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MARKOV AND LAGRANGE SPECTRA

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"It ain't about how hard you hit. It's about how hard you can get hit and keep moving forward; how much you can take and keep moving forward. That's how winning is done!"

Rocky Balboa

To my parents:
Francisca (Fransquinha) and Manoel (Pequeno)

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Abstract

This thesis consists of two parts, both of them related to the study of the Markov and Lagrange spectra.

The first part focuses on the study of some topological properties of dynamical Markov and Lagrange spectra: we relate these sets to the elements that come from periodic orbits in Λ; we prove that generically, in C^1 topology, their interiors are empty; we show that given a horseshoe, there exists an open and dense set of C^1 functions, where $L''(f,\Lambda)=L'(f,\Lambda)$, and we give an example of an open set where such a result can not be true for dynamical Markov spectrum. Also, we give some open sets of the pair (dynamics, function), where we analyze the different beginnings that these spectra can have before their first accumulation points.

The second part focuses in the Bousch's question about the closedness of $M\setminus L$. We show that $M\setminus L$ is not closed, by showing that $1+3/\sqrt{2}$ is a point of the Lagrange spectrum L at which a sequence of elements of the set $M\setminus L$ accumulates. We also analyze the set $M\setminus L$ near the point 3, and we get that 3 might belong to the closure of $M\setminus L$.

Contents

CHAPTER 1

Introduction

The origin of the classical Lagrange and Markov spectra lies in number theory. In 1842, Dirichlet[8] proved that given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the inequality $|\alpha - p/q| < 1/q^2$ has infinitely many rational solutions p/q. In 1879, Markov[23] and in 1891, Hurwitz[14] improved this result by showing that $|\alpha - p/q| < 1/\sqrt{5}q^2$ has infinitely many rational solutions p/q.

On the other hand, for a fixed $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, better results can be expected. For each α, we define the best constant of approximation(Lagrange value of α):

$$\begin{aligned} k(\alpha) &= \sup \left\{ k > 0 : \left| \alpha - \frac{p}{q} \right| < \frac{1}{kq^2} \text{ has infinitely many solutions } \frac{p}{q} \in \mathbb{Q} \right\} \\ &= \limsup_{p \in \mathbb{Z}, q \in \mathbb{N}, |p|, q \to \infty} |q(q\alpha - p)|^{-1} \in \mathbb{R} \cup \{+\infty\}. \end{aligned}$$

Hurwitz's theorem gives us that $k(\alpha) \geq \sqrt{5}$, for all irrational α. Since $k\left((1+\sqrt{5})/2\right) = \sqrt{5}$, the constant $\sqrt{5}$ can not be improved for all irrational numbers at the same time. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be given in terms of continued fraction by $\alpha = [a_0; a_1, a_2, \cdots]$. Define $\alpha_n = [a_n; a_{n+1}, \cdots]$ and $\beta_n = [0; a_{n-1}, a_{n-2}, \cdots, a_1]$, for each $n \in \mathbb{N}$. It can be shown that

$$k(\alpha) = \lim_{n \to \infty} (\alpha_n + \beta_n). \tag{1.1}$$

The classical *Lagrange spectrum* is the set

$$L = \{k(\alpha) : \alpha \in \mathbb{R} \setminus \mathbb{Q} \text{ and } k(\alpha) < \infty\}.$$

Another interesting set that arises from number theory is the classical *Markov spectrum* defined by:

$$M = \left\{ \inf_{(x,y)\in\mathbb{Z}^2\setminus(0,0)} |f(x,y)|^{-1} : f(x,y) = ax^2 + bxy + cy^2, \text{ with } b^2 - 4ac = 1 \right\}.$$

Historical notes, equivalents definitions and classical properties about these sets can be found on the classical book [6] of Cusick and Flahive. The two spectra are closely related: it is possible to prove that L and M are closed subsets of $\mathbb{R}$ such that $L \subset M$.

The study of the geometric structure of L began with Markov [23, 24], who proved in 1879 that $L \cap (-\infty, 3) = \{k_1 < \cdots < k_n < \cdots\}$, where (k_n) is a prescribed increasing sequence converging to 3. By using an approach given in (1.2), it can be shown that $M \cap (-\infty, 3) = L \cap (-\infty, 3)$, according to [2], [6].

On the other hand, Hall[13] showed in 1947 that $C(4) + C(4) = [\sqrt{2} - 1, 4(\sqrt{2} - 1)]$, where $C(4)$ is the regular Cantor set of irrational numbers in $[0, 1]$ of continued fraction with coefficients bounded by 4. Using (1.1) this implies that $[6, +\infty) \subset L \subset M$. In 1975, Freiman[9] determined the biggest half-line $[c_F, +\infty)$ contained in L, where he proved that

$$c_F = \frac{2221564096 + 283748\sqrt{462}}{491993569} \simeq 4,52782956616....$$

Recently in 1996/2016, Moreira[31] studied the intermediate parts $L \cap (3, c_F)$ and $M \cap (3, c_F)$ and proved that the Hausdorff dimension of $L \cap (-\infty, t)$ varies continuously with real t, and the sets $L \cap (-\infty, t)$ and $M \cap (-\infty, t)$ share the same Hausdorff dimension for all $t \in \mathbb{R}$.

Thanks to an approach given by Perron [36], there exists a more dynamical way of interpreting these spectra. Define $\Sigma = (\mathbb{N}^*)^{\mathbb{Z}}$ and $\sigma : \Sigma \to \Sigma$ the shift map defined by $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$. If the height function $f : \Sigma \to \mathbb{R}$ is defined by $f((a_n)_{n\in\mathbb{Z}}) = \alpha_0 + \beta_0 = [a_0; a_1, a_2...] + [0; a_{-1}, a_{-2}, ...]$, then

$$L = \{l(\underline{\theta}) : \underline{\theta} \in \Sigma,\ l(\underline{\theta}) < \infty\} \ \text{ and } \ M = \{m(\underline{\theta}) : \underline{\theta} \in \Sigma,\ m(\underline{\theta}) < \infty\}, \quad (1.2)$$

where $l(\underline{\theta}) = \limsup_{n\to+\infty} f(\sigma^n(\underline{\theta}))$ and $m(\underline{\theta}) = \sup_{n\in\mathbb{Z}} f(\sigma^n(\underline{\theta}))$.

Motivated by this dynamical definition of the classical spectra, Carlos Gustavo Moreira[28] introduced the notion of dynamical spectra in the context of hyperbolic dynamics of compact surfaces. Let $\varphi : M^2 \to M^2$ be a

C^r-diffeomorphism ($r \geq 1$) possessing a horseshoe $\Lambda \subset M^2$ and $f : M^2 \to \mathbb{R}$ be a continuous real function. Then the *Lagrange dynamical spectrum* associated to (f, Λ) is defined by

$$L(f,\Lambda) = \{l_{f,\Lambda}(x) : x \in \Lambda\}, \text{ where } l_{f,\Lambda}(x) = \limsup_{n\to+\infty} f(\varphi^n(x)),$$

and the *Markov dynamical spectrum* associated to (f, Λ) is defined by

$$M(f,\Lambda) = \{m_{f,\Lambda}(x) : x \in \Lambda\}, \text{ where } m_{f,\Lambda}(x) = \sup_{n\in\mathbb{Z}} f(\varphi^n(x)).$$

Since this pioneering definition, many authors have worked on the subject in many contexts. For instance, see [3], [4], [16], [17], [22], [30], [32].

The study of the classical and dynamical Markov and Lagrange spectra are related with dynamical systems and optimization, and it is linked to the problem of simultaneously optimizing an objective function in all the orbits of a given dynamical system. The sets of these optimum values often have an extremely interesting (multi)fractal structure, which is related to the nature of the dynamic system in question.

In this work, we study some topological and fractal properties of these dynamical spectra. It is known that the Markov dynamical spectrum is closed. In order to prove that the Lagrange dynamical spectrum is also closed, we prove the next results that relates the spectra with the values that come from the periodic orbits of the system in the horseshoe:

Theorem 1. $L(f,\Lambda) = \overline{P(f,\Lambda)}$, *where* $P(f,\Lambda) = \{m_{f,\Lambda}(x) : x \in \Lambda$ *is a periodic point of* $\varphi\}$. *In particular, the Lagrange dynamical spectrum is closed.*

Theorem 2. *Let* $B(f,\Lambda) = \{m_{f,\Lambda}(x) : x \in \Lambda$ *is asymptotically periodic*$\}$. *Then,* $M(f,\Lambda) = \overline{B(f,\Lambda)}$.

Corollary. *We have* $L(f,\varphi|_\Lambda) = L(f,\varphi^{-1}|_\Lambda)$, *that is:*

$$\Big\{\limsup_{n\to+\infty} f(\varphi^n(x)) : x \in \Lambda\Big\} = \Big\{\limsup_{n\to-\infty} f(\varphi^n(x)) : x \in \Lambda\Big\}.$$

In [31], Moreira made a deep study of the geometric properties of classical spectra. In particular, it was proved that L' is a perfect set, i.e., $L'' = L'$. Here, we prove a generalization of this result in a general dynamical context:

Theorem 3. *Let* Λ *be a horseshoe for a* C^2*-diffeomorphism* $\varphi : M^2 \to M^2$. *Then, there exists an open and dense set* $H_\Lambda \subset C^1(M,\mathbb{R})$ *such that for all* $f \in H_\Lambda$,

$$L(f,\Lambda)'' = L(f,\Lambda)'.$$

Another natural topological property of dynamical spectra that could be studied is their interior. This study is related to the fractal geometry of regular Cantor sets. Using techniques of stable intersection of two regular C^2-Cantor sets with sum of Hausdorff dimensions greater than 1 (found in [33]), it was proved in [17] that for an open and dense set of real functions on the surface and for 'typical' C^2-horseshoes with Hausdorff dimension greater than one, both the Lagrange and the Markov dynamical spectra have persistently non-empty interior. Using the fact that there are no C^1-stable intersections of regular Cantor sets (see [29]), we prove that this result is not true under any condition on dimension of the horseshoe associated to C^1-diffeomorphism:

Theorem 4. *Let Λ be a horseshoe associated with a C^1-diffeomorphism φ and $\mathcal{U}$ be a C^1-neighbourhood of φ of hyperbolic continuation, such that for each $\psi \in \mathcal{U}$ there exists a hyperbolic continuation Λ_ψ of Λ. Then, there exists a Baire residual set $G \subset \mathcal{U} \times C^1(M, \mathbb{R})$ such that, if $(\psi, f) \in G$ then we have* $\operatorname{int} f(\Lambda_\psi) = \emptyset$. *In particular,*

$$\operatorname{int} L(f, \Lambda_\psi) = \emptyset \text{ and } \operatorname{int} M(f, \Lambda_\psi) = \emptyset.$$

As mentioned before, in 1879 Markov [23] proved that 3 is the first accumulations point of the classical Markov and Lagrange spectra, by showing that the set of numbers less than 3 in these both sets are infinite, countable and discrete, with 3 as its only limit point. The key idea in the proof of this fact is an identity with continued fractions: $[2; 1, 1, \gamma] + [0; 2, \gamma] = 3$, for any $\gamma \geq 1$. This very special identity doesn't look very common, thus a natural question is:

Question: How do the beginnings (before the first accumulation point) of the dynamical spectra behave?

In order to analyze this question, we construct some examples of open sets of pairs (dynamics, function), where many different configurations of the beginning of the dynamical spectra can happen. For example: both the spectra can be equal, both the spectra begin with a finite number of points, the Markov dynamical spectrum beginning with a infinitely many points. See the next propositions:

Proposition 2. *There are open neighborhoods $\mathcal{U}_1 \subset \operatorname{Diff}^2(\mathbb{S}^2)$ and $\mathcal{V}_1 \subset C^1(\mathbb{S}^2; \mathbb{R})$, such that $L(f, \Lambda_\varphi) = M(f, \Lambda_\varphi)$, for every $(\varphi, f) \in (\mathcal{U}_1, \mathcal{V}_1)$. Moreover, the beginning of these set has only one point.*

Proposition 3. *Let n be a positive integer. Then, there are open neighborhoods $\mathcal{U}_n \subset \mathrm{Diff}^2(\mathbb{S}^2)$ and $\mathcal{V}_n \subset C^1(\mathbb{S}^2;\mathbb{R})$, such that $L(f,\Lambda_\varphi)$ and $M(f,\Lambda_\varphi)$ have the same beginning with exactly n elements, for every $(\varphi, f) \in (\mathcal{U}_n, \mathcal{V}_n)$.*

Proposition 4. *There are open neighborhoods $\hat{\mathcal{U}} \subset \mathrm{Diff}^2(\mathbb{S}^2)$ and $\hat{\mathcal{V}} \subset C^1(\mathbb{S}^2;\mathbb{R})$, such that $M(f,\Lambda_\varphi)$ has an infinite beginning, for every $(\varphi, f) \in (\hat{\mathcal{U}}, \hat{\mathcal{V}})$. Moreover, $M'(f,\Lambda_\varphi) \neq M''(f,\Lambda_\varphi)$ and $L(f,\Lambda_\varphi)$ has a finite beginning, for every $(\varphi, f) \in (\hat{\mathcal{U}}, \hat{\mathcal{V}})$.*

In order to understand the general situation of the beginning of the Lagrange dynamical spectra, we first give a different proof of the main result in Kopetzky's paper [19], where the beginning of Dirichlet spectrum [7] is analyzed. This proof allows us to show that persistently, in a neighborhood of the pair in the conservative setting, the beginning of the associated Lagrange spectra are infinite countable, as we can see in the following:

Theorem 5. *There are open neighborhoods $\mathcal{U} \subset \mathrm{Diff}^2_{\omega_0}(\mathbb{S}^2)$ of a given φ_0 with an associated horseshoe Λ_0 and $\mathcal{V} \subset C^1(\mathbb{S}^2;\mathbb{R})$ of a given f_0, such that the beginning of $L(f,\Lambda_\varphi)$ is an infinite set, for every $(\varphi, f) \in \mathcal{U} \times \mathcal{V}$, where Λ_φ is the hyperbolic continuation of Λ_0.*

After, we also built an open in the pair (dynamics, functions) without conservative hypothesis, where the Lagrange spectra has infinitely many points in the beginning. More precisely, we have the following:

Theorem 6. *There are open neighborhoods $\tilde{\mathcal{U}} \subset \mathrm{Diff}^2(\mathbb{S}^2)$ and $\tilde{\mathcal{V}} \subset C^1(\mathbb{S}^2;\mathbb{R})$, such that the beginning of $L(g,\Lambda_\psi)$ has infinitely many points, for every $(\psi, g) \in \tilde{\mathcal{U}} \times \tilde{\mathcal{V}}$, where Λ_ψ is a horseshoe for ψ.*

After this discussion, we show that every possible beginning could occur in both the spectra in a robust form in the pair (dynamics, functions), and thus we cannot expect any general answer for the previous question about the beginning of the dynamical spectra.

