An introduction to Characteristic Classes

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33º Colóquio Brasileiro de Matemática

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Primeira impressão, julho de 2021

Publicado no Brasil / Published in Brazil.

ISBN 978-65-89124-15-3
MSC (2020) Primary: 14C17, Secondary: 55R40, 57R20, 57R25, 58K45

Coordenação Geral Carolina Araujo

Produção Books in Bytes **Capa** Izabella Freitas & Jack Salvador

Realização da Editora do IMPA
IMPA
Estrada Dona Castorina, 110
Jardim Botânico
22460-320 Rio de Janeiro RJ

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editora@impa.br

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Preface

Defining the birth of characteristic classes is not clear.

Who of Pythagoras, Plato, Maurolico, Descartes, Euler, Poincaré, Hopf... can be considered as the creator of the characteristic classes?

This is the reason why I invite you during the course for a cruise in which we will meet these people and others... who will share their contribution with us.

Our cruise starts on the island of Samos, Greece in 570 BC where we meet Pythagoras playing with representations of the tetrahedron, the hexahedron (cube) and the octahedron. Thales comes from Miletus not far from the island to play with us. We leave the island for Athens where we meet Theaetetus, less known than Plato, although he is the finder of the 5 platonic polyhedra.

After playing with the polyhedra, through the Mediterranean Sea we leave Greece for Sicily where we have a walk on the Syracuse beach with Archimedes in 230 BC. Still in Sicily, in Messina, much later, in December 26, 1537, we meet an Italian priest, Francesco Maurolico who, apparently does not care of the war between Charles V and the Pope against the Turks and prefers to spend his time describing the planar representations of the platonic polyhedra. Maurolico tells us that he observed that the 5 platonic polyhedra satisfy the formula:

$$\#\text{vertices} \ - \#\text{edges} \ + \#\text{faces} = +2.$$

The boat takes us to Stockholm, in January 1650, where Descartes is invited by the Queen Christina of Sweden. Descartes is very ill. He entrusts us with his manuscripts, containing among other things a "nice theorem". When Descartes dies, few days later, we take the boat which transports his manuscripts to Paris in a safe. Arriving in Paris, the boat sinks (do you know how to swim?). Fortunately, after 3 days in the river Seine, the safe is recovered, allowing later Leibniz to copy Descartes' manuscripts and to take these copies to Hanover in Germany.

Still in Germany, in 1750 we go to the Jean-Sebastien Bach's funerals. Then in Berlin, on November 14, we meet Euler who just sent a letter to his friend Goldbach saying that he discovered the formula that now bears his name, this for all convex polyhedra in $\mathbb{R}^3$. The formula appears now in fact as a corollary of the theorem that Descartes showed us.

We return in Paris, in 1885, for the Victor Hugo's funerals. Poincaré is there and he says us that he has been able to generalize the Euler characteristic for all dimensions. We stay in Paris to follow the construction of the Eiffel Tower and of the first metro line. We meet again Poincaré in 1899 who tells us that the characteristic, now called Euler–Poincaré, is the obstruction to the construction of a vector field tangent to a compact smooth surface.

Back in Berlin, in 1927, we go to the cinema to see the new silent film "Metropolis" by Fritz Lang. We are sitting next to Hopf who invites us to know how he generalizes the Poincaré result for all dimensions, obtaining the now called Poincaré–Hopf theorem.

Hopf advises his student Stiefel to study the obstruction to construct an r-frame tangent to a smooth manifold. For us, we continue our cruise, this time on the liner "Normandie", on May 29, 1935 for its inaugural crossing to USA. In a festive atmosphere, he wins the "blue ribbon". We reach Weston, where Whitney shows us around his marvelous house. He takes us to climb the peaks of Massachusetts (don't you feel dizzy?). Whitney tells us that he has a similar construction to the Stiefel's one. The Stiefel–Whitney classes are born.

We stay in the United States during WWII and Chern invites us to Princeton in 1946 to read us his poems and tell us how he constructs, in the complex setting, the "Chern classes" in so many ways, not just using the obstruction theory but also, among others, by decomposition of Grassmannian manifolds into Schubert cycles and by differential forms. Chern gets us so excited that we forget our cruise. He suggests us to follow evolution of the theories: Chern–Gauß–Bonnet, Chern–Weil, Chern–Simons.

Back in France, in 1965, we take the train from Paris to Lille in which we meet a woman making strange drawings on sheets of paper. Marie-Hélène Schwartz explains to us that she understood why, in general, the Poincaré–Hopf Theorem does not work for singular varieties and, radiant, she explains to us that we must consider radial fields, making the picture of a pinched torus and folding another sheet of paper. Moreover, she tells us to be able to define Chern classes for singular varieties, using what she calls “Whitney stratifications”.

Four years later, in IHES in Paris, Deligne and Grothendieck conjecture existence and uniqueness of Chern classes for singular varieties, satisfying a system of axioms. The conjecture is proved by Robert MacPherson in 1973. In Paris, Marie-Hélène Schwartz enters in a small clothing store on the Boulevard Saint Michel to buy a shirt for her husband, Laurent. By chance, MacPherson also walks into the same cramped store. No place for us, but we can hear the discussion about characteristic classes ending by a “they must be the same”. They are the same and that has been proved by Marie-Hélène Schwartz and myself. We use one ingredient, defined by MacPherson, the “local Euler obstruction”.

Now, we go to Brasil ! MacPherson gave a lecture during the 9th Colóquio Brasileiro de Matemática, that was in 1973 in Poços de Caldas. We are now at the... 33th Colóquio Brasileiro de Matemática ! The cruise is not finished, there are many researchers, women and men, working on the characteristic classes and on the developments of the local Euler obstruction in Brasil and other countries. The story will continue, with you ?

Readers interested in deepening their knowledge in the subject of characteristic classes can consult, for the smooth case, the books by Dieudonné (1989) and Steenrod (1951), the book by Milnor and Stasheff (1974) “characteristic classes” and, for the case of singular varieties, the forthcoming article by the author to appear in the Handbook of Geometry and Topology of Singularities volume III. Springer.

Eu dedico o curso a Roberto Callejas-Bedregal, falecido do covid em Abril. Jamais esqueceremos sua alegria comunicativa de trabalhar em matemática.

Agradeço ao 33º comitê CBM por me dar a honra de ministrar este curso. Agradeço a todas as pessoas que me ajudaram a escrever o livro do curso e a fazer os vídeos: matemáticos, responsáveis, secretárias, administradores, editores e técnicos.

Muito obrigado a Carolina, Thủy, Paulo, Suely, Leticia, Anderson,…Obrigado a todos por sua ajuda.

Jean-Paul Brasselet
Julho de 2021

1 Introduction

1.1 Manifolds and pseudomanifolds

In the following, we recall basic, and useful, notions about manifolds. In this section and the following ones, unless explicit mention, all considered (pseudo) manifolds are connected.

One of the fundamental notion that we will use is the one of triangulation of the considered spaces, that is done through the notion of simplicial complex.

A simplex is the convex hull of $(k+1)$ linearly independent points in the euclidean space $\mathbb{R}^m$. A ℓ-dimensional face of the simplex is the convex hull of $(\ell+1)$ of these points, $\ell \leqslant k$.

Definition 1.1.1. A (finite) simplicial complex K is a collection of simplexes in some euclidean space $\mathbb{R}^m$ such that

- if $s \in K$ then every face of s belongs to K,
- if $s, t \in K$, then $s \cap t$ is either empty or is a common face of s and t.

Definition 1.1.2. Let us denote by K a (finite) simplicial complex in $\mathbb{R}^m$. The union of simplexes in K is a compact subspace of $\mathbb{R}^m$ denoted by $|K|$ and called *geometric realisation* of K, or *polyhedron* associated to K.

Definition 1.1.3. A topological space X is *triangulable* (or a polyhedron) if there exists a simplicial complex K and a homeomorphism $h : |K| \to X$. Such a pair (K, h), or simply the simplicial complex K, is called a *triangulation* of X.

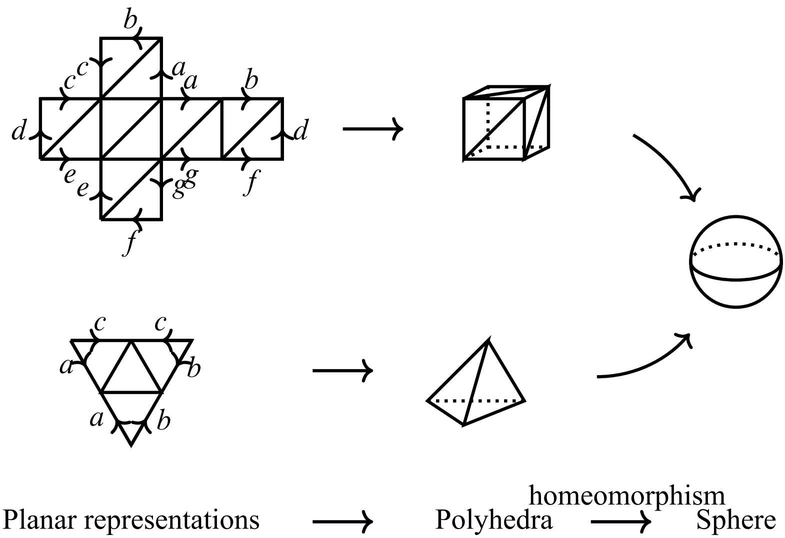


Figure 1.1: Triangulations of the sphere.
In the planar representations, the segments with same name are identified respecting the orientation.

Remark 1.1.4. Not all topological spaces are triangulable (seeVerona (1984)).

Let K be a simplicial complex, and x a point in the polyhedron $|K|$. The simplicial neighbourhood of x in K, denoted by $N_K(x)$ is the set of (closed) simplexes that contain x together with their faces. The link of x, denoted by $\mathrm{Lk}_K(x)$ is the subset of simplexes in $N_K(x)$ that do not contain x. The i-skeleton of K, denoted by $K^{(i)}$, is the set of (K)-simplices whose dimension is less or equal to i.

Definition 1.1.5. (Combinatorial manifold) A polyhedron $|K|$ is called a *combinatorial n-manifold* if for each $x \in |K|$, the link $|\mathrm{Lk}_K(x)|$ is homeomorphic to the sphere $\mathbb{S}^{n-1}$.

Definition 1.1.6. (Topological manifold) A topological space M is called a *(topological) n-manifold* if each point x in M admits a neighbourhood U_x homeomorphic to a ball $\mathbb{B}^n \subset \mathbb{R}^n$ through a homeomorphism $\phi : U_x \to \mathbb{B}^n$ such that $\phi(x) = 0$ and the boundary of U_x, called the link of x, denoted by $\operatorname{Lk}(x)$ is homeomorphic to the sphere $\mathbb{S}^{n-1}$.

The pair (U_x, ϕ) or simply the neighbourhood U_x is called a *local chart* of M. An *atlas* for M is a family $\{U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ of charts which covers M. Let $\{U_\alpha, \phi_\alpha)\}$ and $\{U_\beta, \phi_\beta)\}$ be two charts such that $U_\alpha \cap U_\beta$ is non-empty. The *transition map* $h_{\alpha,\beta}$ is the (homeomorphic) map $h_{\alpha,\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ defined by $h_{\alpha,\beta} = \phi_\beta \circ (\phi_\alpha)^{-1}$.

Examples of n-manifolds are the n-dimensional Euclidean space $\mathbb{R}^n$, the n-dimensional sphere $\mathbb{S}^n$ and the n-dimensional real projective space

$$\mathbb{RP}^n = \mathbb{S}^n/(x \sim -x).$$

Definition 1.1.7. (Differentiable /Analytic manifold) A topological n-manifold is a C^k-differentiable (resp. analytic) n-manifold if for each pair of local charts $\{U_\alpha, \phi_\alpha)\}$ and $\{U_\beta, \phi_\beta)\}$ such that $U_\alpha \cap U_\beta \neq \emptyset$, then the transition map $h_{\alpha,\beta}$ is a C^k-map (resp. analytic map).

One will use suitable triangulations on differentiable (resp. analytic) manifolds:

Definition 1.1.8. Let M be a C^k-differentiable, or analytic manifold, one say that the pair (K, h) is a C^k-differentiable, (resp. analytic) triangulation of M if (K, h) is a triangulation of M and h is a C^k-differentiable, (resp. analytic) embedding on each simplex.

Definition 1.1.9. (Homology $\mathbb{Z}$-manifold) Let K be a triangulation of the triangulable space X. One says that X is an homology $\mathbb{Z}$-manifold of dimension n, or homology n-manifold, if for each $x \in X$ one has $H_*(\operatorname{Lk}_K(x); \mathbb{Z}) \cong H_*(\mathbb{S}^{n-1}; \mathbb{Z})$.

Equivalently, the condition means that

$$H_p(X, X \setminus \{x\}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } p = n \\ 0 & \text{if } p \neq n. \end{cases}$$

Unless specified, in the following, homology groups will be with $\mathbb{Z}$ coefficients and will be omitted.

A topological n-manifold is a homology n-manifold (converse is false).

Definition 1.1.10. A regular point x in a space X is a point which admits a neighbourhood homeomorphic to an open subset in some $\mathbb{R}^k$. One says that k is the dimension of X at x. A singular point is a point which is not regular. If all regular points have the same dimension one says that the space is purely dimensional. The dimension of the (connected) purely dimensional space X is its dimension in its regular points.

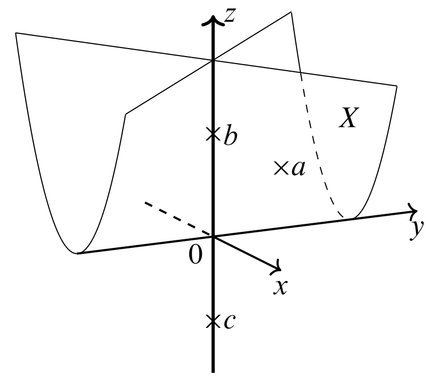


Figure 1.2: The Whitney umbrella.
Here X is the real part in $\mathbb{R}^3$ of the Whitney umbrella whose equation in $\mathbb{C}^3$ is $x^2 - y^2z = 0$. The point a is a regular point of X where the dimension is 2. The point b is a singular point. The point c is a regular point and the dimension of X at c is 1.

We will be interested in singular spaces, (*i.e.* spaces which admit singular points) the simplest example of them being the pinched torus (see Figure 1.3 left). In general, the spaces we will consider will be pseudomanifolds. In fact, the notion of pseudomanifold differs according to the authors. Let us fix the definition we will use in the following.

Definition 1.1.11. (Pseudomanifold – Combinatorial definition) One says that the polyhedron $|K|$ is an *n-pseudomanifold* if the simplicial complex K satisfies the following properties:

(i) $\dim K = n$, *i.e.* the maximal dimension of simplexes in K is n.

(ii) Each simplex is face of an n-simplex.

(iii) Each $(n-1)$-simplex is face of exactly two n-simplexes.

The notion of "simplicial simple n-circuit" (Lefschetz (1942), Poincaré) corresponds to the one of pseudomanifold with the following additional connexity property

(iv) The set of the n and $(n-1)$-simplexes is connected.

The property (iv) means that $|K| \setminus |K^{(n-2)}|$ is connected. Equivalently, given two n simplexes σ and τ in K, there exists a sequence of n-simplexes $\sigma = \sigma_1, \sigma_2, \dots, \sigma_r = \tau$ such that $\sigma_i \cap \sigma_{i+1}$ is an $(n-1)$-simplex.

If properties (i) to (iv) are verified, we will say that $|K|$ is a *simple n-pseudomanifold*.

The topological definition of pseudomanifolds, which is equivalent to the combinatorial one in the case of triangulable topological space, goes as follows:

Definition 1.1.12. (Pseudomanifold – Topological definition) One says that the (paracompact, Hausdorff) topological space X is an *n-pseudomanifold* if there is a subset $\Sigma \subset X$ such that:

(i′) $\dim X = n$.

(ii′) $X \setminus \Sigma$ is a n-topological manifold dense in X.

(iii′) $\dim \Sigma \leqslant n-2$.

The property (iv) is equivalent to the following connexity property

(iv′) The set $X \setminus \Sigma$ is connected.

Notice that, in the triangulated case, one can take for Σ the $(n-2)$-skeleton $|K^{(n-2)}|$.

Examples 1.1.13. *A Thom space, a complex algebraic variety are examples of pseudomanifolds.*
Let K be a triangulation of a connected homology n-manifold, then $|K|$ is an n-pseudomanifold.

Not all n-pseudomanifolds are homology n-manifolds. The pinched torus and the suspension of the torus (see Figure 1.3) are pseudomanifolds, they are not homology n-manifolds.

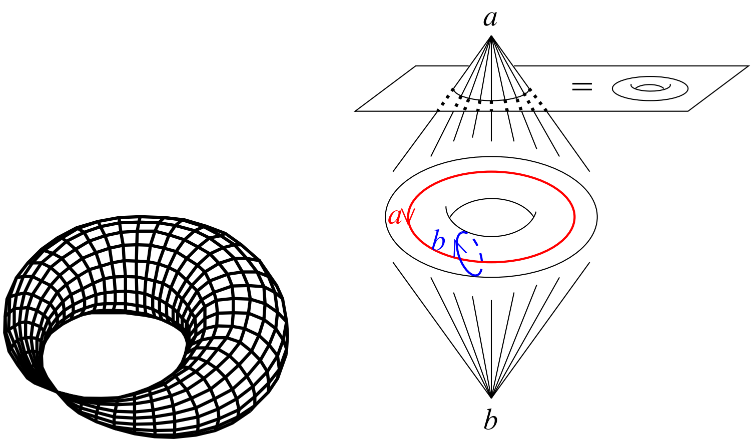


Figure 1.3: The pinched torus and the suspension of the torus

1.2 Orientation

1.2.1 Orientation of pseudomanifolds

We will define the notion of orientation in different settings. Let us start by the combinatorial framework.

An orientation of a simplex is the data of an (equivalence class of) order of its vertices, up to permutation. A simplex has two possible orientations. An orientation of an n-simplex induces an orientation on each of its $(n-1)$-faces.

Let us consider an n-pseudomanifold $X = |K|$ and an $(n-1)$-simplex τ, common face of two oriented n-simplexes σ_1 and σ_2. One says that the given orientations of σ_1 and σ_2 are compatible if they induce opposite orientations on τ.

Definition 1.2.1. One says that the (simple) n-pseudomanifold X is orientable if there exists a triangulation K of X so that one can define a compatible orientation for all n-simplexes in K.

If X is orientable, there are two possible compatible orientations of the n-simplexes. An orientation of X is the choice of one of the possible compatible orientations of the simplexes.

The pinched torus and the suspension of the torus (Figure 1.3) are orientable pseudomanifolds.

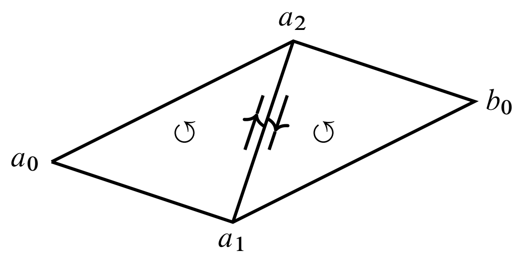


Figure 1.4: Compatible orientations.

Proposition 1.2.2. *A closed (i.e. compact and without boundary) simple pseudomanifold X of dimension n is orientable if and only if $H_n(X) \cong \mathbb{Z}$. If X is non-orientable, then one has $H_n(X) = 0$.*

If X is orientable, a generator of $H_n(X)$ is given by $\sum \sigma_i^n$ where σ_i^n describes the set of n-simplexes in X, endowed with compatible orientations. That is clearly a cycle, called the *fundamental cycle* or *orientation cycle*. Its class, denoted by $[X]$ is the fundamental, or orientation class, of the simple pseudomanifold X.

In the case of a (non simple) pseudomanifold whose the number of connected components of $X \setminus \Sigma$ is k, then if X is orientable one has $H_n(X) \cong \oplus_k \mathbb{Z}$. The double cone is an example of non-simple orientable pseudomanifold and one has $H_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}$

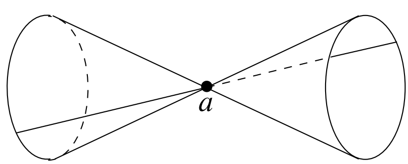


Figure 1.5: The double cone.

1.2.2 Orientation of manifolds

Let us denote by X an n-manifold, a local orientation of X around a point x is a choice of generator of $H_n(X, X \setminus \{x\}; \mathbb{Z}) \cong H_{n-1}(\mathbb{S}^{n-1}) = \mathbb{Z}$. A local orientation at x provides an orientation of the neighbourhood U_x (see Definition 1.1.6).

One says that the atlas $\{U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ of X is an oriented atlas if all transition functions $h_{\alpha,\beta}$ are orientation preserving.

Definition 1.2.3. An n-manifold X is orientable if and only if it admits an oriented atlas.

Examples of orientable n-manifolds are the n-dimensional Euclidean space $\mathbb{R}^n$ and the n-dimensional sphere $\mathbb{S}^n$. The n-dimensional real projective space $\mathbb{RP}^n = \mathbb{S}^n/(x \sim -x)$ is orientable if and only if n is odd.

Lemma 1.2.4. *(see for example Giblin (2010)) A closed (smooth) surface is embedded in $\mathbb{R}^3$ if and only if it is orientable*

Let us denote by M a differentiable n-manifold, TM the tangent bundle of M (see Section 1.6.3) and T^*M the cotangent bundle (bundle of differentiable forms on M). The associated line bundle of n-differentiable forms $\Lambda^n T^*M$ is a line bundle (see Section 1.6.3).

Definition 1.2.5. A differentiable n-manifold M is orientable if and only if the line bundle $\Lambda^n T^*M$ admits a nonzero section, i.e. there is a global non-zero n-differential form on M.

An orientation of TM (hence of M) is an equivalence class of such sections $\omega : M \to \Lambda^n T^*M$ relatively to the following equivalence relation:

$$\omega_1 \sim \omega_2 \quad \Leftrightarrow \quad \exists f \in C^1(M),\ \omega_1 = f\omega_2,\ f(x) > 0 \quad \forall x \in M.$$

Exercise 1.2.6. Show equivalence of Definitions 1.2.1 and 1.2.5 in the case of a triangulable differentiable manifold M.

Examples 1.2.7. (a) The sphere $\mathbb{S}^2$ and the torus are example of 2-dimensional orientable manifolds.

(b) The real projective space $\mathbb{RP}^2$ and the Klein bottle are examples of 2-dimensional nonorientable manifolds.

Lemma 1.2.8. Samelson (1969) *A differentiable compact hypersurface M is orientable.*

Proof: Let us consider a differentiable compact hypersurface M in $\mathbb{R}^{n+1}$, defined by an equation $\{f = 0\}$. At every point $x \in M$, there are two unitary vectors normal to M, *i.e.* normal to the tangent vector space $T_x(M)$. A continuous choice $u(x)$ of one of the unit normal vectors on M determines a nonzero section of $\Lambda^n T^*M$ in the following way (see Seifert and Threlfall (1934)): Let us fix coordinates $(x_1, \dots, x_{n+1})$ in $\mathbb{R}^{n+1}$. One consider the n-differentiable form

$$\omega = i(u)dx_1 \wedge \cdots \wedge dx_{n+1}$$

where $i(u)$ is the interior product, i.e. ω is defined by its value on any n-vector $(v_1, \ldots, v_n)$:

$$[i(u)dx_1 \wedge \cdots \wedge dx_{n+1}]\,(v_1, \ldots, v_n) = [dx_1 \wedge \cdots \wedge dx_{n+1}](u, v_1, \ldots, v_n)$$

One shows that ω is never zero on M (see for instance Seifert and Threlfall (ibid.) §5.3). $\square$

1.2.3 Oriented double covering

Definition 1.2.9. A covering map $\rho : \widetilde{M} \to M$ is a continuous surjective map such that for each $x \in M$, there is an open neighbourhood U_x of x in M such that $\rho^{-1}(U_x)$ is union of disjoint open sets in $\widetilde{M}$, and, on each of them, ρ is an homeomorphism onto U_x.

Examples of coverings are $\rho : \mathbb{S}^n \to \mathbb{RP}^n$ (identification of antipodal points) and the torus as double covering of the Klein bottle.

The following construction will be useful for nonorientable manifolds:

Let M be a n-manifold, we construct a smooth bundle $\widetilde{M}$ over M with fibre the 0-dimensional sphere $\mathbb{S}^0 = \{-1, +1\}$ in the following way: The fibre of the line bundle $\Lambda^n T^*M$ over $x \in M$ is the one-dimensional Euclidean space $\Lambda^n T_x^* M$. The fibre of the bundle $\widetilde{M}$, *i.e.* $\widetilde{M}_x$, will be the unit sphere $\mathbb{S}^0$ in $\Lambda^n T_x^* M$. One obtains a smooth fibre bundle $\rho : \widetilde{M} \to M$ called the oriented double covering of M. The fibre $\widetilde{M}_x$ is composed of two points corresponding to the two possible orientations of $T_x^* M$.

Lemma 1.2.10. *If M is connected, then $\widetilde{M}$ is connected if and only if M is not orientable.*

Proof: Since $\rho^{-1}(x)$ consists of two points, either $\widetilde{M}$ is connected or it has two components $\widetilde{M}_i$, $i = 1, 2$ and the restriction of ρ to each of them is a diffeomorphism. If $\widetilde{M}$ is not connected, $\rho_1^{-1} : M \to \widetilde{M}_1$ is a nonzero section of $\Lambda^n T^* M$, hence M is orientable.

Conversely, if M is orientable, the choice of an orientation in M defines a diffeomorphism $\phi : M \times \mathbb{S}^0 \to \widetilde{M}$ and $\widetilde{M}$ is not connected. $\square$

If M is a connected (orientable or not) n-manifold, then $\widetilde{M}$ is an orientable n-manifold.

Given a triangulation of the (oriented or not) manifold M one defines a triangulation of the double covering $\widetilde{M}$ defining simplexes in $\widetilde{M}$ as inverse image by ρ of simplexes in M. One obtains:

Lemma 1.2.11. *If M is a connected manifold, then $\chi(\widetilde{M}) = 2 \cdot \chi(M)$.*

1.3 Poincaré isomorphism (manifolds)

In Poincaré (1899) Poincaré defined the Poincarré isomorphism using dual decomposition:

We denote by M a triangulated manifold, of real dimension n and by (K) a triangulation of M. A dual cell decomposition of M is obtained in the following way:

We consider a barycentric subdivision (K') of (K). The barycenter of a simplex $\sigma \in K$ will be denoted by $\widehat{\sigma}$. Every simplex in K' can be written as

$$(\widehat{\sigma}_{i_1}, \widehat{\sigma}_{i_2}, \dots, \widehat{\sigma}_{i_p})$$

where $\sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_p}$. Here the symbol $\sigma < \sigma'$ means that σ is a face of σ'.

The dual cell of a simplex σ, denoted by $d(\sigma)$, is the set of all (closed) simplexes τ in (K') such that $\tau \cap \sigma = \{\widehat{\sigma}\}$. That is the set of simplexes on the form $(\widehat{\sigma}, \widehat{\sigma}_{i_1}, \dots, \widehat{\sigma}_{i_k})$ with $\sigma < \sigma_{i_1} < \cdots < \sigma_{i_k}$.

The dual cells satisfy the nice properties (see for instance Munkres (1984)):

Lemma 1.3.1.

1. *The dual cell of a k-simplex is a $(n-k)$-cell, homeomorphic to the unit ball $\mathbb{B}^{n-k} \subset \mathbb{R}^{n-k}$ and its boundary is homeomorphic to the corresponding sphere $\mathbb{S}^{n-k-1}$.*

2. *The set of dual cells provide a cell decomposition of M, called dual cell decomposition associated to the barycentric subdivision (K') of (K).*

The unique intersection point $\widehat{\sigma} = d(\sigma) \cap \sigma$ is the barycenter of σ that we will denote also sometimes by $\widehat{d} = \widehat{d}(\sigma)$.

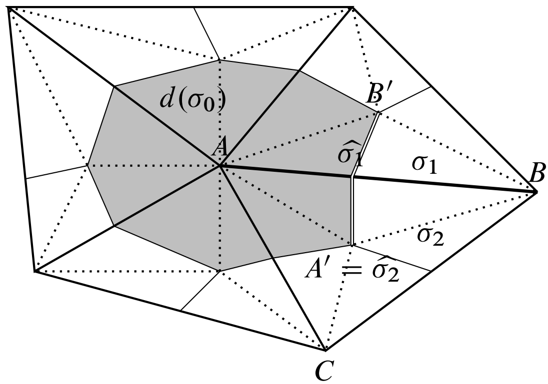


Figure 1.6: Dual cells 1.

In Figure 1.6, the barycenter of $\sigma_0 = A$ is A itself, the dual cell of σ_0 is the gray cell. The dual cell of $\sigma_1 = AB$ is composed of the two segments (double lines) $A'\widehat{\sigma_1}$ and $\widehat{\sigma_1}B'$. The dual cell of the triangle $\sigma_2 = ABC$ is the point A', barycenter of σ_2

The set of dual cells is a cellular decomposition of the manifold M, that provides the manifold a structure of CW-complex:

Definition 1.3.2. A structure of CW-complex on M is a partition of M into open cells such that

1. for each k-dimensional open cell c, there is a continuous function from the k-dimensional closed ball $\mathbb{B}^k$ into M such that the restriction to the interior of $\mathbb{B}^k$ is a homeomorphism onto c,

2. image of boundary of $\mathbb{B}^k$ is contained in the union of a finite number of cells of dimension less than k.

A subset of M is closed if and only if it meets the closure of each cell in a closed set.

The boundary of a cell is a well defined union of cells. The structure of CW complex allows to compute homology as well as cohomology of M. In particular, cohomology is defined as follows:

One considers the cells as elementary cochains and one denotes by $C^k_{(D)}(M)$ the group of k-dimensional D-cochains with integer coefficients.

Let c^p a p-cell, the coboundary of the cochain c^p is the sum

$$\delta(c^p) = \sum [c^p, c_i^{p+1}]\, c_i^{p+1},$$

where the sum is over the $(p+1)$ cells cells c_i^{p+1} whose boundary contains the cell c^p and the incidence sign $[c^p, c_i^{p+1}]$ is ± 1 depending if c^p appears in the boundary of c_i^{p+1} with orientation of the boundary or not.

The coboundary operation

$$\delta : C^k_{(D)}(M) \to C^{k+1}_{(D)}(M)$$

allows to define cocycles and coboundaries, then cohomology groups $H^k_{(D)}(M)$.

