscous approximation. Even though this operational rule is not strictly necessary when $\mathbf{b}(v)$ is isotropic (i.e., $\mathbf{b}(v) = \mathbf{q}(v)I_{\mathbb{R}^N}$), it plays an important role in the investigation of the anisotropic case; see G.-Q. Chen–B. Perthame [27].

In order to apply our averaging techniques, let us unify the hypotheses of Theorems 2.1–2.4 as follows.

Definition 3.2. Let $\mathbf{A} \in \mathscr{C}^1(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$ and $\mathbf{B} \in \mathscr{C}^1(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$, and put $\mathbf{A}'(v) = \mathbf{a}(v)$ and $\mathbf{B}'(v) = \mathbf{b}(v)$.

1. $\mathbf{A}(v)$ and $\mathbf{B}(v)$ are said to satisfy the *one-phase nondegeneracy condition* in some measurable set $X\subset\mathbb{R}$ if the following conditions hold.
   (a) There exists a linear subspace $M\subset\mathbb{R}^N$ such that $R(\mathbf{b}(v))\subset M$ for every $v\in\mathbb{R}$.
   (b) If $(P\mathcal{L})$ is the "principal" symbol $(P\mathcal{L})(i\tau,i\kappa,v) = i(\tau + (P_{M^\perp}\mathbf{a}(v))\cdot\kappa) + \kappa\cdot\mathbf{b}(v)\kappa$, one has that
   $$\text{meas}\{v\in X; (P\mathcal{L})(i\tau,i\kappa,v)=0\} = 0 \ \forall(\tau,\kappa)\in\mathbb{R}\times\mathbb{R}^N \text{ with } \tau^2+|\kappa|^2=1.$$
2. $\mathbf{A}(v)$ and $\mathbf{B}(v)$ are said to satisfy the *two-phase nondegeneracy condition* in some measurable set $X\subset\mathbb{R}$ if the following conditions hold.
   (a) $\mathbf{b}(v)$ has a dichotomous range.
   (b) $\mathbf{b}(v)$ has a satisfies the nontransiency condition in X.
   (c) If $\mathcal{L}(i\tau,i\kappa,v)$ is the symbol $\mathcal{L}(i\tau,i\kappa,v) = i(\tau+\mathbf{a}(v)\cdot\kappa) + \kappa\cdot\mathbf{b}(v)\kappa$, one has that
   $$\text{meas}\{v\in X; \mathcal{L}(i\tau,i\kappa,v)=0\} = 0 \ \forall(\tau,\kappa)\in\mathbb{R}\times\mathbb{R}^N \text{ with } \tau^2+|\kappa|^2=1.$$

We are in conditions to enunciate the main theorem of this chapter.

Theorem 3.1. *Assume that $\mathbf{A}(v)\in\mathscr{C}^{2,\varepsilon}_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$ and $\mathbf{B}(v)\in\mathscr{C}^{2,\varepsilon}_{\text{loc}}(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$ for some $0<\varepsilon\le 1$. Let $\mathscr{I}$ be an arbitrary index set. Assume that $(u_\nu)_{\nu\in\mathscr{I}}$ is a family of entropy solutions to (3.1) in Q, and that there exist $a<b$ such that*

$$-\infty < a \le u_\nu(t,x) \le b < \infty \ \textit{almost everywhere in } Q \tag{3.5}$$

for all $\nu\in\mathscr{I}$. Finally, suppose that $\mathbf{A}(v)$ and $\mathbf{B}(v)$ satisfy either the one- or the two-phase nondegeneracy condition in (a,b).

Then, $(u_\nu)_{\nu\in\mathscr{I}}$ is relatively compact in $L^p_{\text{loc}}(Q)$ for any $1\le p<\infty$. In particular, if Q is of finite measure, $(u_\nu)_{\nu\in\mathscr{I}}$ is relatively compact in $L^p(Q)$ for any $1\le p<\infty$. Furthermore, the limit points of $(u_\nu)_{\nu\in\mathscr{I}}$ are also entropy solutions to (3.1).

Remark 3.2. Let us highlight the importance of the nondegeneracy conditions. Despite its simplicity, the next example can be easily extended to more general settings.

Assume that $\mathbf{B}(v)\equiv 0$, and $Q = \mathbb{R}_t\times\mathbb{R}_x^N$, so that (3.1) transforms into a simple conservation law

$$\frac{\partial u}{\partial t}(t,x) + \text{div}_x\mathbf{A}(u(t,x)) = 0. \tag{3.6}$$

Furthermore, suppose that our nondegeneracy conditions are not observed in a quite dramatic fashion: For some interval, say, $X = [-1, 1]$, presume that there exists a vector $(\tau, \kappa) \in \mathbb{R} \times \mathbb{R}^N$ with $\tau^2 + |\kappa|^2 = 1$ such that

$$\tau + \mathbf{a}(v) \cdot \kappa = 0 \text{ identically in } X = [-1, 1].$$

In this case, it is not difficult to see that, given any $\zeta \in \mathscr{C}^1(\mathbb{R})$ satisfying $\|\zeta\|_{L^\infty(\mathbb{R})} \leq 1$, $u(t,x) = u_\zeta(t,x) = \zeta(\tau t + \kappa \cdot x)$ is an entropy solution to (3.6). On the other hand, it is immediate to verify that such family $(u_\zeta)_{\zeta \in \mathscr{C}^1(\mathbb{R}); \|\zeta\|_\infty \leq 1}$ cannot be relatively compact in L^1_{loc} even though it is bounded in L^∞. As a result, one can thus see how the conclusions of Theorem 3.1 may fail if no nondegeneracy condition is in effect.

3.2 Proof of Theorem 3.1

First of all, let us recall the notion of a kinetic solution introduced by G.-Q. Chen–B. Perthame [27] (see also M. Bendahmane–K.-H. Karlsen [16]). For any ξ and $v \in \mathbb{R}$, define the so-called χ–function $\chi_\xi(v)$ by

$$\chi_\xi(v) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } 0 < v < \xi, \\ -1 & \text{if } \xi < v < 0, \text{ and} \\ 0 & \text{elsewhere;} \end{cases} \tag{3.7}$$

in other words, $\chi_\xi(v) = 1_{(-\infty,\xi)}(v) - 1_{(-\infty,0)}(v)$ almost everywhere.

Remark 3.3. A few of the most fundamental properties of these χ–functions are the following.

1. Given any $\xi \in \mathbb{R}$, $v \mapsto \chi_\xi(v)$ can only assume the values -1, 0, and 1.
2. For all ξ and $v \in \mathbb{R}$, $\text{sign}(v)\chi_\xi(v) = |\chi_\xi(v)|$.
3. It holds that
$$\int_{\mathbb{R}} S'(v)\chi_\xi(v)\, dv = S(\xi) - S(0) \tag{3.8}$$
for all locally absolutely continuous functions $S : \mathbb{R} \to \mathbb{R}$. In particular,
$$\int_{\mathbb{R}} \chi_\xi(v)\, dv = \xi \tag{3.9}$$
for all $\xi \in \mathbb{R}$.
4. Given any ξ_1 and $\xi_2 \in \mathbb{R}$,
$$\int_{\mathbb{R}} |\chi_{\xi_1}(v) - \chi_{\xi_2}(v)|\, dv = |\xi_1 - \xi_2|. \tag{3.10}$$

As a corollary of the properties (1)–(3) above, one may deduce that

$$\int_{\mathbb{R}} |\chi_\xi(v)|^p\, dv = |\xi| \tag{3.11}$$

for all $\xi \in \mathbb{R}$ and $1 \leq p < \infty$.

Some formal manipulations involving the entropy condition (3.2) and the identity in (3.8) give rise to the next notion of a solution to Equation (3.1).

Definition 3.3. Let $u \in L^1_{\text{loc}}(Q)$, and let $f(t,x,v) = \chi_{u(t,x)}(v)$ for $(t,x,v) \in Q \times \mathbb{R}_v$ be its χ–function. One says that u is a *kinetic solution* to (3.1) if the following conditions hold.

1. (Regularity). For any nonnegative function $\psi \in \mathscr{C}_c^\infty(\mathbb{R})$, put $\beta^\psi(v) = \int_0^v \psi(\xi)^{1/2}\sigma(\xi)\,d\xi$. Then,

$$\operatorname{div}_x(\beta^\psi(u)) \in L^2_{\mathrm{loc}}(Q;\mathbb{R}^N).$$

2. (Chain rule). For any nonnegative functions ψ_1 and $\psi_2 \in \mathscr{C}_c^\infty(\mathbb{R})$, it holds that

$$\operatorname{div}_x(\beta^{\psi_1\psi_2}(u)) = \psi_1(u)^{1/2}\operatorname{div}_x(\beta^{\psi_2}(u)) \text{ almost everywhere.}$$

3. (The kinetic equation). There exist two nonnegative measures $\mathfrak{m}(t,x,v)$ and $\mathfrak{n}(t,x,v)$ supported on $Q \times \mathbb{R}_v$ such that

$$\int_{\mathbb{R}_v} \psi(v)\,\mathfrak{n}(t,x,dv) = |\operatorname{div}_x \beta^\psi(u(t,x))|^2$$

for any nonnegative $\psi \in \mathscr{C}_c^\infty(\mathbb{R})$, and the equation

$$\frac{\partial f}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f - \mathbf{b}(v) : D_x^2 f = \frac{\partial \mathfrak{m}}{\partial v} + \frac{\partial \mathfrak{n}}{\partial v} \tag{3.12}$$

is obeyed in the sense of the distributions in $Q \times \mathbb{R}_v$.

4. (Decay estimate). It holds that, for any $Q_0 \subset\subset Q$,

$$\int_{Q_0} (\mathfrak{m}+\mathfrak{n})(dt,dx,v) \le \mu_{Q_0}(v) \tag{3.13}$$

for some $\mu_{Q_0} \in L^\infty(\mathbb{R})$ such that $\mu_{Q_0}(v) \to 0_+$ as $|v| \to \infty$.

As indicated by (3.8), the concepts of entropy and kinetic solutions are almost one and the same. The main difference, however, is that a kinetic solution u may only be locally integrable; hence, while (3.12) still makes sense, (3.2) may not. As a matter of fact, it is not difficult to deduce the next proposition, which expresses the precise relation between these two formulations.

Proposition 3.1. *If $u \in L^\infty_{\mathrm{loc}}(Q)$ is an entropy solution to (3.1), then u is also a kinetic solution. Conversely, if $u \in L^1_{\mathrm{loc}}(Q)$ is a kinetic solution to (3.1), and $u \in L^\infty_{\mathrm{loc}}(Q)$, then u is an entropy solution.*

Likewise, a kinetic solution u is an entropy one if, and only if, for all $Q_0 \subset\subset Q$, there exists some $L_{Q_0} \ge 0$ such that the measures $\mathfrak{m}$ and $\mathfrak{n}$ (as given in Definition 3.3) satisfy $\operatorname{supp}\mathfrak{m}_{|Q_0}$ and $\operatorname{supp}\mathfrak{n}_{|Q_0} \subset Q_0 \times [-L_{Q_0}, L_{Q_0}]$. In this case, one may take $L_{Q_0} = \|u\|_{L^\infty(Q_0)}$.

In this fashion, Theorem 3.1 is an immediate consequence of the next more general compactness principle.

Theorem 3.2. *Assume that $\mathbf{A}(v) \in \mathscr{C}^{2,\varepsilon}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N)$ and $\mathbf{B}(v) \in \mathscr{C}^{2,\varepsilon}_{\mathrm{loc}}(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$ for some $0 < \varepsilon \le 1$. Let $\mathscr{I}$ be an arbitrary index set.*

Suppose that $(u_\nu)_{\nu\in\mathscr{I}}$ is a family of kinetic solutions to (3.1) in Q that enjoys the following uniform integrability property: for all $Q_0 \subset\subset Q$, there exists some function $\lambda_{Q_0} : (0,\infty) \to \mathbb{R}$ such that $\lambda_{Q_0}(A) \to 0_+$ as $A \to \infty$, and

$$\int_{Q_0} \Big[\big(u_\nu(t,x) - A\big)_+ + \big(A - u_\nu(t,x)\big)_-\Big]\,dxdt \le \lambda_{Q_0}(A) \text{ for all } \nu \in \mathscr{I} \text{ and } A > 0, \tag{3.14}$$

where, as usual, $z_+ = \max\{z,0\}$ and $z_- = \max\{-z,0\}$ stand for, respectively, the positive and negative part of a real number z.

Finally, suppose that there exists an interval $X \ne \mathbb{R}$ such that $\operatorname{ess\,ran} u_\nu \subset X$ for all $\nu \in \mathscr{I}$, and that $\mathbf{A}(v)$ and $\mathbf{B}(v)$ satisfy either the one- or the two-phase nondegeneracy condition in X.

Then, $(u_\nu)_{\nu\in\mathscr{I}}$ is relatively compact in $L^1_{\mathrm{loc}}(Q)$, and its limit points satisfy all requirements of a kinetic solution to (3.1) except for possibly the decay estimate (3.13). In particular, if X is bounded, then the limit points of $(u_\nu)_{\nu\in\mathscr{I}}$ are entropy solutions to (3.1).

We should observe that the condition (3.14) above is a quite recurrent a priori estimate in the theory of kinetic solutions; see, *e.g.*, G.-Q. Chen–B. Perthame [27], and B. Gess–M Hofmanová [51]. Additionally, notice that any relatively compact family in L^1_{loc} must be *a fortiori* uniformly integrable in the sense above. Regarding the case $X = \mathbb{R}$, see the next section.

In order to prove this theorem, let us firstly rephrase Theorems 2.2 and 2.4 in a more convenient fashion to our purposes.

Lemma 3.1. *Assume that $\mathbf{A}(v) \in \mathscr{C}^{2,\varepsilon}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N)$ and $\mathbf{B}(v) \in \mathscr{C}^{2,\varepsilon}_{\mathrm{loc}}(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$ for some $0 < \varepsilon \leq 1$. Let also $1 < \ell < 1+\varepsilon$, $1 < p < \infty$, $\mathscr{U} \subset \mathbb{R}_t \times \mathbb{R}^N_x$ be an open set, and $\mathscr{I}$ be an arbitrary index set.*

Suppose that there exist two families $(f_\nu)_{\nu\in\mathscr{I}}$ and $(g_\nu)_{\nu\in\mathscr{I}}$ such that

1. *$(f_\nu)_{\nu\in\mathscr{I}}$ is bounded in $L^p_{\mathrm{loc}}(\mathscr{U} \times \mathbb{R}_v)$,*
2. *$(g_\nu)_{\nu\in\mathscr{I}}$ is relatively compact in $L^p(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v)$, and*
3. *for every $\nu \in \mathscr{I}$, $f = f_\nu$ and $g = g_\nu$ solve the equation*

$$\frac{\partial f}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f - \mathbf{b}(v) : D^2_x f = (-\Delta_{t,x}+1)^{1/2}(-\Delta_v+1)^{\ell/2} g \tag{3.15}$$

in $\mathscr{D}'(\mathscr{U} \times \mathbb{R}_v)$, where $\mathbf{A}'(v) = \mathbf{a}(v)$ and $\mathbf{B}'(v) = \mathbf{b}(v)$.

Finally, let $\eta \in L^\infty(\mathbb{R})$ have compact support, and presume that $\mathbf{A}(v)$ and $\mathbf{B}(v)$ satisfy either the one- or the two-phase nondegeneracy condition in $\operatorname{supp}\eta$.

Then, the averages $(\int_{\mathbb{R}} f_\nu \eta\, dv)_{\nu\in\mathscr{I}}$ form a relatively compact set of $L^p_{\mathrm{loc}}(\mathscr{U})$. In particular, if $\mathscr{U}$ is of finite measure and $(f_\nu)_{\nu\in\mathscr{I}}$ is bounded in $L^q(\mathscr{U}\times\mathbb{R}_v)$ for some $1 < q \leq \infty$, $(\int_{\mathbb{R}} f_\nu \eta\, dv)_{\nu\in\mathscr{I}}$ is a relatively compact set of $L^r(\mathscr{U})$ for all $1 \leq r < q$.

As a further step, let us recall the following classical result in the theory of the Sobolev spaces. For the convenience of the reader, the proof will be provided.

Lemma 3.2. *Let $U \subset\subset \mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v$ be an open set, and $0 < s < 1$. Then, if $\mathfrak{M}(U)$ is the space of the Radon measures supported on U (endowed with the topology of the total variation), then*

$$\mathfrak{M}(U) \subset W^{-s,q}(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v) \text{ with compact injection}$$

for any $1 < q < \frac{N+2}{N+2-s}$.

Proof. The proof is based on the following "fractional" Morrey's theorem: *"If $r > (N+2)/s$, then $W^{s,r}(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v) \subset \mathscr{C}^\alpha(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v)$ with continuous injection for $\alpha = s - (N+2)/r$"*; see, *e.g.*, E. Di Nezza–G. Palatucci–E. Valdinocci [35]. Therefore, by restriction and the Arzelá–Ascoli theorem,

$$W^{s,r}(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v) \subset \mathscr{C}(\overline{U}) \text{ with compact injection,}$$

again, for $r > (N+2)/s$. Per the Schauder's theorem,

$$\mathscr{C}(\overline{U})^\star \subset W^{-s,r'}(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v) \text{ with compact injection.}$$

For $\mathfrak{M}(U) \subset \mathscr{C}(\overline{U})^\star$ with continuous injection, the desired result is thus obtained. □

Proof of Theorem 3.2. For $X \neq \mathbb{R}$, we can evidently suppose by some change of parameters that X is either a bounded interval of the form $[0, L]$ for some $L > 0$, or the half-line $[0,\infty)$.

Given any $\nu \in \mathscr{I}$, let $f_\nu(t,x,v) = \chi_{u_\nu(t,x)}(v)$, and denote by $\mathfrak{m}_\nu$ and $\mathfrak{n}_\nu$ its corresponding measures as stated in Definition 3.3. For any $\nu \in \mathscr{I}$, $1 \le p < \infty$, and $\phi \in \mathscr{C}_c^\infty(Q)$, identity (3.11) yields that

$$\begin{aligned}\int_Q \int_{\mathbb{R}} |\theta(t,x) f_\nu(t,x,v)|^p \, dvdxdt &= \int_Q |\theta(t,x)|^p |u_\nu(t,x)| \, dxdt \\ &\le \|\theta\|^p_{L^\infty(Q)} \|u_\nu\|_{L^1(\operatorname{supp}\theta)}. \end{aligned} \tag{3.16}$$

Observe that, as a consequence of (3.14), $(u_\nu)_{\nu\in\mathscr{I}}$ is bounded in $L^1_{\mathrm{loc}}(Q)$. As a result, $(f_\nu)_{\nu\in\mathscr{I}}$ is bounded in $L^p_{\mathrm{loc}}(Q\times\mathbb{R}_v)$ for all $1 \le p \le \infty$.

Step #1: (A priori estimates for $\mathfrak{m}_\nu$ and $\mathfrak{n}_\nu$). First of all, notice that, for all $\nu \in \mathscr{I}$, $f_\nu(t,x,v) = 0$ for $v < 0$; consequently, the nonegative measure satisfies $\frac{\partial}{\partial v}(\mathfrak{m}_\nu + \mathfrak{n}_\nu)(t,x,v) = 0$ in $Q \times (-\infty, 0)$. In virtue of the decay estimate (3.13), we conclude thus that $(\mathfrak{m}_\nu + \mathfrak{n}_\nu)(t,x,v)$ is supported on $Q \times [0,\infty)$.

Let us now thus bound $(\mathfrak{m}_\nu + \mathfrak{n}_\nu)(t,x,v)$. Given any $Q_0 \subset\subset Q$ and $R > 0$, pick functions $\theta \in \mathscr{C}_c^\infty(Q)$ and $\zeta \in \mathscr{C}^\infty(\mathbb{R}_v)$ such that

$$\begin{cases} \theta \text{ is nonnegative and } \theta \equiv 1 \text{ in } Q_0, \text{ and} \\ \dfrac{d\zeta}{dv} \in \mathscr{C}_c^\infty(\mathbb{R}),\ \dfrac{d\zeta}{dv} \le 0,\ \dfrac{d\zeta}{dv}(v) = -1 \text{ for } |v| < R, \text{ and } \lim\limits_{v\to\infty} \zeta(v) = 0. \end{cases}$$

Again, even though we cannot a priori plug $\varphi(t,x,v) = \theta(t,x)\zeta(v)$ as a test-function into (3.12), one may employ the classical argument of truncations to justify such choice due to the support of f_ν and $(\mathfrak{m}_\nu + \mathfrak{n}_\nu)$. Accordingly,

$$\begin{aligned}(\mathfrak{m}_\nu + \mathfrak{n}_\nu)(Q_0 \times (-R,R)) &\le -\int_Q \int_{\mathbb{R}} \theta \frac{d\zeta}{dv} \, d(\mathfrak{m}_\nu + \mathfrak{n}_\nu)(dt,dx,dv) \\ &= -\int_Q \int_{\mathbb{R}_v} f_\nu \zeta \left(\frac{\partial\theta}{\partial t} + \mathbf{a}(v)\cdot\nabla_x\theta + \mathbf{b}(v) : D_x^2\theta \right) dvdxdt \\ &\le C(\theta, R). \end{aligned} \tag{3.17}$$

Hence, both measures $\mathfrak{m}$ and $\mathfrak{n}$ are locally uniformly bounded. As a consequence, Lemma 3.2 asserts that, given any $\Pi \in \mathscr{C}_c^\infty(Q\times\mathbb{R}_v)$, $1 < \ell < 1+\varepsilon$ and $1 < p < \frac{N+2}{N+2-(\ell-1)}$,

$$\Pi(\mathfrak{m}_\nu + \mathfrak{n}_\nu)_{\nu\in\mathscr{I}} \text{ is relatively compact in } W^{-(\ell-1),p}(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v). \tag{3.18}$$

Step #2: (The localization procedure.) Let $Q_0 \subset\subset Q$, $\theta \in \mathscr{C}_c^\infty(Q)$, and $\zeta \in \mathscr{C}^\infty(\mathbb{R}_v)$ be as in the previous step. It is not difficult to see that $\mathfrak{f}_\nu(t,x,v) = \theta(t,x)\frac{d\zeta}{dv}(v) f_\nu(t,x,v)$ satisfies the equation

$$\frac{\partial \mathfrak{f}_\nu}{\partial t} + \mathbf{a}(v)\cdot\nabla_x \mathfrak{f}_\nu - \mathbf{b}(v) : D_x^2 \mathfrak{f}_\nu = \frac{\partial}{\partial v}\left(\theta \frac{d\zeta}{dv}(\mathfrak{m}_\nu + \mathfrak{n}_\nu)\right) - \theta \frac{d^2\zeta}{dv^2}(\mathfrak{m}_\nu + \mathfrak{n}_\nu) \tag{3.19}$$

in $\mathscr{D}'(Q_0\times\mathbb{R}_v)$. In virtue of Theorem 2.5 and (3.18), it follows that the forcing term in the equation above may be written as

$$\frac{\partial}{\partial v}\left(\theta \frac{d\zeta}{dv}(\mathfrak{m}_\nu + \mathfrak{n}_\nu)\right) - \theta \frac{d^2\zeta}{dv^2}(\mathfrak{m}_\nu + \mathfrak{n}_\nu) = (-\Delta_{t,x} + 1)^{1/2}(-\Delta_v + 1)^{\ell/2} g_\nu,$$

where g_ν is relatively compact in $L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v)$.

Therefore, choosing $\mathscr{U} = Q_0$, Lemma 3.1 assures us that $\int_{\mathbb{R}_v} \mathfrak{f}_\nu 1_{X\cap(-R,R)} \, dv = -\int_{-R}^R f_\nu \, dv$ is relatively compact in $L^r(Q_0)$ for any $1 \le r < \infty$; that is,

$$\int_{-R}^{R} f_\nu \, dv \text{ is relatively compact in } L^r_{\mathrm{loc}}(Q) \tag{3.20}$$

for all $1 \le r < \infty$.

Step #3: (Conclusion.) According to (3.8), the hypothesis of uniform integrability (3.14) yields that, for any $\theta \in \mathscr{C}_c^\infty(Q)$,

$$\int_Q \int_{\mathbb{R}} \int_{|v|>R} |f_\nu(t,x,v)| \theta(t,x) \, dv dx dt \to 0_+ \text{ uniformly as } R \to \infty \text{ for all } \nu \in \mathscr{I}. \tag{3.21}$$

Consequently, due to (3.9), the trivial decomposition

$$\theta(t,x) u_\nu(t,x) = \theta(t,x) \int_{\mathbb{R}} f_\nu(t,x,v)\, dv = \theta(t,x) \left\{ \int_{-R}^{R} + \int_{|v|>R} \right\} f_\nu(t,x,v)\, dv$$

infused with (3.20) and (3.21) yields that $(\theta u_\nu)_{\nu\in\mathscr{I}}$ is totally bounded in $L^1(Q)$. Therefore, indeed $(u_\nu)_{\nu\in\mathscr{I}}$ is relatively compact in $L^1_{\mathrm{loc}}(Q)$.

Lastly, notice that (3.10) implies the transformation $u \mapsto \chi_u$ is an isometry between $L^1(Q_0)$ and $L^1(Q_0 \times \mathbb{R}_v)$ for all $Q_0 \subset \mathbb{R}_t \times \mathbb{R}^N_x$. This evidently shows that the relative compactness of $(u_\nu)_{\nu\in\mathscr{I}}$ in $L^1_{\mathrm{loc}}(Q)$ is equivalent to the relative compactness of the corresponding χ–functions $(f_\nu)_{\nu\in\mathscr{I}}$ in $L^1_{\mathrm{loc}}(Q; L^1(\mathbb{R}_v))$. As a result, for it was shown that the set $(\mathfrak{m}_\nu + \mathfrak{n}_\nu)_{\nu\in\mathscr{I}}$ is bounded in the topology of the σ-finite positive measures, some elementary weak convergence arguments may be applied so as to confirm the claim on the limit points of $(u_\nu)_{\nu\in\mathscr{I}}$. Particularly, if X is bounded, Proposition 3.1 would guarantee that the limit points of $(u_\nu)_{\nu\in\mathscr{I}}$ are indeed entropy solutions to Equation (3.15), as $(\mathfrak{m}_\nu + \mathfrak{n}_\nu)_{\nu\in\mathscr{I}}$ would then be supported on $Q \times [-L, L]$. The theorem is hereby proven. □

Remark 3.4. Let us mention that, even though we could have not established in general the decay estimate (3.13) from our arguments alone, in practice such a property can be easily established from a strengthened, global version of (3.14) and some other particular structure of $(u_\nu)_{\nu\in\mathscr{I}}$; see, *e.g.*, G.-Q. Chen–Perthame [27], B. Gess–M Hofmanová [51], and Subsection 3.3.4 below.

There exists, however, a condition somewhat weaker than the boundedness of X that guarantees that the limit points of $(u_\nu)_{\nu\in\mathscr{I}}$ are indeed kinetic solutions: *If* $(\mathbf{A}(u_\nu))_{\nu\in\mathscr{I}}$ *and* $(\mathbf{B}(u_\nu))_{\nu\in\mathscr{I}}$ *are bounded in, respectively,* $L^1_{\mathrm{loc}}(Q;\mathbb{R}^N)$ *and* $L^1_{\mathrm{loc}}(Q;\mathscr{L}(\mathbb{R}^N))$, *then verily* (3.13) *holds.* This may be seen using test functions of the form $\varphi(t,x)\zeta(v)$, where $\zeta(v)$ is an appropriated mollification and truncation of the Heaviside functions $v \mapsto 1_{(k,\infty)}(v)$ and $v \mapsto 1_{(-\infty,k)}(v)$ with $k \in \mathbb{R}$. We should mention that such an L^1_{loc}–boundedness condition is heavily featured in the works of E. Yu. Panov [92, 93, 94, 95, 96] and H. Holden *et al.* [63]; see also Subsections 3.3.3 below.

Incidentally, (3.14) is equivalent to the next more classical uniform integrability condition: *"For all* $Q_0 \subset\subset Q$*, there exists some function* $\lambda_{Q_0} : (0,\infty) \to \mathbb{R}$ *such that* $\lambda_{Q_0}(A) \to 0_+$ *as* $A \to \infty$*, and*

$$\int_{\{(t,x)\in Q_0; |u_\nu(t,x)|>A\}} |u_\nu(t,x)| \, dx dt \le \lambda_{Q_0}(A) \text{ for all } \nu \in \mathscr{I} \text{ and } A > 0\text{"}. \tag{3.22}$$

What is more, according to the so-called "de la Vallée Poussin criterion", the uniform integrability condition (3.22) is also equivalent to the following assertion: *"For all* $Q_0 \subset\subset Q$*, there exists some increasing, convex real function* $\phi_{Q_0} : [0,\infty) \to [0,\infty)$ *such that*

$$\begin{cases} \phi_{Q_0}(0) = 0, \\ \lim\limits_{v\to\infty} \dfrac{\phi_{Q_0}(v)}{v} = \infty, \text{ and} \\ \big(\phi_{Q_0}(|u_\nu|)\big)_{\nu\in\mathscr{I}} \text{ is bounded in } L^1(Q_0)\text{"}. \end{cases}$$

In particular, (3.14) holds if $(u_\nu)_{\nu\in\mathscr{I}}$ is bounded in $L^q_{\mathrm{loc}}(Q)$ for some $1 < q \le \infty$. This evidently leads to another generalization of Theorem 3.1.

Remark 3.5. In the one-phase case, it is remarkable that only an L^2–version of Lemma 3.1 is necessary to prove Theorem 3.2. The following interpolation argument is inspired by a previous work of G.-Q. Chen–H. Frid [23]; see also W. Neves [86], M. Lazar–D. Mitrovic [78], and H. Frid *et al.* [43].

Keep the notations of the proof of Theorem 3.2—including that of X being either $[0, L]$ or $[0, \infty)$, $Q_0 \subset\subset Q$, $\theta \in \mathscr{C}^\infty_c(Q)$, $\zeta \in \mathscr{C}^\infty(\mathbb{R}_v)$, and $\mathfrak{f}_\nu$—, and let $M \subset \mathbb{R}^N$ be the minimal subspace such that of $R(\mathbf{b}(v)) \subset M$ for all $v \in \mathbb{R}$.

Given any $\Pi \in \mathscr{C}^\infty_c(Q\times\mathbb{R})$, one may apply Theorem 2.5 to (3.18) so as to deduce that

$$\Pi(\mathfrak{m}_\nu + \mathfrak{n}_\nu) = \Big\{\big(-\Delta_{t,x}+1\big)^{1/2} - \Delta_M\Big\}\big(-\Delta_v+1\big)^{\varepsilon/4} h^{(\theta)}_\nu \tag{3.23}$$

where $(h^{(\theta)}_\nu)_{\nu\in\mathscr{I}}$ is relatively compact in $L^p(\mathbb{R}_t\times\mathbb{R}^N_x\times\mathbb{R}_v)$ for $1 < p < \frac{N+2}{N+2-\varepsilon/2}$.

On the other hand, since $\operatorname{supp}(\mathfrak{m}_\nu+\mathfrak{n}_\nu) \subset Q\times[0,\infty)_v$ for all $\nu\in\mathscr{I}$, one may easily justify the formula

$$\begin{aligned}(\mathfrak{m}_\nu+\mathfrak{n}_\nu)(t,x,v) = {}& \frac{\partial}{\partial t}\Big(\int_0^v f_\nu(t,x,w)\,dw\Big) + \operatorname{div}_x\Big(\int_0^v f_\nu(t,x,w)\mathbf{a}(w)\,dw\Big)\\ & - D^2_x : \Big(\int_0^v f_\nu(t,x,w)\mathbf{b}(w)\,dw\Big) \quad \text{in the sense of } \mathscr{D}'(Q\times\mathbb{R}_v)\end{aligned} \tag{3.24}$$

for all $\nu\in\mathscr{I}$. Comparing (3.24) with (3.23), one can verify that $(h^{(\theta)}_\nu)_{\nu\in\mathscr{I}}$ is also uniformly bounded in $L^p(\mathbb{R}_t\times\mathbb{R}^N_x\times\mathbb{R}_v)$ for any $1<p<\infty$. Therefore, by the interpolation inequality, we deduce that $(h^{(\theta)}_\nu)_{\nu\in\mathscr{I}}$ is relatively compact in $L^2(\mathbb{R}_t\times\mathbb{R}^N_x\times\mathbb{R}_v)$.

Accordingly, (3.19) can be written thus as

$$\frac{\partial\mathfrak{f}_\nu}{\partial t} + \mathbf{a}(v)\cdot\nabla_x\mathfrak{f}_\nu - \mathbf{b}(v) : D^2_x\mathfrak{f}_\nu = \Big\{\big(-\Delta_{t,x}+1\big)^{1/2} - \Delta_M\Big\}\big(-\Delta_v+1\big)^{(1+\varepsilon/2)/2} g^{(\theta)}_\nu$$

in $\mathscr{D}'(Q_0\times\mathbb{R}_v)$, where $g^{(\theta)}_\nu$ is relatively compact in $L^2(\mathbb{R}_t\times\mathbb{R}^N_x\times\mathbb{R}_v)$. Consequently, Theorem 2.4 confirms that the averages $\int_{(-R,R)\cap X}\mathfrak{f}_\nu\,dv$ are relatively compact in $L^2_{\mathrm{loc}}(Q)$ for any $R>0$. The rest of the proof is now exactly as before.

Remark 3.6 (Non-Lipschitz coefficients, and a "real analytic nondegeneracy condition"). We could have also assumed in Theorem 3.2 that $\mathbf{A}(v)$ and $\mathbf{B}(v)$ belonged to some class of non-Lipschitz functions; see Remark 2.18. In a similar vein, our results would hold under the following conditions on $\mathbf{A}(v)$ and $\mathbf{B}(v)$:

- There exists a closed set $G\subset\mathbb{R}$ of zero measure, such that
  - $\mathbf{A}\in W^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N)\cap\mathscr{C}^{2,\varepsilon}_{\mathrm{loc}}(\mathbb{R}\setminus G;\mathbb{R}^N)$ for some $0<\varepsilon\le 1$, and
  - $\mathbf{B}\in W^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathscr{L}(\mathbb{R}^N))\cap\mathscr{C}^\infty_{\mathrm{loc}}(\mathbb{R}\setminus G;\mathscr{L}(\mathbb{R}^N))$, with $\mathbf{B}'(v)$ being nonnegative and real analytic outside G.
- If $\mathcal{L}(i\tau,i\kappa,v)$ denotes the symbol $\mathcal{L}(i\tau,i\kappa,v) = i(\tau+\mathbf{A}'(v)\cdot\kappa)+\kappa\cdot\mathbf{B}'(v)\kappa$, then

$$\operatorname{meas}\big\{v\in\mathbb{R};\mathcal{L}(i\tau,i\kappa,v)=0\big\} = 0\ \forall(\tau,\kappa)\in\mathbb{R}\times\mathbb{R}^N \text{ with } \tau^2+|\kappa|^2=1.$$

In this case, it suffices to substitute Lemma 3.1 with Theorem 2.9. Notice that, because $\mathbf{A}'(v)$ and $\mathbf{B}'(v)$ may possess discontinuities in such an averaging lemma, one may modify those coefficients as one wishes outside any interval of interest.

3.3 Generalizations

3.3.1 Equations with source terms

Assume that one adds a source term to the right-hand side of Equation (3.1), transforming it into

$$\frac{\partial u}{\partial t}(t,x) + \operatorname{div}_x \mathbf{A}(u(t,x)) - D_x^2 : \mathbf{B}(u(t,x)) = \mathbf{S}(t,x,u(t,x)) \tag{3.25}$$

where $\mathbf{S}(t,x,v)$ belongs to, say, $\mathscr{C}(Q \times \mathbb{R}_v)$. In this case, the definition of entropy solution would be almost the same as Definition 3.1, but one would need to replace (3.2) with

$$\frac{\partial}{\partial t}\eta(u) + \operatorname{div}_x \mathbf{A}^\eta(u) - D_x^2 : \mathbf{B}^\eta(u) = \mathbf{S}(t,x,u)\eta'(u) - (\mathfrak{m}^{\eta''} + \mathfrak{n}^{\eta''}) \text{ in } \mathscr{D}'(Q).$$

Likewise, the definition of kinetic solution would be identical to Definition 3.3, except that (3.12) should now read

$$\frac{\partial f}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f - \mathbf{b}(v) : D_x^2 f = \mathbf{S}(t,x,v)\delta_u(v) + \left(\frac{\partial \mathfrak{m}}{\partial v} + \frac{\partial \mathfrak{n}}{\partial v}\right).$$

Notice that $\mathbf{S}(t,x,v)\delta_u(v)$ defines a locally finite measure in $Q \times \mathbb{R}_v$. Therefore, one can easily deduce the following extension to Theorem 3.2.

Theorem 3.3. *Assume that* $\mathbf{A}(v) \in \mathscr{C}^{2,\varepsilon}_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$ *and* $\mathbf{B}(v) \in \mathscr{C}^{2,\varepsilon}_{\text{loc}}(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$ *for some* $0 < \varepsilon \le 1$. *Let* $\mathscr{I}$ *be an arbitrary index set, and consider some* $\mathbf{S} \in \mathscr{C}(Q \times \mathbb{R})$.

Suppose that $(u_\nu)_{\nu\in\mathscr{I}}$ *is a family of kinetic solutions to* (3.25) *in* Q *that enjoys the following uniform integrability property: for all* $Q_0 \subset\subset Q$, *there exists some function* $\lambda_{Q_0} : (0,\infty) \to \mathbb{R}$ *such that* $\lambda_{Q_0}(A) \to 0_+$ *as* $A \to \infty$, *and*

$$\int_{Q_0} \Big[\big(u_\nu(t,x) - A\big)_+ + \big(A - u_\nu(t,x)\big)_-\Big]\, dxdt \le \lambda_{Q_0}(A) \text{ for all } \nu \in \mathscr{I} \text{ and } A > 0.$$

Finally, presume that there exists an interval $X \neq \mathbb{R}$ *such that* $\operatorname{ess\,ran} u_\nu \subset X$ *for all* $\nu \in \mathscr{I}$, *and that* $\mathbf{A}(v)$ *and* $\mathbf{B}(v)$ *satisfy either the one- or the two-phase nondegeneracy condition in* X.

Then, $(u_\nu)_{\nu\in\mathscr{I}}$ *is relatively compact in* $L^1_{\text{loc}}(Q)$, *and its limit points satisfy all requirements of a kinetic solution to* (3.25) *except for possibly the decay estimate* (3.13). *In particular, if* X *is bounded, then the limit points of* $(u_\nu)_{\nu\in\mathscr{I}}$ *are entropy solutions to* (3.25).

3.3.2 Equations with varying coefficients

Let us now explore a scenario where the coefficients $\mathbf{A}(v)$ and $\mathbf{B}(v)$ may depend on the indices ν. This situation naturally appears when one employs the vanishing viscosity method so as to establish the existence of entropy solutions (see Remark 3.1).

First, let us introduce some natural restrictions to our analysis. Assume that $\mathscr{I} = (0,1)$, and, for every $\nu \in \mathscr{I}$, u_ν is a kinetic solution to

$$\frac{\partial u_\nu}{\partial t}(t,x) + \operatorname{div}_x \mathbf{A}_\nu(u_\nu(t,x)) - D_x^2 : \mathbf{B}_\nu(u_\nu(t,x)) = 0, \tag{3.26}$$

where $\mathbf{A}_\nu \in \mathscr{C}^1(\mathbb{R};\mathbb{R}^N)$, and $\mathbf{B}_\nu \in \mathscr{C}^1(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$ is such that $\mathbf{B}_\nu'(v) \ge 0$ everywhere.

Definition 3.4. With the notation above, we say that $\mathbf{A}_\nu(v)$ and $\mathbf{B}_\nu(v)$ *converge viscously* to respectively $\mathbf{A}(v) \in \mathscr{C}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$ and $\mathbf{B} \in \mathscr{C}^1_{\text{loc}}(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$ if the following conditions hold.

1. $(\mathbf{A}_\nu')_{0<\nu<1}$ and $(\mathbf{B}_\nu')_{0<\nu<1}$ are bounded in, respectively, $L^\infty_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$ and $L^\infty_{\text{loc}}(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$.
2. $(\mathbf{A}_\nu)'(v) \to \mathbf{A}'(v)$ for all $v \in \mathbb{R}$ as $\nu \to 0_+$.

3. $(\mathbf{B}_\nu)'(v) \to \mathbf{B}'(v)$ uniformly on compact sets of the real line as $\nu \to 0_+$.
4. If M is the minimal linear subspace of $\mathbb{R}^N$ such that $R(\mathbf{B}'_\nu(v)) \subset M$ for all $v \in \mathbb{R}$, then $P_{M^\perp}(\mathbf{B}_\nu)'(v)P_M \equiv 0$ for every $v \in \mathbb{R}$ and $0 < \nu < 1$.

Remark 3.7. Essentially 4. means that, if $\mathbf{B}'(v)$ is everywhere a block matrix

$$\mathbf{B}'(v) = \left(\begin{array}{c|c} \mathbf{b}(v) & 0_{m\times n} \\ \hline 0_{n\times m} & 0_{n\times n} \end{array}\right),$$

where m are n are constant nonnegative integers, then $(\mathbf{B}_\nu)'(v)$ has the form

$$(\mathbf{B}_\nu)'(v) = \left(\begin{array}{c|c} \mathbf{b}^{(1)}_\nu(v) & 0_{m\times n} \\ \hline 0_{n\times m} & \mathbf{b}^{(2)}_\nu(v) \end{array}\right).$$

In a more abstract language, $(\mathbf{B}_\nu)'(v) = P_M(\mathbf{B}_\nu)'(v)P_M + P_{M^\perp}(\mathbf{B}_\nu)'(v)P_{M^\perp}$. Notice that, if $M = \mathbb{R}^N$ or $M = \{0\}$, the definition above brings no restriction into the form of $\mathbf{B}'_\nu$.

Evidently, a simple and common example of a viscous convergence is $\mathbf{A}_\nu(v) = \mathbf{A}(v)$ and $\mathbf{B}_\nu(v) = \mathbf{B}_\nu(v) + \nu v I_{\mathbb{R}^N}$.

The next theorem partially enhances a stability result proposed in P.-L. Lions–B. Perthame–E. Tadmor [82].

Theorem 3.4. *Let $Q \subset \mathbb{R}_t \times \mathbb{R}^N_x$ be an open set, and let $(u_\nu)_{0<\nu<1}$ be a family of kinetic solutions to (3.26) in Q, such that, for every $0 < \nu < 1$, $\mathbf{A}_\nu(v) \in \mathscr{C}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$ and $\mathbf{B}_\nu(v) \in \mathscr{C}^1_{\text{loc}}(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$ with $(\mathbf{B}_\nu)'(v) \geq 0$ for all $v \in \mathbb{R}$. Suppose that $\mathbf{A}_\nu(v)$ and $\mathbf{B}_\nu(v)$ converge viscously to respectively $\mathbf{A} \in \mathscr{C}^{2,\varepsilon}_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$ and $\mathbf{B} \in \mathscr{C}^{2,\varepsilon}_{\text{loc}}(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$, where $0 < \varepsilon \leq 1$.*

Assume that $(u_\nu)_{0<\nu<1}$ enjoys the following uniform integrability property: for all $Q_0 \subset\subset Q$, there exists some function $\lambda_{Q_0} : (0,\infty) \to \mathbb{R}$ such that $\lambda_{Q_0}(A) \to 0_+$ as $A \to \infty$, and

$$\int_{Q_0} \Big[\big(u_\nu(t,x) - A\big)_+ + \big(A - u_\nu(t,x)\big)_-\Big]\,dxdt \leq \lambda_{Q_0}(A) \text{ for all } 0 < \nu < 1 \text{ and } A > 0.$$

Finally, presume that there exists an interval set $X \neq \mathbb{R}$ such that $\operatorname{ess\,ran} u_\nu \subset X$ for all $0 < \nu < 1$, and that $\mathbf{A}(v)$ and $\mathbf{B}(v)$ satisfy either the one- or the two-phase nondegeneracy condition in X.

Then, for any sequence $0 < \nu_n < 1$ with $\nu_n \to 0_+$, $(u_{\nu_n})_{n\in\mathbb{N}}$ is relatively compact in $L^1_{\text{loc}}(Q)$. Furthermore, the limit points of $(u_{\nu_n})_{n\in\mathbb{N}}$ are satisfy all requirements of a kinetic solution to (3.1) with viscous limit coefficients $\mathbf{A}(v)$ and $\mathbf{B}(v)$, except for possibly the decay estimate (3.13). In particular, if X is bounded, then the limit points of $(u_{\nu_n})_{n\in\mathbb{N}}$ are entropy solutions to (3.1).

Before we prove this proposition, let us investigate some of the corollaries of the chain rule for kinetic solutions.

Remark 3.8. Assume that $u \in L^1_{\text{loc}}(Q)$ is a kinetic solution to (3.1), and let $f = \chi_u$ be its χ–function. It is not hard to verify that, for any ϕ and $\zeta \in \mathscr{C}_c(\mathbb{R}_v)$, one has that

$$\begin{cases} \operatorname{div}_x\Big(\displaystyle\int_0^{u(t,x)} \phi(v)\sigma(v)\,dv\Big) = \operatorname{div}_x\Big(\displaystyle\int_{\mathbb{R}} f(t,x,v)\phi(v)\sigma(v)\,dv\Big) \in L^2_{\text{loc}}(Q;\mathbb{R}^N), \text{ and} \\ \operatorname{div}_x\Big(\displaystyle\int_0^{u(t,x)} \phi(v)\zeta(v)\sigma(v)\,dv\Big) = \phi(u(t,x))\operatorname{div}_x\Big(\displaystyle\int_0^{u(t,x)} \zeta(v)\sigma(v)\,dv\Big) \in L^2_{\text{loc}}(Q;\mathbb{R}^N). \end{cases} \tag{3.27}$$

This improvement of the chain rule has three important consequences.

(i) (Chain rule for a discontinuous function). Let $\phi \in \mathscr{C}_c(\mathbb{R}_v)$, and $\xi \in \mathbb{R}$ be arbitrary. Then, by approximating $w \in \mathbb{R} \mapsto 1_{(-\infty,\xi)}(w)$ by uniformly bounded smooth functions that converge pointwisely, it can be shown that

$$\operatorname{div}_x\Big(\int_{-\infty}^{\xi} f(t,x,v)\phi(v)\sigma(v)\,dv\Big) = 1_{(-\infty,\xi)}\big(u(t,x)\big)\operatorname{div}_x\Big(\int_{\mathbb{R}} f(t,x,v)\phi(v)\sigma(v)\,dv\Big) \in L^2_{\mathrm{loc}}(Q)$$

(indeed, apply Lemma 2.2 and dominated convergence theorem). Therefore, given any $Q_0 \subset\subset Q$,

$$\Big\|\operatorname{div}_x\Big(\int_{-\infty}^{\xi} f\phi\sigma\,dv\Big)\Big\|_{L^2(Q_0)} \le \Big\|\operatorname{div}_x\Big(\int_{\mathbb{R}} f\phi\sigma\,dv\Big)\Big\|_{L^2(Q_0)}.$$

In particular,

$$\Big\|\operatorname{div}_x\Big(\int_{-\infty}^{\xi} f\phi\sigma\,dv\Big)\Big\|^2_{L^2(Q_0)} \le \int_{Q_0\times\mathbb{R}} \phi(v)^2\,\mathfrak{n}(dt,dx,dv).$$

(The very same reasoning infused with some weak convergence arguments permits one to conclude that

$$\operatorname{div}_x\Big(\int_{\mathbb{R}} f(t,x,v)\Lambda(v)\phi(v)\sigma(v)\,dv\Big) \in L^2_{\mathrm{loc}}(Q),$$

where $\Lambda \in L^\infty(\mathbb{R}_v)$; furthermore,

$$\Big\|\operatorname{div}_x\Big(\int_{\mathbb{R}} f\Lambda\phi\sigma\,dv\Big)\Big\|_{L^2(Q_0)} \le \|\Lambda\|_{L^\infty(\mathbb{R}_v)}\Big\|\operatorname{div}_x\Big(\int_{\mathbb{R}} f\phi\sigma\,dv\Big)\Big\|_{L^2(Q_0)}$$

for any $Q_0 \subset\subset Q$.)

(ii) (Chain rule for matrix functions). For $\Phi \in \mathscr{C}_c(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$ and $\zeta \in \mathscr{C}_c(\mathbb{R}_v)$, it holds that

$$\operatorname{div}_x\Big(\int_0^u \zeta(v)\Phi(v)\sigma(v)\,dv\Big) = \Phi(u)\operatorname{div}_x\Big(\int_0^u \zeta(v)\sigma(v)\,dv\Big) \in L^2_{\mathrm{loc}}(Q;\mathbb{R}^N), \tag{3.28}$$

and, again,

$$\Big\|\operatorname{div}_x\Big(\int_{\mathbb{R}} f\Phi\zeta\sigma\,dv\Big)\Big\|^2_{L^2(Q_0)} \le C_N\int_{Q_0\times\mathbb{R}} \|\Phi(u(t,x))\|^2_{\mathscr{L}(\mathbb{R}^N)}\zeta(v)^2\,\mathfrak{n}(dt,dx,dv).$$

(This is probably more easily seen using coordinates.)

(iii) Mingling the ideas of the last two remarks, we may deduce that, for every $\Phi \in \mathscr{C}_c(\mathbb{R}_v;\mathscr{L}(\mathbb{R}^N))$, $\zeta \in \mathscr{C}_c(\mathbb{R}_v)$, and $\xi \in \mathbb{R}$,

$$\operatorname{div}_x\Big(\int_{-\infty}^{\xi} \zeta f\Phi\sigma\,dv\Big) = \Phi(u)1_{(-\infty,\xi)}(u)\operatorname{div}_x\Big(\int_0^u \zeta\sigma\,dv\Big) \in L^2_{\mathrm{loc}}(Q;\mathbb{R}^N),$$

and

$$\Big\|\operatorname{div}_x\Big(\int_{-\infty}^{\xi} f\zeta\Phi\sigma\,dv\Big)\Big\|^2_{L^2(Q_0)} \le C_N\int_{Q_0\times\mathbb{R}} \|\Phi(u(t,x))\|^2_{\mathscr{L}(\mathbb{R}^N)}\zeta(v)^2\,\mathfrak{n}(dt,dx,dv). \tag{3.29}$$

Proof of Theorem 3.4. Given any sequence $\nu_n \to 0_+$, the crux of the proof is essentially to write for $f_{\nu_n} = \chi_{u_{\nu_n}(t,x)}$

$$\begin{aligned}\frac{\partial f_{\nu_n}}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f_{\nu_n} - \mathbf{b}(v) : D_x^2 f_{\nu_n} &= \Big(\frac{\partial \mathfrak{m}_{\nu_n}}{\partial v} + \frac{\partial \mathfrak{n}_{\nu_n}}{\partial v}\Big)\\ &\quad + \Big\{(\mathbf{a}(v) - \mathbf{a}_\nu(v))\cdot\nabla_x f_{\nu_n} - (\mathbf{b}(v) - \mathbf{b}_\nu(v)) : D_x^2 f_{\nu_n}\Big\},\end{aligned} \tag{3.30}$$

where

$$\begin{cases} \mathbf{a}(v) = \mathbf{A}'(v), & \mathbf{b}(v) = \mathbf{B}'(v), \\ \mathbf{a}_\nu(v) = (\mathbf{A}_\nu)'(v), \text{ and } & \mathbf{b}_\nu(v) = (\mathbf{B}_\nu)'(v), \end{cases}$$

and demonstrate that the last term in (3.30) is a disappearing perturbation as $\nu \to 0_+$. Given $Q_0 \subset\subset Q$ and $R > 0$, choose some $\theta \in \mathscr{C}_c^\infty(Q)$ such that $\theta \equiv 1$ in Q_0, and $\psi \in \mathscr{C}_c^\infty(\mathbb{R}_v)$ such that $\psi \equiv 1$ in $(-R, R)$. Henceforth, put $\varphi(t,x,v) = \theta(t,x)\psi(v)^3$.

Step #1: The one-phase case. Let us assume initially that $\mathbf{A}(v)$ and $\mathbf{B}(v)$ satisfy the one-phase condition in X.

By letting $\mathfrak{f}_\nu(t,x,v) = \varphi(t,x,v)f_\nu(t,x,v)$, we see that, repeating the same arguments of Theorem 3.2,

$$\begin{aligned} \frac{\partial \mathfrak{f}_{\nu_n}}{\partial t} + \mathbf{a}(v) \cdot \nabla_x f_{\nu_n} - \mathbf{b}(v) : D_x^2 f_{\nu_n} &= (-\Delta_{t,x} + 1)(-\Delta_v + 1)^{\ell/2} g_{\nu_n} \\ &+ \varphi\Big((\mathbf{a}(v) - \mathbf{a}_\nu(v)) \cdot \nabla_x f_{\nu_n} - (\mathbf{b}(v) - \mathbf{b}_\nu(v)) : D_x^2 f_{\nu_n}\Big) \end{aligned} \tag{3.31}$$

in $\mathscr{D}'(Q_0 \times \mathbb{R}_v)$, where ℓ is, say, $1 + \varepsilon/2$, and $p = \frac{N+2}{N+2-\varepsilon/4}$, and $(g_{\nu_n})_{n\in\mathbb{N}}$ is a relatively sequence in $L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v)$.

Furthermore, we may analyze the last terms in the right-hand side of (3.31) similarly to how we did in Theorem 2.2. For instance, notice that

$$\varphi(\mathbf{a}(v) - \mathbf{a}_\nu(v)) \cdot \nabla_x f_{\nu_n} = \theta \mathrm{div}_x(\psi^3 f_{\nu_n}(\mathbf{a}(v) - \mathbf{a}_{\nu_n}(v))),$$

whence hypothesis 1. in Definition 3.4 yields that

$$\varphi(\mathbf{a}(v) - \mathbf{a}_\nu(v)) \cdot \nabla_x f_{\nu_n} = (-\Delta_{t,x} + 1)^{1/2} J_{\nu_n}, \tag{3.32}$$

with $J_{\nu_n} \to 0$ in $L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v)$.

The parabolic terms arising in the right-hand side of (3.31) are somewhat more difficult to investigate. As in Remark 3.7, let us decompose $\mathbf{b}_{\nu_n}$ into two blocks $P_M \mathbf{b}_{\nu_n} P_M + P_{M^\perp} \mathbf{b}_{\nu_n} P_{M^\perp}$, so that its square-root σ_{ν_n} is likewise of the form $\sigma_{\nu_n} = P_M \sigma_{\nu_n} P_M + P_{M^\perp} \sigma_{\nu_n} P_{M^\perp}$. Accordingly, defining $\sigma_{\nu_n}^{(M^\perp)} \overset{\text{def}}{=} P_{M^\perp} \sigma_{\nu_n} P_{M^\perp}$, one has that

1. $\sigma_{\nu_n}(v)^{(M^\perp)}(v) \to P_{M^\perp}\sigma(v)P_{M^\perp} = 0$ uniformly in compact sets of the real line, and,
2. per (3.28), $\mathrm{div}_x \int_0^{u_{\nu_n}} \psi(v)\sigma_{\nu_n}^{(M^\perp)}(v)\,dv = \mathrm{div}_x \int_0^{u_{\nu_n}} \psi(v)P_{M^\perp}\sigma_{\nu_n}(v)\,dv$ is uniformly bounded in $L^2_{\mathrm{loc}}(Q;\mathbb{R})$. (This is where it was necessary to impose Condition 3. in Definition 3.4.)

Hence, $\varphi\big(\mathbf{b}(v) - \mathbf{b}_\nu(v)\big) : D_x^2 f_{\nu_n} = (I)_n + (II)_n$, where

$$\begin{cases} (I)_n = \theta\psi^2 : D_x^2 : \big(\psi(v)(\mathbf{b}(v) - P_M \mathbf{b}_\nu(v) P_M) f_{\nu_n}\big), \text{ and} \\ (II)_n = (\theta\psi) D_x^2 : \big((\psi(v)\sigma_{\nu_n}^{(M^\perp)}(v))^2 f_{\nu_n}\big). \end{cases} \tag{3.33}$$

It is not difficult to see that

$$(I)_n = (-\Delta_M + 1)K_{\nu_n} \tag{3.34}$$

for some K_{ν_n} converging to 0 in $L^r(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v)$. On the other hand, let us transform $(II)_n$ into

$$(II)_n = \theta\psi(\mathrm{div}_x)(\mathrm{div}_x)\Big(\frac{\partial}{\partial v}\Big)\Big(\int_0^v \big(\psi(w)\sigma_{\nu_n}^{(M^\perp)}(w)\big)^2 f_{\nu_n}(t,x,w)\,dw\Big).$$

Since $\psi(v)\sigma_{\nu_n}^{(M^\perp)}(v) \to 0$ uniformly, the chain rule estimate (3.29) asserts that

$$(II)_n = (-\Delta_x + 1)^{1/2}(-\Delta_v + 1)^{1/2} L_{\nu_n}, \tag{3.35}$$

where L_{ν_n} converges to 0 in $L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v)$.

In this fashion, the fusion of (3.31)–(3.35) proves that $\varphi f_{\nu_n} = \mathfrak{f}_{\nu_n}$ satisfies

$$\frac{\partial \mathfrak{f}_{\nu_n}}{\partial t} + \mathbf{a}(v)\cdot\nabla_x \mathfrak{f}_{\nu_n} - \mathbf{b}(v) : D_x^2 \mathfrak{f}_{\nu_n} = \Big\{\big(-\Delta_{t,x}+1\big)^{1/2} - \Delta_M\Big\}\big(-\Delta_v+1\big)^{(1+\varepsilon/2)/2} g_{\nu_n}$$

where $(g_{\nu_n})_{n\in\mathbb{N}}$ is relatively compact in $L^p(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v)$. Hence, Theorem 2.4 implies that $\int_{\mathbb{R}} \mathfrak{f}_{\nu_n} 1_{X\cap(-R,R)}$ is relatively compact in $L^p(Q_0)$. Since $Q_0 \subset\subset Q$ and $R>0$ were arbitrary, the rest of the proof is identical to the proof of Theorem 3.2.

Step #2: The two-phase case. Let us duplicate this investigation under hypothesis of $\mathbf{A}(v)$ and $\mathbf{B}(v)$ observe the two-phase condition in X.

Essentially, the unique modification one needs to perform is the following. Let $\xi_\delta(v)$ be functions as in the proof of Lemma 2.1; *i.e.*,

1. each $\xi_\delta \in \mathscr{C}^\infty(\mathbb{R}_v)$ for all $0<\delta<1$, with $0 \le \xi_\delta(v) \le 1$ everywhere,
2. ξ_δ vanishes near $F =$ boundary of $\big\{v\in\mathbb{R}; \mathbf{b}(v)=0\big\}$, and
3. $\xi_\delta(v) \to 1_{\mathbb{R}\setminus F}(v)$ for all $v\in\mathbb{R}$ as $\delta\to 0_+$.

Define now $\varphi(t,x,v) = \varphi_\delta(t,x,v) = \theta(t,x)\psi(v)^3\xi_\delta(v)$, so that the ideas behind Theorem 2.2 assert that $\mathfrak{f}_{\nu_n}^{(\delta)} = \varphi_\delta f_{\nu_n}$ obeys an equation of the form

$$\begin{aligned}\frac{\partial \mathfrak{f}_{\nu_n}^{(\delta)}}{\partial t} + \mathbf{a}(v)\cdot\nabla_x \mathfrak{f}_{\nu_n}^{(\delta)} - \mathbf{b}(v) : D_x^2 \mathfrak{f}_{\nu_n}^{(\delta)} &= (-\Delta_{t,x}+1)^{1/2}\big(-\Delta_v+1\big)^{(1+\varepsilon/2)/2} g_{\nu_n} \\ &\quad + \sum_{j\in\mathscr{J}} \Pi_j(v)(\Delta_M)\big(-\Delta_v+1\big)^{(1+\varepsilon/2)/2} h_{j,\nu_n} \text{ in } \mathscr{D}'(Q_0\times\mathbb{R}_v),\end{aligned}$$

where ε, p and g_{ν_n} are like in the previous step, $\mathscr{J}$ is a finite index set, and, for all $j\in\mathscr{J}$,

$$\begin{cases}\Pi_j \in \mathscr{C}^{1,\varepsilon}(\mathbb{R}) \text{ is such that } \operatorname{supp}\Pi_j \subset \operatorname{supp}\mathbf{b}, \text{ and} \\ (h_{j,\nu_n})_{n\in\mathbb{N}} \text{ is relatively compact in } L^p(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v).\end{cases}$$

Consequently, Theorem 2.2 guarantees that $\int_{\mathbb{R}} \mathfrak{f}_{\nu_n}^{(\delta)} 1_{X\cap(-R,R)}\,dv = \int_{-R}^{R} \mathfrak{f}_{\nu_n}\eta_\delta\,dv$ is relatively compact in $L^p(Q_0)$. For $1<p<\infty$, and $\mathbf{b}(v)$ observes the nontransiency condition, Lemma 2.2 implies that the sequence $(\int_{-R}^{R} f\,dv)_{n\in\mathbb{N}}$ is totally bounded in $L^1_{\mathrm{loc}}(Q)$. From this point forward, the remainder of the proof becomes once again indistinguishable from the one of Theorem 3.2. □

Notice that the proof above was the sole place in this chapter where we needed the regularity and chain rule assumptions for kinetic solutions. Furthermore, it is worth pointing out that this demonstration required Theorems 2.1–2.4 in their full power.

Incidentally, in the one-phase case, it is clear that we could have weaken hypothesis 2. in Definition to "$P_M(\mathbf{B}_\nu)'(v) \to \mathbf{B}'(v)$ pointwisely, and $P_{M^\perp}(\mathbf{B}_\nu)'(v) \to 0$ uniformly in compact sets of the real line".