In the classical case, a particularly challenging aspect about the structure of these spectra is the description of the nature of the set-theoretical difference $M \setminus L$ between the Lagrange spectrum L and the Markov spectrum M.

In a dynamical setting, elements of $L(f,\Lambda)$ correspond to optimum values (in orbits) of the objective function f that can be achieved as asymptotic optimum values in certain orbits. In the other hand, elements of $M(f,\Lambda) \setminus L(f,\Lambda)$ are optimum values in orbit that are necessarily achieved in the interior of the orbit, and are strictly greater than the corresponding asymptotic optimum values.

The fact that $M \setminus L \neq \emptyset$ was established only in 1968 by Freiman [10]. Until 2017, all that was known about $M \setminus L$ was that the set contained two explicit countable subsets near 3.11 and 3.29 (see Freiman [10], [11] and Flahive [12]).

In a series of three recent articles [26], [27] and [25], Carlos Gustavo Moreira and Carlos Matheus proved that $M \setminus L$ has a rich fractal structure: more concretely, there are three explicit open intervals I_1, I_2 and I_3 nearby 3.11, 3.29 and 3.7 whose boundaries are included in the Lagrange spectrum L such that $(M \setminus L) \cap I_j = M \cap I_j$, $j = 1, 2, 3$ (resp.), are explicit Cantor sets of Hausdorff dimensions at least 0.26, 0.35 and 0.53 (resp.).

In particular, the articles mentioned in the previous paragraph show that the known portions $(M \setminus L) \cap I_j$, $j = 1, 2, 3$ of $M \setminus L$ are closed subsets. This led T. Bousch to ask whether $M \setminus L$ is a closed subset of $\mathbb{R}$.

We answer negatively T. Bousch's question by showing that $M \setminus L$ is not closed. More precisely, we prove that $1 + 3/\sqrt{2} \in L \cap \overline{(M \setminus L)}$. In order to do that we taked a word sequence $\underline{\eta}_k$, given by $\underline{\eta}_k = (2_{2k-1}, 1, 2_{2k}, 1, 2_{2k+1}, 1)$. Consider the periodic word $\theta(\underline{\eta}_k) = \overline{\underline{\eta}_k} \in \{1, 2\}^{\mathbb{Z}}$, and define $\zeta_k^1 \in \{1, 2\}^{\mathbb{Z}}$,

$$\zeta_k^1 := \overline{2_{2k-1}, 1, 2_{2k}, 1, 2_{2k+1}, 1}2^* 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2_{2k-1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 1, \overline{2}.$$

We proved in a work in collaboration with Moreira, Matheus and Lima the following result:

Theorem 8. *The Markov values of $\theta(\underline{\eta}_k)$ and ζ_k^1 satisfy:*

- $m(\theta(\underline{\eta}_k)) < m(\zeta_k^1) < m(\theta(\underline{\eta}_{k-1}))$ *for all* $k \geq 3$*;*
- $\lim\limits_{k\to\infty} m(\theta(\underline{\eta}_k)) = 1 + \frac{3}{\sqrt{2}}$*;*
- $m(\zeta_k^1) \in M \setminus L$ *for all* $k \geq 4$*.*

In particular, $1 + \frac{3}{\sqrt{2}} \in L \cap \overline{(M \setminus L)}$ *and* $M \setminus L$ *is not a closed subset of* $\mathbb{R}$*.*

A priori, we tried to solve negatively T. Bousch's question by giving strong evidence towards the *possibility* that $3 \in L \cap \overline{(M \setminus L)}$. Unfortunately, we could *not* establish that $3 \in \overline{M \setminus L}$ because we were unable to prove the *local uniqueness* property near 3. However, we were able to prove the self-replication mechanism and the local uniqueness in the first four cases, these facts allowed us to construct four new elements $m_4 < m_3 < m_2 < m_1 < 3.11$ of $M \setminus L$ lying in distinct connected components of $\mathbb{R} \setminus L$.

Consider the finite word $\underline{\omega}_k := (2_{2k}, 1_2, 2_{2k+1}, 1_2, 2_{2k+2}, 1_2)$, for each $k \geq 1$, and the bi-infinite word $\gamma_k^1 := (\overline{\omega}_k \underline{\omega}_k^* \underline{\omega}_k \bar{2})$ where the asterisk indicates that the $(2k+2)$-th position occurs in the first 2 in the substring 2_{2k+1} of $\underline{\omega}_k$. We proved also in collaboration with Moreira, Matheus and Lima the next result:

Theorem 7. *The Markov values $m_k = m(\gamma_k^1)$ form a decreasing sequence converging to* 3 *whose first four elements belong to $M \setminus L$. Moreover, these four elements belong to distinct connected components of $\mathbb{R} \setminus L$.*

As mentioned before, until this work the smallest known numbers in $M \setminus L$ were near 3.11, but now we know that m_1, m_2, m_3 and m_4(resp.) are approximately $3.005, 3.0001, 3.000004$ and 3.0000001(resp.).

1.1 Structure of the work

This work is divided in two parts:

- The first one, Chapter 3, is devoted to show all the topological and fractal properties about the dynamical Markov and Lagrange spectra contained in this work.

- The second part, Chapter 4 and 5, we proved the result about the set $M \setminus L$. More precisely, the Chapter 4 are from the paper [21], where we study the set $M \setminus L$ near to 3. The Chapter 5 are from the paper [20], where we proved that $M \setminus L$ is not closed. Both of these paper are made jointly with Carlos Gustavo Moreira, Carlos Matheus and Davi Lima.

These two parts are relatively independent, and can be read separate.

In chapter 2 we give some definitions and preliminary results that will be used in the whole thesis.

CHAPTER 2

Definitions and preliminary results

In this chapter, we establish some definitions, notations and results that will be useful in the rest of the work.

2.1 Preliminaries from dynamical systems

In this section, we give some tools from dynamical systems and we refer to the books [34] and [37] for more details.

Let M^2 be a compact surface and $\varphi : M^2 \to M^2$ be a diffeomorphism. We call $\Lambda \subset M^2$ a hyperbolic set for φ when $\varphi(\Lambda) = \Lambda$ and there exists a decomposition $T_\Lambda M = E^s \oplus E^u$ such that $D\varphi|_{E^S}$ is uniformly contracting and $D\varphi|_{E^u}$ is uniformly expanding. We can check that φ is expansive on Λ, i.e., there exists $\varepsilon_0 > 0$ such that for any pair of distinct points x in M and y in Λ, we have $\sup\limits_{n\in\mathbb{Z}} d(f^n(x), f^n(y)) > \varepsilon_0$, according to [37, pp. 84].

In this work, unless explicitly stated otherwise, we will assume that Λ is a horseshoe: compact, locally maximal, transitive hyperbolic invariant of saddle type, and so it contains a dense subset of periodic orbits.

We recall that the stable and unstable foliations $\mathcal{F}^s(\Lambda)$ and $\mathcal{F}^u(\Lambda)$ are $C^{1+\varepsilon}$, for some $\varepsilon > 0$. Moreover, these foliations can be extended to C^1 foliations defined on a full neighborhood of Λ.

It is well-known that hyperbolic sets have persistence of hyperbolicity under small perturbations. More specifically, let $U \subset M^2$ be an open set such that $\Lambda := \bigcap_{n=-\infty}^{\infty} \varphi^n(U)$ is a hyperbolic set for φ. Then, there is a neighborhood $\mathcal{U}_\varphi$ of φ in $\mathrm{Diff}^k(M^2)$ and a continuous function $\Phi : \mathcal{U}_\varphi \to C^0(\Lambda, M)$ such that $\Lambda_\psi := \Phi(\psi)(\Lambda)$ is a hyperbolic set for $\psi \in \mathcal{U}_\varphi$, which is conjugate to Λ, according to [37, Theorem 8.3]

In the next theorem, we recall a result concerning differentiability of the invariant stable and unstable manifold and foliations themselves of basic set in two dimensions with respect to the diffeomorphism. Let $\Sigma = \mathcal{U}_\varphi$ as in previous paragraph and consider the diffeomorphism $\Psi : \Sigma \times M^2 \to \Sigma \times M^2$ defined by $\Psi(\psi, x) = (\psi, \psi(x))$. According to [34] in the Appendix 1, we have:

Theorem 2.1. *If $\Psi : \Sigma \times M^2 \to \Sigma \times M^2$ is C^2 then there are transverse invariant foliations $\mathcal{F}^s_\psi(x)$, $\mathcal{F}^s_\psi(x)$ defined on U such that the maps $(\psi, x) \to T_x\mathcal{F}^s_\psi(x)$, and $(\psi, x) \to T_x\mathcal{F}^u_\psi(x)$ are $C^{1+\varepsilon}$.*

Now, we recall the definition of a *Markov partition* of a horseshoe Λ for φ. Such a Markov partition consists of a finite set of boxes, i.e, diffeomorphic images of the square $Q = [-1, 1]^2$, say $B_1 = \xi_1(Q), \cdots, B_n = \xi_n(Q)$ such that

(i) $\Lambda \subset \bigcup_{i=1}^{n} B_i$;

(ii) $\mathrm{int}B_i \cap\ \mathrm{int}B_j = \emptyset$, for $i \neq j$;

(iii) $\varphi(\partial_s B_i) \subset \bigcup_{i=1}^{n} \partial_s B_i$ and $\varphi(\partial_u B_i) \subset \bigcup_{i=1}^{n} \partial_u B_i$, where $\partial_s Q_i = \xi_i(\{(x, y) : |x| \leq 1, |y| = 1\})$ and $\partial_u Q_i = \xi_i(\{(x, y) : |y| \leq 1, |x| = 1\})$

(iv) there is a positive integer n such that $\varphi^n(B_i) \cap B_j \neq \emptyset$, for all $1 \leq i, j \leq n$.

Taking the boxes of the Markov partition sufficiently small we can also demand that $\varphi(B_i) \cap B_j$ be either empty or connected. In order to do that and for other uses in this work, according to [34] in the Appendix 2, we recall the next theorem:

Theorem 2.2. *There is a Markov partition for Λ with arbitrarily small diameter.*

Remark 2.1. In the two-dimensional case, we can construct the boxes of the Markov partition for a horseshoe Λ for φ, such that the boundaries consist of pieces of stable and unstable manifolds of finite periodic points of φ. Thus, for a diffeomorphism C^r near φ, we can construct a nearby Markov Partition for the corresponding nearby horseshoe, since the compact parts these manifolds are C^r-close to the original case for φ.

Let Λ be a horseshoe associated with φ. Let us fix a geometric Markov partition $\{R_a\}_{a\in\mathcal{A}}$ of disjoint rectangles of M with sufficiently small diameter, where $R_a \simeq I_a^u \times I_a^s$ is delimited by compact pieces I_a^u, resp. I_a^s, of unstable, resp. stable manifolds of certain points. The set $\mathcal{B} \subset \mathcal{A}^2$ of admissible transitions consists of pairs (a_0, a_1) such that $\varphi(R_{a_0}) \cap (R_{a_1}) \neq \emptyset$. From $\mathcal{B}$ we can induce the following transition matrix B:

$$b_{a_i a_j} = 1 \text{ if } (a_i, a_j) \in \mathcal{B}, \ \ b_{a_i a_j} = 0 \text{ otherwise, } \ \text{for } (a_i, a_j) \in \mathcal{A}^2.$$

Define $\Sigma_{\mathcal{A}} = \{\underline{a} = (a_n)_{n\in\mathbb{Z}} : a_n \in \mathcal{A} \text{ for all } n \in \mathbb{Z}\}$ and the *shift map* $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$, the homeomorphism defined by $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$. In this space, we call a *cylinder* a subset of the form

$$C[m; b_m, \cdots, b_n] := \{\underline{a} \in \Sigma_{\mathcal{A}} : a_j = b_j, \text{ for } m \leq j \leq n\}.$$

Let $\Sigma_B = \{\underline{a} \in \Sigma_{\mathcal{A}} : b_{a_n a_{n+1}} = 1\}$. This set is a closed and σ-invariant subset of $\Sigma_{\mathcal{A}}$. We keep the notation σ to denote the restriction $\sigma|_{\Sigma_B}$. The pair (Σ_B, σ) is called a *subshift of finite type* of $(\Sigma_{\mathcal{A}}, \sigma)$. Given $x, y \in \mathcal{A}$, since $\varphi|_\Lambda$ is transitive, we denote by $n(x, y) \in \mathbb{N}^*$ the minimum length of an admissible string that begins at x and ends at y. We also define $N_0 := \max\{n(x, y) : x, y \in \mathcal{A}\}$.

Subshifts of finite type have a kind of local product structure. First we define the local stable and stable sets:

$$\begin{aligned} W^s_{1/3}(\underline{a}) &= \{\underline{b} \in \Sigma_B : \forall n \geq 0, \ d(\sigma^n(\underline{a}), \sigma^n(\underline{b})) \leq 1/3\} \\ &= \{\underline{b} \in \Sigma_B : \forall n \geq 0, \ a_n = b_n\}, \\ W^u_{1/3}(\underline{a}) &= \{\underline{b} \in \Sigma_B : \forall n \leq 0, \ d(\sigma^n(\underline{a}), \sigma^n(\underline{b})) \leq 1/3\} \\ &= \{\underline{b} \in \Sigma_B : \forall n \leq 0, \ a_n = b_n\}, \end{aligned}$$

where $d(\underline{a}, \underline{b}) = \sum_{n=-\infty}^{\infty} 2^{-(2|n|+1)}\delta_n(\underline{a}, \underline{b})$ and $\delta_n(\underline{a}, \underline{b})$ is 0 when $a_n = b_n$ and 1 otherwise. So, if $\underline{a}, \underline{b} \in \Sigma_B$ and $d(\underline{a}, \underline{b}) < 1/2$, then $a_0 = b_0$ and $W^s_{1/3}(\underline{a}) \cap W^u_{1/3}(\underline{b})$ is a unique point denoted by bracket

$$[\underline{a}, \underline{b}] = (\cdots, b_{-n}, \cdots, b_{-1}; a_0, a_1, \cdots, a_n, \cdots).$$

Thus, $(\varphi|_\Lambda, \Lambda)$ is topologically conjugate to (σ, Σ_B), i.e., there exists a homeomorphism $\Pi : \Sigma_B \to \Lambda$ such that, $\varphi \circ \Pi = \Pi \circ \varphi$.

$$\begin{array}{ccc} \Sigma_B & \xrightarrow{\sigma} & \Sigma_B \\ \Pi \downarrow & & \downarrow \Pi \\ \Lambda & \xrightarrow[\varphi]{} & \Lambda \end{array}$$

Moreover, Π respect the local product structure, that is, $\Pi[\underline{a}, \underline{b}] = [\Pi(\underline{a}), \Pi(\underline{b})]$. Conveniently, sometimes we work thinking about the dynamics either on the horseshoe Λ or on the space of symbols Σ_B. Thus given a $f : M \to \mathbb{R}$, we associate $\tilde{f} = f|_\Lambda \circ \Pi : \Sigma_B \to \mathbb{R}$. In the whole text, by abuse of language, we treat $p \in \Lambda$ and its kneading sequence $(a_n)_{n\in\mathbb{Z}} = (\cdots, a_{-1}; a_0, a_1, \cdots) \in \Sigma_B$ without distinction; we do the same with f and $\tilde{f}$ too.