Let us assume $M = |K|$ oriented, that is all n-simplexes are given a compatible orientation. Other simplexes are arbitrarily oriented. One gives to every cell $d(\sigma)$ the orientation such that orientation of σ followed by orientation of $d(\sigma)$ is orientation of M (see Brasselet (1981)).

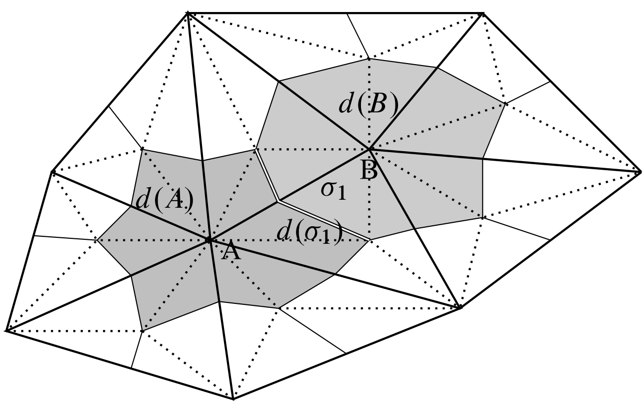


Figure 1.7: Dual cells 2.

In Figure 1.7, the boundary of $\sigma_1 = AB$ is $B - A$. The coboundary of the dual cell $d(\sigma_1)$ is $d(B) - d(A)$.

Let us fix some notations:

- We denote by $d^*(\sigma)$ the elementary (D)-cochain whose value is 1 at the cell $d(\sigma)$ and 0 at other cells of (D).

- We denote by $C_i^{(K)}$ the groups of simplicial K-chains with integer coefficients and by $C^i_{(D)}$ the groups of simplicial D-cochains with integer coefficients.

Let M be a compact oriented n-dimensional manifold, then one has, for every k, a chain isomorphism:

$$C^{n-k}_{(D)}(M;\mathbb{Z}) \longrightarrow C_k^{(K)}(M;\mathbb{Z}), \tag{3.1}$$

that one defines on the elementary elements as

$$d^*(\sigma) \quad \mapsto \quad \sigma$$

and extend linearly. The following Proposition is an easy exercise:

Proposition 1.3.4. *Let $d(\sigma)$ a $(n-k)$-cell, dual of the k-simplex σ. The dual of the coboundary of $d(\sigma)$ is the boundary of σ.*

The following Theorem is one of the possible forms of the Poincaré duality:

Theorem 1.3.5 (Poincaré isomorphism)**.** *(Poincaré (1899)) Let M be a compact oriented n-dimensional manifold, the morphism (3.1) induces, for every k, an isomorphism*

$$H^{n-k}(M;\mathbb{Z}) \longrightarrow H_k(M;\mathbb{Z}),$$

which is the cap-product with the fundamental class $[M] \in H_n(M;\mathbb{Z})$.

$$\begin{array}{ccc} C^{n-k}_{(D)}(M;\mathbb{Z}) & \xrightarrow{D} & C_k^{(K)}(M;\mathbb{Z}) \\ \Big\downarrow \delta & & \Big\downarrow \partial \\ C^{n-k+1}_{(D)}(M;\mathbb{Z}) & \xrightarrow{D} & C_{k-1}^{(K)}(M;\mathbb{Z}) \end{array} \tag{3.2}$$

Remark 1.3.7. Let us remark that the Poincaré duality could be defined in a "dual" way

$$\begin{array}{ccc} C^k_{(K)}(M;\mathbb{Z}) & \xrightarrow{D} & C_{n-k}^{(D)}(M;\mathbb{Z}), \\ \sigma^* & \mapsto & d(\sigma) \end{array}$$

However, in the singular case, the previous way (3.1) will be used and extended, through Alexander duality.

One will see in Section 4.5 how the Poincaré isomorphism extends to singular varieties, as a homomorphism (not any more an isomorphism).

1.4 Boundary

The boundary of an n-dimensional simplicial complex K is the simplicial subcomplex, denoted by ∂K composed of $(n-1)$-simplexes τ such that τ is a face of an unique n-simplex in K, as well as all faces of such simplexes.

Definition 1.4.1. We say that $M = |K|$ is a *n-dimensional manifold with boundary* if for each $x \in |K|$, the link $|\mathrm{Lk}_K(x)|$ is homotopy-equivalent either to $\mathbb{S}^{n-1}$ or to a point. The boundary of M, denoted by ∂M, is the set of points for which $|\mathrm{Lk}_K(x)|$ is homotopy-equivalent to a point.

The boundary ∂M is a $(n-1)$-manifold, which is the geometric realisation of the boundary subcomplex ∂K.

One says that M is a homology n-manifold with boundary if for each $x \in M$ then the reduced homology $\widetilde{H}_*(\mathrm{Lk}_K(x))$ is either $\widetilde{H}_*(\mathbb{S}^{n-1})$ or 0.

One says that M is a n-dimensional differentiable manifold with boundary if every $x \in M$ has a neighbourhood U_x diffeomorphic either to the open ball

$$\mathbb{B}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | \sum_{i=1}^{n} x_i^2 < 1\}$$

and x is called interior point, or to the half space

$$H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | \sum_{i=1}^{n} x_i^2 < 1 \text{ and } x_n \geqslant 0\}$$

whose boundary ∂H^n is

$$\partial H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n = 0\}.$$

The boundary ∂M is the set of points in M that correspond to ∂H^n under such diffeomorphism. It is a differentiable $(n-1)$ manifold.

An orientation of the differentiable manifold with boundary M induces an orientation on ∂M, called boundary orientation, in the following way: At every point $x \in \partial M$ the tangent space to the boundary $T_x(\partial M)$ has codimension 1 in $T_x(M)$. There are two unit vectors in $T_x(M)$ that are orthogonal to $T_x(\partial M)$. One of the vectors points inwards (the half space H^n) and the other u points outwards. Orientation of the boundary is chosen such that orientation of u followed by orientation of ∂M gives orientation of M.

Let M be a compact oriented differentiable manifold with boundary, then ∂M is a compact manifold with orientation and differentiable structure induced from that of M.

1.5 The Gauß map(s)

The Brower topological degree of a continuous map $\gamma : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ is geometrically the number of points in a generic fibre. It is also the number of times the cycle $\gamma(\mathbb{S}^n)$ "covers" $\mathbb{S}^n$, *i.e.* the degree of the map

$$\gamma_* : H_n(\mathbb{S}^n) \cong \mathbb{Z} \longrightarrow H_n(\mathbb{S}^n) \cong \mathbb{Z}.$$

The first mention of the map named Gauß map appears in a draft paper by Carl F. Gauß in 1825, that he published in Gauss (1828). The Gauß map maps a (differentiable) surface in Euclidean space $\mathbb{R}^3$ to the unit sphere $\mathbb{S}^2$. Namely, given a surface X in $\mathbb{R}^3$, the Gauß map is a continuous map

$$\gamma : X \to \mathbb{S}^2$$

such that $\gamma(p)$ is a unit vector in $\mathbb{R}^3$ orthogonal to the tangent space to X at p. One observe that the Gauß map can be defined (globally) if and only if the surface is orientable, but it is always defined locally. There are two choices of such a local map.

More generally, the Gauß map can be defined in the same way for (smooth) hypersurfaces M in $\mathbb{R}^{n+1}$ as the map

$$\gamma : M \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$$

such that $\gamma(p)$ is a unit vector in $\mathbb{R}^{n+1}$ orthogonal to the tangent space to M at p. One obtains a map

$$\gamma_* : H_n(M) \to H_n(\mathbb{S}^n) \cong \mathbb{Z}. \tag{5.3}$$

Definition 1.5.2. The degree of the Gauß map $\gamma : M \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ is defined as $deg(\gamma) = Im(\gamma_*[M])$ where $[M]$ is the fundamental class of M (1.2.2).

In the following, we will use the following property whose proof is an exercise.

Lemma 1.5.3. *Let $M_1 \coprod M_2$ the disjoint union of two smooth hypersurfaces, then*

$$deg(\gamma_{M_1 \coprod M_2}) = deg(\gamma_{M_1}) + deg(\gamma_{M_2})$$

if M_1 and M_2 have the same orientation, otherwise the appropriate signs must be used.

1.5.1 Generalisation of the Gauß map

For an oriented n-submanifold M in $\mathbb{R}^k$ the Gauß map is well defined in the following way: the target space is the oriented Grassmannian manifold, *i.e.* the set $G_n(\mathbb{R}^k)$ of all oriented n-planes in $\mathbb{R}^k$. In this case a point $x \in M$ is mapped to its oriented tangent space $T_x(M)$.

$$\gamma : M \to G_n(\mathbb{R}^k); \qquad \gamma(x) = T_x(M) \subset \mathbb{R}^k.$$

It should be noted that in the (oriented) Euclidean 3-space, an oriented 2-plane is characterised by an unit normal vector, hence this definition is consistent with the above definition.

$$\begin{tikzcd} & G_n(\mathbb{R}^k) \arrow[d] \\ M \arrow[ur, "\gamma"] \arrow[r, hook] & \mathbb{R}^k \end{tikzcd} \tag{5.4}$$

1.6 Fibre bundles.

In this section, we will denote by $\mathbb{K}$ either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. We provide elementary definitions and properties of vector and fibre bundles, as well as a series of examples in the real and complex situations. The reader will find in the literature suitable references for more definitions and properties (see for instance Hirzebruch (1966) and Husemoller (1994)).

1.6.1 Fibre bundles

Let F a topological space and G a topological group which acts effectively and continuously on F, by a left action. The action is effective means that if $g \cdot a = a$ for some $a \in F$ then $g = e$. That action is continuous means that there is a continuous map $G \times F \to F$ such that one has $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ for $g_1, g_2 \in G$ and $a \in F$, and $e \cdot a = a$ if e is the identity element in G.

Definition 1.6.1. A topological space E with a continuous projection $\pi : E \to X$, is called a *fibre bundle* with fibre F and structure group G if G acts effectively and continuously on F and there are a system of coordinates (U_α, ϕ_α) on X and continuous functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that:

- $\{U_\alpha\}$ is an open covering of X and $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a homeomorphism identifying $\pi^{-1}(x)$ with the fibre $\{x\} \times F$,
- $(\phi_\alpha \circ \phi_\beta^{-1},)(x, a) = (x, g_{\alpha\beta}(x) \cdot a)$ for all $x \in U_\alpha \cap U_\beta$ and $a \in F$.

The fibre bundle is said *trivial* if we can take $U_\alpha = X$, that is $E = X \times F$.

Let F' be another topological space with effective and continuous left action of the same group G. The associated bundle to E with fibre F' is the bundle E' for which, with the same covering (U_α) of X, the system of coordinates (U_α, ϕ'_α) satisfies $(\phi'_\alpha \circ \phi'^{-1}_\beta,)(x, a') = (x, g_{\alpha\beta}(x) \cdot a')$ for all $x \in U_\alpha \cap U_\beta$ and $a' \in F'$.

The functions $g_{\alpha\beta}$ are called transition functions. They satisfy

$$g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = \mathrm{id} \qquad \text{for all } \alpha, \beta, \gamma,$$

hence, they define a cocycle in $Z^1(X, G)$, then an element in $H^1(X, G)$. It is well known that isomorphism classes of fibre bundles over X with fibre F and structural group G are in a one-to-one correspondence with the elements of $H^1(X, G)$. The trivial bundle corresponds to the element $1 \in H^1(X, G)$. Fibre bundles in the same isomorphism class $\xi \in H^1(X, G)$ are said associated bundles.

A *section* of the fibre bundle E is a continuous application $s : X \to E$ such that, for every point $x \in X$, one has $s(x) \in E_x = \pi^{-1}(x)$.

The fibre bundle is said differentiable if X is a differentiable manifold and G a real Lie group, the g_{ij} being differentiable functions. The fibre bundle is said complex analytic if X is a complex manifold and G a complex Lie group, the $g_{\alpha\beta}$ being holomorphic functions.

Let $\pi : E \to X$ a bundle on X and $f : Y \to X$ a continuous map. The pullback (or induced) bundle on Y denoted by $f^*(E)$ is the bundle over Y whose fibre on $y \in Y$ is the fiber of E on $f(y)$.

1.6.2 Vector bundles

Vector bundles are examples of fiber bundle with additional structure of vector space on the fibers.

Definition 1.6.2. A *vector bundle* E, over the field $\mathbb{K}$, with base X and rank n is a topological space E, with a continuous map $\pi : E \to X$, the projection, such that for every point $x \in X$, the fibre $E_x = \pi^{-1}(x)$ is a vector space of rank n over $\mathbb{K}$.

A vector bundle satisfies the local triviality condition: for every point $x \in X$, there is an open neighbourhood U_x in X and a homeomorphism $\phi : \pi^{-1}(U_x) \to U_x \times \mathbb{K}^n$ which induces for every $y \in U_x$ an isomorphism $\pi^{-1}(y) \to \mathbb{K}^n$.

The structure group of the vector bundle is the linear group $G = GL_n(\mathbb{K})$.

A trivial bundle is a bundle for which one has "global" triviality, *i.e.* one can take $U_x = X$ in the previous condition.

Given a vector bundle E over X with transition functions $g_{\alpha\beta}$, the dual vector bundle E^* is a vector bundle whose fibers E_x^* are dual of E_x and transition functions are $g_{\alpha\beta}^* = (g_{\alpha\beta}^T)^{-1}$, inverse of the transpose.

1.6.3 Examples of fibre bundles – real case

In order to provide examples of real vector bundles and fibre bundles, we will use the following spaces:

The real projective space $\mathbb{RP}^n$ is the space of lines through the origin of $\mathbb{R}^{n+1}$. The Grassmannian manifold $G_r(\mathbb{R}^n)$ is the space of all vector subspaces of dimension r of $\mathbb{R}^n$. The Grassmannian manifold $G_r(\mathbb{R}^n)$ is identified with the homogeneous space $O(n)/O(r) \times O(n-r)$, where $O(n)$ is the orthogonal group in dimension n.

Let $\mathbb{R}^\infty$ be the vector space of all infinite sequences $(x_1, x_2, \ldots)$ whose elements x_i are real numbers, a finite number of them being nonzero. The infinite Grassmannian manifold $G_r(\mathbb{R}^\infty)$ is the set of all r-dimensional subspaces in $\mathbb{R}^\infty$, *i.e.* the direct limit of the natural sequence of inclusions

$$G_r(\mathbb{R}^r) \subset G_r(\mathbb{R}^{r+1}) \subset G_r(\mathbb{R}^{r+2}) \subset \cdots$$

We consider on $G_r(\mathbb{R}^\infty)$ the topology for which closed subsets are those whose intersections with all $G_r(\mathbb{R}^{r+k})$ are closed.

Examples of real fibre bundles are given by:

1. the *tangent bundle* TM to a differentiable manifold M. That is the set of all pairs (x, v) such that $x \in M$ and v is a vector tangent to M at the point x, *i.e.* an element of T_xM. If M is an n-manifold, then TM is a real vector bundle with rank n over M, the fibre is $\mathbb{R}^n$.

   In particular, one has the bundle $T\mathbb{S}^n$ tangent to the sphere $\mathbb{S}^n$, that is a trivial bundle if $n = 1, 3, 7$, otherwise a non trivial bundle. One has also the bundle $T\mathbb{RP}^n$ tangent to $\mathbb{RP}^n$.

2. the *cotangent bundle* T^*M to a n-differentiable manifold, whose fibre is $(\mathbb{R}^n)^*$.

   The bundle $\Lambda^n T^*M$ is a vector bundle of rank 1 over M, called the orientation bundle (see Definition 1.2.5).

3. the *normal bundle* to a differentiable n-manifold M embedded in $\mathbb{R}^{n+k}$. That is the set of all pairs $(x, v) \in M \times \mathbb{R}^{n+k}$ such that v is orthogonal to the tangent space $T_x M \cong \mathbb{R}^n$ in $T_x(\mathbb{R}^{n+k}) \cong \mathbb{R}^{n+k}$.

4. the *canonical bundle* over $\mathbb{RP}^n$ also called *tautological bundle* and denoted by γ_1^n:
$$\gamma_1^n \to \mathbb{RP}^n \tag{6.5}$$
This line bundle is the set of all pairs $\{(\lambda, v)\}$ where λ is an element of $\mathbb{RP}^n$, i.e. a line passing through the origin of $\mathbb{R}^{n+1}$ and v a vector of λ. The canonical bundle is not trivial, and this fact is the basis for the axiomatic definition of characteristic classes.

5. the canonical bundle γ_r^n over the Grassmannian manifold $G_r(\mathbb{R}^n)$. That is the set of all pairs $\{(P, v)\}$ where P is an element of $G_r(\mathbb{R}^n)$ and v a vector in P. One has the bundle projection
$$\gamma_r^n \to G_r(\mathbb{R}^n)$$
and γ_r^n is a vector bundle with rank r.

   The bundle is also called *universal bundle* for vector bundles of rank r. That means that every bundle ξ with rank r over a (paracompact) topological space X is isomorphic to $f^*(\gamma_r^n)$ for some $f : X \to G_r(\mathbb{R}^n)$ with sufficiently large n.

6. the universal bundle
$$\gamma_r \to G_r(\mathbb{R}^\infty)$$
set of all pairs $\{(P, v)\}$ where P is an element of $G_r(\mathbb{R}^\infty)$ and v a vector of P. It is universal for all rank r-vector bundles.

   In the case $r = 1$, that is the bundle
$$\gamma_1 \to \mathbb{RP}^\infty. \tag{6.6}$$

7. the Stiefel manifold, denoted by $V_r(\mathbb{R}^n)$ is the set of r-frames in $\mathbb{R}^n$, that is the set of ordered r-uples $(v_1, \dots, v_r)$ of r linearly independent vectors in $\mathbb{R}^n$. (see Steenrod (1951) where this manifold is denoted by $V'_{r,n}$).

   One has a homotopy:
$$V_r(\mathbb{R}^n) \cong V_{r,n} = O(n)/O(n-r).$$

The natural map $V_{r,n} \to G_r(\mathbb{R}^n)$ is a principal fibre bundle, i.e. the fibre $O(r)$ coincides with the structural group.

The fibre bundle $V_{r,n} \to G_r(\mathbb{R}^n)$ is an universal bundle for fibre bundles whose basis has dimension $\leqslant n-r-1$.

The vector bundle $\gamma_r^n \to G_r(\mathbb{R}^n)$ is a bundle associated to $V_{r,n} \to G_r(\mathbb{R}^n)$ with fibre $\mathbb{R}^r$.

8. the bundle $V_r(TM)$ of r-frames tangent to a n-differentiable manifold M, i.e. the set of all pairs $(x,(v_1,\dots,v_r))$ where x is a given point of M and $(v_1,\dots,v_r)$ is a r-frame in the fibre T_xM over x. That is the fibre bundle over M whose fibre at x is the manifold $V_r(T_xM)$ of all r-frames in T_xM. The "typical" fibre is the Stiefel manifold $V_r(\mathbb{R}^n)$.

   Note that a section of this bundle is a r-uple of linearly independent sections of the vector bundle TM.

1.6.4 Examples of fibre bundles – complex case

One considers the complex projective space $\mathbb{CP}^n$ whose homogeneous coordinates will be denoted by $(x_0 : x_1 : \ldots : x_i : \ldots : x_n)$. The projective space is covered by open subsets $\{U_i\}_{i=0,\ldots n}$ homeomorphic to $\mathbb{C}^n$ and whose coordinates are $(x_0, x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$.

We will consider the complex Grassmannian manifolds $G_r(\mathbb{C}^n)$ and $G_r(\mathbb{C}^\infty)$ in a similar way than in the real case.

Examples of complex vector bundles are given by:

1. the complex tangent bundle TM to a complex n-dimensional manifold M is a fibre bundle over M. Each fibre T_xM has a complex structure and is isomorphic to $\mathbb{C}^n$.

   In particular, one has the tangent bundle $T\mathbb{CP}^n$ to $\mathbb{CP}^n$.

2. the complex cotangent bundle T^*M is a vector bundle over M whose fibre is $(\mathbb{C}^n)^*$. That is the dual bundle of TM.

   The bundle $\Lambda^n T^*M$ is the canonical bundle on M.

3. the canonical bundle γ_1^n over $\mathbb{CP}^n$. also called tautological or universal bundle and denoted by $\mathcal{O}(-1)$ in algebraic geometry:

$$\gamma_1^n \to \mathbb{CP}^n \tag{6.7}$$

This line bundle is the set of all pairs $\{(\lambda, v)\}$ where λ is an element of $\mathbb{CP}^n$, i.e. a complex line passing through the origin of $\mathbb{C}^{n+1}$ and v a vector in λ. That is the fibre over λ is the line λ.

$$\gamma_1^n = \{(\lambda, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} | v \in \lambda\}.$$

With the previous homogeneous coordinates in $\mathbb{CP}^n$, the transition functions of γ_1^n in $U_i \cap U_j$ are defined by are x_i/x_j.

4. the "hyperplane" bundle $\mathcal{H}$ over $\mathbb{CP}^n$, dual of the canonical bundle. It is denoted by $\mathcal{O}(1)$ in algebraic geometry. With the previous homogeneous coordinates, the transition functions of the hyperplane bundle in $U_i \cap U_j$ are defined by $(x_i/x_j)^{-1}$.

   The hyperplane $H_0 = \{x \in \mathbb{CP}^n | x_0 = 0\}$ with the induced orientation, is homeomorphic to $\mathbb{CP}^{n-1}$. That is a $2(n-1)$-cycle in $H_{2(n-1)}(\mathbb{CP}^n; \mathbb{Z})$. By Poincaré duality isomorphism,

$$H^2(\mathbb{CP}^n; \mathbb{Z}) \to H_{2(n-1)}(\mathbb{CP}^n; \mathbb{Z}), \tag{6.8}$$

   the Poincaré dual cohomology class a of H_0 is a generator of $H^2(\mathbb{CP}^n; \mathbb{Z})$.

5. the universal bundle

$$\gamma_r^n \to G_r(\mathbb{C}^n)$$

   is the set of all pairs $\{(P, v)\}$ where P is an element of $G_r(\mathbb{C}^n)$ and v a vector of P. That is a vector bundle of rank r over $G_r(\mathbb{C}^n)$.

   Every complex vector bundle ξ with rank r over a (paracompact) topological space X is isomorphic to $f^*(\gamma_r^n)$ for some $f : X \to G_r(\mathbb{C}^n)$ with sufficiently large n (see Section 8.2).

6. the universal bundle

$$\gamma_r \to G_r(\mathbb{C}^\infty)$$

   is the set of all pairs $\{(P, v)\}$ where P is an element of $G_r(\mathbb{C}^\infty)$ and v a vector of P. In particular, one has the bundle

$$\gamma_1 \to \mathbb{CP}^\infty$$

   The bundle γ_r is universal for all rank r-vector bundles.

7. In the complex case, one defines the Stiefel manifold $V_r(\mathbb{C}^n)$ which is the set of r-frames in $\mathbb{C}^n$, that is the set of ordered r-uples $(v_1, \dots, v_r)$ of $\mathbb{C}$-linearly independent vectors in $\mathbb{C}^n$ (see Steenrod (1951) where the Stiefel manifold is denoted by $W'_{r,n}$).

   One has a homotopy
$$V_r(\mathbb{C}^n) \cong W_{r,n} = U(n)/U(n-r).$$

   The fibre bundle $W_{r,n} \to G_r(\mathbb{C}^n)$ is a principal bundle with fibre and structural group $U(r)$.

   The fibre bundle $W_{r,n} \to G_r(\mathbb{C}^n)$ is an universal bundle for bundles which basis has dimension $\leqslant 2(n-r)$.

   The vector bundle $\gamma_r^n \to G_r(\mathbb{C}^n)$ is a bundle associated to $W_{r,n} \to G_r(\mathbb{C}^n)$ with fibre $\mathbb{C}^r$.

8. In the complex case, $V_r(TM)$ is the bundle of complex r-frames tangent to the complex n-manifold M, i.e. the set of all pairs $(x, (v_1, \dots, v_r))$ where x is a point of M and $(v_1, \dots, v_r)$ is a r-frame in the fibre T_xM over x. That is the fibre bundle whose fibre at x is the manifold $V_r(T_xM)$ consisting of all complex r-frames in T_xM. The "typical" fibre is the Stiefel manifold $V_r(\mathbb{C}^n)$.

   Note that a section of this bundle is a r-uple of $\mathbb{C}$-linearly independent sections of the complex vector bundle TM.

Exercises

1.1) Show that hte normal bundle to the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is trivial.

1.2) Show that the total space of the bundle γ_1^1 is a Möbius band.

1.3) Let ξ be a vector bundle on B and $\eta \subset \xi$ a subbundle. Verify that the orthogonal $\xi^\perp \subset E$, called normal bundle to ξ, is a subbundle.

1.4) Let $N \subset M$ a submanifold. Verify that $TN \oplus TN^\perp = TM|_N$.

1.5) Let ξ and η two vector bundles on B. Let f be a continuous function in $\mathrm{Hom}(\xi, \eta)$, i.e. there is a collection f_x of linear applications of fibers $f_x : \xi_x \to \eta_x$. Assuming that the rank of linear applications f_x is constant on B, show that $\ker f$ is a subbundle of ξ and $\mathrm{Im} f$ is a subbundle of η.

1.6) Define a function $q : \mathbb{R}^{n+1} \to \mathbb{RP}^n$ by $q(x) = \mathbb{R}x =$ the line through x. Show that the functions $f_{ij}(\mathbb{R}x) = x_i x_j \sum x_k^2$ define a diffeomorphism between

$\mathbb{RP}^n$ and the submanifold of $\mathbb{R}^{(n+1)^2}$ consisting of all symmetric $(n+1)\times(n+1)$ matrices A of trace 1 satisfying $AA = A$.

1.7) Show that, if n is odd, then the unit sphere $\mathbb{S}^n$ admits a vector field which is nowhere zero.

1.8) If $\mathbb{S}^n$ admits a vector field which is nowhere zero, show that the identity map of S^n is homotopic to the antipodal map. For n even show that the antipodal map of S^n is homotopic to the reflection

$$r(x_1, x_2, \ldots, x_{n+1}) = (-x_1, x_2, \ldots, x_{n+1})$$

and therefore has degree -1.

1.9) Show that the line bundle γ_n^1, on $\mathbb{RP}^n$, is locally trivial but is not trivial.

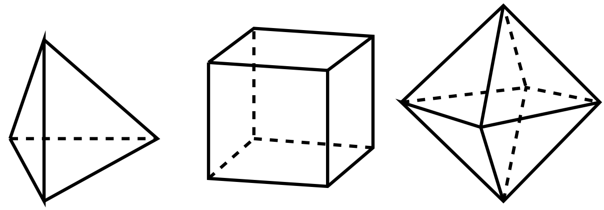
2 The "Euler–Poincaré" characteristic, from Pythagoras to Poincaré.

Since the beginning of mathematical history, men have thought to classify characteristic surfaces by assigning them philosophical and esoterical properties.

2.1 The Greek period

A regular convex polyhedra is a polyhedron whose faces are all identical (regular) and in such a way that the segment liking any two points of the polyhedron is completely included in the polyhedron (convex).

The story begins with Pythagoras of Samos ($\sim$ 570–495 B.C.) who knew three of the regular convex polyhedra: Tetrahedron (4 faces), hexahedron (cube, 6 faces) and octahedron (8 faces)

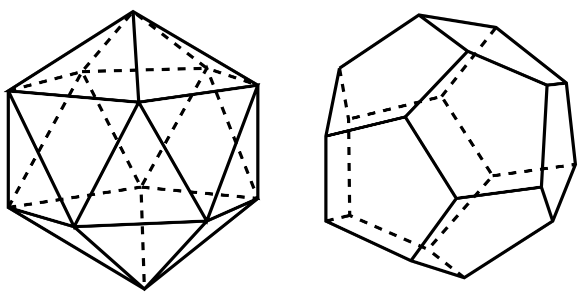


Tetrahedron V=4, E=6, F=4 | Hexahedron V=8, E=12, F=6 | Octahedron V=6, E=12, F=8

Figure 2.1: The Pythagoras polyhedra.

In the Figures 2.1 and 2.2, V is the number of vertices, E is the number of edges, and F is the number of faces.

The following two: Icosahedron(20 faces) and dodecahedron (12 faces)) have been discovered by Theaetetus of Athens ($\sim$ 415–365 B.C.) who gave a mathematical description of all five.



Icosahedron V=12, E=30, F=20 | Dodecahedron V=20, E=30, F=12

Figure 2.2: The Theaetetus polyedra

Some authors say that Theaetetus gave the first known proof that no other convex regular polyhedra exist. The five have been popularized by Plato ($\sim$ 428–348 B.C.) in his philosophical dialogue "Timaeus". Plato associates the element of earth with a cube, of air with an octahedron, of water with an icosahedron, and of fire with a tetrahedron, constituting the physical universe. The fifth element (i.e. the dodecahedron) was taken to represent the shape of the Universe as a whole, possibly because of all the elements it most approximates a sphere, that is the shape into which "God had formed the Universe". It has been suggested that certain carved stone balls created by the late Neolithic people of Scotland represent these shapes.

Euclid of Alexandria ($\sim$ 300 B.C.) completely mathematically describes the five Platonic solids in the manuscript "Elements", XIII.

2.2 Maurolico – Descartes – Euler

In this section, we provide a short history of the contributions of Maurolico, Descartes and Euler.

2.2.1 Maurolico (1494–1575)

Francesco Maurolico was an Italian priest, who lived mainly in Messina, Sicily and studied Mathematics and Astronomy. In an unpublished manuscript *Compaginationes solidorum regularium* (1537), Maurolico stated the so-called "Euler formula" for Platonic solids (see Claudia Addabbo (2015, Pages 291 and 295)):

Formula 2.2.1. *(Maurolico, December 26, 1537) "Item manifestum est in unoquoque regularium solidorum, numerum basium coniunctum cum numero cacuminum conflare numerum, qui binario excedit numerum laterum."*
In the same way it is evident that, in each regular solid, the number of faces added to that of the vertices exceeds by two the number of edges, *i.e.* Consider a Platonic solid with V vertices, E edges and F faces, then

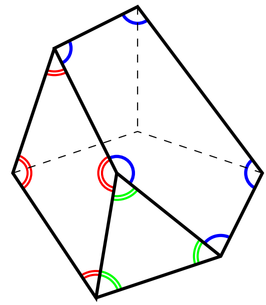
$$V - E + F = 2. \tag{2.1}$$

2.2.2 Descartes (1596–1650)

In his work *De solidorum elementis*, $\sim$ 1625, Descartes proves the following result:

Theorem 2.2.2 (Descartes)**.** *The sum of the angles of all faces of a convex polyhedron is equal to* $2(V-2)\pi$ *where* V *is the number of vertices.*

Here $V = 8$
The sum of the angles
of all faces
is $2(8-2)\pi = 12\pi$.
.