Similarly, in the two-phase case, hypothesis 2. could have been substituted to (i) "$P_M(\mathbf{B}_\nu)'(v) \to \mathbf{B}'(v)$" pointwisely in $\{\mathbf{b}(v)>0\}$, (ii) $P_M(\mathbf{B}_\nu)' \to P_M(\mathbf{B})'(v) = 0$ uniformly over the compact sets of $\operatorname{Int}\{\mathbf{b}(v)=0\}$, and (iii) $P_{M^\perp}(\mathbf{B}_\nu)' \to P_{M^\perp}(\mathbf{B})'(v) = 0$ uniformly over the compact sets of $\mathbb{R}_v$".

At last, we observe that, if $\mathbf{b}_\nu(v) = \mathbf{b}(v) + \nu I_{\mathbb{R}^N}$, some of the calculations above would be considerably easier. Indeed, plugging $\Phi(v) = \nu^{1/2}\mathbf{b}_\nu(v)^{-1/2}$ into (3.28), it follows that $\nu^{1/2}\nabla_x\psi(u)$ is uniformly bounded in $L^2_{\mathrm{loc}}(Q)$ for all $\psi\in\mathscr{C}_c^\infty(\mathbb{R})$ with $\psi\ge 0$. (This shows that, for this particular type of approximation, the conclusions of Theorem 3.4 would be valid if one had assumed the "real analytic" nondegeneracy condition of Remark 3.6.)

3.3.3 The case $X = \mathbb{R}$

Informally speaking, the hypothesis that $X \neq \mathbb{R}$ is a "sign condition". Hence, it is physical, for, in the most applications of (3.15), u generally represents some nonnegative or bounded quantity.

Yet, from a mathematical perspective, one may still wish to consider the case $X = \mathbb{R}$. The meaningful difference between this scenario and the previous one is that the simple estimate on the measure $(\mathfrak{m} + \mathfrak{n})$ on (3.17) is no longer valid, since f may be supported on the entire $Q \times \mathbb{R}_v$. Nevertheless, one moment of reflection on its rationale and on the decay estimate (3.13) shows that such an estimate would in fact hold had one assumed that fluxes $(\operatorname{sign} u)_- \mathbf{A}(u)$ and $(\operatorname{sign} u)_- \mathbf{B}(u)$ belonged to L^1_{loc}. In this fashion, one may deduce the following result, which is very much in the spirit of E. Yu. Panov [92, 93, 94, 95] and H. Holden *et al.* [63]; see also Remarks 3.4 and 3.6.

Theorem 3.5. *Keep the notations and hypotheses on Theorems 3.2–3.4, but let now X possibly be $\mathbb{R}$. Furthermore, for $\pm$ symbolizing either $+$ or $-$, add the following extra conditions.*

- *In Theorems 3.2, assume that $\big((\operatorname{sign} u_\nu)_\pm \mathbf{A}(u_\nu)\big)_{\nu \in \mathscr{I}}$ and $\big((\operatorname{sign} u_\nu)_\pm \mathbf{B}(u_\nu)\big)_{\nu \in \mathscr{I}}$ are bounded in, respectively, $L^1_{\mathrm{loc}}(Q; \mathbb{R}^N)$ and $L^1_{\mathrm{loc}}(Q; \mathscr{L}(\mathbb{R}^N))$.*
- *In Theorem 3.3, assume that $\big((\operatorname{sign} u_\nu)_\pm \mathbf{A}(u_\nu)\big)_{\nu \in \mathscr{I}}$, $\big((\operatorname{sign} u_\nu)_\pm \mathbf{B}(u_\nu)\big)_{\nu \in \mathscr{I}}$, and $\big((\operatorname{sign} u_\nu)_\pm \mathbf{S}(t, x, u_\nu)\big)_{\nu \in \mathscr{I}}$ are bounded in, respectively, $L^1_{\mathrm{loc}}(Q; \mathbb{R}^N)$, $L^1_{\mathrm{loc}}(Q; \mathscr{L}(\mathbb{R}^N))$, and $L^1_{\mathrm{loc}}(Q)$.*
- *In Theorem 3.4, assume that $\big((\operatorname{sign} u_\nu)_\pm \mathbf{A}_\nu(u_\nu)\big)_{\nu \in \mathscr{I}}$ and $\big((\operatorname{sign} u_\nu)_\pm \mathbf{B}_\nu(u_\nu)\big)_{\nu \in \mathscr{I}}$ are bounded in, respectively, $L^1_{\mathrm{loc}}(Q; \mathbb{R}^N)$ and $L^1_{\mathrm{loc}}(Q; \mathscr{L}(\mathbb{R}^N))$.*

Then the conclusions of such Theorems still remain valid.

As a consequence, if the conditions above hold for both $+$ and $-$, the limit points of $(u_\nu)_{\nu \in \mathscr{I}}$ in such Theorems are kinetic solutions to their associated degenerate parabolic-hyperbolic equations.

3.3.4 The case $X = \mathbb{R}$, part II: The whole space and periodic cases

Should the underlying open set Q have the form $Q = (0, T) \times \mathbb{R}^N_x$ (the whole space) or $Q = (0, T) \times \mathbb{T}^N_x$ (the periodic case) for some $0 < T \leq \infty$, one may significantly optimize the conclusions of this chapter. Indeed, then one can modify Definitions 3.1 and 3.3 ever so slightly to better accomodate the expected a priori estimates known for these cases (see, *e.g.*, G.-Q. Chen–B. Perthame [27], and B. Gess–M. Hofmanová [51]). For the sake of clarity, let us restate those concepts. Henceforth, let $\mathscr{O}$ be either $\mathbb{R}^N_x$ or $\mathbb{T}^N_x$.

Definition 3.5. Let $u \in L^\infty((0, T) \times \mathscr{O}) \cap L^\infty(0, T; L^1(\mathscr{O}))$. One says that u is an *entropy solution* to (3.1) in $(0, T) \times \mathscr{O}$ if the following conditions hold.

1. (Regularity). If $\sigma(v) = \mathbf{b}(v)^{1/2}$, and $\beta(v) = \int_0^v \sigma(w)\, dw$, then $\operatorname{div}_x(\beta(u)) \in L^2((0, T) \times \mathscr{O}; \mathbb{R}^N)$.
2. (Chain rule). For any nonnegative function $\psi \in \mathscr{C}(\mathbb{R}_v)$, put $\beta^\psi(v) = \int_0^v \psi(w)^{1/2} \sigma(w)\, dw$, and $\mathfrak{n}^\psi(t, x) = \psi(u(t, x)) |\operatorname{div}_x \sigma(u(t, x))|^2$. Then,
$$\begin{cases} \operatorname{div}_x(\beta^\psi(u)) = \psi(u)^{1/2} \operatorname{div}_x \beta(u) \in L^2((0, T) \times \mathscr{O}; \mathbb{R}^N), \text{ and} \\ \mathfrak{n}^\psi(t, x) = |\operatorname{div}_x \beta^\psi(u(t, x))|^2. \end{cases}$$
3. (The entropy condition). There exists a nonnegative measure $\mathfrak{m}(t, x, v)$ supported on $(0, T) \times \mathscr{O} \times \mathbb{R}_v$, such that, for any function $\eta \in \mathscr{C}^2(\mathbb{R})$, one has that
$$\frac{\partial}{\partial t} \eta(u) + \operatorname{div}_x \mathbf{A}^\eta(u) - D^2_x : \mathbf{B}^\eta(u) = -(\mathfrak{m}^{\eta''} + \mathfrak{n}^{\eta''}) \text{ in } \mathscr{D}'((0, T) \times \mathscr{O}), \tag{3.36}$$
where $\mathbf{A}^\eta(v)$, $\mathbf{B}^\eta(v)$, and $\mathfrak{m}^{\eta''}(t, x)$ are given in (3.3).

Definition 3.6. Let $u \in L^\infty(0,T;L^1(\mathscr{O}))$, and let $f(t,x,v) = \chi_{u(t,x)}(v)$ for $(t,x,v) \in (0,T)\times\mathscr{O}\times\mathbb{R}_v$ be its χ–function. One says that u is a *kinetic solution* to (3.1) in $(0,T)\times\mathscr{O}$ if the following conditions hold.

1. (Regularity). For any nonnegative function $\psi \in \mathscr{C}_c^\infty(\mathbb{R})$, put $\beta^\psi(v) = \int_0^v \psi(w)^{1/2}\sigma(w)\,dw$. Then,
$$\mathrm{div}_x(\beta^\psi(u)) \in L^2((0,T)\times\mathscr{O};\mathbb{R}^N).$$
2. (Chain rule). For any nonnegative functions ψ_1 and $\psi_2 \in \mathscr{C}_c^\infty(\mathbb{R})$, it holds that
$$\mathrm{div}_x(\beta^{\psi_1\psi_2}(u)) = \psi_1(u)^{1/2}\mathrm{div}_x(\beta^{\psi_2}(u)) \text{ almost everywhere.}$$
3. (The kinetic equation). There exist two nonnegative measures $\mathfrak{m}(t,x,v)$ and $\mathfrak{n}(t,x,v)$ supported on $(0,T)\times\mathscr{O}\times\mathbb{R}_v$ such that
$$\int_{\mathbb{R}_v} \psi(v)\,\mathfrak{n}(t,x,dv) = |\mathrm{div}_x\beta^\psi(u(t,x))|^2$$
for any nonnegative $\psi \in \mathscr{C}_c^\infty(\mathbb{R})$, and the equation
$$\frac{\partial f}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f - \mathbf{b}(v) : D_x^2 f = \frac{\partial\mathfrak{m}}{\partial v} + \frac{\partial\mathfrak{n}}{\partial v} \tag{3.37}$$
is obeyed in the sense of the distributions in $(0,T)\times\mathscr{O}\times\mathbb{R}_v$.
4. (Decay estimate). It holds that
$$\int_{(0,T)\times\mathscr{O}} (\mathfrak{m}+\mathfrak{n})(dt,dx,v) \le \mu(v) \tag{3.38}$$
for some $\mu \in L^\infty(\mathbb{R})$ such that $\mu(v) \to 0_+$ as $|v| \to \infty$.

The great advantage we are in possession now over the previous case is that, due to the particular structure of $\mathscr{O}$ and $u(t,x) \in L^\infty(0,T;L^1(\mathscr{O}))$, one can choose test functions whose support are in $(0,T)\times\mathscr{O}$ itself in (3.36) and (3.37). So as to illustrate this point, let $u(t,x)$ be a kinetic solution to (3.1), $f(t,x,v)$ be its χ–function, and $\mathfrak{m}$ and $\mathfrak{n}$ be its corresponding measures. Given any $R > 0$ and any $I \subset\subset (0,T)$, pick $\varphi \in \mathscr{C}_c^\infty(0,T)$ and $\zeta \in \mathscr{C}^\infty(\mathbb{R}_v)$ such that
$$\begin{cases} \varphi \text{ is nonnegative and } \varphi \equiv 1 \text{ in } I\text{, and} \\ \dfrac{d\zeta}{dv} \in \mathscr{C}_c^\infty(-2R,2R),\ \dfrac{d\zeta}{dv} \ge 0,\ \dfrac{d\zeta}{dv}(v) = 1 \text{ in } (-R,R)\text{, and } \displaystyle\lim_{v\to-\infty}\zeta(v) = 0. \end{cases}$$
Thus, since $u \in L^\infty(0,T;L^1(\mathscr{O}))$, it is not difficult to justify that the choice of the test function $\varphi(t,x,v) = \varphi(t)\zeta(v)$, so that
$$\begin{aligned}(\mathfrak{m}_\nu+\mathfrak{n}_\nu)(I\times\mathscr{O}\times(-R,R)) &\le \int_0^T\int_{\mathscr{O}}\int_{\mathbb{R}} \varphi(t)\frac{d\zeta}{dv}(v)\,d(\mathfrak{m}_\nu+\mathfrak{n}_\nu)(dt,dx,dv) \\ &= \int_0^T\int_{\mathscr{O}}\int_{\mathbb{R}_v} f_\nu(t,x,v)\zeta(v)\frac{d\varphi}{dt}(t)\,dvdxdt \\ &\le C(\|\zeta\|_{L^\infty})\int_0^T\int_{\mathscr{O}}\left|\frac{d\varphi}{dt}(t)\right|\big(u(t,x)-2R\big)_+\,dxdt,\end{aligned}$$
which is a quite an improvement over (3.17) for no remainder of integration by parts in x appears. Consequently, one can now estimate $(\mathfrak{m}_\nu+\mathfrak{n}_\nu)$ via the local L^1–norms of u alone. Likewise, trans-

lating and reflecting the function $\zeta(v)$ above, one may also bound the growth of $(\mathfrak{m}_\nu + \mathfrak{n}_\nu)$ for large $|v|$ by such aforementioned norms.

Based on this observation, one can refine the argument of Theorem 3.2 and prove the next result.

Theorem 3.6. *Let $\mathscr{O}$ be either $\mathbb{R}^N_x$ or $\mathbb{T}^N_x$, and $0 < T \le \infty$. Additionally, replace Defintions 3.1 and 3.3 by, respectively, Definitions 3.5 and 3.6.*

In Theorems 3.1–3.4, assume that $(u_\nu)_{\nu\in\mathscr{I}}$ is bounded in $L^\infty(0,T;L^1(\mathscr{O}))$, and substitute the uniform integrability condition (3.14) by the following one: There exists some function $\lambda : (0,\infty) \to \mathbb{R}$ such that $\lambda(A) \to 0_+$ as $A \to \infty$, and

$$\operatorname*{ess\,sup}_{0<t<T} \int_{\mathscr{O}} \Big[\big(u_\nu(t,x) - A\big)_+ + \big(A - u_\nu(t,x)\big)_- \Big]\, dx \le \lambda(A) \text{ for all } \nu \in \mathscr{I} \text{ and } A > 0.$$

Further, in Theorem 3.3, assume that, for all $Q_0 \subset\subset Q$, there exists some $C_{Q_0} > 0$ such that $|\mathbf{S}(t,x,v)| \le C_{Q_0}(1 + |v|)$ for all $(t,x) \in Q_0$ and $v \in \mathbb{R}$.

Then, even if $X = \mathbb{R}$, Theorems 3.1–3.4 of this chapter remain true for $Q = (0,T) \times \mathscr{O}$. Moreover, one can indeed add the conclusion that the limit points of $(u_\nu)_{\nu\in\mathscr{I}}$ are kinetic solutions to their associated degenerate parabolic-hyperbolic equations.

Chapter 4

Strong traces for solutions to multidimensional stochastic scalar conservation laws

4.1 The main result

In this chapter, we will establish the strong trace property for entropy solutions to stochastic scalar conservation laws of the form

$$\frac{\partial u}{\partial t}(t,x) + \operatorname{div}(\mathbf{A}(u(t,x))) = \sum_{k=1}^{\infty} g_k(x,u(t,x))\,\frac{d\beta_k}{dt}(t), \tag{4.1}$$

where (t,x) belongs to some open set $Q \subset \mathbb{R}_t \times \mathbb{R}_x^N$, $\mathbf{A} : \mathbb{R}_u \to \mathbb{R}^N$ is a flux function, $g_k : \mathbb{R}_x^N \times \mathbb{R}_u \to \mathbb{R}$ are diffusion coefficients, and (β_k) is a sequence of mutually independent Brownian motions. Such a result extends the celebrated corresponding deterministic theorem firstly proven by A. Vasseur [110].

Roughly speaking, the main goal here is to show that any entropy solution $u(t,x)$ to (4.1) possesses a legitimate notion of a trace at the "lateral" boundary of Q. Furthermore, we will prove that such trace can be defined as a strong limit in L^1, hence the term "strong trace" (mostly in contrast to the theory of weak traces of G.-Q. Chen–H. Frid [25, 26], in which the trace is only attained in a weak–$\star$ sense). This general and surprising property of entropy solutions will be of fundamental importance in the subsequent chapter.

In order to precisely state our result, we will need first to make some definitions and hypotheses. Thus, let us begin with the definition of a regular deformable Lipschitz boundary introduced by G.-Q. Chen–H. Frid [25].

Definition 4.1 (Deformable Lipschitz boundary)**.** Let $\mathscr{U} \subset \mathbb{R}^N$ be an open set. We say that $\partial\mathscr{U}$ is a *Lipschitz deformable boundary* if the following assertions hold.

(i) For each $x \in \partial\mathscr{U}$, there exist $r = r_x > 0$, a Lipschitz function $\gamma = \gamma_x : \mathbb{R}^{N-1} \to \mathbb{R}$, and a rigid motion $\mathcal{R} = \mathcal{R}_x : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\begin{cases} \mathcal{R}(x) = 0, \text{ and} \\ \mathcal{R}(\mathscr{U}) \cap S(0,r) = \Big\{ y = (y_1,\dots,y_N) \in \mathbb{R}^{N-1}; \gamma(y_1,\cdots,y_{N-1}) < y_N \Big\} \cap S(0,r), \end{cases} \tag{4.2}$$

where $S(z,r) = \big\{ y \in \mathbb{R}^N; |y_i - z_i| \le r \text{ for } i = 1,\cdots,N \big\}$. We denote by $\widetilde{\gamma}$ the "graph map"

$$\widehat{y} = (y_1,\cdots,y_{N-1}) \in \mathbb{R}^{N-1} \mapsto \widetilde{\gamma}(\widehat{y}) = \mathcal{R}^{-1}(\widehat{y},\gamma(\widehat{y})) \in \mathbb{R}^N. \tag{4.3}$$

(ii) There exists a transformation $\Psi : [0,1] \times \partial\mathscr{U} \to \overline{\mathscr{U}}$ such that Ψ is a bi-Lipschitz homeomorphism over its image, and, for all $x \in \partial\mathscr{U}$, $\Psi(0,x) = x$.

For $0 \le s \le 1$, we denote by $\Psi_s : \partial\mathscr{U} \to \overline{\mathscr{U}}$ the function $\Psi_s(x) = \Psi(s,x)$, and set $\partial\mathscr{U}_s \stackrel{\text{def}}{=} \Psi_s(\partial\mathscr{U})$. We call such a map a *Lipschitz deformation* for $\partial\mathscr{U}$.

Definition 4.2 (Regular deformable Lipschitz boundary)**.** Let $\mathscr{U} \subset \mathbb{R}^N$ be an open set with a Lipschitz deformable boundary, and $\Psi : [0,1] \times \partial\mathscr{U} \to \overline{\mathscr{U}}$ a Lipschitz deformation for $\partial\mathscr{U}$. Ψ is said to be *regular* over an open set $\Gamma \subset \partial\mathscr{U}$ if the following condition holds.

- Given any $x \in \Gamma$, let $r > 0$, $\mathcal{R}$, and $\widetilde{\gamma}(\widehat{y})$ be as in (4.2) and (4.3). Diminishing $r > 0$ if necessary so that $\widetilde{\gamma}(\widehat{y})((-r,r)^{N-1}) \subset \Gamma$, then

$$\nabla_{\widehat{y}}\big[\Psi_s(\widetilde{\gamma})\big] \to \nabla_{\widehat{y}}\widetilde{\gamma} \text{ strongly in } L^1((-r,r)^{N-1}) \text{ as } s \to 0_+.$$

In the case of Γ being $\partial\mathscr{U}$, Ψ_s is then simply said to be *regular*, and $\mathscr{U}$ is said to have a *regular Lipschitz deformable boundary.*

Remark 4.1. By a simple argument involving the extension of the unit outward normal field and the theory of the ordinary differential equations, it is clear that any bounded open set $\mathscr{O} \subset \mathbb{R}^N$ of class $\mathscr{C}^{1,1}$ has a regular deformable Lipschitz boundary. Much more generally, G.-Q. Chen–G. E. Comi–M. Torres [22] recently showed that any bounded open set with a Lipschitz boundary possesses a regular Lipschitz deformable boundary in the nomenclature above.

Throughout this chapter, these will be the assumptions tacitly made.

1. *Conditions concerning* Q: The open set $Q \subset \mathbb{R}_t \times \mathbb{R}_x^N$ is bounded and of the cylindrical form $Q = (0,T) \times \mathscr{O}$, where $T > 0$, and $\mathscr{O}$ possesses a regular Lipschitz deformable boundary.

2. *Conditions concerning* $\mathbf{A}$: $\mathbf{A} \in \mathscr{C}^{2,\alpha}_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$ for some $0 < \alpha \le 1$. Denote by $\mathbf{a}(v)$ its derivative: $\mathbf{a}(v) = \mathbf{A}'(v)$.

3. *Conditions concerning* $(\beta_k(t))_{k\in\mathbb{N}}$: Henceforth, $(\Omega, \mathscr{F}, \mathbb{P})$ stands for a probability space endowed with a complete, right-continuous filtration $(\mathscr{F}_t)_{t\ge 0}$. Furthermore, it is assumed the existence of a sequence $(\beta_k(t))_{k\in\mathbb{N}}$ of mutually independent Brownian motions in $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge0}, \mathbb{P})$.

4. *Conditions concerning* $g_k(x,v)$: For any integer $k \ge 1$, we assume that g_k is Carathéodory; *i.e.*, for all $v \in \mathbb{R}$, $x \in \mathscr{O} \mapsto g_k(x,v)$ is measurable in the sense Lebesgue, and, for all $x \in \mathscr{O}$, $v \in \mathbb{R} \mapsto g_k(x,v)$ is continuous. Moreover, we suppose that there exists some constant $C_* > 0$ such that

$$\mathfrak{G}^2(x,v) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} g_k(x,v)^2 \le C_*(1+v^2) \tag{4.4}$$

for all $x \in \mathscr{O}$ and $-\infty < v < \infty$.

Finally, let us state our definition of entropy solution.

Definition 4.3 (Entropy solution)**.** Let $u \in L^\infty(\Omega \times Q)$ be predictable. We say that u is an *entropy solution* to the stochastic conservation law (4.1) if almost surely, given any convex real function

$\eta : \mathbb{R} \to \mathbb{R}$ of class $\mathscr{C}^2$, and any nonnegative test function $\varphi \in \mathscr{C}_c^\infty(Q)$,

$$\begin{aligned}\int_0^T \int_{\mathscr{O}} \eta(u(t,x)) \frac{\partial \varphi}{\partial t}(t,x)\,dxdt + \int_0^T \int_{\mathscr{O}} \mathbf{A}^\eta(u(t,x)) \cdot \nabla_x \varphi(t,x)\,dxdt \\ \geq -\sum_{k=1}^{\infty} \int_0^T \int_{\mathscr{O}} \eta'(u(t,x)) g_k(x,u(t,x)) \varphi(t,x)\,dxd\beta_k(t) \\ -\frac{1}{2}\int_0^T \int_{\mathscr{O}} \eta''(u(t,x)) \mathfrak{G}^2(x,u(t,x)) \varphi(t,x)\,dxdt \end{aligned} \tag{4.5}$$

where $\mathbf{A}^\eta(v) = \int_0^v \eta'(w)\mathbf{a}(w)\,dw$. In other words, it holds almost surely

$$\frac{\partial \eta(u)}{\partial t} + \operatorname{div}_x(\mathbf{A}^\eta(u)) \leq \sum_{k=1}^{\infty} g_k(x,u)\eta'(u)\,\frac{d\beta_k}{dt}(t) + \frac{1}{2}\eta''(u)\mathfrak{G}^2(x,u) \text{ in } \mathscr{D}'(Q) \tag{4.6}$$

for any convex function $\eta \in \mathscr{C}^2(\mathbb{R})$.

Remark 4.2. Notice that, because an entropy solution $u(t,x)$ is predictable—and thus in the space $L^2(\Omega \times (0,T); L^2(\mathscr{O}))$—and the diffusion coefficients satisfy (4.4), all the stochastic integrals make perfect sense (see, *e.g.*, G. Da Prato–J. Zabczyk [29] for a general background on this theory). One could, however, rephrase all equations (4.1), (4.5) and (4.6) in the following more pleasing manner for our theoretical purposes.

First of all, if $\mathscr{H}$ is a separable Hilbert space with a hilbertian basis $(e_k)_{k\in\mathbb{N}}$, by definition, $W(t) = \sum_{k=1}^\infty \beta_k(t)e_k$ defines a cylindrical Wiener process. Fixing such a space $\mathscr{H}$, we may now define the nonlinear operator $\Phi : L^2(\mathscr{O}) \to \mathscr{L}(\mathscr{H}; L^2(\mathscr{O}))$ by

$$(\Phi(f) \cdot h)(x) = \sum_{k=1}^{\infty} g_k(x, f(x))\,(h, e_k)_{\mathscr{H}}$$

whenever $h \in \mathscr{H}$ and $x \in \mathscr{O}$. In the light of (4.4), not only is such $\Phi(f)$ well-defined, but also lies in the Hilbert–Schmidt class $HS(\mathscr{H}; L^2(\mathscr{O}))$. Therefore, given any predictable process $u \in L^2(\Omega \times [0,T]; L^2(\mathscr{O}))$, the stochastic integral

$$t \mapsto \int_0^t \Phi(u(t'))\,dW(t') = \sum_{k=1}^{\infty} \int_0^t g_k(x, u(t',x))\,d\beta_k(t')$$

defines a legitimate $L^2(\mathscr{O})$–valued process. (We will mostly denote $\Phi(u)$ by $\Phi(x,u)$, so as to formally comprehend it as the "matrix" $\Phi(x,u) = \sum_{k=1}^\infty g_k(x,u)\,(\,\cdot\,, e_k)_{\mathscr{H}}$).

Hence, (4.6) can be translated into

$$\frac{\partial \eta(u)}{\partial t} + \operatorname{div}_x(\mathbf{A}^\eta(u)) \leq \eta'(u)\Phi(x,u)\,\frac{dW}{dt} + \frac{1}{2}\eta''(u)\mathfrak{G}^2(x,u). \tag{4.7}$$

Similar technical considerations will also repeated in the next chapter.

Let us mention that, in contrast with the deterministic version of (4.1), the "quadratic variation" term $\frac{1}{2}\eta''(u)\mathfrak{G}^2(x,u)$ appears in (4.7) as it would naturally be expected from the classical Itô's formula.

We are now in conditions to enunciate our theorem concerning the strong traces of entropy solutions to stochastic conservation laws.

Theorem 4.1 (H. Frid *et al.* [43])**.** *Assume the conditions expressed above, and let $u \in L^\infty(\Omega \times Q)$ be an entropy solution to (4.1). Additionally, suppose that there exists some $a < b$ such that*

$$a \leq u(t,x) \leq b \textit{ almost surely in } \mathscr{D}'(Q),$$

and that the following nondegeneracy condition holds:

$$\text{meas}\Big\{v \in [a,b];\ \tau + \mathbf{a}(v)\cdot\kappa = 0\Big\} = 0 \text{ for all } (\tau,\kappa)\in\mathbb{R}\times\mathbb{R}^N \text{ with } \tau^2+|\kappa|^2 = 1. \tag{4.8}$$

Then, there exists a function $u^\tau \in L^\infty(\Omega\times(0,T)\times\partial\mathscr{O})$ such that, for every $\partial\mathscr{O}$–regular Lipschitz deformation $\Psi : [0,1]\times\partial\mathscr{O}\to\overline{\mathscr{O}}$,

$$\operatorname*{ess\,lim}_{s\to 0_+} \mathbb{E}\int_0^T\int_{\partial\mathscr{O}} |u(t,\Psi(s,\widehat{x})) - u^\tau(t,\widehat{x})|\, d\mathcal{H}^{N-1}(\widehat{x})dt = 0, \tag{4.9}$$

where $\mathcal{H}^{N-1}$ denotes the $(N-1)$–dimensional Hausdorff measure. Moreover, we also have that

$$\operatorname*{ess\,lim}_{s\to 0_+} \int_0^T\int_{\partial\mathscr{O}} |u(t,\Psi(s,\widehat{x})) - u^\tau(t,\widehat{x})|\, d\mathcal{H}^{N-1}(\widehat{x})dt = 0, \tag{4.10}$$

almost surely.

Remark 4.3. Evidently, (4.9) is not stronger than (4.10); however, it may come as a surprise that neither (4.10) implies (4.9). Indeed, for one is required to employ “essential limits” in order to state (4.10) (due to the lack of continuity properties of u), the set of s’s in which (4.10) takes place depends a priori implicitly on $\omega\in\Omega$. Therefore, both conclusions (4.9) and (4.10) are dissimilar and possess their own interest.

The proof of this theorem will be divided into several component parts.

4.2 Initial observations

Notice that, modifying u into $u - \frac{b-a}{2}$ and, accordingly, also altering $\mathbf{A}(v)$ and $g_k(x,v)$ by an affine change of coordinates on their arguments, we may assume that

$$a = -L \text{ and } b = L.$$

for some real number $L > 0$. This harmless symmetrization will somewhat facilitate our manipulations.

Let us now deduce the kinetic formulation of Equation (5.1).

Theorem 4.2. *Let $f(t,x,v) = \chi_{u(t,x)}(v) = 1_{v<u(t,x)} - 1_{v<0}$ be the χ–function of $u(t,x)$ (see (3.7)). Then, almost surely, there exists a nonnegative Borel measure $\mathbf{m}(t,x,v)$ supported on $Q\times[-L,L]$ such that*

$$\frac{\partial f}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f = \frac{\partial \mathbf{q}}{\partial v} + \sum_{k=1}^{\infty} g_k(x,v)\delta_{v=u(t,x)}\frac{d\beta_k}{dt} \text{ in } \mathscr{D}'(Q), \tag{4.11}$$

in $\mathscr{D}'(Q\times\mathbb{R})$, where $\mathbf{q}(t,x,v) = \mathbf{m}(t,x,v) - \frac{1}{2}\mathfrak{G}^2(x,v)\delta_{v=u(t,x)}$.

Furthermore, for all $1\le p<\infty$, the mapping $\omega\mapsto\mathbf{m}(t,x,v)$ belongs to $L^p_{\mathfrak{w}}(\Omega;\mathfrak{M}(Q\times\mathbb{R}_v))$—the space of the weakly measurable functions $\omega\mapsto\mathfrak{m}\in\mathfrak{M}(Q\times\mathbb{R}) = \mathscr{C}_0(Q\times\mathbb{R})^\star$ such that $\mathbb{E}\|\mathfrak{m}\|^p_{\mathfrak{M}_{t,x,v}} < \infty$.[1] Indeed, one has that

$$\mathbb{E}\|\mathbf{m}\|^p_{\mathfrak{M}_{t,x,v}} \le C(p,a,b). \tag{4.12}$$

Proof. Step #1: The kinetic formulation. Reprising the argument from Remark 3.1, we see that

[1]Recall that a mapping $\omega\mapsto\mathbf{m}\in\mathfrak{M}_{t,x,v}$ is weakly measurable if, for all $\phi\in\mathscr{C}_0(Q\times\mathbb{R})$, $\omega\in\Omega\mapsto\int_{Q\times\mathbb{R}_v}\phi(t,x,v)\,\mathbf{m}(dt,dx,dv)\in\mathbb{R}$ is measurable. ($\mathscr{C}_0(Q\times\mathbb{R})$, also known as “the space of continuous functions vanishing at infinity”, is the closure of $\mathscr{C}_c(Q\times\mathbb{R})$ in $L^\infty(Q\times\mathbb{R})$.)

linear functional

$$\varphi(t,x)\eta''(v) \in \mathscr{C}_c^\infty(Q\times\mathbb{R}) \mapsto \int_0^T\int_{\mathscr{O}} \eta(u)\frac{\partial\varphi}{\partial t}\,dx\,dt + \int_0^T\int_{\mathscr{O}} \mathbf{A}^\eta(u)\cdot\nabla_x\varphi\,dx\,dt$$
$$+\sum_{k=1}^\infty\int_0^T\int_{\mathscr{O}} \eta'(u)g_k(x,u)\varphi\,dx\,d\beta_k(t) + \frac{1}{2}\int_0^T\int_{\mathscr{O}} \eta''(u)\mathfrak{G}^2(x,u)\varphi\,dx\,dt$$

is almost surely well-defined and nonnegative. Thus, applying conveniently the Riesz representation theorem and the density of tensorial functions $\varphi(t,x)\psi(v)$, it may be extended to σ-finite nonnegative Borel measure $\mathbf{m}(t,x,v)$ in $Q\times\mathbb{R}$; equivalently, almost surely there exists some nonnegative Borel measure $\mathbf{m}(t,x,v)$ in $Q\times\mathbb{R}$ such that

$$\int_0^T\int_{\mathscr{O}}\int_{\mathbb{R}_v}\varphi(t,x)\eta''(v)\,\mathbf{m}(dt,dx,dv) = \int_0^T\int_{\mathscr{O}} \eta(u)\frac{\partial\varphi}{\partial t}\,dxdt + \int_0^T\int_{\mathscr{O}} \mathbf{A}^\eta(u)\cdot\nabla_x\varphi\,dxdt$$
$$+\sum_{k=1}^\infty\int_0^T\int_{\mathscr{O}} \eta'(u)g_k(x,u)\varphi\,dxd\beta_k(t) + \frac{1}{2}\int_0^T\int_{\mathscr{O}} \varphi\eta''(u)\mathfrak{G}^2(x,u)\,dxdt \tag{4.13}$$

for all $\varphi\in\mathscr{C}_c^\infty(Q)$ and $\eta\in\mathscr{C}^2(\mathbb{R}_v)$, as $\|u\|_{L^\infty}\le L$.

Furthermore, as

$$\int_0^T\int_{\mathscr{O}} \eta(u)\frac{\partial\varphi}{\partial t}\,dxdt = \int_0^T\int_{\mathscr{O}}\int_{\mathbb{R}_v} \eta'(v)f(t,x,v)\frac{\partial\varphi}{\partial t}\,dvdxdt, \text{ and}$$
$$\int_0^T\int_{\mathscr{O}} \mathbf{A}^\eta(u)\cdot\nabla_x\varphi\,dxdt = \int_0^T\int_{\mathscr{O}}\int_{\mathbb{R}_v} f(t,x,v)\eta'(v)\mathbf{a}(v)\cdot\nabla_x\varphi\,dvdxdt, \tag{4.14}$$

one can see Equation (4.11) indeed holds once one again recalls the well-known fact that the simple tensors $\varphi(t,x)\eta(v)$ ($\varphi\in\mathscr{C}_c^\infty(Q)$ and $\eta\in\mathscr{C}_c^\infty(\mathbb{R}_v)$) form a dense linear space in $\mathscr{C}_c^\infty(Q\times\mathbb{R})$.

Step #2: The support of $\mathbf{m}(t,x,v)$: Properly speaking, (4.13) only defines $\frac{\partial^2\mathbf{m}}{\partial v^2}$. Therefore, for it is nonnegative, $\mathbf{m}(t,x,v)$ is determined up to a nonnegative measure $c(t,x)$. Let us show indeed $\mathbf{m}(t,x,v)$ is uniquely defined by verifying that it is almost surely supported on $Q\times[-L,L]$.

Pick $\eta''\in\mathscr{C}_c^\infty(L,\infty)$, and put $\eta'(v)=\int_{-\infty}^v\eta''(w)\,dw$ and $\eta(v)=\int_{-\infty}^v\eta'(w)\,dw$. Since $-L\le u(t,x)\le L$ almost surely in $\mathscr{D}'(Q)$, it clear from (4.13) that

$$\int_Q\int_{\mathbb{R}_v}\eta''(v)\varphi(x)\,\mathbf{m}(dt,dx,dv)=0$$

no matter the choice of $\varphi\in\mathscr{C}_c^\infty(Q)$. Consequently, $\mathbf{m}(t,x,v)$ is almost surely supported on $Q\times(-\infty,L]$. Conversely, choosing any $\eta''\in\mathscr{C}_c^\infty(-\infty,-L)$, and letting $\eta'(v)=-\int_v^\infty\eta''(w)\,dw$ and $\eta(v)=-\int_v^\infty\eta'(w)\,dw$, one can inspect that $\mathbf{m}(t,x,v)$ is almost surely supported on $Q\times[-L,L]$.

Step #3: The $L^p_{\mathfrak{w}}$–norms of $\mathbf{m}(t,x,v)$: Finally, let us check (4.12). Notice that, from the kinetic equation itself (4.11), it is clear that $\omega\mapsto\mathbf{m}(t,x,v)$ is weakly measurable.

For $\partial\mathscr{O}$ is a regular Lipschitz boundary, it is not hard to construct a family $\{\theta_\alpha(t,x)\}_{0<\alpha<1}$ of real Lipschitz functions in $\overline{Q}$ satisfying:

1. $0\le\theta_\alpha(t,x)\le1$ for every $(t,x)\in\overline{Q}$ and $\alpha>0$;
2. $\theta_{\alpha_1}(t,x)\le\theta_{\alpha_0}(t,x)$ for every $(t,x)\in\overline{Q}$ and $0<\alpha_0\le\alpha_1\le1$;
3. $\theta_\alpha(t,x)=0$ for every $(t,x)\in\partial Q$ and $\alpha>0$;
4. $\theta_\alpha(t,x)\to1$ for every $(t,x)\in\overline{Q}$ as $\alpha\to0_+$;
5. $|\nabla_{t,x}\theta(t,x)|\le(\text{const.})/\alpha$ for every $(t,x)\in Q$ and $\alpha>0$;
6. measure of $\{(t,x)\in Q;\theta_\alpha(t,x)<1\}\le(\text{const.})\alpha$ for all $\alpha>0$.

(Such family is sometimes known as a "boundary layer" family; see C. Mascia–A. Porreta–A. Terracina [84]). Because $\mathbf{m}(t,x,v)$ has almost surely compact support, it is permitted to choose $\varphi_\alpha(t,x,v) = v\theta_\alpha(t,x)$ as a test function in (5.52). Accordingly,

$$\int_{Q\times[-L,L]} \theta_\alpha(t,x)\,\mathbf{m}(dt,dx,dv) = \int_Q \int_{-L}^{L} vf(t,x,v)\Big\{\frac{\partial\theta_\alpha}{\partial t}(t,x) + \mathbf{a}(v)\cdot\nabla_x\theta_\alpha(t,x)\Big\}\,dvdxdt$$
$$+\frac{1}{2}\int_0^T\int_{\mathscr{O}} \mathfrak{G}^2(x,u(t,x))\theta_\alpha(t,x)\,dxdt + \sum_{k=1}^{\infty}\int_0^T\int_{\mathscr{O}} g_k(x,(u(s,x)))\theta_\alpha(t,x)\,dx\beta_k(t).$$

Applying now the properties of θ_α, the L^∞–bound of $u(t,x)$, and the Burkholder inequality, we deduce thus that

$$\mathbb{E}\Big[\mathbf{m}\Big(\big\{(t,x)\in Q;\theta_\alpha(t,x)=1\big\}\times[-L,L]\Big)^p\Big] \le C(p).$$

Passing $\alpha\to 0_+$, the monotone convergence theorem yields the desired conclusion. □

Remark 4.4. The existence of $\mathbf{m}(t,x,v)$ can be alternatively shown as follows. Choosing a suitable sequence of smooth approximations to the classical entropies $\eta(u;v) = (u-v)_+ - v_+$ and plugging them into the entropy condition (4.5), one can corroborate, for almost $\omega\in\Omega$, that $\mathbf{m}(t,x,v)$ is "explicitly" given by

$$\mathbf{m}(t,x,v) = -\frac{\partial}{\partial t}\eta(u;v) - \mathrm{div}_x(\mathbf{A}^{\eta(\,\cdot\,;v)}(u))$$
$$+\sum_{k=1}^{\infty} g_k(x,u)\frac{\partial\eta}{\partial u}(u;v)\,\frac{d\beta_k}{dt}(t) + \frac{1}{2}\mathfrak{G}^2(x,u)\frac{\partial^2\eta}{\partial u^2}(u;v), \tag{4.15}$$

where $\frac{\partial\eta}{\partial u}(u;v) = 1_{(v,\infty)}(u)$ and $\frac{\partial^2}{\partial u^2}\eta(u;v) = \delta_{u=v}$. A central property of these entropies is that $\frac{\partial\eta}{\partial v}(u;v) = \chi_u(v)$, so that the kinetic formulation (4.11) could have obtained by a differentiation in v of Equation (4.15).

4.3 The existence of weak traces and the criterion for strong traces

Just as A. Vasseur [110] originally argued, we may localize our analysis and assume that $\mathscr{O}$ is initially of the form

$$\mathscr{O}_0 = \big\{x = (\widehat{x}, x_N)\in(-r,r)^{N-1}\times(-r,r);\, x_N > \gamma_0(\widehat{x})\big\}, \tag{4.16}$$

where $r>0$ and $\gamma_0 : (-r,r)^{N-1}\to\mathbb{R}$ is a Lipschitz function satisfying $-r<\gamma_0(\widehat{x})<r$ everywhere. Hence, the boundary we are interested in is

$$\Gamma_0 = \big\{x = (\widehat{x}, x_N)\in(-r,r)^{N-1}\times(-r,r);\, x_N = \gamma_0(\widehat{x})\big\}.$$

Notice that Γ_0 is parametrized in $\widehat{x}$, once that it is the graph of γ_0. Consequently, for any Γ_0–regular Lipschitz deformation $\psi(x,s)$, we can write

$$\begin{cases}\widetilde{\psi}(\widehat{x},s) = \psi(s,(\widehat{x},\gamma_0(\widehat{x}))), \text{ and}\\ f_\psi(t,\widehat{x},s,v) = f(t,\widetilde{\psi}(\widehat{x},s),v) \text{ for every } \widehat{x}\in(-r,r)^{N-1}, \text{ and } 0\le s\le 1.\end{cases} \tag{4.17}$$

In order to facilitate the writing, let us set

$$\Sigma = (0,T)\times(-r,r)^{N-1}.$$

Let us now adapt the ingenious argument of A. Vasseur [110] for the existence of weak traces of the χ–functions $f(t,x,v)$ in Γ; as we will see shortly, this is a decisive step towards Theorem 4.1. Even though the process of writing a generic neighborhood $\mathscr{O}_0$ of $\partial\mathscr{O}$ in the form (4.16) generally requires the usage of a rigid motion, the effect of such a transformation in Equation (4.11) would essentially be that, instead of $\mathbf{a}(v)$, we would have $(\mathscr{Q}\mathbf{a})(v)$ where $\mathscr{Q}$ is some unitary operator in $\mathbb{R}^N$. For simplicity's sake, we will by some abuse of notation neglect this technicality and assume that $f(t,x,v)$ still obeys the very same equation (4.11) in such coordinates (which will still be denoted (t,x,v)).

Lemma 4.1 (Existence of weak traces)**.** *There exists a unique function $f^\tau \in L^\infty(\Omega\times\Sigma\times(-L,L))$ such that, for any Γ_0–regular Lipschitz deformation, we have that*

$$\operatorname*{ess\,lim}_{s\to 0_+} f_\psi(\,\cdot\,,s,\,\cdot\,) = f^\tau(\,\cdot\,,\,\cdot\,) \text{ in the weak-}\star\text{ topology of } L^\infty(\Omega\times\Sigma\times(-L,L)). \tag{4.18}$$

Additionally, there exists a set $\Omega_0 \subset \Omega$ with probability 1 such that, for $\omega\in\Omega_0$,

$$\operatorname*{ess\,lim}_{s\to 0_+} f_\psi(\omega,\,\cdot\,,s,\,\cdot\,) = f^\tau(\omega,\,\cdot\,,\,\cdot\,) \text{ in the weak-}\star\text{ topology in } L^\infty(\Sigma\times(-L,L)). \tag{4.19}$$

Proof. Step #1: Verification of the almost sure weak-$\star$ convergence (4.19)*, part one.* Let us begin by establishing (4.19), which is slightly more subtle than (4.18).

Consider a dense sequence $(h_n)_{n\in\mathbb{N}} \subset \mathscr{C}^1_c(-L,L)$ in $L^1(-L,L)$. Evidently, picking suitable representatives if necessary, there exists some subset $\Omega_0 \subset \Omega$ of probability 1 such that, for all $\omega\in\Omega_0$ and $n\in\mathbb{N}$,

$$\begin{cases} \displaystyle\int_{-L}^{L} h_n(v) f(t,x,v)\,dv, \\ \displaystyle\int_{-L}^{L} h_n(v) f(t,x,v)\mathbf{a}(v)\,dv, \text{ and} \\ h_n'(u(t,x))\mathfrak{G}^2(x,u(t,x)) \end{cases}$$

belong to $L^\infty(Q)$, and the stochastic integrals

$$\sum_{k=1}^{\infty}\int_0^t h_n(u(s,x))g_k(x,u(s,x))\,d\beta_k(s).$$

are elements of $\mathscr{C}([0,T];L^2(\mathscr{O}))$ such that

$$\frac{\partial}{\partial t}\bigg(\int_0^t h_n(u(s,x))g_k(x,u(s,x)\,d\beta_k(s)\bigg) = h_n(u(t,x))g_k(x,u(t,x))\,\frac{d\beta_k}{dt}(t)$$

in $\mathscr{D}'(Q)$ (see Proposition 2.7). Reducing Ω_0 if needed, (4.12) asserts that we may likewise assume that

$$\mathbf{m}\big(Q\times[-L,L]\big) \le C(\omega),$$

for all $\omega\in\Omega_0$.

Fix $\omega\in\Omega_0$ for a moment, and consider the vector fields $F_n : Q\to\mathbb{R}\times\mathbb{R}^N$ given by

$$F_n(t,x) = \bigg(\int_{-L}^{L} h_n(v)f(t,x,v)\,dv - \sum_{k=1}^{\infty}\int_0^t h_n(u(s,x))g_k(x,u(s,x))\,d\beta_k(s),\\ \int_{-L}^{L} h_n(v)f(t,x,v)\mathbf{a}(v)\,dv\bigg). \tag{4.20}$$

Clearly, by our choice of ω's, F_n belongs to $L^2(Q)\times L^\infty(Q;\mathbb{R}^N)$. Moreover, we see that the kinetic formulation equation (4.11) implies that

$$\operatorname{div}_{t,x} F_n(t,x) = -\int_{-L}^{L} h_n'(v)\mathbf{q}(t,x,dv) \in \mathfrak{M}(Q) \text{ in } \mathscr{D}'(Q).$$

As a consequence, since Γ_0 is a strongly regular deformable Lipschitz boundary, we are now in conditions to invoke the following profound normal trace result due to H. FRID [41]; see also G.-Q. CHEN–H. FRID [25, 26], H. FRID–Y. LI [43], and G.-Q. CHEN–G. E. COMI–M. TORRES [22]. (The novelty of the theorem below is that the vector field, like ours, may lie in L^p rather than in L^∞. Notice, however, that the entries F_n are still partially in L^∞, and the component that is not in L^∞ is orthogonal to the normal of the boundary surfaces; consequently, we were able to obtain a weak–$\star$ convergence in L^∞ as asserted below.)

Theorem 4.3. *Let $\mathscr{U} \subset \mathbb{R}^N$ be an open set with a regular deformable Lipschitz boundary, $1 \leq p \leq \infty$, and $F = (F^0, F^1) \in L^p((0,T)\times\mathscr{U}) \times L^\infty((0,T)\times\mathscr{U};\mathbb{R}^N)$ be a vector field such that the distribution $\operatorname{div}_{t,x} F = \partial_t F^0 + \operatorname{div}_x F^1$ is a Radon measure in $(0,T)\times\mathscr{U}$. Then there exists an element $F^{1,\tau}\cdot\nu \in L^\infty((0,T)\times\partial\mathscr{U})$ such that, for every $\partial\mathscr{U}$–strongly regular Lipschitz deformation ψ,*

$$\star\text{–}\operatorname*{ess\,lim}_{s\to 0_+} F^1(\,\cdot\,,\psi(\,\cdot\,,s))\cdot\nu_s(\,\cdot\,) = F^{1,\tau}\cdot\nu \textit{ weakly–}\star \textit{ in } L^\infty((0,T)\times\partial\mathscr{U}), \tag{4.21}$$

where ν_s denotes the unit outward normal vector field of $\psi(\{s\}\times\partial\mathscr{U}) = \partial\mathscr{U}_s$.

Accordingly, there exist a set $\mathcal{S}_n \subset [0,1]$ of total measure and some $F_n^{1,\tau}\cdot\nu \in L^\infty((0,T)\times\Sigma)$, which does not depend on ψ, such that

$$F_n^1(\,\cdot\,,\widetilde{\psi}(\,\cdot\,,s))\cdot\nu_s(\,\cdot\,) \overset{\star}{\rightharpoonup} F_n^{1,b}\cdot\nu \text{ weakly–}\star \text{ in } L^\infty(\Sigma) \text{ as } s\to 0_+ \text{ along } s\in\mathcal{S}_n. \tag{4.22}$$

Write $\mathcal{S} = \cap_{n=1}^\infty \mathcal{S}_n$, so that $\mathcal{S}$ also has total measure in $[0,1]$. We will now check that F_n depends linearly on h_n. For any integer $M \geq 1$ and $\varphi_m \in L^1(\Sigma)$, $1 \leq m \leq M$, the relations (4.3), (4.20), and (4.22) imply that

$$\begin{aligned}\left|\int_\Sigma \sum_{m,n=1}^M (F_n^{1,\tau}.\nu)(t,\widehat{x})\varphi_m(t,\widehat{x})\,dtd\widehat{x}\right| &\leq C\|\mathbf{a}\|_{L^\infty(-L,L)}\int_\Sigma\int_{-L}^{L}\left|\sum_{m,n=1}^M h_n(v)\varphi_m(t,\widehat{x})\right|dvd\widehat{x}dt\\ &= (\text{const.})\left\|\sum_{m,n=1}^M h_n\otimes\varphi_m\right\|_{L^1(\Sigma\times(-L,L))}.\end{aligned}$$

Thus, for $(L^1)^\star = L^\infty$, there exists some $H\cdot\nu \in L^\infty(\Sigma\times(-L,L))$ such that, for all $h\in L^1(-L,L)$ and all $\varphi\in C_c^\infty(\Sigma)$,

$$\begin{aligned}&\int_\Sigma\int_{-L}^{L} h(v)\varphi(t,\widehat{x})f(t,\widehat{x},\widetilde{\psi}(\widehat{x},s))\mathbf{a}(v)\cdot\nu_s(\widehat{x})\,dvd\widehat{x}dt\\ &\qquad\to \int_\Sigma\int_{-L}^{L} h(v)\varphi(t,\widehat{x})(H\cdot\nu)(t,\widehat{x})dvd\widehat{x}dt\end{aligned} \tag{4.23}$$

as $s\to 0_+$ along $s\in\mathcal{S}$. Note that $H\cdot\nu$ is independent on ψ.

Step #2: Verification of the almost sure weak–$\star$ convergence (4.19)*, part two.* So far, we have essentially only shown the existence of the weak trace of $f(\,\cdot\,,\widetilde{\psi}(\,\cdot\,,s),\,\cdot\,)\mathbf{a}(\,\cdot\,)\cdot\nu_s(\,\cdot\,)$, which is not exactly what we wanted—but almost! To conclude, let us observe that, for $\|f_\psi(\,\cdot\,,\,\cdot\,,s,\,\cdot\,)\|_{L^\infty} \leq 1$, the Banach–Alaoglu–Bourbaki theorem asserts that, for every regular Lipschitz deformation ψ and every sequence s_n in $\mathcal{S}$ converging to 0, there exists a subsequence s_{n_k} in $\mathcal{S}$ and some $f_\psi^\tau \in L^\infty(\Sigma\times(-L,L))$ such that

$$f_\psi(\,\cdot\,,s_{n_k},\,\cdot\,) \overset{\star}{\rightharpoonup} f_\psi^\tau \text{ weakly–}\star \text{ in } L^\infty(\Sigma\times(-L,L)) \text{ as } k\to\infty.$$

Thus, from (4.23) and the fact that $\nu_s \to \nu$ strongly in $L^1(\Sigma;\mathbb{R}^N)$, we deduce that

$$\int_\Sigma \int_{-L}^{L} h(v)\varphi(t,\widehat{x}) f_\psi^\tau(t,\widehat{x})\mathbf{a}(v)\cdot\nu(\widehat{x})\,dv\,d\widehat{x}\,dt = \int_\Sigma \int_{-L}^{L} h(v)\varphi(t,\widehat{x})(H\cdot\nu)(t,\widehat{x})\,dv\,d\widehat{x}\,dt,$$

for every $h \in L^1(-L,L)$ and $\varphi \in C_c^\infty(\Sigma)$. Since the right-hand term is independent of ψ and s_n, so must be $\mathbf{a}(v)\cdot\nu(\widehat{x}) f_\psi^\tau(t,\widehat{x},v)$. On the other hand, for the nondegeneracy condition implies that

$$\text{measure of } \big\{v \in (-L,L);\, \mathbf{a}(v)\cdot\nu(\widehat{x}) = 0\big\} = 0,$$

we conclude that f_ψ^τ also does not depend on ψ nor s_n. Consequently, we may denote it by f^τ. This proves (4.19) for $\omega \in \Omega_0$.

Step #3: Verification of the weak–⋆ convergence in the mean (4.18). So as to prove (4.18), all we need to do is to argue precisely as before but employing now the vector fields

$$F_{m,n}(t,x) = \mathbb{E}\Bigg[X_m\Bigg(\int_{-L}^{L} h_n(v)f(t,x,v)\,dv - \sum_{k=1}^{\infty}\int_0^t h_n(u(t,x))g_k(x,u(s,x))\,d\beta_k(s), \\ \int_{-L}^{L} h_n(v)f(t,x,v)a(v)\,dv\Bigg)\Bigg],$$

where $(X_m)_{m\in\mathbb{N}}$ is a sequence in $L^\infty(\Omega)$ that is dense in $L^1(\Omega)$ (notice that we can always suppose that Ω is countably generated), and $(h_n)_{n\in\mathbb{N}}$ are as before. This leads then to the existence of some $f^b \in L^\infty(\Omega\times\Sigma\times(-L,L))$ such that

$$\operatorname*{ess\,lim}_{s\to 0_+} f_\psi(\,\cdot\,,s,\,\cdot\,) = f^b \text{ in the weak–}\star\text{ topology of } L^\infty(\Omega\times\Sigma\times(-L,L)).$$

Step #4: The equivalence between f^τ and f^b. Notice that, since the essential limits in (4.19) depend on $\omega\in\Omega$, it is not obvious that $f^b(\omega,\cdot,\cdot,\cdot) = f^\tau(\omega,\cdot,\cdot,\cdot)$ for almost all $\omega\in\Omega$ in the L^1–sense; as a matter of fact, it is not even clear that f^τ is measurable. These both assertions, however, can be seen from the fact that both f^τ and f^b are the weak–⋆ limit of $\frac{1}{s}\int_0^s f_\psi(\,\cdot\,,\sigma,\,\cdot\,)\,d\sigma$ in $L^\infty(\Omega\times\Sigma\times(-L,L))$ as $s\to 0_+$. Observe that this also shows that $f^\tau \in L^\infty(\Omega\times\Sigma\times(-L,L))$. □

Our task is then to show that one can replace the weak–⋆ convergence above with a strong L^1 one. The simple criterion that we will apply is the next one, whose deterministic counterpart is featured in Vasseur's theory.

Definition 4.4. Let $\widetilde{\Omega}$ be a probability space, (X,μ) be a measure space, and $L>0$. We say that $\phi \in L^\infty(\widetilde{\Omega}\times X\times(-L,L))$ is a *χ–function* if it has a representative $\overline{\phi}$ such that, for almost every $x\in X$, there exists $\mathfrak{a} = \mathfrak{a}(\omega) \in [-L,L]$ satisfying

$$v \in (-L,L) \mapsto \overline{\phi}(x,v) = \chi_{\mathfrak{a}}(v) \text{ almost surely.}$$

(In other words, for almost every $x\in X$, there exists a set of probability one $\widetilde{\Omega}(x)\subset\widetilde{\Omega}$ such that $\overline{\phi}(\omega,x,\,\cdot\,) = \chi_{\mathfrak{a}}(\,\cdot\,)$ for $\omega\in\widetilde{\Omega}(x)$ and some $-L\le\mathfrak{a}(\omega)\le L$.)

Lemma 4.2. *The weak trace f^τ is a χ–function if, and only if, f^τ is a strong trace of f in the sense that, for every regular Lipschitz deformation ψ,*

$$\operatorname*{ess\,lim}_{s\to 0_+} f_\psi(\,\cdot\,,s,\,\cdot\,) = f^\tau \quad \begin{cases} \textit{strongly in } L^1(\Omega\times\Sigma\times(-L,L)), \textit{ and} \\ \textit{strongly in } L^1(\Sigma\times(-L,L)) \textit{ almost surely.}\end{cases} \tag{4.24}$$

(Here f_ψ is as given by (4.17)*).*

The lemma above follows almost immediately from the next general result on the limits of χ–functions.

Proposition 4.1. *Let $\widetilde{\Omega}$ be a probability space, (X,μ) be a finite measure space, and $L>0$. If $f_n \in L^\infty(\widetilde{\Omega}\times X\times(-L,L))$ is a sequence of χ–functions converging weakly–$\star$ to some $f\in L^\infty(\widetilde{\Omega}\times X\times(-L,L))$, one of the following assertions implies the other two.*

(i) f_n converges strongly to f in $L^1(\widetilde{\Omega}\times X\times(-L,L))$.

(ii) $u_n(\,\cdot\,)=\int_{-L}^{L} f_n(\,\cdot\,,v)\,dv$ converges strongly to $u(\,\cdot\,)=\int_{-L}^{L} f(\,\cdot\,,v)\,dv$ in $L^1(\widetilde{\Omega}\times X)$.

(iii) f is a χ–function.

Proof of Proposition 4.1. Let us start with the equivalency between the statements (i) and (ii). Recalling (3.10), the Fubini theorem asserts that, for all m and $n\geq 1$,

$$\mathbb{E}\int_X\int_{-L}^{L}|f_m(x,v)-f_n(x,v)|\,dv\mu(dx)=\mathbb{E}\int_X|u_m(x)-u_n(x)|\,\mu(dx), \tag{4.25}$$

so that it is clear that the strong convergence of (f_n) implies in the strong convergence of (u_n), and *vice versa*. Because the strong limit of (f_n) must be *a fortiori* f, the strong limit of u_n is necessarily $u=\int_{-L}^{L} f\,dv$. Hence (i) and (ii) are logically equivalent.

Moreover, the argument just displayed manifestly demonstrates that the limit function f belongs to the same class of equivalence of $(\omega,x,v)\mapsto\chi_u(v)$. As a result, f is a χ–function, and it follows that (i) and (ii) implies (iii).

Let us turn to assertion that (iii) entails (i), which is verily the important conclusion of this proposition. If f is indeed a χ–function, then $f(x,v)^2=|f(x,v)|=f(x,v)\mathrm{sign}(v)$ almost surely for almost every $x\in X$. Since $\mathrm{sgn}(v)\in L^1(-L,L)$, we may combine the weak–$\star$ convergence $f_n\overset{\star}{\rightharpoonup} f$, the fact each f_n is also a χ–function, and the Fubini theorem to deduce that

$$\lim_{n\to\infty}\mathbb{E}\int_X\int_{-L}^{L}|f_n(x)|^2\,dv\mu(dx)=\mathbb{E}\int_X\int_{-L}^{L}|f(x,v)|^2\,dv\mu(dx). \tag{4.26}$$

On the other hand, it is evident that

$$\underset{n\to\infty}{\text{weak–lim}}\, f_n(\,\cdot\,)=f \text{ weakly in } L^2(\widetilde{\Omega}\times X\times(-L,L)). \tag{4.27}$$

Consequently, harnessing (4.26) and (4.27) to the identity $|f_n-f|^2=f_n^2-2f_nf+f^2$, we conclude that

$$\lim_{n\to\infty}\mathbb{E}\int_X\int_{-L}^{L}|f_n(x,v)-f(x,v)|^2\,dv\mu(dx)=0,$$

as we wanted to show. □

Proof of Lemma 4.2. Taking $\widetilde{\Omega}=\Omega$, it is clear that Proposition 4.1 substantiates the equivalency between f^τ being a χ–function and $\operatorname{ess\,lim}_{s\to 0_+} f_\psi(\,\cdot\,,s,\,\cdot\,)=f^\tau$ strongly in $L^1(\Omega\times\Sigma\times(-L,L))$. On the other hand, if such a strong limit in $L^1(\Omega\times\Sigma\times(-L,L))$ is attained, there exists a sequence $s_n\to 0_+$ such that

$$f_\psi(\,\cdot\,,s_n,\,\cdot\,)\to f^\tau \text{ strongly } L^1(\Sigma\times(-L,L)) \text{ for almost every } \omega\in\Omega.$$

Consequently, resorting again to Proposition 4.1, we see that, for almost all $\omega\in\Omega$, $(t,\widehat{y},v)\in\Sigma\times(-L,L)\mapsto f^\tau(t,\widehat{y},v)$ is a χ–function (with $\widetilde{\Omega}$ being, say, a singleton). Hence, reducing Ω_0 in (4.19) if necessary, we see that, for all $\omega\in\Omega_0$, $\operatorname{ess\,lim}_{s\to 0_+} f_\psi(\,\cdot\,,s,\,\cdot\,)=f^\tau$. This proves the desired conclusion. □

The remainder of this chapter will be devoted to the verification that f^τ is indeed a χ-function.

4.4 The blow-up procedure

Before we initiate this section, let us briefly explicate the spirit of the localization procedure of A. Vasseur. Pick some $(t_0, m_0) \in (0,T) \times \Gamma$. By "flattening Γ_0", we may assume that $(0,T)\times\Gamma_0$ near (t_0, m_0) is indeed Σ; consider thus $\widetilde{f}(t,y,v)$ to be $f(t,x,v)$ in these new coordinates. To better comprehend the behavior of f and f^τ near (t_0,m_0), we may "zoom in" our problem by introducing the scaled functions $\widetilde{f_\varepsilon}(\underline{t},\underline{y},v) = f(t+\underline{t}/\varepsilon, m_0+\underline{y}/\varepsilon, v)$ for $\varepsilon > 0$. Choosing carefully such (t_0,m_0) and a sequence $\varepsilon_n \to 0_+$, it will be verified that the source terms of the kinetic equation (4.11) converge to 0 almost surely in an appropriate negative Sobolev space as $\varepsilon_n \to 0_+$. Hence, the averaging lemma will permit us to conclude $\widetilde{f_{\varepsilon_n}}$ converges strongly in L^1_{loc} as $\varepsilon_n \to 0_+$ (that is, as we "blow up" our variables). On the other hand, from the fact that $\widetilde{f_\varepsilon}$ has a weak trace, it can be shown that $\widetilde{f_{\varepsilon_n}}(\underline{t},\underline{y},v) \rightharpoonup f^\tau(t_0,m_0,v)$ in a weak sense. As a result, $f^\tau(t_0,m_0,v)$ is the strong limit of χ–functions, hence also a χ–function per Proposition 4.1. Lemma 4.2 will then imply Theorem 4.1.