Next, we use the $C^{1+\varepsilon}$-foliations in a neighborhood of Λ to define the projections $\pi_a^u : R_a \to I_a^u \times \{i_a^s\}$ and $\pi_a^s : R_a \to \{i_a^u\} \times I_a^s$ of the rectangles into the connected components $I_a^u \times \{i_a^s\}$ and $\{i_a^u\} \times I_a^s$ of the stable and unstable boundaries of R_a, where $i_a^u \in \partial I_a^u$ an $i_a^s \in \partial I_a^s$ are fixed arbitrarily. Using these projections, we have the stable and unstable Cantor sets

$$K^s = \bigcup_{a\in\mathcal{A}} \pi_a^u(\Lambda \cap R_a) \ \text{ and } \ K^u = \bigcup_{a\in\mathcal{A}} \pi_a^s(\Lambda \cap R_a)$$

associated with Λ.

The stable and unstable Cantor sets K^u and K^s are $C^{1+\varepsilon}$-dynamically defined, i.e., the $C^{1+\varepsilon}$-maps

$$g_s(\pi_{a_1}^u(y)) = \pi_{a_0}^u(\varphi^{-1}(y))$$

for $y \in R_{a_1} \cap \varphi(R_{a_0})$ and

$$g_u(\pi_{a_0}^s(z)) = \pi_{a_1}^s(\varphi(z))$$

for $z \in R_{a_0} \cap \varphi^{-1}(R_{a_1})$ are expanding maps of type $\Sigma_{\mathcal{B}}$ defining K^s and K^u in the sense that

- the domains of g_s and g_u are disjoint unions

$$\bigsqcup_{(a_0,a_1)\in\mathcal{B}} I^s(a_1, a_0) \text{ and } \bigsqcup_{(a_0,a_1)\in\mathcal{B}} I^u(a_0, a_1),$$

  where $I^s(a_1, a_0)$, resp. $I^u(a_0, a_1)$, are compact subintervals of $I_{a_1}^s$, resp. $I_{a_0}^u$;

- for each $(a_0, a_1) \in \mathcal{B}$, the restrictions $g_s|_{I^s(a_1,a_0)}$ and $g_u|_{|I^u(a_0,a_1)}$ are $C^{1+\epsilon}$ diffeomorphisms onto $I^s_{a_0}$ and $I^u_{a_1}$ with $|Dg_s(t)|, |Dg_u(t)| > 1$, for all $t \in I^s(a_1, a_0)$, $t \in I^u(a_0, a_1)$ (for appropriate choices of the parametrization of I^s_a and I^u_a);

- K^s and K^u satisfies

$$K^s = \bigcap_{n\geq 0} g_s^{-n}\left(\bigsqcup_{(a_0,a_1)\in\mathcal{B}} I^s(a_1,a_0)\right) \quad K^u = \bigcap_{n\geq 0} g_u^{-n}\left(\bigsqcup_{(a_0,a_1)\in\mathcal{B}} I^u(a_0,a_1)\right).$$

We will think of the intervals I^u_a, resp. I^s_a, $a \in \mathcal{A}$ inside an abstract line so that it makes sense to say that the interval I^u_a, resp. I^s_a, is located to the left or to the right of the interval I^u_b, resp. I^s_b, for $a, b \in \mathcal{A}$, according to [34] in the Appendix 2. In this setting, given an admissible finite sequence $\alpha = (a_1, \cdots, a_n) \in \mathcal{A}^n$, that is $b_{a_i,a_{i+1}} = 1$ for all $i = 1, \cdots, n-1$, we define

$$I^u(\alpha) := \{x \in K^u : g_u^{i-1}(x) \in I^u(a_i, a_{i+1}),\ \forall i = 1, \cdots, n-1)\}.$$

Analogously, given an admissible finite sequence $\alpha = (a_1, \cdots, a_n) \in \mathcal{A}^n$, we define:

$$I^s(\alpha^T) := \{y \in K^s : g_u^{n-i}(y) \in I^s(a_i, a_{i-1}),\ \forall i = 2, \cdots, n)\}.$$

Here, $\alpha^T = (a_n, \cdots, a_1)$ denotes the *transpose* of α.

The stable and unstable Cantor sets K^s and K^u are closely related to the geometry of the horseshoe Λ. For instance, it is well-known that Λ is locally diffeomorphic to the Cartesian product of the two regular Cantor sets K^s and K^u, and

$$HD(\Lambda) = HD(K^s) + HD(K^u).$$

2.2 Basic features of continued fractions

The continued fraction expansion of an irrational number α is denoted by

$$\alpha = [a_0; a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}},$$

so that the Gauss map $g : (0,1) \to [0,1)$, $g(x) = \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor$ acts on continued fraction expansions by $g([0; a_1, a_2, \dots]) = [0; a_2, \dots]$.

Given $\alpha = [a_0; a_1, \ldots, a_n, a_{n+1}, \ldots]$ and $\tilde{\alpha} = [a_0; a_1, \ldots, a_n, b_{n+1}, \ldots]$ with $a_{n+1} \neq b_{n+1}$, recall that

$$\alpha > \tilde{\alpha} \quad \text{if and only if} \quad (-1)^{n+1}(a_{n+1} - b_{n+1}) > 0. \tag{2.1}$$

For an irrational number $\alpha = \alpha_0$, the continued fraction expansion $\alpha = [a_0; a_1, \ldots]$ is recursively obtained by setting $a_n = \lfloor \alpha_n \rfloor$ and

$$\alpha_{n+1} = \frac{1}{\alpha_n - a_n} = \frac{1}{g^n(\alpha_0)}.$$

The rational approximations

$$\frac{p_n}{q_n} := [a_0; a_1, \ldots, a_n] \in \mathbb{Q}$$

of α satisfy the recurrence relations $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$ (with the convention that $p_{-2} = q_{-1} = 0$ and $p_{-1} = q_{-2} = 1$). Moreover, $p_{n+1}q_n - p_n q_{n+1} = (-1)^n$ and $\alpha = \frac{\alpha_n p_{n-1} + p_{n-2}}{\alpha_n q_{n-1} + q_{n-2}}$. In particular, given $\alpha = [a_0; a_1, \ldots, a_n, a_{n+1}, \ldots]$ and $\tilde{\alpha} = [a_0; a_1, \ldots, a_n, b_{n+1}, \ldots]$, we have

$$\alpha - \tilde{\alpha} = (-1)^n \frac{\tilde{\alpha}_{n+1} - \alpha_{n+1}}{q_n^2(\beta_n + \alpha_{n+1})(\beta_n + \tilde{\alpha}_{n+1})}$$

where $\beta_n := \frac{q_{n-1}}{q_n} = [0; a_n, \ldots, a_1]$.

In general, given a finite string $(a_1, \ldots, a_l) \in (\mathbb{N}^*)^l$, we write

$$[0; a_1, \ldots, a_l] = \frac{p(a_1 \ldots a_l)}{q(a_1 \ldots a_l)}.$$

By Euler's rule,

$$q(a_1 \ldots a_l) = q(a_1 \ldots a_m)q(a_{m+1} \ldots a_l) + q(a_1 \ldots a_{m-1})q(a_{m+2} \ldots a_l)$$

for $1 \leq m < l$, and $q(a_1 \ldots a_l) = q(a_l \ldots a_1)$. In particular, if $(a_1, \ldots, a_l)$ is a palindrome, then $p(a_1 \ldots a_l) = q(a_l, \ldots, a_1)$.

We recall from [6, Chapter 1] the next useful equivalence. For any natural number $n \geq 2$ and real number $\alpha, \beta \geq 1$, we have:

$$[2; 1_n, \alpha] + [0, 2, 1_{n-2}, \beta] \leq 3 \Leftrightarrow \beta \leq \alpha. \tag{2.2}$$

Moreover, the equality holds on the left if and only if $\beta = \alpha$.

Let us establish some notation. We use subscripts to indicate the repetition of a certain character: for example, $1_2 2_4$ is the string 112222. Also, $\overline{a_1, \ldots, a_l}$ is the periodic word obtained by infinite concatenation of the string $(a_1, \ldots, a_l)$. We use the next notation to indicate the transpose of a word: $(a_1, \cdots, a_n)^t := (a_n, \cdots, a_1)$. Moreover, unless explicitly stated otherwise, we indicate the zeroth position a_0 of a string $(a_{-m}, \ldots, a_{-1}, a_0^*, a_1, \ldots, a_n)$ by an asterisk.

CHAPTER 3

Markov and Lagrange dynamical spectra

In this chapter, we will study some topological and fractal properties about the dynamical Markov and Lagrange spectra. Thus, we will consider the spectra associated to a pair given by a continuous real function f and a horseshoe Λ for φ.

3.1 Closedness of the dynamical spectra

We begin by relating these two spectra. More specifically, recall that $L(f,\Lambda) \subset M(f,\Lambda) \subset f(\Lambda)$, according to [17]. In fact, take $a \in L(f,\Lambda)$. Then there exists $x_0 \in \Lambda$ such that $a = \limsup\limits_{n\to+\infty} f(\varphi^n(x_0))$. Since Λ is a compact set, then there exist a subsequence $(\varphi^{n_k}(x_0))_k$ of $(\varphi^n(x_0))_n$ such that $\lim\limits_{k\to+\infty} \varphi^{n_k}(x_0) = y_0 \in \Lambda$ and

$$a = \limsup_{n\to+\infty} f(\varphi^n(x_0)) = \lim_{k\to\infty} f(\varphi^{n_k}(x_0)) = f(y_0).$$

We claim that $f(y_0) = \sup\limits_{n\in\mathbb{Z}} f(\varphi^n(y_0))$, for otherwise there would exist an integer m, such that $f(y_0) < f(\varphi^m(y_0))$. By continuity, given $\varepsilon = f(\varphi^m(y_0)) - f(y_0) > 0$, there exists a neighbourhood U of y_0 such that

$$f(y_0) + \frac{\varepsilon}{2} < f(\varphi^m(z)), \quad \text{for all} \;\; z \in U.$$

Thus, since $\varphi^{n_k}(x_0) \to y_0$, there exists $k_0 \in \mathbb{N}$ such that $\varphi^{n_k}(x_0) \in U$, for all $k \geq k_0$. Therefore,

$$f(y_0) + \frac{\varepsilon}{2} < f(\varphi^{m+n_k}(z)), \text{ for all } k \geq k_0,$$

and this contradicts the definition of $a = f(y_0)$. We get the other inclusion by a similar argument.

It is well known that in the classical case the Markov spectrum is a closed set, and we can find a proof of this in Cusick-Flahive [6]. Using the ideas of the above remark, we prove the same result in the dynamical case:

Proposition 1. *$M(f, \Lambda)$ is a closed set.*

Proof. We claim that if $y = m_{f.\Lambda}(x) = \sup\limits_{n \in \mathbb{Z}} f(\varphi^n(x))$, then there exists x_0 such that $y = f(x_0) = m_{f,\Lambda}(x_0)$. In fact, if the supremum above is attained, then we are done. Otherwise, by an argument similar to that of the remark above we also are done.

Let $(x_k)_k \subset M(f, \Lambda)$ such that $x_k \to x$. We may assume that $x_k = f(y_k) = m_{f,\Lambda}(y_k)$, $y_k \in \Lambda$. Since Λ is a compact set and f is a continuous function, there exists a subsequence (y_{k_j}) such that $y_{k_j} \to y_0 \in \Lambda$ and $f(y_{k_j}) \to f(y_0) = x$. We claim that, $x = f(y_0) = m_{f,\Lambda}(y_0) \in M(f, \Lambda)$. Indeed, suppose that there exists $N \in \mathbb{Z}$ such that $f(\varphi^N(y_0)) > h > f(y_0)$, for some $h \in \mathbb{R}$. By continuity, we have $f(\varphi^N(y_{k_j})) \to f(\varphi^N(y_0))$. If j is large enough, we get

$$f(\varphi^N(y_{k_j})) > h > f(y_{k_j}),$$

and this contradicts the definition of y_{k_j}. □

In this setting it is natural to ask: is $L(f, \Lambda)$ a closed set? Even in the classical case, the proof that the Lagrange spectrum is a closed subset of $\mathbb{R}$ is complicated, and it was proved by Cusick in [5]. This fact follows from the next characterization of L in terms of periodic points:

Proposition. *$L = \overline{P}$, where $P = \{m(\underline{\theta}) : \underline{\theta} \in \Sigma$ is a periodic point$\}$.*

Generalizations of the Cusick's theorem in several contexts can be found in [35]. Rephrasing this proposition in our context, we have the following result:

Theorem 1. *Let $P(f, \Lambda) = \{m_{f,\Lambda}(x) : x \in \Lambda$ is a periodic point for $\varphi\}$. Then, $L(f, \Lambda) = \overline{P(f, \Lambda)}$.*

To prove the previous theorem, we use the next lemma:

Lemma 3.1. *Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in Λ, such that $\lim\limits_{n\to\infty} d(\varphi(y_n), y_{n+1}) = 0$. Then, there exists $z \in \Lambda$, so that $\lim\limits_{n\to\infty} d(\varphi^n(z), y_n) = 0$.*

Proof. Let $\gamma > 0$ be given by the Stable Manifold Theorem for Λ. By the Shadowing Lemma, there exists a $\beta > 0$ such that every β-pseudo-orbit in Λ is $(\gamma/2)$-shadowed by a point of Λ. Take $k \in \mathbb{N}$ such that $d(\varphi(y_m), y_{m+1}) < \beta$, for all $m \geq k$. Consider the β-pseudo-orbit in Λ, given by $(y_m)_{m\geq k}$. Thus, there exists $z_0 \in \Lambda$ whose orbit $(\gamma/2)$-shadows the previous pseudo-orbit, that is:

$$d(\varphi^j(z_0), y_{k+j}) < \frac{\gamma}{2}, \text{ for all } j \geq 0. \tag{3.1}$$

We claim that $\lim\limits_{j\to+\infty} d(\varphi^j(z_0), y_{k+j})) = 0$. Indeed, let $0 < \theta < \gamma/2$. Then there exists $\overline{\beta} > 0$ such that every $\overline{\beta}$-pseudo-orbit in Λ is θ-shadowed by a point of Λ. Let $l > k$ be a natural number large enough, so that $d(\varphi(y_h), y_{h+1}) < \bar{\beta}$, for all $h \geq l$. Consider the $\overline{\beta}$-pseudo-orbit in Λ, given by $(y_h)_{h\geq l}$. Thus, there exists $w_\theta \in \Lambda$ whose orbit θ-shadows the previous pseudo-orbit, that is:

$$d(\varphi^i(w_\theta), y_{l+i}) < \theta, \text{ for all } i \geq 0. \tag{3.2}$$

By (3.1) and (3.2) for all $i \geq 0$, we have $d(\varphi^i(w_\theta), \varphi^{l-k+i}(z_0)) < \gamma$. Thus, $\varphi^{l-k}(z_0) \in W^s_\gamma(w_\theta)$ and so $\lim\limits_{i\to+\infty} d(\varphi^i(w_\theta), \varphi^i(\varphi^{l-k}(z_0))) = 0$. Take $i_0 \in \mathbb{N}$, such that $d(\varphi^i(w_\theta), \varphi^i(\varphi^{l-k}(z_0))) < \theta$, for all $i \geq i_0$. By (3.2), we have that for $i \geq i_0$:

$$d(\varphi^{i+(l-k)}(z_0), y_{k+[i+(l-k)]}) = d(\varphi^i(\varphi^{l-k}(z_0)), y_{l+i}) < 2\theta.$$

This finishes the proof of the claim. Therefore, $z = \varphi^{-k}(z_0)$ satisfies the requirement of the lemma. □

Now, we are able to prove the proposition.