Descartes did not publish the result at that time. Invited by the Queen Christina of Sweden, to be her tutor about his philosophical ideas. Descartes arrives in Stockholm in October 1649 and is hosted by the French Ambassador to Sweden: Pierre Chanut. Descartes had habit to wake up lately, but the Queen asked him to meet on the morning around 5 in the morning. It is unclear if Descartes got a cold or got illness by taking care of Pierre Chanut, who had pneumonia or... was poisoned by a Catholic missionary who opposed his religious views. Descartes died on February, 11, 1650.

In 1653, Chanut sent Descartes' manuscripts by boat in a safe, to Claude Clerselier, his brother-in-law, in Paris. Arriving in Paris the safe sank in the Seine and was recovered 3 days later. Clerselier tried to dry the documents as best he could.

In 1672, Clerselier confided to Leibniz the Descartes' manuscripts in order to copy them because Leibniz intended to publish them. Leibniz came to Hanover in 1676 with a copy of the Descartes' manuscript but he died in 1716 without publishing the manuscript.

On the one hand, after being in the possession of several people, the original documents were eventually lost.

On the other hand, in year 1883, Foucher de Careil, France's Ambassador to Austria–Hungary and author of several works on Descartes and Leibnitz came in Hanover. He discovered between Leibnitz's documents the (copy of) Descartes manuscript *De solidorum elementis* "under the ancient dust that covered them".

Ernest de Fauque de Jonquières published, in 1890 a "Note aux CRAS" de Jonquières (1890a,b,c) in which he published the Descartes Theorem and tells the story. He claims that Descartes discovered Euler's "formula".

2.2.3 Euler (1707–1783)

The legend says that Leonhard Euler was trying to classify convex polyhedra. He proceeded as follows:

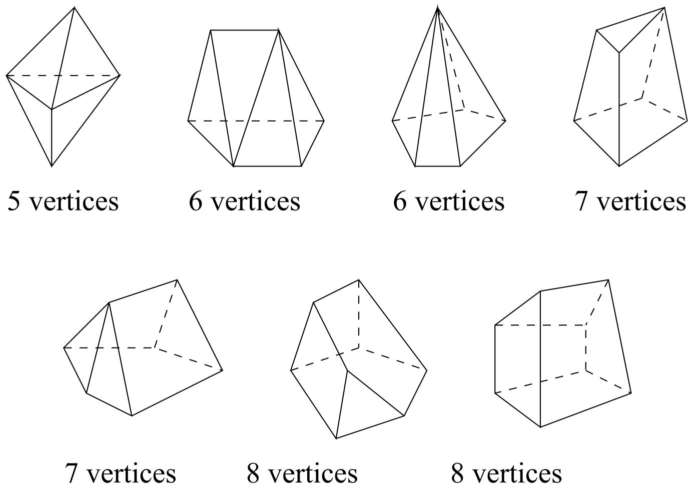


Figure 2.3: Hexahedra, (6 faces) and 5,6,7 ou 8 vertices

Euler classified the convex polyhedra first according to number of faces (for example hexahedra $F = 6$) then classifying by number of vertices (for example $V = 6$ or $V = 8$). When trying to distinguish the polyhedra with same number of faces and vertices according to the number of edges, it was impossible to do that, their number of edges was the same. Euler came to the formula:

Formula 2.2.3. *Consider a convex polyhedron with V vertices, E edges and F faces, then*

$$V - E + F = 2. \tag{2.2}$$

Euler mentioned his discovery in a letter to Christian Goldbach (November 14, 1750). Euler writes "It astonishes me that these general properties of stereometry have not, as far as I know, been noticed by anyone else". Clearly Euler did not know the Maurolico manuscript. He later (1753) published two papers in which he described what he had done in more detail and attempted to give a proof of the formula based on decomposing a polyhedron into simpler pieces. Unfortunately, his argument was not correct. However, results proven later make it possible to use Euler's technique to prove the polyhedral formula.

2.2.4 Descartes' Theorem is equivalent to Euler Formula

Let us show that the Descartes Theorem 2.2.2 is equivalent to the Euler Formula (2.2).

PROOF: Denote by $i = 1, \dots, n_2$ the 2-dimensional faces of a convex polyhedron. For each face i, denote by k_i the number of vertices, which is also the number of edges of the face. In a convex polygon with k_i edges, the sum of all the angles equals $(k_i - 2)\pi$. Since each edge of the polyhedron appears in two faces, then $\sum_{i=1}^{n_2} k_i = 2n_1$. Hence the sum of the angles of all the faces of the polyhedron equals $\sum_{i=1}^{n_2} (k_i - 2)\pi$, that is $(2n_1 - 2n_2)\pi$. We obtain equivalence between Theorem 2.2.2 and formula (2.2). □

2.2.5 Proofs of Euler Formula

Adrien-Marie Legendre (1752–1833) gave a proof (the first correct one) of Euler's formula (1794) that is correct by our standards, using a projection of the polyhedron on a sphere.

Augustin-Louis Cauchy (1789–1857) provided the first combinatorial proof of the formula (1811) and during a long time the Cauchy proof was considered as correct and has been copied in several books.

Imre Lakatos (1976) and Elon Lima (1985b) criticized Cauchy's proof, saying that one cannot find an elementary proof of Euler's formula by Cauchy's method. In fact, in Brasselet and Nguyễn Thị Bích (2021), we provided an elementary proof of the Euler formula using the Cauchy's method.

Richeson (2008, Chapter 12) explains why Cauchy proof does not work as it is.

There are nowadays many proofs of the "Euler formula" by various methods, but (out of Brasselet and Nguyễn Thị Bích (2021)) all of them use results which have been proved after Cauchy time (see the website by Eppstein (n.d.)).

2.2.6 The generalization: Euler–Poincaré characteristic

Theorem 2.2.4 (Poincaré)**.** *Let X be an n-dimensional triangulable topological space and K a triangulation of X. Let us denote by n_i the number of i-dimensional simplexes of K. The alternating sum*

$$\sum_{i=1}^{n} (-1)^{n_i} n_i$$

does not depend on the choice of the triangulation K of X.

The Theorem justifies the following definition:

Definition 2.2.5. Let X be an n-dimensional triangulable topological space and K a triangulation of X. Let us denote by n_i the number of i-dimensional simplexes of K. The Euler–Poincaré characteristic of X is the alternating sum

$$\chi(X) = \sum_{i=1}^{n} (-1)^{n_i} n_i$$

2.3 Poincaré–Hopf Theorem

The starting point of the theory of characteristic classes is the Poincaré–Hopf Theorem. The most beautiful and important results in mathematics are those linking different aspects and viewpoints. The Poincaré–Hopf Theorem is maybe the most famous of them. It is linking two invariants from topology and differential geometry.

The Poincaré–Hopf Theorem Euler–Poincaré characteristic has been proved by Henri Poincaré (1885), in the 2-dimensional case, and by Heinz Hopf (1927) for higher dimensions. The Poincaré–Hopf Theorem is the first apparition of Euler–Poincaré characteristic in differential topology, out of combinatorial topology. One of the motivations of the Poincaré–Hopf Theorem is the study of differential equations in terms of integral curves of an appropriate vector field. The singular points of the vector field are points of equilibrium in dynamical systems. That is the reason for which Poincaré–Hopf Theorem has many applications. The interested reader should experience to search for "Poincaré–Hopf Theorem" on his/her favorite web site.

There are many proofs of the Poincaré–Hopf Theorem and generalizations in the literature. With Thủy Nguyễn Thị Bích, we present (in Portuguese, see Brasselet and Nguyễn Thị Bích (2019)) a proof which allows to understand the meaning of characteristic classes: The Poincaré–Hopf Theorem states that the Euler–Poincaré characteristic of a compact oriented manifold is a measure of the obstruction to the construction of a tangent vector field to the manifold, without singularity. In that sense, the Euler–Poincaré characteristic is the first defined characteristic class.

Theorem 2.3.1. (Poincaré–Hopf) *Let M be a compact manifold with boundary ∂M, and let v be a continuous vector field tangent to M with isolated singularities.*

Denote by $a_i \in Sing(v)$ the singularities of v and $I(v, a_i)$ their indices. Then, if v is pointing outwards of M along ∂M,

$$\chi(M) = \sum_{a_i \in Sing(v)} I(v, a_i)$$

and if v is pointing inwards of M along ∂M,

$$\chi(M) - \chi(\partial M) = \sum_{a_i \in Sing(v)} I(v, a_i).$$

The main consequence of the Poincaré–Hopf Theorem is that the sum of indices of a vector field tangent to a compact manifold, with isolated singularities, does not depend on the vector field.

The first part of this section is devoted to various definitions of the index of a vector field in an isolated singularity. Then we provide a proof of the Poincaré–Hopf Theorem, in such a way to be useful for the following definitions of characteristic classes.

2.3.1 The index of a vector field.

In this section, we gives different ways to define the index of a vector field in an isolated singular point of a manifold. They provide an insight on the different viewpoints concerning the index.

A continuous *vector field* on the n-dimensional smooth manifold M is a section of the tangent bundle TM (see Section 1.6). Giving a local chart (U_a, ϕ) on M, where $\phi : U_a \to \mathbb{B}^n$ (see Definition 1.1.6), a vector field on M is locally expressed as above:

Let us denote by $\underline{x}_i = x_i \circ \phi$ the coordinate functions of ϕ, i.e. the local coordinates in U_a. We denote by $\partial/\partial \underline{x}_i$ the tangent vector at x defined by

$$\frac{\partial}{\partial \underline{x}_i}(h) = \frac{\partial}{\partial x_i}(h \circ \phi^{-1})|_{\phi(x)}$$

for a C^∞ function $h : M \to \mathbb{R}$.

Definition 2.3.2. Let us denote by $\underline{x} = (\underline{x}_1, \dots, \underline{x}_n)$ the local coordinates of the manifold M in the open neighbourhood U_a, a vector field v can be written in terms of the basis $\partial/\partial \underline{x}_i$ of the tangent vector space $T_x M$

$$v = \sum_{i=1}^{n} f_i \frac{\partial}{\partial \underline{x}_i}. \tag{3.3}$$

The functions $(f_1, \dots, f_n)$ are called coordinates of the vector v in U_a. The vector field is said to be continuous, smooth, analytic, according as its components $\{f_1, \dots, f_n\}$ are continuous, smooth, analytic, respectively.

For simplicity, in the following, we will identify coordinates in U_a and $\mathbb{B}^n$, omitting ϕ and we will denote x_i for $\underline{x}_i$.

A singularity ("first type singularity") a of the vector field v is a point at which all coordinate f_i vanish.

Let M be a differentiable manifold of dimension n. The tangent bundle to M, denoted by TM, is a real vector bundle (see Section 1.6) of rank n, whose fibre at a point x of M is the tangent space to M at x, denoted by $T_x(M)$ and is isomorphic to $\mathbb{R}^n$. The vector bundle TM is locally trivial, *i.e.* there is a covering of M by open subsets $\{U_a\}$ such that the restriction of TM to each U_a is homeomorphic to $U_a \times \mathbb{R}^n$.

We denote by s_0 the zero section of TM such that $s_0(x) = 0 \in T_x M \cong \mathbb{R}^n$. We will consider the fiber bundle (not any more a vector bundle) $T^{\times}M = TM \setminus s_0(M)$. Its fibre at a point $x \in M$ is $T_x^{\times}M \cong \mathbb{R}^n \setminus \{0\}$.

Let us consider a ball $B(a)$ centred in a, contained in an open chart U_a over which TM is trivial and sufficiently small so that a is the only singular point of v in $B(a)$. One can think of $B(a)$ as an n-cell, in view of the generalisation we will perform later (1.1). The vector field v defines a section of TM without zero over $S(a) = \partial B(a)$, hence a map

$$S(a) \cong \mathbb{S}^{n-1} \xrightarrow{v} T^{\times}M|_{U_a} \cong U_a \times (\mathbb{R}^n \setminus \{0\}) \xrightarrow{pr_2} \mathbb{R}^n \setminus \{0\} \tag{3.4}$$

where pr_2 is the second projection.

One obtains a map

$$\mathbb{S}^{n-1} \cong \partial B(a) \xrightarrow{pr_2 \circ v} \mathbb{R}^n \setminus \{0\}$$

which defines an element $\lambda(v, a)$ in the homotopy group $\pi_{n-1}(\mathbb{R}^n \setminus \{0\})$. This homotopy group is isomorphic to $\mathbb{Z}$ and allows to define:

Definition 2.3.5. The (local) index $I(v, a)$ of the vector field v at the isolated singularity a is the integer corresponding to $\lambda(v, a)$ by the isomorphism $\pi_{n-1}(\mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z}$.

Let us consider the Gauß map

$$\gamma : \partial B(a) = S(a) \cong \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1} \subset \mathbb{R}^n \setminus \{0\}$$

defined by $\gamma(x) = v(x)/\|v(x)\|$.

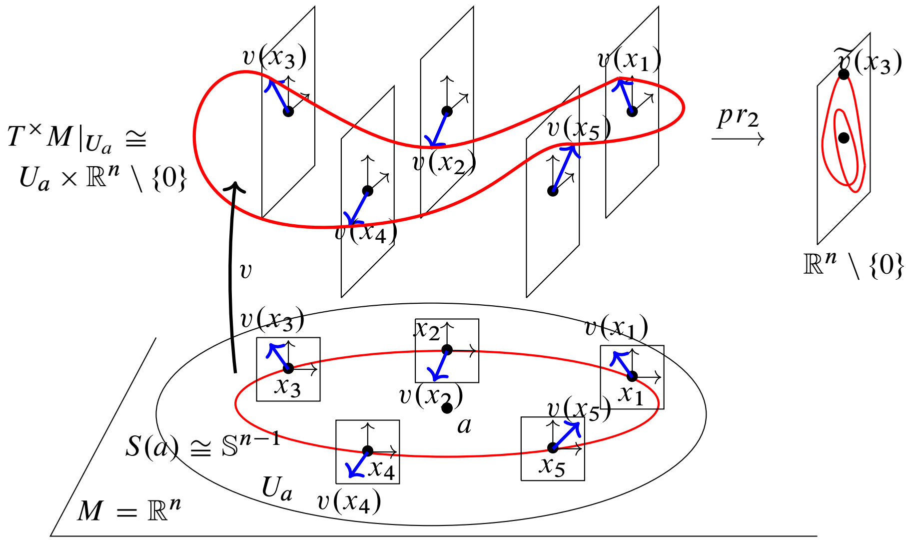


Figure 2.4: The map $\mathbb{S}^{n-1} \to \mathbb{R}^n \setminus \{0\}$.

Proposition 2.3.6. *The index of* v *at* a *is equal to the degree of the Gauß map* $\gamma : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. *That is the degree of the induced map in homology:*

$$\gamma_* : H_{n-1}(\mathbb{S}^{n-1}) = \mathbb{Z} \to H_{n-1}(\mathbb{S}^{n-1}) = \mathbb{Z},$$

i.e. γ_* *is the multiplication by* $I(v, a)$.

The local index does not depend neither on the choice of the small ball $B(a)$ such that there is no other singularity of v within $B(a)$, nor on the choice of coordinates and on the choice of orientation.

By classical homotopy theory, the map

$$\mathbb{S}^{n-1} \cong \partial B(a) \xrightarrow{pr_2 \circ v} \mathbb{R}^n \setminus \{0\}$$

extends to a map

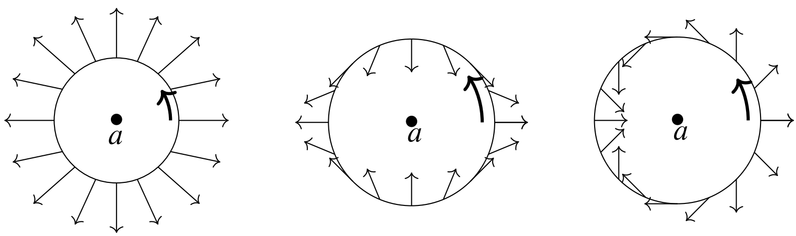
$$B(a) \longrightarrow \mathbb{R}^n \setminus \{0\}$$

if and only if the element $\lambda(v, a)$ is zero in $\pi_{n-1}(\mathbb{R}^n \setminus \{0\})$. In other words, the vector field v extends without singularities within the ball $B(a)$ if and only if the index $I(v, a)$ is zero.

$$\begin{tikzcd}
\partial B(a) \cong \mathbb{S}^{n-1} \arrow[r] \arrow[d, hook] & \mathbb{R}^n \setminus \{0\} \\
B(a) \cong \mathbb{B}^n \arrow[ur, dashed] &
\end{tikzcd} \tag{3.5}$$

That construction is the basis of the obstruction theory, it will be generalised in Chapter 3.

In dimension 2, one recovers the well known index:



Index = $+1$ Index = -1 Index = $+2$

Figure 2.5: Vector fields in dimension 2.

Remark 2.3.8. In the following, we will use also a different kind of singularities for a vector field, that Marie-Hélène Schwartz called *second type* singularities and that we describe now.

Given a vector field v defined on the boundary $S(a)$ of the ball $B(a)$ of radius 1, centered at a, there are many ways to extend the vector field inside $B(a)$. Two are the most natural. Let us denote by $S_\varepsilon(a)$ the sphere of radius ε, $0 < \varepsilon \leqslant 1$. If $x \in S(a)$ the vector $v(\varepsilon x)$ at the point $\varepsilon x \in S_\varepsilon(a)$ is defined either as $v(\varepsilon x) = \varepsilon v(x)$ or as $v(\varepsilon x) = v(x)$.

In the first case, the vector field v will vanish at a, that is the already defined singularity type. We call it, according to Marie-Hélène Schwartz, *singularity of first type*.

In the second case the extension is not defined at a (see Schwartz (1991)), but it defines a cycle $\kappa(v)$ in the fibre $T_a\mathbb{R}^n$ of the tangent bundle to $\mathbb{R}^n$. The cycle

$$\kappa(v) = \{u \in T_a\mathbb{R}^n \mid \exists x \in S(a), u = v(x)\}$$

is the set of limits of vectors $v(x)$ when x tends to a. According to Marie-Hélène Schwartz, we will call this singularity, *singularity of second type*.

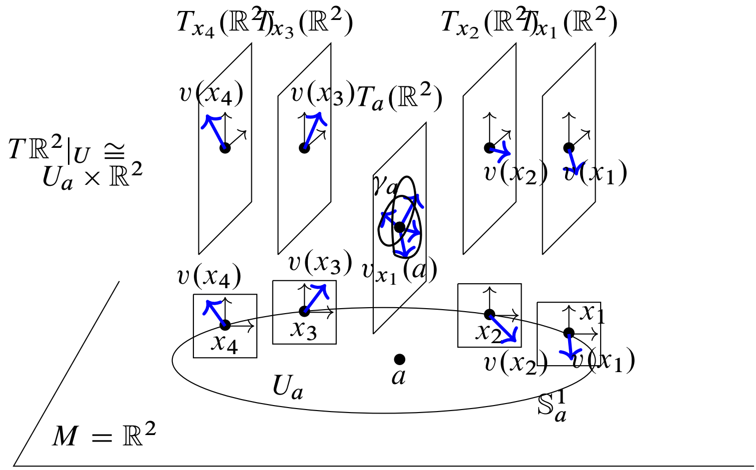


Figure 2.6: Singularity of second type ($n = 2$).

Figure 2.6 shows the vector field (application $v : \mathbb{S}^1_a \to \mathbb{R}^2 \cong T_a(\mathbb{R}^2)$) and the cycle γ_a in $T_a(\mathbb{R}^2)$.

Whatever the type of singularity, the index $I(v, a)$ of v at the isolated singularity a is well defined and does not depend on the involved choices:

Proposition 2.3.9. *Let us consider a second type singularity, then the index of the cycle $\kappa(v)$ in the punctured fibre $T_a\mathbb{R}^n \setminus \{0\}$ is equal to $I(v, a)$.*

That is, by identification $H_{n-1}(T_a\mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z}$, the image of $[\kappa(v)]$ is $I(v, a)$.

PROOF: Let us denote by s_0 the zero section of the tangent vector bundle $T\mathbb{R}^n$. The tangent bundle $T\mathbb{R}^n$ is trivial over $B(a)$, as well as the bundle $T^\times\mathbb{R}^n = T\mathbb{R}^n \setminus \mathrm{Im} s_0$ (not anymore a vector bundle). The fibre of $T^\times\mathbb{R}^n$ at a is $T_a\mathbb{R}^n \setminus \{0\} \cong \mathbb{R}^n \setminus \{0\}$ and, restricted to $B(a)$, the bundle is homeomorphic to $B(a) \times (\mathbb{R}^n \setminus \{0\})$. The vector field v defines a section of $T^\times\mathbb{R}^n$ over $S(a)$ whose image by the second projection $B(a) \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n \setminus \{0\}$ is equal to $\kappa(v)$, by definition. One concludes by the Definition 2.3.5. $\square$

Definition 2.3.10. One says that two vector fields v and v' defined on a subset U and value in TU are homotopic if there is a continuous map $\Phi : (U, [0, 1]) \to TU$ such that $\Phi(x, 0) = v(x)$ and $\Phi(x, 1) = v'(x)$ for all x.

Lemma 2.3.11. *If v and v' are two vector fields with an isolated singularity at a, then their indices at a coincide if and only if they are homotopic through a family of (non zero) vector fields, defined on a neighbourhood of the point a.*

2.3.2 Relation with the Gauß map

Let N be a compact k-manifold with boundary in $\mathbb{R}^k$. The boundary : ∂N is a smooth hypersurface in $\mathbb{R}^k$ and one can consider the Gauß map 5.3

$$\gamma : \partial N \to \mathbb{S}^{k-1}$$

which assigns to each $x \in \partial N$ the outward unit normal vector $\nu(x)$ at x. The degree of the Gauß map is the integer corresponding to the image of the fundamental class $[\partial N]$ of ∂N by the map

$$\gamma_* : H_{k-1}\partial N) \to H_{k-1}(\mathbb{S}^{k-1}) \cong \mathbb{Z}.$$

Proposition 2.3.12. *[Hopf] (Milnor (1965), §6, Lemma 3) If v is a smooth vector field on N with isolated singularities a_i and v points outwards from N along the boundary, then the sum of indices $\sum I(v, a_i)$ equals the degree of the Gauß mapping from ∂N to $\mathbb{S}^{k-1}$.*

Proof: Let us consider small balls $B_\eta(a_i)$ centered at a_i with sufficiently small radius so that the balls lie in the interior of N and so that they do not meet each other. The manifold

$$N \setminus \bigcup_i B_\eta(a_i)$$

is a smooth variety with boundary

$$\partial\left[N \setminus \bigcup_i B_\eta(a_i)\right] = \partial N \bigcup - \left[\bigcup_i \partial B_\eta(a_i)\right],$$

where the sign "−" comes from the fact that the orientation of $\partial B_\eta(a_i)$ as boundary of the ball $B_\eta(a_i)$ is the opposite of the orientation of $\partial B_\eta(a_i)$ as element of the boundary of $N \setminus \bigcup_i B_\eta(a_i)$.

The Gauß map

$$\gamma_v : \partial\left[N \setminus \bigcup_i B_\eta(a_i)\right] \to \mathbb{S}^{k-1}, \quad \text{associated to } \widetilde{v}(x) = \frac{v(x)}{\|v(x)\|},$$

which is well defined on the boundary of $N \setminus \bigcup_i B_\eta(a_i)$, can be extended as an application

$$\gamma_v : N \setminus \bigcup_i B_\eta(a_i) \to \mathbb{S}^{k-1}$$

without singularity. In fact, all singularities of $\widetilde{v}$ are inside the balls $B_\eta(a_i)$.

We know that if a vector field v, defined on the boundary of a ball $B(a)$, extends without singularities within $B(a)$, then the index $I(v, a)$ (its degree) is 0. That is a particular case of an important property of the degree: if a map $f : \partial Y \to \mathbb{S}^{k-1}$ defined on the boundary of a k-dimensional manifold Y in $\mathbb{R}^k$ can be extended without singularities inside Y, then $\deg(f) = 0$. As γ_v is extended without singularity to an application $\gamma_v : N \setminus \bigcup_i B_\eta(a_i) \to \mathbb{S}^{k-1}$, this implies that the degree of γ_v on the boundary of $N \setminus \bigcup_i B_\eta(a_i)$ is 0. In other words, the sum of degrees of γ_v on the components of the boundary of $N \setminus \bigcup_i B_\eta(a_i)$ is 0 and one has (see Lemma 1.5.3):

$$\deg(\gamma_v; \partial N) - \sum_i \deg(\gamma_v; \partial B_\eta(a_i)) = 0,$$

where the sign "–" comes from orientation.

On the one hand, $\widetilde{v}$ is pointing outwards of N along the boundary and $\widetilde{v}$ can be deformed continuously to the vector field ν of normal vectors to the boundary along the boundary and pointing outwards. The deformation can be realized by a continuous homotopy with vectors pointing outwards of N along the boundary. The degree, which is an integer, remains the same and one has $\deg(\gamma_v; \partial N) = \deg(\nu)$. On the other hand, by definition of the index, one has $\deg(\gamma_v; \partial B_\eta(a_i)) = I(v, a_i; N) = I(v, a_i)$. Finally, one has

$$\sum_i I(v, a_i) = \deg(\nu).$$

□

Let us consider a compact n-manifold without boundary $M \subset \mathbb{R}^k$. Let N_ε denote the closed ε-neighbourhood of M (i.e. the set of all $x \in \mathbb{R}^k$ with $\|x - y\| < \varepsilon$ for some $y \in M$). For ε sufficiently small, N_ε is a smooth manifold with boundary.

Theorem 2.3.13. *For any continuous vector field v with isolated singularities a_i, on a compact manifold without boundary $M \subset \mathbb{R}^k$, the index sum $\sum I(v, a_i)$ is* *equal to the degree of the Gauß map*

$$g : \partial N_\varepsilon \to \mathbb{S}^{k-1},$$

where N_ε denotes the closed ε-neighbourhood of M in $\mathbb{R}^k$.

We produce a proof inspired by Marie-Hélène Schwartz (1964) and by John Milnor (1965), §6, Theorem 1. That proof has been delivered by Milnor in December 1963 in lectures in University of Virginia. The procedure used by Milnor is the same as the one developed independently and at the same time by Marie-Hélène Schwartz (1964), in her definition of radial extension in the framework of stratified singular varieties (see Section 5.4). In fact, Milnor gives the proof for non-degenerate vector fields, that is with index either $+1$ or -1. Marie-Hélène Schwartz gives the proof for vector fields with isolated singularities of any index. We will follow her proof.

The idea is to extend a vector field v defined on the manifold M with index $I(v, a; M)$ at the isolated singularity a, as a vector field w in the ambient space $\mathbb{R}^k$ that has also an isolated singularity at a with the same index $I(w, a; \mathbb{R}^k) = I(v, a; M)$. The principle is to sum the parallel extension of v in a neighbourhood of a with a "transverse" vector field.

Proof: For $x \in N_\varepsilon$, let $r(x)$ be the closest point of M. The vector $x - r(x)$ is perpendicular to the tangent space of M at $r(x)$, for otherwise, $r(x)$ would not be the closest point of M. If ε is sufficiently small, then the restriction $r(x)$ is smooth and well defined.

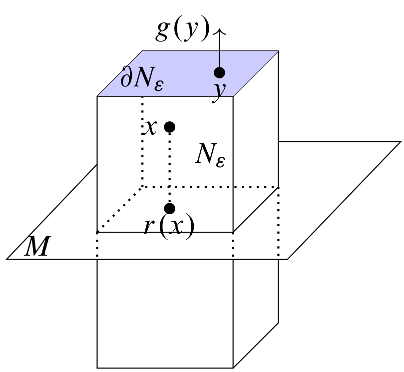


Figure 2.7: The ε-neighbourhood of M

We consider the squared distance function (for the Euclidean metric in $\mathbb{R}^k$):

$$\varphi(x) = \|x - r(x)\|^2$$

whose gradient vector field is

$$\mathrm{grad}\varphi(x) = 2(x - r(x)).$$

On one hand, the gradient vector field is a vector field defined in N_ε that is zero along M, that is transverse to ∂N_ε pointing outwards from N_ε and that increases with the distance to M. For each point x at the level surface $\partial N_\varepsilon = \varphi^{-1}(\varepsilon^2)$, the outward unit normal vector, called transverse vector, is given by

$$g(x) = \mathrm{grad}\varphi(x)/\|\mathrm{grad}\varphi(x)\| = (x - r(x))/\varepsilon.$$

On the other hand, we consider, in each point $x \in N_\varepsilon$, the vector $v_1(x) = v(r(x))$ which is a parallel extension of v.

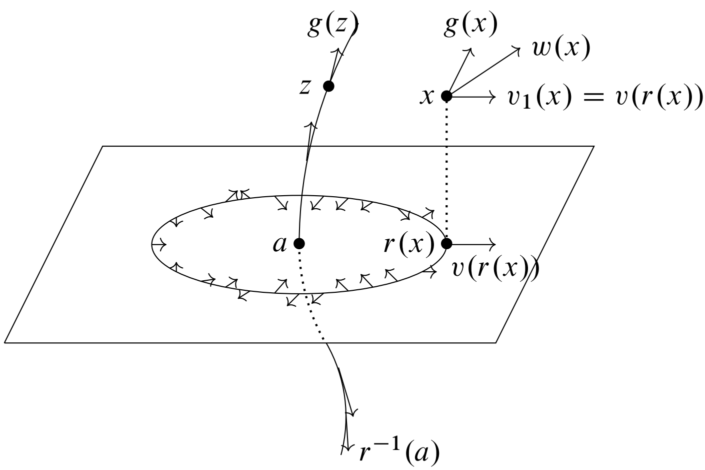


Figure 2.8: The vector field $w(x)$.

We extend v to a vector field w on the neighbourhood N_ε by defining $w(x)$ as the sum of the transverse vector field $g(x)$ and the parallel vector field $v(r(x))$:

$$w(x) = g(x) + v_1(x) = (x - r(x))/\varepsilon + v(r(x)).$$

The vector field w points outwards along the boundary ∂N_ε, since the inner product $w(x) \cdot g(x)$ is equal to $\varepsilon > 0$. In fact w vanish only at the zeros of v in M. That is clear because the two summands $(x - r(x))$ and $v_1(x)$ are orthogonal.