Let us delve into the details of this program. Keep $\mathscr{O}_0$, as in (4.16), fixed.

Since f^τ does not depend on the Γ_0–strongly regular Lipschitz deformation, we may pick the special deformation $\widetilde{\psi}(s,\widehat{x}) = (\widehat{x}, \gamma(\widehat{x}) + s)$, which is trivially strongly regular over Γ_0. Identifying $y_N = s$ and $\widehat{y} = \widehat{x}$, define

$$\widetilde{f}(t,y,v) = f_\psi(t,\widehat{y},y_N,v) = f(t,\widetilde{\psi}(\widehat{y},y_N),v).$$

Notice that there exists an $r_0 > 0$ such that $\widetilde{\psi}(\widehat{y},y_N) \in \mathscr{O}_0$ provided that $(\widehat{y},y_N) \in (-r,r)^{N-1} \times (0,r_0) = \Sigma \times (0,r_0)$.

As a result, we see from (4.11) that $\widetilde{f}$ obeys almost surely the equation in $\mathscr{D}'((0,T)\times\Sigma\times(0,r_0))$

$$\frac{\partial \widetilde{f}}{\partial t} + \widehat{\mathbf{a}}(v)\cdot\nabla_{\widehat{y}}\widetilde{f} + \widetilde{\mathbf{a}_N}(\widehat{y},v)\frac{\partial \widetilde{f}}{\partial y_N} = \frac{\partial \widetilde{\mathbf{q}}}{\partial v} + \sum_{k=1}^\infty \widetilde{g}_k(y,v)\delta_{v=\widetilde{u}(t,y)}\frac{d\beta_k}{dt}. \tag{4.28}$$

In the equation above, we have denoted $\mathbf{a}(v) = (\widehat{\mathbf{a}}(v), \mathbf{a}_N(v)) \in \mathbb{R}^{N-1}\times\mathbb{R}$,

$$\widetilde{\mathbf{a}_N}(\widehat{y},v) = \mathbf{a}_N(v) - \nabla\gamma_0(\widehat{y})\cdot\widehat{\mathbf{a}}(v) = \lambda(\widehat{y})\mathbf{a}(v)\cdot\nu(\widehat{y}), \tag{4.29}$$

where $\lambda(\widehat{y}) = -\sqrt{1+|\nabla\gamma_0(\widehat{y})|^2} \neq 0$, and $\nu(\widehat{y})$ is the outward unit normal at $(\widehat{y},\gamma_0(\widehat{y})) \in \Gamma_0$; moreover, we have also written $\widetilde{\mathbf{q}}(t,x,v) = \widetilde{\mathbf{m}}(t,x,v) - \frac{1}{2}\widetilde{\mathfrak{G}}^2(x,v)\delta_{v=\widetilde{u}(t,y)}$, where

$$\begin{cases} \widetilde{u}(t,y) = u(t,\widetilde{\psi}(\widehat{y},y_N)) = \displaystyle\int_{-L}^{L}\widetilde{f}(t,y,v)\,dv, \\ \widetilde{\mathbf{m}}(t,y,v) = \mathbf{m}(t,\widetilde{\psi}(\widehat{y},y_N),v), \\ \widetilde{g}_k(y,v) = g_k(\widetilde{\psi}(\widehat{y},y_N),v) \text{ for all } k\geq 1, \text{ and} \\ \widetilde{\mathfrak{G}}^2(y,v) = \displaystyle\sum_{k=1}^\infty \widetilde{g}_k(y,v)^2. \end{cases}$$

Before we rescale $\widetilde{f}$, let us recall some preliminary lemmas regarding the "continuity" of some integrals. To facilitate their statements, extend $\widetilde{\mathbf{m}}(t,y,v)$ and $\widetilde{\mathfrak{G}}^2(y,v)$ to be zero outside $(0,T)\times\Sigma\times[-L,L]$. Notice that, in this case, Theorem 4.2 yields

$$\mathbb{E}\|\widetilde{\mathbf{m}}\|^p_{\mathfrak{M}(\mathbb{R}_t\times\mathbb{R}^N_y\times\mathbb{R}_v)} < \infty \tag{4.30}$$

for any $1\leq p<\infty$. Henceforth, $\Omega_0\subset\Omega$ be as in Lemma 4.1.

Lemma 4.3. *There exists a sequence $\varepsilon_n \to 0_+$ and a set of total measure $\mathcal{E}\subset\Sigma$ such that, for* *every* $(t_0, \widehat{y}_0) \in \mathcal{E}$*, and every* $R > 0$*,*

$$\lim_{n\to\infty} \mathbb{E}\frac{1}{\varepsilon_n^N}\widetilde{\mathbf{m}}\Big(\Big\{(t_0,\widehat{y}_0) + (-R\varepsilon_n, R\varepsilon_n)^N\Big\} \times (0, R\varepsilon_n) \times [-L, L]\Big) = 0, \ \textit{and} \tag{4.31}$$

$$\lim_{n\to\infty} \mathbb{E}\frac{1}{\varepsilon_n^N}\int\int_{\{(t_0,\widehat{y}_0)+(-R\varepsilon_n,R\varepsilon_n)^N\}\times(0,R\varepsilon_n)} \frac{1}{2}\widetilde{\mathfrak{G}}^2(y, \widetilde{u}(t,y))\,dydt = 0. \tag{4.32}$$

Consequently, for every $(t_0, \widehat{y}_0) \in \mathcal{E}$ *and every* $R > 0$*,*

$$\lim_{n\to\infty} \mathbb{E}\frac{1}{\varepsilon_n^N}|\widetilde{\mathbf{q}}|\Big(\Big\{(t_0,\widehat{y}_0) + (-R\varepsilon_n, R\varepsilon_n)^N\Big\} \times (0, R\varepsilon_n) \times [-L, L]\Big) = 0,$$

where, as usual, $|\widetilde{\mathbf{q}}|(A)$ *denotes the total variation of* $\widetilde{\mathbf{q}}$ *on the set* A*.*

As another consequence, given $(t_0, \widehat{y}_0) \in \mathcal{E}$*, there exists a subsequence of* $\varepsilon_n = \varepsilon_n(t_0, \widehat{y}_0)$*, still denoted* ε_n*, and a subset* $\Omega_1 = \Omega_1(t_0, \widehat{y}_0) \subset \Omega_0$ *of probability* 1*, such that, for all* $\omega \in \Omega_1$*, and* $R > 0$*,*

$$\begin{cases} \lim_{n\to\infty} \frac{1}{\varepsilon_n^N}\widetilde{\mathbf{m}}\Big(\Big\{(t_0,\widehat{y}_0) + (-R\varepsilon_n, R\varepsilon_n)^N\Big\} \times (0, R\varepsilon_n) \times [-L, L]\Big) = 0, \\ \lim_{n\to\infty} \frac{1}{\varepsilon_n^N}\int\int_{\{(t_0,\widehat{y}_0)+(-R\varepsilon_n,R\varepsilon_n)^N\}\times(0,R\varepsilon_n)} \frac{1}{2}\widetilde{\mathfrak{G}}^2(y, \widetilde{u}(t,y))\,dydt = 0, \ \textit{and} \\ \lim_{n\to\infty} \frac{1}{\varepsilon_n^N}|\widetilde{\mathbf{q}}|\Big(\Big\{(t_0,\widehat{y}_0) + (-R\varepsilon_n, R\varepsilon_n)^N\Big\} \times (0, R\varepsilon_n) \times [-L, L]\Big) = 0. \end{cases} \tag{4.33}$$

Proof. The verification of these limits is not significantly different from those featured in the work of Vasseur; nevertheless, the proof will be presented to show that the same arguments apply to this scenario as well.

Step #1: First, let us examine the limit in (4.31).

For any positive integer $M \geq 1$ and every $\varepsilon > 0$, let us consider the function $\mathcal{M}_\varepsilon^M \in L^1(\Sigma)$ given by

$$\mathcal{M}_\varepsilon^N(t,\widehat{y}) = \mathbb{E}\frac{1}{\varepsilon^N}\widetilde{\mathbf{m}}\Big(\Big\{(t,\widehat{y}) + (-M\varepsilon, M\varepsilon)^N\Big\} \times (0, M\varepsilon) \times [-L, L]\Big),$$

Writing $\widetilde{\mathbf{m}}(dt, d\widehat{y}, dy_N, dv) = \widetilde{\mathbf{m}}(t, \widehat{y}, y_N, v)\,dtd\widehat{y}dy_Ndv$ as it were a function for simplicity, we have thus that

$$\begin{aligned}
&\int_0^T \int_\Sigma \mathcal{M}_\varepsilon^M(t,\widehat{y})\,d\widehat{y}dt \\
&= \int_\Sigma \mathbb{E}\frac{1}{\varepsilon^N}\int_{-L}^{L}\int_0^{M\varepsilon}\int_{(-M\varepsilon,M\varepsilon)^N}\int_{-M\varepsilon}^{M\varepsilon} \widetilde{\mathbf{m}}(t+s, \widehat{y}+\widehat{z}, z_N, v)\,dsd\widehat{z}dz_Ndv \\
&\leq \mathbb{E}\frac{1}{\varepsilon^N}\int_{-M\varepsilon}^{M\varepsilon}\int_{(-M\varepsilon,M\varepsilon)^{N-1}}\int_0^{M\varepsilon} \\
&\qquad \int_{-L}^{L}\int_{(-r-M\varepsilon,r+M\varepsilon)^{N-1}}\int_0^T \widetilde{\mathbf{m}}(s, \widehat{y}, z_N, v)\,dtd\widehat{y}dvdz_Nd\widehat{y}ds \\
&\leq \mathbb{E}\frac{1}{\varepsilon^N}\int_{-M\varepsilon}^{M\varepsilon}\int_{(-M\varepsilon,M\varepsilon)^{N-1}} \widetilde{\mathbf{m}}\big(\Sigma \times (0, M\varepsilon) \times [-L, L]\big)d\widehat{z}ds \\
&\leq M^N\mathbb{E}\,\widetilde{\mathbf{m}}\big(\Sigma \times (0, M\varepsilon) \times [-L, L]\big)
\end{aligned}$$

(the presented calculation is correct, as it only involves the Fubini theorem and the linear change of variables formula—both of which are still valid for measures.) Now, due to (4.30) and the fact that $\widetilde{\mathbf{m}}(\Sigma \times (0, M\varepsilon) \times [-L, L]) \to 0$ almost surely when $\varepsilon \to 0_+$ (this is because $\cap_{\varepsilon>0}(0, M\varepsilon) = \varnothing$), the dominated convergence theorem asserts that

$$\lim_{\varepsilon\to 0_+}\int_0^T\int_\Sigma \mathcal{M}_\varepsilon^M(t,\widehat{y})\,d\widehat{y}dt=0.$$

As a result, there exists a sequence $\varepsilon_n\to 0$ and a set of total measure $\mathcal{E}_M^{(1)}\subset\Sigma$ such that $\mathcal{M}_{\varepsilon_n}^M(t,\widehat{y})\to 0$ for every $(t,\widehat{y})\in\mathcal{E}_M^{(1)}$. By diagonal extraction, we can construct a sequence $\varepsilon_n\to 0_+$ such that $\mathcal{M}_{\varepsilon_n}^M(t,\widehat{y})\to 0$ for every $M\geq 1$ and $(t,\widehat{y})\in\mathcal{E}_M^{(1)}$. The sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ and $\mathcal{E}^{(1)}=\cap_{M=1}^\infty\mathcal{E}_M^{(1)}$ satisfy the required conditions, except for the second limit (4.32).

Step #2: We will now analyze (4.32). If we repeat the previous reasoning for the positive measure $\mu=\widetilde{\mathfrak{G}}^2\delta_{\widetilde{u}=v}$, we can find a subsequence of ε_n, still denoted by ε_n, and $\mathcal{E}^{(2)}\subset\Sigma$, still of total measure, for which (4.32) holds. Taking this novel $(\varepsilon_n)_{n\in\mathbb{N}}$ and $\mathcal{E}=\mathcal{E}^{(1)}\cap\mathcal{E}^{(2)}$ yields the desired conclusion. (Actually, the calculations here are much easier in virtue of (4.4) and the fact that $\|\widetilde{u}(t,y)\|_\infty\leq L$ almost surely.)

Step #3: The statement about $|\widetilde{\mathbf{q}}|$ is immediate, as $\widetilde{\mathbf{q}}=\widetilde{\mathbf{m}}-\frac{1}{2}\widetilde{\mathfrak{G}}^2\delta_{v=\widetilde{u}}$, and this is a decomposition into the difference of two positive measures. At last, all limits in (4.33) are now consequence of the Fisher–Riesz theorem. □

Likewise, extend f^τ and $\widetilde{\mathbf{a}}_N$ to be zero outside of $(0,T)\times\Sigma\times(-L,L)$ in the next lemma.

Lemma 4.4. *There exists a subsequence of ε_n, still denoted by ε_n, and a subset of $\mathcal{E}\subset\Sigma$, also of total measure and still denoted by $\mathcal{E}$, such that, for every $(t_0,\widehat{y}_0)\in\mathcal{E}$, every $R>0$, and every $1\leq p<\infty$,*

$$\int_{-R}^R\int_{(-R,R)^{N-1}}\int_{-L}^L|\widetilde{\mathbf{a}}_N(\widehat{y}_0,v)-\widetilde{\mathbf{a}}_N(\widehat{y}_0+\varepsilon_n\underline{\widehat{y}},v)|^p\,dvd\underline{\widehat{y}}d\underline{t}\to 0,\ \text{and}\tag{4.34}$$

$$\mathbb{E}\int_{-R}^R\int_{(-R,R)^{N-1}}\int_{-L}^L|f^\tau(t_0+\varepsilon_n\underline{t}\,,\widehat{y}_0+\varepsilon_n\underline{\widehat{y}},,v)-f^\tau(t_0,\widehat{y}_0,v)|^p\,dvd\underline{\widehat{y}}\,d\underline{t}\to 0,\tag{4.35}$$

as $n\to\infty$.

Therefore, given $(t_0,\widehat{y}_0)\in\mathcal{E}$, there exists a subsequence of ε_n also denoted $\varepsilon_n=\varepsilon_n(t_0,\widehat{y}_0)$, and a subset of $\Omega_2(t_0,\widehat{y}_0)\subset\Omega_1(t_0,\widehat{y}_0)$, also of probability one, such that, for all $\omega\in\Omega_2$, $1\leq p<\infty$, and $R>0$,

$$\int_{-R}^R\int_{(-R,R)^{N-1}}\int_{-L}^L|f^\tau(t_0+\varepsilon_n\underline{t}\,,\widehat{y}_0+\varepsilon_n\underline{\widehat{y}},v)-f^\tau(t_0,\widehat{y}_0,v)|^p\,dvd\underline{\widehat{y}}d\underline{t}\to 0\ \text{as}\ \varepsilon_n\to 0_+.\tag{4.36}$$

Proof. The demonstration, which is almost identical to the previous one, will be omitted, for the details may be consulted in the original paper of A. Vasseur [110], lemma 3. The only difference here is the power p in (4.35)–(4.36), which is evidently acceptable, as the integrand is uniformly bounded in L^∞. □

We are in conditions to define our scaled functions. Let $\mathcal{E}\subset\Sigma$ be as in the statement of Lemmas 4.3 and 4.4, and pick some $(t_0,\widehat{y}_0)\in\mathcal{E}$. Consider $(\varepsilon_n)_{n\in\mathbb{N}}$ and $\Omega_2(t_0,\widehat{y}_0)$ to be as in such lemmas, and $R=R(t,\widehat{y}_0)$ to be the least number between $|r\pm(\widehat{y}_0)_j|$ $(1\leq j<N)$, r_0, $T-t$ and t. In such a way, we may now introduce

$$\widetilde{f}_\varepsilon(\underline{t},\underline{y},v)=\widetilde{f}(t_0+\varepsilon\underline{t},\widehat{y}_0+\varepsilon\underline{y},v)\tag{4.37}$$

for any $\varepsilon>0$, $\omega\in\Omega$, $-L<v<L$, and

$$(\underline{t},\underline{y})=(\underline{t},\underline{\widehat{y}},\underline{y_N})\in(-R/\varepsilon,R/\varepsilon)\times(-R/\varepsilon,R/\varepsilon)^{N-1}\times(0,R/\varepsilon)\stackrel{\text{def}}{=}\varDelta_\varepsilon.$$

Even though $\widetilde{f}_\varepsilon$ clearly depends on $(t_0, \widehat{y}_0)$, we will omit this dependence once the point in question will be fixed throughout this section.

Clearly, each $\widetilde{f}_\varepsilon$ is still a χ–function, and, in the sense of weak traces,

$$\widetilde{f}_\varepsilon(\underline{t}, \underline{\widehat{y}}, 0, v) = f^\tau(t_0 + \varepsilon \underline{t}, \widehat{y}_0 + \varepsilon \underline{\widehat{y}}, v), \tag{4.38}$$

for $-L < v < L$, and

$$(\underline{t}, \underline{\widehat{y}}) \in (-R/\varepsilon, R/\varepsilon) \times (-R/\varepsilon, R/\varepsilon)^{N-1} \overset{\text{def}}{=} \Sigma_\varepsilon.$$

Finally, let us derive the differential equation $\widetilde{f}_\varepsilon$ satisfies. Pick a test function $\varphi \in \mathscr{C}_c^\infty(\varDelta_\varepsilon \times \mathbb{R}_v)$, so that $(s, z, v) \mapsto \varphi\big(\frac{1}{\varepsilon}(s - t_0), \frac{1}{\varepsilon}(z - \widehat{y}_0), v\big)$ can be applied to (4.28), yielding, almost surely,

$$\begin{aligned}
&\int_{(t_0,\widehat{y}_0)+\varDelta_1} \int_{-L}^{L} \widetilde{f}(s, z, v) \frac{1}{\varepsilon} \left[\frac{\partial \varphi}{\partial s} + \widehat{\mathbf{a}}(v) \cdot \nabla_{\widehat{z}} \varphi + \widetilde{\mathbf{a}_N}(z, v) \frac{\partial \varphi}{\partial z_N} \right] \left(\frac{1}{\varepsilon}(s - t_0), \frac{1}{\varepsilon}(z - \widehat{y}_0), v \right) dv dz ds \\
&\quad = \int_{(t_0,\widehat{y}_0)+\varDelta_1} \int_{-L}^{L} \frac{\partial \varphi}{\partial v} \left(\frac{1}{\varepsilon}(s - t_0), \frac{1}{\varepsilon}(z - \widehat{y}_0), v \right) d\widetilde{\mathbf{q}}(ds, dz, dv) \\
&\qquad + \sum_{k=1}^{\infty} \int_{(t_0,\widehat{y}_0)+\varDelta_1} \int_{-L}^{L} \widetilde{g}_k(s, z, v) \delta_{\widetilde{u}(s,z)}(v) \varphi \left(\frac{1}{\varepsilon}(s - t_0), \frac{1}{\varepsilon}(z - \widehat{y}_0), v \right) dv dz d\beta_k(s).
\end{aligned} \tag{4.39}$$

For every $k \geq 1$, let $\widetilde{\mathfrak{g}}_k(t, x, v)$ be such that, for almost every $(t, x) \in \varDelta_1$,

$$\begin{cases} \dfrac{\partial \widetilde{\mathfrak{g}}_k}{\partial v}(t, x, v) + \widetilde{\mathfrak{g}}_k(t, x, v) = \widetilde{g}_k(x, v) \delta_{\widetilde{u}(t,x)}(v) \quad \text{in } \mathscr{D}'(\mathbb{R}_v), \text{ and} \\ \displaystyle\int_{-\infty}^{\infty} |\widetilde{\mathfrak{g}}_k(t, x, v)|^2 \, dv < \infty; \end{cases}$$

that is, using the basic techniques for Sturm–Liouville problems, $\widetilde{\mathfrak{g}}_k(t, x, v)$ may be expressed via the Green function formula

$$\widetilde{\mathfrak{g}}_k(t, x, v) = 1_{(\widetilde{u}(t,x),\infty)}(v) e^{-(v - \widetilde{u}(t,x))} \widetilde{g}_k(x, \widetilde{u}(t, x)). \tag{4.40}$$

With this new notation, the last term in (4.39) could as well have been phrased as

$$\begin{aligned}
&\sum_{k=1}^{\infty} \int_{(t_0,\widehat{y}_0)+\varDelta_1} \int_{-L}^{L} \widetilde{g}_k(s, z, v) \delta_{\widetilde{u}(s,z)}(v) \varphi \left(\frac{1}{\varepsilon}(s - t_0), \frac{1}{\varepsilon}(z - \widehat{y}_0), v \right) dv dz d\beta_k(s) \\
&\quad = \sum_{k=1}^{\infty} \int_{(t_0,\widehat{y}_0)+\varDelta_1} \int_{\mathbb{R}_v} \widetilde{\mathfrak{g}}_k(s, z, v) \left(-\frac{\partial}{\partial v} + 1 \right) \varphi \left(\frac{1}{\varepsilon}(s - t_0), \frac{1}{\varepsilon}(z - \widehat{y}_0), v \right) dv dz d\beta_k(s) \\
&\quad = -\frac{1}{\varepsilon} \int_{(t_0,\widehat{y}_0)+\varDelta_1} \int_{\mathbb{R}_v} \left[\sum_{k=1}^{\infty} \int_{t_0}^{s} \widetilde{\mathfrak{g}}_k(\xi, z, v) \, d\beta_k(\xi) \right] \\
&\qquad\qquad \frac{\partial}{\partial s} \left(-\frac{\partial}{\partial v} + 1 \right) \varphi \left(\frac{1}{\varepsilon}(s - t_0), \frac{1}{\varepsilon}(z - \widehat{y}_0), v \right) dv dz ds
\end{aligned} \tag{4.41}$$

(Here we are employing the natural convention that, if $X(t)$ is a predictable stochastic process, $\int_{t_0}^{t_1} X(t) \, d\beta_k(t) = -\int_{t_1}^{t_0} X(t) \, d\beta_k(t)$, for all $k \in \mathbb{N}$, and t_0 and $t_1 \geq 0$). In virtue of the explicit formula (4.40), it is clear that the integrals $\sum_{k=1}^{\infty} \int_{t_0}^{s} \widetilde{\mathfrak{g}}_k(\xi, z, v) \, d\beta_k(\xi)$ define a legitimate $L^2(\mathbb{R}_z^N \times \mathbb{R}_v)$–valued stochastic process; thus put

$$\widetilde{\Lambda}(s - t_0, z - \widehat{y}_0, v) = \sum_{k=1}^{\infty} \int_{t_0}^{s} \widetilde{\mathfrak{g}}_k(\xi, z, v) \, d\beta_k(\xi).$$

Inserting (4.41) into (4.39), and realizing the change of variables $\big(\frac{1}{\varepsilon}(s-t_0), \frac{1}{\varepsilon}(z-\widehat{y}_0)\big) \leftrightarrow (\underline{t}, \underline{y})$, we deduce that

$$\int_{\Delta_\varepsilon}\int_{-L}^{L} \widetilde{f}_\varepsilon(\underline{t},\underline{y},v)\left[\frac{\partial\varphi}{\partial\underline{t}} + \widehat{\mathbf{a}}(v)\cdot\nabla_{\widehat{\underline{y}}}\varphi + \widetilde{\mathbf{a}_N}(\widehat{y}_0+\varepsilon\underline{y},v)\frac{\partial\varphi}{\partial\underline{y_N}}\right](\underline{t},\underline{y},\xi)\,dvd\underline{y}d\underline{t}$$
$$= \frac{1}{\varepsilon^N}\int_{(t_0,\widehat{y}_0)+\Delta_1}\int_{-L}^{L}\frac{\partial\varphi}{\partial v}(\underline{t},\underline{y},v)\,d\widetilde{\mathbf{q}}(t+d(\varepsilon\underline{t}),\widehat{y}_0+d(\varepsilon\underline{y}),dv)$$
$$-\int_{\Delta_\varepsilon}\int_{-L}^{L}\widetilde{\Lambda}(\varepsilon\underline{t},\varepsilon\underline{y},v)\frac{\partial}{\partial\underline{t}}\Big(-\frac{\partial}{\partial v}+1\Big)\varphi(\underline{t},\underline{y},v)\,dvd\underline{y}d\underline{t}. \tag{4.42}$$

In order to simplify this formula, we may argue as follows. Define, almost surely, the measure $\widetilde{\mathbf{m}}_\varepsilon(t,x,v)$ by

$$\widetilde{\mathbf{m}}_\varepsilon\Big(\prod_{j=0}^{N}[a_j,b_j]\times[L_1,L_2]\Big)$$
$$= \frac{1}{\varepsilon^N}\widetilde{\mathbf{m}}\Big([t_0+\varepsilon a_0, t_0+\varepsilon b_0]\times\Big[\widehat{y}_0+\prod_{j=1}^{N-1}[\varepsilon a_j,\varepsilon b_j]\Big]\times[\varepsilon a_N,\varepsilon b_N]\times[L_1,L_2]\Big).$$

for every $a_0<b_0$, …, $a_N<b_N$, and $L_1<L_2$. Additionally, if we introduce the new quantities

$$\begin{cases} \widetilde{u}_\varepsilon(\underline{t},\underline{y}) \overset{\text{def}}{=} \int_{-L}^{L}\widetilde{f}_\varepsilon(\underline{t},\underline{y},v)\,dv = \widetilde{u}(t_0+\varepsilon\underline{t},\widehat{y}_0+\varepsilon\underline{y}),\\ \widetilde{g}_{k,\varepsilon}(\underline{y},v) \overset{\text{def}}{=} \widetilde{g}_k(\widehat{y}_0+\varepsilon\underline{y},v),\\ \widetilde{\mathfrak{G}}^2_\varepsilon(\underline{y},v) \overset{\text{def}}{=} \sum_{k=1}^{\infty}\widetilde{g}^2_{k,\varepsilon}(\underline{y},\xi) = \widetilde{\mathfrak{G}}^2(\widehat{y}_0+\varepsilon\underline{y},v),\\ \widetilde{\mathbf{q}}_\varepsilon(\underline{t},\underline{y},v) \overset{\text{def}}{=} \widetilde{\mathbf{m}}_\varepsilon(\underline{t},\underline{y},v) - \widetilde{\mathfrak{G}}^2_\varepsilon(\underline{y},v)\delta_{\widetilde{u}_\varepsilon(\underline{t},\underline{y})}(v), \text{ and}\\ \widetilde{\Lambda}_\varepsilon(\underline{t},\underline{y},v) \overset{\text{def}}{=} \widetilde{\Lambda}(\varepsilon\underline{t},\varepsilon\underline{y},v) = \sum_{k=1}^{\infty}\int_{t_0}^{t_0+\varepsilon\underline{t}}\widetilde{\mathfrak{g}}_k(\xi,y+\varepsilon\underline{y},v)\,d\beta_k(\xi), \end{cases}$$

we may thus convert (4.42) into the pleasing notation

$$\frac{\partial\widetilde{f}_\varepsilon}{\partial\underline{t}} + \widehat{\mathbf{a}}(v)\cdot\nabla_{\widehat{\underline{y}}}\widetilde{f}_\varepsilon + \widetilde{\mathbf{a}_N}(\widehat{y},v)\frac{\partial\widetilde{f}_\varepsilon}{\partial\underline{y}_N} = \frac{\partial}{\partial\underline{y}_N}\Big(\big(\widetilde{\mathbf{a}_N}(\widehat{y},v)-\widetilde{\mathbf{a}_N}(\widehat{y}+\varepsilon\widehat{\underline{y}},v)\big)\widetilde{f}_\varepsilon\Big)$$
$$+\frac{\partial\widetilde{\mathbf{q}}_\varepsilon}{\partial v}(\underline{t},\underline{y},v) + \frac{\partial}{\partial\underline{t}}\Big(\frac{\partial}{\partial v}+1\Big)\widetilde{\Lambda}_\varepsilon(\underline{t},\underline{y},v) \tag{4.43}$$

almost surely in $\mathscr{D}'(\Delta_\varepsilon\times\mathbb{R}_v)$.

In accordance to Lemmas 4.3 and 4.4, we can state a result on the evanescence of $\widetilde{\Lambda}_\varepsilon(\underline{t},\underline{y},v)$. As in the aforementioned propositions, extend $\Lambda_\varepsilon(\underline{t},\underline{y},v)$ to be zero outside of $\Delta_\varepsilon\times\mathbb{R}_v$.

Lemma 4.5. *For every $R>0$, it holds that*

$$\lim_{\varepsilon\to0_+}\mathbb{E}\int_{-R}^{R}\int_{(-R,R)^{N-1}\times(0,R)}\int_{\mathbb{R}_v}|\Lambda_\varepsilon(\underline{t},\underline{y},v)|^2\,dv\,d\underline{y}\,d\underline{t} = 0. \tag{4.44}$$

Therefore, if $\Omega_2(t_0,\widehat{y}_0)\subset\Omega_1(t_0,\widehat{y}_0)$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ are in Lemmas 4.3 and 4.4, there exists a subset $\Omega_3(t_0,\widehat{y}_0)\subset\Omega_2(t_0,\widehat{y}_0)$, still of probability one, and a subsequence of ε_n, still denoted as such, satisfying

$$\lim_{\varepsilon_n\to0_+}\int_{-R}^{R}\int_{(-R,R)^{N-1}\times(0,R)}\int_{\mathbb{R}_v}|\Lambda_\varepsilon(\underline{t},\underline{y},v)|^2\,dv\,d\underline{y}\,d\underline{t} = 0 \tag{4.45}$$

for every $R > 0$, *and* $\omega \in \Omega_3(t_0, \widehat{y}_0)$.

Proof. Let $M \geq 1$ be an integer, and choose $\varepsilon > 0$ sufficiently small, so that $\varDelta_\varepsilon$ contains $(-M,M)\times(-M,M)^{N-1} \times (0,M)$. Then, according to the Itô's isometry, for all $-M \leq \underline{t} \leq M$,

$$\begin{aligned}\mathbb{E}\int_{(-M,M)^{N-1}\times(0,M)}\int_{\mathbb{R}_v} &|\Lambda_\varepsilon(\underline{t},\underline{y},v)|^2\,dv\,d\underline{y}\\ &= \mathbb{E}\int_{(-M,M)^{N-1}\times(0,M)}\int_{\mathbb{R}_v}\left|\sum_{k=1}^{\infty}\int_{t_0}^{t_0+\varepsilon\underline{t}}\widetilde{\mathfrak{g}}_k(\xi,y+\varepsilon\underline{y},v)\,d\beta_k(\xi)\right|^2 dv\,d\underline{y}\\ &= \left|\mathbb{E}\int_{t_0}^{t_0+\varepsilon\underline{t}}\int_{(-M,M)^{N-1}\times(0,M)}\int_{\mathbb{R}_v}\sum_{k=1}^{\infty}|\widetilde{\mathfrak{g}}_k(\xi,y+\varepsilon\underline{y},v)|^2\,dv\,d\underline{y}\,d\xi\right|. \qquad (4.46)\end{aligned}$$

On the other hand, since $|\widetilde{u}(t_0+\varepsilon\underline{t},\widehat{y}_0+\varepsilon\underline{y})| \leq L$, it is clear from the explicit formula (4.40) and from (4.4) that

$$\sum_{k=1}^{\infty}|\widetilde{\mathfrak{g}}_k(\xi,y+\varepsilon\underline{y},v)|^2 \leq (\text{const.})e^{-v}1_{(-L,\infty)}(v)$$

for all $|\xi - t_0| \leq \varepsilon$, $\underline{y} \in (-M,M)^{N-1}\times(0,M)$, and $v \in \mathbb{R}$. Accordingly, (4.46) yields

$$\mathbb{E}\int_{(-M,M)^{N-1}\times(0,M)}\int_{\mathbb{R}_v}|\Lambda_\varepsilon(\underline{t},\underline{y},v)|^2\,dv\,d\underline{y} \leq C2^{N-1}M^N\varepsilon,$$

which as a consequence clearly proves that

$$\mathbb{E}\int_{-M}^{M}\int_{(-M,M)^{N-1}\times(0,M)}\int_{\mathbb{R}_v}|\Lambda_\varepsilon(\underline{t},\underline{y},v)|^2\,dv\,d\underline{y}\,d\underline{t} \leq C2^N M^{N+1}\varepsilon \to 0 \text{ as } \varepsilon \to 0_+. \qquad (4.47)$$

Hence (4.44) is established.

So as to verify (4.45), one may argue as in Lemma 4.3. By the Fischer–Riesz theorem and a diagonal extraction, one may construct a subsequence of $(\varepsilon_n)_{n\in\mathbb{N}}$, which we will still denote by $(\varepsilon_n)_{n\in\mathbb{N}}$, and a set of probability one $\Omega_3(t_0,\widehat{y}_0) \subset \Omega_2(t_0,\widehat{y}_0)$, such that

$$\lim_{\varepsilon_n\to 0_+}\int_{-M}^{M}\int_{(-M,M)^{N-1}\times(0,M)}\int_{\mathbb{R}_v}|\Lambda_{\varepsilon_n}(\underline{t},\underline{y},v)|^2\,dv\,d\underline{y}\,d\underline{t} = 0$$

for any integer $M \geq 1$ and $\omega \in \Omega_3(t_0,\widehat{y}_0)$. This subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$ and this subset $\Omega_3(t_0,\widehat{y}_0)$ are evidently in agreement with the statement of this lemma; the proof is thus complete. □

Finally, we are in conditions to fathom the comportment of $\widetilde{f}_{\varepsilon_n}$ as $\varepsilon_n \to 0_+$. Again, for consistency issues, consider $\widetilde{f}_{\varepsilon_n}$ to be zero outside $\varDelta_\varepsilon \times [-L,L]$.

Lemma 4.6. *Let* $\Omega_3(t_0,\widehat{y}_0)$ *and* $\varepsilon_n = \varepsilon_n(t_0,\widehat{y}_0) \to 0_+$ *be as in Lemma 4.5. Then, for all* $\omega \in \Omega_3(t_0,\widehat{y}_0)$, *it holds that*

$$\widetilde{f}_{\varepsilon_n}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,) \overset{\star}{\rightharpoonup} f^\tau(t_0,\widehat{y}_0,\,\cdot\,) \text{ weakly-}\star \text{ in } L^\infty(\mathbb{R}\times\mathbb{R}^{N-1}\times(0,\infty)\times\mathbb{R}). \qquad (4.48)$$

Proof. Fix $\omega \in \Omega_3(t_0,\widehat{y}_0)$, and let $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}_{\underline{t}}\times\mathbb{R}^N_{\underline{y}}\times\mathbb{R}_v)$ be arbitrary. If $\varrho_\eta \in \mathscr{C}_c^\infty(\mathbb{R})$ $(0<\eta<1)$ are mollifiers in the real line, put

$$\varphi_\eta(\underline{t},\underline{y},v) = \varphi(\underline{t},\underline{y},v)\int_0^{\underline{y}_N}\varrho_\eta(s-2\eta)\,ds,$$

so that $\varphi_\eta \in \mathscr{C}_c^\infty(\mathbb{R}\times\mathbb{R}^{N-1}\times(0,\infty)\times\mathbb{R}_v)$. If ε is sufficiently small, we may plug φ_η into Equation (4.28). Therefore, passing $\eta \to 0_+$ and recalling that $\star\!-\operatorname{ess\,lim}_{\underline{y}_N\to 0_+}\widetilde{f}_\varepsilon(\underline{t},\underline{\widehat{y}},\underline{y}_N,v) = f^\tau(t_0+\varepsilon\underline{t},\widehat{y}_0+$

$\varepsilon\underline{\widehat{y}}, v)$ weakly–$\star$ in $L^\infty(\Sigma_\varepsilon \times \mathbb{R})$ (see Equation (4.38)), we may justify the formula

$$\begin{aligned}\int_{\mathbb{R}}\int_0^\infty\int_{\mathbb{R}^{N-1}}\int_0^\infty\int_{\mathbb{R}} \widetilde{f}_\varepsilon\Big(\frac{\partial\varphi}{\partial\underline{t}} + \widehat{\mathbf{a}}(v)\cdot\nabla_{\underline{\widehat{y}}}\varphi + \widetilde{\mathbf{a}_N}(\widehat{y}_0 + \varepsilon\underline{\widehat{y}}, v)\frac{\partial\varphi}{\partial\underline{y}_N}\Big)\, d\underline{t}\, d\underline{\widehat{y}}\, d\underline{y}_N\, dv \\ + \int_{\mathbb{R}}\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}} \widetilde{\mathbf{a}_N}(\widehat{y}_0 + \varepsilon\underline{\widehat{y}}, v) f^\tau(t_0 + \varepsilon\underline{t}, \widehat{y}_0 + \varepsilon\underline{\widehat{y}}, v)\varphi(\underline{t}, \underline{\widehat{y}}, 0, v)\, d\underline{t}\, d\underline{\widehat{y}}\, dv \\ = \int_{\mathbb{R}}\int_0^\infty\int_{\mathbb{R}^{N-1}}\int_0^\infty\int_{\mathbb{R}} \frac{\partial\varphi}{\partial v}\,\mathbf{q}_\varepsilon(d\underline{t}, d\underline{y}, dv) \\ + \int_{\mathbb{R}}\int_0^\infty\int_{\mathbb{R}^{N-1}}\int_0^\infty\int_{\mathbb{R}} \widetilde{\Lambda}_\varepsilon \frac{\partial}{\partial\underline{t}}\Big(-\frac{\partial}{\partial v} + 1\Big)\varphi\, dv\, d\underline{y}\, d\underline{t}. \end{aligned} \tag{4.49}$$

Let us choose then $\varepsilon = \varepsilon_n(t_0, \widehat{y}_0)$.

Invoking the Banach–Alaoglu–Bourbaki theorem, there exists a subsequence ε_n' of ε_n for which $\widetilde{f}_{\varepsilon_n'} \overset{\star}{\rightharpoonup} f$ in the weak–$\star$ topology of $L^\infty(\mathbb{R}\times\mathbb{R}^{N-1}\times(0,\infty)\times\mathbb{R})$ for some $\widetilde{f} \in L^\infty(\mathbb{R}\times\mathbb{R}^{N-1}\times(0,\infty)\times(-L,L))$. In virtue of Lemma 4.3, $\mathbf{q}_{\varepsilon_n} \rightharpoonup 0$ in the sense of measures, whereas Lemma 4.4 asserts that the coefficient $\widetilde{\mathbf{a}_N}(\widehat{y}_0 + \varepsilon_n\underline{\widehat{y}}, v)$ converges strongly in L^1_{loc} to $\widetilde{\mathbf{a}_N}(\widehat{y}_0, v)$, and that $f^\tau(t_0 + \varepsilon\underline{t}, \widehat{y}_0 + \varepsilon\underline{\widehat{y}}, v) \to f^\tau(t_0, \widehat{y}_0)$ strongly in L^1_{loc}. Finally, as Lemma 4.5 shows that $\widetilde{\Lambda}_{\varepsilon_n} \to 0$ equally in L^1_{loc}, so that the passage $\varepsilon_n' \to 0_+$ transforms (4.49) into

$$\begin{aligned}\int_{\mathbb{R}}\int_0^\infty\int_{\mathbb{R}^{N-1}}\int_0^\infty\int_{\mathbb{R}} \widetilde{f}\Big(\frac{\partial\varphi}{\partial\underline{t}} + \widehat{\mathbf{a}}(v)\cdot\nabla_{\underline{\widehat{y}}}\varphi + \widetilde{\mathbf{a}_N}(\widehat{y}_0, v)\frac{\partial\widetilde{\varphi}}{\partial\underline{y}_N}\Big)\, d\underline{t}\, d\underline{\widehat{y}}\, d\underline{y}_N\, dv \\ + \int_{\mathbb{R}}\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}} \widetilde{\mathbf{a}_N}(\widehat{y}_0, v) f^\tau(t_0, \widehat{y}_0, v)\varphi(\underline{t}, \underline{\widehat{y}}, 0, v)\, d\underline{t}\, d\underline{\widehat{y}}\, dv = 0 \end{aligned} \tag{4.50}$$

In other terms, $\widetilde{f}(\underline{t}, \underline{y}, v)$ is a weak solution to the simple transport equation

$$\begin{cases} \widetilde{\mathbf{a}_N}(\widehat{y}_0, v)\dfrac{\partial\widetilde{f}}{\partial\underline{y}_N} + \dfrac{\partial\widetilde{f}}{\partial t} + \widehat{\mathbf{a}}(v)\cdot\nabla_{\underline{\widehat{y}}}\widetilde{f} = 0 & \text{for } (\underline{t}, \underline{y}, v) \in \mathbb{R}\times\mathbb{R}^{N-1}\times\mathbb{R}, \text{ and } \underline{y}_N > 0, \text{ and} \\ \widetilde{\mathbf{a}_N}(\widehat{y}_0, v)\widetilde{f} = \widetilde{\mathbf{a}_N}(\widehat{y}_0, v) f^\tau(t_0, \widehat{y}_0, v) & \text{for } (\underline{t}, \underline{y}, v) \in \mathbb{R}\times\mathbb{R}^{N-1}\times\mathbb{R}, \text{ and } \underline{y}_N = 0. \end{cases}$$

Notice that, in the light of (4.29) and nondegeneracy condition (4.8), the manifold $\{\,\underline{y}_N = 0\,\}$ is, for almost every $-L \le v \le L$, noncharactheristic. Consequently, it is not difficult to apply the method of the characteristics and deduce that

$$\widetilde{f}\big(t + \underline{y}_N, m + \widehat{\mathbf{a}}_{\widehat{y}}(v)\underline{y}_N, |\widetilde{\mathbf{a}}_N(\widehat{y}_0, v)|\underline{y}_N, v\big) = f^\tau(t_0, \widehat{y}_0, v)$$

for almost every $(s, m, v) \in \mathbb{R}\times\mathbb{R}^{N-1}\times(-L, L)$, and $\underline{y}_N > 0$. (For instance, consider for each $v \in \mathbb{R}$ linear change of variables $(\underline{t}, \widehat{y}, \underline{y}_N) \leftrightarrow (\tau + m_N, \widehat{m} + m_N\widehat{\mathbf{a}}(v), \widetilde{\mathbf{a}}_N(\widehat{y}_0, v)m_N)$ where $(\tau, \widehat{m}) \in \mathbb{R}\times\mathbb{R}^{N-1}$, and $m_N \in (0,\infty)$ if $\widetilde{\mathbf{a}}_N(v) > 0$ or $m_N \in (-\infty, 0)$ if, otherwise, $\widetilde{\mathbf{a}}_N(v) < 0$). Once more, for $\widetilde{\mathbf{a}}_N(\widehat{y}_0, v) \neq 0$ for almost every $-L < v < L$, and due to both $\widetilde{f}$ and f^τ being supported for $-L \le v \le L$, we conclude thus that

$$\widetilde{f}(\underline{t}, \underline{y}, v) = f^\tau(t_0, \widehat{y}_0, v) \text{ for almost every } (\underline{t}, \underline{y}, v) \in \mathbb{R}\times\mathbb{R}^{N-1}\times(0,\infty)\times(-L, L).$$

Because this same conclusion must hold for all limit points of the scaled sequence $\widetilde{f}_{\varepsilon_n}$, it follows that it was not necessary to consider the subsequence ε_n' after all, but indeed $\widetilde{f}_{\varepsilon_n} \overset{\star}{\rightharpoonup} f^\tau(t_0, y_0, \cdot)$ weakly–$\star$ in $L^\infty(\mathbb{R}\times\mathbb{R}^{N-1}\times(0,\infty)\times\mathbb{R}_v)$ for every $\omega \in \Omega_3$. The proof is complete. □

Let us then verify that the weak–$\star$ limit in (4.48) is indeed strong.

Lemma 4.7. *Let $\Omega_3(t_0, \widehat{y}_0)$ and $\varepsilon_n = \varepsilon_n(t_0, \widehat{y}_0) \to 0_+$ be as in Lemma 4.5. Then, for all $\omega \in$*

$\Omega_3(t_0,\widehat{y}_0)$, *it holds that*

$$\widetilde{f}_{\varepsilon_n}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,)\to f^\tau(t_0,\widehat{y}_0,\,\cdot\,)\text{ strongly in }L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^{N-1}\times(0,\infty)\times\mathbb{R}).$$

Proof. We will finally make use of the averaging lemma. Consider any open set $\mathscr{U}\subset\subset\mathbb{R}\times\mathbb{R}^{N-1}\times(0,\infty)$, and consider some $\theta\in\mathscr{C}^\infty_c(\mathbb{R}\times\mathbb{R}^{N-1}\times(0,\infty))$ such that $\theta(\underline{t},\underline{y})=1$ in $\mathscr{U}$. From Equation (4.28), it is clear that each $(\theta\widetilde{f}_\varepsilon)$ obeys

$$\begin{aligned}\frac{\partial(\theta\widetilde{f}_\varepsilon)}{\partial\underline{t}}&+\widehat{\mathbf{a}}(v)\cdot\nabla_{\widehat{\underline{y}}}(\theta\widetilde{f}_\varepsilon)+\widetilde{\mathbf{a}_N}(\widehat{y}_0,v)\frac{\partial(\theta\widetilde{f}_\varepsilon)}{\partial\underline{y}_N}\\&=\frac{\partial}{\partial\underline{y}_N}\Big(\big(\widetilde{\mathbf{a}_N}(\widehat{y}_0,v)-\widetilde{\mathbf{a}_N}(\widehat{y}_0+\varepsilon\widehat{\underline{y}},v)\big)(\theta\widetilde{f}_\varepsilon)\Big)+\frac{\partial(\theta\widetilde{\mathbf{q}}_\varepsilon)}{\partial v}(\underline{t},\underline{y},v)+\frac{\partial}{\partial\underline{t}}\Big(\frac{\partial}{\partial v}+1\Big)(\theta\widetilde{\Lambda}_\varepsilon)\end{aligned}\tag{4.51}$$

in $\mathscr{D}'(\mathscr{U}\times\mathbb{R}_v)$ for all $\varepsilon>0$ sufficiently small.

Fix $\omega\in\Omega_3(t_0,\widehat{y}_0)$ now, and plug $\varepsilon=\varepsilon_n$ in (4.51).

Let $0<\delta<\alpha$ and $1<q<\frac{N+2}{N+2-\delta}$. Due to Lemma 4.4, the first term in the right-hand side of (4.51) symbolizes a vanishing element of $L^q(\mathbb{R}_v;W^{-1,q}(\mathbb{R}_{\underline{t}}\times\mathbb{R}^N_{\underline{y}}))$. On the other hand, as we have argued in the previous chapter, Lemma 4.3 and the Morrey's theorem show that $(\theta\widetilde{\mathbf{q}}_{\varepsilon_n})$ forms a vanishing sequence in $W^{-\delta,q}(\mathbb{R}_{\underline{t}}\times\mathbb{R}^N_{\underline{y}}\times\mathbb{R}_v)$; as a result, we may write $\frac{\partial}{\partial v}(\theta\widetilde{\mathbf{q}}_{\varepsilon_n})=(-\Delta_v+1)^{(1+\delta)/2}(-\Delta_{t,x}+1)^{1/2}\mathbf{Q}_{\varepsilon_n}$ for some $\mathbf{Q}_{\varepsilon_n}\to0$ in $L^q(\mathbb{R}_t\times\mathbb{R}^N_{\underline{y}}\times\mathbb{R}_v)$. At last, Lemma 4.5 guarantees that the last forcing term in (4.51) is a derivative in v of a vanishing sequence in $L^q(\mathbb{R}_v;W^{-1,q}(\mathbb{R}_{\underline{t}}\times\mathbb{R}^N_{\underline{y}}))$.

All in all, we conclude thus that, in $\mathscr{D}'(\mathscr{U}\times\mathbb{R})$,

$$\frac{\partial(\theta\widetilde{f}_{\varepsilon_n})}{\partial\underline{t}}+\widehat{\mathbf{a}}(v)\cdot\nabla_{\widehat{\underline{y}}}(\theta\widetilde{f}_{\varepsilon_n})+\widetilde{\mathbf{a}_N}(\widehat{y}_0,v)\frac{\partial(\theta\widetilde{f}_{\varepsilon_n})}{\partial\underline{y}_N}=(-\Delta_{\underline{t},\underline{y}}+1)^{1/2}(-\Delta_v+1)^{\ell/2}\mathfrak{h}_n,\tag{4.52}$$

where $1<\ell<1+\alpha$ and $\mathfrak{h}_n\to0$ in $L^q(\mathbb{R}_{\underline{t}}\times\mathbb{R}^N_{\underline{y}}\times\mathbb{R}_v)$ for some $1<q<\infty$.

For the vector field $\widetilde{\mathbf{a}}(\widehat{y}_0,v)=(\widehat{\mathbf{a}}(v),\widetilde{\mathbf{a}_N}(\widehat{y}_0,v))$ may be obtained from $\mathbf{a}(v)$ by a simple linear transformation (which is implied in (4.29)), the nondegeneracy condition (4.8) yields that

$$\mathrm{meas}\Big\{v\in[-L,L];\,\tau+\widetilde{\mathbf{a}}(\widehat{y}_0,v)\cdot\kappa=0\Big\}=0\text{ for all }(\tau,\kappa)\in\mathbb{R}\times\mathbb{R}^N\text{ with }\tau^2+|\kappa|^2=1.$$

Therefore, since $(\theta\widetilde{f})$ is uniformly bounded in $(L^1\cap L^\infty)(\mathscr{U}\times\mathbb{R}_v)$, we are in condition to invoke, for instance, the averaging lemma of Lemma 3.1 with $\eta(v)=1_{(-L,L)}(v)$. By doing so, we conclude that $\int_{-L}^L\theta(\underline{t},\underline{y})\widetilde{f}_{\varepsilon_n}(\underline{t},\underline{y},v)\,dv=\widetilde{u}_{\varepsilon_n}(\underline{t},\underline{y})$ defines a relatively compact sequence in $L^p(\mathscr{U})$ for every $1\le p<\infty$.

On the strength of the weak–$\star$ convergence of $\widetilde{f}$, we conclude thus that

$$\widetilde{u}_{\varepsilon_n}(\underline{t},\underline{y})\to\int_{-L}^L f^\tau(t_0,\widehat{y}_0,v)\,dv\text{ strongly in }L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^{N-1}\times(0,\infty))\text{ as }\varepsilon_n\to0_+$$

for every $\omega\in\Omega_3(t_0,\widehat{y}_0)$. Accordingly, Proposition 4.1 now implies the desired convergence of the χ–functions $\widetilde{f}_\varepsilon$. □

Amalgamating Proposition 4.1 and Lemmas 4.6 and 4.7, we see that $v\mapsto f^\tau(t_0,\widehat{y}_0,v)$ is of the form $\chi_{\mathfrak{a}(\omega)}$ for some $\mathfrak{a}\in(-L,L)$ and all $\omega\in\Omega_3(t_0,\widehat{y}_0)$. Since $\Omega_3(t_0,\widehat{y}_0)$ is of probability one, and $(t_0,\widehat{y}_0)$ is an arbitrary element of the set of total measure $\mathcal{E}\subset\Sigma$, we arrive at the following conclusion.

Theorem 4.4. *$f^\tau\in L^\infty(\Omega\times\Sigma\times(-L,L))$ is a χ–function in the sense of Definition 4.4.*

4.5 Proof of Theorem 4.1

For all intents and purposes, Theorem 4.1 is proven—or at least locally proven, for we assumed that $\mathscr{O}$ is locally the epigraph of a Lipschitz function. To pass from this local statement to a global one, let us employ a classical covering argument.

Since $\mathscr{O}$ is bounded and has a regular deformable Lipschitz boundary, we may cover $\partial\mathscr{O}$ with finitely many neighborhoods $\mathscr{U}_1, \ldots, \mathscr{U}_k$, for which, given any integer $1 \le j \le k$, there exists a rigid motion $\mathcal{R}_j : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\begin{cases} \mathcal{R}_j(x_j) = 0, \text{ and} \\ \mathcal{R}_j(\mathscr{U}_j) = \Big\{ y = (y_1, \ldots, y_N) \in \mathbb{R}^{N-1}; \gamma_j(y_1, \cdots, y_{N-1}) < y_N \Big\} \cap S(0, r_j), \end{cases}$$

where $x_j \in \partial\mathscr{O} \cap \mathscr{U}_j$, $r_j > 0$, $\gamma_j : \mathbb{R}^{N-1} \to \mathbb{R}$ is a Lipschitz mapping and, again, $S(z,r) = \big\{ y \in \mathbb{R}^N; |y_i - z_i| \le r \text{ for } i = 1, \cdots, N \big\}$. Let us thus put

$$\begin{cases} \Sigma_j = (0,T) \times (-r_j, r_j)^{N-1}, \text{ and} \\ \Gamma_j = \mathcal{R}_j^{-1}\Big(\Big\{ m \in \mathbb{R}^N; m_N = \gamma_j(m_1, \ldots, m_{N-1}) \Big\} \cap S(0, r_j) \Big). \end{cases}$$

Recall also that $\widetilde{\gamma}_j(\widehat{y}) = \mathcal{R}_j^{-1}(\widehat{y}, \gamma_j(\widehat{y})) \in \mathbb{R}^{N-1} \times \mathbb{R}$.

If ψ is any strongly regular deformation of $\partial\mathscr{O}$, its restriction to each $\partial\mathscr{U}_j$ is trivially a Γ_j–regular Lipschitz deformation. Therefore, on the strength of Lemma 4.2 and Theorem 4.4, we conclude that, for every $1 \le j \le k$, there exists a χ–function $f_j^\tau \in L^\infty(\Omega \times \Sigma_j \times (-L, L))$ such that

$$\operatorname*{ess\,lim}_{s \to 0_+} f_\psi(\,\cdot\,, s, \,\cdot\,) = f_j^\tau \quad \begin{cases} \text{strongly in } L^1(\Omega \times \Sigma_j \times (-L, L)), \text{ and} \\ \text{strongly in } L^1(\Sigma_j \times (-L, L)) \text{ almost surely,} \end{cases} \tag{4.53}$$

where f_ψ is given by (4.17) with r_j and $\widetilde{\gamma}_j$ replacing r and $\widetilde{\gamma}_0$ respectively. We may thus define $u^\tau \in L^\infty(\Omega \times (0,T) \times \partial\mathscr{O})$ by

$$u^\tau(t, m) = \int_{-L}^{L} f_j^\tau(t, \widehat{y}, v)\, dv$$

whenever $0 < t < T$ and $m = (\mathcal{R}_j^{-1}\widetilde{\gamma}_j)(\widehat{y}) \in \Gamma_j$. Thanks to the uniqueness of f_k^τ asserted in Lemma 4.1, this indeed leads to a well-defined measurable function and does not depend on ψ.

Moreover, since $\int_{-L}^{L} f_\psi(t, \widehat{y}, s, v)\, dv = u(t, \psi(s, m))$ if $m = (\mathcal{R}_j^{-1}\widetilde{\gamma}_j)(\widehat{y})$, (4.53) and a simple change of variables in the integral yield

$$\operatorname*{ess\,lim}_{s \to 0_+} u(\,\cdot\,, \psi(s, \,\cdot\,)) = u^\tau(\,\cdot\,, \,\cdot\,) \quad \begin{cases} \text{strongly in } L^1(\Omega \times \Gamma_j), \text{ and} \\ \text{strongly in } L^1(\Gamma_j) \text{ almost surely} \end{cases} \tag{4.54}$$

per Lemma 4.2. Owing to the fact that $\partial\mathscr{O} \subset \cup_{j=1}^k \Gamma_k$, this proves both (4.9) and (4.10). Theorem 4.1 is finally demonstrated. □

Chapter 5

The zero-flux problem for stochastic conservation laws

5.1 The main result

Let us investigate the so-called zero-flux problem for stochastic conservation laws

$$\begin{cases} \dfrac{\partial u}{\partial t} + \operatorname{div}_x(\mathbf{A}(u)) = \displaystyle\sum_{k=1}^{\infty} g_k(x,u)\,\frac{d\beta_k}{dt}(t) & \text{for } (t,x) \in Q, \\ \mathbf{A}(u)\cdot\nu = 0 & \text{for } (t,x) \in (0,T)\times\partial\mathscr{O}, \text{ and} \\ u(0,x) = u_0(x) & \text{for } x \in \mathscr{O}. \end{cases} \tag{5.1}$$

Here $T > 0$ is an arbitrary number, $N \geq 1$ is an integer, $\mathscr{O} \subset \mathbb{R}^N$ is a open set whose outward unit normal at a point $x \in \partial\mathscr{O}$ is $\nu(x)$, $Q = (0,T) \times \mathscr{O}$, $\mathbf{A} : \mathbb{R} \to \mathbb{R}^N$ is a flux function, $\beta_k(t)$ are mutually independent Brownian motions, and $g_k(x,u)$ are diffusion coefficients.

In the absence of the stochastic term $\sum_{k=1}^{\infty} g_k(x,u)\,\frac{d\beta_k}{dt}(t)$, the system (5.1) is a well-known model for many natural phenomena, such as the sedimentation of suspensions in closed vessels, the dispersal of a single species of animals in a finite territory, *etc*—see R. Bürger–H. Frid–K. H. Karlsen [15] and the references therein. One may thus introduce such a random perturbation to take into account uncertainties and fluctuations arising in these applications.

This particular initial–boundary value problem we will delve into is the same previously encountered and successfully solved in H. Frid *et al.* [43] (see also R. Bürger–H. Frid–K.H. Karlsen [15], A. Debussche–J. Vovelle [31], A. Debussche–M. Hofmanová—J. Vovelle [30], H. Frid–Y. Li [42], and B. Gess–M. Hofmanová [51]). The goal of this chapter is to show that, on the strength of the velocity averaging lemmas of this thesis, we can now considerably lighten the collection of assumptions, thus generalizing this aforementioned work. Indeed, the hypotheses we will consider throughout this chapter are the following.

1. *Conditions concerning* $\mathscr{O}$: $\mathscr{O}$ is assumed to be bounded, regular, and of class $\mathscr{C}^{1,1}$.

2. *Conditions concerning* $\mathbf{A}$:

   2.a) (Regularity): There exists some $0 < \alpha \leq 1$ such that

$$\mathbf{A} \in \mathscr{C}^{2,\alpha}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N). \tag{5.2}$$

   2.b) (Existence of saturation states): There exist some $a < b$ such that

$$\mathbf{A}(a) = 0 = \mathbf{A}(b). \tag{5.3}$$

2.c) (Nondegeneracy condition): Putting $\mathbf{a}(v) = \mathbf{A}'(v)$, it holds that

$$\text{meas}\Big\{v \in [a,b];\ \tau + \mathbf{a}(v)\cdot\kappa = 0\Big\} = 0 \quad \text{for all } (\tau,\kappa) \in \mathbb{R}\times\mathbb{R}^N \text{ with } \tau^2 + |\kappa|^2 = 1. \tag{5.4}$$

3. *Conditions concerning* β_k: Like in the previous chapter, $(\Omega, \mathscr{F}, \mathbb{P})$ will denote a probability space endowed with a complete, right-continuous filtration $(\mathscr{F}_t)_{t\geq 0}$. It will be assumed again the existence of a sequence $(\beta_k(t))_{k\in\mathbb{N}}$ of mutually independent Brownian motions in $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Hence, letting $\mathscr{H}$ be a separable Hilbert space with a hilbertian basis $(e_k)_{k\in\mathbb{N}}$,
$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k$$
defines a cylindrical Wiener process.

4. *Conditions on* $g_k(x,u)$: For any integer $k \geq 1$, let $g_k \in \mathscr{C}(\mathscr{O}\times\mathbb{R}_v;\mathbb{R})$ be such that:

4.a) (Growth condition): Defining $\mathfrak{G}^2(x,u) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} g_k(x,u)^2$, there exists some $C_* > 0$ such that
$$\mathfrak{G}^2(x,u) \leq C_*(1+u^2) \tag{5.5}$$
for all $x \in \mathscr{O}$ and $-\infty < u < \infty$.

4.b) (Regularity): For all $\mathscr{U} \subset\subset \mathscr{O}$, there exist some nondecreasing, nonnegative, continuous function $\mathfrak{o}_{\mathscr{U}} : [0,\infty) \to [0,\infty)$ such that $\mathfrak{o}_{\mathscr{U}}(0) = 0$, and
$$\sum_{k=1}^{\infty} |g_k(x,u) - g_k(y,v)|^2 \leq \mathfrak{o}_{\mathscr{U}}(|x-y|)|x-y| + \mathfrak{o}_{\mathscr{U}}(|u-v|)|u-v| \tag{5.6}$$
for all x and $y \in \mathscr{U}$, and all u and $v \in \mathbb{R}$.

4.b) (Existence of saturation states, part II): For the same $a < b$ featured in (5.3), it holds that
$$g_k(x,a) = 0 = g_k(x,b) \tag{5.7}$$
for any $x \in \mathscr{O}$ and integer $k \geq 1$.

Following Remark 4.2, we will now define $\Phi : L^2(\mathscr{O}) \to \mathscr{L}(\mathscr{H}; L^2(\mathscr{O}))$ by
$$(\Phi(f)\cdot h)(x) = \sum_{k=1}^{\infty} g_k(x, f(x))\,(h, e_k)_{\mathscr{H}}$$
whenever $h \in \mathscr{H}$ and $x \in \mathscr{O}$. In the light of (5.5) and (5.6), not only is such $\Phi(f)$ well-defined, but also lies in the Hilbert–Schmidt class $HS(\mathscr{H}; L^2(\mathscr{O}))$. Therefore, given any predictable process $u \in L^2(\Omega\times[0,T]; L^2(\mathscr{O}))$, the stochastic integral
$$t \mapsto \int_0^t \Phi(u(t'))\,dW(t') = \sum_{k=1}^{\infty}\int_0^t g_k(x, u(t',x))\,d\beta_k(t')$$
defines a legitimate $L^2(\mathscr{O})$–valued process.