Proof of Theorem 1. In order to prove $L(f,\Lambda) \subset \overline{P(f,\Lambda)}$, let $l = l_{f,\Lambda}(x)$, $x \in \Lambda$. For any $\varepsilon > 0$, we shall find a periodic point p in Λ such that $|l - m_{f,\Lambda}(p)| < \varepsilon$.

Since Λ is a horseshoe for φ, let $\varepsilon_0 > 0$ be an expansivity constant of φ on Λ. By uniform continuity, we may take $0 < \delta < \varepsilon_0/2$, such that $d(x,y) < \delta$ implies $|f(x) - f(y)| < \varepsilon/2$. According to the Shadowing Lemma, there exists $\alpha > 0$ such that every α-pseudo-orbit in Λ is δ-shadowed by a point of Λ.

By definition of l and compactness there exists a subsequence $(\varphi^{n_k}(x))_k$ such that $f(\varphi^{n_k}(x)) \to l$ and $\varphi^{n_k}(x) \to y$. Take k big enough so that, for all $n \geq n_k$:

$$f(\varphi^n(x)) < l + \frac{\varepsilon}{2}, \; d(\varphi^{n_k}(x), \varphi^{n_{k+1}}(x)) < \alpha \text{ and } |f(\varphi^{n_k}(x)) - l| < \frac{\varepsilon}{2}. \tag{3.3}$$

Consider the following infinite α-pseudo-orbit periodic in Λ:

$$\cdots \underbrace{\varphi^{n_k}(x), \varphi^{n_k+1}(x), \cdots, \varphi^{n_{k+1}-1}(x),}_{\text{period}} \varphi^{n_k}(x), \varphi^{n_k+1}(x), \cdots, \varphi^{n_{k+1}-1}(x) \;\cdots .$$

There exists $p \in \Lambda$, whose orbit δ-shadows the above pseudo-orbit. This means that, for all $j \geq 0$:

$$d(\varphi^j(p), \varphi^{n_k+\bar{j}}(x)) < \delta, \text{ where } 0 \leq \bar{j} < d := n_{k+1} - n_k \textit{ and } \bar{j} \equiv j (\text{mod } d).$$

The case $j < 0$ is similar. Thus, by expansivity, $p = \varphi^d(p)$ is a periodic point, and by uniform continuity of f and (3.3), we have $|l - m_{f,\Lambda}(p)| < \varepsilon$.

In order to prove $\overline{P(f, \Lambda)} \subset L(f, \Lambda)$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of periodic points (each x_n has period p_n) in Λ, such that $r_n = f(x_n) = m_{f,\Lambda}(x_n)$ and $r_n \to s$. We shall to show that $s \in L(f, \Lambda)$. By compactness, we may assume that $x_n \to x$ and so $r_n = f(x_n) \to f(x) = s$. Consider the sequence $(y_n)_{n\in\mathbb{N}}$, given by:

$$x_0, \varphi(x_0), \cdots, \varphi^{p_0-1}(x_0), x_1, \varphi(x_1), \cdots, \varphi^{p_1-1}(x_1), x_2, \cdots .$$

Since $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$, Lemma 3.1 implies that there exists $z \in \Lambda$, such that $\lim_{n\to\infty} d(\varphi^n(z), y_n) = 0$, that is:

$$\lim_{n\to\infty} d(\varphi^n(z), \varphi^{\bar{n}}(x_{r_n})) = 0, \text{ where } 0 \leq \bar{n} < p_{r_n} \textit{ and } n = p_0 + \cdots + p_{(r_n-1)} + \bar{n}.$$

In particular, we get $\lim\limits_{n\to+\infty} d(x_n, \varphi^{p_0+\cdots+p_{n-1}}(z)) = 0$. The uniform continuity of f implies that $\lim\limits_{n\to+\infty} f(\varphi^{p_0+\cdots+p_{n-1}}(z)) = \lim\limits_{n\to+\infty} f(x_n) = f(x) = s$. Now, suppose that $l_{f,\Lambda}(z) = m > s$, so there exists a subsequence $(\varphi^{n_k}(z))$ such that $f(\varphi^{n_k}(z)) \to m$. Taking $\varepsilon = m - s > 0$, by above claim and uniform continuity there exists k sufficiently large such that

$$|f(\varphi^{n_k}(z)) - m| < \frac{\varepsilon}{4}, \; |f(\varphi^{n_k}(z)) - f(\varphi^{\overline{n_k}}(x_{r_k})))| < \frac{\varepsilon}{4} \text{ and } |f(x_{r_k}) - s| < \varepsilon/4,$$

where $0 < \overline{n_k} < p_{r_k}$ and $n_k = p_k + \cdots + p_{(r_{n_k}-1)} + \overline{n_k}$. Thus, $f(\varphi^{\overline{n_k}}(x_{r_k}))) > f(x_{r_k})$ and this contradicts the definition of x_{r_k}. Therefore, $m_{f,\Lambda}(z) = s$. □

As immediate consequences of Theorem 1, we have the following corollaries:

Corollary 3.1. *The set $L(f,\Lambda)$ is closed in $\mathbb{R}$. Let l be a isolated point of $L(f,\Lambda)$, then l is associated to a periodic point, i.e., there exists a periodic point $p \in \Lambda$ such that $m_{f,\Lambda}(p) = l_{f,\Lambda}(p) = l$.*

Corollary 3.2. *We have $L(f,\varphi|_\Lambda) = L(f,\varphi^{-1}|_\Lambda)$, that is:*

$$\Big\{ \limsup_{n\to+\infty} f(\varphi^n(x)) : x \in \Lambda \Big\} = \Big\{ \limsup_{n\to-\infty} f(\varphi^n(x)) : x \in \Lambda \Big\}.$$

In the classical Markov spectrum we have a similar characterization in terms of periodic points, as we can see in [5]:

Proposition. *Let $B = \{m(\underline{\theta}) : \underline{\theta} \in \Sigma$ is eventually periodic on both sides$\}$. Then, $M = \overline{B}$.*

We say that a point x in Λ is *asymptotically periodic* when $\omega(x)$ and $\alpha(x)$ are respectively equal to orbit of p_1 and orbit of p_2 (i.e., $x \in W^s(p_1) \cap W^u(p_2)$), where p_1 and p_2 are periodic points of φ in Λ. We have a result similar to that previous preposition to $M(f,\Lambda)$, more specifically:

Theorem 2. *Let $B(f,\Lambda) = \{m_{f,\Lambda}(x) : x \in \Lambda$ is asymptotically periodic$\}$. Then, $M(f,\Lambda) = \overline{B(f,\Lambda)}$.*

Proof. Since $B(f,\Lambda) \subset M(f,\Lambda)$ and $M(f,\Lambda)$ is closed, we get the inclusion $\overline{B(f,\Lambda)} \subset M(f,\Lambda)$. To prove the inclusion $M(f,\Lambda) \subset \overline{B(f,\Lambda)}$, we consider $x \in \Lambda$ such that $f(x) = m_{f,\Lambda}(x)$. For any $\varepsilon > 0$, we shall construct an asymptotically periodic point $y \in \Lambda$ for which $|m_{f,\Lambda}(x) - m_{f,\Lambda}(y)| < \varepsilon$.

By uniform continuity there exists $0 < \delta < \min\{\varepsilon_0/2, \gamma/2\}$, where ε_0 is an expansivity constant of φ on Λ and γ is given by the Stable Manifold Theorem, such that $d(x,y) < \delta$ implies $|f(x) - f(y)| < \varepsilon$. By the Shadowing Lemma, there exists $\alpha > 0$ for which every α-pseudo-orbit is δ-shadowed by some point of Λ. By compactness, there are convergent subsequences $(\varphi^{n_k}(x))_{k\in\mathbb{N}}$ of $(\varphi^n(x))_{n\geq 0}$ and $(\varphi^{m_k}(x))_{k\in\mathbb{N}}$ of $(\varphi^m(x))_{m<0}$. Thus, there are n_k and $-m_k$ big enough, such that:

$$d(\varphi^{n_k}(x), \varphi^{n_{k+1}}(x)) < \alpha \ \text{ and } \ d(\varphi^{m_k}(x), \varphi^{m_{k+1}}(x)) < \alpha.$$

Take the following eventually periodic on both sides α-pseudo-orbit:

$$\cdots \varphi^{m_k-1}(x), \underbrace{\varphi^{m_{k+1}}(x), \cdots, \varphi^{m_k-1}(x)}_{\text{left period}}, \varphi^{m_k}(x), \varphi^{m_k+1}(x), \cdots, \varphi^{-1}(x), x,$$

$$\varphi(x), \cdots, \varphi^{n_k-1}(x), \underbrace{\varphi^{n_k}(x), \cdots, \varphi^{n_{k+1}-1}(x)}_{\text{right period}}, \varphi^{n_k}(x) \cdots$$

Thus, there exists a $y \in \Lambda$ that δ-shadows the above pseudo-orbit, this means:

$$d(\varphi^j(x), \varphi^j(y)) < \delta \text{ for all } m_k \leq j \leq n_k - 1,$$

$$d(\varphi^l(y), \varphi^{\bar{l}+n_k}(x)) < \delta \text{ for all } l > n_k - 1, \ \bar{l} \in \{0, \cdots, d_1 - 1\},$$
$$d(\varphi^t(y), \varphi^{m_k - \hat{t}}(x)) < \delta \text{ for all } t < m_k, \ \hat{t} \in \{0, \cdots, d_2 - 1\},$$

where $d_1 = n_{k+1} - n_k$, $d_2 = m_k - m_{k+1}$, $l - n_k \equiv \bar{l} (\text{mod } d_1)$ and $t - m_k \equiv -\hat{t} (\text{mod } d_2)$.

By the Shadowing Lemma, we can find p_1 and p_2 periodic points in Λ, such that $\varphi^{n_k}(y) \in W^s_\gamma(p_1)$ and $\varphi^{m_k-1}(y) \in W^u_\gamma(p_2)$. Then, y is asymptotically periodic. Moreover, the uniform continuity gives to us that $\sup_{n\in\mathbb{Z}} f(\varphi^n(y)) < m_{f,\Lambda}(x) + \varepsilon$ and $|f(x) - f(y)| < \epsilon$. Therefore, $|m_{f,\Lambda}(x) - m_{f,\Lambda}(y)| < \varepsilon$. $\square$

We can recover the classical Markov and Lagrange spectra from a dynamical approach, see [22]. In order to do that, let $\varphi : (0,1)^2 \to (0,1)^2$ be a natural extension of the Gauss Map, $g : (0,1) \to (0,1)$ given by $g(x) = \{1/x\}$, defined by

$$\varphi(x,y) = \left(\left\{ \frac{1}{x} \right\}, \frac{1}{\lfloor 1/x \rfloor + y} \right). \tag{3.4}$$

Given $(x,y) \in (0,1)^2$ a pair of irrational numbers, we associate the sequence $\underline{\theta} = (a_n)_{n\in\mathbb{Z}} \in \Sigma := (\mathbb{N}^*)^{\mathbb{Z}}$, where $x = [0; a_0, a_1, \cdots]$ and $y = [0; a_{-1}, a_{-2}, \cdots]$. Note that $\varphi(x,y)$ is associated to $\sigma(\underline{\theta}) = (a_{n+1})_{n\in\mathbb{Z}}$. Thus, we can think of φ as a geometric way to see the shift map.

Define $C(N) := \{x = [0; a_1, a_2, \cdots] : 1 \leq a_n \leq N\}$ and $\Lambda_N := C(N) \times C(N)$. It is possible to see that Λ_N is a horseshoe associate to φ. Let $g : (0,1)^2 \to \mathbb{R}$ be a height function given by $g(x,y) = y + 1/x$. If $\underline{\theta} = (a_i)_{i\in\mathbb{Z}} \in \Sigma$ has some $a_i \geq N+1$, then $m(\underline{\theta}) = \sup_{n\in\mathbb{Z}}([a_n; a_{n+1}, \cdots] + [0; a_{n-1}, a_{n-2}, \cdots]) > N+1$. Thus, $M \cap (-\infty, N) = M(g, \Lambda_N) \cap (-\infty, N)$. Analogously, we have that $L \cap (-\infty, N) = L(g, \Lambda_N) \cap (-\infty, N)$. Therefore, this way to see the classical spectra intersected with semi lines allow us to get back the characterization of the both set in terms of periodic and eventually periodic points from Theorems 1 and 2.