Considering the corresponding Gauß maps for v and w, an easy computation shows that their degrees are equal. Hence the index of w at the zero a, computed in $\mathbb{R}^k$ is equal to the index of v at a, computed in M. That is

$$I(w, a; \mathbb{R}^k) = I(v, a; M)$$

Now, according to the Proposition 2.3.12, the index sum $\sum I(v, a)$ is equal to the degree of g which proves the theorem. $\square$

The Theorem is another way to see that if M is compact, the sum $\sum I(v, a_i)$ for all singularities of v does not depend on v. We will see (Theorem 5.4.6) that, with suitable vector fields, the result extends to the case of singular varieties.

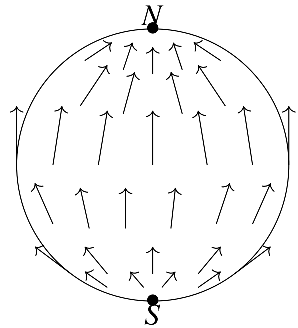
2.4 Proof of Poincaré–Hopf Theorem

There are many ways to prove Poincaré–Hopf Theorem. They correspond to the different viewpoints and definitions of the index. The interested reader can consult Lima (1985a), Milnor (1965) (Hopf and Gauß map), Guillemin and Pollack (1974) (Lefschetz fix points theory), Hirsch (1994) (Intersection numbers).

2.4.1 The smooth case without boundary

Theorem 2.4.1. *[Poincaré–Hopf Theorem] Let M be a compact differentiable manifold without boundary, and let v be a continuous vector field on M with finitely isolated singularities a_i. One has*

$$\chi(M) = \sum_i I(v, a_i)$$



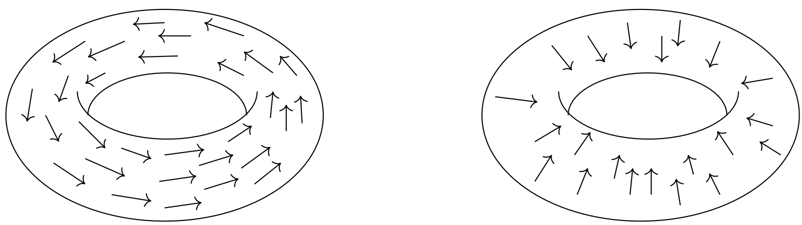


Figure 2.9: Vector fields on the sphere and the torus.

Figure 2.9 illustrates the Poincaré–Hopf Theorem in dimension 2: On the sphere $\mathbb{S}^2$, one has $I(v, N) = I(v, S) = +1$. On the torus, the parallel vector field and the transverse vector field have no singularity.

PROOF: Firstly we prove the Theorem in the orientable case, then in the non-orientable case. We will follow the Milnor proof which is close to the generalisation to singular varieties that we will provide in the next chapters.

1) Orientable case.

The idea of the proof is the following: In a first step, one shows that the sum of indices of a continuous tangent vector field with isolated singularities does not depend of the choice of the vector field. The second step of the proof consists in describing a particular vector field for which the sum of indices is equal to $\chi(M)$.

For the first step, Theorem 2.3.13 provides directly the result.

For the second step, such a vector field is given, for example, by the Hopf vector field H of which we recall the construction (see Steenrod (1951), p. 202). Let us consider a triangulation K of M and a barycentric subdivision K' of K. The Hopf vector field will be tangent to simplexes of K', with a singularity at every vertex of K', i.e. at every barycenter of K. An n dimensional simplex will be denoted by σ^n and its barycenter by $\widehat{\sigma}^n$. On every 1-simplex $[\widehat{\sigma}^i, \widehat{\sigma}^j]$ of K', where $\widehat{\sigma}^i$ is barycenter of σ^i, and $i < j$, the vector field H is going in the direction from $\widehat{\sigma}^i$ to $\widehat{\sigma}^j$. For example it is going outwards from all vertices of K. The Figure 2.11 illustrates the case $n = 2$ and provides an idea of the general case.

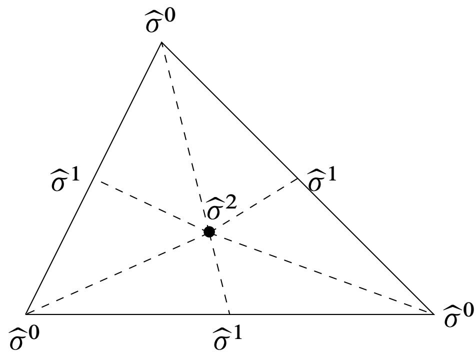


Figure 2.10: Barycenters and barycentric subdivision K'.

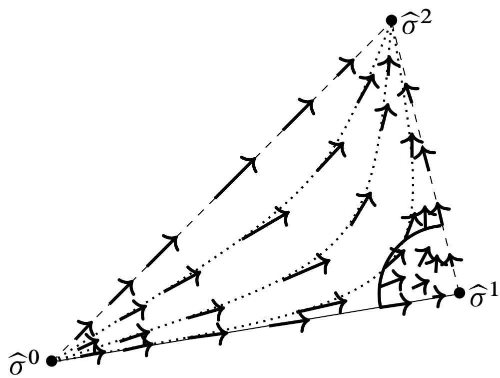


Figure 2.11: The Hopf vector field H in a triangle $[\widehat{\sigma}^0, \widehat{\sigma}^1, \widehat{\sigma}^2]$.

The Hopf vector field H has a singularity of index $(-1)^i$ at the barycenter of every i-simplex of K. The sum of indices of H at all singularities is $\sum_{i=0}^{n}(-1)^i k_i$ where k_i is the number of i-dimensional simplexes of K, so it is equal to $\chi(M)$.

2) The non-orientable case:

Let us consider the oriented double covering $\pi : \widetilde{M} \to M$. On the one side, if v is a continuous vector field on M with isolated singular points a_i of index $I(v; a_i)$, then on can define a lifting $\widetilde{v}$ of v which is a continuous vector field on $\widetilde{M}$ with isolated singular points a_i^j, $j = 1, 2$ such that $\pi(a_i^j) = a_i$. As π is a local homeomorphism, one has $I(v; a_i^j) = I(v; a_i)$ for $j = 1, 2$. One obtains $\sum_{i,j} I(v; a_i^j) = 2 \sum_i I(v; a_i)$. On the other side, $\chi(\widetilde{M}) = 2\chi(M)$ (cf Lemma 1.2.11). One conclude the Poincaré–Hopf Theorem :

$$\chi(M) = 1/2 \cdot \chi(\widetilde{M}) = 1/2 \sum_{i,j} I(v; a_i^j) = \sum_i I(v; a_i). \qquad \square$$

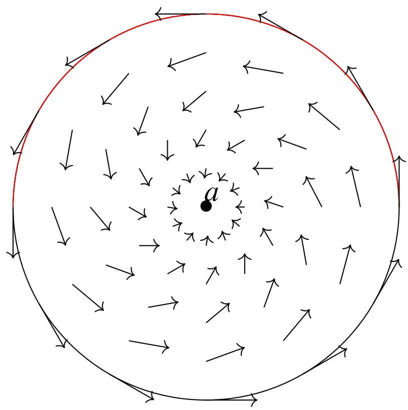


Figure 2.12: Poincaré–Hopf Theorem for the real projective plane.

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Figure 2.12 illustrates the Poincaré–Hopf Theorem for the real projective plane. That is a non-orientable surface with $\chi(\mathbb{RP}^2) = 1$. For the pictured vector field v, one has $I(v, a) = +1$

Consequences of Poincaré–Hopf Theorem

As an important consequence of the Poincaré–Hopf Theorem, one has the following

Corollary 2.4.2. *Let M be a compact smooth manifold, if $\chi(M) \neq 0$, then any continuous vector field tangent to the manifold M admits at least a singular point. Reciprocally, every compact manifold such that $\chi(M) = 0$ admits a continuous tangent vector field without singularities.*

The unitary sphere $\mathbb{S}^n$ with odd n satisfies $\chi(\mathbb{S}^n) = 0$ and admits continuous tangent vector fields without singularities. If n is even, $\chi(\mathbb{S}^n) = 2$ and in that case every continuous vector field tangent to $\mathbb{S}^n$ admits at least one singularity.

Corollary 2.4.3. *Every compact odd dimensional manifold admits a continuous tangent vector field without singularity.*

The torus and the Klein bottle are the only one compact 2-dimensional surfaces admitting a non-zero continuous tangent vector field.

Lemma 2.4.4. *For even-dimensional hypersurfaces, the Euler–Poincaré characteristic $\chi(M)$ equals twice the degree of the Gauß map γ.*

PROOF: Take the projection $\pi : \mathbb{S}^n \to \mathbb{RP}^n$ and a regular value $p \in \mathbb{RP}^n$ of the composed map $\pi \circ \gamma : M \to \mathbb{RP}^n$. Take a differentiable vector field w on $\mathbb{S}^n$ with isolated singularities in $\{a, b\} = \pi^{-1}(p)$ of indices $+1$. The vector field v on M such that $v(x) = w(\gamma(x))$ has a finite number of isolated singularities $\{a_1, \ldots, a_r\} = \gamma^{-1}(a)$ and $\{b_1, \ldots, b_s\} = \gamma^{-1}(b)$. One one hand, one has $\deg(\gamma) = \sum_{i=1}^{r} I(v; a_i) = \sum_{j=1}^{s} I(v; b_j)$, on the other hand $\chi(M) = \sum_{i=1}^{r} I(v; a_i) + \sum_{j=1}^{s} I(v; b_j)$. That gives the Lemma. □

Note that for odd-dimensional hypersurfaces, one has $\chi(M) = 0$. That is the case of the sphere $\mathbb{S}^n$, of the real projective space $\mathbb{RP}^n$, with n odd.

2.4.2 The smooth case with boundary

Let M be an oriented manifold with boundary, one has a similar theorem:

Theorem 2.4.5. *[Poincaré–Hopf Theorem with boundary] Let M be a compact manifold with boundary ∂M embedded in an oriented differentiable manifold N. Let v be a non-singular continuous vector field tangent to M, strictly pointing outwards (resp. inwards) of M along the boundary ∂M. Then:*

1. *v can be extended to the interior of M as a vector field tangent to M with finitely many isolated singularities a_i.*

2. *The total index of v in M is independent of the way we extend it to the interior of M. In other words, the total index of v is determined by its behaviour near the boundary.*

3. *If v is everywhere transverse to the boundary and pointing outwards from M, then one has*

$$\chi(M) = \sum_i I(v, a_i). \tag{4.6}$$

If v is everywhere transverse to ∂M and pointing inwards then

$$\chi(M) - \chi(\partial M) = \sum_i I(v, a_i). \tag{4.7}$$

PROOF: The first statement is proved by obstruction theory (Section 2.3.1). The vector field can be extended without singularities to the $(n-1)$-skeleton of M. Then we extend it to the n-cells introducing (if necessary) a singular point for each n-cell.

The second statement is also a general result in obstruction theory, consequence of statement 3. (see also Section 3.1 and Steenrod (1951)).

A proof of the third statement goes in the following way: As in Theorem 2.3.13, on consider the closed ε-neighbourhood of M, denoted by N_ε. If the vector field is pointing outwards of M along ∂M, then it can be extended over the neighbourhood N_ε so that the extended one points outwards of N_ε along ∂N_ε. The extension w is defined as before by $w(x) = (x - r(x)) + v(r(x))$ and is a continuous vector field near ∂M. In this case, N_ε is not necessarily of class $\mathcal{C}^\infty$, but only a $\mathcal{C}^1$-manifold. Nevertheless, the same argument as in the case "without boundary" can be carried out (see Milnor (1965) §6), that gives (4.6).

If the vector field is pointing inwards along ∂M, one can extend v inside M with finitely many isolated singularities a_i of index $I(v, a_i)$.

One proceeds to the following construction: the boundary ∂M admits a neighbourhood $\partial M \times [0,1]$ in M and one can extend this neighbourhood as $\partial M \times [0,2]$. Let us call M' the new manifold $M \cup (\partial M \times [0,2])$. One has $\chi(M') = \chi(M)$ and $\partial M' \cong \partial M$. Let us call C the "collar" $\partial M \times [1,2]$. One has $\chi(C) = \chi(\partial M)$.

At the level $C_1 = \partial M \times \{1\}$, one has the vector field v pointing inwards of M and outwards of C. At the level $C_2 = \partial M \times \{2\}$, one considers any vector field v' pointing outwards of M' along $\partial M'$. Let us call w the vector field defined on ∂C which is equal to v and v' on C_1 and C_2 respectively. The vector field w is defined on the boundary of C and pointing outwards of C along the boundary. By (4.6) on C, one can extend w inside C with finitely many isolated singularities b_j and one has

$$\chi(C) = \chi(\partial M) = \sum_j I(w, b_j).$$

On M' one consider the vector field v', which is v on M and w on C. It has isolated singularities a_i and b_j and it is pointing outwards from M'. Again one can apply (4.6) (on M') and one has

$$\chi(M) = \chi(M') = \sum_i I(v, a_i) + \sum I(w, b_j) = \sum_i I(v, a_i) + \chi(\partial M)$$

and the result. □

Corollary 2.4.8. *Let us suppose that M is odd-dimensional, then*

$$\chi(\partial M) = 2 \cdot \chi(M)$$

PROOF: Let us denote by v a vector field pointing outwards M along the boundary, as in 4.6 and let us consider the vector field $w = -v$. Then, w has same singularities than v and, as M is odd-dimensional, in each singularity a_i, one has $I(w, a_i) = -I(v, a_i)$. Equations (4.6) and (4.7) provide the result. □

The interested reader will find in Elon Lima (1985a), Lefschetz (1930) and Alexandroff and Hopf (1935) other expressions of the Euler–Poincaré characteristic for instance in terms of curvature (Gauß–Bonnet Theorem).

Exercises

2.1) Recover classical examples of indices $+1$, -1, $+2$ in $\mathbb{R}^2$ by obstruction theory.

3 Characteristic classes : smooth case

In 1935 and independently, Stiefel , who was student of Hopf, and Whitney . defined characteristic classes in cohomology for real manifolds. Stiefel considers the obstruction point of view (for the construction of r-frames tangent to the manifold), computing homotopy groups of so-called Stiefel manifolds. Whitney considers sphere bundles on a manifold M and defines cohomology classes with coefficients in $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. The Stiefel and Whitney methods are similar and represent the basis of obstruction theory. We call Stiefel–Whitney classes of a vector bundle or of the associated sphere bundle, the classes obtained in that way.

In 1942 Pontrjagyn defined classes for Grassmannian manifolds, using a decomposition of these manifolds in terms of Schubert varieties, due to Ehresmann.

In his fundamental 1946 paper S.-s. Chern (1946), Chern gave several constructions of characteristic classes for Hermitian Manifolds. The paper provides basement for the relationship between obstruction theory, Schubert varieties, differential forms and transgression.

Contribution of Wu Wen Tsün in the history of characteristic classes is important. The Stiefel–Whitney classes are the Steenrod squares of the Wu classes defined by Wu Wen Tsün in 1955. Wu Wen Tsün proved the product formula for Stiefel–Whitney and Chern classes, he gave also a simple formulation of the decomposition of the Grassmannian manifold of oriented vector subspaces and he extended the definition of Chern classes for any complex vector space on any finite simplicial complex.

As it happens often in Mathematics, one object, here the Chern classes, has (at least) two definitions: the geometric definition allows to understand the signification of classes, but it is difficult to proceed to effective computations in this context. The axiomatic definition provides easy ways to compute effectively the classes but is less suitable to understanding the origin and the meaning of the classes. We give in Sections 3.2.1 and 3.2.2 the axiomatic definition of characteristic classes.

A trivial bundle is induced from a map to a point, so all its characteristic classes (except the zero dimensional one) should be zero. More generally, equality of all characteristic classes of two bundles is a necessary (and in some circumstances sufficient) test for their equivalence. That is one of the important applications of characteristic classes.

The interested reader will find all wished references in the Dieudonné book Dieudonné (1989, §3, IV).

3.1 General obstruction theory

Let us recall the idea of the construction of characteristic classes by obstruction theory, following Steenrod (1951, part III).

We have seen that the meaning of Poincaré–Hopf Theorem is that the Euler–Poincaré characteristic of a manifold M is a measure of the obstruction for the construction of a vector field tangent to M. In a more general way, the aim of the obstruction theory is to define invariants providing a measure of the obstruction to the construction of linearly independent sections of vector bundles. In a more precise way, the objective is to answer to questions of the following type:

Let E be a vector bundle of rank n on a variety X and fix r such that $1 \leqslant r \leqslant n$, is it possible to construct r sections of E, linearly independent everywhere?

It is obviously possible to define such sections on the 0-skeleton of a triangulation of X. So, the question becomes the following:

Let us consider a triangulation of X. Performing the construction of r independent sections by increasing dimension of the simplexes, up to what dimension can we proceed? Arriving to this obstruction dimension, is it possible to evaluate the obstruction?

At that point, let us make a comment: Classical obstruction theory uses a triangulation of the considered space. In the following we will use a slightly different viewpoint, taking into account the fact that, later on, we will deal with the singular case. It appears that in the singular case, the good decomposition to be taken into account for the construction of the sections is not a triangulation of the space but a dual cell decomposition in the ambient space. That is the reason for which, we will work on a cell decomposition, already in the non-singular case.

In a first step, we study the case of the (real) tangent bundle to a differentiable smooth manifold or the (complex) tangent bundle to an analytic complex manifold. We will denote by $\mathbb{K}$ the field $\mathbb{R}$ or $\mathbb{C}$, according to the situation.

Let M be a manifold of dimension n, over $\mathbb{K}$, endowed with an euclidean (or hermitian) metric. The tangent bundle to M, denoted by TM, is a vector bundle of rank n over $\mathbb{K}$, whose fibre in a point x of M is the tangent vector space to M in x, denoted by $T_x(M)$ and is isomorphic to $\mathbb{K}^n$. The vector bundle TM is locally trivial, *i.e.* there is a covering of M by open subsets such that the restriction of TM to U is isomorphic to $U \times \mathbb{K}^n$.

The objective is to evaluate the obstruction to the construction of r sections of TM linearly independent (over $\mathbb{K}$) at each point, *i.e.* an r-frame:

Definition 3.1.1. An *r-field* on a subset A of M is a set $v^{(r)} = \{v_1, \dots, v_r\}$ of r continuous vector fields tangent to M, defined on A. A singular point of $v^{(r)}$ is a point where the vectors (v_i) fail to be linearly independent. A non-singular r-field is called an *r-frame*.

The r-frames are sections of the fibre bundle $V_r(TM)$ over M. That is the fibre bundle associated to TM and whose fibre at the point x of M is the set of r-frames of $T_x(M)$. The fibre is the Stiefel manifold denoted by $V_r(\mathbb{K}^n)$ that we described in Section 1.6.3 (8) in the real case and in Section 1.6.4 (8) in the complex case.

To construct r linearly independent sections of TM over a subset A of M is equivalent to construct a section of $V_r(TM)$ over A.

Let us consider the following situation: (K) is a cell decomposition of M sufficiently small so that every cell d is included in an open subset U over which $V_r(TM)$ is trivial. One remarks that trivialisation open sets for $V_r(TM)$ are the same that the ones of $T(M)$. There exists always such a cell decomposition (taking subdivision if necessary).

Let us consider the following question:

Let us suppose that one has a section $v^{(r)}$ of $V_r(TM)$ on the boundary ∂d of the k-cell d. Is it possible to extend this section in the interior of d ? Is the answer is no, can we evaluate the obstruction for such an extension ?

In order to answer the question, we need to define the notion of index of an r-field in a singular point and we need some notions and results on general obstruction theory. That is aim of the following sections. Then we will apply these results to the real and complex case, that is to define Stiefel–Whitney and Chern classes.

3.1.1 Index of an r-frame

Let us consider an r-field $v^{(r)}$ defined on the boundary ∂d of a k-cell of the cell decomposition (D) of M. In the same way than in Section 2.3.1, $v^{(r)}$ is a section of the bundle $V_r(TM)$, defined on the boundary of d. It provides a map

$$\partial d \xrightarrow{v^{(r)}} V_r(TM)|_U \cong U \times V_r(\mathbb{K}^n) \xrightarrow{pr_2} V_r(\mathbb{K}^n), \tag{1.1}$$

where pr_2 is the second projection.

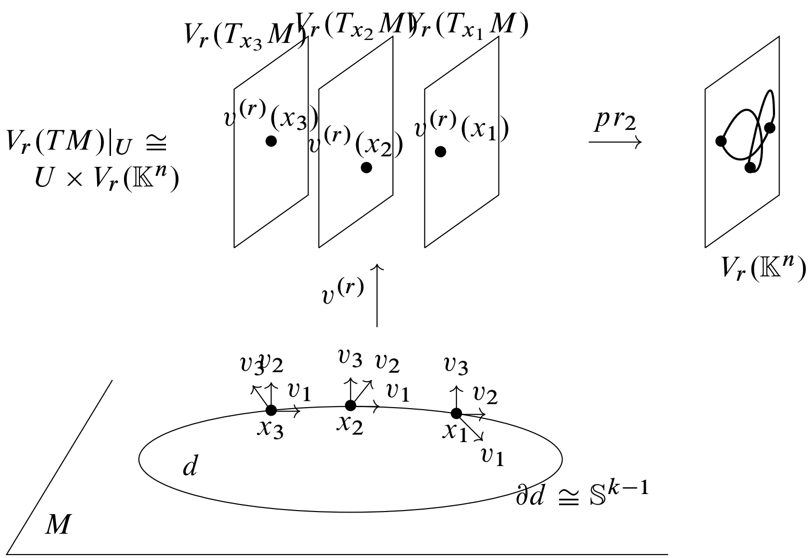


Figure 3.1: The map $pr_2 \circ v^{(r)} : \mathbb{S}^{k-1} \to V_r(\mathbb{K}^n)$.

One obtains a map

$$\mathbb{S}^{k-1} \cong \partial d \xrightarrow{pr_2 \circ v^{(r)}} V_r(\mathbb{K}^n) \tag{1.2}$$

which defines an element of $\pi_{k-1}(V_r(\mathbb{K}^n))$ denoted by $[\xi(v^{(r)}, d)]$.

Let us suppose that $[\xi(v^{(r)}, d)] = 0$, then, by classical homotopy theory, the map $\mathbb{S}^{k-1} \to V_r(\mathbb{K}^n)$ defined on the boundary $\mathbb{S}^{k-1}$ of the ball $\mathbb{B}^k$ can be extended inside the ball. In another words, if $[\xi(v^{(r)}, d)] = 0$, then the map $\partial d \to V_r(\mathbb{K}^n)$, *i.e.* the r-frame, can be extended inside the cell d without singularity. This means that there is no obstruction to the extension of the section $v^{(r)}$ inside d. This happens for example in the case $\pi_{k-1}(V_r(\mathbb{K}^n)) = 0$.

In order to answer to the previous question, we need to know the homotopy groups of $V_r(\mathbb{K}^n)$. The homotopy groups $\pi_i(V_r(\mathbb{K}^n))$ have been computed by Stiefel and by Whitney (see Stiefel (1935)) in the two cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$.

Let $V_r(\mathbb{R}^n)$ be the Stiefel manifold of r-frames in $\mathbb{R}^n$, one has:

$$\pi_i(V_r(\mathbb{R}^n)) = \begin{cases} 0 & \text{for } i < n-r \\ \mathbb{Z} & \text{for } i = n-r \text{ even or } i = n-1 \text{ if } r = 1 \\ \mathbb{Z}_2 & \text{for } i = n-r \text{ odd and } r > 1 \end{cases} \tag{1.3}$$

For the Stiefel manifold $V_r(\mathbb{C}^n)$ of (complex) r-frames in $\mathbb{C}^n$, one has:

$$\pi_i(V_r(\mathbb{C}^n)) = \begin{cases} 0 & \text{for } i < 2n-2r+1 \\ \mathbb{Z} & \text{for } i = 2n-2r+1 \end{cases} \tag{1.4}$$

One obtain the following results:

Real case.

Proposition 3.1.6. *Let us consider an r-frame $v^{(r)}$ defined on the boundary of the k-cell d.*

- *If $k < n-r+1$, one has $[\xi(v^{(r)}, d)] = 0$ and one can extend the r-frame defined on ∂d inside d without singularity.*
- *If $r = 1$ and $k = n$, one can extend the vector field $v^{(1)} = v$ defined on ∂d inside d with an isolated singularity for example at the barycenter $\widehat{d}$ of the cell, with index $[\xi(v, d)] = I(v, \widehat{d})$. That is the index we defined in Definition 2.3.5.*
- *If $r > 1$ and $k = n-r+1$, then one can extend the r-frame $v^{(r)}$ defined on ∂d inside d with an isolated singularity at the barycenter $\widehat{d}$ of the cell. In that case, $[\xi(v^{(r)}, d)]$ is an integer if k is odd and an integer* mod 2 *if k*

is even. Reducing modulo 2, one obtains an index $I(v^{(r)}, \widehat{d})$ that measures the obstruction to the extension of $v^{(r)}$ inside the k-cell d.

The dimension $p = n - r + 1$ is called the *obstruction dimension* for the construction of an r-frame tangent to M.

Complex case.

Proposition 3.1.7. *Let us consider a complex r-frame $v^{(r)}$ defined on the boundary ∂d of the k-cell d.*

- *If $k < 2(n - r + 1)$, one has $[\xi(v^{(r)}, d)] = 0$ and one can extend the r-frame $v^{(r)}$ inside d without singularity.*
- *If $k = 2(n - r + 1)$, then one can extend the r-frame $v^{(r)}$ inside d with an isolated singularity at the barycenter $\widehat{d}$ of the cell. In that case, one obtain an index $[\xi(v^{(r)}, d)] \in \mathbb{Z}$ that we define as $I(v^{(r)}, \widehat{d})$. The index measures the obstruction to the extension of $v^{(r)}$, defined on the boundary ∂d, inside d.*

The dimension $2p = 2(n - r + 1)$ is called the *obstruction dimension* for the construction of a complex r-frame tangent to M.

3.1.2 General obstruction theory

In this subsection, one provides a general presentation of obstruction theory. Aim is to show that the obstruction process produces a cocycle, hence a well defined characteristic class. Among possible references, see for example Davis and Kirk (2001)).

Let us consider a (simplicial or cellular) complex K and a subcomplex L. We will denote by $X = |K|$ and $Y = |L|$ the respective geometric realisations. The q-skeleton of K is denoted by K^q, that is the subcomplex consisting of all simplexes (or cells) whose dimension is less or equal to q. Let us denote $X_q = |K^q|$ the associated space.

We consider a fibre bundle E with base space X and fibre F. To give a section on a subset included in an trivialisation open for the bundle provides a map with target F, as we already seen, see for instance (1.2).

Aim of obstruction theory is to describe the problem of extension of maps $f : Y \to F$ to all of X, by successive extensions of the map from X_q to X_{q+1}. Let us suppose that the function $f : X \to F$ is already known on X_{q-1} and let us denote it by f_{q-1}. Let d^q an oriented q-cell, f_{q-1} is well defined on the boundary ∂d^q and determines an element $[f_{q-1}|_{\partial d^q}] \in \pi_{q-1}(F)$.

Definition 3.1.8. The relative cochain denoted by $c(f_{q-1}) \in C^q(K, L; \pi_{q-1}(F))$ and defined by

$$c(f_{q-1})(d^q) = [f_{q-1}|_{\partial d^q}] \in \pi_{q-1}(F) \tag{1.5}$$

is called *obstruction cochain* (for the extension of f_{q-1} to X_q).

Here we are using cohomology with local coefficients $H^q(K; \{\pi_{q-1}(F)\})$, *i.e.* bundle of abelian groups which associate to each point x of X the coefficient group $\pi_{q-1}(F_x)$.

The function f_{q-1} can be extended to X_q if and only if $c(f_{q-1}) = 0$. In particular, if $\pi_i(F) = 0$ for $i = 1, \ldots, j-1$, then every function $f_Y : Y \to F$ can be extended to $f_j : X_j \to F$.

Lemma 3.1.10. *If f_{q-1} is homotopic to g_{q-1}, then $c(f_{q-1}) = c(g_{q-1})$.*

PROOF: In fact, as $f_{q-1}|_{\partial d^q} \cong g_{q-1}|_{\partial d^q}$, one has $[f_{q-1}|_{\partial d^q}] = [g_{q-1}|_{\partial d^q}]$. □

Theorem 3.1.11. *$c(f_{q-1})$ is a cocycle.*

PROOF: Let τ^{q+1} a $(q+1)$-cell. One has to show that image by the coboundary $\delta : C^q(K, L; \pi_{q-1}(F)) \to C^{q+1}(K, L; \pi_{q-1}(F))$ satisfies $\delta[c(f_{q-1})](\tau^{q+1}) = 0$. One has

$$\delta[c(f_{q-1})](\tau^{q+1}) = c(f_{q-1})[\partial\tau^{q+1}] = c(f_{q-1})(\sum[\tau^{q+1} : d_i^q]d_i^q) =$$

$$\sum[\tau^{q+1} : d_i^q]c(f_{q-1})(d_i^q) = \sum[\tau^{q+1} : d_i^q][f_{q-1}|_{\partial d_i^q}]$$

where the sum is taken on all cells d_i^q which are faces of τ^{q+1}. Let us suppose that incidence of all faces d_i^q of τ^{q+1} with τ^{q+1} is positive, then denoting $f_{q-1}|_{\partial d_i^q} = \alpha_i$, one has $\sum \alpha_i = 0$, and the result. If incidence is not positive, then $[\tau^{q+1} : d_i^q][f_{q-1}|_{\partial d_i^q}] = \alpha_i$ is the element of $\pi_{q-1}(F)$ obtained from the function f_{q-1} restricted to the boundary of the face d_i^q with the orientation induced from τ^{q+1} and one has $\sum[\tau^{q+1} : d_i^q][f_{q-1}|_{\partial d_i^q}] = \sum \alpha_i = 0$. □

The difference cochain

Let f_{q-1} and g_{q-1} two extensions on X_{q-1} of the same $f_{q-2} : X_{q-2} \to F$. We intend to provide a "measure" of their difference. Let d^{q-1} a $(q-1)$-cell in (K). The two functions $f_{q-1}|_{d^{q-1}}$ and $g_{q-1}|_{d^{q-1}}$ coincide on the boundary ∂d^{q-1}. The cell d^{q-1} is homeomorphic both to the north hemisphere D^{q-1}_{+}, and to the south hemisphere D^{q-1}_{-}, of the sphere $\mathbb{S}^{q-1}$. One can interpret $f_{q-1}|_{d^{q-1}}$, resp. $g_{q-1}|_{d^{q-1}}$, as a function from D^{q-1}_{+}, resp. D^{q-1}_{-} to F. These functions coincide on the equator $\mathbb{S}^{q-2}$, homeomorphic to the boundary ∂d^{q-1}, hence they define a function $\gamma : \mathbb{S}^{q-1} \to F$. Having homotopy class $[\gamma] \in \pi_{q-1}(F) = 0$ is a necessary and sufficient condition to deform $g_{q-1}|_{d^{q-1}}$ in $f_{q-1}|_{d^{q-1}}$.