5. *Conditions on* u_0:

5.a) (Mensurability): $u_0 \in L^2(\Omega; L^2(\mathscr{O}))$ is $\mathscr{F}_{t=0}$–measurable.

5.b) (Existence of saturation states, part III): If a and b are same ones as in (5.3) and (5.7), then

$$a \le u_0(x) \le b \text{ almost surely in } \mathscr{D}'(\mathscr{O}). \tag{5.8}$$

At last, let state the concept of solution employed, and the main result of this chapter. Henceforth, the constants a and b will be as in (5.3), (5.7), and (5.8).

Definition 5.1 (Entropy solution)**.** A predictable function $u \in L^2(\Omega \times [0,T]; L^2(\mathscr{O}))$ is said to be an entropy solution to (5.1), if the following conditions are met.

1. (L^∞–bound): Almost surely,
$$a \le u(t,x) \le b \text{ in } \mathscr{D}'(Q).$$

2. (The entropy condition): Almost surely, for all convex functions $\eta \in \mathscr{C}^2(\mathbb{R})$, and nonnegative $\varphi \in \mathscr{C}_c^\infty((-\infty,T)\times\mathscr{O})$,
$$\begin{aligned}\int_0^T\int_{\mathscr{O}} \eta(u(t,x))\frac{\partial\varphi}{\partial t}(t,x)\,dtdx + \int_{\mathscr{O}}\eta(u_0(x))\varphi(0,x)\,dx + \int_0^T\int_{\mathscr{O}}\mathbf{A}^\eta(u(t,x))\cdot\nabla_x\varphi(t,x)\,dx\,dt \\ \ge -\sum_{k=1}^\infty\int_0^T\int_{\mathscr{O}}\eta'(u(t,x))g_k(x,u(t,x))\varphi(t,x)\,dx\,d\beta_k(t) \\ -\frac{1}{2}\int_0^T\int_{\mathscr{O}}\varphi(t,x)\eta''(u(t,x))\mathfrak{G}^2(x,u(t,x))\,dx\,dt,\end{aligned} \tag{5.9}$$
where $\mathbf{A}^\eta(v) = \int_0^v \eta'(w)\mathbf{a}(w)\,dw$.

3. (The boundary condition): Almost surely, for all $\theta \in \mathscr{C}_c^\infty((0,T)\times\mathbb{R}^N)$, it holds that
$$\begin{aligned}\int_0^T\int_{\mathscr{O}} u(t,x)\frac{\partial\theta}{\partial t}(t,x)\,dxdt + \int_0^T\int_{\mathscr{O}}\mathbf{A}(u(t,x))\cdot\nabla\theta(t,x)\,dxdt \\ +\sum_{k=1}^\infty\int_0^T\int_{\mathscr{O}} g_k(x,u(t,x))\theta(t,x)\,dx\,d\beta_k(t) = 0.\end{aligned} \tag{5.10}$$

Remark 5.1 (On a, b, $\mathbf{A}(v)$, and $\Phi(x,u)$). In the applications of Equation (5.1), $u(t,x)$ quantifies some concentration, hence it can only attain values in a bounded interval $[a,b]$; see, *e.g.*, M. C. Bustos *et al.* [17]. The extreme values a and b are then stationary solutions, a property that can be mathematically translated to (5.1) if one imposes (5.3) and (5.7).

Theoretically, such conditions are not superficial either, once they are employed to obtain the L^∞–bound of the entropy solutions (as expressed in 1. above). This property is to a great extent utilized in both the deduction of the strong trace property of u, as well as the boundedness of the hyperbolic entropy dissipation measure $\mathbf{m}(t,x,v)$.

In any event, it is not hard generate a flux function $\mathbf{A}(v)$ satisfying the conditions imposed. For instance, pick N linearly independent real-analytic functions $\mathbf{a}_1,\dots,\mathbf{a}_N : \mathbb{R}\to\mathbb{R}$ such that $\int_a^b \mathbf{a}_j(w)\,dw = 0$ for all $1 \le j \le N$; then, it is clear that

$$\mathbf{A}(v) = \left(\int_a^v \mathbf{a}_1(w)\,dw,\dots,\int_a^v \mathbf{a}_N(w)\,dw\right)$$

possesses the desired properties.

In conclusion, we may point out that continuity conditions expressed in (5.6) are not only considerably weaker than of H. Frid *et al.* [43], but also of A. Debussche–J. Vovelle [31], A. Debussche–M. Hofmanová–J. Vovelle [30], and B. Gess–M. Hofmanová [51]. For instance, $g_k(x,v)$ is free to oscillate rapidly as x reaches the boundary $\partial\mathscr{O}$.

We are now in conditions to state our generalization of the well-posedness result of H. Frid *et al.* [43].

Theorem 5.1. *Under the hypotheses expressed above, there exists a unique entropy solution* $u \in L^2(\Omega \times (0,T); L^2(\mathscr{O})) \cap L^\infty(\Omega \times Q)$ *to the initial–boundary value problem* (5.1).

Moreover, let u *and* v *be entropy solutions to* (5.1) *with, respectively,* $\mathscr{F}_{t=0}$*–measurable initial data* u_0 *and* $v_0 \in L^\infty(\Omega \times \mathscr{O})$*. Then* u *and* v *possess representatives, respectively,* $\mathbf{u}$ *and* $\mathbf{v}$ *belonging to* $L^p(\Omega; \mathscr{C}([0,T]; L^p(\mathscr{O})))$ *for all* $1 \le p < \infty$*. Additionally, the comparison principle holds:*

$$\mathbb{E}\int_{\mathscr{O}} (\mathbf{u}(t,x) - \mathbf{v}(t,x))_+ \, dx \le \mathbb{E}\int_{\mathscr{O}} (u_0(x) - v_0(x))_+ \, dx \text{ for all } 0 \le t \le T.$$

The verification of Theorem 5.1 will be performed in the next two sections. In the last section of this chapter, we will investigate the Sobolev regularity of the solution $u(t,x)$ obtained above.

Let us terminate this motivating paragraph with some equivalent definitions of entropy solution. The next pivotal concept is due to A. Debussche–J. Vovelle [31]; see also A. Debussche–M. Hofmanová–J. Vovelle [30].

Definition 5.2 (Kinetic measure)**.** A map $\mathbf{m} : \Omega \to \mathfrak{M}_+\big([0,T] \times \mathscr{O} \times \mathbb{R}\big)$ (the set of the nonnegative measures in $[0,T] \times \mathscr{O} \times \mathbb{R}$) is said to be a *kinetic measure* if the following conditions are met.

1. (Weak mensurability): Understanding $\mathfrak{M}\big([0,T] \times \mathscr{O} \times \mathbb{R}\big)$ (the set of the measures defined in $[0,T] \times \mathscr{O} \times \mathbb{R}$) as the dual space of $\mathscr{C}_0\big([0,T] \times \mathscr{O} \times \mathbb{R}\big)$ (the closure of $\mathscr{C}_c\big([0,T] \times \mathscr{O} \times \mathbb{R}\big)$ in $L^\infty(Q \times \mathbb{R})$), $\mathbf{m}$ is *weakly measurable.* In other words, given any $\phi \in \mathscr{C}_0\big([0,T] \times \mathscr{O} \times \mathbb{R}\big)$,

$$\omega \in \Omega \mapsto \big\langle \mathbf{m}, \phi \big\rangle_{\mathfrak{M},\mathscr{C}_0} = \int_{Q \times \mathbb{R}} \phi(t,x,v) \, \mathbf{m}(dt,dx,dv) \in \mathbb{R}$$

is measurable;

2. (Decay at the infinity): $\mathbf{m}$ *vanishes for large* v; i.e., if $B_R^c = \{v \in \mathbb{R} : |v| \ge R\}$, then

$$\lim_{R \to \infty} \mathbb{E}\mathbf{m}\big([0,T] \times \mathscr{O} \times B_R^c\big) = 0. \tag{5.11}$$

3. (Predictability): Given any $\zeta \in \mathscr{C}_c^\infty(\mathscr{O} \times \mathbb{R})$, the process

$$t \in [0,T] \mapsto \int_{[0,t] \times \mathscr{O} \times \mathbb{R}} \zeta(x,v) \, \mathbf{m}(ds,dx,dv) \tag{5.12}$$

possesses a predictable representative.

Theorem 5.2. *Let* $u \in L^2(\Omega \times (0,T); L^2(\mathscr{O})) \cap L^\infty(\Omega \times Q)$ *be such that* $a \le u(t,x) \le b$ *almost surely in* $\mathscr{D}'(Q)$*. Consider also some* $\mathscr{F}_{t=0}$*–measurable* $u_0 \in L^\infty(\Omega \times \mathscr{O})$*, and assume that* $u(t,x)$ *satisfies the boundary condition* (5.10).

One of the following statements implies the other two.

a) (The entropy condition). $u(t,x)$ *is an entropy solution to* (5.1) *with initial data* $u(0,x) = u_0(x)$.

b) (The A. Debussche–J. Vovelle [31] kinetic formulation). *If* $f(t,x,v) = 1_{(-\infty,u(t,x))}(v) = 1_{v<u(t,x)}$ *and* $f_0(x,v) = 1_{v<u_0(x)}$*, there exists some nonnegative kinetic measure* $\mathbf{m}(t,x,v)$

such that

$$\int_0^T \int_{\mathscr{O}} \int_{\mathbb{R}_v} f(t,x,v) \frac{\partial \varphi}{\partial t}(t,x,v)\, dv dx dt + \int_{\mathscr{O}} \int_{\mathbb{R}_v} f_0(x,v)\varphi(0,x,v)\, dv dx$$
$$+ \int_0^T \int_{\mathscr{O}} \int_{\mathbb{R}_v} f(t,x,v)\mathbf{a}(v) \cdot \nabla_x \varphi(t,x,v)\, dv dx dt$$
$$= -\sum_{k=1}^{\infty} \int_0^T \int_{\mathscr{O}} g_k(x,u(t,x))\varphi(t,x,u(t,x))\, dx\, d\beta_k(t)$$
$$- \frac{1}{2} \int_0^T \int_{\mathscr{O}} \frac{\partial \varphi}{\partial v}(t,x,u(t,x))\mathfrak{G}^2(x,u(t,x))\, dx\, dt$$
$$+ \int_0^T \int_{\mathscr{O}} \int_{\mathbb{R}_v} \frac{\partial \varphi}{\partial v}(t,x,v)\, \mathbf{m}(dt,dx,dv) \tag{5.13}$$

almost surely for every $\varphi \in \mathscr{C}_c^\infty((-\infty,T) \times \mathscr{O} \times \mathbb{R}_v)$.

c) (A kinetic formulation *à* P.L. Lions–B. Perthame–E. Tadmor [82]). *Let* $\mathbf{f}(t,x,v) = \chi_{u(t,x)}(v) = 1_{(-\infty,u(t,x))}(v) - 1_{(-\infty,0)}(v)$ *and* $\mathbf{f}_0(x,v) = \chi_{u_0(x)}(v)$ *be the* χ*–function related to* $u(t,x)$ *and* $u_0(x)$ *respectively (see* (3.7)*). Then there exists some nonnegative kinetic measure* $\mathbf{m}(t,x,v)$ *such that*

$$\int_0^T \int_{\mathscr{O}} \int_{\mathbb{R}_v} \mathbf{f}(t,x,v) \frac{\partial \varphi}{\partial t}(t,x,v)\, dv dx dt + \int_{\mathscr{O}} \int_{\mathbb{R}_v} \mathbf{f}_0(x,v)\varphi(0,x,v)\, dv dx$$
$$+ \int_0^T \int_{\mathscr{O}} \int_{\mathbb{R}_v} \mathbf{f}(t,x,v)\mathbf{a}(v) \cdot \nabla_x \varphi(t,x,v)\, dv dx dt$$
$$= -\sum_{k=1}^{\infty} \int_0^T \int_{\mathscr{O}} g_k(x,u(t,x))\varphi(t,x,u(t,x))\, dx\, d\beta_k(t)$$
$$- \frac{1}{2} \int_0^T \int_{\mathscr{O}} \frac{\partial \varphi}{\partial v}(t,x,u(t,x))\mathfrak{G}^2(x,u(t,x))\, dx\, dt$$
$$+ \int_0^T \int_{\mathscr{O}} \int_{\mathbb{R}_v} \frac{\partial \varphi}{\partial v}(t,x,v)\, \mathbf{m}(dt,dx,dv) \tag{5.14}$$

almost surely for every $\varphi \in \mathscr{C}_c^\infty((-\infty,T) \times \mathscr{O} \times \mathbb{R}_v)$.

Moreover, if $L = \max\{|a|,|b|\}$*, the kinetic measure* $\mathbf{m}$ *given in* (5.13) *and* (5.14) *is almost surely supported on* $Q \times [-L,L]$ *and belongs to* $L^p_{\mathfrak{w}}(\Omega; \mathfrak{M}(Q \times \mathbb{R}_v))$ *for all* $1 \leq p < \infty$.

Remark 5.2. The reason to be of this theorem is as follows. While the entropy condition is very easily verifiable, it is not well-suited to prove the comparison principle in the stochastic setting. On the other hand, the Debussche–Vovelle kinetic condition is perfect for this goal. Finally, the more classical kinetic condition (5.14) is the one appropriate for applications of velocity averaging lemmas. Notice that, taking $\varphi \in \mathscr{C}_c^\infty(Q \times \mathbb{R}_v)$, then both (5.13) and (5.14) could have been written more concisely as

$$\frac{\partial \mathfrak{f}}{\partial t} + \mathbf{a}(v) \cdot \nabla_x \mathfrak{f} = \frac{\partial \mathbf{q}}{\partial v} + \sum_{k=1}^{\infty} g_k(x,v)\delta_{v=u(t,x)} \frac{d\beta_k}{dt} \text{ in } \mathscr{D}'(Q), \tag{5.15}$$

where $\mathbf{q}(t,x,v) = \mathbf{m}(t,x,v) - \frac{1}{2}\mathfrak{G}^2(x,v)\delta_{v=u(t,x)}$.

Proof of Theorem 5.2. Evidently, the conclusion that the entropy condition (5.9) implies both Equations (5.13) and (5.14) has completely parallel proof to the one of Theorem 4.2. Furthermore, it is obvious that (5.13) and (5.14) are equivalent. For the entropy inequality (5.9) can be obtained via (5.13) by choosing a test-function of the form $\varphi(t,x,v) = \eta'(v)\phi(t,x)$, we therefore conclude that all equations (5.9), (5.13), and (5.14) are the one and the same. As the quantitative properties of $\mathbf{m}(t,x,v)$ were already deduced in Theorem 4.2, all that remains to be established is that $\mathbf{m}(t,x,v)$ is kinetic.

First of all, the fact that $\mathbf{m}(t,x,v)$ is weakly measurable is contained in the assertion that $\mathbf{m} \in L^p_{\mathfrak{w}}(\Omega;\mathfrak{M}(Q\times\mathbb{R}))$. Since it is also supported almost surely in $Q\times[-L,L]$, the decay at the infinite is valid for trivial reasons.

Finally, the predictability condition may be seen as follows. Given any $\zeta \in \mathscr{C}_c^\infty(Q\times\mathbb{R})$, put

$$\Lambda_\zeta(t) = \int_{[0,t]}\int_{\mathscr{O}}\int_{\mathbb{R}} \zeta(x,v)\,\mathbf{m}(dt,dx,dv).$$

Plugging $\psi(t)\int_{-\infty}^v \zeta(x,w)\,dw = \psi(t)Z(x,v)$ as a test-function in (5.14) where $\psi \in \mathscr{C}_c^\infty(-\infty,T)$, we may conclude that

$$\begin{aligned}\int_0^T\int_{\mathscr{O}}\int_{\mathbb{R}_v} \zeta(x,v)\psi(t)\,\mathbf{m}(dt,dx,dv) &= \int_0^T\int_{\mathscr{O}}\int_{\mathbb{R}_v} \mathbf{f}(t,x,v)Z(x,v)\frac{\partial\psi}{\partial t}(t)\,dvdxdt \\ &+ \int_{\mathscr{O}}\int_{\mathbb{R}_v} \mathbf{f}_0(x,v)Z(x,v)\psi(0)\,dvdx \\ &+ \int_0^T\int_{\mathscr{O}}\int_{\mathbb{R}_v} \mathbf{f}(t,x,v)\mathbf{a}(v)\cdot\nabla_x Z(x,v)\psi(t)\,dvdxdt \\ &+ \sum_{k=1}^\infty\int_0^T\int_{\mathscr{O}}\int_{\mathbb{R}_v} g_k(x,u(t,x))Z(x,u(t,x))\psi(t)\,dxd\beta_k(t) \\ &+ \frac{1}{2}\int_0^T\int_{\mathscr{O}}\int_{\mathbb{R}_v} \zeta(x,u(t,x))\mathfrak{G}^2(x,u(t,x))\psi(t)\,dxdt.\end{aligned}$$

Therefore, by letting $\psi(t)$ approximate $1_{(-\infty,t^*]}$ where the t^*'s are the Lebesgue points of $\mathbf{f} \in L^2(0,T;L^2(\Omega\times\mathscr{O}))$, it follows that

$$\begin{aligned}\Lambda_\zeta(t) = &-\int_{\mathscr{O}}\int_{\mathbb{R}_v} \mathbf{f}(t,x,v)Z(x,v)\,dvdxdt + \int_{\mathscr{O}}\int_{\mathbb{R}_v} \mathbf{f}_0(x,v)Z(x,v)\,dvdx \\ &+ \int_0^t\int_{\mathscr{O}}\int_{\mathbb{R}_v} \mathbf{f}(t,x,v)\mathbf{a}(v)\cdot\nabla_x Z(x,v)\,dvdxdt \\ &+ \sum_{k=1}^\infty\int_0^t\int_{\mathscr{O}}\int_{\mathbb{R}_v} g_k(x,u(t,x))Z(x,u(t,x))\,dx\,d\beta_k(t) \\ &+ \frac{1}{2}\int_0^t\int_{\mathscr{O}}\int_{\mathbb{R}_v} \zeta(x,u(t,x))\mathfrak{G}^2(x,u(t,x))\,dxdt\end{aligned}$$

almost every $(\omega,t)\in\Omega\times(0,T)$. Once the right-hand side is predictable, $\Lambda_\zeta(t)$ indeed possesses a predictable representative. The proof is thus complete. □

5.2 Uniqueness

We will now establish the uniqueness of entropy solutions to problem (5.1) via the techniques introduced by A. Debussche–J. Vovelle [31]. Such an approach was also employed successfully by M. Hofmanová [62], A. Debussche–M. Hofmanová–J. Vovelle [30], and B. Gess–M. Hofmanová [51] to prove similar results regarding degenerate parabolic-hyperbolic equations.

Let us first recall some of the crucial concepts of their theory.

Definition 5.3. Let (X,λ) be a finite measure space.

1. (Young measure). Denote by $\mathfrak{M}_1(\mathbb{R})$ the set of Borel probability measures on the real line. A

mapping $\mu : X \to \mathfrak{M}_1(\mathbb{R})$ is said to be a *Young measure* if it is weakly measurable in sense that, for all $\phi \in \mathscr{C}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the real function $x \in X \mapsto \int_{\mathbb{R}} \phi(v)\, \mu_x(dv)$ is measurable.

Moreover, a Young measure μ is said to *vanish at infinity* if, for any $1 \le p < \infty$,

$$\int_X \int_{\mathbb{R}} |v|^p \, \mu_x(dv)\, \lambda(dx) < \infty.$$

2. (Kinetic function). A measurable function $f : X \to [0,1]$ is said to be a *kinetic function* if there exists a Young measure vanishing at infinity such that, for almost every $x \in X$ and all $v \in \mathbb{R}$,

$$f(x,v) = \mu_x\big((v,\infty)\big).$$

Furthermore, f is said to be an *equilibrium* if one can take $\mu_x = \delta_{u(x)}$ for almost every $x \in X$ and for some measurable function $u : X \to \mathbb{R}$. (In other words, f is an equilibrium if $f(x,v) = 1_{v<u(x)}$.)

Finally, if f is a kinetic function, its conjugate function $\overline{f}$ is defined as $\overline{f} = 1 - f$.

To set the stage for the doubling of variables, let us thus state a result that recovers some a priori "weak continuity" of entropy solutions. The proof of this proposition may be found in A. Debussche–J. Vovelle [31], once it is virtually identical to the corresponding result in this reference; see also A. Debussche–M. Hofmanová–J. Vovelle [30].

Lemma 5.1. *Let u be an entropy solution to (5.1). Then its kinetic function $f = 1_{u>v}$ admits representatives f^- and f^+ that are, respectively, almost surely left- and right-continuous at all points $0 \le t^* \le T$ in sense of the distributions in $\mathscr{O} \times \mathbb{R}_v$. More precisely, for all $0 \le t^* \le T$, there are $f^{*,\pm}$ on $\Omega \times \mathscr{O} \times \mathbb{R}$ such that, putting $f^\pm(t^*) = f^{*,\pm}$, then $f^\pm = f$ almost everywhere, and, for some set $\Omega_0 \subset \Omega$ of probability one,*

$$\big\langle f^\pm(t^* \pm \varepsilon), \varphi \big\rangle \to \big\langle f^\pm(t^*), \varphi \big\rangle \text{ as } \varepsilon \to 0_+$$

for every $\varphi \in \mathscr{C}_c^\infty(\mathscr{O} \times \mathbb{R})$ and $0 \le t^* \le T$. Moreover, almost surely, the set of $t^* \in [0,T]$ such that $f^+(t^*) \ne f^-(t^*)$ is countable.

Endowed of this fact, we may now state the version of Kruzhkov's doubling of variables technique due to A. Debussche–J. Vovelle [31], to which again we refer the proof. Observe that, while x and v are duplicated, t is not.

Lemma 5.2 (Doubling of variables)**.** *Let u_1 and u_2 be kinetic solutions, and let $f_1 = 1_{v<u_1}$ and $f_2 = 1_{v<u_2}$ be their kinetic functions. Let also $f_1^\pm$ and $f_2^\pm$ be the representatives given by Lemma 5.1, and denote by $f_{1,0} = 1_{v<u_{1,0}}$ and $f_{2,0} = 1_{v<u_{2,0}}$ the kinetic functions associated to, respectively, $u_{1,0}(x)$ and $u_{2,0}(x) \in L^\infty(\Omega \times \mathscr{O})$, the initial data of u_1 and u_2.*

Then, for all $0 \le t \le T$, and non-negative test functions $\psi \in \mathscr{C}_c^\infty(\mathbb{R}_v)$, $\rho \in \mathscr{C}_c^\infty(\mathbb{R}^N)$, and $\varphi \in \mathscr{C}_c^\infty(\mathscr{O})$ such that $\rho(x-y)\varphi((x+y)/2) \in \mathscr{C}_c^\infty(\mathscr{O}_x \times \mathscr{O}_y)$, we have

$$\begin{aligned}
&\mathbb{E} \int_{\mathscr{O}_x} \int_{\mathscr{O}_y} \int_{\mathbb{R}_v} \int_{\mathbb{R}_w} \rho(x-y)\psi(v-w)\varphi\Big(\frac{x+y}{2}\Big) f_1^\pm(t,x,v)\overline{f_2^\pm}(t,y,w)\, dv\, dw\, dx\, dy \\
&\qquad \le \mathbb{E} \int_{\mathscr{O}_x} \int_{\mathscr{O}_y} \int_{\mathbb{R}_v} \int_{\mathbb{R}_w} \rho(x-y)\psi(v-w)\varphi\Big(\frac{x+y}{2}\Big) f_{1,0}(x,v)\overline{f_{2,0}}(y,w)\, dw\, dv\, dx\, dy \\
&\qquad\quad + I_\rho + I_\phi + I_\psi, \qquad (5.16)
\end{aligned}$$

where, letting $\mu^1_{s,x}(v) = \delta_{u_1(s,x)}(v)$ *and* $\mu^2_{s,y}(w) = \delta_{u_2(s,y)}(w)$*,*

$$I_\rho = \mathbb{E}\int_0^t \int_{\mathscr{O}_x}\int_{\mathscr{O}_y}\int_{\mathbb{R}_v}\int_{\mathbb{R}_w} f_1(s,x,v)\overline{f_2}(s,y,w)\varphi\left(\frac{x+y}{2}\right)\psi(v-w) \\ (\mathbf{a}(v)-\mathbf{a}(w))\cdot\nabla_x\rho(x-y)\,dwdvdydxds,$$

$$I_\varphi = \frac{1}{2}\mathbb{E}\int_0^t \int_{\mathscr{O}_x}\int_{\mathscr{O}_y}\int_{\mathbb{R}_v}\int_{\mathbb{R}_w} f_1(s,x,v)\overline{f_2}(s,y,w)\psi(v-w)\rho(x-y) \\ (\mathbf{a}(v)+\mathbf{a}(w))\cdot\nabla_x\varphi\left(\frac{x+y}{2}\right)dwdvdydxds, \text{ and}$$

$$I_\psi = \frac{1}{2}\mathbb{E}\int_0^t \int_{\mathscr{O}_x}\int_{\mathscr{O}_y}\int_{\mathbb{R}_v}\int_{\mathbb{R}_w} \rho(x-y)\varphi\left(\frac{x+y}{2}\right)\psi(v-w) \\ \sum_{k=1}^\infty |g_k(x,v)-g_k(y,w)|^2\,\mu^2_{s,y}(dw)\mu^1_{s,x}(dv)dydxds.$$

We can now finally deduce the so-called Kruzhkov's inequality, whose verification proposition will be provided, once our hypotheses on the diffusion coefficients $\Phi(x,v)$ are somewhat weaker than those of A. Debussche–J. Vovelle [31].

Lemma 5.3. *Let* u_1 *and* u_2 *be kinetic solutions, and let* $f_1 = 1_{v<u_1}$ *and* $f_2 = 1_{v<u_2}$ *be their kinetic functions. Let also* $f_1^\pm$ *and* $f_2^\pm$ *be the representatives given by Lemma 5.1, and denote by* $u_{1,0}(x)$ *and* $u_{2,0}(x) \in L^\infty(\Omega\times\mathscr{O})$ *the initial data of* u_1 *and* u_2 *respectively.*

Then, for all $0\le t\le T$ *and for every nonnegative* $\varphi\in\mathscr{C}_c^\infty(\mathscr{O})$*, it holds that*

$$\mathbb{E}\int_{\mathscr{O}}\int_{\mathbb{R}_v} f_1^\pm(t,x,v)\overline{f_2^\pm}(t,x,v)\varphi(x)\,dv\,dx \le \mathbb{E}\int_{\mathscr{O}}(u_{1,0}(x)-u_{2,0}(x))_+\,\varphi(x)\,dx \\ +\mathbb{E}\int_0^t\int_{\mathscr{O}} \operatorname{sign}(u_1(s,x)-u_2(s,x))_+(\mathbf{A}(u_1(s,x))-\mathbf{A}(u_2(s,x)))\cdot\nabla\varphi(x)\,dx\,ds \quad (5.17)$$

Proof. Let $\rho\in\mathscr{C}_c^\infty(\mathbb{R}^N)$ and $\psi\in\mathscr{C}_c^\infty(\mathbb{R})$ be symmetric nonnegative functions such that $\int_{\mathbb{R}^N}\rho\,dx = 1$, and $\int_{\mathbb{R}}\psi(v)\,dv = 1$, and define thus the "mollifiers"

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^N}\rho\left(\frac{x}{\varepsilon}\right), \text{ and } \psi_\delta(v) = \frac{1}{\delta}\psi\left(\frac{v}{\delta}\right)$$

for any $\varepsilon>0$ and $\delta>0$. Given any $\varphi\in\mathscr{C}_c^\infty(\mathscr{O})$, we may choose $\varepsilon>0$ sufficiently small so that we may plug $\rho=\rho_\varepsilon$ and $\psi=\psi_\delta$ in (5.16). Since such functions formally converge to Dirac deltas, we infer that, for every $0<t<T$,

$$\mathbb{E}\int_{\mathscr{O}}\int_{\mathbb{R}_v} f_1^\pm(t,x,v)\overline{f_2^\pm}(t,x,v)\varphi(x)\,dv\,dx = \mathbb{E}\int_{\mathscr{O}_x}\int_{\mathscr{O}_y}\int_{\mathbb{R}_v}\int_{\mathbb{R}_w}\rho_\varepsilon(x-y)\psi_\delta(v-w)\varphi\left(\frac{x+y}{2}\right) \\ f_1^\pm(t,x,v)\overline{f_2^\pm}(t,y,w)\,dv\,dw\,dx\,dy - \mathfrak{r}_t(\delta,\varepsilon) \quad (5.18)$$

where the error term $\mathfrak{r}_t(\delta,\varepsilon)\to 0$ as δ and $\varepsilon\to 0_+$. Similarly, we infer that

$$\mathbb{E}\int_{\mathscr{O}}(u_{0,1}(x)-u_{0,2}(x))_+\varphi(x)\,dx = \mathbb{E}\int_{\mathscr{O}}\int_{\mathbb{R}_v} f_{0,1}(x,v)\overline{f_{0,2}}(x,v)\varphi(x)\,dv\,dx \\ = \mathbb{E}\int_{\mathscr{O}_x}\int_{\mathscr{O}_y}\int_{\mathbb{R}_v}\int_{\mathbb{R}_w}\rho_\varepsilon(x-y)\psi_\delta(v-w)\varphi\left(\frac{x+y}{2}\right) \\ f_{0,1}(t,x,v)\overline{f_{0,2}}(t,y,w)\,dv\,dw\,dx\,dy + \widetilde{\mathfrak{r}_0}(\delta,\varepsilon) \quad (5.19)$$

with $\mathfrak{r}_0(\delta,\varepsilon)\to 0$ again as δ and $\varepsilon\to 0_+$.

Let us now analyze each individual term I_ρ, I_ψ and I_ϕ arising in (5.16).

Since $\mathbf{a} \in \mathscr{C}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$ and both u_1 and u_2 are essentially bounded, I_ρ can be thus estimated as

$$I_\rho \le C(\|\mathbf{a}'\|_{L^\infty_{\text{loc}}}, \|u_1\|_{L^\infty}, \|u_2\|_{L^\infty}) \int_0^t \int_{\mathscr{O}_x} \int_{\mathscr{O}_y} \int_{\mathbb{R}_v} \int_{\mathbb{R}_w} f_1(s,x,v)\overline{f_2}(s,y,w) \\ \left|\varphi\left(\frac{x+y}{2}\right)\right| \|\nabla_x \rho_\varepsilon(x-y)\| |v-w| |\psi_\delta(v-w)|\, dv\, dw\, dx\, dy\, ds \le Ct\frac{\delta}{\varepsilon} \tag{5.20}$$

as $|z\psi_\delta(z)| \le C\delta$ and $\|\nabla_x \rho_\varepsilon(x)\|_{L^1} \le C/\varepsilon$. In a similar note, I_φ can be written as

$$\begin{aligned} I_\varphi &= \mathbb{E}\int_0^t \int_{\mathscr{O}_x} \int_{\mathbb{R}_v} f_1(s,x,v)\overline{f_2}(s,y,v)\mathbf{a}(v) \cdot \nabla\varphi(x)\, dv\, dx\, ds + \widehat{\mathfrak{r}}_t(\delta,\varepsilon) \\ &= \mathbb{E}\int_0^t \int_{\mathscr{O}_x} \int_{u_2(s,x)\le v< u_1(s,x)} \mathbf{a}(v) \cdot \nabla\varphi(x)\, dv\, dx\, ds + \widehat{\mathfrak{r}}_t(\delta,\varepsilon) && (5.21) \\ &= \mathbb{E}\int_0^t \int_{\mathscr{O}} \operatorname{sign}(u_1-u_2)_+(\mathbf{A}(u_1)-\mathbf{A}(u_2)) \cdot \nabla\varphi(x)\, dx\, ds + \widehat{\mathfrak{r}}_t(\delta,\varepsilon), && (5.22) \end{aligned}$$

where the remainder again satisfies $\widehat{\mathfrak{r}}_t(\delta,\varepsilon) \to 0$ as δ and $\varepsilon \to 0_+$

Finally, due to (5.6), we infer that there exists nondecreasing, nonnegative, continuous function $\mathfrak{o} : [0,\infty) \to [0,\infty)$ with $\mathfrak{o}(0) = 0$ that allows I_ψ to be bounded by

$$\begin{aligned} I_\psi &\le \mathbb{E}\int_0^t \int_{\mathscr{O}_x} \int_{\mathscr{O}_y} \varphi\left(\frac{x+y}{2}\right)\rho_\varepsilon(x-y)|x-y|\mathfrak{o}(|x-y|)\psi_\delta(u_1-u_2)\, dx\, dy\, ds \\ &\quad + \mathbb{E}\int_0^t \int_{\mathscr{O}_x} \int_{\mathscr{O}_y} \varphi\left(\frac{x+y}{2}\right)\rho_\varepsilon(x-y)\psi_\delta(u_1-u_2)|u_1-u_2|\mathfrak{o}(|u_1-u_2|)\, dy\, dx\, ds \\ &\le Ct\frac{\varepsilon}{\delta}\mathfrak{o}(\varepsilon) + Ct\mathfrak{o}(\delta). && (5.23) \end{aligned}$$

Gathering (5.18)–(5.23), we deduce that

$$\begin{aligned} \mathbb{E}\int_{\mathscr{O}} \int_{\mathbb{R}_v} f_1^\pm(t,x,v)\overline{f_2^\pm}(t,x,v)\varphi(x)\, dv\, dx &\le \mathbb{E}\int_{\mathscr{O}} (u_{1,0}(t,x) - u_{2,0}(t,x))_+\varphi(x)\, dx \\ &\quad + \mathbb{E}\int_0^t \int_{\mathscr{O}} \operatorname{sgn}(u_1-u_2)_+(\mathbf{A}(u_1)-\mathbf{A}(u_2)) \cdot \nabla\varphi(x)\, dx\, ds \\ &\quad + \mathfrak{r}_t(\delta,\varepsilon) + \widetilde{\mathfrak{r}_0}(\delta,\varepsilon) + \widehat{\mathfrak{r}}_t(\delta,\varepsilon) + CT\left(\frac{\delta}{\varepsilon} + \frac{\varepsilon}{\delta}\mathfrak{o}(\varepsilon) + \mathfrak{o}(\delta)\right). \end{aligned}$$

Hence, in order to obtain (5.17), it suffices to take $\delta = \varepsilon\mathfrak{o}(\varepsilon)^{1/2}$ and let $\varepsilon \to 0_+$. □

At last, we deduce the comparison principle and hence the uniqueness of solutions.

Theorem 5.3 (The comparison principle). *Let u_1 and u_2 be entropy solutions to (5.1) with initial data $u_{1,0}(x)$ and $u_{2,0}(x) \in L^\infty(\Omega \times \mathscr{O})$ respectively.*

Then there exist representatives $u_1^\pm$ and $u_2^\pm$ to respectively of u_1 and u_2, such that $f_1^\pm = 1_{v<u_1^\pm}$ and $f_2^\pm = 1_{v<u_2^\pm}$, where $f_1^\pm$ and $f_2^\pm$ are the kinetic functions given by Lemma 5.1.

Moreover, for all $0 \le t \le T$,

$$\mathbb{E}\int_{\mathscr{O}} (u_1^\pm(t,x) - u_2^\pm(t,x))_+\, dx \le \mathbb{E}\int_{\mathscr{O}} (u_{1,0}(x) - u_{2,0}(x))_+\, dx. \tag{5.24}$$

Proof. Essentially, the idea is to choose φ in (5.17) to be a "boundary layer sequence" in the nomenclature of C. Mascia–A. Porreta–A. Terracina [84]; that is, we wish to consider a sequence φ_n in $\mathscr{C}_c^\infty(\mathscr{O})$ that increases to 1 everywhere in $\mathscr{O}$. It can be shown that in a weak sense $\nabla\varphi_n$ converges to $\nu(x)\, d\sigma(x)$, the unit outward normal times the superficial measure in $\partial\mathscr{O}$; thus, at least formally, the strong trace theorem proves that the term

$$\mathbb{E}\int_0^t\int_{\mathscr{O}} \operatorname{sgn}(u_1-u_2)_+(\mathbf{A}(u_1)-\mathbf{A}(u_2))\cdot\nabla\varphi_n(x)\,dx\,ds$$
$$\to \mathbb{E}\int_0^t\int_{\partial\mathscr{O}} \operatorname{sgn}(u_1-u_2)_+(\mathbf{A}(u_1)-\mathbf{A}(u_2))\cdot\nu(x)\,d\sigma(x)\,ds = 0$$

since $\mathbf{A}(u_1)\cdot\nu = 0 = \mathbf{A}(u_2)\cdot\nu$ in almost everywhere $(0,T)\times\mathscr{O}$. In this way, (5.24) is basically obtained by getting rid of the boundary term, letting $\varphi\equiv 1$, and performing a simple analysis of the final result. So as to justify this reasoning, we have to consider a convenient boundary layer sequence.

Step #1: Notice that, in virtue of Definition 5.1 and the hypotheses made in this chapter, Theorem 4.1 applies and asserts that any entropy solution u possesses a strong strace u^τ in $(0,T)\times\partial\mathscr{O}$. If one writes boundary condition (5.10) as

$$\int_0^T\int_{\mathscr{O}}\left(u(t,x)-\int_0^t\Phi(x,u(s,x))\,dW(s)\right)\frac{\partial\theta}{\partial t}(t,x)\,dxdt$$
$$+\int_0^T\int_{\mathscr{O}}\mathbf{A}(u(t,x))\cdot\nabla\theta(t,x)\,dxdt = 0,$$

then the Green–Gauss formulas arising from Chen–Frid theory [25, 26] (see Theorem 4.21 and [41]) show that the traces u^τ observe $\mathbf{A}(u^\tau)\cdot\nu = 0$ almost everywhere in $\Omega\times(0,T)\times\partial\mathscr{O}$.

Step #2: Given any strongly regular deformation $\Psi:\partial\mathscr{O}\times[0,1]\to\overline{\mathscr{O}}$, let the Lipschitz function $\mathfrak{h}:\overline{\mathscr{O}}\to\mathbb{R}$ be given by

$$\mathfrak{h}(x) = \begin{cases} s & \text{if } x\in\Psi(\partial\mathscr{O}\times\{s\}) \text{ for some } 0\le s\le 1, \text{ and}\\ 1 & \text{if } x\notin\Psi(\partial\mathscr{O}\times[0,1]),\end{cases}$$

and define, for any $\varepsilon>0$,

$$\varphi_\varepsilon(x) = \min\left\{1,\ \frac{1}{\varepsilon}\mathfrak{h}(x)\right\}. \tag{5.25}$$

As Inequality (5.17) evidently extends to Lipschitz functions φ vanishing at $\partial\mathscr{O}$, we may insert $\varphi=\varphi_\varepsilon$ in it. Before we pass $\varepsilon\to 0_+$, notice that

$$\nabla\varphi_\varepsilon(x) = \begin{cases} -\frac{1}{\varepsilon}\theta(\Psi(x,\mathfrak{h}(x)))\nu(\Psi(x,\mathfrak{h}(x))) & \text{if } x\in\Psi([0,\varepsilon]\times\partial\mathscr{O}), \text{ and}\\ 0, & \text{otherwise,}\end{cases}$$

where $\theta(y)$ is a real Lipschitz function, and $\nu(\Psi(x,\mathfrak{h}(x)))$ denotes the unit outward normal at $x\in\Psi(\partial\mathscr{O}\times\{\mathfrak{h}(x)\})$. Thanks to the regularity of this deformation, $\nu(\Psi_s(x))\to\nu(x)$ in $L^1(\partial\mathscr{O})$, as $s\to 0_+$, and, as a result,

$$\mathbb{E}\int_0^t\int_{\mathscr{O}}\operatorname{sign}(u_1-u_2)_+\left(\mathbf{A}(u_1)-\mathbf{A}(u_2)\right)\cdot\nabla\varphi_\varepsilon(x)\,dx\,ds$$
$$\to\mathbb{E}\int_0^t\int_{\mathscr{O}}\operatorname{sign}(u_1^\tau-u_2^\tau)_+\left(\mathbf{A}(u_1^\tau)-\mathbf{A}(u_2^\tau)\right)\cdot\nu(x)\theta(x)\,d\sigma(x)\,ds = 0 \text{ as } \varepsilon\to 0_+.$$

(Observe that the factor $1/\varepsilon$ does not bring problems, as it is compensated by the fact the integral above is taken in $(0,T)\times\Psi([0,\varepsilon]\times\partial\mathscr{O})$. Furthermore, note that $(u,v)\in\mathbb{R}\times\mathbb{R}\mapsto\operatorname{sign}(u-v)_+(\mathbf{A}(u)-\mathbf{A}(v))\in\mathbb{R}^N$ is a continuous function.) Since $0\le\varphi_\varepsilon(x)\le 1$ and $\varphi_\varepsilon(x)\to 1$ for all

$x \in \mathscr{O}$, we see that passing $\varepsilon \to 0_+$

$$\mathbb{E}\int_{\mathscr{O}}\int_{\mathbb{R}_v} f_1^{\pm}(t,x,v)\overline{f_2^{\pm}}(t,x,v)\,dx \le \mathbb{E}\int_{\mathscr{O}} \left(u_{1,0}(x) - u_{2,0}(x)\right)_+ \,dx. \tag{5.26}$$

Step #3: Let us investigate (5.26). Choosing $f_1 = f_2$, we deduce that

$$\mathbb{E}\int_{\mathscr{O}}\int_{\mathbb{R}_v} f_1^{\pm}(t,x,v)\overline{f_1^{\pm}}(t,x,v)\,dx = 0$$

for all $0 \le t \le T$. Consequently, for almost every $(\omega,t,x) \in \Omega \times Q$, $f_1^{\pm}(t,x,v)$ is either 0 or 1. Thence, for $-\frac{\partial f_1^{\pm}}{\partial v}$ is a Young measure, we conclude that $f_1^{\pm}$ is an equilibrium. Once the very same argument holds for $f_2^{\pm}$, we may thus write $f_1^{\pm} = 1_{v<u_1^{\pm}}$ and $f_2^{\pm} = 1_{v<u_2^{\pm}}$.

Accordingly, for

$$\mathbb{E}\int_{\mathscr{O}}\int_{\mathbb{R}_v} f_1^{\pm}(t,x,v)\overline{f_2^{\pm}}(t,x,v)\,dv\,dx = \mathbb{E}\int_{\mathscr{O}}\int_{u_2^{\pm}(t,x)\le v<u_1^{\pm}(t,x)} dv\,dx = \mathbb{E}\int_{\mathscr{O}}(u_1^{\pm}(t,x) - u_2^{\pm}(t,x))_+\,dx,$$

the desired identity (5.24) follows from (5.26). □

Remark 5.3. Notice that, as a consequence of (5.3) and (5.7), the constant functions $w \equiv a$ and $w \equiv b$ are entropy solutions to (5.1). As a result, if the initial data u_0 obeys (5.8), then necessarily $a \le u(t,x) \le b$ almost surely in $\mathscr{D}'(Q)$. This provides some consistency to the hypotheses and definitions of this chapter.

We will close this section deducing that an entropy solution to $u(t,x)$ to (5.1) has almost surely continuous paths, which allows us to drop the cumbersome $\pm$–notation.

Corollary 5.1. *Let u be an entropy solution to (5.1) with an initial data $u_0 \in L^\infty(\Omega \times \mathscr{O})$. Then u possesses a representative in the class $L^p(\Omega;\mathscr{C}([0,T];L^p(\mathscr{O})))$ for all $1 \le p < \infty$.*

Proof. It suffices to see that the representative u^+ given in Theorem 5.3 has almost surely continuous paths. Notice that u^+—as well as u^-—has finite $L^p(\Omega;L^\infty(0,T;L^p(\mathscr{O})))$–norms for all $1 \le p < \infty$.

Step #1: Let us initially show that u^+ has almost surely right-continuous paths. Let $0 \le t^* < T$ be given, and consider any sequence $t_n \to t^*_+$. According to Lemma 5.1, there exists a set of probability one $\Omega_0 \subset \Omega$—which does not depend on t^*—such that $f^+(t_n) \overset{\star}{\rightharpoonup} f^+(t)$ weakly–$\star$ in $L^\infty(\mathscr{O} \times \mathbb{R})$ for all $\omega \in \Omega_0$. On the other hand, since $f^+(t^*)$ is an equilibrium, reducing Ω_0 if necessary and adapting the techniques of Proposition 4.1, we see that indeed $u^+(t_n) \to u^+(t)$ strongly in $L^p(\mathscr{O})$ for any $1 \le p < \infty$ and $\omega \in \Omega_0$. In particular, $u^+(t) \to u_0$ in $L^p(\mathscr{O})$ for $\omega \in \Omega_0$ as $t \to 0+$.

Similarly, one can verify that the representative u^-, also stated in Theorem 5.3, has almost surely left-continuous paths in $L^p(\mathscr{O})$ for all $1 \le p < \infty$.

Step #2: Given any $0 < t^* < T$, the entropy solution (defined in $\Omega \times [0, T - t^*] \times \mathscr{O}$) with initial data $u^-(t^*)$ must belong to the same equivalence class as $u(t^* + \cdot)$. Consequently, by Step #1, $u^-(t^*) = \lim_{t\to t^*_+} u^+(t) = u^+(t^*)$ for all $\omega \in \Omega_0$. According to Lemma 5.1, this shows that $u^+(t)$ indeed belongs to $L^p(\Omega;\mathscr{C}([0,T];L^p(\mathscr{O})))$ for any $1 \le p < \infty$. □

5.3 Existence

5.3.1 The vanishing viscosity method

First of all, one needs to construct certain approximate solutions to (5.1); as is traditional in the field of nonlinear problems, we will thus employ the vanishing viscosity method. So as to apply such procedure, let us manufacture some appropriate mollified versions of $\mathbf{A}(v)$ and $\Phi(x,v)$.

Proposition 5.1. *There exist* $\widetilde{\mathbf{A}} : \mathbb{R} \to \mathbb{R}^N$, *and, for any* $0 < \varepsilon < 1$, $\Phi^{(\varepsilon)} : L^2(\mathscr{O}) \to \mathscr{L}(\mathscr{H}; L^2(\mathscr{O}))$ *enjoying the following properties.*

1. $\widetilde{\mathbf{A}} \in (\mathscr{C}^1 \cap W^{1,\infty})(\mathbb{R}; \mathbb{R}^N)$, *and* $\widetilde{\mathbf{A}}(v) = \mathbf{A}(v)$ *for* $a \le v \le b$.
2. *Writing* $\Phi^{(\varepsilon)}(x,u) = \sum_{k=1}^{\infty} g_k^{(\varepsilon)}(x,u)\,(\,\cdot\,, e_k)_{\mathscr{H}}$, *then:*
   (a) *For all* $k \ge 1$, *it holds that* $\operatorname{supp} g_k^{(\varepsilon)} \subset \mathscr{O} \times [a,b]$;
   (b) *There exists some constant* $C_{**} > 0$ *such that, for every* $0 < \varepsilon < 1$, $x \in \mathscr{O}$, *and* $v \in \mathbb{R}$,

$$(\mathfrak{G}^{(\varepsilon)})^2(x,v) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} |g_k^{(\varepsilon)}(x,v)|^2 \le C_{**}(1+v^2). \tag{5.27}$$

   (c) *For all* $\mathscr{U} \subset\subset \mathscr{O}$, *it holds that*

$$\lim_{\varepsilon \to 0+} \max_{x \in \overline{\mathscr{U}}, a \le u \le b} \sum_{k=1}^{\infty} |g_k^{(\varepsilon)}(x,u) - g_k(x,u)|^2 = 0. \tag{5.28}$$

   (d) *Each* $g_k^{(\varepsilon)} \in \mathscr{C}(\overline{\mathscr{O}} \times \mathbb{R})$. *Moreover,* $\frac{\partial g_k^{(\varepsilon)}}{\partial v}$ *exists, and belongs to* $\mathscr{C}(\overline{\mathscr{O}} \times \mathbb{R})$, *and there are* $\gamma_k^{(\varepsilon)} \ge 0$ *such that, for any* $(x,v) \in \overline{\mathscr{O}} \times \mathbb{R}$,

$$\left| \frac{\partial g_k^{(\varepsilon)}}{\partial v}(x,v) \right| \le \gamma_k^{(\varepsilon)},$$

   and $\sum_{k=1}^{\infty} (\gamma_k^{(\varepsilon)})^2 \le C(\varepsilon) < \infty$.

Proof. Evidently, $\widetilde{\mathbf{A}}$ can be obtained by truncating the original $\mathbf{A}(v)$ outside $[a,b]$; on the other hand, the fabrication of $\Phi^{(\varepsilon)}(x,u)$ is somewhat more complicated and depends fundamentally on the following claim. Notice that we may suppose that each $g_k(x,v)$ is supported on $\overline{\mathscr{O}} \times [a,b]$.

Claim: For all $\mathscr{U} \subset\subset \mathscr{O}$, the series $\sum_{k=1}^{\infty} g_k(x,v)^2$ converges uniformly in $\overline{\mathscr{U}} \times [a,b]$.

Indeed, Estimate (5.5) ensures that this series is uniformly bounded, whereas Condition (5.6) forces the sequence of the partial sums to be equicontinuous. Thus, the desired assertion follows from the classical Arzelá–Ascoli theorem.

With this claim in our possession, we may argue as follows. Let $0 < \varepsilon < 1$. If $\mathscr{U}_\varepsilon = \{x \in \mathscr{O}; \operatorname{dist}(x, \partial\mathscr{O}) > \varepsilon\}$, pick some $\theta_\varepsilon \in \mathscr{C}_c^\infty(\mathscr{O})$ such that $\theta_\varepsilon(x) = 1$ for $x \in \mathscr{U}_\varepsilon$ and $0 \le \theta_\varepsilon(x) \le 1$ everywhere. For the series $\sum_{k=1}^{\infty} \theta_\varepsilon(x)^2 g_k(x,v)^2$ converges uniformly, there exists an integer $K_\varepsilon \ge 1$ such that

$$\sum_{k=K_\varepsilon+1}^{\infty} \theta_\varepsilon(x)^2 |g_k(x,v)|^2 < \varepsilon/3$$

for all $(x,v) \in \overline{\mathscr{O}} \times [a,b]$. Without loss of generality, we may presume that $[a,b] = [-1,1]$, so that there exists some $1 < \lambda_\varepsilon \le 2$ such that

$$\sum_{k=1}^{K_\varepsilon} \theta_\varepsilon(x)^2 |g_k(x,v) - g_k(x, \lambda_\varepsilon v)|^2 < \varepsilon/3 \text{ for all } (x,v) \in \mathscr{O} \times [-1,1].$$

If $(\varrho_\varepsilon)_{\varepsilon>0}$ is a mollifier family in the real line, and pick some $0 < \delta_\varepsilon < 1 - 1/\lambda_\varepsilon$ such that

$$\max_{x \in \overline{\mathscr{O}}, v \in \mathbb{R}} \sum_{k=1}^{K_\varepsilon} \theta_\varepsilon(x)^2 |(\varrho_{\delta_\varepsilon} \star_v g_k)(x,v) - g_k(x,v)|^2 < \varepsilon/3.$$

In this fashion, it suffices to choose

$$g_k^{(\varepsilon)}(x,v) = \begin{cases} \theta_\varepsilon(x)(\varrho_{\delta_\varepsilon} \star_v g_k)(x, \lambda_\varepsilon \,\cdot\,) & \text{if } 1 \le k \le K_\varepsilon, \text{ and} \\ 0 & \text{if } k > K_\varepsilon, \end{cases}$$

Since the verification that $\Phi^{(\varepsilon)}(x,u) = \sum_{k=1}^{\infty} g_k^{(\varepsilon)}(x,u)\ \ (\,\cdot\,, e_k)_{\mathscr{H}}$ satisfies the required impositions is immediate, the proposition is hereby proven. □

Henceforth, we will tacitly presume that $\widetilde{\mathbf{A}}(v)$, $g_k^{(\varepsilon)}(x,v)$, and $\Phi^{(\varepsilon)}(x,v)$ are as in the proposition above.

On the grounds of Theorem A.1 in the Appendix A, we may thus assert the following result.

Lemma 5.4. *For any $0 < \varepsilon < 1$, there exists a unique solution $u^{(\varepsilon)} \in L^2(\Omega;\, \mathscr{C}([0,T]; L^2(\mathscr{O}))) \cap L^2(0,T; H^1(\mathscr{O})))$ to*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \operatorname{div}_x \widetilde{\mathbf{A}}(u) - \varepsilon \Delta_x u = \Phi^{(\varepsilon)}(x,u)\, \dfrac{dW}{dt} & \text{for } 0 < t < T \text{ and } x \in \mathscr{O}, \\ \widetilde{\mathbf{A}}(u) \cdot \nu = \varepsilon \dfrac{\partial u}{\partial \nu} & \text{for } 0 < t < T \text{ and } x \in \partial\mathscr{O}, \text{ and} \\ u(0,x) = u_0(x) & \text{for } t = 0 \text{ and } x \in \mathscr{O} \end{cases} \tag{5.29}$$

in the sense that

$$\int_0^T \int_{\mathscr{O}} u(t,x) \frac{\partial \varphi}{\partial t}(t,x)\, dxdt + \int_{\mathscr{O}} u_0(x)\varphi(0,x)\, dx + \int_0^T \int_{\mathscr{O}} \widetilde{\mathbf{A}}(u(t,x)) \cdot \nabla_x \varphi(t,x)\, dxdt$$
$$- \varepsilon \int_0^T \int_{\mathscr{O}} \nabla_x u(t,x) \cdot \nabla_x \varphi(t,x)\, dxdt = - \sum_{k=1}^{\infty} \int_0^T \int_{\mathscr{O}} g_k^{(\varepsilon)}(x, u(t,x))\varphi(t,x)\, dxd\beta_k(t)$$

almost surely for all $\varphi \in \mathscr{C}_c^\infty((-\infty,T) \times \mathbb{R}^N)$.

Furthermore, such a solution has the following properties.

1. (L^∞–bound). *For any $0 < \varepsilon < 1$, one has almost surely that*

$$a \le u^{(\varepsilon)}(t,x) \le b \text{ in } \mathscr{D}'(Q). \tag{5.30}$$

2. (Energy estimate). *For all $1 \le p < \infty$, there exists a constant $C_p = C_p(a,b)$, independent of $0 < \varepsilon < 1$, such that*

$$\mathbb{E}\Big[\Big(\int_0^T \int_{\mathscr{O}} \varepsilon |\nabla u^{(\varepsilon)}(t,x)|^2\, dxdt\Big)^p\Big] \le C_p. \tag{5.31}$$

3. (Entropy formulation). *Almost surely, for any function $\eta \in \mathscr{C}^2(\mathbb{R})$ with $\eta'' \in L^\infty(\mathbb{R})$, and any $\phi \in \mathscr{C}_c^1((-\infty,T) \times \mathscr{O})$, it holds that*

$$\begin{aligned} \int_0^T \int_{\mathscr{O}} \Big(\eta(u^{(\varepsilon)}) \frac{\partial \phi}{\partial t} + \mathbf{A}^\eta(u^{(\varepsilon)}) \cdot \nabla_x \phi\Big)\, dxdt = &- \int_{\mathscr{O}} \eta(u_0(x))\phi(0,x)\, dx \\ &+ \int_0^T \int_{\mathscr{O}} \Big(\varepsilon \nabla_x \eta(u^{(\varepsilon)}) \cdot \nabla_x \phi + \varepsilon \eta''(u^{(\varepsilon)}) |\nabla u^{(\varepsilon)}|^2 \phi\Big)\, dxdt \\ &- \int_0^T \int_{\mathscr{O}} \eta'(u^{(\varepsilon)}) \Phi^{(\varepsilon)}(x, u^{(\varepsilon)}) \phi\, dxdW(t) \\ &- \frac{1}{2} \int_0^T \int_{\mathscr{O}} \eta''(u^{(\varepsilon)}) (\mathfrak{G}^{(\varepsilon)})^2(x, u^{(\varepsilon)})\, dxdt, \end{aligned} \tag{5.32}$$

where we have denoted by $(\mathfrak{G}^{(\varepsilon)})^2(x,u) = \sum_{k=1}^{\infty} g_k^{(\varepsilon)}(x,u)^2$.

4. (Kinetic formulation). *If* $\mathbf{f}^{(\varepsilon)}(t,x,v) = 1_{(-\infty,u(t,x))}(v) - 1_{(0,\infty)}(v)$ *is the* χ–function associated to $u^{(\varepsilon)}(t,x)$*, then it satisfies almost surely in* $\mathscr{D}'(Q)$

$$\frac{\partial \mathbf{f}^{(\varepsilon)}}{\partial t} + \mathbf{a}(v)\cdot\nabla_x \mathbf{f}^{(\varepsilon)} - \varepsilon\Delta_x \mathbf{f}^{(\varepsilon)} = \frac{\partial \mathbf{q}^{(\varepsilon)}}{\partial v} + \delta_{u^{(\varepsilon)}(t,x)}(v)\Phi^{(\varepsilon)}(x,v)\,\frac{dW}{dt}, \tag{5.33}$$

where we have written

$$\begin{cases}\mathbf{m}^{(\varepsilon)}(t,x,v) = \varepsilon|\nabla u^{(\varepsilon)}(t,x)|^2\delta_{u^{(\varepsilon)}(t,x)}(v), \text{ and}\\ \mathbf{q}^{(\varepsilon)}(t,x,v) = \mathbf{m}^{(\varepsilon)}(t,x,v) - \frac{1}{2}(\mathfrak{G}^{(\varepsilon)})^2(x,v)\delta_{u^{(\varepsilon)}(t,x)}(v).\end{cases}$$

5. (The boundary condition). *Almost surely, for all* $\theta \in \mathscr{C}_c^\infty((0,T)\times\mathbb{R}^N)$*, it holds that*

$$\int_0^T\int_{\mathscr{O}} u^{(\varepsilon)}(t,x)\frac{\partial\theta}{\partial t}(t,x)\,dxdt + \int_0^T\int_{\mathscr{O}}\big(\mathbf{A}^{(\varepsilon)}(u(t,x)) - \varepsilon\nabla u^{(\varepsilon)}(t,x)\big)\cdot\nabla\theta(t,x)\,dxdt + \sum_{k=1}^\infty\int_0^T\int_{\mathscr{O}} g_k^{(\varepsilon)}(x,u(t,x))\theta(t,x)\,dx\,d\beta_k(t) = 0. \tag{5.34}$$

Notice that, once $a \le u^{(\varepsilon)}(t,x) \le b$ almost surely, one could write $\mathbf{A}^\eta(v)$ in (5.32) and $\mathbf{a}(v)$ in (5.33) rather than $\widetilde{\mathbf{A}}^\eta(v)$ and $\widetilde{\mathbf{A}}'(v)$ respectively.

5.3.2 The compactness argument, part I: a priori estimates

In the purely deterministic case $\Phi^{(\varepsilon)} \equiv 0$, one could conclude the existence of entropy solutions to (5.1) as follows. In virtue of the nondegeneracy condition (5.4) and the L^∞–bound (5.30), Theorem 3.4 would imply that $u^{(\varepsilon)}$ belongs to a compact of $L^1(Q)$, and that its limit points obey the entropy condition (5.9). Since the boundary condition (5.10) follows directly from the L^1–convergence and (5.34), indeed the limit points of $\{u^{(\varepsilon)}\}$ would be entropy solutions to (5.1). By the uniqueness of solutions (see Theorem 5.3), we would have then established that $u^{(\varepsilon)}$ converges as $\varepsilon \to 0_+$ to the unique entropy solution to (5.1) with initial data u_0.

However, in the stochastic case, the situation becomes sensitively more intricate, as $\mathbf{f}^{(\varepsilon)}$ does not converge strongly in $L^1_\omega H^{-1}_{t,x,v}$ a priori, and so one is initially hindered from invoking local averaging lemmas. Likewise, one cannot argue by "diagonal extraction" in $\omega \in \Omega$, as such a set is uncountable; furthermore, once Ω is in principle devoid of any topological structure, no Kolmogorov–M. Riesz–Fréchet theorem should be available.

Fortunately, there exists a simple compactness argument based on a famous work of T. Yamada–S. Watanabe [113], which enables us to somewhat reproduce the "deterministic" proof in a probabilistic setting. The heart of the matter is the next proposition of I. Gyöngy–N. Krylov [60]. Recall that a Polish space is nothing more than a separable, complete metric space.

Theorem 5.4 (Gyöngi–Krylov's criterion for convergence in probability)**.** *Let* $(\Omega,\mathscr{F},\mathbb{P})$ *be a probability space,* (X_n) *be a sequence of random elements with values in a Polish space* M *(equipped with the Borel* σ*-algebra).*

Then (X_n) *converges in probability if, and only if, for every pair of subsequences* $(X_{n'})$ *and* $(X_{n''})$*, there exists a subsequence* $v_k = (X_{n'(k)}, X_{n''(k)})$ *converging weakly to a random element* v *supported on the diagonal* $\{(x,y)\in M\times M; x = y\}$*.*

Endowed with this criterion, we will roughly proceed as follows. We will show that the laws of $u^{(\varepsilon)}$, called momentarily $\mu^{(\varepsilon)}$, are tight in some convenient negative Sobolev space; thence, the Prohorov's compactness theorem asserts that such laws are relatively compact. Therefore, given any two sequences ε_n and $\varepsilon_n' \to 0_+$, the laws $(\mu^{(\varepsilon_n)}, \mu^{(\varepsilon_n')})$ possess a subsequence, still denoted as such, that converges weakly to some μ. According to Skorohod's representation theorem, there is another probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}})$ with random elements $(\widetilde{\mathbf{u}}_n, \widehat{\mathbf{u}}_{n'})$, which have same laws as $\mu_{\varepsilon_n,\varepsilon_n'}$, and converge pointwisely to some $(\widetilde{\mathbf{u}}, \widehat{\mathbf{u}})$ whose law is identical to μ's.