Another spectrum that comes from number theory is the *Dirichlet Spectrum.* In [19], this is defined as the following set:

$$D = \{\tilde{\gamma}(\underline{\theta}) : \underline{\theta} = (a_n)_{n\in\mathbb{Z}} \in \Sigma = (\mathbb{N}^*)^{\mathbb{Z}}\},$$

where $\tilde{\gamma}(\underline{\theta}) := \limsup_{n\to\infty}([a_{n+1}; a_{n+2}, \cdots]\cdot[a_n; a_{n-1}, \cdots])$. Using the dynamic defined in (3.4) and the function $f_0 : (0,1)^2 \to \mathbb{R}$ defined by $f(x,y) = 1/xy$, we can see this spectrum as a dynamical spectrum. More specifically, $D \cap (-\infty, N) = L(f_0, \Lambda_N) \cap (-\infty, N)$, for every $N \in \mathbb{N}$. By Theorem 1, we get that D is a closed set and also get the characterization of D in terms of the periodic sequences in Σ. In particular, this fact shows that the right end point in the gap $((5-\sqrt{21})/2, (3-\sqrt{3})/6)$ of D belong to D, what wasn't known in [19], according to Theorem 2[19].

3.2 Generic properties

In this section, instead of studying properties of Lagrange and Markov spectra associated to any horseshoe and any potential, we discuss properties of a typical spectra.

In [31], Moreira made a deep study of geometric properties of classical spectra. In particular, it was proved that the Hausdorff dimensions of intersections of both spectra with half-lines $(-\infty, t)$ always coincide and they vary continuously with t, and also it was established that L' is a perfect set, i.e., $L'' = L'$. In [3], a generalization of the first result was proved in a conservative setting for dynamical spectra. We prove a generalization of the second result for a dynamical Lagrange spectrum:

Theorem 3. *Let Λ be a horseshoe associated to a C^2-diffeomorphism φ. Then, there exists an open and dense set $H_\Lambda \subset C^1(M,\mathbb{R})$, such that for all $f \in H_\Lambda$,*

$$L(f,\Lambda)'' = L(f,\Lambda)'.$$

The proof of this theorem follows the same lines as in the proof of Moreira's theorem. The main idea is to put a Cantor set near to any accumulation point. In order to do that, we use as tools the subsequent results. The next lemma gives us the subset H_Λ of functions:

Lemma 3.2. *The set*

$$H_\Lambda = \{f \in C^1(M,\mathbb{R}) : Df_z(e_z^u) \neq 0 \text{ or } Df_z(e_z^s) \neq 0,\ \forall z \in \Lambda\}$$

is open and dense in $C^1(M,\mathbb{R})$, where $e_z^{s,u}$ are unit vectors in $E_z^{s,u}$ as in the definition of hyperbolicity, respectively. Moreover, for every $f \in H_\Lambda$, the set $M_{f|_\Lambda} := \{z \in \Lambda : f(z) \geq f(y),\ \forall y \in \Lambda\}$ is contained in a finite family of C^2-curves $\alpha_f = \{\alpha_i : [0,1] \to M :\ i = 1, \cdots, m\}$.

Proof. It is clear that H_Λ is open in in $C^1(M,\mathbb{R})$. It remains to prove that this set is C^1-dense. Since the set $\mathcal{M}$ of C^2-Morse function are open is dense in $C^2(M,\mathbb{R})$ it is enough to show that H_Λ is dense in $\mathcal{M}$. In particular, it is dense in $C^1(M,\mathbb{R})$. Let $g \in \mathcal{M}$, then $\mathrm{Crit}(g)$ is a discrete set, because the the critical points in g are non-degenerate. Since Λ is compact, we have $\#(\mathrm{Crit}(g)\cap\Lambda) < \infty$, and as $\mathrm{int}\,\Lambda = \emptyset$, we can perturb g to a C^2-close function f, such that $\mathrm{Crit}(f) \cap \Lambda = \emptyset$. Thus, either $Df_z(e^s_z) \neq 0$ or $Df_z(e^u_z) \neq 0$, for every $z \in \Lambda$. This finishes the first part.

In order to check the second part, given $f \in H_\Lambda$, by continuity of ∇f, we take a Markov partition $\{R_a\}_{a\in\mathcal{A}}$ with diameter sufficiently small, such that $Df_z(e^s_z) \neq 0$ for every $z \in R_a$ or $Df_z(e^u_z) \neq 0$ for every $z \in R_a$, where $R_a \simeq I^s_a \times I^u_a$ are rectangles defined by bounded compact pieces I^s_a and I^u_a of stable and unstable manifolds, respectively, of certain points of Λ. Thus, the set of maximal point of $f|_\Lambda$ is contained in the finite family α of C^2-curves given by $\bigcup_{a\in\mathcal{A}}(\partial I^s_a \times I^u_a) \cup (I^s_a \times \partial I^u_a)$, where $\partial I^{s,u}_a = \{r^{s,u}_a, s^{s,u}_a\}$. $\square$

The following lemma from [15], gives us sub-horseshoes that avoid a finite number of C^1-curves:

Lemma 3.3 ([15]). *Let $\alpha = \{\alpha_i : [0,1] \to M : i \in \{1,\cdots,m\}\}$ be a finite family of C^1-curves. Then for all $\varepsilon > 0$ there are sub-horseshoes $\Lambda^s_\alpha, \Lambda^u_\alpha$ of Λ such that $\Lambda^{s,u}_\alpha \cap \alpha_i([0,1]) = \emptyset$ for any $i = 1,\cdots,m$ and*

$$HD(K^s_\alpha) \geq HD(K^s) - \varepsilon \text{ and } HD(K^u_\alpha) \geq HD(K^u) - \varepsilon,$$

where K^s_α, K^s are the stable regular Cantor sets associated to Λ^s_α, Λ respectively, and K^u_α, K^u are the unstable regular Cantor sets associated to Λ^u_α, Λ, respectively.

In the following, for the completeness of the text, we reproduce an argument of Moreira and Ibarra's paper[17], where it is put in the spectra the image by a function f a diffeomorphic part of a big part of the horseshoe.

Let Λ be a horseshoe of φ, considering a Markov partition $\{R_a\}_{a\in\mathcal{A}}$ as in the proof of the Lemma 3.2, we can conjugate $\varphi : \Lambda \to \Lambda$ to $\sigma : \Sigma_B \to \Sigma_B$ (a subshift of finite type), by a map $\Pi : \Sigma_B \to \Lambda$, thus given a function $f : M \to \mathbb{R}$ we associate to it $\tilde{f} = f\circ\Pi$. Let $f \in H_\Lambda$. By Lemmas 3.2 and 3.3, there is a sub-horseshoe $\tilde{\Lambda}$ such that $\tilde{\Lambda} \cap \alpha_f = \emptyset$, we can take $\tilde{\Lambda} = \Lambda^s_{\alpha_f}$ or $\Lambda^u_{\alpha_f}$ as in Lemma 3.3. Let say that, $HD(K^s) \sim HD(\tilde{K}^s) > 0$, since Λ is a horseshoe there exist $C, \beta > 0$ such that for any admissible finite word γ, we have: $C^{-1}|I^s(\gamma^t)|^\beta < |I^u(\gamma)| < C|I^s(\gamma^t)|^{1/\beta}$. Therefore, $HD(\tilde{K}^u) > 0$.

Fixe $x_M \in M_{f|_\Lambda} \subset \alpha_f$ with kneading sequence $\underline{x}_M = (a_n)_{n\in\mathbb{Z}} \in \Sigma_B$. By compactness, if $\varepsilon > 0$ is small enough, we can take $s \in \mathbb{N}$ such that $\Sigma_{|n|\geq s} 2^{-(2|n|+1)} < \varepsilon$ and $\underline{a}_s := (a_{-s}, \cdots, a_0, \cdots, a_s)$ such that $x_M \in R_{\underline{a}_s} := \bigcap_{i=-s}^{s} \varphi^{-i}(R_{a_i})$ and

$$\sup \tilde{f}|_{\Pi^{-1}(\tilde{\Lambda})_\varepsilon} < \inf \tilde{f}|_{\Pi^{-1}(R_{\underline{a}_s}\cap\Lambda)}, \tag{3.5}$$

where $\Pi^{-1}(\tilde{\Lambda})_\varepsilon := \{\underline{x} \in \Sigma_B : d(\underline{x}, \Pi^{-1}(\tilde{\Lambda})) < \varepsilon\}$.

Given $d \in \tilde{\Lambda}$, with kneading sequence $\underline{d} = (d_n)_n$, define the relative cylinder in Σ_B:

$$C_{\underline{d}_s,B} = \{\underline{w} \in \Sigma_B : w_i = d_i,\ i = -s, \cdots, s\}.$$

Let $l > \max\{s, N_0\}$, where $N_0 := \max\{n(x,y) : x, y \in \mathcal{A}\}$, according to Section 2.1, and then define $\underline{\alpha} = (a_{-l}, \cdots, a_l)$. By transitivity, there are admissible strings $\underline{e} := (e_1, \cdots, e_{k_0})$ and $\underline{f} = (f_1, \cdots, f_{j_0})$ joining d_0 with a_{-l} and a_l with d_1, respectively, with $k_0, j_0 \leq N_0$. Define $A : C_{\underline{d}_s,B} \to \Sigma_B$, given by:

$$A(\underline{x}) := (\cdots, x_{-2}, x_{-1}, x_0, \underline{e}, \underline{\alpha}, \underline{f}, x_1, x_2, \cdots),$$

where a_0 in the middle of α is the zero position of $A(\underline{x})$.

We may characterize $\sup_{n\in\mathbb{Z}} \tilde{f}(\sigma^n(A(\underline{x})))$, for $\underline{x} \in C_{\underline{d}_s,B} \cap \Pi^{-1}(\tilde{\Lambda})$. By choice of s, we have $d(\sigma^{l+j_0+2s+n}(A(\underline{x})), \sigma^{2s+n}(\underline{x})) < \varepsilon$ and $d(\sigma^{-(l+k_0+2s+n)}(A(\underline{x})), \sigma^{(2s+n)}(\underline{x})) < \varepsilon$, for all $n \geq 0$. Since $\Pi^{-1}(\tilde{\Lambda})$ is σ-invariant, if $\underline{x} \in \Pi^{-1}(\tilde{\Lambda})$, then

$$\tilde{f}(\sigma^{l+j_0+2s+n}(A(\underline{x}))),\ \tilde{f}(\sigma^{-(l+k_0+2s+n)}(A(\underline{x}))) < \inf \tilde{f}|_{\Pi^{-1}(R_{\underline{a}_s}\cap\Lambda)},\ \forall n \geq 0.$$

Hence, for $\underline{x} \in C_{\underline{d}_s,B} \cap \Pi^{-1}(\tilde{\Lambda})$ we have $\sup_{n\in\mathbb{Z}} \tilde{f}(\sigma^n(A(\underline{x}))) = \tilde{f}(\sigma^j(A(\underline{x})))$, for some $j \in \{-(l+k_0+2s), \cdots, (l+j_0+2s)\}$. Let $\Pi(\underline{x}) = x$, define the set $\tilde{\Lambda}_j := \{x \in \tilde{\Lambda} \cap \Pi(C_{d_s,B}) : \sup_{n\in\mathbb{Z}} \tilde{f}(\sigma^n(A(\underline{x}))) = \tilde{f}(\sigma^j(A(\underline{x})))\}$. Since,

$$\tilde{\Lambda} \cap \Pi(C_{\underline{d}_s,B}) = \bigcup_{j=-(l+k_0+2s)}^{l+j_0+2s} \tilde{\Lambda}_j,$$

by Baire's theorem, some $\tilde{\Lambda}_{m_0}$ has non-empty interior in $\tilde{\Lambda} \cap \Pi(C_{\underline{d}_s,B})$, so:

$$HD(\tilde{\Lambda}) = HD(\tilde{\Lambda} \cap \Pi(C_{\underline{d}_s,B}) = HD(\tilde{\Lambda}_{m_0}).$$

Therefore, for every $\underline{x} \in \Pi^{-1}(\tilde{\Lambda}_{m_0})$:

$$\sup_{n\in\mathbb{Z}} \tilde{f}(\sigma^n(A(\underline{x}))) = \tilde{f}(\sigma^{m_0}(A(\underline{x}))).$$

Define $\tilde{A} := \Pi \circ A \circ \Pi^{-1}$. According to [17], it can be shown that $\tilde{A}$ extends to a C^1-diffeomorphism defined in a neighborhood U_d of d. In order to do that, first we use the symbolic language and the fact that Π is a morphism of the local product structure (i.e., Π commutes the brackets) to extend $\tilde{A}$ to a local diffeomorphism in local stable and unstable local manifolds of d, $W^s_{\text{loc}}(d)$ and $W^u_{\text{loc}}(d)$. After, we use that the stable and unstable laminations of the horseshoe Λ can be extended to a C^1 invariant foliations defined on a full neighborhood of Λ to extend $\tilde{A}$ to a diffeomorphism in a neighborhood of d.

Therefore, we have $\{f(\varphi^{m_0}(\tilde{A}(x))) : x \in \tilde{\Lambda}_{m_0}\} \subset M(f, \Lambda)$.

We can do an analogue construction to prove the same for the Lagrange spectrum. Then, using the above notations, given $x \in \tilde{\Lambda} \cap \Pi(C_{\underline{d}_s,B})$ and $\Pi^{-1}(x) = \underline{x} = (\cdots, x_1; x_0, x_1, \cdots)$. There are $E_i = (e^i_1, \cdots, e^i_{s_i})$ joining x_i and x_{-i} and $s_i \leq N_0$ for each $i \in \mathbb{N}$. Define $A_1 : C_{\underline{d}_s,B} \to \Sigma_B$ by

$$\begin{aligned} A_1(\underline{x}) = (&\cdots, x_2, x_1, x_0, \beta^*, x_1, E_1, x_{-1}, x_0, \beta, x_1, x_2, E_2, x_{-2}, x_{-1}, x_0, \beta, \\ &x_1, x_2, x_3, E_3, x_{-3}, x_{-2}, x_{-1}, x_0, \beta, x_1, x_2, x_3, x_4, E_4, x_{-4}, x_{-3}, \cdots), \end{aligned}$$

where $\beta = \underline{e\alpha f}$ and the position zero of the sequence $A_1(\underline{x})$ is in the a_0 in the middle of $\underline{\alpha}$ in the β^*.