Definition 3.1.12. The *difference cochain* $d(f_{q-1}, g_{q-1}) \in C^{q-1}(K, L; \pi_{q-1}(F))$ is defined by

$$d(f_{q-1}, g_{q-1})(d^{q-1}) = (-1)^q [\gamma] \in \pi_{q-1}(F).$$

The difference cochain is a relative cochain of K modulo L. It vanishes if and only if $f_{q-1} \cong g_{q-1}$ relatively to X_{q-2}. If h_{q-1} is a third extension of f_{q-2} then one has

$$d(f_{q-1}, h_{q-1}) = d(f_{q-1}, g_{q-1}) + d(g_{q-1}, h_{q-1}).$$

If f_{q-1} is an extension of f_{q-2} and $c^{q-1} \in C^{q-1}(K, L; \pi_{q-1}(F))$ is a relative cochain, then there is an extension g_{q-1} of f_{q-2} such that $d(f_{q-1}, g_{q-1}) = c^{q-1}$.

Theorem 3.1.13. *One has*

$$\delta d(f_{q-1}, g_{q-1}) = c(f_{q-1}) - c(g_{q-1}).$$

That means that the difference of the obstruction cocycles of two extensions of f_{q-2} is a coboundary.

PROOF: Let τ^q be a q-cell. One has:

$$\begin{aligned}\delta d(f_{q-1}, g_{q-1})(\tau^q) &= d(f_{q-1}, g_{q-1})(\partial \tau^q) \\ &= d(f_{q-1}, g_{q-1}) \left(\sum_i [\tau^q; \sigma_i^{q-1}] \sigma_i^{q-1} \right)\end{aligned}$$

where the sum is taken on simplices σ_i^{q-1} in the boundary of τ^q. Each σ_i^{q-1} can be written as $\sigma_{i,+}^{q-1} \cup \sigma_{i,-}^{q-1}$ where $\sigma_{i,+}^{q-1} \cap \sigma_{i,-}^{q-1}$ is $\partial \sigma_i^{q-1}$ is in the $q-2$-skeleton of K and f_{q-1} is defined on $\sigma_{i,+}^{q-1}$ and g_{q-1} is defined on $\sigma_{i,-}^{q-1}$. They coincide on $\sigma_{i,+}^{q-1} \cap \sigma_{i,-}^{q-1} = \partial\sigma_i^{q-1}$. By definition of the difference cochain, one has:

$$\begin{aligned}
\delta d(f_{q-1}, g_{q-1})(\tau^q) &= \left(\sum_i [\tau^q; \sigma_i^{q-1}] f_{q-1}(\sigma_{i,+}^{q-1})\right) - \left(\sum_i [\tau^q; \sigma_i^{q-1}] g_{q-1}(\sigma_{i,-}^{q-1})\right) \\
&= f_{q-1}\left(\sum_i [\tau^q; \sigma_i^{q-1}](\sigma_i^{q-1})\right) - g_{q-1}\left(\sum_i [\tau^q; \sigma_i^{q-1}](\sigma_i^{q-1})\right) \\
&= f_{q-1}(\partial\tau^q) - g_{q-1}(\partial\tau^q) \\
&= c(f_{q-1})(\tau^q) - c(g_{q-1})(\tau^q)
\end{aligned}$$

□

Lemma 3.1.14. *If f_{q-2} can be extended as a function $f_{q-1} : X_{q-1} \to F$, then all obstruction cocycles $c(f_{q-1})$ for extension of f_{q-2} to X_{q-1} belong to the same cohomology class*

$$\bar{c}(f) \in H^q(K, L; \pi_{q-1}(F)).$$

Theorem 3.1.15. *Let $f_{q-1} : X_{q-1} \to F$, then f_{q-2} extends to $f_q : X_q \to F$ if and only if $\bar{c}(f) = 0$.*

The obstruction classes

Let us suppose that $\pi_i(F) = 0$ for $i \leqslant q-2$. Then one can construct a function f_i without singularity for $1 \leqslant i \leqslant q-1$.

Definition 3.1.16. Let us suppose that $\pi_i(F) = 0$ for $i \leqslant q-2$. The *primary obstruction class* is the class of the obstruction cocycle $[c(f_{q-1})]$, that is

$$\bar{c}(f) \in H^q(K, L; \pi_{q-1}(F)).$$

Let us remark that in general, the system of coefficients $\{\pi_{q-1}(F)\}$ is twisted.

3.2 Applications

We apply the previous construction to the cases of r-fields tangent to a manifold, in the real and the complex case.

3.2.1 Stiefel–Whitney classes

The Stiefel–Whitney classes have been defined by obstruction theory (see Stiefel (1935),Whitney (1935)). In fact, Whitney used the same strategy than Stiefel, applying it to arbitrary sphere bundles. We use the Steenrod presentation (Steenrod (1951), part III).

The p^{th} Stiefel–Whitney class of M, denoted by $w^p(M)$, is defined as the obstruction to constructing an r-frame over M, that is a section of $V_r(TM)$ or a set of r linearly independent vector fields tangent to M, with $p = n-r+1$. More precisely, we perform the following construction:

Using the result in (1.3) one can construct an r-frame by choosing any r-frame $v^{(r)}$ on the 0-skeleton of the cell decomposition (D), then extending it without zeroes till the obstruction dimension $p = n - r + 1$. That means that $v^{(r)}$ has no singularity on the $(p-1)$-skeleton and isolated singularities on the p-skeleton of (D). Given the r-frame $v^{(r)}$ on the boundary of each p-cell d, one extend $v^{(r)}$ on d with a singularity at the barycenter $\widehat{d}$ of index

$$I(v^{(r)}, \widehat{d}) = [(v^{(r)})_{p-1}|_{\partial d^p}] \in \pi_{p-1}(V_r(\mathbb{R}^n))$$

where

$$\pi_{p-1}(V_r(\mathbb{R}^n)) = \begin{cases} \mathbb{Z} & \text{for } p = n - r + 1 \text{ odd or } p = n \text{ if } r = 1 \\ \mathbb{Z}_2 & \text{for } p = n - r + 1 \text{ even and } r > 1, \end{cases}$$

using the notation in Equation (1.5).

Since $\pi_{p-1}(V_r(\mathbb{R}^n))$ is either infinite-cyclic or isomorphic to $\mathbb{Z}_2$, there is a non trivial homomorphism from $\pi_{p-1}(F)$ to $\mathbb{Z}_2$. hence we can reduce the coefficients modulo 2 obtaining $I(v^{(r)}, \widehat{d}) \in \mathbb{Z}_2$.

We define the p-cochain $\sum I(v^{(r)}, \widehat{d})\, d^*$ in $C^p(D, \mathbb{Z}_2)$, its value on each p-cell d is $I(v^{(r)}, \widehat{d})$. According to Theorem 3.1.11, the cochain is in fact a cocycle and defines an element $w^p(M)$ in $H^p(M; \mathbb{Z}_2)$. The Definition 3.1.16 provides the following:

Definition 3.2.1. The p-th *Stiefel–Whitney class* of M, denoted by $w^p(M) \in H^p(M; \mathbb{Z}_2)$ is the class of the primary obstruction cocycle corresponding to constructing an r-frame tangent to M.

By the general obstruction theory, the obtained classes do not depend on the choices we make in the construction.

In the particular case $r = 1$, one can use integer coefficients. The evaluation of $w^n(M) \in H^n(M;\mathbb{Z})$ on the fundamental class $[M]$ of M is the Euler–Poincaré characteristic of M.

Let us suppose that the cell decomposition (D) is obtained by duality of a triangulation (K) of M. Each p-cell $d = d(\sigma)$ in (D) is dual of an $(r-1)$-simplex σ in (K). By Poincaré duality (cap-product by the fundamental class),

$$H^{n-r+1}(M;\mathbb{Z}_2) \longrightarrow H_{r-1}(M;\mathbb{Z}_2)$$

the image of d^* is σ and image of $w^p(M)$ is the so-called $(r-1)$-homology Stiefel–Whitney class, denoted by $w_{r-1}(M)$. A cycle representing $w_{r-1}(M)$ is given (mod 2) by

$$\sum_{\dim\sigma = r-1} I(v^{(r)}, \widehat{d}(\sigma))\sigma.$$

Combinatorial definition

A combinatorial definition of the Stiefel–Whitney classes was already conjectured by E. Stiefel Stiefel (1935). Then H. Whitney wrote a proof for a book, that unfortunately, never appeared. G. Cheeger (1968) provided a proof using different techniques and the complete proof appeared in a paper by Halperin and Toledo (1972).

Let M be a differentiable n-manifold without boundary and K a differentiable triangulation of M. Let K' denote the first barycentric subdivision of K. Each K'-simplex τ is written in an unique way as $\tau = \langle \widehat{\sigma}_0, \cdots, \widehat{\sigma}_k \rangle$ where $\sigma_0 < \cdots < \sigma_k \in K$ (see Section 1.3). Each simplex τ is given the orientation for which $\langle \widehat{\sigma}_0, \cdots, \widehat{\sigma}_k \rangle$ is a positive ordering of the vertices.

An infinite integral simplicial k-chain on M means a formal infinite integral combination $\sum \lambda_\sigma \sigma$ where the sum runs over the k-simplexes of K', oriented with the previous order.

Theorem 3.2.2. *Halperin and Toledo (ibid.) Let M be a smooth n-manifold without boundary and K a smooth triangulation of M. The infinite chain*

$$\widehat{w}_k(M) = \sum_{\sigma_0 < \cdots < \sigma_k} (-1)^{|\sigma_0| + \cdots + |\sigma_k|} \langle \widehat{\sigma}_0, \cdots, \widehat{\sigma}_k \rangle \qquad (0 \leqslant k < n)$$

is an integral cycle if $n-k$ is odd or $k = 0$. It is a mod 2 *cycle if $n-k$ is even. Its class is the Stiefel–Whitney class w_k.*

Corollary 3.2.3. *Halperin and Toledo (1972) The infinite chain*

$$w_k(M) = \sum_{\sigma_0 < \cdots < \sigma_k} \langle \widehat{\sigma}_0, \cdots, \widehat{\sigma}_k \rangle$$

is a (mod 2)*-cycle. It represents the k^{th}* (mod 2) *Stiefel–Whitney homology class of M.*

Application: The Thom Theorem

Two manifolds M and N are called cobordant if there is a compact manifold W whose boundary is the disjoint union of M and N, *i.e.* $\partial W = M \sqcup N$. All manifolds cobordant to a fixed given manifold M form the cobordism class of M. Cobordism is a fundamental equivalence relation on the class of compact manifolds of the same dimension.

The Stiefel–Whitney numbers of an (unoriented) closed n-dimensional manifold M are defined as

$$\Big\langle w^{i_1}(M) \cup \cdots \cup w^{i_k}(M), [M] \Big\rangle \in \mathbb{Z}_2$$

for any collection $(i_1, \cdots, i_k)$ of integers such that $i_1 + \cdots + i_k = n$.

These numbers are known to be cobordism invariants. It was proved by Lev Pontryagin that if M is the boundary of a smooth compact $(n+1)$–dimensional manifold, then the Stiefel–Whitney numbers of M are all zero. Later on, it was proved by René Thom that if all the Stiefel–Whitney numbers of M are zero then M can be realised as the boundary of some smooth compact manifold. So we have:

Theorem 3.2.4. *Thom (1954) A smooth compact manifold M is the boundary of some smooth compact (unoriented) manifold if and only if all the Stiefel–Whitney numbers of M vanish.*

Axiomatic definition

We defined classes of a manifold as obstruction classes of the tangent bundle $E = TM$ and the associated bundles of frames $V_r(TM)$. The obstruction theory applies as well to any real vector bundle E over a triangulated space X, with n-dimensional fiber. Note that X does not need to be smooth and can be a CW-complex.

In the same way than the tangent bundle of a manifold, we construct r everywhere independent sections of E without obstruction on the $(n-r)$-skeleton of the given CW-structure of X and with singularities of index $I(v^{(r)}, \widehat{d}) \in \pi_{p-1}(V_r(\mathbb{R}^n))$ on the $n-r+1$ cells d. Let us denote $p = n - r + 1$, the data

$$d \mapsto I(v^{(r)}, \widehat{d})$$

define a cochain in $C^p(X; \pi_{p-1}(V_r(\mathbb{R}^n))$. This cochain is actually a cocycle and defines a class $\widehat{w}^p(E)$ in the p-th simplicial (or cellular) cohomology of X with twisted coefficients, the coefficient system being the homotopy group $\pi_{p-1}(V_r(\mathbb{R}^n))$:

$$\widehat{w}^p(E) \in \begin{cases} H^p(X; \mathbb{Z}) & \text{if } p \text{ is odd or } p = n, \\ H^p(X; \mathbb{Z}_2) & \text{if } p \text{ is even and } p < n. \end{cases}$$

Whitney proved that $\widehat{w}^p(E) = 0$ if and only if E, when restricted to the p-th skeleton of X, admits $r = (n - p + 1)$ linearly-independent sections.

Definition 3.2.5. The Stiefel–Whitney classes of the real vector bundle E with n-dimensional fiber, on the triangulated (or CW) space X are the reduced classes modulo 2 of $\widehat{w}^p(E) = 0$:

$$w^p(E) \in H^p(X; \mathbb{Z}_2).$$

In his 1940 paper, Whitney states (for sphere bundles) the formula providing classes of the sum of two vector bundles E and E' over the same base space B:

$$w^p(E \oplus E') = \sum_{i+j=p} w^i(E) \smile w^j(E') \tag{2.6}$$

In 1948, Chern Shiing Shen and Wu Wen-Tsün published the first complete proofs of the formula (2.6), both in the same volume of Annals of Mathematics S.-s. Chern (1948) and Wu (1948).

The formula Equation (2.6) is one of the axioms entering in the axiomatic definition of Stiefel–Whitney classes, given by Friedrich Hirzebruch (1966):

Definition 3.2.7. Axiomatic definition of Stiefel–Whitney classes.

Let E be a real vector bundle of (finite) rank n over a (paracompact)space X, there is an unique class $w(E) \in H^*(X; \mathbb{Z}_2)$ satisfying the following properties:

1. One has $w(E) = 1 + w^1(E) + \cdots + w^n(E)$, where $w^i(E) \in H^i(X; \mathbb{Z}_2)$ and $w^i(E) = 0$ if $i > n$.
2. (Naturality) If $f : Y \to X$ is a continuous map, then $f^*(w(E)) = w(f^*(E))$ where $f^*(E)$ is the “pull-back” vector bundle on Y.
3. (Whitney–Wu sum) If E and E' are two bundles over M, then
$$w(E \oplus E') = w(E) \cup w(E').$$
4. let γ_1^1 be the canonical line bundle over $\mathbb{RP}^1 = \mathbb{S}^1$ (see Equation (6.5)), then $w^1(\gamma_1^1)$ is non zero in $H^1(\mathbb{S}^1; \mathbb{Z}_2)$.

Note that, by definition, $w^0(E) = 1$

Examples and applications

Among properties and applications of the obstruction theory and axiomatic definitions of the Stiefel–Whitney classes we have:

Proposition 3.2.8. • *If E is a trivial bundle, then $w^i(E) = 0$ for $i \geqslant 1$.*
• *$w^1(E) = 0$ if and only if the bundle E is orientable, in particular the first Stiefel–Whitney class $w^1(M)$ of a manifold is zero if and only if M is orientable Steenrod (1951).*

Example 3.2.9. • The Stiefel–Whitney (total) class of the sphere is $w(\mathbb{S}^n) = 1$.
• $w(\gamma_n^1) = 1 + \alpha_n$ where $\alpha_n \in H^1(\mathbb{RP}^n)$ is non zero class, and $w(\mathbb{RP}^n) = (1 + \alpha_n)^{n+1}$.

3.2.2 Chern classes

In his fundamental paper S.-s. Chern (1946), Chern provides several equivalent definitions of Chern classes for complex hermitian manifolds, among them the definition by obstruction theory, (as we made for the real case) using the cell decomposition of the complex Grassmannian manifold $G_{n,m}(\mathbb{C})$ by Schubert cells, using differential forms and transgression.

The definition of Chern classes by obstruction theory in the complex case is similar to the real case, even simpler.

Let M denote an analytic complex manifold of (complex) dimension n and TM the complex tangent bundle to M. The p^{th} Chern class of M, denoted by

$c^p(M)$, is defined as the obstruction to constructing a complex r-frame over M, that is a section of $V_r(TM)$ or a set of r linearly independent vector fields tangent to M, with $p = n - r + 1$.

Using the result in (1.4) one can construct an r-frame by choosing any r-frame $v^{(r)}$ on the 0-skeleton of the cell decomposition (D), then extending it without zeroes till the obstruction dimension

$$2p = 2(n - r + 1). \tag{2.7}$$

That means that $v^{(r)}$ has no singularity on the $(2p-1)$-skeleton and isolated singularities on the $2p$-skeleton of (D). Given the r-frame $v^{(r)}$ on the boundary of each $2p$-cell d, one can extend $v^{(r)}$ on d with a singularity at the barycenter $\widehat{d}$ of index

$$I(v^{(r)}, \widehat{d}) = [(v^{(r)})_{2p-1}|_{\partial d^{2p}}] \in \pi_{2p-1}(V_r(\mathbb{C}^n)) = \mathbb{Z}$$

using the notation in Equation (1.5).

The generators of $\pi_{2p-1}(V_r(\mathbb{C}^n))$ being consistent (see Steenrod (1951)), one can define the $2p$-cochain $\sum I(v^{(r)}, \widehat{d})\, d^*$ in $C^{2p}(D, \mathbb{Z})$, its value on each $2p$-cell d is $I(v^{(r)}, \widehat{d})$. According to Theorem 3.1.11, the cochain is in fact a cocycle and defines an element $c^p(M)$ in $H^{2p}(M; \mathbb{Z})$. The Definition 3.1.16 provides the following:

Definition 3.2.11. The p-th *Chern class* of M, denoted by $c^p(M) \in H^{2p}(M; \mathbb{Z})$ index[std]classes!Chern is the class of the obstruction cocycle corresponding to the construction of a complex r-frame tangent to M.

By the general obstruction theory, the obtained classes do not depend on the choices we make in the construction.

In the particular case $r = 1$, the evaluation of $c^m(M)$ on the fundamental class $[M]$ of M yields the Euler–Poincaré characteristic of M.

Let us suppose that the cell decomposition (D) is obtained by duality of a triangulation (K) of M. Each $2p$-cell $d = d(\sigma)$ in (D) is dual of an $2(r-1)$-simplex σ in (K). By Poincaré duality (cap-product by the fundamental class),

$$H^{2(n-r+1)}(M; \mathbb{Z}) \longrightarrow H_{2(r-1)}(M; \mathbb{Z})$$

the image of d^* is σ and image of $c^p(M)$ is the so-called $2(r-1)$-homology Chern class, denoted by $c_{r-1}(M)$. A cycle representing $c_{r-1}(M)$ is given by

$$\sum_{\dim \sigma = 2(r-1)} I(v^{(r)}, \widehat{d}(\sigma))\sigma.$$

Remark 3.2.12. Unlike the real case, there is no combinatorial definition of Chern classes, the coefficient of elementary cochains can be any real number.

Axiomatic definition of Chern classes

Let E be a complex vector bundle of (complex) rank n over a space X, in the same way than in the real case, we define Chern classes $c^p(E) \in H^{2p}(X;\mathbb{Z})$, for $p = 1, \dots, n$ by obstruction theory. The total Chern class of E is denoted

$$c(E) = 1 + c^1(E) + \cdots + c^n(E).$$

In his thesis, Wu Wen-Tsün extended the product formula (2.6) to Chern classes:

$$c(E \oplus E') = c(E) \smile c(E'), \tag{2.8}$$

The formula Equation (2.8) is one of the axioms entering in the axiomatic definition of Chern classes, due to Friedrich Hirzebruch (1966):

Definition 3.2.13. Axiomatic definition of Chern classes.

Let E be a complex vector bundle of rank n over a space X, there is a class $c(E) \in H^*(X;\mathbb{Z})$ satisfying the following properties:

1. One has $c(E) = 1 + c^1(E) + \cdots + c^n(E)$, where $c^i(E) \in H^{2i}(X;\mathbb{Z})$ and $c^i(E) = 0$ if $i > n$.
2. (Naturality) If $f : Y \to X$ is a continuous map, then $f^*(c(E)) = c(f^*(E))$ where $f^*(E)$ is the "pull-back" complex vector bundle on Y.
3. (Whitney–Wu) If E and E' are two bundles over X, then

$$c(E \oplus E') = c(E) \cup c(E').$$

4. let γ_1^1 be the canonical line bundle over $\mathbb{CP}^1$ (see Equation (6.7)) then $c^1(\gamma_1^1)$ is non zero in $H^2(\mathbb{CP}^1;\mathbb{Z})$.

Examples and applications

Proposition 3.2.14. *If E is a trivial bundle, then $c^i(E) = 0$ for $i \geqslant 1$.*

The first Chern class $c^1(\mathbb{CP}^n)$ is the generator a of $H^2(\mathbb{CP}^n;\mathbb{Z})$ (see (6.8)). The total Chern class c of $\mathbb{CP}^n$ is

$$c(\mathbb{CP}^n) = c(T\mathbb{CP}^n) = c(\mathcal{O}(1))^{n+1} = (1+a)^{n+1}.$$

(see the Section 1.6.4) and its class c^k is

$$c^k(\mathbb{CP}^n) = \binom{n+1}{k} a^k.$$

The first Chern class of the canonical (tautological) bundle (1.6.4 item 3) is

$$c^1(\mathcal{O}(-1)) = -a.$$

Chern classes have many applications in mathematics for instance in knot theory, Chern–Weil and Chern–Simons theories, theory of Calabi–Yau manifolds, and in physics for instance in string theory, quantum field theory etc. (see Section 8.6).

Exercises

3.1) Use Pontrjagyn–Thom's theorem to show that a non-orientable n-manifold can never bound an $(n+1)$-manifold.

3.2) Show that if $\mathbb{RP}^n$ is parallelizable (i.e. $T\mathbb{RP}^n$ is trivial), then $n = 2^k, \quad k \geqslant 0$.

3.3) Show that if there is an immersion of $\mathbb{RP}^9$ into $\mathbb{RP}^{9+k}$, then $k > 0$.

3.4) Show that if there is an immersion of $\mathbb{RP}^{2^r}$ into $\mathbb{RP}^{2^r+k}$, then $k > 2r-1$.

3.5) Show that a real bundle $E \to B$ is orientable if and only if $w_1(E) = 0$.

3.6) If an n-manifold M can be immersed in $\mathbb{R}^{n+1}$, show that the Stiefel–Whitney classes must be of the form $w_k(TM) = w_1(TM)^k$ for all k. Show that if $\mathbb{RP}^n$ can be immersed into $\mathbb{R}^{n+1}$, then n must be of the form $2^r - 1$ or $2^r - 2$.

3.7) Show that the change of coefficient homomorphism

$$H^*(B;\mathbb{Z}) \to H^*(B;\mathbb{Z}_2)$$

maps the total Chern class of a complex vector bundle to the total Stiefel–Whitney class of the underlying real vector

4 Singular varieties

A singular variety is a variety which contains points which do not satisfy the property in Definition 1.1.6. Examples of singular varieties are the following: The pinched torus (Figure 4.1, left): the pinched point a does not admit any neighbourhood satisfying the property 1.1.6. In that case, the link of an "elementary neighbourhood" of a is the union of two not connected circles c_1 and c_2).

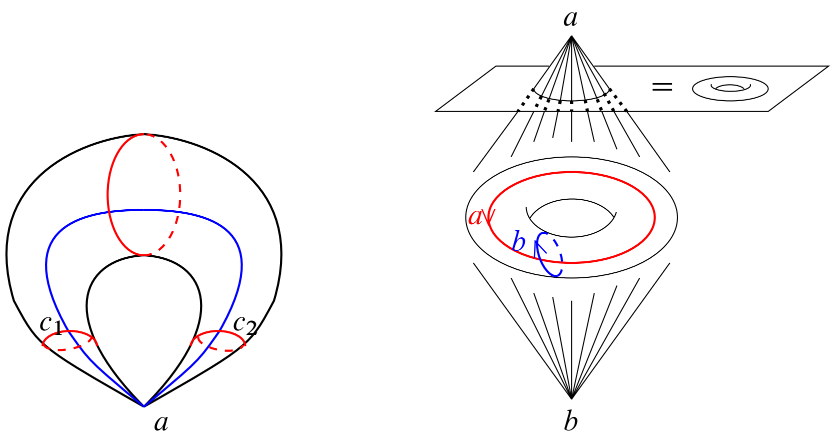


Figure 4.1: The pinched torus and the suspension of the torus

Another example is provided by the suspension of the torus (Figure 4.1, right). The two points a and b of the suspension of the torus are singular points, in that case, the link of a (or b) is a torus, it is not a sphere.

In order to extend the notion of characteristic classes to singular varieties, it is necessary to know the local structure of the singular variety. That is given by the structure of stratified space that is, the way the variety is cut into smooth pieces (the strata) and the way these strata are glued together.

4.1 Stratifications

Definition 4.1.1. Let X be a topological space, we denote by $\mathcal{X}$ a *filtration* of X by closed subsets

$$\mathcal{X} \qquad \emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{n-2} \subset X_{n-1} \subset X = X_n \tag{1.1}$$

A *topological stratification* of X is the data of a filtration $\mathcal{X}$ of X such that each difference $V_\alpha = X_\alpha - X_{\alpha-1}$ is either empty or a topological manifold of pure dimension α. The connected components of the V_α are called the *strata*.

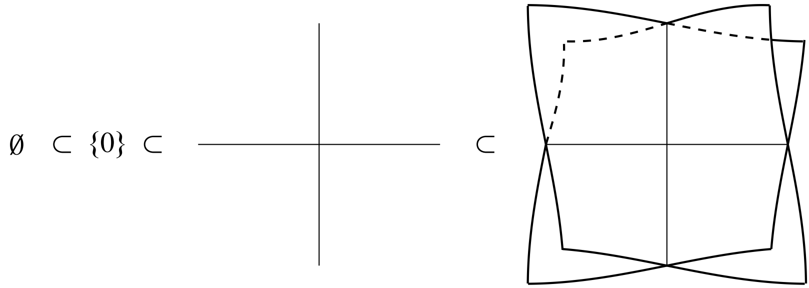


Figure 4.2: A stratification.

The stratifications that we will consider will be locally finite partitions of X into locally closed submanifolds, the strata, satisfying the *frontier condition*:

$$V_\alpha \cap \overline{V}_\beta \neq \emptyset \Rightarrow V_\alpha \subset \overline{V}_\beta$$

If X is a closed subset of a differentiable (resp. analytic) manifold M, a *differentiable stratification* (resp. *analytic stratification*) of X is a topological stratification $\mathcal{X}$ of X such that each stratum in V_k is a differentiable (resp. analytic) submanifold of M.

In order to work with, the considered stratification should satisfy conditions which precise the way the strata are glued together. On the one hand, there are many ways to define these conditions, according to the specific problem. On the other hand, given conditions on the stratification, one has to know which kind of singular variety admits a stratification satisfying these conditions. In the following, we considers the stratifications which will be useful for the construction of characteristic classes. One refer to Trotman (2020) for more information on the different types of stratifications.

4.2 Angles

Let us consider on M a Riemannian metric. Given a point $x \in M$, a vector $v(x) \in T_x^*(M)$ and a vector subspace $E \subset T_x(M)$, the angle between $v(x)$ and E is denoted by $\alpha(v(x), E) \in [0, \pi/2[$.

Given two vectorial subspaces E and F in $T_x(M)$, such that $\dim F \leqslant \dim E$, one defines the angle of E and F by

$$\alpha(F, E) = \sup_{v(x) \in F} (v(x), E).$$

4.3 Whitney stratifications

Definition 4.3.1. We says that the Whitney conditions are satisfied for a stratification if, for any pair of strata (V_α, V_β) such that V_α is in the closure of V_β, one has:

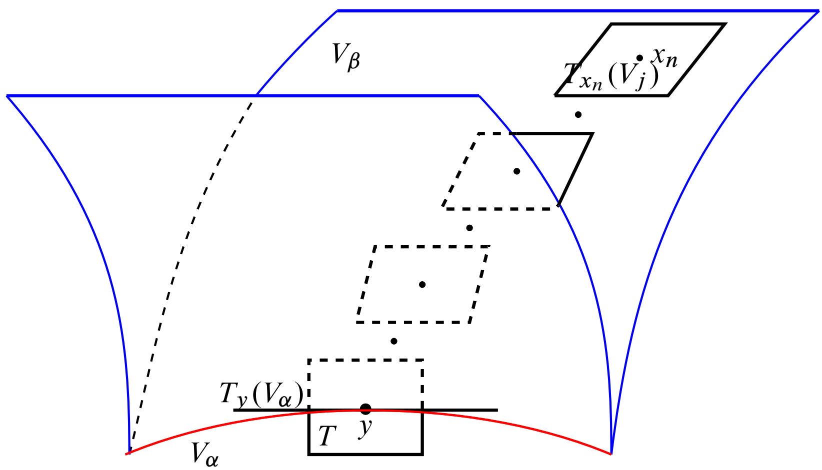


Figure 4.3: Whitney condition (a)

a) if (x_n) is a sequence of points in V_β with limit $y \in V_\alpha$ and if the sequence of tangent spaces $T_{x_n}(V_\beta)$ admits a limit T (in the suitable Grassmannian space) when n goes to $+\infty$, then $T_y(V_\alpha)$ is included in T.