Applying the averaging lemma conveniently, we will then verify that both $\widetilde{\mathbf{u}}$ and $\widehat{\mathbf{u}}$ are martingale entropy solutions to (5.1) (*i.e.*, they are entropy solutions to (5.1) in some probability space with some Wiener process). Since such a problem has unique entropy solutions, we will conclude that $\widetilde{\mathbf{u}} \equiv \widetilde{\mathbf{u}}$, establishing that the law of $(\widetilde{\mathbf{u}}, \widehat{\mathbf{u}})$, and consequently μ, is supported on the diagonal of such negative Sobolev space. As a result, Theorem 5.4 implies that the original approximate solutions $u^{(\varepsilon)}$ converge in probability in this negative Sobolev space to some u. Repeating the previous reasonings (which allows us to employ the averaging lemma), we will thus prove that u is a entropy solution to (5.1) in its original probability space.[1]

With this in mind, let us state some priori estimates related to $u^{(\varepsilon_n)}$. Let us thus introduce

$$\mathbf{G}^{(\varepsilon)} = \operatorname{div}_x\big(\varepsilon \nabla_x \mathbf{f}^{(\varepsilon)}\big) + \Psi^{(\varepsilon)} \frac{dW}{dt} + \frac{\partial \mathbf{q}^{(\varepsilon)}}{\partial v} \tag{5.35}$$

so that (5.33) reads now

$$\frac{\partial \mathbf{f}^{(\varepsilon)}}{\partial t} + \mathbf{a}(v) \cdot \nabla_x \mathbf{f}^{(\varepsilon)} = \mathbf{G}^{(\varepsilon)},$$

in such a way that (5.33) becomes a "system" of deterministic kinetic equations labeled in $\omega \in \Omega$.

Our first aim is to embed $\mathbf{G}^{(\varepsilon)}$ into some appropriate negative Sobolev space with satisfactory "compactness" estimates. In order to do so, let us begin by investigating the stochastic forcing term. In what follows, let us put

$$\Psi^{(\varepsilon)}(t,x,v) \stackrel{\text{def}}{=} \delta_{u^{(\varepsilon)}(t,x)}(v)\Phi^{(\varepsilon)}(x,v).$$

By $\Psi^{(\varepsilon)} \frac{dW}{dt}$, we understand the "almost sure" distribution (see Proposition 2.7)

$$\phi \in \mathscr{S}(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v) \mapsto \left\langle \Psi^{(\varepsilon)} \frac{dW}{dt}, \phi \right\rangle_{\mathscr{S}', \mathscr{S}} = \int_0^T \int_{\mathscr{O}} \int_{\mathbb{R}_v} \phi(t,x,v) \Psi^{(\varepsilon)}(t,x,v)\, dv dx dW(t).$$

Lemma 5.5. *Let $\frac{1}{2} < s \le 1$, and consider some bounded open interval I such that $(a,b) \subset\subset I$. For all $0 < \varepsilon < 1$, the distributions $\Psi^{(\varepsilon)}(t,x,v) \frac{dW}{dt}$ belong to a bounded set of $L^2(\Omega; H^{-s}(Q;\, H^{-s}(I)))$.*

Remark 5.4. We refer to J.-L. Lions–E. Magenes [80] (especially its chapter 1), T. Cazenave–A. Haraux [20], H. Amann [4], and T. Hytönen *et al.* [68] for details regarding the vector-valued Sobolev spaces $W^{s,p}(\mathscr{U}; E)$. For the convenience of the reader, we will enunciate below some of their attributes that will play an important role in the subsequent arguments. Henceforth, $\mathscr{U}$ denotes an arbitrary open set in $\mathbb{R}^d$ $(d \ge 1)$, $1 \le p < \infty$ is an exponent, and E stands for a Banach space.

1. Let $m \ge 0$ be some integer. The Sobolev space $W^{m,p}(\mathscr{U}; E)$ is composed of the "functions" $u \in L^p(\mathscr{U}; E)$ with the following property: Given any multi-index α with $|\alpha| \le m$, there exists some $g_\alpha \in L^p(\mathscr{U}; E)$ such that

$$\int_{\mathscr{U}} (D^\alpha \varphi)(y) u(y)\, dy = (-1)^{|\alpha|} \int_{\mathscr{U}} \varphi(y) g_\alpha(y)\, dy$$

for all $\varphi \in \mathscr{C}_c^\infty(\mathscr{U})$, where the integral above is understood in the Bochner sense. Of course, each $g_\alpha(y)$, if it exists, is determined by $u(y)$, hence we may write $g_\alpha(y) = (D^\alpha u)(y)$. This allows us to introduce the norms

$$\|u\|^p_{W^{m,p}(\mathscr{U};E)} = \sum_{|\alpha| \le m} \big\|D^\alpha u\big\|^p_{L^p(\mathscr{U};E)},$$

[1] Notice that this scheme of the proof informally shows that (existence of solutions in some probability space) + (uniqueness of solutions) ⇒ (existence of solutions). As it will be clear in a few moments, establishing the existence of such a "generalized" solution is essentially the result of the extraction of some "compactness" estimates. In this fashion, this strategy closely resembles the Riesz–Fredholm theory, in which, under some compactness hypotheses, (uniqueness) ⇒ (existence).

which transform each $W^{m,p}(\mathscr{U};E)$ into a Banach space.

On the other hand, if $0 < s < 1$, the fractional Sobolev space $W^{s,p}(\mathscr{U};E)$, also known as the Sobolev–Slobodetskii space, is defined as the set of elements $u \in L^p(\mathscr{U};E)$ such that

$$[u]^p_{W^{s,p}(\mathscr{U};E)} \overset{\text{def}}{=} \int_{\mathscr{U}} \int_{\mathscr{U}} \frac{\|u(y) - u(y')\|^p_E}{|y-y'|^{d+sp}}\, dydy' < \infty.$$

In this case, one introduces the norm

$$\|u\|^p_{W^{s,p}(\mathscr{U})} = \|u\|^p_{L^p(\mathscr{U};E)} + [u]^p_{W^{s,p}(\mathscr{U};E)}.$$

Again, each $W^{s,p}(\mathscr{U};E)$ is also a Banach space. Such spaces could also be obtained as the real or complex interpolation of $L^p(\mathscr{U};E)$ with $W^{1,p}(\mathscr{U};E)$ under some natural smoothness hypotheses on the boundary of $\mathscr{U}$ (*e.g.*, if $\mathscr{U} = \mathbb{R}^d$, or if $\mathscr{U}$ is bounded, regular, and Lipschitz).

Finally, for some general real number $s \geq 0$, set $m = \lfloor s \rfloor$, and $z = s - \lfloor s \rfloor$. Then $W^{s,p}(\mathscr{U};E)$ is simply the space of functions $u \in W^{m,p}(\mathscr{U};E)$ such that $D^\alpha u \in W^{z,p}(\mathscr{U};E)$ for $|\alpha| = m$. Once more, $W^{s,p}(\mathscr{U};E)$ is a Banach space under the norm

$$\|u\|^p_{W^{s,p}(\mathscr{U};E)} = \sum_{|\alpha|<m} \|D^\alpha u\|^p_{L^p(\mathscr{U};E)} + \sum_{|\alpha|=m} \|D^\alpha u\|^p_{W^{z,p}(\mathscr{U};E)}.$$

These Sobolev spaces $W^{s,p}(\mathscr{U};E)$ inherit several important properties from the usual Sobolev spaces and the codomain E. For instance,

- if E is separable, so is $W^{s,p}(\mathscr{U};E)$ (recall that $1 \leq p < \infty$), and
- if E is reflexive and $1 < p < \infty$, then $W^{s,p}(\mathscr{U};E)$ is reflexive as well.

Both these statements may be derived from the general theory of the Bochner spaces $L^p(\mathscr{U};E)$. (*References:* [20], and [68]).

2. Let $s \geq 0$. By $W^{s,p}_0(\mathscr{U};E)$, we will understand the closure of $\mathscr{C}^\infty_c(\mathscr{U};E)$ in $W^{s,p}(\mathscr{U};E)$. Applying the classical techniques of regularization and truncation, it is not difficult to see that $W^{s,p}_0(\mathbb{R}^d;E) = W^{s,p}(\mathbb{R}^d;E)$. Moreover, assuming for instance that $\mathscr{U}$ is either $(0,T)$ or $Q = (0,T) \times \mathscr{O}$ (recollect that $\mathscr{O}$ is of class $\mathscr{C}^{1,1}$), it is not difficult to see that $W^{s,p}_0(\mathscr{U};E)$ may be interpreted as the elements in $W^{s,p}(\mathbb{R}^d;E)$ whose supports are contained in $\mathscr{U}$ if $0 \leq s \leq 1$.

   Suppose now that E is reflexive for simplicity's sake. We will set

$$W^{-s,p'}(\mathscr{U};E^\star) \overset{\text{def}}{=} W^{s,p}(\mathscr{U};E)^\star,$$

   where p' is the conjugate of p. Again, one may argue that $W^{-s,p'}(\mathscr{U};E^\star)$ can be canonically identified with the subspace of the elements of $W^{-s,p'}(\mathbb{R}^d;E^\star)$ whose support lie in $\mathscr{U}$. (These definitions are inspired by [80]).

3. Let $m \geq 0$ be an integer, and E be a reflexive Banach space. Then, any $\Lambda \in W^{-m,p'}(\mathscr{U};E^\star)$ may be represented in a nonunique fashion as

$$\Lambda = \sum_{|\alpha|\leq m} D^\alpha f_\alpha, \tag{5.36}$$

   where $f_\alpha \in L^{p'}(\mathscr{U};E^\star)$ and

$$\|\Lambda\|^{p'}_{W^{-m,p'}(\mathscr{U};E^\star)} = \sum_{|\alpha|\leq m} \left\|D^\alpha f_\alpha\right\|^{p'}_{L^{p'}(\mathscr{U};E^\star)}.$$

(By (5.36), we mean that

$$\langle \Lambda, u \rangle_{W^{-m,p'}(\mathscr{U};E^\star);W_0^{m,p}(\mathscr{U};E)} = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} \int_{\mathscr{U}} \langle f_\alpha(y), (D^\alpha u)(y) \rangle_{E^\star,E}\, dy$$

for all $u \in W_0^{m,p}(\mathscr{U};E)$).

The representation expressed in (5.36) follows from elementary arguments and the fact that one may understand $L^p(\mathscr{U};E)^\star = L^{p'}(\mathscr{U};E^\star)$ if E is reflexive (or if $E^\star$ is separable). Notice that, conversely, if Λ has the form (5.36), then it belongs to $W^{-m,p'}(\mathscr{U};E^\star)$. This provides a quite tangible way to comprehend the assertion that "$W^{-m,p'}(\mathscr{U};E)$ are the elements of $W^{-m,p'}(\mathbb{R}^d;E^\star)$ whose supports are in $\mathscr{U}$". (*References:* [80], [4], and [68]).

4. The case $p = 2$ is evidently special, and we write $H^s(\mathscr{U};E) = W^{s,2}(\mathscr{U};E)$, $H_0^s(\mathscr{U};E) = W_0^{s,2}(\mathscr{U};E)$, and $H^{-s}(\mathscr{U};E) = W^{-s,2}(\mathscr{U};E)$ wherever $s \geq 0$. If E is a Hilbert space, all such spaces are likewise Hilbert spaces.

   Furthermore, (still assuming that E is Hilbert) one may charactherize $H^s(\mathbb{R}^d;E)$ by means of the Fourier transform as follows. Let $\mathscr{S}(\mathbb{R}^d;\mathbb{C})$ be the usual Schwartz space. By "the space of the tempered distributions with values in E", we understand $\mathscr{S}'(\mathbb{R}^d;E) = \mathscr{L}(\mathscr{S}(\mathbb{R}^d;\mathbb{C});E)$ (the set of the continuous linear transformations from $\mathscr{S}(\mathbb{R}^d;\mathbb{C})$ into E).

   One can then define the Fourier transform of any element $u \in \mathscr{S}'(\mathbb{R}^d;E)$ as

$$\langle \mathfrak{F}_y u, f \rangle_{\mathscr{S}',\mathscr{S}} = \langle u, \mathfrak{F}_y f \rangle_{\mathscr{S}',\mathscr{S}}.$$

   As in the scalar case, it can be shown that $\mathfrak{F}_y$ defines a unitary map in $L^2(\mathscr{U};E)$; moreover, one can also prove that, for all $s \in \mathbb{R}$, $H^s(\mathbb{R}^d;E)$ is the space

$$\left\{ u \in \mathscr{S}'(\mathbb{R}^d;E); (1+|\xi|^2)^{s/2}(\mathfrak{F}_y u)(\xi) \in L^2(\mathbb{R}^d_\xi;E) \right\}$$

   endowed with the equivalent norm $\|u\|_{*H^s(\mathbb{R}^d;E)} = \|(1+|\xi|^2)^{s/2}(\mathfrak{F}_y u)(\xi)\|_{L^2(\mathbb{R}^d_\xi;E)}$.

   Supposing now that $\mathscr{U}$ is either $\mathbb{R}^d$, $(0,T)$ or $Q = (0,T)\times\mathscr{O}$, this observation has the following two important consequences.

   - If $0 \leq s < 1/2$, then $H_0^s(\mathscr{U};E) = H^s(\mathscr{U};E)$.
   - For all $0 \leq s < 1/2$ and any $e \in \mathbb{R}^d$ with $|e| = 1$ ($d = 1$ if $\mathscr{U} = (0,T)$, and $d = N+1$ if $\mathscr{U} = (0,T)\times\mathscr{O}$), then the differential operator $\frac{\partial}{\partial e}$ maps $H^s(\mathscr{U};E) = H_0^s(\mathscr{U};E)$ continuously into $H^{s-1}(\mathscr{U};E)$.

   (*References:* [80], and [68]).

5. Let us provide some interesting applications of such remarks.

   - If Ω is a probability space, $s \geq 0$, and I is an open interval, then $L^2(\Omega;H^{-s}(Q;H^{-s}(I))) = L^2(\Omega;H_0^s(Q;H_0^s(I)))^\star$.
   - Let $1 < q < \infty$ and $s \geq 0$. Then $W^{-1,q}(\mathbb{R}_t\times\mathbb{R}_x^N;W^{-s,q}(\mathbb{R}_v))$ may be understood as the set of the distributions of the form

$$(-\Delta_{t,x}+1)^{1/2}(-\Delta_v+1)^{s/2}g,$$

     where $g \in L^q(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v)$. Note that this is the appropriate form of source term to applying velocity averaging lemmas such as Theorems 2.2 and 2.4.

Remark 5.5. Throughout this section, an indispensable instrument is a celebrated continuity criterion by A. N. Kolmogorov. For the reader's convenience, we reproduce the statement of this result as it is enunciated in the book of D. Stroock [106].

Theorem 5.5 (Kolmogorov's continuity criterion). *Let $\{X(t)\}_{0\leq t\leq T}$ be a stochastic process defined on $(\Omega,\mathscr{F},\mathbb{P})$ taking values on a Banach space E. Assume that, for some $1\leq p<\infty$, $C>0$, and some $0<r\leq 1$,*

$$\Big[\mathbb{E}^{\mathbb{P}}\|X(t)-X(s)\|_E^p\Big]^{1/p}\leq C|t-s|^{1/p+r}\ \text{for all } 0\leq s \text{ and } t\leq T.$$

Then there exists a stochastic process $\{\widetilde{X}(t)\}_{0\leq t\leq T}$ defined in Ω and taking values in E such that $X(t)=\widetilde{X}(t)$ $\mathbb{P}$-almost surely for each $0\leq t\leq T$, and $t\in[0,T]\mapsto\widetilde{X}(\omega,t)$ is continuous for all $\omega\in\Omega$. In fact, for each $0<\alpha<r$,

$$\left[\mathbb{E}^{\mathbb{P}}\sup_{0\leq s<t\leq T}\left(\frac{\|\widetilde{X}(t)-\widetilde{X}(s)\|_E}{(t-s)^\alpha}\right)^p\right]^{1/p}\leq\frac{5CT^{1/p+r-\alpha}}{(1-2^{-r})(1-2^{\alpha-r})}.$$

Proof of Lemma 5.5. Essentially, we are going to revisit the manufacturing procedure behind the coefficients $\mathfrak{g}_k(t,x,v)$ (Equation (4.40)) in the previous chapter. Notice that, as distributions, each $(t,x,v)\mapsto g_k^{(\varepsilon)}(x,v)\delta_{u^{(\varepsilon)}(t,x)}(v)=\Psi(t,x,v)e_k$ is supported on $Q\times I$, where I is as in the statement of this lemma.

Step #1: Let $0<\varepsilon<1$ and $\frac{1}{2}<s\leq 1$ be given. For any $\phi(x,v)\in L^2(\mathscr{O};H_0^s(I))$, $0\leq t\leq T$, and $k\geq 1$, we see that

$$\begin{aligned}\big\langle\Psi^{(\varepsilon)}(t)e_k,\phi\big\rangle_{L^2(\mathscr{O};H^{-s}(I));L^2(\mathscr{O};H_0^s(I))}&=\int_{\mathscr{O}}g_k^{(\varepsilon)}(x,u^{(\varepsilon)}(t,x))\phi(x,u^{(\varepsilon)}(t,x))\,dx\\&\leq\int_{\mathscr{O}}|g_k^{(\varepsilon)}(x,u(t,x))|\|\phi(x,\cdot)\|_{L^\infty(\mathbb{R}_v)}\,dx\\&\leq C(s)\|g_k(\cdot,u(t,\cdot))\|_{L^2(\mathscr{O})}\|\phi(x,v)\|_{L^2(\mathscr{O},H_0^s(I))},\end{aligned}$$

almost surely, since $s>\frac{1}{2}$ and thus $H_0^s(I)\subset\mathscr{C}(\overline{I})$ continuously. In other words, as $L^2(\mathscr{O};H^{-s}(I))=L^2(\mathscr{O};H_0^s(I))^\star$,

$$\|\Psi^{(\varepsilon)}(t)e_k\|^2_{L^2(\mathscr{O},H^{-s}(I))}\leq C\|g_k(x,u(t,x))\|^2_{L^2(\mathscr{O})}.$$

Consequently, we may sum the former estimate in $k\geq 1$, apply Condition (5.27), and recall the L^∞–bound (5.30) in order to deduce

$$\operatorname*{ess\,sup}_{(\omega,t)\in\Omega\times[0,T]}\big\|\Psi^{(\varepsilon)}(t,x,v)\big\|_{HS(\mathscr{H};L^2(\mathscr{O};H^{-s}(I)))}\leq C,\tag{5.37}$$

for some constant $C=C(a,b)$ not depending on $0<\varepsilon<1$.

Step #2: Consider any $2<p<\infty$, and $0\leq s$ and $t\leq T$. Per Equation (5.37), the Burkholder inequality yields

$$\begin{aligned}\mathbb{E}\left\|\int_s^t\Psi^{(\varepsilon)}(r)\,dW(r)\right\|^p_{L^2(\mathscr{O};H^{-s}(I))}&\leq C\mathbb{E}\left[\left(\int_s^t\big\|\Psi^{(\varepsilon)}(r)\big\|^2_{HS(\mathscr{H};L^2(\mathscr{O};H^{-s}(I)))}\,dr\right)^{p/2}\right]\\&\leq C_p|t-s|^{p/2},\end{aligned}$$

Therefore, the Kolmogorov's continuity criterion (Theorem 5.5) forces the process $t\in[0,T]\mapsto\int_0^t\Psi^{(\varepsilon)}\,dW$ to be uniformly bounded in $L^2(\Omega;\mathscr{C}^\sigma([0,T];L^2(\mathscr{O};H^{-s}(I)))$ for any $0<\sigma<\frac{1}{2}$. Thus, we see that $\int_0^t\Psi^{(\varepsilon)}\,dW$ is likewise uniformly bounded in $L^2(\Omega;H^\sigma(0,T;L^2(\mathscr{O};H^{-s}(I)))=L^2(\Omega;H_0^\sigma(0,T;L^2(\mathscr{O};H^{-s}(I)))$ for any $0<\sigma<\frac{1}{2}$; see Remark 5.4.

For Proposition 2.7 implies that

$$\Psi^{(\varepsilon)}\frac{dW}{dt}=\frac{\partial}{\partial t}\left(\int_0^t\Psi^{(\varepsilon)}(t',x,v)\,dW(t')\right)\tag{5.38}$$

almost surely in sense of the weak derivatives, we conclude that $\Psi^{(\varepsilon)}(t,x,v)\frac{dW}{dt}$ is bounded in $L^2(\Omega; H^{-s}(0,T;\ L^2(\mathscr{O}; H^{-s}(I)))$. Because $H^{-s}(0,T; L^2(\mathscr{O}; H^{-s}(I))$ is the dual space of $H_0^s(0,T;$ $L^2(\mathscr{O}; H_0^s(I)))$, which contains continually $H_0^s(Q; H_0^s(I))$, the desired assertion follows. □

Theorem 5.6. *The following assertions hold.*

1. *The laws of* $(\mathbf{f}^{(\varepsilon)})_{0<\varepsilon<1}$ *are tight in*

$$X_f = H^{-1}(\mathbb{R}_t \times \mathbb{R}_x^N \times \mathbb{R}_v).$$

Moreover, $\mathbb{E}\|\mathbf{f}^{(\varepsilon)}\|_{X_f}^2 \le C$, *for some constant that does not depend on* $0<\varepsilon<1$.

2. *Let* $0 < z < \frac{\alpha}{2}$ *(where* α *is the same as in Section 5.1), and* $1 < q < \frac{N+2}{N+2-z}$. *Then, the laws of* $(\mathbf{G}^{(\varepsilon)})_{0<\varepsilon<1}$ *are tight in the separable Banach space*

$$X_G = W^{-1,q}(\mathbb{R}_t \times \mathbb{R}_x^N; W^{-(1+z),q}(\mathbb{R}_v)).$$

Furthermore, $\mathbb{E}\|\mathbf{G}^{(\varepsilon)}\|_{X_G}^2 \le C$ *for some constant independent on* $0<\varepsilon<1$.

3. *The laws of* $(u^{(\varepsilon)})_{0<\varepsilon<1}$ *are tight in*

$$X_u = \Big\{ u \in \mathscr{C}([0,T]; H^{-2}(\mathscr{O}));\ u(0) \in L^2(\mathscr{O}) \Big\}.$$

Additionally, for all $0<\varepsilon<1$, *there exists a constant* C, *depending solely on* $\mathbb{E}\|u_0\|_{L^2(\mathscr{O})}^2$ *and* $\mathbb{E}\|u^{(\varepsilon)}\|_{\mathscr{C}([0,T];L^2(\mathscr{O}))}^2$, *such that* $\mathbb{E}\|u^{(\varepsilon)}\|_{X_u}^2 = \mathbb{E}\|u^{(\varepsilon)}(0)\|_{L^2(\mathscr{O})}^2 + \mathbb{E}\|u^{(\varepsilon)}\|_{\mathscr{C}([0,T];H^{-2}(\mathscr{O}))}^2 \le C$.

4. *The law of the cylindrical Wiener process* W *is tight in the separable Banach space*

$$X_W = \mathscr{C}_{\mathrm{loc}}([0,\infty); \mathscr{H}_0),$$

where $\mathscr{H}_0$ *is linear space* $\mathscr{H}_0 = \big\{ h \in \mathscr{H}; \sum_{k=1}^\infty \frac{1}{k^2}|(h,e_k)_{\mathscr{H}}|^2 < \infty \big\}$ *endowed with the norm* $\|h\|_{\mathscr{H}_0}^2 = \sum_{k=1}^\infty \frac{1}{k^2}|(h,e_k)_{\mathscr{H}}|^2$.

Proof. Step #1: Once $\|\mathbf{f}^{(\varepsilon)}\|_{L^2(Q\times(a,b))}^2 \le LT|\mathscr{O}|$ almost surely and for all $\varepsilon > 0$, the first statement is an immediate consequence of the Rellich–Kondrachov theorem.

Step #2.1: Let us inspect now the assertion about $\mathbf{G}^{(\varepsilon)}$. We begin by analyzing $\varepsilon\Delta_x \mathbf{f}^{(\varepsilon)}(t,x,v)$. Given any $\theta \in \mathscr{C}_c^\infty(Q\times\mathbb{R}_v)$ and any $1 \le j \le k$, perceive that, almost surely,

$$\begin{aligned}
\left\langle \frac{\partial \mathbf{f}^{(\varepsilon)}}{\partial x_j}, \theta \right\rangle_{\mathscr{D}'(Q\times\mathbb{R}_v),\mathscr{C}_c^\infty(Q\times\mathbb{R}_v)} &= -\int_Q \int_{\mathbb{R}_v} \frac{\partial \theta}{\partial x_j}(t,x,v)\mathbf{f}^{(\varepsilon)}(t,x,v)\,dvdxdt \\
&= -\int_Q \left(\frac{\partial \Theta}{\partial x_j}\right)(t,x,u^{(\varepsilon)}(t,x))\,dxdt \\
&= \int_Q \theta(t,x,u^{(\varepsilon)}(t,x))\frac{\partial u^{(\varepsilon)}}{\partial x_j}(t,x)\,dxdt
\end{aligned} \tag{5.39}$$

where $\Theta(t,x,v) = \int_0^v \theta(t,x,w)\,dw$. As a result, besides justifying the formal formula

$$\frac{\partial \mathbf{f}^{(\varepsilon)}}{\partial x_j}(t,x,v) = \delta_{u(t,x)}(v)\frac{\partial u^{(\varepsilon)}}{\partial x_j}(t,x),$$

Equation (5.39) also implies that

$$\begin{aligned}\left|\left\langle \frac{\partial \mathbf{f}^{(\varepsilon)}}{\partial x_j}, \theta \right\rangle_{\mathscr{D}'(Q\times\mathbb{R}_v),\mathscr{C}_c^\infty(Q\times\mathbb{R}_v)}\right| &\leq \int_Q \|\theta(t,x,\,\cdot\,)\|_{L^\infty(\mathbb{R}_v)} \left|\frac{\partial u^{(\varepsilon)}}{\partial x_j}(t,x)\right| dxdt \\ &\leq C\int_Q \|\theta(t,x,\,\cdot\,)\|_{W^{1,q'}(\mathbb{R}_v)} \left|\frac{\partial u^{(\varepsilon)}}{\partial x_j}(t,x)\right| dxdt \\ &\leq C\|\theta\|_{L^2(Q;W^{1,q'}(\mathbb{R}_v))} \left\|\frac{\partial u^{(\varepsilon)}}{\partial x_j}\right\|_{L^2(Q)},\end{aligned}$$

where we have utilized the Sobolev inequality $W^{1,q'}(\mathbb{R}_v) \subset L^\infty(\mathbb{R}_v)$. From the duality relation

$$L^2(Q;W^{1,q'}(\mathbb{R}_v))^\star = L^2(Q;W^{-1,q}(\mathbb{R}_v)), \tag{5.40}$$

we conclude that, according to (5.31),

$$\mathbb{E}\left\|\varepsilon\frac{\partial \mathbf{f}^{(\varepsilon)}}{\partial x_j}\right\|^2_{L^2(Q;W^{-1,q}(\mathbb{R}_v))} \leq \varepsilon C\mathbb{E}\int_Q \varepsilon|\nabla u^{(\varepsilon)}(t,x)|^2\,dx \leq C\varepsilon. \tag{5.41}$$

Therefore, $\varepsilon\Delta_x\mathbf{f}^{(\varepsilon)} \to 0$ in $L^2(\Omega;H^{-1}(Q;W^{-1,q}(\mathbb{R}_v)))$ as $\varepsilon\to 0_+$.

For $H^{-1}(Q;W^{-1,q}(\mathbb{R}_v)) \subset X_G$ continuously, given any $\lambda>0$, it is not difficult to construct a compact set $K_\lambda^{(1)} \subset X_G$ such that

$$\mathbb{P}\Big(\mathrm{div}_x(\varepsilon\nabla_x\mathbf{f}^{(\varepsilon)}) \in K_\lambda^{(1)}\Big) \geq 1-\frac{1}{\lambda^2},$$

provided that one reprises the classical arguments used in the theory of the weak convergence of probability measures; see, *e.g.*, D. Stroock [106].

Step #2.2: We will now examine $\Psi^{(\varepsilon)}\frac{dW}{dt}$.

In virtue of Lemma 5.5, $\Psi^{(\varepsilon)}\frac{dW}{dt}$ is bounded in $L^2(\Omega;\,H^{-s}(Q;H^{-s}(I)))$ for some $s<1$ and open interval I containing $[a,b]$. Accordingly, the Tchebychev's inequality asserts that, for any $\lambda>0$,

$$\mathbb{P}\left(\left\|\Psi^{(\varepsilon)}\frac{dW}{dt}\right\|_{H^{-s}(Q;H^{-s}(I))} \geq \lambda\right) \leq \frac{1}{\lambda^2}\sup_{0<\varepsilon<1}\mathbb{E}\left\|\Psi^{(\varepsilon)}\frac{dW}{dt}\right\|^2_{H^{-s}(Q;H^{-s}(I))} = \frac{C}{\lambda^2}.$$

On the other hand, it is evident from the Rellich–Kondrachov theorem and a duality argument that $H^{-s}(Q;H^{-s}(I)) \subset X_G$ with compact injection. (Indeed, it is clear that

$$H_0^1(Q;H_0^1(I)) \subset L^2(Q;L^2(I)) = L^2(Q\times I) \text{ with compact injection.}$$

By interpolation thus,

$$H_0^1(Q;H_0^1(I)) \subset H_0^s(Q;H_0^s(I)) \text{ with compact injection.}$$

As a result, Schauder's theorem asserts that

$$H^{-s}(Q;H^{-s}(I)) \subset H^{-1}(Q;H^{-1}(I)) \text{ with compact injection as well;}$$

see also H. Amann [4]. On the other hand, using the representation formulas for elements in the negative Sobolev spaces, $H^{-1}(Q;H^{-1}(I)) \subset X_G$ with continuous injection; see Remark 5.4. This proves the claim).

Hence, for any $\lambda>0$, the subset

$$K_\lambda^{(2)} = \big\{\Lambda\in X_G;\,\|\Lambda\|_{H^{-s}(Q;H^{-s}(I))} \leq \lambda\big\}$$

is compact in X_G and observes

$$\mathbb{P}\Big(\Psi\frac{dW}{dt} \in K_\lambda^{(2)}\Big) \geq 1 - \frac{C_2}{\lambda^2},$$

for some $C_2 > 0$.

Step #2.3: Let us repeat this analysis to $\frac{\partial \mathbf{q}^{(\varepsilon)}}{\partial v}$.

Thanks Proposition 5.2, $\mathbf{q}^{(\varepsilon)}$ is bounded in $L^2_{\mathfrak{w}}(\Omega; \mathfrak{M}(Q\times[-L,L]))$. Thus, again by the Tchebychev's inequality,

$$\mathbb{P}\Big(\|\mathbf{q}^{(\varepsilon)}\|_{\mathfrak{M}} > \lambda\Big) \leq \frac{\mathbb{E}\|\mathbf{q}^{(\varepsilon)}\|^2_{\mathfrak{M}}}{\lambda^2}.$$

Per Lemma 3.2, $\mathfrak{M}(Q \times [-L, L]) \subset W^{-z,q}(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v)$ with compact injection. For this reason, $\mathbf{q}^{(\varepsilon)}$ is bounded in $L^2(\Omega; W^{-z,q}(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v))$—as Pettis' theorem implies that the reflexiveness and separability of $W^{-z,q}$ eliminates the necessity of weak mensurability; see, *e.g.*, T. Cazenave–A. Haraux [20]. Being $\frac{\partial}{\partial v}$ a bounded linear transformation from $W^{-z,q}(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v)$ into X_G, given any $\lambda > 0$, the set

$$K_\lambda^{(3)} = \frac{\partial}{\partial v}\Big\{\Lambda \in \mathfrak{M}; \|\Lambda\|_{\mathfrak{M}} \leq \lambda\Big\}$$

is a compact set of X_G, and, for any $0 < \varepsilon < 1$,

$$\mathbb{P}\Big(\frac{\partial \mathbf{q}^{(\varepsilon)}}{\partial v} \in K_\lambda^{(3)}\Big) \geq 1 - \frac{C_3}{\lambda^2},$$

for some constant $C_3 > 0$.

Step #2: (Conclusion). For any $\lambda > 0$, $K_\lambda = K_\lambda^{(1)} + K_\lambda^{(2)} + K_\lambda^{(3)}$ is compact, and, since

$$\Big\{\mathbf{G}^{(\varepsilon)} \in K_\lambda\Big\} \supset \Big\{\mathrm{div}_x(\varepsilon\nabla_x \mathbf{f}^{(\varepsilon)}) \in K_\lambda^{(1)}\Big\} \cap \Big\{\frac{\partial \mathbf{q}^{(\varepsilon)}}{\partial v} \in K_\lambda^{(2)}\Big\} \cap \Big\{\Psi^{(\varepsilon)}\frac{dW}{dt} \in K_\lambda^{(3)}\Big\},$$

one has that

$$\mathbb{P}\Big(\mathbf{G}^{(\varepsilon)} \notin K_\lambda\Big) \leq \frac{1 + C_2 + C_3}{\lambda^2},$$

which establishes first assertion in (2). Nonetheless, the second one is a direct corollary to (5.41), Lemma 5.5, and the a priori estimates (5.30) and (5.31).

Step #3: Statement (3) can be proven as follows (see H. Frid *et al.* [43]). Per the theory of Appendix A (more specifically, Proposition A.5), one can see that

$$u^{(\varepsilon)}(t) = u_0 + \int_0^t \Lambda_{u^{(\varepsilon)}(s)}\,ds + \int_0^t \Phi^{(\varepsilon)}\big(u^{(\varepsilon)}(s)\big)\,dW(s)$$

almost surely in $H^1(\mathscr{O})^\star$, where

$$\big\langle \Lambda_u, \phi\big\rangle_{H^1(\mathscr{O})^\star, H^1(\mathscr{O})} = \int_{\mathscr{O}} \big(\varepsilon\nabla_x u - \mathbf{A}(u)\big)\cdot\nabla\phi\,dx.$$

Notice that the L^∞–bound (5.30) and the energy estimate (5.31) imply that $\Lambda_{u^{(\varepsilon)}}$ is uniformly bounded in $L^2(\Omega\times[0,T]; H^{-1}(\mathscr{O}))$. Furthermore, the Kolmogorov's continuity criterion (Theorem 5.5) ensures that $\int_0^t \Phi^{(\varepsilon)}(u^{(\varepsilon)})\,dW$ is uniformly bounded in, say, $L^2(\Omega; \mathscr{C}^{1/3}([0,T]; L^2(\mathscr{O})))$ (see Lemma 5.5).

Therefore, $u^{(\varepsilon)}$ is uniformly bounded in $L^2(\Omega; \mathscr{C}^{1/3}([0,T]; H^{-1}(\mathscr{O}))$. The desired conclusion follows once again from the Tchebychev's inequality, the fact that $u^{(\varepsilon)}(0) = u_0$ for all $0 < \varepsilon < 1$, and the compact inclusion $\mathscr{C}^{1/3}([0,T]; H^{-1}(\mathscr{O})) \subset \mathscr{C}([0,T]; H^{-2}(\mathscr{O}))$ (which is a corollary of the Arzelá–Ascoli theorem).

Step #4: Finally, the last assertion follows directly from the theory of weak convergence of measures—we refer again to the book of D. Stroock [106]. □

Let us rephrase the previous theorem as follows.

Theorem 5.7. *The joint laws of the septets* $\big(\mathbf{f}^{(\varepsilon)}, u^{(\varepsilon)}, \mathbf{G}^{(\varepsilon)}, \mathbf{f}^{(\varepsilon')}, u^{(\varepsilon')}, \mathbf{G}^{(\varepsilon')}, W\big)$ *for* $0 < \varepsilon$ *and* $\varepsilon' < 1$ *are tight in the "doubled" path-space*

$$X = (X_f \times X_u \times X_G) \times (X_f \times X_u \times X_G) \times X_W,$$

which is a separable, complete metric space. Thus, according to Prohorov's theorem, such laws are relatively compact in sense of the weak convergence of probability measures.

Denote by $\mu^{(\varepsilon,\varepsilon')}$ the law of $\big(\mathbf{f}^{(\varepsilon)}, u^{(\varepsilon)}, \mathbf{G}^{(\varepsilon)}, \mathbf{f}^{(\varepsilon')}, u^{(\varepsilon')}, \mathbf{G}^{(\varepsilon')}, W\big)$, and consider two arbitrary sequences ε_n and $\varepsilon_n \to 0_+$. Passing to subsequences $\varepsilon_{n(\ell)}$ and $\varepsilon_{n'(\ell)}$, we may assume that $\mu^{(\varepsilon_{n(\ell)},\varepsilon_{n'(\ell)})} \rightharpoonup \mu$ to some probability measure μ in sense of the weak convergence of measures. Applying the Skorohod's representation theorem, we can infer the next result.

Theorem 5.8. *There exists a probability space* $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}})$ *with random variables* $\big(\mathfrak{f}_\ell, \mathbf{u}_\ell, \mathscr{G}_\ell, \mathfrak{g}_l, \mathbf{v}_\ell, \mathscr{K}_\ell, \mathfrak{W}\big)$ *and* $\big(\mathfrak{f}, \mathbf{u}, \mathscr{G}, \mathfrak{g}, \mathbf{v}, \mathscr{K}, \mathfrak{W}\big)$ *for which the following statements hold.*

1. *The laws of* $\big(\mathfrak{f}_\ell, \mathbf{u}_\ell, \mathscr{G}_\ell, \mathfrak{g}_\ell, \mathbf{v}_\ell, \mathscr{K}_\ell, \mathfrak{W}\big)$ *and of* $\big(\mathfrak{f}, \mathbf{u}, \mathscr{G}, \mathfrak{g}, \mathbf{v}, \mathscr{K}, \mathfrak{W}\big)$ *coincide with, respectively,* $\mu^{(\varepsilon_{n(\ell)},\varepsilon_{n'(\ell)})}$ *and* μ*. In other words, for any Borel set* $B \subset X$*,*

$$\begin{aligned}\widetilde{\mathbb{P}}\Big(\big(\mathfrak{f}_\ell, \mathbf{u}_\ell, \mathscr{G}_\ell, \mathfrak{g}_\ell, \mathbf{v}_\ell, \mathscr{K}_\ell, \mathfrak{W}\big) \in B\Big) \\ &= \mathbb{P}\Big(\big(\mathbf{f}^{(\varepsilon_{n(\ell)})}, u^{(\varepsilon_{n(\ell)})}, \mathbf{G}^{(\varepsilon_{n(\ell)})}, \mathbf{f}^{(\varepsilon_{n'(\ell)})}, u^{(\varepsilon_{n'(\ell)})}, \mathbf{G}^{(\varepsilon_{n'(\ell)})}, W\big) \in B\Big) \\ &= \mu^{(\varepsilon_{n(\ell)},\varepsilon_{n'(\ell)})}(B), \text{ and} \\ \widetilde{\mathbb{P}}\Big(\big(\mathfrak{f}, \mathbf{u}, \mathscr{G}, \mathfrak{g}, \mathbf{v}, \mathscr{K}, \mathfrak{W}\big) \in B\Big) = \mu(B) \end{aligned} \tag{5.42}$$

2. $\big(\mathfrak{f}_\ell, \mathbf{u}_\ell, \mathscr{G}_\ell, \mathfrak{g}_\ell, \mathbf{v}_\ell, \mathscr{K}_\ell, \mathfrak{W}\big)$ *converges almost surely to* $\big(\mathfrak{f}, \mathbf{u}, \mathscr{G}, \mathfrak{g}, \mathbf{v}, \mathscr{K}, \mathfrak{W}\big)$ *in* X *as* $\ell \to \infty$*.*

5.3.3 The compactness argument, part II: an auxiliary problem

Theorem 5.9. *Keep the notations of Theorem 5.8.* $\mathbf{u}$ *and* $\mathbf{v}$ *are martingale entropy solutions to* (5.1) *with the same initial data. Consequently,* $\mathbf{u} \equiv \mathbf{v}$*.*

Proof. The proof will be divided into several steps.

Step #1: (Preliminary arguments). Let us first investigate how each $(\mathfrak{f}_\ell, \mathbf{u}_\ell, \mathscr{G}_\ell)$ looks like. We begin by recalling the next profound theorem due to N. N. Lusin and M. Y. Souslin. For the proof, we refer to the books of A. S. Kechris [73], theorem 15.1, and S. M. Srivastava [102], theorem 4.5.4.

Theorem 5.10 (Lusin–Souslin)**.** *Let* M_0 *and* M_1 *be Polish spaces,* $A \subset M_0$ *be a Borel set, and* $T : M_0 \to M_1$ *be an injective continuous function. Then the set* $T(A)$ *is Borel.*

With this in mind, we will recover several properties of $\mathfrak{f}_\ell$ and $\mathbf{u}_\ell$.

Step #1.1: Each $\mathfrak{f}_\ell$ is χ–function supported on $Q \times [-L, L]$ with probability one. (Recall that L is the greatest number between $|a|$ and $|b|$.) In particular, it holds that

$$\widetilde{\mathbb{E}}\|\mathfrak{f}_\ell\|^2_{L^2(Q\times\mathbb{R})} \le C. \tag{5.43}$$

Indeed, let $M_0 = L^2(Q \times \mathbb{R})$, $M_1 = X_f = H^{-1}(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v)$, and A be the set of the χ–functions in M_0 supported on $Q \times [-L, L]$. Since M_0 and M_1 are Polish, and A is a closed set (see Proposition 4.1), Theorem 5.10 implies that we may understand A as a Borel set of M_1. Hence,

$$\widetilde{\mathbb{P}}\big(\mathfrak{f}_\ell \in A\big) = \mathbb{P}\Big(\mathbf{f}^{(\varepsilon_{n(\ell)})} \in A\Big) = 1,$$

and the claim follows.

Step #1.2: Each $\mathbf{u}_\ell$ belongs to $\mathscr{C}([0,T];L^2(\mathscr{O}))$ with probability one; moreover,

$$a \le \mathbf{u}_\ell(t,x) \le b \text{ almost surely in } \mathscr{D}'(Q). \tag{5.44}$$

In this case, we will choose $M_0 = \mathscr{C}([0,T];L^2(\mathscr{O}))$, $M_1 = X_u = \mathscr{C}([0,T];H^{-2}(\mathscr{O}))$, and A to be set of the functions $\varphi \in M_0$ such that $a \le \varphi(t) \le b$ in the sense of the distributions for all $0 \le t \le T$. Again, M_0 and M_1 are Polish, and A is closed in M_0. Hence, for $u^{(\varepsilon)} \in A$ for all $0 < \varepsilon < 1$ with probability one, $\mathbf{u}_\ell \in A$ with probability one as well.

Step #1.3: Each $\mathfrak{f}_\ell$ is the χ–function associated to $\mathbf{u}_\ell$ with probability one.

This can be shown via duality. Let $\eta \in \mathscr{C}_c^\infty(\mathbb{R})$ be identically 1 in $(-L,L)$. For any $\varphi \in \mathscr{C}_c^\infty(Q)$, and any continuous mapping $\gamma : X_u \times X_f \to [0,1]$,

$$\begin{aligned}\widetilde{\mathbb{E}}\Big\{\gamma(\mathbf{u}_\ell,\mathfrak{f}_\ell)\int_Q \varphi\Big[\Big(\int_{\mathbb{R}} \mathfrak{f}_\ell \eta\, dv\Big) - \mathbf{u}_\ell\Big]\, dxdt\Big\}&\\ = \mathbb{E}\Big\{\gamma\Big(u^{(\varepsilon_{n(\ell)})}, f^{(\varepsilon_{n(\ell)})}\Big)\int_Q \varphi\Big[\Big(\int_{\mathbb{R}} \mathbf{f}^{(\varepsilon_\ell)}\eta\, dv\Big) - u^{(\varepsilon_\ell)}\Big]\, dxdt\Big\}&\\ = \mathbb{E}\Big\{\gamma\Big(u^{(\varepsilon_{n(\ell)})}, f^{(\varepsilon_{n(\ell)})}\Big)\int_Q \varphi\Big[u^{(\varepsilon_\ell)} - u^{(\varepsilon_\ell)}\Big]\, dxdt\Big\} = 0.&\end{aligned} \tag{5.45}$$

Hence, Lusin's theorem guarantees that $\int_{\mathbb{R}} \mathfrak{f}_\ell\, dv = \mathbf{u}_\ell$ almost surely. Since $\mathfrak{f}_\ell$ is a χ–function, the result follows.

Step #1.4: The sequence $(\mathscr{G}_\ell)$ is bounded in $L^2(\widetilde{\Omega};X_G)$.

In order to see this, for any $t>0$, consider the sets $C_t = \{\Lambda \in X_G; \|\Lambda\|_{X_G} > t\}$. Evidently, C_t is an open set of X_G, thence

$$\widetilde{\mathbb{P}}\{\|\mathscr{G}_\ell\|_{X_G} > t\} = \widetilde{\mathbb{P}}(\mathscr{G}_\ell \in C_t) = \mathbb{P}\Big(\mathbf{G}^{(\varepsilon_{n(\ell)})} \in C_t\Big) = \mathbb{P}\Big\{\big\|\mathbf{G}^{(\varepsilon_{n(\ell)})}\big\|_{X_G} > t\Big\}.$$

Thus, by the theory of the distribution functions,

$$\widetilde{\mathbb{E}}\|\mathscr{G}_\ell\|_{X_G}^2 = 2\int_0^\infty \widetilde{\mathbb{P}}\{\|\mathscr{G}_\ell\|_{X_G} > t\}t\, dt = 2\int_0^\infty \mathbb{P}\Big\{\big\|\mathbf{G}^{(\varepsilon_{n(\ell)})}\big\|_{X_G} > t\Big\}t\, dt = \mathbb{E}\big\|\mathbf{G}^{(\varepsilon_{n(\ell)})}\big\|_{X_G}^2, \tag{5.46}$$

and the result follows.

Step #1.5: Every $\mathfrak{f}_\ell$ obeys almost surely the equation

$$\frac{\partial \mathfrak{f}_\ell}{\partial t} + \mathbf{a}(v)\cdot\nabla_x \mathfrak{f}_\ell = \mathscr{G}_\ell \text{ in } \mathscr{D}'(Q\times\mathbb{R}_v). \tag{5.47}$$

This last statement can be deduced by an argument parallel to the ones already presented; thus, we will omit its proof.

Step #2: (The averaging lemma). Due to (5.43) and (5.46), the Egorov's theorem asserts that we actually have that

$$\begin{cases}\mathfrak{f}_\ell \to \mathfrak{f} & \text{strongly in } L^r(\widetilde{\Omega};X_f), \text{ and}\\ \mathscr{G}_\ell \to \mathscr{G} & \text{strongly in } L^r(\widetilde{\Omega};X_G)\end{cases} \tag{5.48}$$

for any $1 \le r < 2$ as $\ell \to \infty$. Therefore, letting $\mathbf{g}_\ell \in L^2(\Omega;L^q(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v))$ being such that

$$\mathscr{G}_\ell = (-\Delta_v+1)^{(1+z)/2}(-\Delta_{t,x}+1)^{1/2}\mathbf{g}_\ell,$$

$(\mathbf{g}_\ell)$ is a convergent sequence in $L^r(\widetilde{\Omega};L^q(\mathbb{R}_t\times\mathbb{R}_x^N\times\mathbb{R}_v))$ for any $1\le r<2$ (see Remark 5.4).

In a nutshell, let us recapitulate what was deduced so far: $(\mathfrak{f}_\ell)_{\ell\in\mathbb{N}}$ is a bounded sequence in $L^2(\widetilde{\Omega};L^2(Q\times\mathbb{R}_v))$ that is convergent in $L^1(\widetilde{\Omega};H^{-1}(Q\times\mathbb{R}_v))$ and obeys almost surely the kinetic equation

$$\frac{\partial \mathfrak{f}_\ell}{\partial t} + \mathbf{a}(v)\cdot\nabla_x \mathfrak{f}_\ell = (-\Delta_v+1)^{(1+z)/2}(-\Delta_{t,x}+1)^{1/2}\mathbf{g}_\ell,$$

where $(\mathbf{g}_\ell)$ is a convergent sequence in $L^1(\Omega; L^q(\mathbb{R}_t\times\mathbb{R}^N_x\times\mathbb{R}_v))$.

On the light of the nondegeneracy condition (5.4) and that $0<z<\alpha$, we are thus in conditions to apply Theorem 2.2 with the weight function $\eta(v)=1_{(a,b)}$. As $\int_a^b \mathfrak{f}_\ell\, dv = \mathbf{u}_\ell + c$, where $c=c(a,b)$ is constant factor, we may conclude the next crucial result.

Lemma 5.6. *For any* $1\le p<\infty$,

$$\mathbf{u}_\ell \to \mathbf{u} \text{ strongly in } L^p(\widetilde{\Omega}\times Q) \text{ as } \ell\to\infty. \tag{5.49}$$

Remark 5.6. Indeed, Theorem 2.2 only asserts the relative compactness of $(\mathbf{u}_\ell)$. However, since $\mathfrak{f}_\ell$ is convergent in $L^r(\widetilde{\Omega}; H^{-1}_{t,x,v})$ for $1<r<2$, it is clear that the set of limit points of $(\mathbf{u}_\ell)$ is a singleton.

Notice that, as a corollary, Proposition 4.1 asserts that $\mathfrak{f}_\ell\to\mathfrak{f}$ in $L^r(\widetilde{\Omega}\times Q\times\mathbb{R})$ for any $1\le r<\infty$. In particular, $\mathfrak{f}$ is a χ–function.

Step #2: (The introduction of the stochastic basis). We will now introduce a stochastic framework that will allow us to ascertain that $\mathbf{u}$ is indeed a martingale entropy solution.

For any $0\le t<\infty$, let $\mathfrak{r}_t$ be the restriction operator to $[0,t]$; *i.e.*, $\mathfrak{r}_t\phi=\phi_{|[0,t]}$. Since each $\mathbf{u}_\ell\in L^2(\widetilde{\Omega};\mathscr{C}([0,T];L^2(\mathscr{O})))$, we may extend $\mathbf{u}_\ell$ to $L^2(\widetilde{\Omega};\mathscr{C}_{\text{loc}}([0,\infty);L^2(\mathscr{O})))$ by allowing $\mathbf{u}_\ell(t)=\mathbf{u}_\ell(T)$ for $t>T$. With these conventions in mind, let us introduce the filtration $(\widetilde{\mathscr{F}}_t)_{t\ge0}$ to be the augmented counterpart of

$$\widehat{\mathscr{F}_t}=\sigma\big(\mathfrak{r}_t\mathbf{u}_\ell,\mathfrak{r}_t\mathfrak{W}\,;\ \ell\in\mathbb{N}\big); \tag{5.50}$$

that is, more explicitly, each $\widetilde{\mathscr{F}}_t$ is the coarsest complete, right-continuous σ–algebra that contains the σ–algebra generated by $(\mathfrak{r}_t\mathbf{u}_\ell)_{\ell\in\mathbb{N}}$ and $\mathfrak{r}_t\mathfrak{W}$.

Notice that each $\mathbf{u}_\ell$ is adapted to this filtration and possesses almost surely continuous paths; as a result, all $\mathbf{u}_\ell$'s are predictable. Furthermore, as $\mathfrak{f}_\ell$ may be obtained from $\mathbf{u}_\ell$, again every $\mathfrak{f}_\ell$ is a predictable process with values in, say, $L^2(\mathscr{O}\times\mathbb{R}_v)$.

Lemma 5.7. $\mathfrak{W}(t)$ *is an* $(\widetilde{\mathscr{F}}_t)$*–cylindrical Wiener process; i.e., there exists a collection of mutually independent real-valued* $(\widetilde{\mathscr{F}}_t)$*–Brownian motions* $(\widetilde{\beta}_k)_{k\ge1}$ *such that* $\mathfrak{W}(t)=\sum_{k=1}^\infty\widetilde{\beta}_k(t)e_k$.

Proof. We will adapt some of the ideas of M. Hofmanová [62] as follows. Evidently, reprising the analysis of the previous steps, $\mathfrak{W}$ is a cylindrical Wiener process with values in $\mathscr{H}$; that is, $\mathfrak{W}(t)=\sum_{k=1}^\infty\beta_k(t)e_k$, where the family $(\widetilde{\beta}_k)_{k\in\mathbb{N}}$ is independent and, for all $k\in\mathbb{N}$,

1. $\widetilde{\beta}_k(0)=0$,
2. $\widetilde{\beta}_k$ has independent increments,
3. $\widetilde{\beta}_k(t+s)-\widetilde{\beta}_k(s)$ is normally distributed with zero mean and variance t for all $s\ge0$ and $t>0$, and
4. the paths of $t\mapsto\widetilde{\beta}_k(t)$ are almost surely continuous.

Therefore, all it remains to be shown is the martingale property, which will be achieved in stages.

Step #A: Consider the σ–algebras $(\widehat{\mathscr{F}_t})$ given in (5.50). We claim that, for all $t\ge0$, each $\widehat{\mathscr{F}_t}$ enjoys the following property: *"Given any* $A\in\widehat{\mathscr{F}_t}$ *and any* $\varepsilon>0$*, there exists an integer* $N_0\ge1$ *and some* $B\in\sigma\big(\mathfrak{r}_t\mathbf{u}_1,\dots,\mathfrak{r}_t\mathbf{u}_{N_0},\mathfrak{r}_t\mathfrak{W}\big)$ *such that*

$$\widetilde{\mathbb{P}}\big[(A\setminus B)\cup(B\setminus A)\big]<\varepsilon.\text{"}$$

Indeed, let H be class of the sets G such that $G \in \sigma\big(\mathfrak{r}_t\mathbf{u}_1, \ldots, \mathfrak{r}_t\mathbf{u}_m, \mathfrak{r}_t\mathfrak{W}\big)$ for some integer $m \geq 1$. Then H is a Boolean algebra of $\widetilde{\Omega}$ (that is, a nonempty family of subsets of $\widetilde{\Omega}$ that is closed under complement and finite unions), and it is not difficult to see that $\sigma(H) = \widehat{\mathscr{F}_t}$.

Additionally, consider Λ to be class of sets A of $\widehat{\mathscr{F}_t}$ that can be approximated by elements of H in the sense that, for all $\varepsilon > 0$, there exists some $G \in H$ such that $\widetilde{P}\big[(A \setminus G) \cup (G \setminus A)\big] < \varepsilon$. A routine inspection verifies that

(i) $\widetilde{\Omega} \in \Lambda$,

(ii) if $G_1 \subset G_2$ belong to Λ, then so does $G_2 \setminus G_1$ (this follows from a basic property of the symmetric differences; see, *e.g.*, W. Rudin [99], chapter 11);

(iii) if $G_1 \subset G_2 \subset \ldots$ all belong to Λ, then so does $\cup_{n=1}^{\infty} G_n$ (indeed $\widetilde{\mathbb{P}}(\cup_{n=1}^{\infty} G_n) \leq 1$, hence one may approximate the countable union by finite unions).

Therefore, Λ is a so-called λ–system. For $H \subset \Lambda$, the celebrated Dynkin's lemma asserts that $\widehat{\mathscr{F}_t} = \sigma(H) \subset \Lambda$. Hence all elements of $\widehat{\mathscr{F}_t}$ has the desired approximation property.

Step #B: Let $0 \leq s \leq t$. We claim that the increment $\mathfrak{W}(t) - \mathfrak{W}(s)$ is independent of $\widehat{\mathscr{F}_s}$.

Indeed, let $N_0 \geq 1$, and let $\gamma : (\mathscr{C}([0,s]; H^{-2}(\mathscr{O})))^{N_0} \times \mathscr{C}([0,s]; \mathscr{H}_0) \to [0,1]$ be a continuous function. Then, as integrals with values in $\mathscr{H}_0$,

$$
\begin{aligned}
&\widetilde{\mathbb{E}}\Big\{\gamma(\mathfrak{r}_s\mathbf{u}_1, \ldots, \mathfrak{r}_s\mathbf{u}_{N_0}, \mathfrak{r}_s\mathfrak{W})\big[\mathfrak{W}(t) - \mathfrak{W}(s)\big]\Big\} \\
&\qquad = \mathbb{E}\Big\{\gamma(\mathfrak{r}_s u^{\varepsilon_n(1)}, \ldots, \mathfrak{r}_s u^{\varepsilon_n(N_0)}, \mathfrak{r}_s W)\big[W(t) - W(s)\big]\Big\} = 0.
\end{aligned}
$$

As a result, Lusin's theorem and Step #A yield the desired conclusion.

Step #C: Finally, we will prove the martingale property of $\mathfrak{W}(t)$.

Recall that $\widetilde{\mathscr{F}_t}$ can be written as

$$
\widetilde{\mathscr{F}_t} = \cap_{s>t} \widehat{\mathscr{F}}_s^{(0)},
$$

where $\widehat{\mathscr{F}}_s^{(0)}$ is the union of $\widehat{\mathscr{F}_s}$ with the null sets of $\widetilde{\Omega}$. Obviously, any increment $\mathfrak{W}(t) - \mathfrak{W}(s)$ with $t \geq s$ is still independent of $\widehat{\mathscr{F}}_s^{(0)}$.

In any event, the martingale property will be confirmed once it is verified that

$$
\widetilde{\mathbb{P}}\Big(\big\{\mathfrak{W}(t) - \mathfrak{W}(s) \in A\big\} \cap B\Big) = \widetilde{\mathbb{P}}\big\{\mathfrak{W}(t) - \mathfrak{W}(s) \in A\big\}\, \widetilde{\mathbb{P}}(B)
$$

for any $0 \leq s \leq t$, any closed set $A \subset \mathscr{H}_0$, and $B \in \widetilde{\mathscr{F}_s}$. However, because $\mathfrak{W}(t)$ has almost surely continuous paths, and $\mathfrak{W}(t) - \mathfrak{W}(s+\delta)$ is independent of $\widehat{\mathscr{F}}_{s+\delta}^{(0)} \supset \widetilde{\mathscr{F}_s}$ for all $\delta > 0$, one can see that

$$
\begin{aligned}
\widetilde{\mathbb{P}}\Big(\big\{\mathfrak{W}(t) - \mathfrak{W}(s) \in A\big\} \cap B\Big) &= \widetilde{\mathbb{E}}\Big(1_A \circ \big\{\mathfrak{W}(t) - \mathfrak{W}(s) \in A\big\} 1_B\Big) \\
&= \lim_{n\to\infty} \widetilde{\mathbb{E}}\Big(\big(1 - n \operatorname{dist}(W(t) - W(s))\big)_+ 1_B\Big) \\
&= \lim_{n\to\infty} \lim_{\delta\to 0_+} \widetilde{\mathbb{E}}\Big(\big(1 - n \operatorname{dist}(W(t) - W(s+\delta))\big)_+ 1_B\Big) \\
&= \lim_{n\to\infty} \lim_{\delta\to 0_+} \widetilde{\mathbb{E}}\Big(\big(1 - n \operatorname{dist}(W(t) - W(s+\delta))\big)_+\Big)\, \widetilde{\mathbb{P}}(B) \\
&= \widetilde{\mathbb{P}}\big\{\mathfrak{W}(t) - \mathfrak{W}(s) \in A\big\}\, \widetilde{\mathbb{P}}(B),
\end{aligned}
$$

as we desired to show. □

Step #3: (The introduction of the stochastic integral). Once we are in possession of a stochastic basis and a Wiener process, and all the functions of interest $\mathbf{u}_\ell$ and $\mathfrak{f}_\ell$ are predictable, we may now consider stochastic integrals. As it would be expected, one has the following result.

Lemma 5.8. *Let us write* $\mathfrak{W}(t) = \sum_{k=1}^{\infty} \widetilde{\beta}(t)$ *as in Lemma 5.7. Let* $(\mathfrak{h}_k)_{k\in\mathbb{N}}$ *be a sequence of real continuous functions defined in* $\mathscr{O} \times \mathbb{R}$ *such that* $\sum_{k=1}^{\infty} |\mathfrak{h}_k(x,v)|^2 \le C(1+v^2)$ *for some universal constant* $C>0$ *and all* $(x,v) \in \mathscr{O} \times \mathbb{R}_v$.

Then, for all $\ell \in \mathbb{N}$*, the stochastic processes with values in* $L^2(\mathscr{O})$

$$\begin{cases} t \mapsto \displaystyle\sum_{k=1}^{\infty} \int_0^t \mathfrak{h}_k(x, \mathbf{u}_\ell(s,x))\, d\widetilde{\beta}_k(s), \ \textit{and} \\ t \mapsto \displaystyle\sum_{k=1}^{\infty} \int_0^t \mathfrak{h}_k\Big(x, u^{(\varepsilon_{n(\ell)})}(s,x)\Big)\, d\beta_k(s) \end{cases}$$

have the same laws.

Proof. Let us fix $\ell \in \mathbb{N}$, and write $\mathbf{u}$ and u instead of $\mathbf{u}_\ell$ and $u^{(\varepsilon_{n(\ell)})}$ for simplicity. Notice that, repeating the arguments of the previous steps, $\mathbf{u} \in L^2(\widetilde{\Omega}; \mathscr{C}([0,T]; L^2(\mathscr{O})))$ and $u \in L^2(\Omega; \mathscr{C}([0,T]; L^2(\mathscr{O})))$ have same laws. In particular, for all $t \ge 0$, $\mathbf{u}(t) \in L^2(\widetilde{\Omega}; L^2(\mathscr{O}))$ and $u(t) \in L^2(\Omega; L^2(\mathscr{O}))$ again possess the same laws.

Therefore, we may argue via "Riemann sums" as follows. Given any $t>0$, consider a partition $\mathcal{P} = \big\{0 = s_0 < s_1 < \ldots < s_m = t\big\}$, and define the simple predictable functions

$$\begin{cases} \mathbf{u}^{(\mathcal{P})}(s) = \displaystyle\sum_{k=0}^{m-1} \mathbf{u}(s_k) 1_{(s_k, s_{k+1}]}(s), \ \text{and} \\ u^{(\mathcal{P})}(s) = \displaystyle\sum_{k=0}^{m-1} u(s_k) 1_{(s_k, s_{k+1}]}(s). \end{cases}$$

Evidently, both $\mathbf{u}^{(\mathcal{P})}$ and $u^{(\mathcal{P})}$ have the same laws as processes taking values in $L^2(\mathscr{O})$. On the other hand, by the definition of the stochastic integral, $\mathcal{I}^{(\mathcal{P})}(s) \overset{\text{def}}{=} \sum_{k=1}^{\infty} \int_0^s \mathfrak{h}_k(x, \mathbf{u}^{(\mathcal{P})})\, d\widetilde{\beta}_k$ and $I^{(\mathcal{P})}(s) \overset{\text{def}}{=} \sum_{k=1}^{\infty} \int_0^s \mathfrak{h}_k(x, u^{(\mathcal{P})})\, d\beta_k$ also have the same laws.