By an analogous argument, according to [17], we can show that

$$\limsup_{n\to+\infty} \tilde{f}(\sigma^n(A_1(\underline{x}))) = \tilde{f}(\sigma^{j_0}(A(\underline{x}))),$$

for every $\underline{x} \in \Pi^{-1}(\Lambda'_{j_0})$, where $\Lambda'{}_{j_0}$ has non-empty interior in $\tilde{\Lambda} \cap \Pi(C_{\underline{d}_s,B})$, and so:

$$HD(\tilde{\Lambda}) = HD(\tilde{\Lambda} \cap \Pi(C_{\underline{d}_s,B})) = HD(\Lambda'{}_{j_0}).$$

Therefore,

$$\{f(\varphi^{j_0}(\tilde{A}(x))) : x \in \Lambda'{}_{j_0}\} \subset L(f, \Lambda).$$

Remark 3.1. From a remark in [17], if $Df_{\varphi^{j_0}(\tilde{A}(x))}(e^*_{\varphi^{j_0}(\tilde{A}(x))}) \neq 0$, then we have $D(f \circ \varphi^{j_0} \circ \tilde{A})_x(e^*_x) \neq 0$, for $* = s$ or u and for every $x \in \Lambda'{}_{j_0}$. Indeed, since $D\varphi^{j_0}_{\tilde{A}(x)}(e^{s,u}_{\tilde{A}(x)}) \in E^{s,u}_{\varphi^{j_0}(\tilde{A}(x))}$ and by construction of $\tilde{A}$, $\partial\tilde{A}/\partial e^{s,u}_x$ is parallel to $e^{s,u}_{\tilde{A}(x)}$, we get get the result by chain rule.

In the proof the theorem we also use the following combinatorial lemma:

Lemma 3.4. *Let $\mathcal{A}$ be a finite alphabet. Given two finite words in this alphabet γ and $\tilde{\gamma}$ such that $(\gamma)^{l_1} = (\tilde{\gamma})^{l_2}$, for some $l_1, l_2 \in \mathbb{N}$. Then, there exist a finite word ω and $c_1, c_2 \in \mathbb{N}$, such that $\gamma = (\omega)^{c_1}$ and $\tilde{\gamma} = (\omega)^{c_2}$.*

Proof. We may assume that $\gcd(l_1, l_2) = 1$. Otherwise, let $b = l_1|\gamma| = l_2|\tilde{\gamma}|$, given $\gcd(l_1, l_2) = d$, we have $b/d = (l_1/d)|\gamma| = (l_2/d)|\tilde{\gamma}|$ and then, $(\gamma)^{l_1/d} = (\tilde{\gamma})^{l_2/d}$ with $\gcd(l_1/d, l_2/d) = 1$.

Let $k = |\tilde{\gamma}|/l_1 = |\gamma|/l_2$, thus we can split γ in l_2 subwords of length k, as been $\gamma = \sigma_1\sigma_2\cdots\sigma_{l_2}$, where $|\sigma_i| = k$. Analogously, $\tilde{\gamma} = \tilde{\sigma}_1\tilde{\sigma}_2\cdots\tilde{\sigma}_{l_1}$, where $|\tilde{\sigma}_i| = k$.

Given $j \in \{1, \cdots, l_1\}$, since $\gcd(l_1, l_2) = 1$, the equation $l_2 \cdot x \equiv j \pmod{l_1}$ has a solution $\bar{x} \in \{1, \cdots, l_1\}$, i.e., there is $q \in \{1, \cdots, l_2\}$ such that $l_2 \cdot \bar{x} - l_1 \cdot q = j$. Thus, $\gamma^{\bar{x}} = \tilde{\gamma}^q\tilde{\sigma}_1\cdots\tilde{\sigma}_j$, and then $\tilde{\sigma}_j = \sigma_{l_2}$, $\forall j \in \{1, \cdots, l_1\}$. Analogously, $\sigma_m = \tilde{\sigma}_{l_1}$, $\forall m \in \{1, \cdots, l_2\}$. Since $(\gamma)^{l_1} = (\tilde{\gamma})^{l_2}$, we have $\sigma_{l_2} = \tilde{\sigma}_{l_1} =: \omega$. Therefore, $\gamma = \omega^{l_2}$ and $\tilde{\gamma} = \omega^{l_1}$. □

Finally, we are able to prove the theorem.

Proof of Theorem 3. Let $f \in H_\Lambda \subset C^1(M, \mathbb{R})$ and $\ell \in L(f, \Lambda)'$. Take a sequence of distinct elements $\ell_n \in L(f, \Lambda)$ converging to ℓ, such that $\ell_n = l_{f,\Lambda}(y_n) = \limsup_{k\to\infty} f(\varphi^k(y_n))$, where $y_n \in \Lambda$ has the kneading sequence $\underline{y}_n = (b^n_k)_{k\in\mathbb{Z}} \in \Sigma_B$.

For a fixed $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, we have:

$$|l_{f,\Lambda}(y_n) - \ell| < \delta \text{ and thus, } |f(\varphi^k(y_n)) - \ell| < \delta,\ \forall k \in \mathbb{N}_n,$$

where $\mathbb{N}_n$ is an infinite subset of $\mathbb{N}$. Rewriting this last inequality in symbolic language, we obtain:

$$|\tilde{f}(\cdots, b^n_{k-1}, (b^n_k)^*, b^n_{k+1}, \cdots) - \ell| < \delta,\ \forall k \in \mathbb{N}_n, \tag{3.6}$$

where the asterisk $*$ indicates the 0'th position.

Take $N \in \mathbb{N}$ large enough so that $\sum_{n=N+1}^{\infty} 2^{-(2n+1)} < \gamma/2$, where γ is such that: $d(\underline{a}, \underline{b}) < \gamma$ implies $|\tilde{f}(\underline{a}) - \tilde{f}(\underline{b})| < \delta$. Consider the following strings $S(j, n) := (b^n_{j-N}, \cdots, b^n_j \cdots b^n_{j+N})$. By the pigeonhole principle, there exist a $S = (s_N, \cdots, s_0, \cdots, s_N) \in \mathcal{A}^{2N+1}$ and an infinite subset $\mathbb{N}^* \subset \mathbb{N}$ such that for $n \in \mathbb{N}^*$, there are infinitely many $j_1(n) < j_2(n) < \cdots$ in $\mathbb{N}_n$, with $\lim_{i\to\infty}(j_{i+1}(n) - j_i(n)) = \infty$ and $S(j_i(n), n) = S$.

For every $n \in \mathbb{N}^*$ and $i \geq 1$, define:

$$C(i, n) := (b^n_{j_i(n)+N+1}, b^n_{j_i(n)+N+2}, \cdots, b^n_{j_{i+1}(n)+N}).$$

Taking distinct $m, n \in \mathbb{N}^*$, let $k_0 \in \mathbb{N}$ be such that $k \geq k_0$, we have $f(\varphi^k(y_*)) < \ell_* + \delta$, where $* \in \{m, n\}$. Since $x_m \neq x_n$, by the Lemma 3.4, there are i_1 and i_2 with $j_{i_1}(m)$, $j_{i_2}(n) > k_0$, such that there exists no sequence γ such that $C(i_1, m)$ and $C(i_2, n)$ are concatenations of γ. Hence, the set $C = \{C(i_1, m)C(i_2, n), C(i_2, n)C(i_1, m)\}$ defines a complete subshift $\Sigma(C) \subset \Sigma_B \subset \mathcal{A}^{\mathbb{Z}}$, and this one is associated with a subhorseshoe $\Lambda_C \subset \Lambda$.

Claim: $d(L(f, \Lambda_C), \ell) < 2\delta$.

Let $y \in \Lambda_C$ with kneading sequence $\underline{y} = (y_i)_{i\in\mathbb{Z}}$. For each $k \in \mathbb{Z}$, there exists $p \geq k_0$ such that $\sigma^k((y_i)_{i\in\mathbb{Z}})$ belongs to the cylinder $C[-N; b^*_{p-N}, \cdots, b^*_{p+N}]$, where $* \in \{m, n\}$. See Figure 3.1. So, by uniform continuity of f, we have $f(\varphi^k(y)) < \ell + 2\delta$.

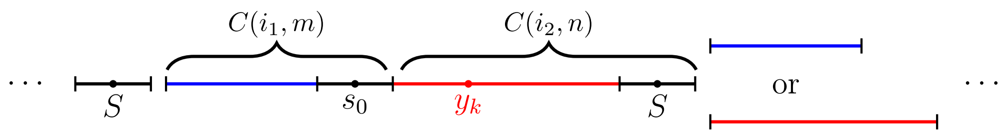

Figure 3.1: Representation of the kneading sequence of y.

In particular, for each one of the infinitely many $q \in \mathbb{Z}$ such that $y_q = s_0$ in the middle of S, by uniform continuity of f and (3.6), we get $|f(\varphi^q(y)) - \ell| < 2\delta$. Hence,

$$l_{f,\Lambda_C}(y) \in (\ell - 2\delta, \ell + 2\delta).$$

This finishes the proof of the claim.

Finally, by the previous discussion, for each $f \in H_\Lambda$ there exists a subhorseshoe $\tilde{\Lambda}_C$ (as in the previous discussion such that $HD(\tilde{K}^u_C), HD(\tilde{K}^s_C) > 0$), a subset $\Lambda'_{C,j_0} \subset \tilde{\Lambda}_C$ with relative interior non-empty in $\tilde{\Lambda}_C$, a local C^1-diffeomorphism $\tilde{A}$, such that $\{f(\varphi^{j_0}(\tilde{A}(x))) : x \in \Lambda'_{C,j_0}\} \subset L(f, \Lambda_C)$. Let x_0 be a point in the interior of Λ'_{C,j_0}. By the previous remark, if for $* = s$ or u we have $Df_{\varphi^{j_0}(\tilde{A}(x_0))}(e^*_{\varphi^{j_0}(\tilde{A}(x_0))}) \neq 0$, then $D(f \circ \varphi^{j_0} \circ \tilde{A})_{x_0}(e^*_{x_0}) \neq 0$. Define $K^*(x_0) := W^*_{loc,\varphi|_{\tilde{\Lambda}_C}}(x_0) = W^*_{loc,\varphi|_\Lambda}(x_0) \cap \tilde{\Lambda}_C$ a regular Cantor set, since $D(f \circ \varphi^{j_0} \circ \tilde{A})_{x_0}(e^*_{x_0}) \neq 0$, we get that $(f \circ \varphi^{j_0} \circ \tilde{A})(K^*(x_0)) \subset L(f, \Lambda_C) \subset \mathbb{R}$ is also a Cantor set. This concludes the proof of theorem. $\square$

The next result follows from the proof of the previous theorem:

Corollary 3.3. *Let Λ be a horseshoe for a C^2-diffeomorphism φ. Then, for all $f \in H_\Lambda$, we have:*

$$\inf L'(f, \Lambda) = \inf\{b \in \mathbb{R} : HD(L(f, \Lambda) \cap (-\infty, b)) > 0\}.$$

The property in Theorem 3 could be inquired about for the Markov dynamical spectrum, i.e., do we have that $M(f,\Lambda)'' = M(f,\Lambda)'$ under some generic conditions on the dynamics and the function? In the classical case it is unknown whether $M'' = M'$. But, in Section 3.3, we answer this question. In order to do that, we build an example, which is a set open in the pair (dynamical, function), where these set are distinct.

Another natural topological property about the dynamical spectra that could be studied is the interior of those sets. This study is related to the fractal geometry of regular Cantor sets. Using the fact that a generic pair of regular Cantor set in the C^2-topology whose sum of Hausdorff dimension is larger than 1 have translations which get stable intersection, it was proved in [17] that:

Theorem. *Let Λ be a horseshoe associates to a C^2-diffeomorphism φ such that $HD(\Lambda) > 1$. Then, arbitrarily close to φ, there exist a diffeomorphism φ_0 and a C^2-neighbourhood W of φ_0 such that, if Λ_ψ denotes the continuation of Λ associated to $\psi \in W$, there exists an open and dense set $H_\psi \subset C^1(M,\mathbb{R})$ such that for all $f \in H_\psi$,*

$$\operatorname{int} L(f,\Lambda_\psi) \neq \emptyset \ \text{ and } \ \operatorname{int} M(f,\Lambda_\psi) \neq \emptyset.$$

Note that if $HD(\Lambda) < 1$ the last result is not true, because if f is Lipschitz then $HD(f(\Lambda)) < 1$ and so $\operatorname{int} f(\Lambda) = \emptyset$. Using the fact that there are no C^1-stable intersection of regular Cantor sets, according to [29], we can prove that previous theorem doesn't work under any condition on the dimension of the horseshoe associated to C^1-diffeomorphism. More specifically,

Theorem 4. *There is a Baire residual set $\mathcal{G} \subset \mathrm{Diff}^1(M) \times C^1(M,\mathbb{R})$ such that, for every $(\varphi, f) \in \mathcal{G}$, we have $\operatorname{int} f(\Lambda) = \emptyset$, for any horseshoe Λ of φ. In particular,*

$$\operatorname{int} L(f,\Lambda) = \emptyset \ \text{ and } \ \operatorname{int} M(f,\Lambda) = \emptyset.$$

Proof. We start by proving a local version of this result. More specifically, we have the following:
Claim: Let $\varphi \in \mathrm{Diff}^1(M)$ having a horseshoe Λ and $\mathcal{U}$ be a C^1-neighbourhood of φ of hyperbolic continuation. Then, there is a Baire residual set $G \subset \mathcal{U} \times C^1(M,\mathbb{R})$ such that, for every $(\psi, f) \in G$, we have $\operatorname{int} f(\Lambda_\psi) = \emptyset$, where Λ_ψ is the hyperbolic continuation of Λ for ψ.

Indeed, define $G_r = \{(\psi, f) \in \mathcal{U} \times C^1(M, \mathbb{R}) :\ r \notin f(\Lambda_\psi)\}$, for any $r \in \mathbb{Q}$. We will prove that G_r is open and dense, then $G := \cap_{r\in\mathbb{Q}} G_r$ is a residual set and $(\psi, f) \in G$ implies that $f(\Lambda_\psi) \cap \mathbb{Q} = \emptyset$ and so, int $f(\Lambda_\psi) = \emptyset$.

First we prove that G_r is open. Let $(\psi, f) \in G_r$. By compactness we have $d(f(\Lambda_\psi), r) = \varepsilon$. Take an open neighborhood $\mathcal{V} \subset B_{C^1}(f, \varepsilon/3)$ such that there exists $K \in \mathbb{R}$ for which

$$|g(x) - g(y)| \leq K d(x, y),\ \forall g \in \mathcal{V},\ \forall x, y \in M. \tag{3.7}$$

Now, let $\tilde{\mathcal{U}}$ be an open neighborhood of ψ in $\mathcal{U}$ such that $d_{C^0}(\Phi(\tilde{\psi}), \Phi(\psi) < \varepsilon/3K$, $\forall \tilde{\psi} \in \tilde{\mathcal{U}}$, where $\Phi(\cdot) : \Lambda \to M$ is the conjugate map from the hyperbolic continuation of Λ, according to [37, Theorem 8.3]. Thus, $d_{\text{Hausd}}(\Lambda_\psi, \Lambda_{\tilde{\psi}}) < \varepsilon/3K$. Given $\tilde{x} \in \Lambda_{\tilde{\psi}}$ take $x \in \Lambda_\psi$ so that $d(x, \tilde{x}) < \varepsilon/3K$. By (3.7), we have for $g \in \mathcal{V}$:

$$d(g(\tilde{x}), r) \geq d(f(x), r) - d(f(x), g(x)) - d(g(x), g(\tilde{x})) \geq \varepsilon - \varepsilon/3 - K \cdot \varepsilon/3K = \varepsilon/3.$$

Therefore, $r \notin g(\Lambda_{\tilde{\psi}})$, for every $(\tilde{\psi}, g) \in \tilde{\mathcal{U}} \times \mathcal{V}$, and thus G_r is open.