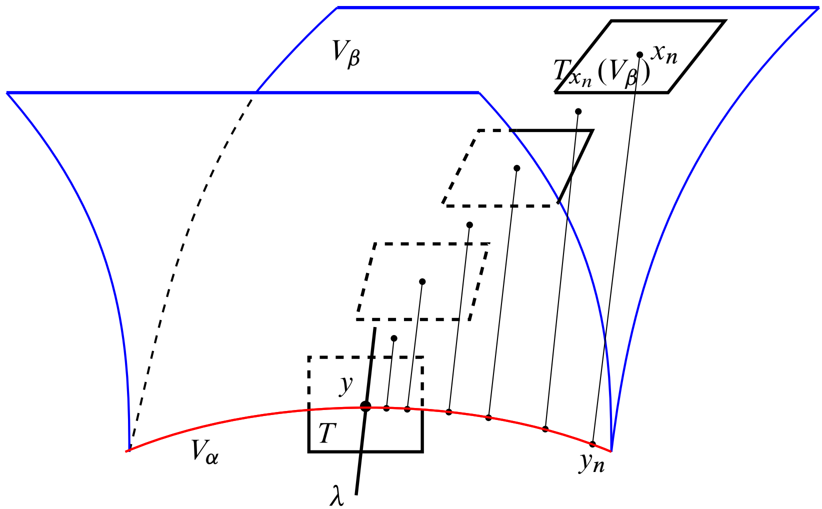


Figure 4.4: Whitney condition (b)

b) if (x_n) is a sequence of points in V_β with limit $y \in V_\alpha$ and if (y_n) is a sequence of points in V_α with limit y, such that the sequence of tangent spaces $T_{x_n}(V_\beta)$ admits a limit T for n going to $+\infty$ and such that the sequence of directions $\overline{x_n\, y_n}$ admits a limit λ when n goes to $+\infty$, then λ lies in T.

Example The conditions (a) and (b) are not satisfied for the stratification of the cone X consisting of a generatrix $D = V_\alpha$ and $V_\beta = X \setminus D$. This is clear taking for (x_n) a sequence of points going to the vertex y of the cone, along a generatrix (different from D), and for y_n a sequence of points such that the segment $x_n\, y_n$ has always the same direction. Adding the vertex of the cone as a supplementary 0-dimensional stratum, the new stratification satisfies the Whitney conditions.

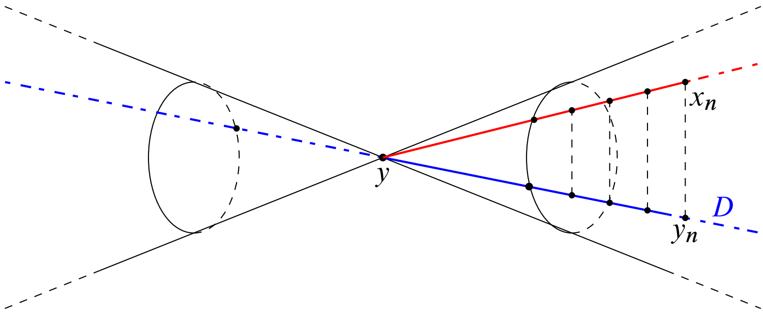


Figure 4.5: Stratification of the cone.

Example Let V be the variety whose equation in $\mathbb{C}^3$ is $y^2 - x^3 - t^2x^2 = 0$, stratified by the t-axis $Y = V_1$ and $V_2 = V - V_1$, then the (a)-condition is satisfied but not (b). Adding the vertex $\{0\}$ of $\mathbb{C}^n$ as a new stratum V_0, the Whitney conditions are verified.

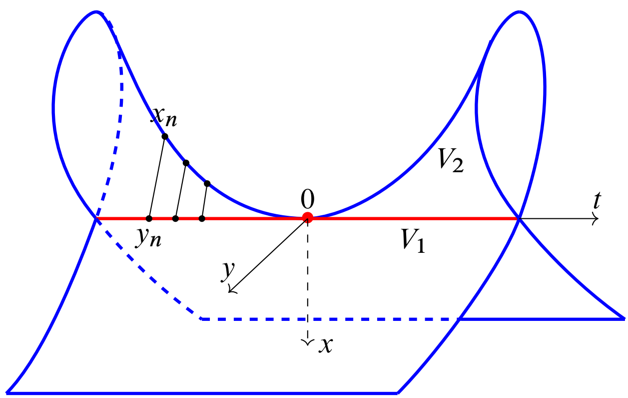


Figure 4.6: Stratifications of the (real part of the) variety V.

4.4 Fundamental properties of Whitney stratifications

Whitney stratifications are very important for several reasons, some of them will become apparent along this text. Not all singular spaces admit suitable stratifications. Not all singular stratified spaces admit triangulation compatible with the stratification. That is the reason for which we will work in the situation of analytic and semi-analytic spaces.

Definition 4.4.1. Łojasiewicz (1993, §II, 1) Let M be a real analytic manifold, a subspace X in M is called *semi-analytic* if each point in M admits an open neighbourhood U such that $X \cap U$ is defined by a finite family of inequalities of the form $f > 0$ or $f \geqslant 0$ where f is an analytic map in U.

Property 4.4.2. Some important facts about Whitney stratifications are:
• Any closed analytic subset of an analytic manifold admits a Whitney stratification whose strata are analytic (see Trotman (2020, Theorem 4.2.10)).
• Whitney stratified spaces can be triangulated compatibly with the stratification (see Trotman (ibid., Triangulation, page 237)).

Theorem 4.4.3 (Thom–Mather Theorem). *(see Trotman (ibid., Theorem 4.2.17)) Let M a real analytic manifold equipped with a Whitney stratification and X a subspace union of strata $\{V_\alpha\}$. For every point x in a stratum V_α, there is a* distinguished neighbourhood *$U_x \subset X$ and a homeomorphism*

$$\psi_x : U_x \to \mathbb{B}^\alpha \times cL_x$$

where $\mathbb{B}^\alpha$ is the standard open α-dimensional ball and cL_x is the cone over the link L_x. The link L_x is independent of the point x in V_α and is stratified

$$\emptyset = L_{-1} \subset L_0 \subset \cdots \subset L_{n-\alpha-2} \subset L_x = L_{n-\alpha-1}$$

moreover, the homeomorphism ψ_x preserves the stratifications of U_x (induced by the one of X) and the one of the product $\mathbb{B}^\alpha \times cL_x$ respectively, that is there are restriction homeomorphisms

$$\phi_x|_{X_\beta} : U_x \cap X_\beta \to \mathbb{B}^\alpha \times \mathring{c}(L_{\beta-\alpha-1}), \quad \textit{for } \alpha \leqslant \beta.$$

Here, the strata of the cone $cL_x = L_x \times [0, 1[/L_x \times \{0\}$ are the punctured cones $c(L_\gamma)$ where L_γ are the strata of L_x and $\{0\}$, the vertex of the cone. The strata of the product $\mathbb{B}^\alpha \times cL_x$ are products of $\mathbb{B}^\alpha$ by the strata of the cone cL_x. By definition, $c(\emptyset) = \{pt\}$.

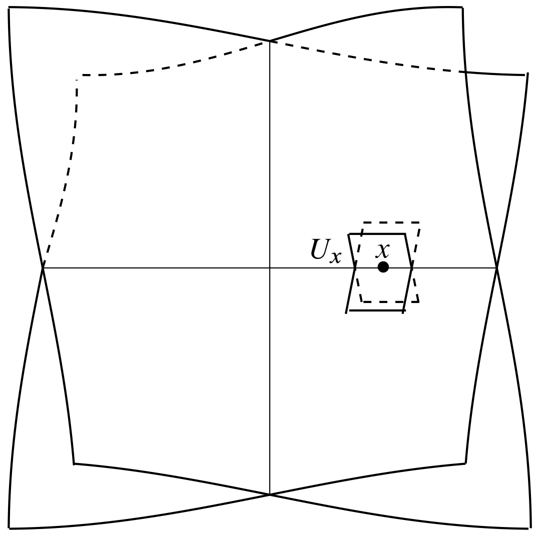


Figure 4.7: Distinguished neighbourhood.

Property 4.4.4. The conditions (a) and (b) of Whitney stratifications are the ones which allow M.-H. Schwartz to construct radial extension of stratified vector fields, using two consequences of the conditions :

- The Whitney condition a) allows to show that one can extend a vector field defined on a stratum as a stratified vector field "parallel" to the initial one in a suitable tubular neighbourhood of the stratum (see Section 5.4.1 a)).

- The Whitney condition b) allows to show that one can construct a stratified vector field "transverse" to a stratum in a suitable neighbourhood of the stratum (see Section 5.4.1 b)).

4.5 Poincaré homomorphism

In the case of a singular variety, there is no more Poincaré isomorphism (1.3) however, the Poincaré homomorphism $H^{n-i}(X) \to H_i(X)$ can be described in the following way:

Let us suppose X is a triangulable oriented singular n-dimensional pseudovariety in a topological oriented m-dimensional manifold M.

Any stratification of X defines a stratification of M adding $M - X$ as regular stratum. Let us denote by (K) a locally finite triangulation of M compatible with the stratification and by (K') a barycentric subdivision of (K). The chain (or cochain) complexes relatively to (K) or (K') will be denoted by $C_*^{(K')}(X)$, $C^*_{(K)}(X)$ for example.

Providing to all n-dimensional simplexes of (K') the orientation of the regular part of X, the sum of these simplexes is a cycle in $C_n^{(K')}(X)$. Its class in $H_n(X)$ is the fundamental class of X, denoted by $[X]$.

For every $(n-i)$-simplex σ in (K), the dual cell of σ in M, denoted by $d(\sigma)$ has dimension $m-(n-i)$. It is constructed with simplexes in (K'). It is transverse to X, i.e. to every stratum $X_\alpha - X_{\alpha-1}$ of X. The intersection $d(\sigma) \cap X$ is an oriented i-dimensional (K')-chain in X.

The Poincaré homomorphism

$$H^{n-i}(X) \to H_i(X)$$

is given by the chain map

$$C^{n-i}_{(K)}(X) \to C_i^{(K')}(X)$$

which maps the elementary $(n-i)$-cochain σ^*, dual of the simplex σ in K, to the i-chain $d(\sigma) \cap X$ of K'.

4.6 Alexander isomorphism

Let M an m-dimensional real analytic manifold equipped with a Whitney stratification and X a subspace union of strata $\{V_\alpha\}$. We consider a triangulation (K) of M compatible with the stratification.

4.6.1 Cellular tubes

Definition 4.6.1. A *cellular tube* $\mathcal{T}$ around X in M is the union of cells (D) which are dual of simplexes in X for the triangulation (K).

This notion generalizes the concept of tubular neighbourhood of a submanifold X. If X is a submanifold, then $\mathcal{T}$ is a bundle around X, whose fibers are discs. In general (in the singular situation), that is not the case.

Remark 4.6.2. A cellular tube $\mathcal{T}$ around X has the following properties :
i) $\mathcal{T}$ is a compact neighbourhood of X, containing X in its interior and $\partial\mathcal{T}$ is a retract of $\mathcal{T} \setminus X$.
ii) $\mathcal{T}$ is a *regular* neighbourhood of X, thus $\mathcal{T}$ retracts to X.

The Alexander isomorphism

$$H^{m-i}(M, M - X) \to H_i(X)$$

is defined in the following way: We denote by $\mathcal{T}$ the neighbourhood of X which is the union of all dual cells $d(\sigma)$ of simplexes σ in X. The boundary $\partial\mathcal{T}$ of $\mathcal{T}$ is the union of dual cells $d(\tau)$ in $\mathcal{T}$ such that τ is not a simplex in X. The correspondence

$$C^{m-i}_{(D)}(\mathcal{T}, \mathcal{T} \setminus \partial\mathcal{T}) \to C_i^{(K)}(X)$$

which associates to a (D)-cochain $(d^{m-i})^*$ such that $d^{m-i} \cap X \neq \emptyset$ the K-chain σ_i such that $d^{m-i} = d(\sigma_i)$ is an isomorphism and induces the isomorphism

$$H^{m-i}(\mathcal{T}, \partial\mathcal{T}) \to C_i(X).$$

We have isomorphisms

$$H^{m-i}(\mathcal{T}, \partial\mathcal{T}) \cong H^{m-i}(\mathcal{T}, \mathcal{T} \setminus X) \cong H^{m-i}(M, M - X)$$

the first one, by retraction of $\mathcal{T} \setminus X$ on $\partial\mathcal{T}$ and the second by excision.

The Alexander isomorphism will play an important role in the study of Chern classes for singular varieties.

Exercises

4.1) Let $Z = Z_2 = \{y^2 = t^2x^2 + x^3\} \subset \mathbb{R}^3$. Set $Z_1 = \{(0,0,t) | t \in \mathbb{R}\}$ and $Z_0 = \emptyset$. Show that $\emptyset = Z_0 \subset Z_1 \subset Z_2$ is a filtration defining a $\mathcal{C}^\infty$ stratification with 4 strata of dimension 2 and one stratum $Y = Z_1$ of dimension 1.

Denote the strata as

$$X_1 = (Z_2 - Z_1) \cap \{t > 0\} \cap \{x < 0\}, \qquad X_2 = (Z_2 - Z_1) \cap \{t < 0\} \cap \{x < 0\}$$

$$X_3 = (Z_2 - Z_1) \cap \{y < 0\} \cap \{x > 0\}, \qquad X_4 = (Z_2 - Z_1) \cap \{y > 0\} \cap \{x > 0\}.$$

Verify that the pairs (X_3, Y) and (X_4, Y) are (b)-regular. Verify that the pairs (X_1, Y) and (X_2, Y) are not (b)-regular. at $(0,0,0)$, although they are (a)-regular.

Show that the frontier property does not hold for the pairs (X_1, Y) and (X_2, Y).

Join X_1 and X_2 into one connected stratum by turning Y into a circle, Show that now the frontier condition holds. But (b) still fails.

4.2) We want to show that Whitney conditions do not imply that there is a $\mathcal{C}^1$-diffeomorphism mapping neighbourhoods of a point y_1 on a stratum Y to neighbourhoods of another point y_2 on the same stratum Y. That is an example from Whitney.

Let $Z = \{(x, y, t) | xy(x - y)(x - ty) = 0, \quad t \neq 1\} \subset \mathbb{R}^3$, stratified by $Z = Z_2 \supset Z_1 = (Ot)$. This is a family of 4 lines parametrised by t. Show that the stratification is (b)-regular. Show that there is no $\mathcal{C}^1$-diffeomorphism mapping mapping Z_{t_1} to Z_{t_2} where $Z_t = Z \cap (\mathbb{R}^2 \times \{t\})$.

5 Poincaré–Hopf Theorem (singular varieties)

5.1 Introduction

The first proof of Poincaré–Hopf Theorem for singular varieties and the first definition of Chern class for singular varieties have been given in 1964 by Marie-Hélène Schwartz in the preprint Schwartz (1964) (Lille University), then in 1965 in two "Notes aux CRAS" Schwartz (1965).

In the following, M will be a real analytic manifold equipped with a real semi-analytic stratification $\{V_\alpha\}$: for every stratum V_α, the closure $\bar{V_\alpha}$ and the boundary $\dot{V_\alpha} = \bar{V_\alpha} \setminus V_\alpha$ are semi-analytic sets, union of strata. We denote by $X \subset M$ a real analytic compact subset stratified by $\{V_\alpha\}$.

If X is a singular variety, the Poincaré–Hopf Theorem fails to be true, the main reason is that there is no more tangent space at singular points. The definition of the index of a vector field at one of its singular points takes sense on a smooth manifold only, the reason being that, in a smooth manifold, the link of a point is a sphere.

In order to obtain a Poincaré–Hopf Theorem, one can think to consider a stratification of the singular variety (see Section 4.1), and consider continuous vector fields which are stratified in the following sense:

Definition 5.1.1. A *stratified vector field* v on X is a (continuous) section of the tangent bundle of M, $T(M)$, such that, for every $x \in X$, then one has $v(x) \in T(V_{\alpha(x)})$ where $V_{\alpha(x)}$ is the stratum containing x.

In that case, we can define the index of a stratified vector field with isolated singularities, computing the index at a singular point either in the stratum of the given point, or in the ambient manifold. Unfortunately, in general, that definition does not provide a Poincaré–Hopf Theorem. In the following section, we give counterexamples. The main reason for being a counterexample is that the index computed in the stratum and the index computed in the ambient manifold are different.

The idea, developed by M.-H. Schwartz, is to consider particular stratified vector fields defined in the manifold M containing X and called *radial vector fields*. They satisfy two main properties: in a neighbourhood of X, they have same isolated singularities than their restriction to each stratum and their index computed in the stratum coincide with their index computed in the ambient manifold. Moreover, the vector fields are pointing outwards of suitable neighbourhoods of the strata. The construction as well as main properties of the constructed vector fields are provided in Section 5.3.

Moreover, in the same way that the radial vector fields allow to recover the Poincaré–Hopf Theorem, the construction of characteristic classes for singular varieties will consist in a construction of stratified vector frames adapted to the singular situation and generalizing the notion of radial vector fields.

5.2 Why the radial vector fields ?

Let us consider M a real analytic manifold equipped with a real semi-analytic stratification $\{V_\alpha\}$ and $X \subset M$ a real analytic compact subset stratified by $\{V_\alpha\}$.

Let v be a stratified vector field on M with isolated singularities a_k. We could define the index of the stratified vector field v at a singular point a situated in the stratum V_α as the index of the restriction $I(v|_{V_\alpha}, a)$. The natural generalization of the Poincaré–Hopf Theorem to singular varieties would be the following formula:

$$\chi(X) = \sum_{a_k} I(v|_{V_{\alpha(a_k)}}, a_k), \tag{2.1}$$

with $a_k \in \mathrm{Sing}(v) \cap X$.

In general, the formula(2.1) is not true. Let us provide firstly the counterexample given by Marie-Hélène Schwartz (1991, p. 6.2.1):

Example 5.2.2. In a first step, in $\mathbb{R}^2$ with coordinates (x, y), one considers the (closed) balls centered at the origin, B with radius 1 and B' with radius 2 (see Figure 5.1 (i)). We have $\chi(B') = +1$.

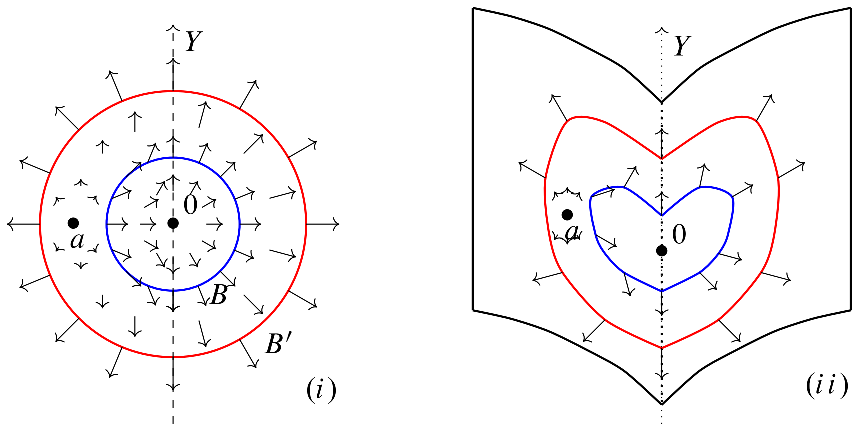


Figure 5.1: M.-H. Schwartz's counterexample.

Inside the ball B, we consider the continuous vector field $v_1(x, y) = (|x|, y)$. One has $v_1(0) = 0$, the point 0 is an isolated singularity of v_1 with index $I(v_1, 0) = 0$.

On the boundary $\partial B'$, we consider the vector field $v_2(x, y) = (x, y)$ pointing radially outwards. We can extend v_2 inside B' as a continuous vector field v which is v_2 along $\partial B'$, v_1 inside B and which is tangent to the y-axis Y along Y. For instance, the vector field defined by

$$v(x, y) = \begin{cases} \left(2|x| - x + (x - |x|)\sqrt{x^2 + y^2}, y\right) & \text{on } B' \setminus B \\ v_1(x, y) = (|x|, y) & \text{inside } B \end{cases}$$

satisfies the conditions (Figure 5.1 (i)).

The vector field v has an isolated singular point of index 0 at 0 and another isolated singular point at $a = (-3/2, 0) \in B' \setminus B$. By Poincaré–Hopf Theorem with boundary(2.4.2), we have

$$\chi(B') = +1 = I(v, 0) + I(v, a),$$

that implies $I(v, a) = +1$.

Let us remark that while $I(v, 0) = 0$, one has $I(v|_Y, 0) = +1$.

In a second step, fold the picture along the y-axis, in order to obtain a singular surface $x^2 - z^3 = 0$ in $\mathbb{R}^3$ (see Figure 5.1 (ii)). In that case, B' becomes a singular variety X, with boundary and stratified by Y and $X \setminus Y$. The vector field v in B' defines a stratified vector field, still denoted by v on X. It has two isolated singular points: 0 and a. We have $I(v|_Y, 0) = +1$ and $I(v, a) = +1$. One has:

$$\chi(X) = +1 \neq I(v, a) + I(v|_Y, 0) = 1 + 1 = 2.$$

So, the formula (2.1) is not true.

We remark that the vector field v is not "radial" at the singular point 0, in the sense that it is not pointing outwards the unit ball centered at 0 in $\mathbb{R}^3$.

We provide another example showing that we cannot take any vector field in order to prove a Poincaré–Hopf Theorem for singular varieties.

Example 5.2.3. We consider the pinched torus X in $\mathbb{R}^3$, obtained from the 2-dimensional torus T by identification of a meridian S_a into the point a. The pinched point a is a singular point of X, that is the singular set of the pinched torus.

We consider a small ball $\mathbb{B}^3(a) \subset \mathbb{R}^3$ centered at $\{a\}$. It intersects the pinched torus along two meridians. We can consider that the surface joining the two meridians, inside the ball, is either a cylinder (in that case, we obtain the torus), or a double cone, to obtain the pinched torus.

We can assume that the vector field v is defined in the small ball $\mathbb{B}^3(a) \subset \mathbb{R}^3$ with an isolated singularity at $\{a\}$ and that its restriction to $X \setminus \{a\}$ is tangent to $X \setminus \{a\}$.

On the one hand, such a vector field is non singular on the boundary $\partial\mathbb{B}^3(a)$, so the way to define its index is to consider the index $I(v, a)$ in $\mathbb{R}^3$.

On the other hand, such a vector field can be obtained from a continuous vector field tangent to the torus T and vanishing on the meridian S_a.

Let us consider two examples of such a vector field:

a) Firstly let us consider the unit vector field on the torus, tangent to the parallels of the torus, it has no singular point on the torus. This vector field can be extended in a neighbourhood of the torus, by parallel extension, in order to be defined on the boundary $\partial\mathbb{B}^3(a)$ of the ball $\mathbb{B}^3(a)$. The vectors $v(x)$ for $x \in \partial\mathbb{B}^3(a)$ are all unit and parallel vectors, so the index $I(v, a)$ is zero.

Now, pinch the torus along S_a. The vector field v does not change outside the ball $\mathbb{B}^3(a)$. Inside the ball, the length of the vector goes to zero with the distance

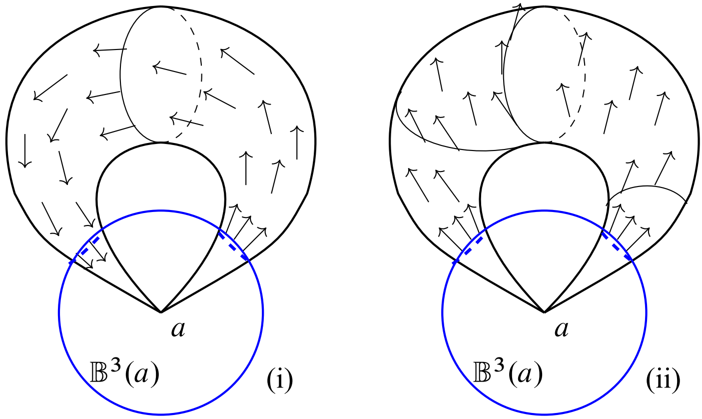


Figure 5.2: Vector fields on the pinched torus

to the point a. We obtain a vector field on the pinched torus with only one singular point with index $I(v, a) = 0$ (see Figure 5.2 (i)).

In this case, the Poincaré–Hopf Theorem is not satisfied, indeed one has

$$\chi(X) = 1 \neq 0 = I(v, a).$$

b) Let us consider now a radial vector field ρ, *i.e.* a vector field with an isolated singularity at $\{a\}$, pointing outwards the ball $\mathbb{B}^3(a)$ along $\partial\mathbb{B}^3(a)$ and tangent to the pinched torus X along the intersection $X \cap \partial\mathbb{B}^3(a)$. On the one hand, the vector field ρ has index $I(\rho, a) = +1$ at a. On the other hand, ρ can be extended on the pinched torus as a continuous vector field without other singularity. Indeed, one can define an extension of ρ in $X \setminus \mathbb{B}^3(a)$ such that the angle of $\rho(x)$ with the tangent line to the meridian containing x decreases with the distance to a until being 0 for the meridian opposed to a. This angle is $\pi/2$ on $X \cap \partial\mathbb{B}^3(a)$, (see Figure 5.2 (ii)). In that case, the Poincaré–Hopf Theorem is valid:

$$\chi(X) = 1 = \sum_{a_k \in \mathrm{Sing}(\rho)} I(\rho, a_k) = I(\rho, a)$$

The vector field ρ is the first example of Marie-Hélène Schwartz radial vector field, of which we will perform a systematic study in the next chapters.

5.3 Why the dual cells decomposition ?

Let us consider M an m-dimensional real analytic manifold equipped with a real semi-analytic Whitney stratification $\{V_\alpha\}$ and $X \subset M$ an n-dimensional real analytic compact subset stratified by $\{V_\alpha\}$.

We denote by (K) a triangulation of M compatible with the stratification, *i.e.* each open simplex is contained in a stratum.

The first observation of Marie-Hélène Schwartz concerns the triangulations:

One knows (2.7) that m is the obstruction dimension to the construction of a vector field tangent to M. In the same way, s is the obstruction dimension to the construction of a vector field tangent to the s-dimensional stratum V_α. That means that if one intends to construct a stratified vector field tangent to X using the triangulation (K), then one will use simplexes of different dimensions according to the dimension of the considered stratum. If we want to define an obstruction cocycle in that way, it will have different dimension according to the strata. That is an obstacle for the use of the triangulation (K), in order to obtain a global Poincaré–Hopf Theorem.

The M.-H. Schwartz observation is the following: Let us denote by (D) the dual cell decomposition of (K) associated to a barycentric subdivision (K') (see Section 1.3). Each (D)-cell is transverse to the strata. In particular, if d is an m-dimensional (D)-cell and if V_α is a stratum of dimension s, then the dimension of the cell $d \cap V_\alpha$ is

$$\dim(d \cap V_\alpha) = s$$

that is precisely the obstruction dimension for the construction of a vector field tangent to V_α.

This observation leads naturally to the construction of a stratified vector field by induction on the dimension of the strata, using the dual cell decomposition (D) and not the triangulation (K).

However, Example 5.2.2 shows that this is not sufficient to obtain a Poincaré–Hopf Theorem. The second observation of M.-H. Schwartz, based on that example and Example 5.2.3, is that one has to consider stratified vector fields which are radial in a sense to be made clear. That is the M.-H. Schwartz construction of radial extension of vector fields that we explain below.

5.4 Radial vector fields

The idea of the construction of radial vector fields is very simple. The construction has to be made in suitable tubes, that makes the M.-H. Schwartz proof delicate. The details of the construction of the tubular neighbourhood is provided in Schwartz (1991).

The construction of a radial vector field goes in two steps: the local (5.4.1) and the global (5.4.2) construction. One obtains the Poincaré–Hopf Theorem for singular varieties (Theorem 5.4.6).

5.4.1 Radial vector fields – Local construction

We consider a neighbourhood $U_\alpha \subset V_\alpha$ of a point a in the stratum V_α and a vector field v tangent to V_α on U_α with possibly an isolated singularity at a. In the same way than in the proof of Theorem 2.3.13, the local radial extension of the vector field v is obtained as the sum of two vector fields defined in a suitable tubular neighbourhood of V_α in the ambient manifold M: the parallel extension and the transverse vector field. In the case of a stratified singular variety, the construction must be made in such a way as to obtain a stratified vector field. This is possible thanks to the conditions (a) and (b) of Whitney.

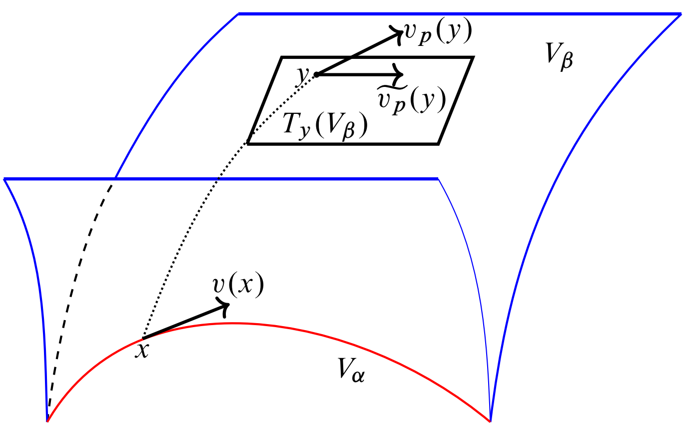


Figure 5.3: Parallel extension.

a) The *parallel extension* of the vector field v in a neighbourhood of U_α in M is defined in the following way: If y is a point on the fibre in x of a sufficiently small tube $\mathcal{T}_\varepsilon(U_\alpha)$, of "ray" ε, then the Whitney condition (a) implies that the vector $v_p(y)$ parallel to $v(x)$ can be projected perpendicularly as a non-zero vector $\widetilde{v_p}(y)$ on the tangent space in y to the stratum containing the point y. We extend v in that way inside the tube $\mathcal{T}_\varepsilon(U_\alpha)$ as a stratified vector field "parallel" to v. Of course, if v admits (isolated) singular points, the vector field $\widetilde{v_p}$ will have "disks" of singular locus corresponding to singularities of v, Schwartz (1964, §3) and Schwartz (1991, Théorème 1.1).

b) The *transverse vector field* ρ is defined in the following way: the vector ρ gradient of the "distance to V_α" (relatively to a suitable metric) vanishes on V_α, it is transverse to the boundary of every sufficiently small "geodesic" tube $\mathcal{T}_\varepsilon(U_\alpha)$ composed of the geodesic rays in M normal to V_α. The Whitney condition (b) guarantees that for every point $y \in \mathcal{T}_\varepsilon(U_\alpha)$, the vector $\rho(y)$ can be projected as a non-zero vector $\widetilde{\rho}(y)$ on the tangent space at y to the stratum containing y, providing a stratified vector field in $\mathcal{T}_\varepsilon(U_\alpha)$, Schwartz (ibid., Théorème 2.3.1).