Because $\mathbf{u} \in L^2(\widetilde{\Omega}; \mathscr{C}([0,t];\, L^2(\mathscr{O})))$ and $u \in L^2(\Omega; \mathscr{C}([0,t]; L^2(\mathscr{O})))$, it is clear that, as $|\mathcal{P}| = \max |s_{k+1} - s_k| \to 0_+$,

$$\begin{cases} \mathbf{u}^{(\mathcal{P})} \to \mathbf{u} \text{ in } L^2(\widetilde{\Omega} \times [0,t]; L^2(\mathscr{O})), \ \text{and} \\ u^{(\mathcal{P})} \to u \text{ in } L^2(\Omega \times [0,t]; L^2(\mathscr{O})). \end{cases}$$

Accordingly, the Itô isometry guarantees the convergence of both $\{\mathcal{I}^{(\mathcal{P})}\}$ and $\{I^{(\mathcal{P})}\}$ in, respectively, $L^2(\widetilde{\Omega}; \mathscr{C}([0,t]; L^2(\mathscr{O})))$ and $L^2(\Omega; \mathscr{C}([0,t]; L^2(\mathscr{O})))$. The limits must have same laws and must coincide with, respectively, $\sum_{k=1}^{\infty} \int_0^s \mathfrak{h}_k(x, \mathbf{u})\, d\widetilde{\beta}_k$ and $\sum_{k=1}^{\infty} \int_0^s \mathfrak{h}_k(x, u)\, d\beta_k$, hence the desired conclusion. □

Notice that, employing the ideas of Proposition 2.7, for all $\varphi \in \mathscr{C}_c^\infty((-\infty, T) \times \mathbb{R}_x^N)$, and any integer $k \ge 1$,

$$\begin{aligned} &\int_0^T \varphi(s,x) \mathfrak{h}_k(x, \mathbf{u}_\ell)\, d\widetilde{\beta}_k(s) = -\int_0^T \frac{\partial \varphi}{\partial s}(s,x) \bigg[\int_0^s \mathfrak{h}_k(x, \mathbf{u}_\ell(\xi, x))\, d\widetilde{\beta}_k(\xi) \bigg] ds, \ \text{and} \\ &\int_0^T \varphi(s,x) \mathfrak{h}_k\Big(x, u^{(\varepsilon_n(\ell))}\Big)\, d\beta_k(s) = -\int_0^T \frac{\partial \varphi}{\partial s}(s,x) \bigg[\int_0^s \mathfrak{h}_k\Big(x, u^{(\varepsilon_{n(\ell)})}(\xi, x)\Big)\, d\beta_k(\xi) \bigg] ds. \end{aligned}$$

As a corollary, Lemma 5.8 implies that the integrals $\sum_{k=1}^{\infty} \int_0^T \varphi(s,x) \mathfrak{h}_k(x, \mathbf{u}_\ell(s,x))\, d\widetilde{\beta}_k(s)$ and $\sum_{k=1}^{\infty} \int_0^T \varphi(s,x) \mathfrak{h}_k\big(x, u^{(\varepsilon_n(\ell))}(s,x)\big)\, d\beta_k(s)$ have the same laws.

Step #4: ($\mathbf{u}$ is a martingale entropy solution). First of all, notice for all $\ell \in \mathbb{N}$, $\mathbf{u}_\ell(0) = \mathbf{u}_1(0) \overset{\text{def}}{=} \mathbf{u}_0$. Evidently, $\mathbf{u}_0$ has the same law as u_0's, hence $\mathbf{u}_0 \in L^\infty(\widetilde{\Omega} \times \mathscr{O})$ and $a \le \mathbf{u}_0 \le b$ almost surely in $\mathscr{D}'(Q)$.

Thanks to Lemma 5.6, the bound (5.27), and the local convergence (5.28), it is clear that the dominated convergence theorem yields

$$\sum_{k=1}^{\infty}\widetilde{\mathbb{E}}\int_0^T\int_{\mathscr{O}}\left|\eta'(\mathbf{u}_\ell)g_k^{(\varepsilon_n(\ell))}(x,\mathbf{u}_\ell)-\eta'(\mathbf{u})g_k(x,\mathbf{u})\right|^2 dxdt\to 0 \text{ as } \ell\to\infty$$

for any $\mathscr{C}^2$ real function $\eta:\mathbb{R}\to\mathbb{R}$. In a similar vein, we have that

$$\eta''(\mathbf{u}_\ell)\left(\mathfrak{G}^{(\varepsilon_{n(\ell)})}\right)^2(x,\mathbf{u}_\ell)\to\eta''(\mathbf{u})\mathfrak{G}^2(x,\mathbf{u}) \text{ as } \ell\to\infty$$

in $L^p(\widetilde{\Omega}\times Q)$ for all $1\le p<\infty$. Thence, the Itô isometry confirms that, for any test function $\varphi\in\mathscr{C}_c^\infty((-\infty,T)\times\mathscr{O})$,

$$\sum_{k=1}^{\infty}\int_0^T\varphi(s,x)\eta'(\mathbf{u}_\ell)g_k^{(\varepsilon_n(\ell))}(x,\mathbf{u}_\ell)\,d\widetilde{\beta}_k(s)\to\sum_{k=1}^{\infty}\int_0^T\varphi(s,x)\eta'(\mathbf{u})g_k(x,\mathbf{u})\,d\widetilde{\beta}_k(s)$$

in $L^2(\widetilde{\Omega})$ as $\ell\to\infty$.

Based on the previous discussions, let us indeed verify the entropy condition (5.9) for $\mathbf{u}$. Given any nonnegative $\phi\in\mathscr{C}_c^\infty((-\infty,T)\times\mathscr{O})$, any convex function $\eta:\mathbb{R}\to\mathbb{R}$ of class $\mathscr{C}^2$, and any continuous function $\gamma: X_u\times X_W\to[0,1]$, (5.32) yields

$$\begin{aligned}
&\widetilde{\mathbb{E}}\gamma(\mathbf{u}_\ell,\mathfrak{W})\Bigg[\int_Q\Bigg\{\eta(\mathbf{u}_\ell)\frac{\partial\phi}{\partial t}+\mathbf{A}^\eta(\mathbf{u}_\ell)\cdot\nabla_x\phi\\
&\qquad+\frac{1}{2}\eta''(\mathbf{u}_\ell)\left(\mathfrak{G}^{(\varepsilon_{n(\ell)})}\right)^2(x,\mathbf{u}_\ell)\phi\Bigg\}\,dxdt+\int_{\mathscr{O}}\eta(\mathbf{u}_0(x))\phi(0,x)\,dx\\
&\qquad+\sum_{k=1}^{\infty}\int_0^T\int_{\mathscr{O}}\eta'(\mathbf{u}_0(x))\mathbf{\Phi}^{(\varepsilon_{n(\ell)})}(x,\mathbf{u}_\ell)\phi\,dxd\mathfrak{W}(t)\Bigg]\\
&=\mathbb{E}\gamma(u^{(\varepsilon_{n(\ell)})},W)\Bigg[\int_Q\Bigg\{\eta(u^{(\varepsilon_{n(\ell)})})\frac{\partial\phi}{\partial t}+\mathbf{A}^\eta(u^{(\varepsilon_{n(\ell)})})\cdot\nabla_x\phi\\
&\qquad+\frac{1}{2}\eta''(u^{(\varepsilon_{n(\ell)})})\left(\mathfrak{G}^{(\varepsilon_{n(\ell)})}\right)^2(x,u^{(\varepsilon_{n(\ell)})})\phi\Bigg\}\,dxdt+\int_{\mathscr{O}}\eta(u_0(x))\phi(0,x)\,dx\\
&\qquad+\sum_{k=1}^{\infty}\int_0^T\int_{\mathscr{O}}\eta'(u^{(\varepsilon_{n(\ell)})})\mathbf{\Phi}^{(\varepsilon_{n(\ell)})}(x,u^{(\varepsilon_{n(\ell)})})\phi\,dxdW(t)\Bigg]\\
&=\varepsilon_{n(\ell)}\mathbb{E}\gamma(u^{(\varepsilon_{n(\ell)})},W)\Bigg[\int_Q\left\{\eta'(u^{(\varepsilon_{n(\ell)})})\nabla u^{(\varepsilon_{n(\ell)})}\cdot\nabla_x\phi+\eta''(u^{(\varepsilon_{n(\ell)})})|\nabla u^{(\varepsilon_{n(\ell)})}|^2\phi\right\}dxdt\Bigg]\\
&\ge-\|\eta'\|_{L^\infty(a,b)}\|\nabla_x\phi\|_{L^2(Q)}\,\varepsilon_{n(\ell)}^{1/2}\left\{\mathbb{E}\int_Q\varepsilon_{n(\ell)}\left|\nabla u^{(\varepsilon_{n(\ell)})}\right|^2dxdt\right\}^{1/2}.
\end{aligned}$$

Letting $\varepsilon_{n(\ell)}\to 0_+$, we conclude from (5.31) that

$$\begin{aligned}
&\int_0^T\int_{\mathscr{O}}\eta(\mathbf{u}(t,x))\frac{\partial\varphi}{\partial t}(t,x)\,dtdx+\int_{\mathscr{O}}\eta(\mathbf{u}_0(x))\varphi(0,x)\,dx+\int_0^T\int_{\mathscr{O}}\mathbf{A}^\eta(\mathbf{u}(t,x))\cdot\nabla_x\varphi(t,x)\,dtdx\\
&\qquad\ge-\sum_{k=1}^{\infty}\int_0^T\int_{\mathscr{O}}\eta'(\mathbf{u}(t,x))g_k(x,\mathbf{u}(t,x))\varphi(t,x)\,dx\,d\widetilde{\beta}_k(t)\\
&\qquad\quad-\frac{1}{2}\int_0^T\int_{\mathscr{O}}\varphi(t,x)\eta''(\mathbf{u}(t,x))\mathfrak{G}^2(x,\mathbf{u}(t,x))\,dx\,dt
\end{aligned}$$

almost surely. Therefore, the entropy condition (5.9) is obtained by considering a countable dense class of η's and ϕ's.

In an analogous fashion, the boundary condition (5.10) can be also be justified. For clearly $a \leq \mathbf{u}(t,x) \leq b$ almost surely in $\mathscr{D}'(Q)$, it is shown that $\mathbf{u}$ is a martingale entropy solution to (5.1).

Step #5: (Conclusion). Of course, reproducing the arguments of the former four steps, it can be proven that $\mathbf{v}$ is also a martingale entropy solution with an initial data $\mathbf{v}_0 = \mathbf{v}_1(0)$. On the strength of

$$\widetilde{\mathbb{P}}\big(\mathbf{u}_0 = \mathbf{v}_0\big) = \mathbb{P}\big(u^{(\varepsilon_n(1))} = u^{(\varepsilon'_n(1))}\big) = 1,$$

the comparison principle (Theorem 5.3) implies that $\mathbf{u} \equiv \mathbf{v}$. The lemma is hereby proven. □

Remark 5.7. Notice that both $\mathscr{G}$ and $\mathscr{K} \in X_G$ are almost surely supported on $Q \times [-L, L]$. Because of Equation (5.47) (and its corresponding relation to $\mathscr{K}$), they are completely determined by $\mathfrak{f}$ and $\mathfrak{g}$. Hence, $\mathscr{G} \equiv \mathscr{K}$.

5.3.4 Conclusion

As a consequence of the previous subsection, the measure μ given by Theorem 5.7 is supported on the diagonal

$$\varDelta = \Big\{\big(f_1, u_1, G_1, f_2, u_2, G_2, W\big) \in X; f_1 = f_2, G_1 = G_2, \text{ and } u_1 = u_2\Big\}.$$

Since the sequences ε_n and ε'_n were arbitrary, the Gyöngi–Krylov criterion (Theorem 5.4) asserts that there exists some $(\mathbf{f}, u, \mathbf{G})$ such that

$$(\mathbf{f}^{(\varepsilon)}, u^{(\varepsilon)}, \mathbf{G}^{(\varepsilon)}) \to (\mathbf{f}, u, \mathbf{G}) \text{ in probability in } X_f \times X_u \times X_G \text{ as } \varepsilon \to 0_+.$$

From the a priori estimates from Subsections 5.3.1 and 5.3.2, it is clear that

$$\begin{cases} \mathbf{f}^{(\varepsilon)} \to \mathbf{f} & \text{strongly in } L^r(\Omega; X_f), \text{ and} \\ \mathbf{G}^{(\varepsilon)} \to \mathbf{G} & \text{strongly in } L^r(\Omega; X_G) \end{cases}$$

as $\varepsilon \to 0_+$ for all $1 \leq r < \infty$. Comparing this to (5.48), we thus see that the averaging lemma (*e.g.*, Theorem 2.2) implies

$$u^{(\varepsilon)} \to u \text{ in } L^p(\Omega \times Q)$$

as $\varepsilon \to 0_+$ for any $1 \leq p < \infty$. Therefore, reprising the ideas of Lemma 5.9, we conclude the following theorem.

Theorem 5.11 (Existence of solutions)**.** *As $\varepsilon \to 0_+$, the approximate solutions $u^{(\varepsilon)}$ given by Lemma 5.4 converge in $L^p(\Omega \times Q)$ to some $u \in L^\infty(\Omega \times Q)$ for all $1 \leq p < \infty$. Moreover, u is an entropy solution to (5.1) with initial data $u(0) = u_0$.*

Amalgamating Theorems 5.3 and 5.11, Theorem 5.1 is consequently formed.

5.4 Regularity

Let us now inspect the Sobolev regularity of the solutions given by Theorem 5.1. As it is traditional in the study of kinetic solutions (see, *e.g.*, P.-L. Lions–B. Perthame–E. Tadmor [82], E. Tadmor–T. Tao [107], and B. Gess–M. Hofmanová [51]), one needs to impose a certain uniformity on the nondegeneracy condition (5.4). More precisely, one supplements it with the next hypothesis:

2.c*) There exist some $\eta \in \mathscr{C}_c^\infty(\mathbb{R})$ satisfying $\eta \equiv 1$ in $[a, b]$, a constant $C > 0$, an exponent $0 < \mathfrak{e} \leq 1$, such that, for all $\delta > 0$,

$$\operatorname{meas}\Big\{v \in \operatorname{supp} \eta;\ |\tau + \mathbf{a}(v) \cdot \kappa| \leq \delta\Big\} \leq C\delta^{\mathfrak{e}}$$
$$\text{for all } (\tau, \kappa) \in \mathbb{R} \times \mathbb{R}^N \text{ with } \tau^2 + |\kappa|^2 = 1. \tag{5.51}$$

Remark 5.8. A simple example of a flux-function $\mathbf{A}(v)$ satisfying this new condition (5.51) is

$$\mathbf{A}(v) = \Big(\lambda_1(v-a)^{\ell_1+1}(v-b)^{\ell_1+1}, \cdots, \lambda_N(v-a)^{\ell_N+1}(v-b)^{\ell_N+1}\Big),$$

where λ_1, ... and λ_N are nonzero reals, and ℓ_1, ..., ℓ_N are pairwise distinct positive integers. In this case, $\mathfrak{e} = 1/(\max \ell_k)$.

Thus, under the same hypotheses of the first section of this chapter and the novel assumption (c*) above, we will establish the next result.

Theorem 5.12. *Let $u(t,x)$ be the solution obtained in Theorem 5.1, and assume* (c*)*. Letting*

$$0 \le s < \frac{\mathfrak{e}}{2(2+\mathfrak{e})} \text{ and } r = \frac{4+\mathfrak{e}}{2+\mathfrak{e}},$$

we have for any $1 \le p < \infty$ that

$$\begin{cases} u \in L^p\big(\Omega; W^{s,r}(Q)\big), \text{ and} \\ \mathbb{E}\|u\|^p_{W^{s,r}(Q)} \le C(p,s,a,b). \end{cases}$$

Notice that the regularity of Theorem 5.12 is exactly one-half of the one obtained by P.-L. Lions–B. Perthame–T. Tadmor [82] for the deterministic case, which is in accordance with the principle that stochastic equations possess one-half of the smoothing effect its deterministic counterparts would display. Furthermore, it is worth pointing out that one also gains some Sobolev regularity in t, and that the smoothing effect takes place near the boundary.

The theorem above is a consequence of the next averaging lemma in the spirit of P.-L. Lions–B. Perthame–T. Tadmor [82] and T. Tadmor–T. Tao [107] that also improves the regularity exponent of B. Gess–M. Hofmanová [51].

Lemma 5.9. *Let $s_0 \ge 0$, and $1 \le p < \infty$. Suppose that the hypotheses 2.a), 3.) and 4.b) of the beginning of this chapter hold.*

Let $\mathbf{f} \in L^p(\Omega \times \mathbb{R}_v; H^{s_0}(Q)))$, $u \in L^p(\Omega \times [0,T]; L^1(\mathscr{O}))$, and $\mathbf{q} \in L^p_{\mathfrak{w}}(\Omega; \mathfrak{M}(Q \times \mathbb{R}_v))$, and suppose that the equation

$$\frac{\partial \mathbf{f}}{\partial t} + \mathbf{a}(v) \cdot \nabla_x \mathbf{f} = \frac{\partial \mathbf{q}}{\partial v} + \delta_{u(t,x)}(v)\Phi(x,v)\,\frac{dW}{dt}, \tag{5.52}$$

is obeyed almost surely in $\mathscr{D}'(Q)$. Let $\eta \in \mathscr{C}^\infty_c(\mathbb{R}_v)$ be such that (5.51) holds.

Finally, introduce the exponents

$$0 \le \mathfrak{s} < \frac{\mathfrak{e}}{2(4+\mathfrak{e})} + \frac{4s_0}{4+\mathfrak{e}} \text{ and } 1 \le r < \frac{4+\mathfrak{e}}{2+\mathfrak{e}},$$

and the average $\mathfrak{v} = \int_{\mathbb{R}} \mathbf{f}\eta\, dv$.

Then, for every $\varphi \in \mathscr{C}^\infty_c(Q)$, $\varphi\mathfrak{v} \in L^p(\Omega; W^{\mathfrak{s},r}(\mathbb{R}_t \times \mathbb{R}^N_x))$. Moreover, given any $\zeta \in \mathscr{C}^\infty_c(\mathbb{R}_v)$ such that $\zeta \equiv 1$ on $\operatorname{supp}\eta$, there exists some $\mathfrak{r} = \mathfrak{r}(s)$ such that

$$\mathbb{E}\|\varphi\mathfrak{v}\|^p_{W^{\mathfrak{s},r}_{t,x}} \le C_{p,\mathfrak{s},r,\eta}\Bigg\{\mathbb{E}\|\varphi\zeta\mathbf{f}\|^p_{L^2_v H^{s_0}_{t,x}} + \mathbb{E}\|\varphi(|\zeta|+|\zeta'|)\mathbf{q}\|^p_{\mathfrak{M}_{t,x,v}} + \mathbb{E}\Big\|\Big(\frac{\partial\varphi}{\partial t} + \mathbf{a}(v)\cdot\nabla_x\varphi\Big)\zeta\mathbf{f}\Big\|^p_{L^1_{t,x,v}}$$
$$+ \Big(\mathbb{E}\sup_{t\in\mathbb{R}}\Big[\int_{\mathscr{O}} \varphi(t,x)^2\zeta(u(t,x))^2\mathfrak{G}^2(x,u(t,x))\,dx\Big]^{\mathfrak{r}}\Big)^{p/(2\mathfrak{r})}\Bigg\}. \tag{5.53}$$

Proof of Lemma 5.9. Let us argue inspired on the Littlewood–Paley decomposition of T. Tadmor–T. Tao [107]. (Even though we reckon that an argument based on the K-method is possible and perhaps shorter, the following proof is nonetheless certainly more elementary.) Consider $\psi_1(z)$ and $\psi_2(z) \in \mathscr{C}^\infty_c(\mathbb{C};\mathbb{R})$ such that

1. $\operatorname{supp}\psi_0 \subset \{|z| \le 1\}$, and $\psi_0(z) \ge 0$ everywhere,
2. $\operatorname{supp}\psi_1 \subset \{\frac{1}{2} \le |z| \le 2\}$, and $\psi_1(z) \ge 0$ everywhere,
3. for all $z \in \mathbb{C}$,
$$\psi_0(z) + \sum_{m=0}^{\infty} \psi_1(2^{-m}z) = 1, \text{ and} \tag{5.54}$$
4. for all $z \in \mathbb{C} \setminus \{0\}$,
$$\sum_{m=-\infty}^{\infty} \psi_1(2^{-m}z) = 1. \tag{5.55}$$

Henceforth, J will denote a dyadic number (i.e., $J = 2^m$ for some $m \in \mathbb{Z}$), and $\psi_J(z)$ will be set as $\psi_J(z) = \psi_2(J^{-1}z)$. In this fashion, (5.54) and (5.55) now read

$$\begin{cases} \psi_0(z) + \displaystyle\sum_{J \text{ dyadic},\ J \ge 1} \psi_J(z) = 1 & \text{for all } z \in \mathbb{C}, \text{ and} \\ \displaystyle\sum_{J \text{ dyadic}} \psi_J(z) = 1 & \text{for all } z \neq 0. \end{cases}$$

Finally, given any tempered distribution $\Lambda \in \mathscr{S}'(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}_v)$, let us write

$$\begin{cases} \Lambda_0(t,x,v) = \mathfrak{F}^{-1}_{t,x}\Big[\psi_0(\sqrt{\tau^2 + |\kappa|^2})\,(\mathfrak{F}_{t,x}\Lambda)(\tau,\kappa,v)\Big], \text{ and} \\ \Lambda_J(t,x,v) = \mathfrak{F}^{-1}_{t,x}\Big[\psi_J(\sqrt{\tau^2 + |\kappa|^2})\,(\mathfrak{F}_{t,x}\Lambda)(\tau,\kappa,v)\Big], \end{cases}$$

where $(\tau,\kappa) \in \mathbb{R} \times \mathbb{R}^N$ are the frequency variables associated to (t,x). Notice that each parcel above is compactly supported on the Fourier space, and that

$$\Lambda = \Lambda_0 + \sum_{J \ge 1} \Lambda_J.$$

Moreover, in virtue of Proposition 2.3,

$$\|\Lambda_J\|_{L^p(\mathbb{R}^N)} \le (\text{const. independent of } J \text{ and } p)\|\Lambda\|_{L^p(\mathbb{R}^N)}$$

for any $J \ge 1$, $1 \le p \le \infty$, and $\Lambda \in L^p(\mathbb{R}_t \times \mathbb{R}^N_x)$.

Before we initiate our investigation of $\mathfrak{v}$, let us enunciate the characterization of the fractional Sobolev spaces by means of the Littlewood–Paley expansion. For the proof and a throughout discussion, we refer to the book of H. Triebel [108]; see also R. A. Adams–J. J. F. Fournier [1]. One should compare this theorem with the classical definition of the $H^s(\mathbb{R}^n)$–spaces.

Theorem 5.13. *Let $s \ge 0$ and $1 < p < \infty$. There exists constants $c = c(N,s,p)$ and $C = C(N,s,p)$ such that, for any $f \in \mathscr{S}(\mathbb{R}_t \times \mathbb{R}^N_x)$,*

$$c\,\|f\|_{W^{s,p}(\mathbb{R}_t \times \mathbb{R}^N_x)} \le \left\| \left\{ |f_0|^2 + \sum_{J \ge 1} J^{2s}|f_J|^2 \right\}^{1/2} \right\|_{L^p(\mathbb{R}_t \times \mathbb{R}^N_x)} \le C\,\|f\|_{W^{s,p}(\mathbb{R}_t \times \mathbb{R}^N_x)}.$$

Therefore, if $f \in L^p(\mathbb{R}_t \times \mathbb{R}^N_x)$, and $\|f_J\|_{L^p(\mathbb{R}_t \times \mathbb{R}^N_x)} \le (\text{const.})J^{-\sigma}$ for some $\sigma > 0$ and all $J \ge 1$, then $f \in W^{s,p}(\mathbb{R}_t \times \mathbb{R}^N_x)$ for any $0 \le s < \sigma$.

Step #1: (The equation satisfied by $\varphi\zeta\mathbf{f}$). Let $\varphi \in \mathscr{C}^\infty_c(Q)$, and let $\zeta \in \mathscr{C}^\infty_c(\mathbb{R}_v)$ be as in the statement of this Lemma, so that $\varphi\zeta\mathbf{f}$ observes the equation

$$\Big(\frac{\partial}{\partial t}+\mathbf{a}(v)\cdot\nabla_x\Big)(\varphi\zeta\mathbf{f})=\frac{\partial}{\partial v}(\varphi\zeta\mathbf{q})-\varphi\zeta'\mathbf{q}+\Big(\frac{\partial\varphi}{\partial t}+\mathbf{a}(v)\cdot\nabla_x\varphi\Big)\zeta\mathbf{f}$$
$$+\,\delta_{u(t,x)}(v)\varphi(t,x)\zeta(v)\Phi(x,v)\,\frac{dW}{dt}. \tag{5.56}$$

Let us now transform the right–hand side of the equation above to our liking. Pick some $0<\varepsilon<\min\{\alpha/4,2/p\}$.

Step #1.1: (Analysis of the "deterministic" terms). As we have argued before in Theorem 5.6, for any $1<q_\varepsilon<\frac{N+2}{N+2-\varepsilon}$,

$$L^1(Q\times\mathbb{R}_v)\subset\mathfrak{M}(Q\times\mathbb{R}_v)\subset W^{-\varepsilon,q_\varepsilon}(\mathbb{R}_t\times\mathbb{R}^N_x\times\mathbb{R}_v)\text{ continuously}$$

(notice that $q_\varepsilon\to1$ as $\varepsilon\to0_+$). Hence, for these q_ε's,

$$\frac{\partial}{\partial v}(\varphi\mathbf{q})+\varphi\zeta'\mathbf{q}+\Big(\frac{\partial\varphi}{\partial t}+\mathbf{a}(v)\cdot\nabla_x\varphi\Big)\zeta\mathbf{f}=\big(-\Delta_{t,x}+1\big)^{\varepsilon/2}\big(-\Delta_v+1\big)^{(1+\varepsilon)/2}\mathbf{G}^{(\varepsilon)}, \tag{5.57}$$

where $\mathbf{G}^{(\varepsilon)}\in L^p(\Omega;L^{q_\varepsilon}(\mathbb{R}_t\times\mathbb{R}^N\times\mathbb{R}_v))$ obeys

$$\mathbb{E}\|\mathbf{G}^{(\varepsilon)}\|^p_{L^{q_\varepsilon}_{t,x,v}}\le C_\varepsilon\Big\{\mathbb{E}\|\varphi(|\zeta|+|\zeta'|)\mathbf{q}\|^p_{\mathfrak{M}}+\mathbb{E}\Big\|\Big(\frac{\partial\varphi}{\partial t}+\mathbf{a}(v)\cdot\nabla_x\varphi\Big)\zeta\mathbf{f}\Big\|^p_{L^1_{t,x,v}}\Big\}. \tag{5.58}$$

Step #1.2: (Analysis of the "stochastic" terms). As for the stochastic term, let us reprise the ideas of Lemma 5.5. Arguing as then, one can conclude that, almost surely,

$$\|\delta_{u(t,x)}(v)\varphi(t,x)\zeta(v)\Phi(x,v)\|^2_{HS(\mathscr{H};L^2(\mathscr{O}_x;H^{-(1/2+\varepsilon)}(I)))}$$
$$\le C_\varepsilon\int_{\mathscr{O}}\varphi(t,x)^2\zeta(u(t,x))^2\mathfrak{G}^2(x,u(t,x))\,dx$$

where $I\subset\mathbb{R}$ is an open interval containing the range of $\zeta(v)$. (Notice that the integral on the right-hand side poses no problems, as it is indeed the integral of a bounded function.) As a result, for any $2<\mathfrak{r}<\infty$, and any $-\infty<s<t<\infty$, the Burkholder inequality asserts that

$$\mathbb{E}\Big\|\int_s^t\delta_{u(r,x)}(v)\varphi(r,x)\zeta(v)\Phi(x,v)\,dW(r)\Big\|^{\mathfrak{r}}_{L^2(\mathscr{O}_x;H^{-(1/2+\varepsilon)}(I))}$$
$$\le C_{\mathfrak{r}}\mathbb{E}\Big[\Big(\int_s^t\int_{\mathscr{O}}\varphi(r,x)^2\zeta(u(r,x))^2\mathfrak{G}^2(x,u(r,x))\,dxdr\Big)^{\mathfrak{r}/2}\Big]$$
$$\le C_{\mathfrak{r}}|t-s|^{\mathfrak{r}/2}\mathbb{E}\Big(\sup_{-\infty<r<\infty}\Big[\int_{\mathscr{O}}\varphi(r,x)^2\zeta(u(r,x))^2\mathfrak{G}^2(x,u(r,x))\,dx\Big]^{\mathfrak{r}/2}\Big).$$

Therefore, choosing any $\mathfrak{r}=2/\varepsilon$ (remember that $\varepsilon<2/p$, so that $\mathfrak{r}\ge p$), Kolmogorov's continuity criterion (Theorem 5.5) ensures that

$$t\mapsto\int_0^t\delta_{u(r,x)}(v)\varphi(r,x)\zeta(v)\Phi(x,v)\,dW(r)\in L^p(\Omega;\mathscr{C}^{1/2-\varepsilon}([0,T];L^2(\mathscr{O};H^{-(1/2+\varepsilon)}(I))))$$

and its norm is $\le C_\varepsilon\mathbb{E}[(\sup_t\int_{\mathscr{O}}\varphi(t,x)^2\zeta(u(t,x))^2\mathfrak{G}^2(x,u(t,x))\,dx)^{1/\varepsilon}]^{\varepsilon/2}$.

Recalling that $\frac{\partial}{\partial t}\int_0^t\{\dots\}\,dW=\{\dots\}\,\frac{dW}{dt}$ (Proposition 2.7), we conclude that indeed

$$\delta_{u(t,x)}(v)\varphi(t,x)\zeta(v)\Phi(x,v)\,\frac{dW}{dt}\in L^p(\Omega;H^{-(1/2+\varepsilon)}(0,T;L^2(\mathscr{O}_x;H^{-(1/2+\varepsilon)}(I)))).$$

Consequently, making usage of the compact support of the functions involved, we finally conclude that

$$\delta_{u(t,x)}(v)\varphi(t,x)\zeta(v)\Phi(x,v)\,\frac{dW}{dt} = (-\Delta_{t,x}+1)^{(1/2+\varepsilon)/2}(-\Delta_v+1)^{(1/2+\varepsilon)/2}\mathbf{H}^{(\varepsilon)} \tag{5.59}$$

where

$$\mathbb{E}\|\mathbf{H}^{(\varepsilon)}\|^p_{L^{q_\varepsilon}_{t,x,v}} \le C_\varepsilon\left(\mathbb{E}\sup_{t\in\mathbb{R}}\left[\int_{\mathscr{O}}\varphi(t,x)^2\zeta(u(t,x))^2\mathfrak{G}^2(x,u(t,x))\,dx\right]^{1/\varepsilon}\right)^{p\varepsilon/2}. \tag{5.60}$$

Step #1: (Conclusion). All in all, in the light of (5.57)–(5.58) and (5.59)–(5.60), (5.56) can be converted into

$$\left(\frac{\partial}{\partial t}+\mathbf{a}(v)\cdot\nabla_x\right)(\varphi\zeta\mathbf{f}) = \big(-\Delta_{t,x}+1\big)^{(1/2+\varepsilon)/2}\left(\frac{\partial}{\partial v}(-\Delta_v)^{\varepsilon/2}+1\right)\mathbf{F}^{(\varepsilon)} \tag{5.61}$$

where $0<\varepsilon<\alpha/4$, and

$$\begin{aligned}\mathbb{E}\|\mathbf{F}^{(\varepsilon)}\|^p_{L^{q_\varepsilon}_{t,x,v}} \le C_\varepsilon\mathbb{E}\Bigg\{&\|\varphi(|\zeta|+|\zeta'|)\mathfrak{m}\|^p_{\mathfrak{M}_{t,x,v}} + \left\|\left(\frac{\partial\varphi}{\partial t}+\mathbf{a}(v)\cdot\nabla_x\varphi\right)\zeta\mathbf{f}\right\|^p_{L^1_{t,x,v}}\\ &+\left(\sup_{t\in\mathbb{R}}\left[\int_{\mathscr{O}}\varphi(t,x)^2\zeta(u(t,x))^2\mathfrak{G}^2(x,u(t,x))\,dx\right]^{1/\varepsilon}\right)^{p\varepsilon/2}\Bigg\}.\end{aligned} \tag{5.62}$$

Step #2: (The Littlewood—Paley decompositions). One has that

$$\begin{aligned}\varphi\mathfrak{v} &= \varphi\int_{\mathbb{R}}\mathbf{f}\eta\,dv\\ &= \int_{\mathbb{R}}(\varphi\zeta\mathbf{f})_0\eta\,dv + \sum_{J\ \text{dyadic},J\ge 1}\int_{\mathbb{R}}(\varphi\zeta\mathbf{f})_J\eta\,dv\\ &= \int_{\mathbb{R}}(\varphi\zeta\mathbf{f})_0\eta\,dv + \sum_{J\ \text{dyadic},J\ge 1}\int_{\mathbb{R}}(\varphi\zeta\mathbf{f})_J\eta\,dv\\ &= (\varphi\mathfrak{v})_0 + \sum_{J\ \text{dyadic},J\ge 1}(\varphi\mathfrak{v})_J.\end{aligned}$$

Therefore, our task is reduced to the estimation of the L^r–norm of each average expressed above. Since the Paley–Wiener theorem asserts that $(\varphi\mathfrak{v})_0$ and $(\varphi\mathfrak{v})_1$ lies in $L^2(\Omega;W^{k,r}_{t,x})$ for any $k\ge 0$ and $1\le r<\infty$ with

$$\mathbb{E}\|(\varphi\mathfrak{v})_0\|^p_{W^{k,r}(Q)} + \mathbb{E}\|(\varphi\mathfrak{v})_1\|^p_{W^{k,r}(Q)} \le C_{k,r}\mathbb{E}\|\varphi\zeta\mathbf{f}\|^p_{L^2(\mathbb{R}_v;L^2(\mathbb{R}_t\times\mathbb{R}^N_x))},$$

we may restrict our attention to $(\varphi\mathfrak{v})_J$ for $J\ge 2$.

For this purpose, let us introduce a second Littlewood—Paley to each $(\varphi\mathfrak{v})_J$ in terms of the symbol $\mathcal{L}(i\tau,i\kappa,v)=i(\tau+\mathbf{a}(v)\cdot\kappa)$. Putting

$$M = \max_{\substack{\tau^2+|\kappa|^2=1\\ v\in\operatorname{supp}\eta}}|\mathcal{L}(i\tau,i\kappa,v)|,$$

subdivide $(\varphi\zeta\mathbf{f})_J$ as

$$(\varphi\zeta\mathbf{f})_J = \sum_{K\ \text{dyadic},K\le 2M}(\varphi\zeta\mathbf{f})_{J,K}, \tag{5.63}$$

where each component $(\varphi\mathbf{f})_{J,K}$ is given by

$$(\varphi\mathbf{f})_{J,K} \stackrel{\text{def}}{=} \mathfrak{F}_{t,x}^{-1}\left[\psi_1\left(\frac{\mathcal{L}(i\tau,i\kappa,v)}{K\sqrt{\tau^2+|\kappa|^2}}\right)(\mathfrak{F}_{t,x}(\varphi\zeta\mathbf{f}))_J\right]$$
$$= \mathfrak{F}_{t,x}^{-1}\left[\psi_1\left(\frac{\mathcal{L}(i\tau,i\kappa,v)}{K\sqrt{\tau^2+|\kappa|^2}}\right)\psi_1\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{J}\right)(\mathfrak{F}_{t,x}(\varphi\zeta\mathbf{f}))\right].$$

Hence,

$$(\varphi\mathfrak{v})_J = \sum_{K \text{ dyadic},K\le 2M}\int_{\mathbb{R}}(\varphi\zeta\mathbf{f})_{J,K}\eta\,dv \stackrel{\text{def}}{=} \sum_{K \text{ dyadic},K\le 2M}(\varphi\mathfrak{v})_{J,K}. \tag{5.64}$$

Step #3.1: (The L^2–estimate of $(\varphi\mathfrak{v})_{J,K}$). This is the counterpart of Lemma 2.37, but with the following subtlety. On the strength of the Plancherel identity, the Cauchy–Schwarz inequality, and the hypothesis in (5.51), it holds that almost surely that

$$\|(\varphi\mathfrak{v})_{J,K}\|^2_{L^2(\mathbb{R}_t\times\mathbb{R}^N_x)} = \left\|\int_{\mathbb{R}_v}\psi_1\left(\frac{\mathcal{L}(i\tau,i\kappa,v)}{K\sqrt{\tau^2+|\kappa|^2}}\right)\mathfrak{F}_{t,x}(\varphi\zeta\mathbf{f})_J\eta\,dv\right\|_{L^2(\mathbb{R}_\tau\times\mathbb{R}^N_\kappa)}$$
$$\le \left\{\int_{\{v\in\operatorname{supp}\eta;|\mathcal{L}(i\tau,i\kappa,v)|\le 2K\sqrt{\tau^2+|\kappa|^2}\}}\eta(v)^2\,dv\right\}$$
$$\left\{\int_{\mathbb{R}_\tau}\int_{\mathbb{R}^N_\kappa}\int_{\mathbb{R}_v}\left|\psi_1\left(\frac{\mathcal{L}(i\tau,i\kappa,v)}{K\sqrt{\tau^2+|\kappa|^2}}\right)\mathfrak{F}_{t,x}(\varphi\zeta\mathbf{f})_J\right|^2 dvd\kappa d\tau\right\}$$
$$\le CK^{\mathfrak{c}}\int_{\mathbb{R}_\tau}\int_{\mathbb{R}^N_x}\int_{\mathbb{R}_v}|(\varphi\zeta\mathbf{f})_J(t,x,v)|^2\,dvdxdt.$$

Since, as Theorem 5.13 asserts, $\|(\varphi\zeta\mathbf{f})_J\|_{L^2_{t,x,v}} \le \frac{C}{J^{s_0}}\|\varphi\zeta\mathbf{f}\|_{L^2_vH^{s_0}_{t,x}}$, we conclude that

$$\left\{\mathbb{E}\|(\varphi\mathfrak{v})_{J,K}\|^p_{L^2(\mathbb{R}_t\times\mathbb{R}^N_x)}\right\}^{1/p} \le C\frac{K^{\mathfrak{c}/2}}{J^{s_0}}\left\{\mathbb{E}\|\varphi\zeta\mathbf{f}\|^p_{L^2_vH^{s_0}_{t,x}}\right\}^{1/p}. \tag{5.65}$$

Step #3.2: (The L^{q_ε}–estimate of $(\varphi\mathfrak{v})_{J,K}$). From (5.61), it is not only clear that

$$\left(\frac{\partial}{\partial t}+\mathbf{a}(v)\cdot\nabla_x\right)(\varphi\zeta\mathbf{f})_J = \big(-\Delta_{t,x}+1\big)^{(1/2+\varepsilon)/2}\left(\frac{\partial}{\partial v}(-\Delta_v)^{\varepsilon/2}+1\right)\mathbf{F}^{(\varepsilon)}_J, \tag{5.66}$$

but also that, via the Fourier transform,

$$\mathfrak{F}_{t,x}(\varphi\zeta\mathbf{f})_{J,K} = \frac{\big(\sqrt{\tau^2+|\kappa|^2}+1\big)^{1/2+\varepsilon}}{\mathcal{L}(i\tau,i\kappa,v)}\psi_1\left(\frac{\mathcal{L}(i\tau,i\kappa,v)}{K\sqrt{\tau^2+|\kappa|^2}}\right)$$
$$\psi_1\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{J}\right)\left(\frac{\partial}{\partial v}(-\Delta_v)^{\varepsilon/2}+1\right)(\mathfrak{F}_{t,x}\mathbf{F}^{(\varepsilon)}_J). \tag{5.67}$$

In order to simplify the calculations a little, notice that, since $J\ge 2$, one can introduce

$$(\mathfrak{F}_{t,x}\widetilde{\mathbf{F}}^{(\varepsilon)}_J) = \frac{\big(\sqrt{\tau^2+|\kappa|^2}+1\big)^{1/2+\varepsilon}}{\big(\sqrt{\tau^2+|\kappa|^2}\big)^{1+\varepsilon}}\Big(1-\psi_0\big(\sqrt{\tau^2+|\kappa|^2}\big)\Big)(\mathfrak{F}_{t,x}\mathbf{F}^{(\varepsilon)}_J),$$

so that again

$$\big\{\mathbb{E}\|\widetilde{\mathbf{F}}^{(\varepsilon)}_J\|^p_{L^{q_\varepsilon}_{t,x,v}}\big\}^{1/p} \le C\big\{\mathbb{E}\|\mathbf{F}^{(\varepsilon)}_J\|^p_{L^{q_\varepsilon}_{t,x,v}}\big\}^{1/p} \tag{5.68}$$

and (5.67) becomes

$$\mathfrak{F}_{t,x}(\varphi\zeta\mathbf{f})_{J,K} = \frac{1}{J^{1/2-\varepsilon}K}\widetilde{\psi_1}\left(\frac{\mathcal{L}(i\tau,i\kappa,v)}{K\sqrt{\tau^2+|\kappa|^2}}\right)\widehat{\psi_\varepsilon}\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{J}\right)\left(\frac{\partial}{\partial v}(-\Delta_v)^{\varepsilon/2}+1\right)(\mathfrak{F}_{t,x}\widetilde{\mathbf{F}}_J^{(\varepsilon)}),$$

where $\widetilde{\psi_1}(z) = z^{-1}\psi_1(z)$ and $\widehat{\psi_\varepsilon}(z) = |z|^{-(1/2-\varepsilon)}\psi_1(z)$ both belong to $\mathscr{C}_c^\infty(\mathbb{C};\mathbb{C})$.

Consequently, we deduce that

$$(\varphi\mathfrak{v})_{J,K}(t,x) = \int_{\mathbb{R}_v}\mathfrak{F}_{t,x}^{-1}\left[\left(\mathfrak{F}_{t,x}\widetilde{\mathbf{F}}^{(\varepsilon)}\right)(t,x,v)\left(-(-\Delta_v)^{\varepsilon/2}\frac{\partial}{\partial v}+1\right)\left\{m_{J,K}^{(\varepsilon)}(\tau,\kappa,v)\eta(v)\right\}\right]dv, \quad (5.69)$$

where we have abbreviated

$$m_{J,K}(\tau,\kappa,v) = \frac{1}{J^{1/2-\varepsilon}K}\widetilde{\psi}_1\left(\frac{\mathcal{L}(i\tau,i\kappa,v)}{K\sqrt{\tau^2+|\kappa|^2}}\right)\widehat{\psi_\varepsilon}\left(\frac{\sqrt{\tau^2+|\kappa|^2}}{J}\right).$$

Thus we may reprise the arguments previously employed in Chapter 2. Due to the definition of the fractional Laplacean operator (see (2.58)), we infer that

$$\begin{aligned}\left(-(-\Delta_v)^{\varepsilon/2}\frac{\partial}{\partial v}+1\right)\left\{m_{J,K}^{(\varepsilon)}\eta\right\} &= m_{J,K}^{(\varepsilon)}\eta + (-\Delta_v)^{\varepsilon/2}\left(\eta\frac{\partial m_{J,K}^{(\varepsilon)}}{\partial v}\right) + (-\Delta_v)^{\varepsilon/2}\left(\eta' m_{J,K}^{(\varepsilon)}\right)\\ &= (I)+(II)+(III)+(IV)+(V), \quad (5.70)\end{aligned}$$

in which it is tacitly written

$$\begin{cases}(I) = m_{J,K}^{(\varepsilon)}(\tau,\kappa,v)\eta(v),\\ (II) = c_\varepsilon\eta(v)\displaystyle\int_{\mathbb{R}_w}\frac{\frac{\partial m_{J,K}^{(\varepsilon)}}{\partial v}(\tau,\kappa,v+w)-\frac{\partial m_{J,K}}{\partial v}(\tau,\kappa,w)}{|w|^{1+\varepsilon}}\,dw,\\ (III) = c_\varepsilon\displaystyle\int_{\mathbb{R}_w}\frac{\partial m_{J,K}^{(\varepsilon)}}{\partial v}(\tau,\kappa,v+w)\frac{\eta(v+w)-\eta(v)}{|w|^{1+\varepsilon}}\,dw\\ (IV) = c_\varepsilon\eta'(v)\displaystyle\int_{\mathbb{R}_w}\frac{m_{J,K}^{(\varepsilon)}(\tau,\kappa,v+w)-m_{J,K}(\tau,\kappa,w)}{|w|^{1+\varepsilon}}\,dw, \text{ and}\\ (V) = c_\varepsilon\displaystyle\int_{\mathbb{R}_w}m_{J,K}^{(\varepsilon)}(\tau,\kappa,v+w)\frac{\eta'(v+w)-\eta'(v)}{|w|^{1+\varepsilon}}\,dw.\end{cases}$$

Thanks to Theorem 2.5, it is not difficult to verify that $(-(-\Delta_v)^{\varepsilon/2}\frac{\partial}{\partial v}+1)\{m_{J,K}^{(\varepsilon)}\eta\}$ is an $L^z_{t,x}$–multiplier for all $1<z<\infty$ and $v\in\mathbb{R}_v$, and, moreover,

$$\left\|\mathfrak{F}_{t,x}^{-1}\left[(\mathfrak{F}_{t,x}f)\left(-(-\Delta_v)^{\varepsilon/2}\frac{\partial}{\partial v}+1\right)\left\{m_{J,K}^{(\varepsilon)}(\,\cdot\,,\,\cdot\,,v)\eta(\,\cdot\,)\right\}\right]\right\|_{L^z_{t,x}} \le \frac{C_{\varepsilon,M,p}}{J^{1/2-\varepsilon}K^{2+\varepsilon}(1+|v|)^{1+\varepsilon}}\|f\|_{L^z_{t,x}}$$

for all $f\in\mathscr{S}(\mathbb{R}_t\times\mathbb{R}_x^N)$. Accordingly, returning to (5.69) and applying the Hölder inequality, we deduce that

$$\left\{\mathbb{E}\|(\varphi\mathfrak{v})_{J,K}\|^p_{L^{q_\varepsilon}_{t,x}}\right\}^{1/p} \le \frac{C_{\varepsilon,\eta}}{K^{2+\varepsilon}J^{1/2-\varepsilon}}\left\{\mathbb{E}\|\widetilde{\mathbf{F}}_J^{(\varepsilon)}\|^p_{L^{q_\varepsilon}_{t,x,v}}\right\}^{1/p}. \quad (5.71)$$

(Analogously, one could have deduced this estimate via the L. Grafakos–S. Oh's Kato–Ponce inequality [54] given in (2.78)).

Step #4: (The L^r–estimate of $(\varphi\mathfrak{v})_J$). Consequently, if $0 \le \theta \le 1$ and

$$\frac{1}{z} = \frac{1-\theta}{2} + \frac{\theta}{q_\varepsilon},$$

the Hölder inequality applied to (5.65) and (5.71) yields

$$\Big\{\mathbb{E}\|(\varphi\mathfrak{v})_{J,K}\|^p_{L^z_{t,x}}\Big\}^{1/p} \le C_{\varepsilon,\eta}\frac{K^{(1-\theta)\mathfrak{e}/2-\theta(2+\varepsilon)}}{J^{s_0(1-\theta)+\theta(1/2-\varepsilon)}}\Big\{\mathbb{E}\|\varphi\zeta\mathbf{f}\|^p_{L^2_vH^{s_0}_{t,x}}\Big\}^{(1-\theta)/p}\Big\{\mathbb{E}\|\widetilde{\mathbf{F}}^{(\varepsilon)}_J\|^p_{L^{q_\varepsilon}_{t,x,v}}\Big\}^{\theta/p}.$$

In order to "almost" minimize the exponent of K, let us choose $(1-\theta_\varepsilon)\mathfrak{e} = \theta_\varepsilon(4+4\varepsilon)$, *i.e.*,

$$\theta_\varepsilon = \frac{\mathfrak{e}}{4(1+\varepsilon)+\mathfrak{e}}.$$

In this case,

$$\Big\{\mathbb{E}\|(\varphi\mathfrak{v})_{J,K}\|^p_{L^{r_\varepsilon}_{t,x}}\Big\}^{1/p} \le C_{\varepsilon,\eta}\frac{K^{\varepsilon\theta_\varepsilon}}{J^{\frac{4(1+\varepsilon)s_0}{4(1+\varepsilon)+\mathfrak{e}}+\frac{(1/2-\varepsilon)\mathfrak{e}}{4(1+\varepsilon)+\mathfrak{e}}}}\big(\mathbb{E}\|\varphi\zeta\mathbf{f}\|^p_{L^2_vH^{s_0}_{t,x}}\big)^{(1-\theta_\varepsilon)/p}\big(\mathbb{E}\|\widetilde{\mathbf{F}}^{(\varepsilon)}_J\|^p_{L^{q_\varepsilon}_{t,x,v}}\big)^{\theta_\varepsilon/p}.$$

For such terms are summable as K ranges over the dyadics $\le 2M$, we conclude that

$$\Big\{\mathbb{E}\|(\varphi\mathfrak{v})_J\|^p_{L^{r_\varepsilon}_{t,x}}\Big\}^{1/p} \le \frac{C_{\varepsilon,\eta}}{J^{\frac{4(1+\varepsilon)s_0}{4(1+\varepsilon)+\mathfrak{e}}+\frac{(1/2-\varepsilon)\mathfrak{e}}{4(1+\varepsilon)+\mathfrak{e}}}}\big(\mathbb{E}\|\varphi\zeta\mathbf{f}\|^p_{L^2_vH^{s_0}_{t,x}}\big)^{(1-\theta_\varepsilon)/p}\big(\mathbb{E}\|\widetilde{\mathbf{F}}^{(\varepsilon)}_J\|^p_{L^{q_\varepsilon}_{t,x,v}}\big)^{\theta_\varepsilon/p}$$

for any dyadic $J \ge 2$. Therefore, thanks to Theorem 5.13 and Estimates (5.62) and (5.68), the desired conclusion is reached by letting ε be closer and closer to 0. The proof of the lemma is complete. □

Proof of Theorem 5.12. As it would be expected, the proof will depend on the kinetic formulation. For this purpose, let us employ the notations of Theorem 5.2.

Step #1: (The passage to the limit). First of all, since $\mathscr{O}$ is of class $\mathscr{C}^{1,1}$, let us repeat the ideas of Theorem 4.2 and construct a family $\{\theta_\ell\}_{0<\ell<1}$ in $\mathscr{C}^\infty_c(\mathscr{O})$ and other $\{\tau_\ell\}_{0<\ell<1}$ in $\mathscr{C}^\infty_c(0,T)$ satisfying:

(i) $0 \le \theta_\ell(x) \le 1$ and $0 \le \tau_\ell(t) \le 1$ for every $(t,x) \in Q$ and $0 < \ell < 1$;

(ii) $\big\{\varphi_\ell < 1\big\} \subset \big\{x \in \mathscr{O}; \operatorname{dist}(x;\partial\mathscr{O}) < \ell\big\}$ and $\operatorname{meas}\big\{\varphi_\ell < 1\big\} \le 2\ell$ for all $\ell > 0$;

(iii) $\big\{\tau_\ell < 1\big\} \subset (0,\ell)\cup(T-\ell,T)$, hence $\operatorname{meas}\big\{\tau_\ell < 1\big\} \le (\text{const.})\ell$, for all $\ell > 0$; and

(iv) $|(\nabla_x\theta_\ell)(x)|$ and $|\tau'_\ell(t)| \le (\text{const.})/\ell$ for every $(t,x) \in Q$ and $\ell > 0$.

Let also ζ be any $\mathscr{C}^\infty_c(\mathbb{R}_v)$ such that $\zeta \equiv 1$ in $\operatorname{supp}\eta$.

If $\mathbf{f}$ is the χ–function of $u(t,x)$, let us initially assume that $\mathbf{f} \in L^p(\Omega; L^2(\mathbb{R}_v; H^{s_0}(Q)))$ for some $0 \le s_0 < \frac{1}{2}$ (which can be surely chosen to be 0). Invoking (5.53) with $\varphi(t,x) = \varphi_\ell(t,x) = \tau_\ell(t)\theta_\ell(x)$, we obtain for $\mathfrak{v} = \int_{\mathbb{R}}\eta\mathbf{f}\,dv = u + c(a,b)$ (where $c(a,b)$ is a numerical constant)

$$\mathbb{E}\|\varphi_\ell u\|^p_{W^{\mathfrak{s},r}_{t,x}} \le C_{\mathfrak{s},r,\eta}\mathbb{E}\bigg\{1 + (b-a)^p + \|\varphi_\ell\mathbf{f}\|^p_{L^2_vH^{s_0}_{t,x}} + \|\mathbf{q}\|^p_{\mathfrak{M}_{t,x,v}} + \bigg\|\Big(\frac{\partial\varphi_\ell}{\partial t} + \mathbf{a}(v)\cdot\nabla_x\varphi_\ell\Big)\mathbf{f}\bigg\|^p_{L^1_{t,x,v}}\bigg\}. \tag{5.72}$$

for any $1 \le p < \infty$, $0 \le \mathfrak{s} < \frac{1}{2}\frac{\mathfrak{e}}{4+\mathfrak{e}} + \frac{4s_0}{4+\mathfrak{e}}$ and $1 \le r < \frac{4+\mathfrak{e}}{2+\mathfrak{e}}$. Notice that we already made use of Estimate (5.5) and of the L^∞–bound (5.30)—as a result, we have also employed that the fact that $\mathbf{q}$ is almost surely supported on $\mathscr{O}\times[a,b]$.

Our desire is to let $\ell \to 0_+$. Due to the properties of $\varphi_\ell(t,x)$ listed previously,

$$\left\| \left(\frac{\partial \varphi_\ell}{\partial t} + \mathbf{a}(v) \cdot \nabla_x \varphi_\ell \right) \mathbf{f} \right\|^2_{L^1_{t,x,v}} \leq C(a,b) \text{ for all } 0 < \ell < 1. \tag{5.73}$$

If $s_0 = 0$, then clearly $\mathbb{E}\|\varphi_\ell \mathbf{f}\|^p_{L^2_v H^{s_0}_{t,x}} \leq \mathbb{E}\|\mathbf{f}\|^p_{L^2_{t,x,v}} = \mathbb{E}\|\mathbf{f}\|^p_{L^2_v H^{s_0}_{t,x}}$. On the other hand, the scenario $0 < s_0 < \frac{1}{2}$ is slightly more delicate. Define the operator $T_\ell : L^2(Q) \to L^2(Q)$ by $T_\ell f = \varphi_\ell f$. Evidently, one has that

$$\begin{cases} \|T_\ell\|_{\mathscr{L}(L^2(Q))} = 1, \text{ and} \\ T_\ell f \to f \text{ in } L^2(Q) \text{ strongly for all } f \in L^2(Q) \text{ as } \ell \to 0_+. \end{cases}$$

Additionally, per the properties of the H^1_0–functions (*e.g.*, the Hardy's inequality of theorem 11.3 in chapter 1 of J.-L. Lions–E. Magenes [77]), one may easily inspect that

$$\begin{cases} \|T_\ell\|_{\mathscr{L}(H^1_0(Q))} \leq C_{\mathscr{O}}, \text{ and} \\ T_\ell f \to f \text{ strongly in } H^1_0(Q) \text{ for all } f \in H^1_0(Q) \text{ as } \ell \to 0_+. \end{cases}$$

As a result, for $0 < s < \frac{1}{2}$ (thence $H^s_0 = H^s$), an interpolation argument à J.-L. Lions–E. Magenes [77], vol. I, shows that

$$\|T_\ell\|_{\mathscr{L}(H^s(Q))} \leq 1^s (C_{\mathscr{O}})^{1-s} \leq C. \tag{5.74}$$

What is more, another classic argument of density and strong convergence of operators now leads to

$$T_\ell f \to f \text{ strongly in } H^s(Q) \text{ for all } f \in H^s(Q) \text{ as } \ell \to 0_+. \tag{5.75}$$

In a nutshell, the mingling of (5.74), (5.75) and the dominated convergence theorem implies that

$$\mathbb{E}\|\varphi_\ell \mathbf{f}\|^p_{L^2_v H^{s_0}_{t,x}} \to \mathbb{E}\|\mathbf{f}\|^p_{L^2_v H^{s_0}_{t,x}} \tag{5.76}$$

for all $1 \leq p < \infty$ provided that $0 \leq s_0 < \frac{1}{2}$ and $\ell \to 0_+$.

Therefore, with (5.73) and (5.76) at our disposal, we may return to (5.72) and conclude via the Fatou's lemma and Theorem 5.2 that

$$\mathbb{E}\|u\|^p_{W^{\mathfrak{s},r}(Q)} \leq C_{\mathfrak{s},r,\eta} \mathbb{E}\left\{ C(a,b) + \|\mathbf{f}\|^p_{L^2_v H^{s_0}_{t,x}} \right\} \tag{5.77}$$

for any $0 \leq s_0 < \frac{1}{2}$, $0 \leq \mathfrak{s} < \frac{1}{2}\frac{\mathfrak{e}}{4+\mathfrak{e}} + \frac{4s_0}{4+\mathfrak{e}}$, and $1 \leq r < \frac{4+\mathfrak{e}}{2+\mathfrak{e}}$.

Step #2: (The bootstrap argument). So as to obtain the final result, we will now engage the iterative procedure of P.-L. Lions–B. Perthame–E. Tadmor [82] (see also E. Tadmor–T. Tao [107]). Let us first apply (5.77) with $s_0 = 0$, so that

$$\mathbb{E}\|u\|^p_{W^{\mathfrak{s},r}(Q)} \leq C_{p,\mathfrak{s},r,\eta}, \tag{5.78}$$

for any $0 \leq \mathfrak{s} < \frac{\mathfrak{e}}{4+\mathfrak{e}} \stackrel{\text{def}}{=} s_1$ and $1 \leq r < \frac{4+\mathfrak{e}}{2+\mathfrak{e}}$, so that, in particular, $u \in L^p_\omega W^{\mathfrak{s},1}_{t,x}(Q)$.

As $\mathbf{f}$ is a χ–function, one may inspect that

$$\int_{\mathbb{R}_v} |\mathbf{f}(t,x,v) - \mathbf{f}(t',x',v)|^2 \, dv = \int_{\mathbb{R}_v} |\mathbf{f}(t,x,v) - \mathbf{f}(t',x',v)| \, dv = |u(t,x) - u(t',x')|$$

for all (t,x) and $(t',x') \in Q$. Hence the definition of the $H^s(Q)$–norm yields

$$\|\mathbf{f}\|^2_{L^2(\mathbb{R}_v;H^s(Q))} \le \|u\|_{L^1_{t,x}} + \int_{\mathbb{R}_v}\int_Q\int_Q \frac{|\mathbf{f}(t,x,v)-\mathbf{f}(t',x',v)|^2}{|(t-t',x-x')|^{N+1+2s}}\,dtdxdt'dx'dv$$
$$= \|u\|_{L^1_{t,x}} + \int_Q\int_Q \frac{|u(t,x)-u(t',x')|}{|(t-t',x-x')|^{N+1+2s}}\,dtdxdt'dx' \le C\|u\|_{W^{2s,1}(Q)}. \tag{5.79}$$

As a result, $\mathbf{f} \in L^p_\omega L^2_v H^s_{t,x}$ for any $0 \le s < \frac{s_1}{2} = \frac{1}{4}\frac{\mathfrak{e}}{4+\mathfrak{e}} < \frac{1}{2}$, and $\mathbb{E}\|\mathbf{f}\|^p_{L^2_v H^{s_1}_{t,x}} \le C_{\mathfrak{s},r,\eta}$.

We are thus allowed to reapply (5.77) with $0 \le s_0 < \frac{s_1}{2}$, implying that indeed (5.78) holds for any $0 \le \mathfrak{s} < s_2 \stackrel{\text{def}}{=} \frac{1}{2}\frac{\mathfrak{e}}{4+\mathfrak{e}} + \frac{2s_1}{4+\mathfrak{e}} \le \frac{1}{2}\frac{\mathfrak{e}}{4+\mathfrak{e}} + \frac{1}{2}\frac{\mathfrak{e}}{4+\mathfrak{e}} < \frac{1}{2}$.

Repeating the procedure *ad infinitum* by induction, we can infer the validity of (5.78) for any $0 \le s < s_n$, where $s_1 \le s_2 \le s_3 \le \ldots < \frac{1}{2}$ are defined iteratively as

$$\begin{cases} s_1 = \dfrac{1}{2}\dfrac{\mathfrak{e}}{4+\mathfrak{e}}, \\ s_{n+1} = \dfrac{1}{2}\dfrac{\mathfrak{e}}{4+\mathfrak{e}} + \dfrac{2s_n}{4+\mathfrak{e}}. \end{cases}$$

In conclusion, (5.78) holds true for all $0 \le s < \lim s_n = \frac{1}{2}\frac{\mathfrak{e}}{2+\mathfrak{e}}$ and $1 \le r < \frac{4+\mathfrak{e}}{2+\mathfrak{e}}$.