Next we prove that G_r is dense. Let $(\psi_0, f) \in \mathcal{U} \times C^1(M, \mathbb{R})$. We approximate ψ_0 in C^1-topology by a C^∞-diffeomorphism ψ. Thus, the laminations $\mathcal{F}^s_\psi$, $\mathcal{F}^u_\psi$ of Λ_ψ are $C^{1+\varepsilon}$ and can be extended to a neighbourhood of Λ_ψ as $C^{1+\varepsilon}$ invariant foliations, as we can see in [34]. Consider a finite Markov partition $\{P_i\}_{i=1}^N$ with small diameter, so that in coordinates (ξ_i), we have that f is C^1-close to a $\tilde{f}$, where in these coordinates $\tilde{f}(x, y) = a_i x + b_i y + c_i$ in a neighborhood of P_i, and the foliations $\mathcal{F}^s_\psi(z) \cap P_i$ and $\mathcal{F}^s_\psi(z) \cap P_i$ of ψ restricted to $z \in P_i$ are C^1-close to the linear linear foliations of P_i given respectively by straight lines parallel to $E^s_i = (1, \nu_i)$ and $E^u_i = (\mu_i, 1)$. Now, up to a C^1-perturbation of ψ, we can assume that the stable (unstable) foliation of ψ in coordinates(ξ_i) in the pieces P_i is the foliation by straight lines parallel to E^s_i (E^u_i). Indeed, changing the coordinates to the coordinates given by stable and unstable foliations, the diffeomorphism on P_i has the form $\psi(x, y) = (g_i(x), h_i(y))$. We replace the foliations $F^s_\psi \cap P_i$ and $F^u_\psi \cap P_i$ with the foliations given in each P_i respectively for the linear foliations parallel to E^s_i and E^u_i, and define $\tilde{\psi}(x, y) = (g_i(x), h_i(y))$ in the coordinates given by these linear foliations. Since the $F^{s,u}_\psi \cap P_i$ is C^1-close to the linear foliation parallel to $E^{s,u}_i$, the map $\tilde{\psi}$ is C^1-close to ψ. From now on, we shall assume that stable (unstable) foliation of ψ restricted to P_i is the linear foliation parallel to E^s_i (E^u_i).

Thus, in the system of coordinates given by linear foliations parallel to $E_i^{s,u}$ in each P_i, we can write $\psi(x,y) = (g_i(x), h_i(y))$, and in the coordinates (ξ_i), we have that $\Lambda_\psi \cap P_i$ is $(1,\nu_i)K_i^s + (\mu_i, 1)K_i^u$, where $K_i^{s,u}$ are regular Cantor sets in $\mathbb{R}$.

In this setting, $r \in \tilde{f}(\Lambda_\psi \cap P_i)$ if and only if,

$$(r - c_i) \in (a_i + b_i\nu_i)K_i^s + (a_i\mu_i + b_i)K_i^u.$$

Since C^1-stable intersections of regular Cantor sets do not exist, according to [29], there are C^1-perturbations of these Cantor sets obtained by replacing in P_i the expression $\psi(x,y) = (g_i(x), h_i(y))$ with a C^1-close expression $\hat{\psi}(x,y) = (\hat{g}_i(x), \hat{h}_i(y))$ and keeping the linear foliations, so that $(r - c_i) \notin (a_i + b_i\nu_i)\hat{K}_i^s + (a_i\mu_i + b_i)\hat{K}_i^u$. Thus, G_r is dense. This concludes the proof of the Claim.

If a diffeomorphism φ of M has a horseshoe Λ, we say that an open set $U \subset M$ is *good* for Λ if $\Lambda \subset U$ and Λ is the maximal invariant of $\overline{U}$, i.e., $\Lambda = \bigcap_{n\in\mathbb{Z}} \varphi^n(\overline{U})$. This condition is equivalent to the existence of an open set V such that $\overline{U} \subset V$ and Λ is the maximal invariant of V, and then is an C^1-open condition, that is, there is an open subset $\mathcal{U} \subset \text{Diff}^1(M)$ of hyperbolic continuation of Λ, where $\Lambda_\psi = \bigcap_{n\in\mathbb{Z}} \varphi^n(\overline{U}) = \bigcap_{n\in\mathbb{Z}} \varphi^n(V)$, for every $\psi \in \mathcal{U}$.

Finally, we fix a countable basis of open sets of M. Let Λ be a horseshoe associated with φ, then we can take a good open U for Λ which is a finite union of open sets of the basis. Given an open set $\tilde{U}$ of M, there is a generic set of $(\varphi, f) \in \text{Diff}^1(M) \times C^1(M,\mathbb{R})$, such that if the maximal invariant of φ in $\tilde{U}$ is a horseshoe $\tilde{\Lambda}$, and $\tilde{U}$ is good for it, then $\text{int} f(\tilde{\Lambda}) = \emptyset$. Since there are only countable many finite unions of open sets in the fixed basis, by Baire's theorem, we finish the proof of the theorem. □

3.3 Beginning of spectra

The study of the geometry of the classical Markov and Lagrange spectra began with the study of the first accumulation point of this set, in 1879 by Markov [23]. In this paper, Markov showed that the set of numbers less than 3 in the Markov and Lagrange spectra is countable and discrete, with 3 as its only limit point.

A proof of this result can be found on the first chapter of the book [6]. Though Cusick and Flahive used continued fractions, the ideas go back to

Markov. The main tool in their proof is the very following special identity involving continued fractions, as we can see in (2.2):

$$[2;1,1,\gamma]+[0;2,\gamma]=3, \text{ for any } \gamma \geq 1. \tag{3.8}$$

Using this identity and some corollaries, they do a renormalization process in the sequences $\underline{\theta} \in \Sigma$ associated with Markov numbers less than 3, and obtained that $\underline{\theta}$ must be periodic and this finished the proof.

In a second paper, Markov [24] noticed a relationship between certain binary quadratic forms and rational approximations of certain irrational numbers. This allowed him to make a more detailed characterization of the spectra until the number 3. More precisely, Markov showed that

$$L \cap (-\infty, 3) = M \cap (-\infty, 3) = \{k_1 < k_2 < k_3 < \cdots\},$$

where $k_n := \sqrt{9 - \dfrac{4}{m_n^2}}$ and m_n is the n-th Markov number, where a Markov number is the largest coordinate of a Markov triple (x, y, z), i.e, an integral solution of $x^2 + y^2 + z^2 = 3xyz$. In [2], Bombieri also gave a interesting proof of this theorem, just using theory of continued fractions.

Next, let us define:

Definition 3.1. Given a closed set $X \subset \mathbb{R}$ bounded from below, we define the **beginning** of X as been the set $X \cap (-\infty, \inf X')$, that is, the set of points before the first accumulation point of X.

Thus, Markov's papers consist of a complete study of the beginning of the classical Markov and Lagrange spectra. In the dynamical context, Moreira [30] proved that typically the minima of the corresponding Markov and Lagrange dynamical spectra coincide and is an isolated point given by a periodic orbit. Therefore, a natural question is:

Question: How is the behaviour of the beginning of dynamical spectra?

By Theorem 1 and 2, we know that the isolated points in the beginning of the Markov and Lagrange spectra are associated respectively with periodic and eventually periodic points of the dynamics.

In this section, we analyse the beginning of the dynamical spectra, and we see that every possible beginning could occur in both the spectra in a robust form in the pair (dynamics, functions), and thus we cannot expect any general (in a generic context) answer for the previous question about the beginning of the dynamical spectra.

3.3.1 Equality of spectra and finite beginning spectra

In this subsection, we build an open set in the pair (dynamic, function) where both spectra are equal and there exists just one point before the first accumulation point. Using the same kind of argument, given n a natural number, we build another open set in the pair where both spectra have the same n points before the first accumulation point.

Let k be odd. Define $g_k : [0, 1/k] \cup [2/k, 3/k] \cup \cdots \cup [(k-1)/k, 1] \to [0, 1]$ to be the expanding map given by $g_k(x) := kx - j$, if $x \in [j/k, (j+1)/k]$, for $j = 0, 2, \cdots, k-1$. See Figure 3.2. Denote the inverse branches of g_k by $h_{k,j} : [0, 1] \to [j/k, (j+1)/k]$, where $h_{k,j}(y) = (y+j)/k$, for every $j = 0, 2, \cdots, k-1$.

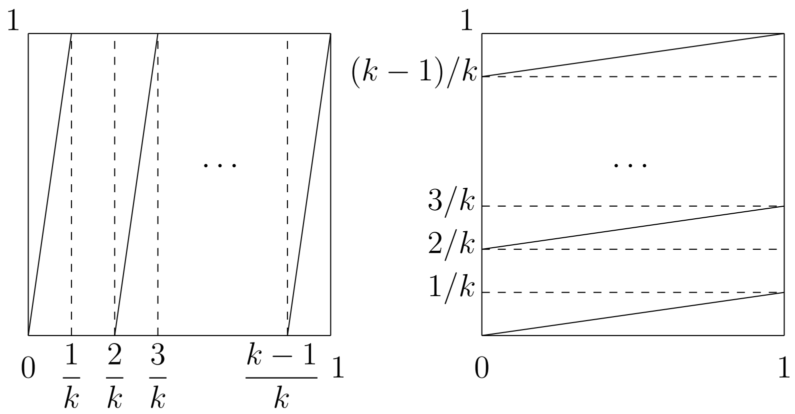

Figure 3.2: On the left the graph of g_k and on the right the graphs of $h_{k,j}$.

Given the following two vertical strips $R_0 = [0, 1/3] \times [0, 1]$ and $R_2 = [2/3, 1] \times [0, 1]$ in the unit square $Q = [0, 1] \times [0, 1]$, define the diffeomorphism on its image $\varphi_0 : R_0 \cup R_2 \to \varphi_0(R_0) \cup \varphi_0(R_2) \subset Q$, given by

$$\varphi_0(x, y) = \begin{cases} (g_3(x), h_{3,0}(y)), & \text{if } x \in [0, 1/3] \\ (g_3(x), h_{3,2}(y)), & \text{if } x \in [2/3, 1] \end{cases}. \tag{3.9}$$

As in the Smale's horseshoe this map φ_0 can be extended to a C^2–diffeomorphism on all of M^2(where $M^2 = \mathbb{S}^2$ or $\mathbb{T}^2$), which gives a maximal invariant horseshoe $\Lambda_0 = K_3 \times K_3$, where $K_3 = \bigcap_{n \geq 0} g_3^{-n}([0, 1/3] \cup [2/3, 1])$. Also:

i) the horizontal(vertical) lines are the local unstable(stable) manifolds of points in Λ_0, that is, $\operatorname{comp}_p(W^u_{\Lambda_0}(p) \cap Q)$ is a vertical line and $\operatorname{comp}_p(W^u_{\Lambda_0}(p) \cap Q)$ is a horizontal line;

ii) the expanding maps g_s^0, g_u^0 in the definitions of (uns)stable Cantor sets $K_s^{\varphi_0} = K_u^{\varphi_0} = K_3$ are increasing ($(g_{s,u}^0)' = 3 > 1$);

iii) the rectangles $\{R_0, R_2\}$ are a Markov partition for Λ_0, that induce a coding with $\Sigma_2 = \{0,2\}^{\mathbb{Z}}$. This coding is given by a conjugate map $\Pi : \Sigma_2 \to \Lambda_0$, such that for $\theta = (a_n) \in \Sigma_2$, we have $\Pi(\theta) = p$, where $f^j(p) \in R_{a_j}$, for all j.

Now, define the function $f_0 : [0,1] \times [0,1] \to \mathbb{R}$, given by $f_0(x,y) = x + y$. Thus, we have $\langle \nabla f_0(z), e_z^{s,u} \rangle = 1$, where $e_z^s = (0,1) \in T_z^1(\mathrm{comp}_p(W^s_{\Lambda_0}(p) \cap Q))$ or $e_z^s = (1,0) \in T_z^1(\mathrm{comp}_p(W^u_{\Lambda_0}(p) \cap Q))$, for any $p \in \Lambda_0$.

Consider $\mathcal{U} \subset \mathrm{Diff}^2(M^2)$ a neighborhood of φ_0, where for every $\varphi \in \mathcal{U}$ we have a hyperbolic continuation of Λ_0 to a Λ_φ associated with φ. Since the foliations maps $(\varphi, x) \to \mathcal{F}_\varphi^{s,u}(x)$ are C^1, according to Theorem 2.1, we can shrink $\mathcal{U}$, so that:

a) we have a nearby Markov partition $\{R_1^\varphi, R_2^\varphi\}$ for the corresponding Λ_φ, that induces a coding $\Pi : \Sigma_2 \to \Lambda_\varphi$, as in Section 2.1. For each point $p \in \Lambda_\varphi$ denote its kneading sequence by $\theta_\varphi = (\cdots, a_{-1}; a_0, a_1, \cdots)_\varphi$, whenever $\Pi(\theta_\varphi) = p$

b) the expanding maps g_φ^u and g_φ^s in the definitions of resp. K_φ^u and K_φ^s are increasing, more specifically, $7/2 > (g_\varphi^{u,s})' > 5/2 > 0$

c) for a C^1-neighborhood $\mathcal{V}$ of f_0, we have $3/2 > \langle \nabla f(z), e_z^{s,u} \rangle > 1/2$, where $e_z^s \in T_z^1(\mathrm{comp}_p(W^s_{\Lambda_\varphi}(p) \cap Q_\varphi))$ orientated from down to up or $e_z^s \in T_z^1(\mathrm{comp}_p(W^u_{\Lambda_\varphi}(p) \cap Q_\varphi))$ orientated from left to right, for any $p \in \Lambda_\varphi$.