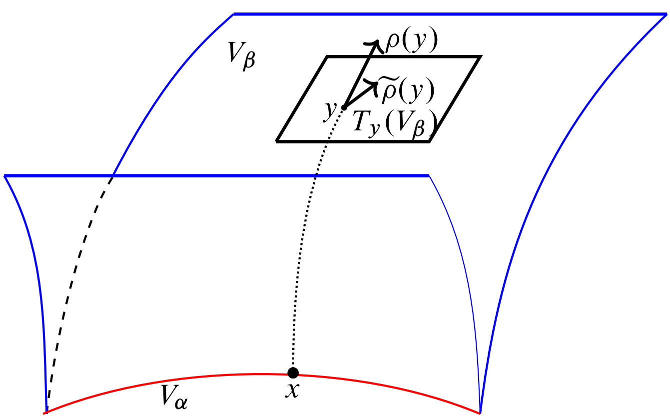
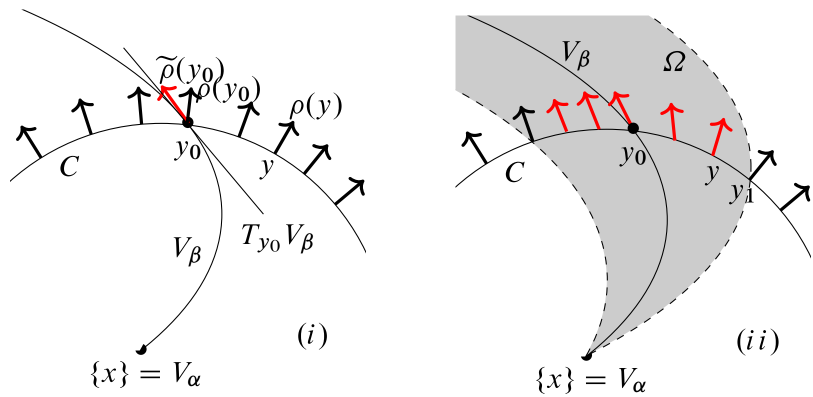


Figure 5.4: transverse vector field.

It is clear that the obtained vector fields $\widetilde{v}_p(y)$ and $\widetilde{\rho}(y)$ are not continuous, as vector fields tangent to M. Indeed, let us look at the case of the field $\widetilde{\rho}(y)$: in the Figure 5.5 (i), V_α is a singleton $\{x\}$, the stratum V_β is a curve and M is the plan. Consider a point y_0 of V_β, intersection of a small circle C centered at x with V_β. In order to obtain a vector field tangent to the strata, we have seen that we replace

$\rho(y_0)$ by its projection $\widetilde{\rho}(y_0)$ on $T_{y_0}(V_\beta)$. But then the field is not continuous: $\widetilde{\rho}(y_0)$ is not limit of the vectors $\rho(y)$ as y approaches y_0 along the circle C.



The gradient vector field τ. The transverse vector field t.

Figure 5.5: A “tapered” neighbourhood.

To overcome this drawback, Marie-Hélène Schwartz considers “tapered” neighbourhoods Ω of the strata (here of the stratum V_β) in which she modifies the vector field $\rho(y)$ so as to obtain a field $\widetilde{\rho}(y)$, called the “transverse” vector field, tangent to the strata and also continuous. More precisely, the “transverse” vector field is built as follows: denote by $\lambda \in [0,1]$ the parameter of the portion of the curve C going from $y_0 = V_\beta \cap C$ at point y_1 intersection of C and the boundary of Ω. At the point y of the curve, of parameter λ, the field $\widetilde{\rho}(y)$ is equal to

$$\widetilde{\rho}(y) = \lambda\rho(y) + (1-\lambda)\widetilde{\rho}_y(y_0) \tag{4.2}$$

where $\widetilde{\rho}_y(y_0)$ is the vector parallel to $\widetilde{\rho}(y_0)$ at the point y. (see Figure 5.5 (ii)).

Similarly, in the “tapered” neighbourhoods of the strata, we build a “parallel” vector field $\widetilde{v_p}(y)$ tangent to the strata and also continuous, from the field $v_p(y)$.

The (local) radial extension of the vector field v is the vector field defined in $\mathcal{T}_\varepsilon(U_\alpha)$ by:

$$\widetilde{v}(y) = \widetilde{v_p}(y) + \widetilde{\rho}(y)$$

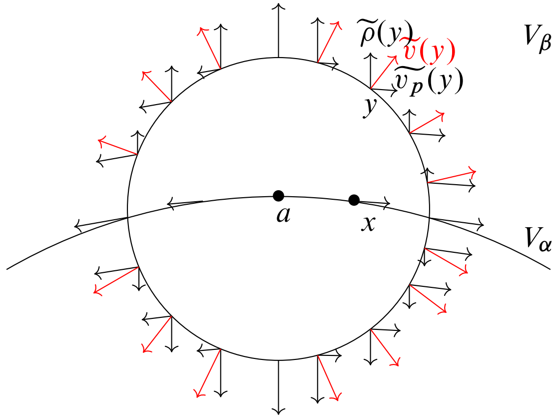


Figure 5.6: Local radial extension of a vector field

Proposition 5.4.2. (Local radial extension of a vector field) *The local radial extension $\widetilde{v} = \widetilde{v_p} + \widetilde{\rho}$ satisfies the following properties:*

1. *the vector field $\widetilde{v}$ is pointing outwards of the tube $\mathcal{T}_\varepsilon(U_\alpha)$ along $\partial\mathcal{T}_\varepsilon(U_\alpha) \setminus \mathcal{T}_\varepsilon(\partial U_\alpha)$,*

2. *If $a \in U_\alpha$ is an isolated singularity of v, it is also an isolated singularity of $\widetilde{v}$ and the index of $\widetilde{v}$ at a as a vector field tangent to the ambient manifold M is the same than the index of v at a, computed as a vector field tangent to V_α:*
$$I(v, a; V_\alpha) = I\widetilde{v}, a; M). \tag{4.3}$$

3. *if two vector fields v_1 and v_2, tangent to V_α are homotopic as sections of $T(V_\alpha)$ over U_α, then their extensions $\widetilde{v}_1 = \widetilde{v_1}_p + \widetilde{\rho}$ and $\widetilde{v}_2 = \widetilde{v_2}_p + \widetilde{\rho}$ are homotopic as sections of TM over $\mathcal{T}_\varepsilon(U_\alpha)$.*

5.4.2 Radial vector fields – Global construction

The "global" construction of vector fields by radial extension goes as follows. The stratification (see Equation (1.1)) is denoted by

$$\emptyset = X_{-1} \subset X_{\alpha_0} = V_{\alpha_0} \subset X_\beta \subset X_\gamma \subset \cdots \subset X_{n-2} \subset X_{n-1} \subset X = X_n \tag{4.4}$$

where the lowest dimensional stratum can be a 0-dimensional one or a stratum V_{α_0} of dimension $2s > 0$. If V_{α_0} is 0-dimensional, i.e. a set V_0 of finitely many points a_i, then one consider a radial vector field v in a ball $B_\varepsilon(a_i)$ centered in each of these points. If the lowest dimensional stratum is a stratum of dimension $2s > 0$, then we construct a vector field v on V_{α_0} with finitely many isolated singularities a_i. We notice that V_{α_0} is a manifold and it has to be compact if X is compact. In this case the total Poincaré–Hopf index of v on V_{α_0} is $\chi(V_{\alpha_0})$. Denote by $\varepsilon(\alpha_0) = \inf \varepsilon_i$ where $\mathcal{T}_{\varepsilon_i}(U_{\alpha_0})$ is the local tubular neighbourhood we constructed around the singular point a_i. The vector field v is well defined by radial extension in the tubular neighbourhood $\mathcal{T}_{\varepsilon(\alpha_0)}(V_{\alpha_0})$ of V_{α_0} with same singularities a_i and their indices satisfy the Equation (4.3).

The radial vector field $\widetilde{v}$ is now defined in a tubular neighbourhood $\mathcal{T}_{\varepsilon(\alpha_0)}(V_{\alpha_0})$ of the lowest dimensional stratum V_{α_0} and it is pointing outwards from $\mathcal{T}_{\varepsilon(\alpha_0)}(V_{\alpha_0})$ (Figure 5.7).

We show now how to extend the vector field $\widetilde{v}$ in the next strata in $X_\beta \setminus X_{\alpha_0}$ *i.e.* the lowest dimensional strata V_β such that $V_{\alpha_0} \subset \partial V_\beta$. We denote by V_β the (finite) union of these strata and

$$W_\beta = V_\beta \setminus \mathcal{T}_{\varepsilon(\alpha_0)}(V_{\alpha_0}).$$

Then W_β is a manifold such that the vector field $\widetilde{v}$ is well defined and pointing inwards of W_β on the boundary $\partial W_\beta = V_\beta \cap \partial \mathcal{T}_{\varepsilon(\alpha_0)}(V_{\alpha_0})$. We can extend $\widetilde{v}$ inside V_β with finitely many isolated singular points b_j. The Poincaré–Hopf Theorem with boundary (Equation (4.7)) implies

$$\chi(W_\beta) - \chi(\partial W_\beta) = \sum_{b_j \in V_\beta} I(\widetilde{v}, b_j).$$

where

$$\chi(\partial W_\beta) = \sum_{a_i \in V_{\alpha_0}} I(\widetilde{v}, a_i).$$

We obtain

$$\chi(X_\beta) = \sum_{a_i \in V_{\alpha_0}} I(\widetilde{v}, a_i) + \sum_{b_j \in V_\beta} I(\widetilde{v}, b_j).$$

The strata V_β admit a tubular neighbourhood $T_{\varepsilon(\beta)}(V_\beta)$ in which we construct a radial extension of $\widetilde{v}$.

In Figure 5.7, the radial vector field is constructed in the order : red, blue, black.

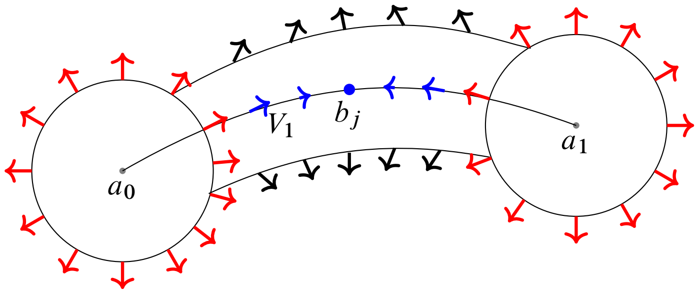


Figure 5.7: The radial vector field

The process continues by increasing dimension of the strata. Note that, for the next dimensional strata V_γ we have to consider

$$W_\gamma = V_\gamma \setminus \left(\mathcal{T}_{\varepsilon(\alpha_0)}(V_{\alpha_0}) \cup \mathcal{T}_{\varepsilon(\beta)}(V_\beta)\right).$$

At the end of the process, the construction provides a "tubular neighbourhood"

$$\mathcal{T}_\varepsilon(X) = \bigcup \mathcal{T}_{\varepsilon(\kappa)}(V_\kappa) \tag{4.5}$$

where κ describes all indices of strata and a radial vector field $\widetilde{v}$ defined on the variety X. We have:

$$\chi(X) = \sum_{a_k \in X} I(\widetilde{v}, a_k)$$

for all singularities a_k of $\widetilde{v}$.

5.4.3 Poincaré–Hopf Theorem for singular varieties.

Theorem 5.4.6. (Schwartz (1991, Théorème 6.2.2)) *Let X be an analytic subset of the analytic manifold M and $\{V_\alpha\}$ a Whitney stratification of the pair (M, X). Let $\widetilde{v}$ be a radial vector field defined on X. There is a finite number of zeroes a_k of $\widetilde{v}$ whose index $I(\widetilde{v}, a_k)$ is the same in the stratum of a_k and in M. We have:*

$$\chi(X) = \sum_{a_k \in X} I(\widetilde{v}, a_k) \tag{4.6}$$

where, if $\dim V_{i(a)} = 0$, *then by construction* $I(\widetilde{v}|_{V_{i(a)}}, a) = +1$.

Exercises

5.1) Take a torus and pinch the torus at 3 points. What is the Euler-Poincaré characteristic of the obtained figure ? Compute Euler-Poincaré characteristic by triangulations and by radial vectorfields. Show that the Poincaré-Hopf Theorem is verified for radial vector fields.

5.2) Take a 2-dimensional sphere and, keeping the north N and south S poles, pinch along a meridian in order to obtain two bananas glued along the line NS. Compute Euler-Poincaré characteristic by triangulations and by radial vectorfields. Show that the Poincaré-Hopf Theorem is verified for radial vector fields.

5.3) We know that compact 2-dimensional smooth surfaces are classified by their Euler-Poincaré characteristic. Is that true for compact 2-dimensional singular surfaces ? (Compare 5.1 and 5.2).

6 Schwartz classes

The construction of Schwartz classes follows the same general principle as that of the construction of radial vector fields for the Poincaré–Hopf Theorem. However, the context is now that of complex and no longer real analytic varieties and that of r-frames and no longer of vector fields. Let X be an analytic subset of the analytic manifold M and $\{V_\alpha\}$ a Whitney stratification of the pair (M, X). The *complex* dimensions of M and X will be denoted by m and n.

As for the Poincaré–Hopf Theorem, the first idea of Marie-Hélène Schwartz is to consider consider stratified (but here complex) vector fields for a (complex) Whitney stratification. That means that she considers the space (not anymore a bundle)

$$\bigcup_{V_\alpha \subset X} T(V_\alpha) \subset T(M)$$

as a substitute to the tangent bundle to X when X is a singular variety whose V_α are the strata.

6.1 Radial extension of frames

The main observation of Marie-Hélène Schwartz concerns the obstruction dimensions (see Proposition 3.1.7 and Section 5.3):

On the one hand, the obstruction dimension to the construction of an r-frame tangent to M is equal to $2p = 2(m - r + 1)$. The obstruction dimension to the construction of an r-frame tangent to a stratum V_α of complex dimension s is equal to $2q = 2(s - r + 1)$. As we have seen in Section 5.3, this property justifies to consider the cell decomposition (D) dual of a triangulation (K) compatible with the given stratification: In that case, the dimension of intersection of a $2p$-cell with V_α is equal to the obstruction dimension $2q = 2(s - r + 1)$ for the construction of an r-frame tangent to V_α.

On the other hand, the obstruction dimension to the construction of an $(r-1)$-frame tangent to M is equal to $2p + 2 = 2(m - r + 2)$. This means that we can construct an $(r-1)$-frame $v^{(r-1)} = (v_1, v_2, \dots, v_{r-1})$ without singularities on the $2p$-cells in (D). In this case, the $(r-1)$ vectors in $v^{(r-1)}$ are $\mathbb{C}$-linearly independent on the $2p$-cells d_i^{2p}. The singularities of an r-frame $v^{(r)} = (v^{(r-1)}, v_r)$ in a $2p$-cells d_i^{2p} will be isolated points at which the last vector v_r either vanish or belongs to the $(r-1)$ complex plane generated by the vectors in $v^{(r-1)}$.

More precisely, if V_α is a stratum of complex dimension s, we will construct an $(r-1)$-frame $v^{(r-1)}$ without singularities on the $2q$-cells in $(D)^{2p} \cap V_\alpha$, then v_r will be a vector field $\mathbb{C}$-linearly independent of $v^{(r-1)}$ with an isolated singularity at the barycenter $\widehat{d_i^{2p}}$ situated in $d^{2p} \cap V_\alpha$. Note that $\widehat{d_i^{2p}}$ is also barycenter of the (K)-simplex $\sigma_i^{2(r-1)} \subset V_\alpha$ where $2m - 2p = 2n - 2s = 2(r-1)$.

We will denote by Δ_α^{2q} the intersection $(D)^{2p} \cap V_\alpha$ and by δ_i^{2q} the cells $d_i^{2p} \cap V_\alpha$, dual of the simplexes $\sigma_i^{2(r-1)} \subset V_\alpha$.

6.1.1 Local radial extension of r-frames

We consider now a stratified r-frame $v^{(r)} = (v^{(r-1)}, v_r)$, section of $V_r(TM)$ over $\Delta^{2q} \subset V_\alpha^{2s}$ (with $q = s - r + 1$), with isolated singularities which are zeroes of the last vector v_r. We define in the tube $\mathcal{T}_\varepsilon(\delta_i^{2q})$ the parallel extension $(\widetilde{v}_p^{(r-1)}, (\widetilde{v}_r)_p)$ of $v^{(r)}$, by the same method than in Section 5.4.1 and we consider the transverse vector field $\widetilde{\rho}$ as defined in Equation (4.2)

Proposition 6.1.1. (Local radial extension for a frame) *If ε is sufficiently small, the radial extension of $v^{(r)}$, defined by $\widetilde{v}^{(r)} = (\widetilde{v}_p^{(r-1)}, (\widetilde{v}_r)_p + \widetilde{\rho})$ satisfies the* *following conditions:*

i) *the radial extension* $\widetilde{v}_r = (\widetilde{v}_r)_p + \widetilde{\rho}$ *of* v_r *satisfies the* Proposition 5.4.2,

ii) *if the* $(r-1)$*-frame* $v^{(r-1)}$ *has no singularity on* $\delta_i^{2q} = d_i^{2p} \cap V_\alpha^{2s}$ *and if* $v^{(r)}$ *admits an isolated singularity at the barycenter* $a \in \delta_i^{2q}$ *which is a zero of* v_r*, then* $\widetilde{v}^{(r)} = (\widetilde{v}_p^{(r-1)}, (\widetilde{v}_r))$ *satisfies the same properties in* $\mathcal{T}_\varepsilon(\delta_i^{2q})$*. In that case, if the* $(r-1)$*-complex plane generated by* $v^{(r-1)}(a)$ *is linearly independent of the tangent plane* $T_a(\Delta^{2q})$ *in* $T_a(V_\alpha^{2s})$*, then the index* $I(\widetilde{v}^{(r)}, a; M)$ *of the extension* $\widetilde{v}^{(r)}$ *at* a*, considered as an* r*-frame tangent to* M *is equal to the index* $I(v^{(r)}, a; V_\alpha)$ *of* $v^{(r)}$ *at* a *considered as an* r*-frame tangent to* V_α^{2s}*.*

iii) *In the same hypothesis than (ii), if* $q = 0$ *(i.e;* $s = r-1$*), and if* a *is a zero of* v_r*, then the index of* $\widetilde{v}^{(r)}$ *in* a *is* $+1$*.*

We will denote by $I(v^{(r)}, a)$ the index of $\widetilde{v}^{(r)}$ at the isolated singularity a.

6.1.2 Global radial extension of r-frames

As in the case of the Poincaré–Hopf Theorem, we will construct $v^{(r)}$ over the subsets $\Delta_\alpha^{2q} = (D)^{2p} \cap V_\alpha^{2s}$, by increasing dimensions of the strata V_α. We will construct $v^{(r)}$ at each step over $\overline{\Delta_\alpha}$ and a tube $\mathcal{T}_\varepsilon(\overline{\Delta_\alpha})$, neighbourhood of $\overline{\Delta_\alpha}$ in $D^{(2p-1)}$.

i) If V_α^{2r-2} is a stratum whose real dimension is $2r-2 = 2(m-p)$, the obstruction dimension to the construction of a section of $V_r(TV_\alpha)$ is zero. One takes any $(r-1)$-frame $v^{(r-1)}$ tangent to V_α^{2r-2} at the vertices $a_j = \Delta_j^0$ of Δ located in $(D)^{2p} \cap V_\alpha^{2r-2}$ and the last vector v_r zero at these points.

One construct the radial extension of the r-frame in the tubes $\mathcal{T}_\varepsilon(\Delta_j^0)$ as an r-frame still denoted by $v^{(r)}$. According to Proposition 6.1.1 (iii), one has $I(v^{(r)}, a_j) = +1$.

ii) Let us suppose $s > r-1$ and the construction already performed on all strata V_β whose dimension is less than $2s$. That means that the construction has been performed on the sets $\overline{\Delta_\beta}$ and the tubes $T_\varepsilon(\overline{\Delta_\beta})$. We constructed an r-frame pointing outwards of the $2p$-skeleton of a tubular neighbourhood of V_β^{2t} for all strata V_β with dimension $2t < 2s$.

We consider a $2s$-dimensional stratum V_α that contains a stratum V_β^{2t} in its closure. The r-frame is constructed on a tubular neighbourhood of the boundary of

V_α within V_α. We extend the r-frame inside V_α, more precisely in the $2q$-skeleton of Δ_α^{2q}, with $2q = 2(s - r + 1)$ and with isolated singularities at the barycenters of cells δ_i^{2q} which are zeroes of the last vector v_r.

In summary, an r-frame already known on a neighborhood of the boundary of a stratum is extended with isolated singularities inside (a suitable skeleton) of the stratum and then extended with property (ii) of the Proposition 6.1.1 to a regular neighbourhood of this stratum.

The number of singularities of $\widetilde{v}$ is finite. We consider a "sufficiently small" triangulation K of M compatible with the stratification and such that
i) The singularities of $\widetilde{v}$ are barycenters of simplexes of K,
ii) The cellular tube $\mathcal{T}$ around X lies in the tube $\mathcal{T}_\varepsilon(X)$ (see Equation (4.5)).

We still denote by $\mathcal{T}$ the tubular neighborhood of X in M consisting of the (D)-cells which meet X (see Definition 4.6.1).

The constructed r-frame satisfies

Theorem 6.1.2. *(Brasselet and Schwartz (1981), Schwartz (1965), Schwartz (2000)) Let X be an analytic subset of the analytic manifold M and $\{V_\alpha\}$ a Whitney stratification of the pair (M, X). We can construct, on the $2p$-skeleton $(D)^{2p}$, a stratified r-frame $v^{(r)}$, called radial frame, whose singularities satisfy the following properties:*

i) $v^{(r)}$ has only isolated singular points, which are zeroes of the last vector v_r. On $(D)^{2p-1}$, the r-frame $v^{(r)}$ has no singular point and on $(D)^{2p}$ the $(r-1)$-frame $v^{(r-1)}$ has no singular point.

ii) Let $a \in V_\alpha \cap (D)^{2p}$ be a singular point of $v^{(r)}$ in the $2s$-dimensional stratum V_α. If $s > r-1$, the index of $v^{(r)}$ at a, denoted by $I(v^{(r)}, a)$, is the same as the index of the restriction of $v^{(r)}$ to $V_\alpha \cap (D)^{2p}$ considered as an r-frame tangent to V_α. If $s = r-1$, then $I(v^{(r)}, a) = +1$.

iii) Inside a $2p$-cell d which meets several strata, the only singularities of $v^{(r)}$ are inside the lowest dimensional one (in fact located in the barycenter of d).

iv) The r-frame $v^{(r)}$ is pointing outwards of a regular (cellular) neighbourhood $\mathcal{T}$ of X in M. It has no singularity on $\partial\mathcal{T}$.

6.2 Obstruction cocycles and classes

Let us recall that d^* is the elementary (D)-cochain whose value is 1 at d and 0 at all other cells. We can define a $2p$-dimensional (D)-cochain in $C^{2p}(\mathcal{T}, \partial\mathcal{T})$ by:

$$\widehat{c} = \sum_{\substack{d(\sigma)\subset\mathcal{T} \\ \dim d(\sigma)=2p}} I(v^{(r)}, \widehat{\sigma})\, d^*(\sigma)\,. \tag{2.1}$$

This cochain actually is a cocycle whose class $c^p(X)$ lies in

$$H^{2p}(\mathcal{T}, \partial\mathcal{T}) \cong H^{2p}(\mathcal{T}, \mathcal{T} \setminus X) \cong H^{2p}(M, M \setminus X),$$

where the first isomorphism is given by retraction along the rays of $\mathcal{T}$ and the second by excision (by $M \setminus \mathcal{T}$).

Definition 6.2.2. (Schwartz (1965),Schwartz (2000)) The p-th Schwartz class of X, denoted by $c_S^p(X)$ is the class

$$c^p(X) \in H^{2p}(M, M \setminus X).$$

The Schwartz class does not depend of any of the choices: stratification, triangulation, r-frame...

Exercises

See Chapter 7.

7 MacPherson classes

The MacPherson construction of classes answers a conjecture by Deligne and Grothendieck which associates homology classes c_M to constructible functions on algebraic complex varieties and satisfying suitable properties. With Marie-Hélène Schwartz, I proved that the MacPherson class c_M is dual of the Schwartz class c_S. These classes are now named Schwartz–MacPherson classes C_{SM}.

The MacPherson's idea is to substitute the Nash bundle to the tangent bundle in the singular case.

7.1 Nash transformation

Let M be an complex analytic manifold, of complex dimension m. Let X be a n-dimensional semi-analytic complex variety, $X \subset M$. We denote by $\Sigma = X_{\text{sing}}$ the singular part of X and by $X_{\text{reg}} = X \setminus \Sigma$ its regular part.

The Grassmannian manifold of complex n-planes in $\mathbb{C}^m$ is denoted by $G_n(\mathbb{C}^m)$. We consider the Grassmann bundle of n (complex) planes in TM, denoted by $G_n(TM)$. The fibre $G_n(T_xM)$ over $x \in M$ is the set of n-planes in $T_x(M)$ and is isomorphic to $G_n(\mathbb{C}^m)$. An element of $G_n(TM)$ is denoted by (x, P) where $x \in M$ and $P \in G_n(T_xM)$.

On the regular part of X, one can define the Gauss map

$$\gamma : X_{\mathrm{reg}} \longrightarrow G_n(TM) \qquad \gamma(x) = (x, T_x(X_{\mathrm{reg}})).$$

Definition 7.1.1. The *Nash transformation* $\widetilde{X}$ is defined as the closure of the image of γ in $G_n(TM)$.

$$\begin{tikzcd} & G_n(TM) \arrow[d] \\ X_{\mathrm{reg}} \arrow[ur, "\gamma"] \arrow[r, hook] & M \end{tikzcd} \qquad \begin{tikzcd} \widetilde{X} = \overline{\mathrm{Im}\gamma} \arrow[r, hook] \arrow[d, "\nu"] & G_n(TM) \arrow[d] \\ X \arrow[r, hook] & M \end{tikzcd} \tag{1.1}$$

In general, $\widetilde{X}$ is not smooth, nevertheless, it is an analytic variety and the restriction $\nu : \widetilde{X} \to X$ of the bundle projection $G_n(TM) \to M$ is analytic.

We denote by E the tautological bundle over $G_n(TM)$. The fibre E_P at a point $(x, P) \in G_n(TM)$ is the set of the vectors v in the n-plane $P \in G_n(T_xM)$.

$$E_P = \{v(x) \in T_xM \ : \ v(x) \in P\}$$

We consider the restriction $\widetilde{E} = E|_{\widetilde{X}}$. On the inverse image

$$\widetilde{X}_{\mathrm{reg}} = \nu^{-1}(X_{\mathrm{reg}}) \cong X_{\mathrm{reg}}$$

the restriction $\widetilde{E}|_{\widetilde{X}_{\mathrm{reg}}}$ can be identified with $T(X_{\mathrm{reg}})$ and

$$\widetilde{E} = E \times_{G_n(TM)} \widetilde{X} = \{(v(x), \widetilde{x}) \in E \times \widetilde{X} \ : v(x) \in \widetilde{x}\}$$

where $\widetilde{x} \in \widetilde{X}$ is a n-complex plane in $T_x(M)$ and $x = \nu(\widetilde{x})$.

We have a diagram:

$$\begin{tikzcd} \widetilde{E} \arrow[r, hook] \arrow[d] & E \arrow[d] \\ \widetilde{X} \arrow[r, hook] \arrow[d, "\nu"'] & G_n(TM) \arrow[d] \\ X \arrow[r, hook] & M \end{tikzcd}$$

An element in $\widetilde{E}$ is written (x, P, v) where $x \in X$, P is an n-plane in $\nu^{-1}(x)$ and v is a vector in P. If $x \in X_{\mathrm{reg}}$, then $P = T_x(X_{\mathrm{reg}})$.

7.2 local Euler obstruction

We denote by $\{V_\alpha\}$ a complex analytic stratification of (M, X) satisfying the *Whitney conditions* (see Section 4.3).

The following lemma is fundamental for the understanding of the geometrical definition of the local Euler obstruction. The proof is a direct application of the Whitney condition (a).

Lemma 7.2.1. (Brasselet and Schwartz (1981, Proposition 9.1)) *A stratified vector field v defined on $A \subset X$ admits a canonical lifting $\widetilde{v}$ on $\nu^{-1}(A)$ as a section of $\widetilde{E}$.*

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\nu_*} & TM|_X \\ \widetilde{v}\upharpoonleft\downharpoonright & & v\upharpoonleft\downharpoonright \\ \widetilde{X} & \xrightarrow{\nu} & X \end{array} \qquad \nu_*(x, \widetilde{x}, v(x)) = (x, v(x))$$

PROOF: Let us consider a stratified vector field v on $A \subset X$ and a point $\widetilde{x} \in \widetilde{X}$. Let us denote $x = \nu(\widetilde{x})$.

(i) If $x \in X_{\mathrm{reg}}$, then $v(x) \in T_x(X_{\mathrm{reg}}) = \widetilde{x}$ with $\widetilde{x} = \nu^{-1}(x)$. We define $\widetilde{v}(\widetilde{x}) = (x, T_x(X_{\mathrm{reg}}), v(x))$.

(ii) If $x \in V_\alpha$, then $v(x) \in T_x(V_\alpha)$. Each $\widetilde{x} \in \nu^{-1}(x)$ is in the closure of the image of γ (see diagram 1.1), i.e. there is a sequence $(\widetilde{x_n})$ of points of $\widetilde{X}_{\mathrm{reg}}$ such that $\widetilde{x} = \lim \widetilde{x_n}$, $\nu(\widetilde{x_n}) = x_n \in X_{\mathrm{reg}}$ and $\widetilde{x_n} = T_{x_n}(X_{\mathrm{reg}})$. Then one has $\lim(x_n) = x$ and $\lim T_{x_n}(X_{\mathrm{reg}}) = \widetilde{x}$. By the Whitney condition (a), one has $T_x(V_\alpha) \subset \widetilde{x}$ that implies $v(x) \in \widetilde{x}$ and we can define $\widetilde{v}(\widetilde{x}) = (x, \widetilde{x}, v(x))$. □

The definition of local Euler obstruction was firstly defined by MacPherson (1974) using differential forms. Here, we give the equivalent definition, see Brasselet and Schwartz (1981), using vector fields. We now consider a *radial* stratified vector field v in a neighbourhood of the point $\{0\} \in X$ so that there exists $\varepsilon_0 > 0$ such that for all ε, $0 < \varepsilon < \varepsilon_0$, the vector $v(x)$ is pointing outwards of the ball $\mathbb{B}_\varepsilon$ over the boundary $\mathbb{S}_\varepsilon = \partial\mathbb{B}_\varepsilon$. By the Bertini–Sard theorem, (see for instance Verdier (1976)) $\mathbb{S}_\varepsilon$ is transverse to the strata V_α if ε is small enough, so the following definition takes sense.