At last, in order to obtain the limit case $\mathfrak{r} = \frac{4+\mathfrak{e}}{2+\mathfrak{e}}$, it suffices to interpolate these estimates for $L^p_\omega W^{s,r}_{t,x}$ with the L^∞–bound (5.30). Consequently, all the assertions made in Theorem 5.12 were verified, and the proof is complete. □

Remark 5.9. Beyond a shadow of doubt, the regularity analysis provided is much simpler than that of B. Gess–M. Hofmanová [51] (in the hyperbolic case). On the other hand, their result, under some conditions (for instance, if $\mathbf{A}(v)$ behaves polynomially), may be also valid for weight-functions η whose support is the entire line. This characteristic is very attractive, once in general L^∞–bounds are not available for stochastic degenerate parabolic–hyperbolic equations. We reckon, however, that our argument may translate well to an unbounded situation such as this by substituting (5.63) with

$$(\varphi\zeta\mathbf{f})_J = \sum_{K \text{ dyadic}, K\le 1} (\varphi\zeta\mathbf{f})_{J,K} + \mathfrak{F}^{-1}_{t,x}\left[(1-\psi_0)\left(\frac{\mathcal{L}(i\tau,i\kappa,v)}{K\sqrt{\tau^2+|\kappa|^2}}\right)(\mathfrak{F}_{t,x}(\varphi\zeta\mathbf{f}))_J\right].$$

Notice that the novel term is actually beneficial, as Equation (5.66) is nondegenerate in this support. Evidently, several other little modifications would be necessary, but we will not pursue this direction here, seeing that it is beyond our purposes.

Remark 5.10. In Lemma 5.9—more specifically in (5.59)—, we have treated the stochastic source term as we have in the existence proof: by reducing it to some deterministic one of order $(1/2+\varepsilon)$ in (t,x) via the Kolmogorov's continuity criterion. Even though this is consistent with the arguments of this Chapter, it may not be optimal.

Indeed, using the methods of Sturm–Liouville problems (see (4.40)), we could have written

$$\varphi(t,x)\zeta(v)\Phi(x,v)\delta_{v=u}\frac{dW}{dt} = \left(\frac{\partial}{\partial v}+1\right)\left\{\Psi(t,x,v)\frac{dW}{dt}\right\},$$

where $\Psi \in L^2(\Omega\times[0,T];HS(\mathscr{H};L^2(\mathscr{O}\times\mathbb{R}_v)))$ is predictable, and satisfies

$$\mathbb{E}\|\Psi(t,x,v)\|^2_{HS(\mathscr{H};L^2(\mathscr{O}\times\mathbb{R}_v))} \le C\int_{\mathscr{O}} \varphi(t,x)^2\zeta(u(t,x))^2\mathfrak{G}^2(x,u(t,x))\,dx.$$

On the other hand, according to the idea expressed in Remark 2.11,

$$\mathbb{E}\Big\| \int_{\mathbb{R}} \eta(v)\mathfrak{F}_{t,x}^{-1}\Big[\psi_1\Big(\frac{\mathcal{L}(i\tau, i\kappa, v)}{K\sqrt{\tau^2+|\kappa|^2}}\Big)\psi_1\Big(\frac{\sqrt{\tau^2+|\kappa|^2}}{J}\Big)\Big(\frac{\partial}{\partial v}+1\Big)\mathfrak{F}_{t,x}\Big(\Psi\frac{dW}{dt}\Big)\Big]\,dv\Big\|^2_{L^2(\mathbb{R}_t\times\mathbb{R}_x^N)}$$
$$\leq \frac{C_\varepsilon}{J^{(1/2-\varepsilon)}K}\mathbb{E}\int_0^T \|\Psi(t,x,v)\|^2_{HS(\mathscr{H};L^2(\mathscr{O}\times\mathbb{R}_v))}\,dt$$

for all $0<\varepsilon<1/2$. This inequality should be compared with (5.71).

In this fashion, one can get a much better picture of the contribution of the stochastic forcing. Nevertheless, this simple argument has the difficulty of producing an $L^2(\mathbb{R}_t\times\mathbb{R}_x^N)$–estimate, which may not mingle nicely with the $L^{q_\varepsilon}(\mathbb{R}_t\times\mathbb{R}_x^N)$–estimate of the purely deterministic term. Although this issue was overcome in B. Gess–M. Hofmanová [51] by some rather difficult and long interpolation argument, we reckon this would eliminate any sort of simplicity this new estimate had brought to the table, and thus it is beyond the scope of this thesis. (Notwithstanding, we are under the impression that considering our problem in some homothetic version of the torus $\mathbb{T}_t\times\mathbb{T}_x^N$ would significantly remedy some of these issues).

In any event, it would be very desirable to deduce (5.53) without the exponent $\mathfrak{r}$ appearing in the "quadratic variation" parcel, even if this should force $1\leq p\leq 2$.

Appendix A

The viscous approximation

A.1 Hypotheses and the main result

In this supplementary chapter, we will delve into the parabolic approximation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \operatorname{div}_x \mathbf{A}(u) - \varepsilon \Delta_x u = \Phi(x,u)\,\dfrac{dW}{dt} & \text{for } 0 < t < T \text{ and } x \in \mathscr{O}, \\ \mathbf{A}(u) \cdot \nu = \varepsilon \dfrac{\partial u}{\partial \nu} & \text{for } 0 < t < T \text{ and } x \in \partial\mathscr{O}, \text{ and} \\ u(0,x) = u_0(x) & \text{for } t = 0 \text{ and } x \in \mathscr{O}, \end{cases} \tag{A.1}$$

where $T > 0$ is an arbitrary number, $\varepsilon > 0$ is a viscosity coefficient, and $\nu(x)$ denotes the outward unit normal at a point $x \in \partial\mathscr{O}$. Quite similarly to how we have written Chapter 5, we begin by enumerating the hypotheses tacitly made here.

1. *Conditions concerning* $\mathscr{O}$: $\mathscr{O}$ is assumed to be bounded, regular, and of class $\mathscr{C}^{1,1}$.
2. *Conditions concerning* $\mathbf{A}$:
   (a) (Regularity): $\mathbf{A} : \mathbb{R} \to \mathbb{R}^N$ is a continuously differentiable Lipschitz vector function, *i.e.*,
   $$\mathbf{A} \in (\mathscr{C}^1 \cap W^{1,\infty})(\mathbb{R}; \mathbb{R}^N). \tag{A.2}$$
   (b) (Existence of saturation states): There exist some $a < b$ such that
   $$\mathbf{A}(a) = 0 = \mathbf{A}(b). \tag{A.3}$$
3. *Conditions concerning* W: $(\Omega, \mathscr{F}, \mathbb{P})$ denotes a probability space endowed with a complete, right-continuous filtration $(\mathscr{F}_t)_{t\geq 0}$. Furthermore, it is assumed the existence of a sequence $(\beta_k(t))_{k\in\mathbb{N}}$ of mutually independent Brownian motions in $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Hence, letting $\mathscr{H}$ be a separable Hilbert space with a hilbertian basis $(e_k)_{k\in\mathbb{N}}$, $W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k$ defines a cylindrical Wiener process.
4. *Conditions concerning* $\Phi(x,u)$: For any integer $k \geq 1$, $g_k \in \mathscr{C}(\mathscr{O} \times \mathbb{R}; \mathbb{R})$ is such that:
   (a) (Regularity): $(x,v) \mapsto \frac{\partial g_k}{\partial v}(x,v)$ exists and lies in $\mathscr{C}(\mathscr{O} \times \mathbb{R}; \mathbb{R})$. Moreover, there exists a sequence of constants $\alpha_k \geq 0$ such that
   $$\left| \frac{\partial g_k}{\partial v}(x,v) \right| \leq \alpha_k \ \forall (x,v) \in \mathscr{O} \times \mathbb{R}, \tag{A.4}$$

and $\sum_{k=1}^{\infty}\alpha_k^2 = D < \infty$. Consequently, for any $x \in \mathscr{O}$, and u and $v \in \mathbb{R}$, it holds that

$$\sum_{k=1}^{\infty} |g_k(x,u) - g_k(x,v)|^2 \leq D|u-v|^2. \tag{A.5}$$

(b) (Existence of saturation states, part II): For the same $a < b$ featured in (A.3), it holds that

$$g_k(a) = 0 = g_k(b) \tag{A.6}$$

for any integer $k \geq 1$. Therefore, an amalgamation of (A.4) and (A.6) shows that

$$\mathfrak{G}^2(x,v) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} g_k(x,v)^2 = \sum_{k=1}^{\infty} |g_k(x,v) - g_k(x,a)|^2 \leq C(1+v^2) \tag{A.7}$$

for all $x \in \mathscr{O}$ and $v \in \mathbb{R}$.

Thus, let us again define $\Phi : L^2(\mathscr{O}) \to \mathscr{L}(\mathscr{H}; L^2(\mathscr{O}))$ by

$$(\Phi(f)\cdot h)(x) = \sum_{k=1}^{\infty} g_k(x, f(x))\,(h, e_k)_{\mathscr{H}}$$

whenever $h \in \mathscr{H}$ and $x \in \mathscr{O}$. In the light of (A.7), $\Phi(f)$ is well-defined, and is in the Hilbert–Schmidt class $HS(\mathscr{H}; L^2(\mathscr{O}))$. Therefore, given any predictable process $u \in L^2(\Omega \times [0,T]; L^2(\mathscr{O}))$, the stochastic integral

$$t \mapsto \int_0^t \Phi(u(t'))\,dW(t') = \sum_{k=1}^{\infty}\int_0^t g_k(x, u(t',x))\,d\beta_k(t')$$

defines an $L^2(\mathscr{O})$–valued process.

5. *Conditions on u_0*:

(a) (Mensurability): $u_0 \in L^2(\Omega; L^2(\mathscr{O}))$ is $\mathscr{F}_{t=0}$–measurable.

(b) (Existence of saturation states, part III): If a and b are same ones in (A.3) and (A.6), then

$$a \leq u_0(x) \leq b \text{ almost surely in } \mathscr{D}'(\mathscr{O}). \tag{A.8}$$

Henceforth, we will understand the measure space $\Omega \times [0,T]$ as endowed with its predictable σ–algebra.

We are now in conditions to define the natural notion of weak solution to (A.1), and enunciate the main result of this chapter.

Definition A.1. A predictable process $u \in L^2(\Omega; \mathscr{C}([0,T]; L^2(\mathscr{O}))) \cap L^2(\Omega \times [0,T]; H^1(\mathscr{O}))$ is said to be a *weak solution* to (A.1) if, given any $\varphi \in \mathscr{C}_c^\infty((-\infty,T)\times\mathbb{R}^N)$, it holds almost surely that

$$\int_0^T\int_{\mathscr{O}} u(t,x)\frac{\partial\varphi}{\partial t}(t,x)\,dxdt + \int_{\mathscr{O}} u_0(x)\varphi(0,x)\,dx + \int_0^T\int_{\mathscr{O}} \mathbf{A}(u(t,x))\cdot\nabla_x\varphi(t,x)\,dxdt$$
$$- \varepsilon\int_0^T\int_{\mathscr{O}} \nabla_x u(t,x)\cdot\nabla_x\varphi(t,x)\,dxdt = -\sum_{k=1}^{\infty}\int_0^T\int_{\mathscr{O}} g_k(x,u(t,x))\varphi(t,x)\,dxd\beta_k(t). \tag{A.9}$$

Theorem A.1. *There exists a unique weak solution $u \in L^2(\Omega; \mathscr{C}([0,T]; L^2(\mathscr{O})) \cap L^2(0,T; H^1(\mathscr{O})))$ to (3.1). Furthermore, this solution has the following properties.*

1. (L^∞–bound). *One has almost surely that*

$$a \leq u(t,x) \leq b \textit{ in } \mathscr{D}'((0,T)\times\mathscr{O}). \tag{A.10}$$

2. (Energy estimate). *For all* $1 \le p < \infty$*, there exists a constant*

$$C = C\Big(p, a, b, T, \sup_{a \le v \le b} |\mathbf{A}(v)|, \mathbb{E}\Big[\Big(\int_0^T \int_{\mathscr{O}} \mathfrak{G}^2(x, u(t,x))\, dxdt\Big)^{p/2}\Big]\Big),$$

independent of $0 < \varepsilon < 1$*, such that*

$$\mathbb{E}\Big[\Big(\int_0^T \int_{\mathscr{O}} \varepsilon |\nabla_x u(t,x)|^2\, dxdt\Big)^p\Big] \le C. \tag{A.11}$$

3. (Entropy formulation). *Almost surely, for any function* $\eta \in \mathscr{C}^2(\mathbb{R})$ *with* $\eta'' \in L^\infty(\mathbb{R})$*, and any* $\phi \in \mathscr{C}_c^1((-\infty, T) \times \mathscr{O})$*, it holds that*

$$\begin{aligned}\int_0^T \int_{\mathscr{O}} \Big(\eta(u)\frac{\partial \phi}{\partial t} + \mathbf{A}^\eta(u)\cdot\nabla_x\phi\Big)\, dxdt = &- \int_{\mathscr{O}} \eta(u_0(x))\phi(0,x)\, dx \\ &+ \int_0^T \int_{\mathscr{O}} \Big(\varepsilon\nabla_x\eta(u)\cdot\nabla_x\phi + \varepsilon\eta''(u)|\nabla u|^2\phi\Big)\, dxdt \\ &- \int_0^T \int_{\mathscr{O}} \eta'(u)\Phi(x,u)\phi\, dxdW(t) \\ &- \frac{1}{2}\int_0^T \int_{\mathscr{O}} \eta''(u)\mathfrak{G}^2(x,u)\, dxdt,\end{aligned} \tag{A.12}$$

where, if $\mathbf{a}(v) = \mathbf{A}'(v)$*, we have written*

$$\begin{cases}(\mathbf{A}^\eta)'(u) = \eta'(u)\mathbf{a}(u), \text{ and} \\ \mathfrak{G}^2(x,u) = \sum_{k=1}^\infty g_k(x,u)^2.\end{cases}$$

4. (Kinetic formulation). *If* $f(t,x,v) = 1_{(-\infty, u(t,x))}(v) - 1_{(0,\infty)}(v)$ *is the* χ*–function associated to* $u(t,x)$*, then it satisfies almost surely in* $\mathscr{D}'((0,T)\times\mathscr{O})$

$$\frac{\partial f}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f - \varepsilon\Delta_x f = \frac{\partial \mathbf{q}}{\partial v} + \delta_{u(t,x)}(v)\Phi(x,v)\,\frac{dW}{dt}, \tag{A.13}$$

where we have abbreviated

$$\begin{cases}\mathbf{m}(t,x,v) = \varepsilon|\nabla u(t,x)|^2\delta_{u(t,x)}(v), \text{ and} \\ \mathbf{q}(t,x,v) = \mathbf{m}(t,x,v) - \frac{1}{2}\mathfrak{G}^2(x,v)\delta_{u(t,x)}(v).\end{cases}$$

Theorem A.1 will be proven by translating Problem (A.1) into an adequate abstract setting. Thus, before properly presenting this proof's steps, let us recall some basic facts from Spectral Theory.

A.2 The diagonalization method

First of all, let us enunciate the celebrated spectral theorem in its multiplicative operator form, whose statement we quote from M. Reed–B. Simon [98]:

Proposition A.1. *Let* Λ *be a self-adjoint operator on a separable Hilbert space* $\mathfrak{U}$ *with domain* $D(\Lambda)$*. Then there is a measure space* (X, μ) *with* μ *a finite measure, a unitary operator* $\mathfrak{T} : \mathfrak{U} \to L^2(X, d\mu)$*, and a real-valued function* λ *on* X *which is finite a.e. so that*

1. $u \in D(\Lambda)$ *if and only if* $\lambda(\,\cdot\,)(\mathfrak{T}u)(\,\cdot\,) \in L^2(X, d\mu)$*;*

2. *If* $\varphi \in \mathfrak{T}(D(\Lambda))$*, then* $(\mathfrak{T}\Lambda\mathfrak{T}^{-1}u)(m) = \lambda(m)u(m)$*.*

In the remainder of this section, we will preserve the notations and assumptions of this spectral theorem. Furthermore, we will assume that the operator Λ is *nonnegative*, which allows us to characterize its generated semigroup $\mathfrak{S}(t) = \exp\{-t\Lambda\}$ by means of the operational calculus as

$$(\mathfrak{T} \exp\{-t\Lambda\}u)(m) = \exp\{-t\lambda(m)\}(\mathfrak{T}u)(m).$$

The nonnegativeness of Λ also allows us to describe the so-called "intermediate spaces" (see J.-L. Lions–E. Magenes [80]) via

$$\mathfrak{U}^\alpha_\Lambda \stackrel{\text{def}}{=} D(\Lambda^\alpha) = \mathfrak{T}^{-1}(L^2(X, (1+\lambda(m))^{2\alpha} d\mu)) \stackrel{\text{def}}{=} \mathfrak{T}^{-1}(\mathfrak{V}^\alpha) \text{ (with equivalent norms)}$$

if $\alpha \geq 0$. However, if $\alpha < 0$, we will put $\mathfrak{U}^\alpha_\Lambda = (\mathfrak{U}^{-\alpha}_\Lambda)^\star$, which may still be naturally identified with $\mathfrak{V}^\alpha = L^2(X, (1+\lambda(m))^{2\alpha} d\mu)$. Consequently, $(I+\Lambda)^\beta$ defines an isometric isomorphism between $\mathfrak{U}^\alpha_\Lambda$ and $\mathfrak{U}^{\alpha-\beta}_\Lambda$ for all α and $\beta \in \mathbb{R}$. Additionally, given any $\alpha < \beta$,

$$\mathfrak{U}^\beta_\Lambda \subset \mathfrak{U}^\alpha_\Lambda \text{ with dense and continuous inclusion.}$$

Observe that $\mathfrak{S}(t)$ is still a contraction, self-adjoint semigroup of operators on such $\mathfrak{U}^\alpha_\Lambda$ spaces.

Let $T > 0$. For any $\alpha \in \mathbb{R}$, we may define the Duhamel convolution operator

$$(\mathcal{I}_\Lambda f)(t) = \int_0^t \mathfrak{S}(t-t') f(t')\, dt'$$

for $f \in L^2(0,T;\mathfrak{U}^\alpha_\Lambda)$. Of course, for $\mathfrak{S}(t)$ is a contraction semigroup, given any $0 \leq t \leq T$,

$$\begin{cases} \|(\mathcal{I}_\Lambda f)(t)\|^2_{\mathfrak{U}^\alpha_\Lambda} \leq t \displaystyle\int_0^t \|f(t')\|^2_{\mathfrak{U}^\alpha_\Lambda}\, dt', \text{ hence} \\ \displaystyle\int_0^T \|(\mathcal{I}_\Lambda f)(t)\|^2_{\mathfrak{U}^\alpha_\Lambda}\, dt \leq \frac{T^2}{2} \int_0^T \|f(t)\|^2_{\mathfrak{U}^\alpha_\Lambda}\, dt. \end{cases} \tag{A.14}$$

Nonetheless, one may say much more regarding the regularization of such an operator.

Proposition A.2. *Conserve the notations above, and let* $\alpha \in \mathbb{R}$.

1. $\mathcal{I}_\Lambda$ *maps* $L^2(0,T;\mathfrak{U}^\alpha_\Lambda)$ *into* $L^2(0,T;\mathfrak{U}^{\alpha+1}_\Lambda)$ *continuously: Given any* $f \in L^2(0,T;\mathfrak{U}^\alpha_\Lambda)$,
$$\int_0^T \left\|\mathcal{I}_\Lambda f(t)\right\|^2_{\mathfrak{U}^{\alpha+1}_\Lambda} dt \leq C \int_0^T \left\|f(t)\right\|^2_{\mathfrak{U}^\alpha_\Lambda} dt, \tag{A.15}$$
for some absolute constant C *depending only on* T *and* α.

2. *Additionally,* $\mathcal{I}_\Lambda$ *also maps* $L^2(0,T;\mathfrak{U}^\alpha_\Lambda)$ *continuously into* $\mathscr{C}([0,T];\mathfrak{U}^{\alpha+1/2}_\Lambda)$*: Given any* $f \in \mathscr{C}([0,T];\, \mathfrak{U}^\alpha_\Lambda)$,
$$\max_{0\leq t\leq T} \left\|\mathcal{I}_\Lambda f(t)\right\|^2_{\mathfrak{U}^{\alpha+1/2}_\Lambda} \leq C \int_0^T \left\|f(t)\right\|^2_{\mathfrak{U}^\alpha_\Lambda} dt, \tag{A.16}$$
for another absolute constant C *depending only on* T *and* α.

3. *Therefore,* $\mathcal{I}_\Lambda$ *defines a continuous linear transformation from* $L^2(\Omega \times [0,T];\mathfrak{U}^\alpha_\Lambda)$ *into* $L^2(\Omega;\mathscr{C}([0,T];\mathfrak{U}^{\alpha+1/2}_\Lambda)) \cap L^2(\Omega\times[0,T];\mathfrak{U}^{\alpha+1}_\Lambda)$.

Proof. Step #1: (The proof of the first assertion.) Evidently, by the remarks above, one may assume that $\alpha = 0$. Let $f \in L^2(0,T;\mathfrak{U})$. For

$$\int_0^T \left\|\mathcal{I}_\Lambda f(t)\right\|^2_{\mathfrak{U}^1_\Lambda} ds \leq C\Big\{ \int_0^T \left\|\mathcal{I}_\Lambda f(t)\right\|^2_{\mathfrak{U}} dt + \int_0^T \left\|\Lambda(\mathcal{I}_\Lambda f)(t)\right\|^2_{\mathfrak{U}} dt \Big\},$$

and the first term was already estimated in (A.14), we may concentrate only on estimating the second one. Combining the spectral theorem A.1, its operational calculus and the Cauchy–Schwarz inequality, one arrives at

$$\begin{aligned}\int_0^T \left\|\Lambda(\mathcal{I}_\Lambda f)(t)\right\|_{\mathfrak{U}}^2 dt &= \int_0^T \int_X \lambda(m)^2 \left| \int_0^t \exp\{-(t-t')\lambda(m)\} \big(\mathfrak{T} f(t')\big)(m)\, dt' \right|^2 d\mu(m) dt \\ &= \int_0^T \int_X \left| \int_0^t \exp\{-(t-t')\lambda(m)\}\lambda(m) \big(\mathfrak{T} f(t')\big)(m)\, dt' \right|^2 d\mu(m) dt \\ &\le \int_0^T \int_X \int_0^t \exp\{-(t-t')\lambda(m)\}\lambda(m) \left|\big(\mathfrak{T} f(t')\big)(m)\right|^2 dt' d\mu(m) dt.\end{aligned}$$

Since $\mathfrak{T} f \in L^2(0,T;X)$, the Tonelli theorem thus yields

$$\begin{aligned}\int_0^T \left\|\Lambda(\mathcal{I}_\Lambda f)(t)\right\|_{\mathfrak{U}}^2 dt &\le \int_X \int_0^T \int_{t'}^T \exp\{-(t-t')\lambda(m)\}\lambda(m) \left|\big(\mathfrak{T} f(t')\big)(m)\right|^2 dt dt' d\mu(m) \\ &\le \int_X \int_0^T \left|\big(\mathfrak{T} f(t')\big)(m)\right|^2 dt d\mu(m) = \|f\|_{L^2(0,T;\mathfrak{U})}^2.\end{aligned}$$

This shows the validity of the first assertion.

Step #2: (The proof of the second assertion.) Likewise, in virtue of (A.14), so as to demonstrate (A.16), it suffices to inspect $\Lambda^{1/2}(\mathcal{I}_\Lambda f)(t)$. On the other hand, essentially the same argument of the previous step ensures that

$$\begin{aligned}\left\|\Lambda^{1/2}(\mathcal{I}_\Lambda f)(t)\right\|_{\mathfrak{U}}^2 &= \int_X \left| \int_0^t \exp\{-(t-t')\lambda(m)\}\lambda(m)^{1/2} \big(\mathfrak{T} f(t')\big)(m)\, dt' \right|^2 d\mu(m) \\ &\le \int_X \int_0^t \left|\big(\mathfrak{T} f(t')\big)(m)\right|^2 dt' d\mu(m) = \|f\|_{L^2(0,T;\mathfrak{U})}^2,\end{aligned}$$

hence (A.16).

It remains, however, to verify that $(\mathcal{I}_\Lambda f) \in \mathscr{C}([0,T];\mathfrak{U}_\Lambda^{1/2})$. Notice that, were f in, say, $\mathscr{C}([0,T];\mathfrak{U}_\Lambda^{1/2})$, the assertion that $(\mathcal{I}_\Lambda f) \in \mathscr{C}([0,T];\mathfrak{U}_\Lambda^{1/2})$ would constitute a simple corollary of the strong continuity of $\mathfrak{S}(t)$ and the fact that the range of f is compact in $\mathfrak{U}_\Lambda^{1/2}$. In the general case, once one is in possession of (A.16), one can argue by density.

Step #3: (The proof of the third assertion.) Finally, the last conclusion of this proposition follows from the previous two assertions and the constatation that $\mathcal{I}_\Lambda$ maps predictable processes into predictable processes (which can be immediately seen by approximations via simple functions). □

Concerning the next result, recall the probabilistic assumptions of the first section of this appendix chapter. Under these conditions, we may introduce the stochastic Duhamel operator

$$(\mathcal{I}_W \Psi)(t) = \int_0^t \mathfrak{S}(t-t')\Psi(t')\, dW(t')$$

for predictable processes $\Psi \in L^2(\Omega \times [0,T]; HS(\mathscr{H};\mathfrak{U}_\Lambda^\alpha))$. Concerning such an operator, we have the following result.

Proposition A.3. *Conserve the notations above, and let $\alpha \in \mathbb{R}$.*

1. *$\mathcal{I}_\Lambda$ maps $L^2(\Omega \times [0,T]; HS(\mathscr{H};\mathfrak{U}_\Lambda^\alpha))$ into $L^2(\Omega \times [0,T]; \mathfrak{U}_\Lambda^{\alpha+1/2})$ continuously: Given any predictable process $\Psi \in L^2(\Omega \times [0,T]; HS(\mathscr{H};\mathfrak{U}_\Lambda^\alpha))$,*

$$\mathbb{E}\int_0^T \left\|\mathcal{I}_W\Psi(t)\right\|_{\mathfrak{U}_\Lambda^{\alpha+1/2}}^2 dt \le C\mathbb{E}\int_0^T \left\|\Psi(t)\right\|_{HS(\mathscr{H};\mathfrak{U}_\Lambda^\alpha)}^2 dt, \qquad (A.17)$$

for some absolute constant C depending only on T and α.

2. *Additionally, $\mathcal{I}_\Lambda$ also maps $L^2(\Omega\times[0,T];HS(\mathscr{H};\mathfrak{U}^\alpha_\Lambda))$ continuously into $L^2(\Omega;\mathscr{C}([0,T];\mathfrak{U}^\alpha_\Lambda))$: Given any predictable process $L^2(\Omega\times[0,T];HS(\mathscr{H};\mathfrak{U}^\alpha_\Lambda))$,*

$$\mathbb{E}\max_{0\le t\le T}\left\|\mathcal{I}_W\Psi(t)\right\|^2_{\mathfrak{U}^\alpha_\Lambda}\le C\mathbb{E}\int_0^T\left\|\Psi(t)\right\|^2_{HS(\mathscr{H};\mathfrak{U}^\alpha_\Lambda)}dt, \tag{A.18}$$

for another absolute constant C depending only on T and α.

Proof. Step #1: (The proof of the first assertion). Once again, we may assume that $\alpha=0$. As one could have expected, the verification of (A.17) is very akin to one of (A.15); nevertheless, due to the fact that it involves a stochastic integral, one needs to employ the Itô isometry, hence the weaker smoothing effect.

Let $\Psi\in L^2(\Omega\times[0,T];HS(\mathscr{H};\mathfrak{U}))$, let (e_k) be a Hilbert basis of $\mathscr{H}$, and put $\Psi_k(t)=\Psi(t)\cdot e_k\in L^2(\Omega\times[0,T];\mathfrak{U})$. Then, by the aforementioned Itô isometry and Theorem A.1,

$$\begin{aligned}\mathbb{E}\int_0^T\|\mathcal{I}_W\Psi(t)\|^2_{\mathfrak{U}^{1/2}_\Lambda}dt&=\mathbb{E}\int_0^T\left\|\int_0^t\mathfrak{S}(t-t')\Psi(t')\,dW(t')\right\|^2_{\mathfrak{U}^{1/2}_\Lambda}dt\\&=\mathbb{E}\int_0^T\int_0^t\left\|\mathfrak{S}(t-t')\Psi(t')\right\|^2_{HS(\mathscr{H};\mathfrak{U}^{1/2}_\Lambda)}dt'dt\\&=\sum_{k=1}^\infty\mathbb{E}\int_0^T\int_0^t\left\|\mathfrak{S}(t-t')\Psi_k(t')\right\|^2_{\mathfrak{U}^{1/2}_\Lambda}dt'dt\\&=\sum_{k=1}^\infty\mathbb{E}\int_X\int_0^T\int_0^t e^{-2(t-t')\lambda(m)}(1+\lambda(m))\left|(\mathfrak{T}\Psi_k(t')(m)\right|^2dt'dtd\mu(m)\\&\le C\sum_{k=1}^\infty\mathbb{E}\int_X\int_0^T\left|(\mathfrak{T}\Psi_k(t')(m)\right|^2dt'd\mu(m)=C\mathbb{E}\int_0^T\left\|\Psi(t)\right\|^2_{HS(\mathscr{H};\mathfrak{U})}dt.\end{aligned}$$

Therefore, (A.17) is proven. The verification that $\mathcal{I}_W\Psi$ is predictable may be seen via approximation by simple processes.

Step #2: (The proof of the second assertion). The estimate in (A.18) follows from the next argument due to L. Tubaro [109]. Let again $\Psi\in L^2(\Omega\times[0,T];HS(\mathscr{H};\mathfrak{U}))$. Putting $\mathfrak{v}(t)=(\mathcal{I}_W\Psi)(t)$, the previous step asserts that $\mathfrak{v}\in L^2(\Omega\times[0,T];\mathfrak{U}^{1/2}_\Lambda)$. We claim that it actually holds that

$$d\mathfrak{v}(t)=-\Lambda\mathfrak{v}(t)\,dt+\Psi(t)\,dW\text{ almost surely in }\mathfrak{U}^{-1/2}_\Lambda. \tag{A.19}$$

Indeed, let $\phi\in\mathfrak{U}^1_\Lambda$, and $0<t<T$ be arbitrary. The stochastic Fubini theorem and the symmetry

of $\mathfrak{S}(t)$ imply that, almost surely,

$$\begin{aligned}
-\left\langle \int_0^t \Lambda\mathfrak{v}(t')\,dt', \phi \right\rangle_{\mathfrak{U}_\Lambda^{-1},\mathfrak{U}_\Lambda^1} &= -\int_0^t \left\langle \Lambda\mathfrak{v}(t'), \phi \right\rangle_{\mathfrak{U}_\Lambda^{-1},\mathfrak{U}_\Lambda^1} dt' \\
&= -\int_0^t \big(\Lambda\phi, \mathfrak{v}(t')\big)_{\mathfrak{U}}\, dt' \\
&= -\int_0^t \int_0^{t'} \big(\mathfrak{S}(t'-s)\Lambda\phi, \Psi(s)\,dW(s)\big)_{\mathfrak{U}_\Lambda}\, dt' \\
&= -\int_0^t \left(\int_s^t \mathfrak{S}(t'-s)\Lambda\phi\,dt', \Psi(s)\,dW(s) \right)_{\mathfrak{U}} \\
&= \int_0^t \big(\mathfrak{S}(t-s)\phi, \Psi(s)\,dW(s)\big)_{\mathfrak{U}_\Lambda} - \int_0^t \big(\phi, \Psi(s)\,dW(s)\big)_{\mathfrak{U}} \\
&= \left\langle \mathfrak{v}(t) - \int_0^t \Psi(s)\,dW(s), \phi \right\rangle_{\mathfrak{U}_\Lambda^{-1},\mathfrak{U}_\Lambda^1}.
\end{aligned}$$

This, combined with the density of $\mathfrak{U}_\Lambda^1$ in $\mathfrak{U}_\Lambda^{1/2}$, yields (A.19).

So as to prove (A.18), let $I_\delta = (I + \delta\Lambda)^{-1}$ for $\delta > 0$. Then, for any $\alpha \in \mathbb{R}$,

1. $I_\delta \in \mathscr{L}(\mathfrak{U}_\Lambda^\alpha; \mathfrak{U}_\Lambda^{\alpha+1})$ with norm $\leq C/\delta$; and
2. for any $\phi \in \mathfrak{U}_\Lambda^\alpha$, $I_\delta\phi \to \phi$ in $\mathfrak{U}_\Lambda^\alpha$ as $\delta \to 0_+$.

Put $\mathfrak{v}_\delta(t) = I_\delta\mathfrak{v}(t)$, in a fashion that $\mathfrak{v}_\delta \in L^2(\Omega \times [0,T]; \mathfrak{U}_\Lambda^1)$, $\mathfrak{v}_\delta(0) = 0$, and

$$d\mathfrak{v}_\delta(t) = -\Lambda\mathfrak{v}_\delta(t)\,dt + I_\delta\Psi(t)\,dW \text{ almost surely in } \mathfrak{U}.$$

Thus, by the usual Itô formula with $F(Y) = \|Y\|_{\mathfrak{U}}^2$, and the fact that $\Lambda \geq 0$, we obtain that, for all $0 \leq t \leq T$,

$$\begin{aligned}
\|\mathfrak{v}_\delta(t)\|_{\mathfrak{U}}^2 &= -2\int_0^t \big(\Lambda\mathfrak{v}_\delta(t'), \mathfrak{v}_\delta(t')\big)_{\mathfrak{U}}\,dt' + 2\int_0^t \big(\mathfrak{v}_\delta(t'), I_\delta\Psi(t')\,dW(t')\big)_{\mathfrak{U}} + \int_0^t \|I_\delta\Psi(t')\|_{HS(\mathscr{H};\mathfrak{U})}^2\,dt' \\
&\leq 2\int_0^t \big(\mathfrak{v}_\delta(t'), I_\delta\Psi(t')\,dW(t')\big)_{\mathfrak{U}} + \int_0^t \|I_\delta\Psi(t')\|_{HS(\mathscr{H};\mathfrak{U})}^2\,dt' \text{ almost surely.}
\end{aligned}$$

On the other hand, the Burkholder inequality (see M. Ondrejat [89]) guarantees that

$$\begin{aligned}
\mathbb{E}\sup_{0\leq t\leq T} \|\mathfrak{v}_\delta(s)\|_{\mathfrak{U}}^2 &\leq C\mathbb{E}\left(\left[\int_0^T \big\|\mathfrak{v}_\delta(t')\big\|_{\mathfrak{U}}^2 \big\|I_\delta\Psi(t')\big\|_{HS(\mathscr{H};\mathfrak{U})}^2\,dt' \right]^{1/2} \right) + 2\mathbb{E}\int_0^T \big\|I_\delta\Psi(t')\big\|_{HS(\mathscr{H};\mathfrak{U})}^2\,dt' \\
&\leq \frac{1}{2}\mathbb{E}\sup_{0\leq t\leq T} \|\mathfrak{v}_\delta(s)\|_{\mathfrak{U}}^2 + C\mathbb{E}\int_0^T \big\|I_\delta\Psi(t')\big\|_{HS(\mathscr{H};\mathfrak{U})}^2\,dt'.
\end{aligned}$$

Therefore,

$$\mathbb{E}\sup_{0\leq t\leq T} \|\mathfrak{v}_\delta(t)\|_{\mathfrak{U}}^2 \leq C\mathbb{E}\int_0^T \big\|I_\delta\Psi(t')\big\|_{HS(\mathscr{H};\mathfrak{U})}^2\,dt'.$$

Due to this inequality's linearity, we conclude that $(\mathfrak{v}_\delta)_{0<\delta<1}$ converges in $L^2(\Omega; \mathscr{C}([0,T]; \mathfrak{U}))$ as $\delta \to 0_+$, hence the validity of (A.18). The proposition is proven. □

Lastly, let us investigate the action of the semi-group $\mathfrak{S}(t)$ itself. Once the next proposition's verification follows closely the arguments of diagonalization we have displayed previously, we will omit it.

Proposition A.4. *Conserve the notations above, and let $\alpha \in \mathbb{R}$.*

1. *If $u_0 \in \mathfrak{U}^\alpha_\Lambda$, then $t \in [0,T] \mapsto \mathfrak{S}(t)u_0$ belongs to $\mathscr{C}([0,T];\mathfrak{U}^\alpha_\Lambda)$, and it holds that*

$$\max_{0\le t\le T} \|\mathfrak{S}(t)u_0\|^2_{\mathfrak{U}^\alpha_\Lambda} \le \|u_0\|^2_{\mathfrak{U}^\alpha_\Lambda}.$$

2. *Additionally, $t \in [0,T] \mapsto \mathfrak{S}(t)u_0$ also belongs to $L^2(0,T;\mathfrak{U}^{\alpha+1/2}_\Lambda)$, and it holds that*

$$\int_0^T \|\mathfrak{S}(t)u_0\|^2_{\mathfrak{U}^{\alpha+1/2}_\Lambda}\,dt \le C\|u_0\|^2_{\mathfrak{U}^\alpha_\Lambda} \tag{A.20}$$

   for some absolute constant C depending only on T and α.

3. *Therefore, if $u_0 \in L^2(\Omega;\mathfrak{U}^\alpha_\Lambda)$ is $\mathscr{F}_0$–mensurable, then $t \in [0,T] \mapsto \mathfrak{S}(t)u_0$ is predictable and lies in $L^2(\Omega;\mathscr{C}([0,T];\mathfrak{U}^\alpha_\Lambda)) \cap L^2(\Omega\times[0,T];\mathfrak{U}^{\alpha+1/2}_\Lambda))$.*

Inspired by the last propositions, let us now enunciate a result that explicates the reason for introducing the operators $\mathcal{I}_\Lambda$ and $\mathcal{I}_W$.

Proposition A.5. *Let $u \in L^2(\Omega;\mathscr{C}([0,T];\mathfrak{U})) \cap L^2(\Omega\times[0,T];\mathfrak{U}^{1/2}_\Lambda)$, $u_0 \in L^2(\Omega;\mathfrak{U})$ be $\mathscr{F}_0$–mensurable, and let $f \in L^2(\Omega\times[0,T];\mathfrak{U}^{-1/2}_\Lambda)$ and $\Psi \in L^2(\Omega\times[0,T];HS(\mathscr{H};\mathfrak{U}))$ be predictable. One of the following three statements implies the others two:*

(a) For all $\varphi \in \mathscr{C}^1_c((-\infty,T);\mathfrak{U}^{1/2}_\Lambda)$,

$$\begin{aligned}\int_0^T \big(\varphi'(t),u(t)\big)_{\mathfrak{U}}\,dt &= -\big(\varphi(0),u_0\big)_{\mathfrak{U}} + \int_0^T \big(\Lambda^{1/2}\varphi(t),\Lambda^{1/2}u(t)\big)_{\mathfrak{U}}\,dt\\ &\quad - \int_0^T \big(\varphi(t),f(t)\big)_{\mathfrak{U}}\,dt - \int_0^T \big(\varphi(t),\Psi(t)\,dW(t)\big)_{\mathfrak{U}} \text{ almost surely.}\end{aligned} \tag{A.21}$$

(b) u can be written as

$$u(t) = u_0 - \int_0^t \Lambda u(t')\,dt' + \int_0^t f(t')\,dt' + \int_0^t \Psi(t')\,dW(t') \text{ almost surely in } \mathfrak{U}^{-1/2}_\Lambda.$$

(c) u possesses the Duhamel representation formula

$$u(t) = \mathfrak{S}(t)u_0 + \int_0^t \mathfrak{S}(t-t')f(t')\,dt' + \int_0^t \mathfrak{S}(t-t')\Psi(t')\,dW(t') \text{ almost surely.}$$

Proof. Given $\phi \in \mathfrak{U}^{1/2}_\Lambda$, and $\psi \in \mathscr{C}^\infty_c(-\infty,T)$, plug $\varphi(t) = \psi(t)\phi$ in (A.21) so as to obtain

$$\begin{aligned}\int_0^T \psi'(t)\big(\phi,u(t)\big)_{\mathfrak{U}}\,dt &= -\psi(0)\big(\varphi,u_0\big)_{\mathfrak{U}} + \int_0^T \psi(t)\big(\Lambda^{1/2}\phi,\Lambda^{1/2}u(t)\big)_{\mathfrak{U}}\,dt\\ &\quad - \int_0^T \psi(t)\big(\phi,f(t)\big)_{\mathfrak{U}}\,dt - \int_0^T \psi(t)\big(\phi,\Psi(t)\,dW(t)\big)_{\mathfrak{U}}.\end{aligned}$$

Thus, picking any countable dense subset $(\tau_n)_{n\in\mathbb{N}}$ in $(0,T)$, and letting $\psi(t)$ be a smooth approximation of $t \mapsto 1_{(-\infty,\tau_n)}(t)$, one can apply the Itô isometry and the dominated convergence theorem to verify that

$$\begin{aligned}\big(\phi,u(t)\big)_{\mathfrak{U}} &= \big(\phi,u_0\big)_{\mathfrak{U}} - \int_0^t \big(\Lambda^{1/2}\phi,\Lambda^{1/2}u(s)\big)_{\mathfrak{U}}\,ds\\ &\quad + \int_0^t \big(\phi,f(s)\big)_{\mathfrak{U}}\,ds + \int_0^t \big(\phi,\Psi(s)\,dW(s)\big)_{\mathfrak{U}}\end{aligned}$$

almost surely whenever $t = \tau_n$ $(n = 1, 2, \ldots)$. Since the functions appearing in the equation above are almost surely continuous, it is indeed valid almost surely for all $0 \le t \le T$. This proves that (a) implies (b).

Assume (b) holds, and pick $\phi \in \mathfrak{U}^1_\Lambda$ and $0 < t \le T$. Letting $\varphi(s) = \mathfrak{S}(t-s)\phi$, one can see that

$$\begin{cases} \varphi \in \mathscr{C}^1([0,t]; \mathfrak{U}^1_\Lambda), \text{ with } \varphi'(s) = \Lambda\varphi(s), \\ \varphi(0) = \mathfrak{S}(t)\phi, \text{ and} \\ \varphi(t) = \phi. \end{cases}$$

As a result, one can easily verify via the Itô formula that, almost surely,

$$\begin{aligned} \big(\varphi(t), u(t)\big)_{\mathfrak{U}} &= \big(\varphi(0), u_0\big)_{\mathfrak{U}} + \int_0^t \big(\varphi(s), f(s)\big)_{\mathfrak{U}}\, ds + \int_0^t \big(\varphi(s), \Psi(s)\, dW(s)\big)_{\mathfrak{U}} \\ &= \big(\mathfrak{S}(t)\phi, u_0\big)_{\mathfrak{U}} + \int_0^t \big(\mathfrak{S}(t-s)\phi, f(s)\big)_{\mathfrak{U}}\, ds + \int_0^t \big(\mathfrak{S}(t-s)\phi, \Psi(s)\, dW(s)\big)_{\mathfrak{U}} \\ &= \big(\phi, \mathfrak{S}(t)u_0\big)_{\mathfrak{U}} + \left(\phi, \int_0^t \mathfrak{S}(t-s)f(s)\, ds\right)_{\mathfrak{U}} + \left(\phi, \int_0^t \mathfrak{S}(t-s)\Psi(s)\, dW(s)\right)_{\mathfrak{U}}, \end{aligned}$$

where we have applied the symmetry of the semigroup $\mathfrak{S}(t)$. For $\big(\varphi(t), u(t)\big)_{\mathfrak{U}} = \big(\phi, u(t)\big)_{\mathfrak{U}}$, and $\mathfrak{U}^1_\Lambda$ is dense in $\mathfrak{U}$, (c) is consequently shown.

Finally, by a very similar argument to the one appearing in the proof of Proposition A.3 (albeit in the opposite direction), (c) implies (a) as well. Thus, the proposition is proven. □

A.3 Proof of Theorem A.1

In order to apply the theory of the previous section, let us put $\mathfrak{U} = L^2(\mathscr{O})$, and introduce the operator $\Lambda : D(\Lambda) \subset L^2(\mathscr{O}) \to L^2(\mathscr{O})$

$$\begin{cases} D(\Lambda) = \left\{ u \in H^2(\mathscr{O}); \dfrac{\partial u}{\partial \nu} = 0 \text{ in the sense of traces in } L^2(\partial\mathscr{O}) \right\}, \text{ and} \\ \Lambda u = -\Delta u. \end{cases} \tag{A.22}$$

Theorem A.2. $\Lambda : D(\Lambda) \subset L^2(\mathscr{O}) \to L^2(\mathscr{O})$, *as defined above, is a nonnegative self-adjoint operator. Moreover,*

$$\mathfrak{U}^{1/2}_\Lambda = D(\Lambda^{1/2}) = H^1(\mathscr{O}) \textit{ isometrically,} \tag{A.23}$$

in the sense that

$$\big(\Lambda^{1/2}u, \Lambda^{1/2}v\big)_{L^2(\mathscr{O})} = \int_{\mathscr{O}} \nabla u \cdot \nabla v\, dx \textit{ for all } u \textit{ and } v \in H^1(\mathscr{O}). \tag{A.24}$$

Hence,

$$\mathfrak{U}^{-1/2}_\Lambda = (\mathfrak{U}^{1/2}_\Lambda)^\star = (H^1(\mathscr{O}))^\star \textit{ isometrically as well.} \tag{A.25}$$

Proof. Step #1: For any $f \in L^2(\mathscr{O})$, there exists a unique $u \in H^1(\mathscr{O})$ such that

$$\int_{\mathscr{O}} \nabla u \cdot \nabla \varphi\, dx + \int_{\mathscr{O}} u\varphi\, dx = \int_{\mathscr{O}} f\varphi\, dx \tag{A.26}$$

whenever $\varphi \in H^1(\mathscr{O})$.

Indeed, this is an application of Lax–Milgram theorem; see, *e.g.*, H. Brézis [13], proposition 9.24.

Step #2: The function $u \in H^1(\mathscr{O})$, characterized by (A.26), is actually in $H^2(\mathscr{O})$; furthermore, there exists a constant $C > 0$, depending only on $\mathscr{O}$, such that

$$\|u\|_{H^2(\mathscr{O})} \leq C\|f\|_{L^2(\mathscr{O})}. \tag{A.27}$$

For this fact, we refer again to H. Brézis [13], theorem 9.26.

Step #3: More precisely, the function $u \in H^1(\mathscr{O})$, characterized by (A.26), lies in $D(\Lambda)$.

First, pick $\varphi \in \mathscr{C}_c^\infty(\mathscr{O})$. An integration by parts in (A.26) implies that

$$\int_{\mathscr{O}} (-\Delta u + u - f)\varphi \, dx = 0, \tag{A.28}$$

hence $-\Delta u + u = f$ in the L^2–sense. Consequently, choosing $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^N)$, the same argument and (A.28) lead to

$$\int_{\mathscr{O}} f\varphi \, dx = \int_{\mathscr{O}} \nabla u \cdot \nabla\varphi \, dx + \int_{\mathscr{O}} u\varphi \, dx = \int_{\partial\mathscr{O}} \frac{\partial u}{\partial\nu}\varphi \, d\sigma + \int_{\mathscr{O}} (-\Delta u + u)\varphi \, dx,$$

in a fashion that $\int_{\partial\mathscr{O}} \frac{\partial u}{\partial\nu}\varphi \, d\sigma = 0$. Arguing by density, *a fortiori* $\frac{\partial u}{\partial\nu} = 0$ in the sense of traces.

Step #4: We are now in condition to verify that Λ is self-adjoint.

It is immediate to see that Λ is symmetric and nonnegative; in particular, $\|u\|_{L^2(\mathscr{O})} \leq \|\Lambda u + u\|_{L^2(\mathscr{O})}$ for all $u \in D(\Lambda)$. In virtue of (A.27), Λ is closed as well. Thus, according to the previous steps, $I + \Lambda : D(\Lambda) \to L^2(\mathscr{O})$ is invertible, *i.e.*, 1 is in the resolvent of $-\Lambda$. Consequently, the second corollary of theorem X.1 in M. Reed–B. Simon [98], vol. 2, and theorem 13.11 in W. Rudin [100] assure that Λ is self-adjoint.

Step #5: Finally, let us establish (A.23). Notice that, for all u and $v \in D(\Lambda)$,

$$\left(\Lambda^{1/2}u, \Lambda^{1/2}v\right)_{L^2(\mathscr{O})} = (\Lambda u, v)_{L^2(\mathscr{O})} = -\int_{\mathscr{O}} (\Delta u)v \, dx = \int_{\mathscr{O}} \nabla u \cdot \nabla v \, dx.$$

Therefore, once $D(\Lambda) \subset \mathfrak{U}_\Lambda^{1/2}$ densely, we see that $\mathfrak{U}_\Lambda^{1/2} \subset H^1(\mathscr{O})$ isometrically. In order to prove its equality, assume that $f \in H^1(\mathscr{O})$ is orthogonal to all $u \in \mathfrak{U}_\Lambda^{1/2}$. In particular, if $u \in D(\Lambda)$,

$$0 = \int_{\mathscr{O}} (\nabla u \cdot \nabla f + uf) \, dx = (\Lambda u + u, f)_{L^2(\mathscr{O})}.$$

Since $R(I + \Lambda) = L^2(\mathscr{O})$, $f = 0$; as a consequence, (A.23) and (A.24) follow. In contrast, (A.25) is a simple duality relation. The theorem is hereby proven. □

Recalling (A.2), let us state the following trivial assertions with a view to recovering the conclusions of Proposition A.5. Once their proofs are immediate, they will be omitted.

Proposition A.6. *Let* $\mathbf{A} : \mathbb{R} \to \mathbb{R}^N$ *and* $\Phi : L^2(\mathscr{O}) \to HS(\mathscr{H}; L^2(\mathscr{O}))$ *be as in Section 1 of this chapter. Then:*

1. *The mapping* $f : L^2(\mathscr{O}) \to H^1(\mathscr{O})^\star = \mathfrak{U}_\Lambda^{-1/2}$ *given by*

$$\left\langle f(u), \phi\right\rangle_{H^1(\mathscr{O})^\star, H^1(\mathscr{O})} = \int_{\mathscr{O}} \mathbf{A}(u) \cdot \nabla\phi \, dx$$

   is a well-defined Lipschitz continuous mapping.

2. *Likewise, the function* $\Phi : L^2(\mathscr{O}) \to HS(\mathscr{H}; L^2(\mathscr{O})) = HS(\mathscr{H}; \mathfrak{U})$ *is a well-defined Lipschitz continuous mapping.*

Therefore, fix $0 < \varepsilon < 1$, and put

$$\begin{aligned} \mathscr{E} &= L^2(\Omega; \mathscr{C}([0,T]; L^2(\mathscr{O}))) \cap L^2(\Omega \times [0,T]; H^1(\mathscr{O})) \\ &= L^2(\Omega; \mathscr{C}([0,T]; \mathfrak{U})) \cap L^2(\Omega \times [0,T]; \mathfrak{U}_\Lambda^{1/2}). \end{aligned}$$

In the light of Theorem A.2 (especially (A.24)), a comparison between Definition A.1 and Proposition A.5 yields that a function $u \in \mathscr{E}$ is weak solution to (A.1) if, and only if, u is a fixed point to the operator $\mathcal{K} : \mathscr{E} \to \mathscr{E}$ defined by

$$(\mathcal{K}v)(t) = \mathfrak{S}(t)u_0 + \int_0^t \mathfrak{S}(t-t')f(u(t'))\,dt' + \int_0^t \mathfrak{S}(t-t')\Phi(u(t'))\,dW(t') \quad \big[v \in \mathscr{E}, 0 \le t \le T\big],$$

where $\mathfrak{S}(t)$ is the contraction semigroup associated to $\varepsilon\Lambda$, and f is as in the previous proposition. Notice that Propositions A.2–A.4, in conjunction with Theorem A.2, assert that not only is $\mathcal{K} : \mathscr{E} \to \mathscr{E}$ well-defined but it is indeed continuous.

Lemma A.1. *There exists an universal constant $B = B(\varepsilon) > 0$ such that, for all v_1 and $v_2 \in \mathscr{E}$, and $0 \le t \le T$,*

$$\mathbb{E} \sup_{0\le s\le t} \left\{ \big\|(\mathcal{K}v_1)(s) - (\mathcal{K}v_2)(s)\big\|^2_{L^2(\mathscr{O})} + \int_0^s \varepsilon \big\|\nabla_x(\mathcal{K}v_1)(s) - \nabla(\mathcal{K}v_2)(s)\big\|^2_{L^2(\mathscr{O})}\,ds' \right\} \le B\,\mathbb{E}\int_0^t \big\|v_1(s) - v_2(s)\big\|^2_{L^2(\mathscr{O})}\,ds \tag{A.29}$$

Proof. The reasoning here is very similar to the one applied in the second half of the proof of Proposition A.3; nevertheless, we will repeat the main line of argument to fix ideas. Let I_δ be the regularizing operators as in such a proof. According to Proposition A.5, we see that, for all v_1 and $v_2 \in \mathscr{E}$, $0 \le t' \le t \le T$, and all $0 < \delta < 1$,

$$I_\delta(\mathcal{K}v_1)(t') - I_\delta(\mathcal{K}v_2)(t') = -\varepsilon \int_0^{t'} \big(\Lambda I_\delta(\mathcal{K}v_1)(s) - \Lambda I_\delta(\mathcal{K}v_2)(s)\big)\,ds + \int_0^{t'} \big((I_\delta f)(v_1(s)) - (I_\delta f)(v_2(s))\big)\,ds + \int_0^{t'} \big((I_\delta\Phi)(v_1(s)) - (I_\delta\Phi)(v_2(s))\big)\,dW(s).$$

Hence, applying Itô formula with $F(Y) = \|Y\|^2_{L^2(\mathscr{O})}$,

$$\begin{aligned}\big\|I_\delta(\mathcal{K}v_1)(t') - I_\delta(\mathcal{K}v_2)(t')\big\|^2_{L^2(\mathscr{O})} &= -\varepsilon \int_0^{t'}\int_{\mathscr{O}} \big|(\nabla I_\delta \mathcal{K}v_1)(s,x) - (\nabla I_\delta K v_2)(s,x)\big|^2\,dxds \\ &+ \int_0^{t'}\int_{\mathscr{O}} \big(\mathbf{A}(v_1(s,x)) - \mathbf{A}(v_2(s,x))\big)\cdot\nabla_x\big(I_\delta^2(\mathcal{K}v_1)(s,x) - (I_\delta^2\mathcal{K}v_2)(s,x)\big)\,dxds \\ &+ \sum_{k=1}^\infty \int_0^{t'}\int_{\mathscr{O}} \big(g_k(x,v_1(s,x)) - g_k(x,v_2(s,x))\big)\big(I_\delta^2(\mathcal{K}v_1)(s,x) - (I_\delta^2\mathcal{K}v_2)(s,x)\big)\,dxd\beta_k(s) \\ &+ \sum_{k=1}^\infty \int_0^{t'}\int_{\mathscr{O}} \big((I_\delta g_k)(x,v_1(s,x)) - (I_\delta g_k)(s,v_2(s,x))\big)^2\,dxds.\end{aligned}$$

As a result, passing $\delta \to 0_+$, and employing the Itô isometry and the Young inequality,

$$\begin{aligned}&\big\|(\mathcal{K}v_1)(t') - (\mathcal{K}v_2)(t')\big\|^2_{L^2(\mathscr{O})} + \varepsilon\int_0^{t'}\int_{\mathscr{O}} \big|(\nabla\mathcal{K}v_1)(s,x) - (\nabla K v_2)(s,x)\big|^2\,dxds \\ &\le C\int_0^{t'} \big\|v_1(s) - v_2(s)\big\|^2_{L^2(\mathscr{O})}\,ds + \sum_{k=1}^\infty \int_0^{t'}\int_{\mathscr{O}} \big(g_k(x,v_1) - g_k(x,v_2)\big)(\mathcal{K}v_1 - \mathcal{K}v_2)\,dxd\beta_k(s)\end{aligned}$$

almost surely. Finally, take the supremum for $0 \le t' \le t$ in the expression above. As the Burkholder inequality and (A.4) assert that

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t'\le t}\left|\sum_{k=1}^{\infty}\int_0^{t'}\int_{\mathscr{O}}\big(g_k(x,v_1)-g_k(x,v_2)\big)(\mathcal{K}v_1-\mathcal{K}v_2)\,dxd\beta_k(s)\right|\\
&\quad\le C\mathbb{E}\left[\left(\int_0^{t'}\sum_{k=1}^{\infty}\left|\int_{\mathscr{O}}(g_k(x,v_1)-g_k(x,v_2(x)))((\mathcal{K}v_1)(s,x)-(\mathcal{K}v_2)(s,x))\,dx\right|^2 ds\right)^{1/2}\right]\\
&\quad\le C\mathbb{E}\left[\sup_{0\le t'\le t}\big\|\mathcal{K}v_1(t')-\mathcal{K}v_2(t')\big\|_{L^2(\mathscr{O})}\left\{\int_0^{t'}\|v_1(s)-v_2(s)\|^2_{L^2(\mathscr{O})}\,ds\right\}^{1/2}\right],
\end{aligned}$$

the desired estimate (A.29) follows immediately from another usage of the Young inequality. □

Theorem A.3 (Existence of solutions)**.** *$\mathcal{K}:\mathscr{E}\to\mathscr{E}$ possesses a unique fixed point, which may be obtained via the method of successive approximations. Consequently,* (A.1) *has a unique solution.*

Proof. Let $0<\alpha<1$ be arbitrary, and define the equivalent norm in $\mathscr{E}$ given by

$$\|u\|^2_{*\mathscr{E}}=\sup_{0\le t\le T}e^{-Bt/\alpha}\left[\mathbb{E}\sup_{0\le s\le t}\left\{\|u(s)\|^2_{L^2(\mathscr{O})}+\int_0^s\varepsilon\|\nabla_x u(s')\|^2_{L^2(\mathscr{O})}\,ds'\right\}\right].$$

Accordingly, (A.29) asserts that, given any two v_1 and $v_2\in\mathscr{E}$,

$$\begin{aligned}
\|\mathcal{K}v_1-\mathcal{K}v_2\|^2_{*\mathscr{E}}&\le\sup_{0\le t\le T}e^{-Bt/\alpha}\left[B\mathbb{E}\int_0^t\big\|v_1(s)-v_2(s)\big\|^2_{L^2(\mathscr{O})}\,ds\right]\\
&\le\sup_{0\le t\le T}e^{-Bt/\alpha}\left[B\int_0^t e^{Bs/\alpha}\left\{e^{-Bs/\alpha}\mathbb{E}\sup_{0\le s\le t}\|v_1(s)-v_2(s)\|^2_{L^2(\mathscr{O})}\right\}ds\right]\\
&\le\|v_1-v_2\|^2_{*\mathscr{E}}\sup_{0\le t\le T}\left[e^{-Bt/\alpha}B\int_0^t e^{Bs/\alpha}\,ds\right]\\
&\le\alpha\|v_1-v_2\|^2_{*\mathscr{E}}.
\end{aligned}$$

Because $0<\alpha<1$, $\mathcal{K}$ is a contraction under the new norm $\|\quad\|_{*\mathscr{E}}$. The desired conclusion is now a corollary to the classical Banach fixed point theorem. □

So as to obtain the other properties of the solution $u(t,x)$ to (A.1), let us state two chain rules, whose demonstrations are completely parallel to the proof of Lemma A.1.

Lemma A.2. *Let $\eta\in\mathscr{C}^2(\mathbb{R})$ be such that $\eta''\in L^\infty(\mathbb{R})$. Furthermore, let u and v be solutions to* (A.1) *with initial data, respectively, u_0 and $v_0\in L^\infty(\Omega\times\mathscr{O})$, where both of the latter are $\mathscr{F}_0$–mensurable.*

Then:

(a) It holds almost surely that, for all $\varphi\in\mathscr{C}^1(\overline{\mathscr{O}})$ and $0\le t\le T$:

$$\begin{aligned}
\int_{\mathscr{O}}\eta(u(t,x))\varphi(x)\,dx&=\int_{\mathscr{O}}\eta(u_0(x))\varphi(x)\,dx-\varepsilon\int_0^t\int_{\mathscr{O}}\nabla_x\eta(u(s,x))\cdot\nabla_x\varphi(x)\,dxds\\
&+\int_0^t\int_{\mathscr{O}}\mathbf{A}(u(s,x))\cdot\nabla_x\big(\eta'(u(s,x))\varphi(x)\big)\,dxds+\int_0^t\int_{\mathscr{O}}\eta'(u(s,x))\Phi(x,u(s,x))\varphi(x)\,dW(s)\\
&+\int_0^t\eta''(u(s,x))\left\{\frac{1}{2}\mathfrak{G}^2(u(s,x))-\varepsilon|\nabla_x u(s,x)|^2\right\}\varphi(x)\,dxds.
\end{aligned}\tag{A.30}$$

(b) It holds almost surely that, for all $\varphi \in \mathscr{C}^1(\overline{\mathscr{O}})$ and $0 \le t \le T$:

$$\begin{aligned}
\int_{\mathscr{O}} \eta(u(t,x) - v(t,x))\varphi(x)\,dx &= \int_{\mathscr{O}} \eta(u_0(x) - v_0(x))\varphi(x)\,dx \\
&- \varepsilon \int_0^t \int_{\mathscr{O}} \big(\nabla_x u(s,x) - \nabla_x v(s,x)\big) \cdot \nabla_x \big(\eta'(u(s,x) - v(s,x))\varphi(x)\big)\,dxds \\
&+ \int_0^t \int_{\mathscr{O}} \big(\mathbf{A}(u(x,x)) - \mathbf{A}(v(s,x))\big) \cdot \nabla_x \big(\eta'(u(s,x) - v(s,x))\varphi(x)\big)\,dxds \\
&+ \int_0^t \int_{\mathscr{O}} \eta'(u(s,x) - v(s,x))\big(\Phi(x,u(s,x)) - \Phi(s,x)\big)\varphi(x)\,dW(s) \\
&+ \frac{1}{2}\sum_{k=1}^{\infty} \int_0^t \eta''(u(s,x) - v(s,x))\big(g_k(x,u(s,x)) - g_k(x,u(s,x))\big)^2 \varphi(x)\,dxds. \qquad \text{(A.31)}
\end{aligned}$$

Notice that the entropy formulation (A.12) and the kinetic formulation (A.13) can be extracted from (A.30) by a standard argument—see Propositions 2.7 or A.5. With (A.31) at hand, let us prove the comparison principle.