Definition 7.2.2. (Brasselet and Schwartz (1981)) Let v be a basic radial stratified vector field over $X \cap \mathbb{S}_\varepsilon$ and $\widetilde{v}$ the lifting of v on $\nu^{-1}(X \cap \mathbb{S}_\varepsilon)$. The local Euler obstruction $\mathrm{Eu}_0(X)$ is the obstruction to extend $\widetilde{v}$ as a nowhere zero section of $\widetilde{E}$

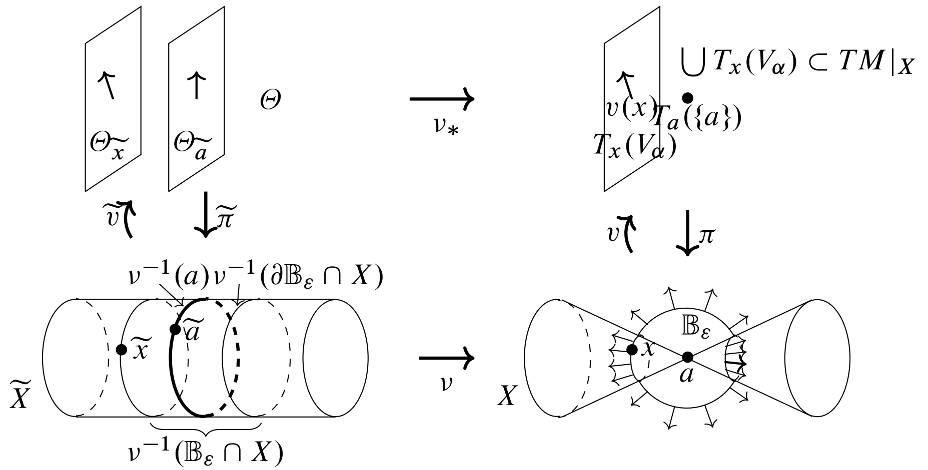


Figure 7.1: The Nash transformation.

over $\nu^{-1}(X \cap \mathbb{B}_\varepsilon)$, evaluated on the orientation class $\mathcal{O}_{\nu^{-1}(\mathbb{B}_\varepsilon),\nu^{-1}(\mathbb{S}_\varepsilon)}$:

$$\mathrm{Eu}_0(X) = \mathrm{Obs}(\widetilde{v}, \widetilde{E}, \nu^{-1}(X \cap \mathbb{B}_\varepsilon)).$$

The local Euler obstruction satisfies the following properties:
i) $Eu_x(X) = 1$ if x is a regular point of X.
ii) Constructibility:

Proposition 7.2.3. (MacPherson (1974),Brasselet and Schwartz (1981) and other authors)*: The local Euler obstruction is constant along the strata of a Whitney stratification of X.*

iii) Proportionality Theorems (Brasselet and Schwartz (ibid.), Théorème 11.1):

Theorem 7.2.4. (Proportionality Theorem for vector fields). *Let v be any radial vector field with an isolated singularity at the point $a \in V_\alpha$, with index $I(v,a) = I(v|_{V_\alpha}, a)$, and let b a small ball centered at a without other singularity of v, then*

$$Obs(\widetilde{v}, \widetilde{E}, \nu^{-1}(b \cap X)) = Eu_a(X) \cdot I(v,a) \tag{2.2}$$

We denote by $\widetilde{E}^r$ the bundle on $\widetilde{X}$ associated to $\widetilde{E}$ whose fiber on $\widetilde{x}$ is the set of r-frames whose vectors belong to $\widetilde{E}|_{\widetilde{x}}$. An element in $\widetilde{E}^r$ is written $(x, P, v^{(r)})$ where $x \in X$, P is a n-plane in $\nu^{-1}(x)$ and $v^{(r)}$ is an r-frame in P.

Theorem 7.2.6. (Proportionality Theorem for frames). *Let $v^{(r)}$ be a radial r-frame with isolated singularities on the $2p$-cells d_i^{2p} with index $I(v^{(r)}, \hat{\sigma}_i)$ at the barycenter $\{\hat{\sigma}_i\} = d_i^{2p} \cap \sigma_i$ (see Theorem 6.1.2 (ii)). Then the obstruction to the extension of $\widetilde{v}^{(r)}$ as a section of $\widetilde{E}^r$ on $\nu^{-1}(d_i^{2p} \cap X)$ is*

$$Obs(\widetilde{v}^{(r)}, \widetilde{E}^r, \nu^{-1}(d_i^{2p} \cap X)) = Eu_{\hat{\sigma}_i}(X) \cdot I(v^{(r)}, \hat{\sigma}_i). \qquad (2.3)$$

7.3 Constructible sets and functions

A *constructible set* in a variety X is a subset obtained by finitely many unions, intersections and complements of subvarieties. A *constructible function* $\varphi : X \to \mathbb{Z}$ is a function such that $\varphi^{-1}(n)$ is a constructible set for all n. The constructible functions on X form a group denoted by $\mathbb{F}(X)$. If $A \subset X$ is a subvariety, we denote by $\mathbf{1}_A$ the characteristic function whose value is 1 over A and 0 elsewhere.

If X is triangulable, φ is a constructible function if and only if there is a triangulation (K) of X such that φ is constant on the interior of each simplex of (K). Such a triangulation of X is called φ-adapted.

The correspondence $\mathbb{F} : X \to \mathbb{F}(X)$ defines a contravariant functor when considering the usual pull-back $f^* : \mathbb{F}(Y) \to \mathbb{F}(X)$ for a morphism $f : X \to Y$. One interesting fact is that it can be made a covariant functor when considering the pushforward defined on characteristic functions by:

$$f_*(\mathbf{1}_A)(y) = \chi(f^{-1}(y) \cap A), \qquad y \in Y$$

for a morphism $f : X \to Y$, and linearly extended to elements of $\mathbb{F}(X)$. The following result was conjectured by Deligne and Grothendieck in 1969 in the framework of algebraic complex varieties.

Conjecture 7.3.1. Let $\mathbb{F}$ be the covariant functor of constructible functions and let $H_*(\ ;\mathbb{Z})$ be the usual covariant $\mathbb{Z}$-homology functor. Then there exists a unique natural transformation

$$c_* : \mathbb{F} \to H_*(\ ;\mathbb{Z})$$

satisfying $c_*(\mathbf{1}_X) = c^*(X) \cap [X]$ if X is a manifold.

The conjecture means that for every algebraic complex variety, one has a functor $c_* : \mathbb{F}(X) \to H_*(X;\mathbb{Z})$ satisfying the following properties:

1. $c_*(\varphi + \psi) = c_*(\varphi) + c_*(\psi)$ for φ and ψ in $\mathbb{F}(X)$,

2. $c_*(f_*\varphi) = f_*(c_*(\varphi))$ for $f : X \to Y$ morphism of algebraic varieties and $\varphi \in \mathbb{F}(X)$,
3. $c_*(\mathbf{1}_X) = c^*(X) \cap [X]$ if X is a manifold.

7.4 Mather classes

The first approach to the proof of the Deligne–Grothendieck's conjecture is to think to the Nash bundle as a substitute to the tangent bundle in the case of singular varieties. That approach leads to the construction of Mather classes. Let X a possibly singular algebraic complex variety embedded in a smooth one M. We define the Nash transformation $\widetilde{X}$ of X, and the Nash bundle $\widetilde{E}$ on $\widetilde{X}$ as in Section 7.1.

Definition 7.4.1. The Mather class of X is defined by:

$$c_{Ma}(X) = \nu_*(c^*(\widetilde{E}) \cap [\widetilde{X}]) \tag{4.4}$$

where $c^*(\widetilde{E})$ denotes the usual (total) Chern class of the bundle $\widetilde{E}$ in $H^*(\widetilde{X})$ and the cap-product with $[\widetilde{X}]$ is the Poincaré duality homomorphism (in general not an isomorphism).

The Mather classes do not satisfy the Deligne–Grothendieck's conjecture.

7.5 MacPherson classes

The MacPherson idea is to give a different weight to the contribution of strata in the Mather construction, depending on the local Euler obstruction. The construction uses both the constructions of Mather classes and local Euler obstruction. We consider a Whitney stratification of the ambient complex algebraic manifold M such that X is union of strata.

Proposition 7.5.1. MacPherson (1974) *There is a isomorphism T between algebraic cycles on X and constructible functions, given by*

$$T\left(\sum n_\alpha V_\alpha\right)(x) = \sum n_\alpha \mathrm{Eu}_x(\overline{V_\alpha})$$

There are integers n_α such that, for every point $x \in X$, we have:

$$\sum_\alpha n_\alpha \mathrm{Eu}_x(\overline{V_\alpha}) = 1. \tag{5.5}$$

Definition 7.5.3. (MacPherson (ibid.)) The MacPherson's functor

$$c_* : \mathbb{F} \to H_*(\,;\mathbb{Z}) \quad \text{is defined by} \quad c_M(\mathbf{1}_X) = \sum_\alpha n_\alpha \, i_* c_{Ma}(\overline{V_\alpha})$$

where i denotes the inclusion $\overline{V_\alpha} \hookrightarrow X$, and is defined for all constructible functions φ on X by linearity. The (total) MacPherson class of X is defined by $c_M(X) = c_M(\mathbf{1}_X)$.

Theorem 7.5.4. *(MacPherson (ibid.)) The MacPherson functor satisfies the Deligne–Grothendieck Conjecture 7.3.1*

Note that we have the following relation : $c_{Ma}(X) = c_M(\mathrm{Eu}_X)$.
In Brasselet and Schwartz (1981) we proved the following result:

Theorem 7.5.5. *(Brasselet and Schwartz (ibid.)) The MacPherson class $c_M(X)$ is the image of the Schwartz class $c_S(X)$ by the Alexander duality isomorphism Section 4.6*

$$H^{2(m-r+1)}(M, M \setminus X) \xrightarrow{\cong} H_{2(r-1)}(X).$$

The classes are now named Chern–Schwartz–MacPherson classes or Schwartz–MacPherson classes.

Proof: Using the notations of Chapter 6, and precisely the formula (2.1) the r-frame $v^{(r)}$ determines a cocycle of the M. H. Schwartz class:

$$\widehat{c} = \sum_{d_i^{2p} \cap X \neq \emptyset} I(v^r, \widehat{\sigma}_i)(d_i^{2p})^* \,. \tag{5.6}$$

It determines also a cocycle $\widetilde{c}$ of the Chern class $c^p(\widetilde{E})$ such that (see Equation (2.3))

$$< \widetilde{c}.\nu^{-1}(d_i^{2p} \cap X) >= Eu_{a_i}(X) I(v^r, \widehat{\sigma}_i).$$

We will denote $\mu_i = I(v^r, \widehat{\sigma}_i)$ for $\sigma_i = \sigma_i^{2r-2}$, simplex whose the cell d_i^{2p} is dual, i.e. $d_i^{2p} = d(\sigma_i^{2r-2})$ and a_i will be any point of σ_i^{2r-2}. The Proposition 7.2.3 shows that $Eu_{a_i}(X)$ does not depend on the point in σ_i^{2r-2}.

Using Equation (4.4), the homology Chern–Mather class $c_{r-1}^{Ma}(X)$ of (real) degree $2(r-1)$ is represented by the cycle:

$$\sum_{\sigma_i^{2r-2} \subset X} Eu_{a_i}(X) \mu_i \sigma_i^{2r-2}.$$

The previous result, written for each $\overline{V}_\alpha$ instead of X, says that the homology Chern–Mather class $c^{Ma}_{r-1}(\overline{V}_\alpha)$ of (real) degree $2(r-1)$ is represented by the cycle:

$$\sum_{\sigma_i^{2r-2}\subset\overline{V}_\alpha} Eu_{a_i}(\overline{V}_\alpha)\mu_i\sigma_i^{2r-2}.$$

By Definition 7.5.3, the MacPherson class $c_M(X)$ of (real) degree $2(r-1)$ is represented by the cycle:

$$\sum_\alpha n_\alpha\left(\sum_{\sigma_i^{2r-2}\subset\overline{V}_\alpha} Eu_{a_i}(\overline{V}_\alpha)\mu_i\sigma_i^{2r-2}\right)=\sum_{\sigma_i^{2r-2}\subset X}\left(\sum_{\alpha\in A_i} n_\alpha Eu_{a_i}(\overline{V}_\alpha)\right)\mu_i\sigma_i^{2r-2}$$

In this expression, the coefficient of $\mu_i\sigma_i^{2r-2}$ is (see formula Equation (5.5)):

$$\sum_{\alpha\in A_i} n_\alpha Eu_{a_i}(\overline{V}_\alpha)=1,\quad \text{with}\quad A_i=\{\alpha:\sigma_i^{2r-2}\subset\overline{V}_\alpha\}=\{\alpha:a_i\in\overline{V}_\alpha\}.$$

We obtain a cycle of the MacPherson class of X of the form:

$$\gamma=\sum_{\sigma_i^{2r-2}\subset X}\mu_i\sigma_i^{2r-2}.$$

Let us recall (4.6) that the Alexander isomorphism $H^{2p}(M,M-X)\to H_{2r-2}(X)$ is induced by the isomorphism:

$$C^{2p}_{(D)}(M,M-\overset{\circ}{T})\to C_{2r-2,(K)}(X)$$

which associates to a (D)-cochain $(d_i^{2p})^*$ such that $d_i^{2p}\cap X\neq\emptyset$ the (K)-chain σ_i^{2r-2} such that $d_i^{2p}=d(\sigma_i^{2r-2})$. By this isomorphism, the cycle γ is image of the cocycle of the M. H. Schwartz class (cf Equation (5.6))

$$\widehat{c}=\sum_{d_i^{2p}\cap X\neq\emptyset}\mu_i(d_i^{2p})^*$$

which proves the theorem. $\square$

We observe that we determined a cycle of the MacPherson class. In fact, one has the following corollary:

Corollary 7.5.7. *Let (K) be a simplicial triangulation of M compatible with a Whitney stratification of the pair (M, X) and $v^{(r)}$ a r-radial frame defined on the $2p$-skeleton $D^{(2p)}$ of a cellular decomposition (D) dual of (K). The $(r-1)$-st MacPherson class $c_{r-1}(X)$ is represented by the cycle*

$$\sum_{\sigma \in X} I(v^{(r)}, \hat{d}(\sigma))\, \sigma$$

with $\dim \sigma = 2(r-1)$.

Exercises

7.1) The Thom space.

Let Y be a manifold in $\mathbb{P}_N$ and denote by L the restriction of the hyperplane bundle of $\mathbb{P}_N$ to Y.

Denote $E = \mathbb{P}(L \oplus 1_Y)$ where 1_Y is the trivial bundle of complex rank 1 on Y.

Show that the canonical projection $p\colon E \to Y$ admits two sections, zero and infinite, with images $Y_{(0)}$ and $Y_{(\infty)}$.

The projective cone KY is defined as a quotient of E by contraction of $Y_{(\infty)}$ into a point $\{s\}$. It is the Thom space associated to the bundle L, with basis Y.

7.2) The Thom space associated to the Segre embedding.

Consider the image Y of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, defined in homogeneous coordinates by

$$\varphi_S : (x_0 : x_1) \times (y_0 : y_1) \mapsto (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1),$$

Show that that is an embedding whose bidegree is $(1, 1)$ and image $\varphi_S(\mathbb{P}^1 \times \mathbb{P}^1)$ is a non degenerate quadric Q provided with two families of generatices d_1 and d_2.

Show that the Euler class of the bundle E in $H^2(\mathbb{P}^1_x \times \mathbb{P}^1_y) = H^2(\mathbb{P}^1_x) \oplus H^2(\mathbb{P}^1_y) = \mathbb{Z} \oplus \mathbb{Z}$ is $c^1(E) = (\eta_x, \eta_y)$ where η_x is Euler class of the hyperplane bundle of $\mathbb{P}^1_x$, i.e. such that $\eta_x \cap [\mathbb{P}^1_x] = 1$.

Show that the Chern–Schwartz–MacPherson class of the Thom space is

$$c_*(KY) = \underbrace{[KY]}_{H_6(KY)} + \underbrace{3([Kd_1] + [Kd_2])}_{H_4(KY)} + \underbrace{8[Ka]}_{H_2(KY)} + \underbrace{5[s]}_{H_0(KY)},$$

where a is a point in Y and K means taking the cone with vertex s.

7.3) The Thom space associated to the Veronese embedding.
Consider the image of the Veronese embedding $\mathbb{P}_2 \hookrightarrow \mathbb{P}_5$ defined by

$$\varphi_V : (x_0 : x_1 : x_2) \mapsto (x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2).$$

Show that that is an embedding whose degree is 2 and image $\varphi_V(\mathbb{P}^2)$ is smooth and has degree 4. It is called Veronese surface.

Show that the Euler class of the bundle E in $H^2(\mathbb{P}^2)$ is $c^1(E) = 2\eta_L$ where H is a hyperplane in $\mathbb{P}^5$, and $H \cap V \cong 2L$ is a divisor in $\mathbb{P}^2$, L being hyperplane in $\mathbb{P}^2$. Show that $\eta_L = c^1(E_L)$ is generator of $H^2(\mathbb{P}^2)$, and $\eta_L \cap [L] = 1$.

Show that the Chern–Schwartz–MacPherson class of the Thom space is

$$c_*(KY) = \underbrace{[KY]}_{H_6(KY)} + \underbrace{5[Kd]}_{H_4(KY)} + \underbrace{9[Ka]}_{H_2(KY)} + \underbrace{4[s]}_{H_0(KY)}$$

where d is a projective line in Y.

8 Developments and perspectives

The last chapter is devoted to remarks and complements on the previous chapters and to developments and perspectives about characteristic classes.

In the presentation of the characteristic classes and more particularly of the Chern classes we have privileged the definition by obstruction theory. As we have pointed out, this is not the only one possible. From a topological point of view, the obstruction theory has the advantage of giving a clear meaning of the characteristic classes: a measure of the obstruction to the construction of a field of r-frames. However, this theory has a drawback: the difficulty of effective calculation in many situations. This explains why most authors prefer an axiomatic definition, which allows explicit calculations but hides the primitive beginnings of the knowledge and the meaning of classes.

8.1 Remarks and complements.

In the first section of the last chapter, we provide some remarks and complements concerning the previous chapters.

The first remark is that some authors write that Archimedes, and even Pappus of Alexandria (~ 290, ~ 350), knew of Euler's formula. This does not seem to be confirmed in a certain way.

The first papers by Stiefel, Whitney and Chern (first definition) are written in the context of the sphere bundle associated with the tangent bundle, instead of the tangent bundle itself. The vectors considered are then of length $+1$, but the reasoning is similar and gives the same results as those presented in this course (see S.-s. Chern (1946)).

In 1947, Lev Pontryagin, a Russian mathematician, defined another type of classes: the Pontryagin classes, by obstruction theory. On an n-dimensional manifold M Pontryagin considers $(n-2k)+2$ vector fields in general position. The set of points where they span a subspace of $T_x M$ of dimension less or equal to $(n-2k)$ is a cycle of codimension $4k$. By Poincaré duality, one obtains a class which is the Pontryagin class $p_k(M) \in H^{4k}(M;\mathbb{Z})$. For a real vector bundle E, the Pontryagin classes of E are related to the Chern classes of the complexification $E \otimes \mathbb{C} = E \oplus iE$ of E by the formula

$$p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C}) \in H^{4k}(M;\mathbb{Z}).$$

8.2 About the fundamental Chern article.

In his fundamental article S.-s. Chern (ibid.), Chern gave, in particular, the definitions of his classes in terms of Schubert cycles, differential forms, obstruction cocycles, differential forms of transgression.

The context of his first definition is the one of the complex sphere bundle. The context of his second definition is the one of fibre bundles in which he considers sections which are ordered sets of r linearly independent complex vectors (our context in Section 3.2.2). The context of his third definition is the one of sections which are ordered sets of r mutually perpendicular vectors of the sphere. Chern's observation (see S.-s. Chern (ibid., P. 101 and 103)) is that the three contexts are equivalent for the definition of characteristic classes of a complex manifold.

The first definition given by Chern uses two results: the first one is the result of Charles Ehresmann (Theorem 3) describing suitable Schubert varieties as the basis of cycles for Grassmannian manifolds. The second one (proved by Chern in the Theorems 1 and 2) shows that the Grassmannian of suitable dimension is a classifying space for (sphere) bundles of given rank (see Section 1.6.4 item 5).

Here we allow ourselves a small change of notation. Chern denotes by $H(n,N)$ the Grassmannian manifold of complex n planes in $\mathbb{C}^{n+N}$ which is therefore the one we noted previously $G_n(n+N)$.

Theorem 8.2.1 (Chern's Theorem 1). *To every bundle $\mathcal{F}$ of complex spheres $S(n)$ over a finite polyhedron B of topological dimension d, there exits a continuous mapping f of B into $H(n, N)$ with $N \geqslant d/2$, such that $\mathcal{F}$ is equivalent to the bundle induced by f.*

Theorem 8.2.2 (Chern's Theorem 2). *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two bundles of complex spheres $S(n)$ over a finite polyhedron B of topological dimension d induced by the mappings f_1, f_2 respectively of B into $H(n, N)$ with $N \geqslant d/2$. The bundles $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent when and only when the mappings f_1 and f_2 are homotopic.*

Chern deduces from these a first definition of classes (Theorem 5). Chern considers suitable Schubert varieties Z_r of dimension $2(Nn - n + r - 1)$ and defines invariant differential forms Φ_r of degree $2p = 2(n - r + 1)$ such that, for any cycle ζ of dimension $2p$ one has:

$$KI(\zeta, Z_r) = \int_\zeta \Phi_r \tag{2.1}$$

where the Lefschetz's notation KI means the intersection Kronecker index, for transverse cycles of complementary dimensions.

Let M be a complex manifold of complex dimension n, if $f : M \to H(n, N)$ is the classifying map defined in Chern's Theorem 1, then the Chern classes of M are image, by the map in cohomology $f^* : H^{2p}(H(n, N)) \to H^{2p}(M)$, of the classes of the cocycles defined by the invariant differential forms Φ_r.

The second Chern's definition (S.-s. Chern (ibid., Theorem 7)) is the one we provided in Section 3.2.2, using the obstruction theory.

The third Chern's definition (S.-s. Chern (ibid., Theorem 8)) introduces for a bundle of complex spheres $S(n)$ over the complex manifold M the associated fibre bundles $\mathcal{F}^{(r)*}$ over M whose fibre at each point is the manifold $U^*(n, r)$ of all ordered sets of $r(1 \leqslant r \leqslant n)$ complex mutually perpendicular vectors of $S(n)$.

Theorem 8.2.4 (Chern's Theorem 8). *Each of the cocycles γ of the Chern $2p = 2(n - r + 1)$-cohomology class of M, has the following property: Under the projection $\pi : \mathcal{F}^{(r)*} \to M$, the cocycle $\gamma^* = \pi^*(\gamma)$ satisfy:*

- *there exists on $\mathcal{F}^{(r)*}$ a $(2n - 2r + 1)$-cochain β^*, such that $\delta\beta^* = \gamma^*$*
- *on each fibre of $\mathcal{F}^{(r)*}$ over a point $x \in M$, one has, for each $(2n - 2r + 1)$-cycle λ,*

$$\beta^*(\lambda) = I(\lambda) \tag{2.2}$$

where $I(\lambda)$ is the index of λ in $H_{2n-2r+1}(\mathcal{F}^{(r)*}|_x) \cong \mathbb{Z}$.

In chapter III, section 3 of his article, Chern provides a version of the third definition where γ^* and β^* are explicit differential forms. Then property (2.2) is written

$$\int_{\lambda} \beta^* = I(\lambda). \tag{2.3}$$

This property appears to be very useful in the papers by M.-H. Schwartz.

8.3 The polar varieties and Mather classes

In 1953, Revaz Valerianovic Gamkrelidze (1924-), Georgian mathematician published, in Russian, "Computation of Chern cycles of Algebraic Manifolds" mainly using the decomposition of Grassmannian manifolds in terms of Schubert cycles Gamkrelidze (1953, 1956).

A similar method was developed by Wu Wen-Tsün in 1965. The Wu's method applies also in the case of singular complex varieties. The paper, written in Chinese did not have the success it deserved. Jianyi Zhou showed that the Mather's classes, defined by MacPherson (Section 7.4) are the same as Wu's classes.

The Chern classes of complex manifolds can be expressed in terms of polar varieties. This fact has been developed by Ragni Piene, in the line of Severi and Todd. In the singular case, polar varieties are defined as well, and the same definition (the same formula) corresponds to the Mather classes. In this context Lê Dũng Tráng and Bernard Teissier provide a nice definition of Chern–Schwartz–MacPherson classes in the singular situation Tráng and Teissier (1981).

8.4 More developments of Chern classes for singular varieties

8.4.1 Bivariant classes

Robert MacPherson and William Fulton (1981) developed a new formalism called bivariant theories. These are simultaneous generalizations of covariant group valued "homology-like" theories and contravariant ring valued "cohomology-like" theories. They showed existence and unicity of Stiefel–Whitney classes in this formalism and conjectured the same for Chern classes.

Jean-Paul Brasselet (1983) and Claude Sabbah (1986) have shown the existence of bivariant Chern classes in two papers, using different methods. Jianyi

Zhou (2000, n.d.) showed that the two obtained classes are the same and finally, Shoji Yokura has proven its uniqueness.

8.4.2 Other generalizations of classes in the singular case.

As we have seen, the Schwartz classes use a generalization of the tangent bundle in the singular case, it is the union of tangent bundles to the strata of a Whitney stratification (and no longer a bundle). The Mather classes, introduced by MacPherson in his definition, use the Nash bundle (which is the tangent bundle over regular points of the singular variety). The Fulton method is another way to generalize the tangent bundle and obtain characteristic classes.

The Fulton classes

The definition of Fulton classes (1984)

$$c^F(X) = c(TM|_X) \cap s(X, M)$$

uses the Segre classes $s(X, M)$ of the proper subvariety X of the manifold M Fulton (1984). In the case of local complete intersections, the normal bundle of the regular part X_{reg} canonically extends to X as a vector bundle $N_X M$. The virtual tangent bundle of X is then defined as $\tau_X = TM|_X - N_X M$ (defined in the Grothendieck group of vector bundles on X) and one has

$$c^F(X) = c(\tau_X) \cap [X].$$

These classes have been well studied by Paolo Aluffi (1994) who proposes alternative ways to define the Chern classes in the singular situation. Fulton classes have been generalized to arbitrary singular varieties in terms of Segre classes of coherent sheaves, to produce so called Fulton–Johnson classes (1980) Fulton and Johnson (1980).

The Milnor classes

The difference between the Schwartz–MacPherson classes and the Fulton classes have been (and continue to be) the subject of many papers.

The starting point is a paper by Tatsuo Suwa (1997) who has shown that if X is a compact local complete intersection wi th isolated singularities then the difference

$$\mu_*(X) = (-1)^n (c^F(X) - c_{SM}(X)$$

is localized in degree 0 and is the sum of Milnor numbers at the singular points.

The difference μ_* between Schwartz–MacPherson classes and Fulton classes has been called Milnor class of X and many authors studied this class providing different characterizations and equivalent definitions, using different notions of indices of vector frames at singular points, for example the GSV-index. Among them: Paolo Aluffi, Jean-Paul Brasselet, Daniel Lehmann, Adam Paruziński, Piotr Pragacz, José Seade, Tatsuo Suwa, Shoji Yokura...

8.4.3 Hirzebruch formalism

In his famous book "Topological methods in Algebraic Geometry" (translated from German), Friedrich Hirzebruch unified the theories of Chern classes, Todd classes and L_* classes. Jean-Paul Brasselet, Jörg Schürmann and Shoji Yokura (2007) use motivic theory to obtain a generalization of the result of Hirzebruch in the case of singular varieties. They unify the theories of Schwartz–MacPherson classes and generalizations of Todd classes and L_* classes in the singular case.

8.5 The Euler local obstruction

The original definition of Euler local obstruction was given by MacPherson (1974), using differential forms. The definition we provided (Brasselet and Schwartz (1981) and Section 7.2) is dual in the sense that we have used vector fields. Many authors generalized the local Euler obstruction, for collections of differential forms, for maps... or computed and gave appropriate formulae in the case of toric varieties, determinantal varieties... These generalizations and calculations continue to be studied and involve many Brazilian mathematicians. I listed 19 of them mainly in São Carlos and also Itajubá, João Pessoa, Maringá, Rio Claro, Uberaba...

8.6 Some applications in other mathematical domains and in theoretical physics.

This section would require one ot two more courses. I just mention some of the applications.

The Chern–Weil theory, named after Shiing-Shen Chern and André Weil, considers topological invariants of vector bundles on a smooth manifold M in terms of connections and curvature. Characteristic classes are represented in the de Rham cohomology ring of M.

The theory allows to prove the Chern–Gauß–Bonnet theorem (Shiing-Shen Chern, Carl Friedrich Gauß, and Pierre Ossian Bonnet) which states that the Euler–Poincaré characteristic of a closed even-dimensional Riemannian manifold is equal to the integral of a certain polynomial of its curvature form S.-s. Chern (1944).

In theoretical physics, Chern classes appear mainly through the notion of Calabi–Yau manifolds, named after mathematicians Eugenio alabi and Shing-Tung Yau (1957). These are complex compact Kähler manifolds with a vanishing first Chern class and metric properties (Ricci flat metric). These manifolds are important in particular in superstring theory.

Chern classes appear in Hall effect, particle physics, superstring theory, in brane models, gauge theory, condensed matter physics, topological quantum field theories, etc.

Chern–Simons theory, named after Shiing-Shen Chern and James Harris Simons (1974) is applied in mathematics to knot invariants and three-manifold invariants. In theoretical physics, the theory leads to a 3-dimensional topological quantum field theory mainly developed by Edward Witten (1988).

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