Theorem A.4 (The comparison principle)**.** *Let u and v be solutions to* (A.1) *with initial data, respectively, u_0 and $v_0 \in L^\infty(\Omega \times \mathscr{O})$, where both of the latter are $\mathscr{F}_0$–mensurable. Then, for all $0 \le t \le T$,*

$$\mathbb{E} \int_{\mathscr{O}} \big(u(t,x) - v(t,x)\big)_+ \,dx \le \mathbb{E} \int_{\mathscr{O}} \big(u_0(x) - v_0(x)\big)_+ \,dx. \qquad \text{(A.32)}$$

Proof. Let $\psi \in \mathscr{C}_c^\infty(-\infty,\infty)$ be such that $\psi \ge 0$, $\operatorname{supp}\psi \subset (-1,1)$, and $\int_{-\infty}^{\infty} \psi(w)\,dw = 1$. If $\psi_\delta(v) = \frac{1}{\delta}\psi(\delta^{-1}v)$ ($\delta > 0$), put

$$\operatorname{sign}_\delta^+(u) = \int_{-\infty}^{u} \psi_\delta(w)\,dw.$$

Define also $\eta_\delta(u) = \int_{-\infty}^{u} \operatorname{sign}_\delta^+(v)\,dv$. Notice that η_δ is a smooth convex approximation of the "positive part" function $u \mapsto u_+$.

Plug such $\eta(v) = \eta_\delta(v)$ and $\varphi(x) \equiv 1$ on (A.31), so as to obtain

$$\begin{aligned}
\int_{\mathscr{O}} \eta_\delta(u(t,x) - v(t,x))\,dx &= \int_{\mathscr{O}} \eta_\delta(u_0(x) - v_0(x))\,dx - \varepsilon \int_0^t \int_{\mathscr{O}} \eta_\delta''(u-v)|\nabla_x u - \nabla_x v|^2\,dxds \\
&+ \int_0^t \int_{\mathscr{O}} \eta_\delta''(u-v)(\mathbf{A}(u) - \mathbf{A}(v)) \cdot (\nabla_x u - \nabla_x v)\,dxds + \int_0^t \int_{\mathscr{O}} \eta_\delta'(u-v)\big(\Phi(u) - \Phi(v)\big)\,dW(s) \\
&+ \frac{1}{2}\sum_{k=1}^{\infty} \int_0^t \int_{\mathscr{O}} \eta_\delta''(u-v)\big(g_k(x,u) - g_k(x,v)\big)^2\,dxds \qquad \text{(A.33)}
\end{aligned}$$

almost surely for any $0 \le t \le T$. First of all, since $\eta_\delta'' \ge 0$, it holds that

$$-\varepsilon\mathbb{E} \int_0^t \int_{\mathscr{O}} \eta_\delta''(u-v)|\nabla_x u - \nabla_x v|^2\,dxds \le 0. \qquad \text{(A.34)}$$

Moreover, notice that

$$\begin{aligned}
\big|\eta_\delta''(u-v)(\mathbf{A}(u) - \mathbf{A}(v)) \cdot (\nabla_x u - \nabla_x v)\big| &\le \|\mathbf{A}'\|_{L^\infty} \psi\Big(\frac{u-v}{\delta}\Big)\Big|\frac{u-v}{\delta}\Big||\nabla_x u - \nabla_x v| \\
&\le C|\nabla_x u - \nabla_x v| \in L^1(\Omega \times \mathscr{O}),
\end{aligned}$$

and $\big|\eta_\delta''(u-v)(\mathbf{A}(u) - \mathbf{A}(v)) \cdot (\nabla_x u - \nabla_x v)\big| \to 0$ everywhere as $\delta \to 0_+$. Therefore, the dominated convergence theorem guarantees that

$$\mathbb{E}\int_0^t\int_{\mathscr{O}}\eta_\delta''(u-v)(\mathbf{A}(u)-\mathbf{A}(v))\cdot(\nabla_x u-\nabla_x v)\,dxds\to 0 \text{ as } \delta\to 0_+. \tag{A.35}$$

Once the properties of the stochastic integral yield that

$$\mathbb{E}\int_0^t\int_{\mathscr{O}}\eta_\delta'(u-v)\big(\Phi(u)-\Phi(v)\big)\,dW(s)=0, \tag{A.36}$$

it only remains to investigate the last term in (A.33). However, as we have argued before,

$$\begin{aligned}\mathbb{E}\frac{1}{2}\sum_{k=1}^{\infty}\int_0^t\eta_\delta''(u-v)\big(g_k(x,u)-g_k(x,u)\big)\big)^2\,dxds &\le C\mathbb{E}\int_0^t\int_{\mathscr{O}}\psi\Big(\frac{u-v}{\delta}\Big)\frac{(u-v)^2}{\delta}\,dxds\\ &\le C\delta\to 0 \text{ as } \delta\to 0_+.\end{aligned} \tag{A.37}$$

Agglutinating (A.34)–(A.37) and amalgamating their conclusions with (A.33), we deduce (A.32). The theorem is hereby proven. □

Corollary A.1. *Let $u(t,x)$ be a solutions to (A.1) with initial data u_0 satisfying the mensurability and boundedness conditions expressed in the first section of this chapter. Then, almost surely,*

$$a\le u(t,x)\le b \text{ in } \mathscr{D}'((0,T)\times\mathscr{O}).$$

Proof. It suffices to observe that the constant states $(\omega,t,x)\mapsto a$ and $(\omega,t,x)\mapsto b$ are solutions to (A.1) in virtue of (A.3) and (A.6). Consequently, the result follows from the comparison principle and the hypothesis in (A.8). □

We close this chapter with the proof of the energy estimate in (A.11).

Proposition A.7. *Let $u(t,x)$ be a solutions to (A.1) with initial data u_0 satisfying the mensurability and boundedness conditions expressed in the first section of this chapter. Then, for all $1\le p<\infty$,*

$$\begin{aligned}&\mathbb{E}\bigg[\Big(\int_0^T\int_{\mathscr{O}}\varepsilon|\nabla_x u(t,x)|^2\,dxdt\Big)^p\bigg]\\ &\qquad\le C\bigg(p,a,b,T,\sup_{a\le v\le b}|\mathbf{A}(v)|,\mathbb{E}\bigg[\Big(\int_0^T\int_{\mathscr{O}}\mathfrak{G}^2(x,u(t,x))\,dxdt\Big)^{p/2}\bigg]\bigg).\end{aligned}$$

Proof. Let $\eta(v)=\frac{1}{2}v^2$ and $\varphi(x)\equiv 1$ in (A.30), and apply the Burkholder inequality to obtain

$$\begin{aligned}&\mathbb{E}\bigg[\Big(\int_0^T\int_{\mathscr{O}}\varepsilon|\nabla_x u(s,x)|^2\,dxds\Big)^p\bigg]\\ &\quad\le C_p\mathbb{E}\bigg[\Big(\int_{\mathscr{O}}u_0(x)^2\,dx\Big)^p+\Big|\int_0^T\int_{\mathscr{O}}\mathbf{A}(u)\cdot\nabla u\,dxds\Big|^p+\Big(\int_0^T\int_{\mathscr{O}}\mathfrak{G}^2(x,u)\,dxds\Big)^{p/2}\bigg].\end{aligned}$$

The only term above we need to be preoccupied with is the hyperbolic term, as the Young inequality may not be applied here. Nevertheless, if one lets $\mathbf{G}(v)=\int_a^v\mathbf{A}(w)\,dw$, the divergence theorem asserts that

$$\int_0^T\int_{\mathscr{O}}\mathbf{A}(u)\cdot\nabla u\,dxds=\int_0^T\int_{\mathscr{O}}\operatorname{div}_x\mathbf{G}(u)\,dxds=\int_0^T\int_{\partial\mathscr{O}}\mathbf{G}(u)\cdot\nu\,d\sigma ds$$

The desired conclusion can be now obtained in a routine fashion from the L^∞–bound. □

Appendix B

The Sobolev regularity of entropy solutions to a parabolic–hyperbolic equation

Based on the ideas exposed in this thesis, let us revisit degenerate parabolic–hyperbolic equation (1.7)

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left\{\frac{1}{\ell+1}u^{\ell+1}\right\}-\frac{\partial^2}{\partial y^2}\left\{\frac{1}{n+1}|u|^n u\right\}=0, \tag{B.1}$$

and demonstrate that one can completely dismiss the artificial constraint that $n\geq 2\ell$ previously imposed in E. Tadmor–T. Tao [107]. Henceforward, we will employ the notations, definitions, and conventions of Chapter 3.

Theorem B.1. *Let $N_h\geq 1$ and $N_p\geq 1$ be integers, $Q\subset\mathbb{R}_t\times\mathbb{R}_x^{N_h}\times\mathbb{R}_y^{N_p}$ be an open set, $\mathbf{A}\in\mathscr{C}^{2,\alpha}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{N_h})$ and $\mathbf{B}\in\mathscr{C}^{2,\alpha}_{\mathrm{loc}}(\mathbb{R};\mathscr{L}(\mathbb{R}^{N_p}))$ for some $0<\alpha\leq 1$. Put $\mathbf{a}(v)=\mathbf{A}'(v)$ and $\mathbf{b}(v)=\mathbf{B}'(v)$, and suppose that $\mathbf{b}(v)\geq 0$ everywhere.*

Let $u=u(t,x,y)\in L^\infty(Q)$ be an entropy solution to

$$\frac{\partial u}{\partial t}(t,x,y)+\mathrm{div}_x\mathbf{A}\big(u(t,x,y)\big)-D_y^2:\mathbf{B}(u(t,x,y))=0 \text{ in } \mathscr{D}'(Q), \tag{B.2}$$

and let $a\leq b$ be such that

$$a\leq u(t,x,y)\leq b \text{ in } \mathscr{D}'(Q).$$

Finally, assume that there exist some $\eta\in\mathscr{C}_c^\infty(\mathbb{R}_v)$, and exponents $0\leq\mathfrak{e}_h$ and $\mathfrak{e}_p\leq 1$ such that $\eta(v)=1$ for $a\leq v\leq b$, and, for all $\delta>0$,

$$\begin{cases}\mathrm{meas}\Big\{v\in\mathrm{supp}\,\eta;\ |\tau+\mathbf{a}(v)\cdot\kappa_h|\leq\delta\Big\}\leq C\delta^{\mathfrak{e}_h}\\ \qquad\qquad\text{for all } (\tau,\kappa_h)\in\mathbb{R}\times\mathbb{R}^{N_h} \text{ with } \tau^2+|\kappa_h|^2=1, \text{ and}\\ \mathrm{meas}\Big\{v\in\mathrm{supp}\,\eta;\ \kappa_p\cdot\mathbf{b}(v)\kappa_p\leq\delta\Big\}\leq C\delta^{\mathfrak{e}_p}\\ \qquad\qquad\text{for all } \kappa_p\in\mathbb{R}^{N_p} \text{ with } |\kappa_p|^2=1.\end{cases} \tag{B.3}$$

Then, for any open sets $\mathscr{U}_h\subset\mathbb{R}_t\times\mathbb{R}_x^{N_h}$ and $\mathscr{U}_p\subset\mathbb{R}_y^{N_p}$ such that $\mathscr{U}_h\times\mathscr{U}_p\subset\subset Q$, we have that

$$u\in L^{\mathfrak{r}_p}(\mathscr{U}_h;W^{s_p,\mathfrak{r}_p}(\mathscr{U}_p))\cap L^{\mathfrak{r}_h}(\mathscr{U}_p;W^{s_h,\mathfrak{r}_h}(\mathscr{U}_h))$$

with

$$\|u\|_{L^{\mathfrak{r}_p}(\mathscr{U}_h;W^{s_p,\mathfrak{r}_p}(\mathscr{U}_p))}+\|u\|_{L^{\mathfrak{r}_h}(\mathscr{U}_p;W^{s_h,\mathfrak{r}_h}(\mathscr{U}_h))}\leq C_{\mathscr{U}_p,\mathscr{U}_h,s_p,s_h}(a,b),$$

where the exponents satisfy

$$\begin{cases} 0 \le s_p < \mathfrak{s}_p = \dfrac{2\mathfrak{e}_h\mathfrak{e}_p}{2\mathfrak{e}_h + \mathfrak{e}_p(\mathfrak{e}_h+1)}, & \mathfrak{r}_p = \dfrac{4\mathfrak{e}_h + \mathfrak{e}_p(2+\mathfrak{e}_h)}{2\mathfrak{e}_h + \mathfrak{e}_p(1+\mathfrak{e}_h)}, \\ 0 \le s_h < \mathfrak{s}_h = \dfrac{\mathfrak{e}_h\mathfrak{e}_p}{2\mathfrak{e}_p + \mathfrak{e}_h(\mathfrak{e}_p+1)}, \ \textit{and} & \mathfrak{r}_h = \dfrac{4\mathfrak{e}_p + \mathfrak{e}_h(2+\mathfrak{e}_p)}{2\mathfrak{e}_p + \mathfrak{e}_h(1+\mathfrak{e}_p)}. \end{cases} \tag{B.4}$$

(Informally speaking, "$u(t,x,y)$ has s_p $L^{\mathfrak{r}_p}$–derivatives in y and s_h $L^{\mathfrak{r}_h}$–derivatives in (t,x)".) In particular, for $\mathfrak{s} = \min\{s_h, s_p\}$ and $\mathfrak{r} = \min\{\mathfrak{r}_h, \mathfrak{r}_p\}$, $u \in W^{\mathfrak{s},\mathfrak{r}}_{\mathrm{loc}}(Q)$.

Notice that Equation (B.2) falls in the cathegory considered in the "one-phase" averaging lemma of Chpater 2, assuming, without loss of generality, that $M = \{0\} \times \mathbb{R}^{N_p} \subset \mathbb{R}^{N_h} \times \mathbb{R}^{N_p}$.

Proof. Our argument will be deeply inspired by the one we have already made use of in Theorem 5.12. For simplicity, one may assume that, for some $L > 0$,

$$a = -L \text{ and } b = L.$$

Step #1: The localization procedure. Let $f : Q \times \mathbb{R}_v \to \mathbb{R}$ be the χ–function associated to $u(t,x,y)$, and let $\mathbf{m}(t,x,y,v)$ be a σ–finite Borel measure such that

$$\frac{\partial f}{\partial t} + \mathbf{a}(v)\cdot\nabla_x f - \mathbf{b}(v) : D^2_y f = \frac{\partial \mathbf{m}}{\partial v} \text{ in } \mathscr{D}'(Q \times \mathbb{R}_v). \tag{B.5}$$

Since $u \in L^\infty(Q)$, we can and will assume that $\operatorname{supp}\mathbf{m} \subset Q \times [-L, L]$. Given any $\mathscr{U}_h \subset \mathbb{R}_t \times \mathbb{R}^{N_h}_x$ and $\mathscr{U}_p \subset \mathbb{R}^{N_p}_y$ such that $\mathscr{O} = \mathscr{U}_h \times \mathscr{U}_p \subset\subset Q$, pick some nonnegative $\varphi \in \mathscr{C}^\infty_c(Q)$ with $\varphi(t,x,y) = 1$ in $\mathscr{O}$, and some nonnegative $\phi \in \mathscr{C}^\infty_c(Q)$ such that $\phi(t,x,y) = 1$ in $\operatorname{supp}\varphi$. If $\zeta \in \mathscr{C}^\infty_c(\mathbb{R}_v)$ is nonnegative everywhere and $\zeta(v) = 1$ for $-L \le v \le L$, then $\mathfrak{f}(t,x,y,v) = \varphi(t,x,y)\zeta(v)^2 f(t,x,y,v)$ obeys the equation

$$\begin{aligned} \frac{\partial \mathfrak{f}}{\partial t} + \mathbf{a}(v)\cdot\nabla_x\mathfrak{f} - \mathbf{b}(v) : D^2_y\mathfrak{f} = \frac{\partial(\varphi\zeta^2\mathbf{m})}{\partial v} + \zeta^2 f\left(\frac{\partial\varphi}{\partial t} + \mathbf{a}(v)\cdot\nabla_x\varphi - \mathbf{b}(v) : D^2_y\varphi\right) \\ + 2\nabla_y(\varphi\zeta)\cdot \operatorname{div}_y(\zeta(v)\sigma(v)\sigma(v)f) \end{aligned} \tag{B.6}$$

where $\sigma(v) = \mathbf{b}(v)^{1/2}$. Of course, $(\varphi\zeta^2\mathbf{m})$ is a bounded Borel measure in $\mathbb{R}_t \times \mathbb{R}^{N_h}_x \times \mathbb{R}^{N_p}_y \times \mathbb{R}_v$, and so is the second term $\zeta f\big(\frac{\partial\varphi}{\partial t} + \mathbf{a}(v)\cdot\nabla_x\varphi - \mathbf{b}(v) : D^2_y\varphi\big) \in L^1(\mathbb{R}_t \times \mathbb{R}^{N_h}_x \times \mathbb{R}^{N_p}_y \times \mathbb{R}_v)$. Similarly, since

$$2\nabla_y(\varphi\zeta)\cdot\nabla_y(\zeta(v)\sigma(v)\sigma(v)f) = 2\nabla_y(\varphi\zeta)\cdot\frac{\partial}{\partial v}\left\{\operatorname{div}_y\left(\int_{-\infty}^v \zeta(w)\sigma(w)\sigma(w)f(t,x,w)\,dw\right)\right\},$$

the chain-rule (3.29) implies that $2\nabla_y(\varphi\zeta)\cdot\nabla_y(\zeta(v)\sigma(v)\sigma(v)f)$ can be also thought as some derivative in v of a measure.

Therefore, Theorem 2.5 and the Morrey's theorem assert that, for any $0 < \varepsilon < \min\{\alpha, \mathfrak{e}_h, \mathfrak{e}_p\}/5$ and $1 < q_\varepsilon < \frac{N+2}{N+2-\varepsilon}$, there exists some $\mathbf{F}^{(\varepsilon)} \in L^{q_\varepsilon}(\mathbb{R}_t \times \mathbb{R}^{N_h}_x \times \mathbb{R}^{N_p}_y \times \mathbb{R}_v)$ such that

$$\frac{\partial\mathfrak{f}}{\partial t} + \mathbf{a}(v)\cdot\nabla_x\mathfrak{f} - \mathbf{b}(v) : D^2_y\mathfrak{f} = (-\Delta_{t,x} + \Delta^2_y + 1)^{\varepsilon/2}\left(\frac{\partial}{\partial v}(-\Delta_v)^{\varepsilon/2} + 1\right)\mathbf{F}^{(\varepsilon)} \text{ in } \mathscr{D}'(Q\times\mathbb{R}),$$

and observing

$$\big\|\mathbf{F}^{(\varepsilon)}\big\|_{L^{q_\varepsilon}_{t,x,y,v}} \le C_{\varepsilon,\mathscr{O}}\left\{\big\|\phi\mathbf{m}\big\|_{\mathfrak{M}_{t,x,y,v}} + \left\|f\left(\frac{\partial\varphi}{\partial t} + \mathbf{a}(v)\cdot\nabla_x\varphi - \mathbf{b}(v) : D^2_y\varphi\right)\right\|_{L^1_{t,x,y,v}}\right\}. \tag{B.7}$$

Step #2.1: The Littlewood–Paley decomposition. Let us now employ the same scheme of proof of Theorem 5.12. The difference here is that we will apply two Littlewood–Paley decompositions:

one in the "hyperbolic" (t, x)–variables, and other in the "parabolic" y–variables.

Let $\psi_1(z)$ and $\psi_2(z) \in \mathscr{C}_c^\infty(\mathbb{C}; \mathbb{R})$ be again such that

1. $\operatorname{supp} \psi_0 \subset \{|z| \le 1\}$, and $\psi_0(z) \ge 0$ everywhere,
2. $\operatorname{supp} \psi_1 \subset \{\frac{1}{2} \le |z| \le 2\}$, and $\psi_1(z) \ge 0$ everywhere,
3. for all $z \in \mathbb{C}$,
$$\psi_0(z) + \sum_{m=0}^{\infty} \psi_1(2^{-m} z) = 1, \text{ and} \tag{B.8}$$
4. for all $z \in \mathbb{C} \setminus \{0\}$,
$$\sum_{m=-\infty}^{\infty} \psi_1(2^{-m} z) = 1. \tag{B.9}$$

Thus, if J denotes a dyadic number, and $\psi_J(z)$ is defined as $\psi_J(z) \overset{\text{def}}{=} \psi_2(J^{-1} z)$, (B.8) and (B.9) are then translated into

$$\begin{cases} \psi_0(z) + \displaystyle\sum_{J \text{ dyadic}, J \ge 1} \psi_J(z) = 1 & \text{for all } z \in \mathbb{C}, \text{ and} \\ \displaystyle\sum_{J \text{ dyadic}} \psi_J(z) = 1 & \text{for all } z \neq 0. \end{cases}$$

Let $(\tau, \kappa_h) \in \mathbb{R} \times \mathbb{R}^{N_h}$ be the frequency variables associated to (t, x), and $\kappa_p \in \mathbb{R}^{N_p}$ be the one related to y. Henceforth, let us understand by $\mathscr{J}$ the set $\{J; J = 0 \text{ or } J \text{ is a dyadic number } \ge 1\}$. Given any tempered distribution $\Lambda \in \mathscr{S}'(\mathbb{R}_t \times \mathbb{R}_x^{N_h} \times \mathbb{R}_y^{N_p} \times \mathbb{R}_v)$, and any J_h and $J_p \in \mathscr{J}$, write

$$\Lambda_{J_h, J_p}(t, x, y, v) = \mathfrak{F}_{t,x,y}^{-1}\Big[\psi_{J_h}(\sqrt{\tau^2 + |\kappa_h|^2})\psi_{J_p}(|\kappa_p|)\,(\mathfrak{F}_{t,x,y}\Lambda)(\tau, \kappa_h, \kappa_p, v)\Big],$$

so that

$$\Lambda = \sum_{J_p \text{ and } J_p \in \mathscr{J}} \Lambda_{J_h, J_p}.$$

Since all the symbols $(\tau, \kappa_h) \mapsto \sqrt{\tau^2 + |\kappa_h|^2}$ and $\kappa_p \mapsto |\kappa_p|$ have the truncation property (Proposition 2.3), the functions $\mathfrak{f}_{J_h, J_p}$ all lie in L^p for all $1 \le p \le \infty$. Furthermore, notice that each $\mathfrak{f}_{J_h, J_p}$ is governed by the equation

$$\begin{aligned} &\frac{\partial \mathfrak{f}_{J_h, J_p}}{\partial t} + \mathbf{a}(v) \cdot \nabla_x \mathfrak{f}_{J_h, J_p} - \mathbf{b}(v) : D_y^2 \mathfrak{f}_{J_h, J_p} \\ &\qquad = (-\Delta_{t,x} + \Delta_y^2 + 1)^{\varepsilon/2}\left(\frac{\partial}{\partial v}(-\Delta_v)^{\varepsilon/2} + 1\right)\mathbf{F}_{J_h, J_p}^{(\varepsilon)} \text{ in } \mathscr{D}'(Q \times \mathbb{R}). \end{aligned} \tag{B.10}$$

Likewise, we have as before that

$$\begin{cases} \varphi(t, x, y) u(t, x, y) = \displaystyle\int_{\mathbb{R}} \mathfrak{f}(t, x, y, v) \eta(v)\, dv, \\ (\varphi u)_{J_h, J_p} = \displaystyle\int_{\mathbb{R}} \mathfrak{f}_{J_h, J_p} \eta \, dv. \end{cases}$$

Per Theorem 5.13, in order to secure the Sobolev regularity of φu, one needs to estimate the L^r–norm of functions

$$\begin{cases} (\varphi u)_{\cdot, J_p} = \displaystyle\sum_{J_h \in \mathscr{J}} (\varphi u)_{J_h, J_p}, \text{ and} \\ (\varphi u)_{J_h, \cdot} = \displaystyle\sum_{J_p \in \mathscr{J}} (\varphi u)_{J_h, J_p}, \end{cases}$$

and this is what we are going to do now.

Step #2.2: The Littlewood–Paley decomposition, part II. So as to better understand the behavior of Equation (B.10), let us also introduce a second partition of the frequency space in terms of the symbols of Equation (B.10). Define

$$\begin{cases} \mathcal{H}(i\tau, i\kappa_h, v) = i\left(\dfrac{\tau}{\sqrt{\tau^2+|\kappa_h|^2}} + \mathbf{a}(v)\cdot\dfrac{\kappa_h}{\sqrt{\tau^2+|\kappa_h|^2}}\right), \text{ and} \\ \mathcal{P}(i\kappa_p, v) = \dfrac{\kappa_p\cdot\mathbf{b}(v)\kappa_p}{\kappa_p\cdot\kappa_p}, \end{cases}$$

and let $M>0$ be given by

$$M = \max\left\{\max_{v\in\operatorname{supp}\eta,\sqrt{\tau^2+|\kappa_h|^2}=1} |\mathcal{H}(i\tau,i\kappa_h,v)|, \max_{v\in\operatorname{supp}\eta,|\kappa_p|=1} |\mathcal{P}(i\kappa_p,v)|\right\}.$$

Then, given any dyadic numbers K_h and $K_p < 2M$, we may write

$$\mathfrak{f}^{K_h,K_p}_{J_h,J_p}(t,x,y,v) = \mathfrak{F}^{-1}_{t,x,y}\left[\psi_1\left(\frac{\mathcal{H}(i\tau,i\kappa_h,v)}{K_h}\right)\psi_1\left(\frac{\mathcal{P}(i\kappa_p,v)}{K_p}\right)(\mathfrak{F}_{t,x,y}\mathfrak{f}_{J_h,J_p})(\tau,\kappa_h,\kappa_p,v)\right], \tag{B.11}$$

implying once more that

$$\mathfrak{f} = \sum_{J_p \text{ and } J_p\in\mathscr{J}}\ \sum_{K_h \text{ and } K_p \text{ dyadic } \leq 2M} \mathfrak{f}^{K_h,K_p}_{J_h,J_p}.$$

Hence, let

$$(\varphi u)^{K_h,K_p}_{J_h,J_p} = \int_{\mathbb{R}} \mathfrak{f}^{K_h,K_p}_{J_h,J_p}\eta\, dv, \tag{B.12}$$

in such a way that

$$\begin{cases} (\varphi u)_{J_h,J_p} = \displaystyle\sum_{K_h \text{ and } K_p \text{ dyadic } \leq 2M} \int_{\mathbb{R}} \mathfrak{f}^{K_h,K_p}_{J_h,J_p}\eta\, dv, \\ (\varphi u)_{\cdot,J_p} = \displaystyle\sum_{J_h\in\mathscr{J}}\ \sum_{K_h \text{ and } K_p \text{ dyadic } \leq 2M} \int_{\mathbb{R}} \mathfrak{f}^{K_h,K_p}_{J_h,J_p}\eta\, dv, \text{ and} \\ (\varphi u)_{J_h,\cdot} = \displaystyle\sum_{J_p\in\mathscr{J}}\ \sum_{K_h \text{ and } K_p \text{ dyadic } \leq 2M} \int_{\mathbb{R}} \mathfrak{f}^{K_h,K_p}_{J_h,J_p}\eta\, dv. \end{cases} \tag{B.13}$$

Let us now estimate each component $(\varphi u)^{K_h,K_p}_{J_h,J_p}$ and sum them accordingly to prove the desired conclusion.

Once both variables can be, aside from minor technicalities, analyzed in a selfsame manner, we will only present the analysis of the regularity in y. From this point forward, we will tacitly presume that $\mathfrak{f} \in L^2(\mathbb{R}_v\times\mathbb{R}_t\times\mathbb{R}^{N_h}_x; H^{s_p}(\mathbb{R}^{N_p}_y))$ for some $s_p \geq 0$—clearly, we can always assume that $s_p = 0$.

Step #3.1: The $L^2_{t,x,y}$–norm of $(\varphi u)^{K_h,K_p}_{J_h,J_p}$. Imitating the arguments of (5.65), it is clear that

$$\left\|(\varphi u)^{K_h,K_p}_{J_h,J_p}\right\|_{L^2_{t,x,y}} \leq C\frac{K_h^{\mathfrak{e}_h/2}}{J_p^{s_p}}\|\varphi f\|_{L^2_{v,t,x}H^{s_p}_y}. \tag{B.14}$$

Step #3.2: The $L^{q_\varepsilon}_{t,x,y}$–norm of $(\varphi u)^{K_h,K_p}_{J_h,J_p}$. From (B.10), (B.11), and (B.12), we have that

$$(\varphi u)^{K_h,K_p}_{J_h,J_p} = \mathfrak{F}^{-1}_{t,x,y}\Bigg[\int_{\mathbb{R}_v} \eta(v)\psi_1\Big(\frac{\mathcal{H}(i\tau,i\kappa_h,v)}{K_h}\Big)\psi_1\Big(\frac{\mathcal{P}(i\kappa_p,v)}{K_p}\Big) \frac{\sqrt{1+\tau^2+|\kappa_h|^2+|\kappa_p|^4}^{\varepsilon}}{i(\tau+\mathbf{a}(v)\cdot\kappa_h)+\kappa_p\cdot\mathbf{b}(v)\kappa_p}\Big(\frac{\partial}{\partial v}(-\Delta_v)^{\varepsilon/2}+1\Big)\Bigg](\mathfrak{F}_{t,x,y}\mathbf{F}^{(\varepsilon)})\,dv\Bigg].$$

Applying a decomposition akin to (5.70) and employing some simple inequalities such as

$$(J_h+J_p^2)\le (K_hJ_h+K_pJ_p^2)/\min\{K_h,K_p\},$$

one may use Theorem 2.5 to argue that

$$\big\|(\varphi u)^{K_h,K_p}_{J_h,J_p}\big\|_{L^{q_\varepsilon}_{t,x,y}} \le \frac{C_{\varepsilon,\eta}}{\min\{K_h,K_p\}^{1+2\varepsilon}}\frac{1}{(K_hJ_h+K_pJ_p^2)^{1-\varepsilon}}\|\mathbf{F}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}} \tag{B.15}$$

According to the Hölder estimate

$$c_\varepsilon(K_hJ_h)^\varepsilon(K_pJ_p^2)^{1-\varepsilon}\le K_hJ_h+K_pJ_p^2,$$

(B.15) may be then translated into

$$\big\|(\varphi u)^{K_h,K_p}_{J_h,J_p}\big\|_{L^{q_\varepsilon}_{t,x,y}} \le \frac{C_{\varepsilon,\eta}}{\min\{K_h,K_p\}^{1+2\varepsilon}}\frac{1}{(K_hJ_h)^{(1-\varepsilon)\varepsilon}(K_pJ_p^2)^{(1-\varepsilon)^2}}\|\mathbf{F}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}}. \tag{B.16}$$

Keep in mind that, once K_h and K_p both lie in the interval $(0,2M)$, we can always utilize the bound $\min\{K_h,K_p\}\ge c_MK_hK_p$.

Step #3.3: The $L^{z_\varepsilon}_{t,x,y}$–norm of $(\varphi u)^{K_h,K_p}_{J_h,J_p}$. Let us now interpolate (B.14) and (B.16). Given any $0\le\theta\le 1$, define $z=z(\theta)$ by

$$\frac{1}{z}=\frac{1-\theta}{2}+\frac{\theta}{q_\varepsilon},$$

so that the interpolation inequality yields

$$\big\|(\varphi u)^{K_h,K_p}_{J_h,J_p}\big\|_{L^{z}_{t,x,y}} \le \frac{C_{\varepsilon,\eta}}{J_p^{s_p(1-\theta)+2\theta(1-\varepsilon)^2}K_p^{\theta(1-\varepsilon)^2}}\frac{1}{(K_hJ_h)^{\theta\varepsilon(1-\varepsilon)}}\frac{K_h^{(1-\theta)\mathfrak{e}_h/2}}{\min\{K_h,K_p\}^{\theta(1+2\varepsilon)}} \Big\{\|\varphi f\|_{L^2_{v,t,x}H^{s_p}_y}+\|\mathbf{F}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}}\Big\}.$$

for J_h and $J_p\ge 1$. Evidently, a similar—and easier—estimate is available for $J_h=0$.

In order to make the exponent of J_p the greatest while keeping such terms summable in both K_h and J_h, let us choose $\theta=\theta_\varepsilon$ such that $(1-\theta)\mathfrak{e}_h/2>(1+3\varepsilon)\theta$, say, $(1-\theta_\varepsilon)\mathfrak{e}_h/2=(1+4\varepsilon)\theta_\varepsilon$. Consequently, for $J_p\ge 1$,

$$\begin{aligned}\big\|(\varphi u)^{\cdot,K_p}_{\cdot,J_p}\big\|_{L^{z_\varepsilon}_{t,x,y}} &\le \sum_{J_h,K_h}\big\|(\varphi u)^{K_h,K_p}_{J_h,J_p}\big\|_{L^{z_\varepsilon}_{t,x,y}}\\ &\le \frac{C_{\varepsilon,\eta,M}}{J_p^{s_p(1-\theta_\varepsilon)+2\theta_\varepsilon(1-\varepsilon)^2}}\frac{1}{K_p^{\theta_\varepsilon(1-\varepsilon)^2+\theta_\varepsilon(1+2\varepsilon)}}\Big\{\|\varphi f\|_{L^2_{v,t,x}H^{s_p}_y}+\|\mathbf{F}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}}\Big\}.\end{aligned} \tag{B.17}$$

Notice that, with this choice of θ_ε,

$$\begin{cases}\theta_\varepsilon=\dfrac{\mathfrak{e}_h/2}{1+4\varepsilon+\mathfrak{e}_h/2}, \text{ and}\\[2mm] \dfrac{1}{z_\varepsilon}=\dfrac{1-\theta_\varepsilon}{2}+\dfrac{\theta_\varepsilon}{q_\varepsilon},\end{cases}$$

in such a fashion that, as $\varepsilon \to 0_+$,

$$\begin{cases} \theta_\varepsilon \to \theta_0 \stackrel{\text{def}}{=} \dfrac{\mathfrak{e}_h}{2+\mathfrak{e}_h}, \text{ and} \\ z_\varepsilon \to z_0 \stackrel{\text{def}}{=} \dfrac{2+\mathfrak{e}_h}{1+\mathfrak{e}_h}. \end{cases} \tag{B.18}$$

Step #3.3: The $L^{r_\varepsilon}_{t,x,y}$–norm of $(\varphi u)^{\cdot,K_p}_{\cdot,J_p}$. Evidently, just as (5.65) was derived, it is clear that

$$\big\|(\varphi u)^{\cdot,K_p}_{\cdot,J_p}\big\|_{L^2_{t,x,y}} \leq C\frac{K_h^{\mathfrak{e}_p/2}}{J_p^{s_p}}\|\varphi f\|_{L^2_{v,t,x}H^{s_p}_y}. \tag{B.19}$$

Therefore, given any $0 \leq \vartheta \leq 1$ and putting

$$\frac{1}{r} = \frac{1-\vartheta}{2} + \frac{\vartheta}{z_\varepsilon},$$

we conclude thus from (B.17) that

$$\big\|(\varphi u)^{\cdot,K_p}_{\cdot,J_p}\big\|_{L^r_{t,x,y}} \leq \frac{C_{\varepsilon,\eta,M}}{J_p^{s_p((1-\vartheta)+\vartheta(1-\theta_\varepsilon))+2\vartheta\theta_\varepsilon(1-\varepsilon)^2}} \frac{K_p^{(1-\vartheta)\mathfrak{e}_p/2}}{K_p^{\vartheta(\theta_\varepsilon(1-\varepsilon)^2+\theta_\varepsilon(1+2\varepsilon))}}\Big\{\|\varphi f\|_{L^2_{v,t,x}H^{s_p}_y} + \|\mathbf{F}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}}\Big\}.$$

Again, in order to reach the greatest exponent of J_p while keeping the sequence summable in K_p, let us pick $\vartheta = \vartheta_\varepsilon$ satisfying $(1-\vartheta_\varepsilon)\mathfrak{e}_p/2 = \vartheta_\varepsilon\theta_\varepsilon(2+2\varepsilon)$. Thence,

$$\begin{aligned} \big\|(\varphi u)_{\cdot,J_p}\big\| &\leq \sum_{K_p}\big\|(\varphi u)^{\cdot,K_p}_{\cdot,J_p}\big\|_{L^r_{t,x,y}} \\ &\leq \frac{C_{\varepsilon,\eta,M}}{J_p^{s_p((1-\vartheta_\varepsilon)+\vartheta_\varepsilon(1-\theta_\varepsilon))+2\vartheta_\varepsilon\theta_\varepsilon(1-\varepsilon)^2}}\Big\{\|\varphi f\|_{L^2_{v,t,x}H^{s_p}_y} + \|\mathbf{F}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}}\Big\}. \end{aligned}$$

It is time thus to pass $\varepsilon \to 0_+$ and observe what was accomplished. Since

$$\begin{cases} \vartheta_\varepsilon = \dfrac{\mathfrak{e}_p/2}{\theta(2+2\varepsilon)+\mathfrak{e}_p/2}, \text{ and} \\ \dfrac{1}{r_\varepsilon} = \dfrac{1-\vartheta_\varepsilon}{2} + \dfrac{\vartheta_\varepsilon}{z_\varepsilon}, \end{cases}$$

the previous relations (B.18) guarantee that

$$\begin{cases} \vartheta_\varepsilon \to \vartheta_0 \stackrel{\text{def}}{=} \dfrac{\mathfrak{e}_p}{4\theta_0+\mathfrak{e}_p} = \dfrac{\mathfrak{e}_p}{4\frac{\mathfrak{e}_h}{\mathfrak{e}_h+2}+\mathfrak{e}_p}, \\ r_\varepsilon \to r_0 \stackrel{\text{def}}{=} \dfrac{4\mathfrak{e}_h+\mathfrak{e}_p(2+\mathfrak{e}_h)}{2\mathfrak{e}_h+\mathfrak{e}_p(1+\mathfrak{e}_h)}, \text{ and} \\ s_p((1-\vartheta_\varepsilon)+\vartheta_\varepsilon(1-\theta_\varepsilon)) + 2\vartheta_\varepsilon\theta_\varepsilon(1-\varepsilon)^2 \to s_p^\star \stackrel{\text{def}}{=} s_p\left(\dfrac{4\mathfrak{e}_h+2\mathfrak{e}_p}{4\mathfrak{e}_h+\mathfrak{e}_p(\mathfrak{e}_h+2)}\right) + \dfrac{2\mathfrak{e}_h\mathfrak{e}_p}{4\mathfrak{e}_h+\mathfrak{e}_p(\mathfrak{e}_h+2)}. \end{cases}$$

In other words, the conclusion is that

$$\begin{aligned}
&u \in L^r(\mathscr{U}_h; W^{s,r}(\mathscr{U}_p)), \text{ and} \\
&\|u\|_{L^r(\mathscr{U}_h;W^{s,r}(\mathscr{U}_p))} \leq C_{s,r}\Big\{ \|\phi\mathbf{m}\|_{\mathfrak{M}_{t,x,y,v}} + \Big\| f\Big(\frac{\partial\varphi}{\partial t} + \mathbf{a}(v)\cdot\nabla_x\varphi - \mathbf{b}(v) : D_y^2\varphi\Big)\Big\|_{L^1_{t,x,y,v}} \\
&\qquad\qquad + \|\varphi f\|_{L^2_{t,x,v}H_y^{s_p}}\Big\}.
\end{aligned} \tag{B.20}$$

for exponents obeying

$$\begin{cases} 0 \leq s < s_p^\star, \text{ and} \\ 1 \leq r < \dfrac{4\mathfrak{e}_h + \mathfrak{e}_p(2+\mathfrak{e}_h)}{2\mathfrak{e}_h + \mathfrak{e}_p(1+\mathfrak{e}_h)}. \end{cases}$$

Step #4: The bootstrap argument and the final regularity estimate in y. Since f is a χ–function and $0 \leq s < 1$, the same rationale supporting (5.79) still holds true regarding the y–regularity of f. Thus, if u is locally in $L^r_{t,x}W^{s,r}_y$, then f is locally in $L^2_{t,x,v}H_y^{s/2}$, allowing us to reiterate (B.20). By doing so, it is not hard to deduce that, for any

$$\begin{cases} 0 \leq s < \mathfrak{s}_p \overset{\text{def}}{=} \dfrac{2\mathfrak{e}_h\mathfrak{e}_p}{2\mathfrak{e}_h + \mathfrak{e}_p(\mathfrak{e}_h+1)}, \text{ and} \\ 1 \leq r < \mathfrak{r}_p \overset{\text{def}}{=} \dfrac{4\mathfrak{e}_h + \mathfrak{e}_p(2+\mathfrak{e}_h)}{2\mathfrak{e}_h + \mathfrak{e}_p(1+\mathfrak{e}_h)}, \end{cases}$$

one indeed has that $u \in L^r(\mathscr{U}_h; W^{s,r}(\mathscr{U}_p))$. Evidently, one can reach $r = \mathfrak{r}_p$ in these estimates by interpolating them with the hypothesized L^∞–bound of $u(t,x,y)$. The proof is complete. □

We do not claim that the exponents $\mathfrak{s}_p$ and $\mathfrak{s}_h$ in (B.4) are sharp; quite on the contrary, we believe we have employed some overly rough bounds in our deduction. Nevertheless, it is curious to perceive that one can recuperate the well-known smoothness exponents of P.-L. Lions–B. Perthame–E. Tadmor [82] and of E. Tadmor–T. Tao [107] for purely degenerate parabolic and purely hyperbolic equations by formally letting $\mathfrak{e}_h \to \infty$ or $\mathfrak{e}_p \to \infty$; this, of course, can be rigorously justified. Let us also point out that it may be difficult even to conjecture which are the optimal regularity exponents for Equation (B.2). We should mention, however, two very riveting works:

- C. De Lellis–M. Westdickenberg [32] showed that for "$\mathfrak{e}_p = \infty$" and $\mathfrak{e}_h = 1$, the order $\mathfrak{s}_h = 1/3$ can be understood to be optimal for this method of velocity averaging.
- On the other hand, B. Gess [50] and B. Gess–J. Sauer–E. Tadmor [53] recently established optimal regularity theorem for the porous media equation via velocity averaging lemmas.

It is fit to compare Theorem B.1 with the authoritative general theory of E. Tadmor–T. Tao [107], from which the argument above clearly derives. Notice that the symbol associated to Equation (B.1) is

$$\mathcal{L}(i\tau, i\kappa_h, i\kappa_p, v) = i(\tau + v^\ell\kappa_h) + |v|^n\kappa_p^2.$$

It is not difficult to verify (see E. Tadmor–T. Tao [107], or B. Gess–M. Hofmanová [51]) that, for any $\delta > 0$ and any compact interval $J \subset \mathbb{R}$,

$$\begin{cases} \sup\limits_{\tau^2+\kappa_h^2=1} \operatorname{meas}\Big\{v \in J;\ \big|\tau + v^\ell\kappa_h\big| \leq \delta\Big\} \leq C\delta^{1/\ell}, \text{ and} \\ \sup\limits_{\kappa_p^2=1} \operatorname{meas}\Big\{v \in J;\ |v|^n\kappa_p^2 \leq \delta\Big\} \leq C\delta^{1/n}; \end{cases}$$

consequently, Theorem B.1 readily applies with $\mathfrak{e}_h = 1/\ell$ and $\mathfrak{e}_p = 1/n$ for any $\ell \geq 1$ and $n > 1$. (The case $n = 1$ is possible if one supposes that u is nonnegative). While we do not wish to juxtapose the exact order of smoothness of both results,[1] it is important to bring attention to the fact that our result has the great advantage of being quite permissive in the coefficients of (B.2), which are somewhat constrained in E. TADMOR–T. TAO [107].

Additionally, it is worth observing that their argument entirely falls apart if one introduces a first-order term in y, even though the new associated symbol still observes a nondegeneracy condition. On the other hand, such a difficulty does not appear in our method, as we see below.

Theorem B.2. *Let $N_h \geq 1$ and $N_p \geq 1$ be integers, $Q \subset \mathbb{R}_t \times \mathbb{R}_x^{N_h} \times \mathbb{R}_y^{N_p}$ be an open set, $\mathbf{A}_h \in \mathscr{C}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{N_h})$, $\mathbf{A}_p \in \mathscr{C}^{2,\alpha}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{N_p})$, and $\mathbf{B} \in \mathscr{C}^{2,\alpha}_{\mathrm{loc}}(\mathbb{R};\mathscr{L}(\mathbb{R}^{N_p}))$ for some $0 < \alpha \leq 1$. Put $\mathbf{a}_h(v) = \mathbf{A}'_h(v)$, $\mathbf{a}_p(v) = \mathbf{A}'_p(v)$, and $\mathbf{b}(v) = \mathbf{B}'(v)$, and suppose that $\mathbf{b}(v) \geq 0$ everywhere.*

Let $u = u(t,x,y) \in L^\infty(Q)$ be an entropy solution to

$$\frac{\partial u}{\partial t}(t,x,y) + \mathrm{div}_x \mathbf{A}_h\big(u(t,x,y)\big) + \mathrm{div}_y \mathbf{A}_p\big(u(t,x,y)\big) - D_y^2 : \mathbf{B}(u(t,x,y)) = 0 \text{ in } \mathscr{D}'(Q), \tag{B.21}$$

and let $a \leq b$ be such that

$$a \leq u(t,x,y) \leq b \text{ in } \mathscr{D}'(Q).$$

Finally, assume that there exist some $\eta \in \mathscr{C}^\infty_c(\mathbb{R}_v)$ and exponents $0 < \mathfrak{e}_h$ and $\mathfrak{e}_p \leq 1$ such that $\eta(v) = 1$ for $a \leq v \leq b$, and, for all $\delta > 0$,

$$\begin{cases} \mathrm{meas}\Big\{v \in \mathrm{supp}\,\eta;\ |\tau + \mathbf{a}_h(v)\cdot\kappa_h| \leq \delta\Big\} \leq C\delta^{\mathfrak{e}_h} \\ \qquad\qquad \text{for all } (\tau,\kappa_h) \in \mathbb{R}\times\mathbb{R}^{N_h} \text{ with } \tau^2 + |\kappa_h|^2 = 1, \text{ and} \\ \mathrm{meas}\Big\{v \in \mathrm{supp}\,\eta;\ \kappa_p\cdot\mathbf{b}(v)\kappa_p \leq \delta\Big\} \leq C\delta^{\mathfrak{e}_p} \\ \qquad\qquad \text{for all } \kappa_p \in \mathbb{R}^{N_p} \text{ with } |\kappa_p|^2 = 1. \end{cases}$$

Then, for any open sets $\mathscr{U}_h \subset \mathbb{R}_t \times \mathbb{R}_x^{N_h}$ and $\mathscr{U}_p \subset \mathbb{R}_y^{N_p}$ such that $\mathscr{U}_h \times \mathscr{U}_p \subset\subset Q$, we have that

$$u \in L^{\mathfrak{r}_p}(\mathscr{U}_h; W^{s_p,\mathfrak{r}_p}(\mathscr{U}_p)) \cap L^{\mathfrak{r}_h}(\mathscr{U}_p; W^{s_h,\mathfrak{r}_h}(\mathscr{U}_h)) \tag{B.22}$$

with

$$\|u\|_{L^{\mathfrak{r}_p}(\mathscr{U}_h;W^{s_p,\mathfrak{r}_p}(\mathscr{U}_p))} + \|u\|_{L^{\mathfrak{r}_h}(\mathscr{U}_p;W^{s_h,\mathfrak{r}_h}(\mathscr{U}_h))} \leq C_{\mathscr{U}_p,\mathscr{U}_h,s_p,s_h}(a,b),$$

where the exponents satisfy

$$\begin{cases} 0 \leq s_p < \mathfrak{s}_p = \dfrac{2\mathfrak{e}_h\mathfrak{e}_p}{4\mathfrak{e}_h + \mathfrak{e}_p(2+\mathfrak{e}_h)}, & \mathfrak{r}_p = \dfrac{4\mathfrak{e}_p + \mathfrak{e}_h(2+\mathfrak{e}_p)}{2\mathfrak{e}_p + \mathfrak{e}_h(1+\mathfrak{e}_p)}, \\ 0 \leq s_h < \mathfrak{s}_h = \dfrac{1+\mathfrak{s}_p}{2}\dfrac{\mathfrak{e}_h\mathfrak{e}_p}{2\mathfrak{e}_p + \mathfrak{e}_h(\mathfrak{e}_p+1)}, \text{ and} & \mathfrak{r}_h = \dfrac{4\mathfrak{e}_h + \mathfrak{e}_p(2+\mathfrak{e}_h)}{2\mathfrak{e}_h + \mathfrak{e}_p(1+\mathfrak{e}_h)}. \end{cases} \tag{B.23}$$

In particular, for $\mathfrak{s} = \min\{s_h, s_p\}$ and $\mathfrak{r} = \min\{\mathfrak{r}_h, \mathfrak{r}_p\}$, $u \in W^{\mathfrak{s},\mathfrak{r}}_{\mathrm{loc}}(Q)$.

Proof. The argumentation here is a variant of the previous one; namely, the crux of our reasoning is this: If $f : Q \times \mathbb{R}_v \to \mathbb{R}$ is the χ–function of $u(t,x,y)$, and $\mathbf{m}(t,x,y,v)$ is its entropy production measure, then f obeys the equation

$$\frac{\partial f}{\partial t} + \mathbf{a}_h(v)\cdot\nabla_x f - \mathbf{b}(v) : D_y^2 f = \frac{\partial \mathbf{m}}{\partial v} - \mathrm{div}_y(\mathbf{a}_p(v) f) \text{ in } \mathscr{D}'(Q\times\mathbb{R}_v), \tag{B.24}$$

[1] Probably the exponent given in [107] is not correct, as, if $n = 2$ and $\ell = 1$, it implies a smoothness of order > 1. This seems impossible, as solutions $u(t,x)$ to the hyperbolic equation $u_t + (u^2/2)_x = 0$ must be solutions to Equation (B.1), and solutions to these equations can develop shocks in finite time.

turning thus the "perturbation" $\mathbf{a}_p(v)\cdot\nabla_y f$ into a forcing term of negative order in y. Since far away from degeneracy set $\{(\tau,\kappa_h,\kappa_p,v); i(\tau+\mathbf{a}_h(v)\cdot\kappa_h)+\kappa_p\cdot\mathbf{b}(v)\kappa_p=0\}$ one expects a regularization of second-order in y, one can still obtain some smoothness for u.

Thus, let us sketch the main differences between this proof and the last one, as the remaining details are essentially tedious calculations. Conserve the former conventions on L, φ, ϕ, $\mathfrak{f}=\varphi f$, $\mathfrak{f}_{J_h,J_p}$, *etc.*

Step #1: The regularity estimate on y. Let us once more assume initially that $\phi f\in L^2_{t,x,v}H^{s_p}_y$ for some $0\le s_p<1$. Notice that, given any $0<\varepsilon<\min\{\alpha,\mathfrak{e}_h,\mathfrak{e}_p,1-s_p\}/5$ and $1<q_\varepsilon<\frac{N+2}{N+2-\varepsilon}$, we may rewrite (B.24) as

$$\frac{\partial\mathfrak{f}}{\partial t}+\mathbf{a}_h(v)\cdot\nabla_x\mathfrak{f}-\mathbf{b}(v):D_y^2\mathfrak{f}=\Big\{(-\Delta_{t,x}+\Delta_y^2+1)^{\varepsilon/2}\Big\}\Big(\frac{\partial}{\partial v}(-\Delta_v)^{\varepsilon/2}+1\Big)\mathbf{F}^{(\varepsilon)}$$
$$+(-\Delta_y+1)^{(1-s_p)/2}\mathbf{G}^{(\varepsilon)}\text{ in }\mathscr{D}'(Q\times\mathbb{R}),\tag{B.25}$$

where $\mathbf{F}^{(\varepsilon)}$ and $\mathbf{G}^{(\varepsilon)}\in L^{q_\varepsilon}_{t,x,y,v}$ satisfy

$$\begin{cases}\|\mathbf{F}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}}\le C_{\varepsilon,\mathscr{O}}\Big\{\|\phi\mathbf{m}\|_{\mathfrak{M}_{t,x,y,v}}\\ \qquad+\Big\|f\Big(\dfrac{\partial\varphi}{\partial t}+\mathbf{a}_h(v)\cdot\nabla_x\varphi+\mathbf{a}_p(v)\cdot\nabla_y\varphi-\mathbf{b}(v):D_y^2\varphi\Big)\Big\|_{L^1_{t,x,y,v}}\Big\},\text{ and}\\ \|\mathbf{G}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}}\le C_\varepsilon\|\phi f\|_{L^2_{t,x,v}H^{s_p}_y},\end{cases}$$

In this way, while (B.14) still holds true, the new version of (B.15) should read

$$\|(\varphi u)^{K_h,K_p}_{J_h,J_p}\|_{L^{q_\varepsilon}_{t,x,y}}\le C_{\varepsilon,\eta}\Big[\frac{C_{\varepsilon,\eta}}{\min\{K_h,K_p\}^{1+2\varepsilon}}\frac{1}{(K_hJ_h+K_pJ_p^2)^{(1-\varepsilon)}}\|\mathbf{F}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}}$$
$$+\frac{J_p^{1-s_p}}{K_hJ_h+K_pJ_p^2}\|\mathbf{G}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}}\Big].\tag{B.26}$$

Adapting conveniently the previous ideas, we conclude thus that the new L^{q_ε}–estimate on $(\varphi u)^{K_h,K_p}_{J_h,J_p}$ is

$$\|(\varphi u)^{K_h,K_p}_{J_h,J_p}\|_{L^{q_\varepsilon}_{t,x,y}}\le\frac{C_{\varepsilon,\eta}}{\min\{K_h,K_p\}^{1+2\varepsilon}}$$
$$\frac{1}{(K_hJ_h)^{\varepsilon(1-\varepsilon)}(K_pJ_p^{1+s_p-2\varepsilon})^{1-\varepsilon}}\Big\{\|\mathbf{F}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}}+\|\mathbf{G}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}}\Big\}.\tag{B.27}$$

One can inspect that the same choices of θ_ε and ϑ_ε not only are consistent with the general philosophy we exposed, but also work fine in this new scenario. The unique difference between the previous statement and this is that, instead of $2(1-\varepsilon)$ being the exponent of J_p, this time is $1+s_p-2\varepsilon$. Hence, it follows that

$$\|(\varphi u)_{\cdot,J_p}\|_{L^r}\le\frac{C_{s,r}}{J_p^s}\Big\{\|\varphi f\|_{L^2_{v,t,x}H^{s_p}_y}+\|\mathbf{F}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}}+\|\mathbf{G}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}}\Big\}$$

for any choice of

$$\begin{cases}0\le s<s_p^\star=s_p\Big(\dfrac{4\mathfrak{e}_h+2\mathfrak{e}_p+\mathfrak{e}_h\mathfrak{e}_p}{4\mathfrak{e}_h+\mathfrak{e}_p(\mathfrak{e}_h+2)}\Big)+\dfrac{\mathfrak{e}_p\mathfrak{e}_h}{4\mathfrak{e}_h+\mathfrak{e}_p(\mathfrak{e}_h+2)},\text{ and}\\ 1\le r<\mathfrak{r}_p=\dfrac{4\mathfrak{e}_h+\mathfrak{e}_p(2+\mathfrak{e}_h)}{2\mathfrak{e}_h+\mathfrak{e}_p(1+\mathfrak{e}_h)}.\end{cases}$$

The desired conclusion now follows from Theorem 5.13 and the bootstrap argument.

Step #2: The regularity estimate on (t,x). Let us now investigate the regularity in the genuinely hyperbolic variables (t,x). Assume initially that $\varphi f \in L^2_{y,v}H^{s_h}_{t,x}$ for some $0 \le s_h < 1$.

Notice that, by the previous step, (B.26) holds true for any $0 \le s_p < \mathfrak{s}_p$, where $\mathfrak{s}_p$ is given in (B.23). In this way, once the Young's inequality asserts that

$$c_\varepsilon\Big\{(K_hJ_h)^{\frac{1+s_p}{2}-2\varepsilon}(K_pJ_p^2)^{\frac{1-s_p}{2}+2\varepsilon}\Big\} \le K_hJ_h + K_pJ_p^2,$$

(B.26) yields

$$\big\|(\varphi u)^{K_h,K_p}_{J_h,J_p}\big\|_{L^{q_\varepsilon}_{t,x,y}} \le \frac{C_{\varepsilon,\eta}}{\min\{K_h,K_p\}^{1+2\varepsilon}} \frac{1}{\Big(K_hJ_h^{\frac{1+s_p}{2}-2\varepsilon}\Big)^{1-\varepsilon}(K_pJ_p^2)^{\varepsilon(1-\varepsilon)}}\Big\{\|\mathbf{F}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}} + \|\mathbf{G}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}}\Big\}. \tag{B.28}$$

Therefore, we observe again that the only distinction between (B.28) and the estimate obtained in the previous case is that one has a smaller index $(1+s_p)/2 - 2\varepsilon$ exponentiating J_h (in contrast to $1-\varepsilon$). As a result, one can show that

$$\big\|(\varphi u)_{J_h,\cdot}\big\|_{L^r} \le \frac{C_{s,r}}{J_h^s}\Big\{\|\varphi f\|_{L^2_{v,y}H^{s_p}_{t,x}} + \|\mathbf{F}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}} + \|\mathbf{G}^{(\varepsilon)}\|_{L^{q_\varepsilon}_{t,x,y,v}}\Big\}$$

for any choice of

$$\begin{cases} 0 \le s < s_h^\star = s_h\left(\dfrac{4\mathfrak{e}_p + 2\mathfrak{e}_h}{4\mathfrak{e}_p + \mathfrak{e}_h(\mathfrak{e}_p+2)}\right) + \dfrac{1+\mathfrak{s}_p}{2}\dfrac{\mathfrak{e}_p\mathfrak{e}_h}{4\mathfrak{e}_p + \mathfrak{e}_h(\mathfrak{e}_p+2)}, \text{ and} \\ 1 \le r < \mathfrak{r}_h = \dfrac{4\mathfrak{e}_p + \mathfrak{e}_h(2+\mathfrak{e}_p)}{2\mathfrak{e}_p + \mathfrak{e}_h(1+\mathfrak{e}_p)}. \end{cases}$$

Once more, the assertion in (B.22) is obtained by Theorem 5.13 and the bootstrap procedure. The proof is complete. □

Extensions to stochastic versions of Equations (B.2) and (B.21) are possible, as Theorem 5.12 hints. In such cases, one expects one–half of the smoothness orders $\mathfrak{s}_p$ and $\mathfrak{s}_h$ of this Appendix.

Furthermore, Theorems B.1 and B.2 may be employed to prove versions of Theorem 3.4 that are more in line with what P.-L. Lions–B. Perthame–E. Tadmor [82] initially envisioned. Despite requiring much stricter regularity and nondegeneracy conditions, such variants would allow diffusion matrices of more general forms. A deceptively simple statement based on Remark 2.23 is this.

Proposition B.1. *Let* $Q \subset \mathbb{R}_t \times \mathbb{R}^N_x$ *be an open set, and let* $(u_\nu)_{0<\nu<1}$ *be such that, for every* $0<\nu<1$, u_ν *is an entropy solution to*

$$\frac{\partial u_\nu}{\partial t}(t,x) + \operatorname{div}_x \mathbf{A}_\nu(u_\nu(t,x)) - D_x^2 : \mathbf{B}_\nu(u_\nu(t,x)) = 0 \tag{B.29}$$

in Q. *Assume that, for some* $0<\varepsilon\le 1$, $\mathbf{A}_\nu(v)$ *and* $\mathbf{B}_\nu(v)$ *are uniformly bounded in, respectively,* $\mathscr{C}^{2,\varepsilon}_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$ *and* $\mathscr{C}^{2,\varepsilon}_{\text{loc}}(\mathbb{R};\mathscr{L}(\mathbb{R}^N))$, *and that* $\mathbf{B}'_\nu(v) \ge 0$ *everywhere.*

Additionally, suppose that there exist $a<b$, *and* $0<\mathfrak{e}\le 1$ *with the following properties.*

1. $a \le u_\nu(t,x) \le b$ *in* $\mathscr{D}'(Q)$ *for all* $0<\nu<1$.
2. *For all* $0<\nu<1$, $\mathbf{B}_\nu(v)$ *is real analytic in a neighborhood* I *of* $[a,b]$.
3. *For the same* I *as above, there exists some constant* $C>0$ *such that, for all* $\delta>0$, $0<\nu<1$,

and $(\tau, \kappa) \in \mathbb{R} \times \mathbb{R}^N$ *with* $\tau^2 + |\kappa|^2 = 1$,

$$\mathrm{meas}\Big\{v \in I;\ |\tau + (P_{X_\nu}\mathbf{A}_\nu)'(v) \cdot \kappa| \leq \delta,\ \textit{and}\ \kappa \cdot (\mathbf{B}_\nu)'(v)\kappa \leq \delta\Big\} \leq C\delta^{\mathfrak{c}}$$

where $X_\nu = \cap_{v \in I} N(\mathbf{B}'_\nu(v))$. *(*$N(\mathbf{B}'_\nu(v))$ *denotes the null space, or kernel, of* $\mathbf{B}'_\nu(v)$*).*

Then, the set $(u_\nu)_{0<\nu<1}$ *is relatively compact in* $L^1_{\mathrm{loc}}(Q)$. *In particular, if respectively* $\mathbf{A}_\nu(v)$ *and* $\mathbf{B}_\nu(v)$ *converge pointwisely to some* $\mathbf{A}(v)$ *and* $\mathbf{B}(v)$ *as* $\nu \to 0_+$, *then the limit points of any sequence* $(u_{\nu_n})_{n\in\mathbb{N}}$ *with* $\nu_n \to 0_+$ *are entropy solutions to* $\frac{\partial u}{\partial t}(t,x) + \mathrm{div}_x \mathbf{A}(u(t,x)) - D^2_x : \mathbf{B}(u(t,x)) = 0$ *in* Q.

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