Thus, for every $(\varphi, f) \in (\mathcal{U}, \mathcal{V})$, we have some constants $c_1, c_2, c_3, c_4 > 0$ such that the following estimates hold:

$$\begin{aligned} \frac{c_1}{2}|I_\varphi^u(a_0,...,a_n)| &< f(\underline{\theta}_1; a_0, ..., a_n, 2, \underline{\theta}_2) - f(\underline{\theta}_1; a_0, ..., a_n, 0, \underline{\theta}_2') \\ &< \frac{3c_2}{2}|I_\varphi^u(a_0,...,a_n)|, \end{aligned} \tag{3.10}$$

$$\begin{aligned} \frac{c_3}{2}|I_\varphi^s(a_{-1},...,a_m)| &< f(\underline{\theta}_3, 2, a_m, ..., a_{-1}; \underline{\theta}_4) - f(\underline{\theta}_3', 0, a_m, ..., a_{-1}; \underline{\theta}_4) \\ &< \frac{3c_4}{2}|I_\varphi^s(a_{-1},...,a_m)|, \end{aligned} \tag{3.11}$$

for every $\underline{\theta}_1, \underline{\theta}_3, \underline{\theta}_3' \in \Sigma_2^-$ and $\underline{\theta}_4, \underline{\theta}_2, \underline{\theta}_2' \in \Sigma_2^+$.

In this setting we are able to prove the following proposition:

Proposition 2. *There are open neighborhoods $\mathcal{U}_1 \subset \mathrm{Diff}^2(\mathbb{S}^2)$ of φ_0 and $\mathcal{V}_1 \subset C^1(\mathbb{S}^2;\mathbb{R})$ of f_0, such that $L(f,\Lambda_\varphi) = M(f,\Lambda_\varphi)$, for every $(\varphi, f) \in (\mathcal{U}_1, \mathcal{V}_1)$. Moreover, the beginning of these set has only one point.*

Proof. Consider $\mathcal{U}_1 \subset \mathrm{Diff}^2(\mathbb{S}^2)$ and $\mathcal{V}_1 \subset C^1(\mathbb{S}^2;\mathbb{R})$ as in the above discussion. Let $(\varphi, f) \in (\mathcal{U}_1, \mathcal{V}_1)$ and $m \in M(f,\Lambda_\varphi)$, where $m = \sup_n f(\sigma^n(\underline{\theta})) = f(\underline{\theta})$ and $\underline{\theta} = (a_n)_\varphi \neq (\bar{0})_\varphi$ is associated to $p \in \Lambda_\varphi$. In order to prove that $L(f,\Lambda_\varphi) = M(f,\Lambda_\varphi)$, it is sufficient to prove that $m \in L(f,\Lambda_\varphi)$. To this end, we analyse $\underline{\theta}$ in four cases:

I) $\underline{\theta} \notin W^s_\sigma(\bar{0}) \cup W^u_\sigma(\bar{0})$;

II) $\underline{\theta} \in W^s_\sigma(\bar{0}) \cap W^u_\sigma(\bar{0})$;

III) $\underline{\theta} \in W^u_\sigma(\bar{0}) \setminus W^s_\sigma(\bar{0})$;

IV) $\underline{\theta} \in W^s_\sigma(\bar{0}) \setminus W^u_\sigma(\bar{0})$.

In case I), for every n natural number, we set

$$B^1_n := (0, 0_{r_n}, a_{-n}, \cdots, a_{-1}; a_0, \cdots a_n, 0_{s_n}, 0),$$

where r_n is the number of zeros between a_{-n} and the next symbol 2 on the left of a_{-n} in the sequence $\underline{\theta}$ and s_n is the number of zeros between a_n and the next symbol 2 on the right of a_n in the sequence $\underline{\theta}$. Define, $\underline{\theta}^{(1)} = (\bar{0}; B^1_1, B^2_2, \cdots)_\varphi$, where $\Pi(\underline{\theta}^{(1)}) \in \Lambda_\varphi$. Given n and $j \in \{-n, \cdots n\}$, we call $k_1(j,n)$ the position in $\underline{\theta}^{(1)}$ of a_j with respect to the block B^1_n. By (3.10) and (3.11), we get that:

$$\begin{aligned}
f(\sigma^{k_1(j,n)}(\underline{\theta}^{(1)})) &= f(\cdots, B^1_{n-1}, \underbrace{0, 0_{r_n}, a_{-n}, \cdots, a_{j-1}; a_j, \cdots a_n, 0_{s_n}, 0}_{B^1_n}, B^1_{n+1}, \cdots) \\
&< f(\cdots, B^1_{n-1}, 0, 0_{r_n}, a_{-n}, \cdots, a_{j-1}; \underbrace{a_j, \cdots a_n, 0_{s_n}, 2, \cdots}_{\underline{\theta}^+_j}) \\
&< f(\underbrace{\cdots, 2, 0_{r_n}, a_{-n}, \cdots, a_{j-1}}_{\underline{\theta}^-_j}; \underbrace{a_j, \cdots a_n, 0_{s_n}, 2, \cdots}_{\underline{\theta}^+_j}) = f(\sigma^j(\underline{\theta})) \leq m,
\end{aligned}$$

where $\sigma^l(\underline{\theta}) = (\cdots, a_{j-1}; a_j, a_{j+1}, \cdots) := (\underline{\theta}^-_l; \underline{\theta}^+_l)$, for every $l \in \mathbb{Z}$. By continuity, $\lim_{n\to\infty} f(\sigma^{k_1(0,n)}(\underline{\theta}^{(1)})) = f(\underline{\theta}) = m$. Therefore, in this case

$$m = \limsup_{n\to\infty} f(\sigma^n(\underline{\theta}^{(1)})) \in L(f,\Lambda_\varphi).$$

In case II), if $\underline{\theta} \in W^s_\sigma(\bar{0}) \cap W^u_\sigma(\bar{0})$, then $\underline{\theta} = (\bar{0}, a_r, \cdots, a_{-1}; a_0, \cdots, a_s, \bar{0})_\varphi$. For every n, we define:

$$B^2_n := (0_n, a_r, \cdots, a_{-1}; a_0, \cdots a_s, 0_n),$$

and $\underline{\theta}^{(2)} = (\bar{0}; B^2_1, B^2_2, \cdots)_\varphi$, where $\Pi(\underline{\theta}^{(2)}) \in \Lambda_\varphi$. Given n and $j \in \{r, \cdots, s\}$, we call $k_2(j,n)$ the position in $\underline{\theta}^{(2)}$ of a_j with respect to the block B^2_n. By continuity, $\lim_{n\to\infty} f(\sigma^{k_2(j,n)}(\underline{\theta}^{(2)})) = f(\sigma^j(\underline{\theta}))$, for every $j \in \{r, \cdots, s\}$. Therefore, in the second case

$$m = \limsup_{n\to\infty} f(\sigma^n(\underline{\theta}^{(2)})) \in L(f, \Lambda_\varphi).$$

In case III), if $\underline{\theta} \in W^u_\sigma(\bar{0}) \backslash W^s_\sigma(\bar{0})$, then $\underline{\theta} = (\bar{0}, a_r, \cdots, a_{-1}; a_0, \cdots, a_s, \cdots)_\varphi$. Given $n \in \mathbb{N}$, we define s_n as the number of zeros between a_n and the next symbol 2 on the right of a_n in the sequence $\underline{\theta}$, and we set

$$B^3_n := (0_{N_n}, a_r, \cdots, a_{-1}; a_0, \cdots, a_n, 0_{s_n}, 0),$$

where N_n is big enough so that for every $j \in \{r, \cdots, n\}$, we get:

$$|I^s_\varphi(a_{j-1}, \cdots, a_r, 0_{N_n})| < \frac{c_1}{3c_4} |I^u_\varphi(a_j, \cdots, a_n, 0_{s_n})|. \tag{3.12}$$

We define $\underline{\theta}^{(3)} = (\bar{0}; B^3_1, B^3_2, \cdots)_\varphi$, where $\Pi(\underline{\theta}^{(3)}) \in \Lambda_\varphi$. Let $n \in \mathbb{N}$ and $j \in \{r, \cdots, n\}$, we denote by $k_3(j,n)$ the position in $\underline{\theta}^{(3)}$ of a_j with respect to the block B^3_n. By (3.10), (3.11) and using (3.12), we have:

$$
\begin{aligned}
f(\sigma^{k_3(j,n)}(\underline{\theta}^{(3)})) &= f(\cdots, B^3_{n-1}, \underbrace{0_{N_n}, a_r, \cdots, a_{j-1}; a_j, \cdots a_n, 0_{s_n}, 0}_{B^3_n}, B^3_{n+1}, \cdots) \\
&< f(\cdots, B^3_{n-1}, 0_{N_n}, a_r, \cdots, a_{j-1}; \underbrace{a_j, \cdots a_n, 0_{s_n}, 2, \cdots}_{\underline{\theta}^+_j}) - \frac{c_1}{2} |I^u_\varphi(a_j, ..., a_n, 0_{s_n})| \\
&< f(\underbrace{\bar{0}, a_r, \cdots, a_{j-1}}_{\underline{\theta}^-_j}; \underbrace{a_j, \cdots a_n, 0_{s_n}, 2, \cdots}_{\underline{\theta}^+_j}) + \frac{3c_4}{2} |I^s_\varphi(a_{j-1}, ..., a_r, 0_{N_n})| - \\
&- \frac{c_1}{2} |I^u_\varphi(a_j, ..., a_n, 0_{s_n})| \\
&< f(\sigma^j(\underline{\theta})) \leq m,
\end{aligned}
$$

where $\sigma^j(\underline{\theta}) = (\cdots, a_{j-1}; a_j, a_{j+1}, \cdots) =: (\underline{\theta}^-_j; \underline{\theta}^+_j)$, for every $l \in \mathbb{Z}$. By continuity, $\lim_{n\to\infty} f(\sigma^{k_3(0,n)}(\underline{\theta}^{(3)})) = f(\underline{\theta}) = m$. Therefore, in this case

$$m = \limsup_{n\to\infty} f(\sigma^n(\underline{\theta}^{(3)})) \in L(f, \Lambda_\varphi).$$

For case IV), the analysis is analogous to the previous case.

Therefore it follows that $M(f,\Lambda_\varphi)=L(f,\Lambda_\varphi)$.

Finally, we show the following:

Claim: $\inf M'(f,\Lambda_\varphi)=m_0':=m_{f,\Lambda_\varphi}((\bar{0};2,\bar{0})_\varphi)$ and $M(f,\Lambda_\varphi)\cap(-\infty,m_0')=\{f((\bar{0})_\varphi)\}$. Indeed, by (3.10) and (3.11), we have that

$$m_0':=m_{f,\Lambda_\varphi}((\bar{0};2,\bar{0})_\varphi)=\sup_{n\in\mathbb{Z}} f(\sigma^n((\bar{0};2,\bar{0})_\varphi)=\max\{f((\bar{0};2,\bar{0})_\varphi),f((\bar{0},2;\bar{0})_\varphi)\}.$$

We can assume that $m_0'=f((\bar{0};2,\bar{0})_\varphi)$. Define $\theta_n:=(\bar{0};2,0_n,2,\bar{0})_\varphi$, thus

$$\lim_{n\to\infty} m_{f,\Lambda_n}(\theta_n)=\lim_{n\to\infty} f(\theta_n)=m_0'.$$

Hence, m_0' is an accumulation point of $M(f,\Lambda_\varphi)$. Moreover, let $\underline{\hat{\theta}}=(b_n)_\varphi$ such that there are two integers $m>l$ with $b_m=b_l=2$ and $b_{l+1}=\cdots=b_{m-1}=0$. By (3.10) and (3.11), we get:

$$m_{f,\Lambda_\varphi}(\underline{\hat{\theta}})\geq f(\sigma^l(\hat{\theta}))=f(\cdots,b_{l-1};2,0,\cdots,0,2,b_{m+1},\cdots)\geq f((\bar{0};2,\bar{0})_\varphi)=m_0'.$$

Therefore, $M(f,\Lambda_\varphi)\cap(-\infty,m_0')=\{f((\bar{0})_\varphi)\}$ and so m_0' is the first accumulation point of $M(f,\Lambda_\varphi)$. This concludes the proof of the Claim and thus, we finished the proof of proposition. □

Now, using the same ideas as before, given a natural number n, we build an example of neighbourhood in the pair (dynamics, function), where the beginnings of the both dynamical spectra coincide and is a set having n elements.

Proposition 3. *Let n be a positive integer. Then, there are open neighborhoods $\mathcal{U}_n\subset \mathrm{Diff}^2(\mathbb{S}^2)$ and $\mathcal{V}_n\subset C^1(\mathbb{S}^2;\mathbb{R})$, such that $L(f,\Lambda_\varphi)$ and $M(f,\Lambda_\varphi)$ have the same beginning with exactly n elements, for every $(\varphi,f)\in(\mathcal{U}_n,\mathcal{V}_n)$.*

Proof. Given a natural number n, we define in the square $Q=[0,1]\times[0,1]$ the following vertical strips: $Q_i=\left[\dfrac{i}{4n-1},\dfrac{i+1}{4n-1}\right]$, for $i=0,2,\cdots 4n-2$. Now, as we did before in (3.9), we define a diffeomorphism on its image $\psi_n:R_0\cup\cdots\cup R_{4n-2}\to\psi_n(R_0)\cup\cdots\cup\psi_n(R_{4n-2})\subset Q$, given by:

$$\psi_n(x,y):=(g_{4n-1}(x),h_{4n-1,i}(y)),\ \text{when}\ x\in R_i.$$

We can extend ψ_n to a C^2 diffeomorphism on all of $\mathbb{S}^2$, which gives a maximal invariant horseshoe $\Lambda_0 = K_{2n} \times K_{2n}$ associated with a full shift $\sigma : \Sigma_{2n} \to \Sigma_{2n}$ by the Markov partition $\{R_0, R_2, \cdots, R_{4n-2}\}$, where

$$K_{2n} = \bigcap_{k\geq 0} g_{4n-1}^{-k}\left(\left[0, \frac{1}{4n-1}\right] \cup \cdots \cup \left[\frac{4n-2}{4n-1}, 1\right]\right) \text{ and } \Sigma_{2n} := \{0, 2, \cdots, 4n-2\}^{\mathbb{Z}}.$$

Now, we define n subhorseshoes of Λ_0:

$$\Lambda_0^i := \Lambda_0(i, i+2) = \bigcap_{k\in\mathbb{Z}} \psi_n^k(R_i \cup R_{i+2}), \quad \text{for each } i = 0, 4, \cdots, 4n-4.$$

Note that each subhorseshoe above is associated with a full shift of two symbols $\sigma : \Sigma(i, i+2) \to \Sigma(i, i+2)$, where $\Sigma(i, i+2) := \{i, i+2\}^{\mathbb{Z}}$. For each $i = 0, 4, \cdots, 4n-4$, let

$$C_0^i := (R_i \cap \psi_n(R_i)) \cup (R_{i+2} \cap \psi_n(R_{i+2})) \cup (R_i \cap \psi_n(R_{i+2})) \cup (R_{i+2} \cap \psi_n(R_i)).$$

Note that $\Lambda_0^i \subset C_i$, for every $i = 0, 4, \cdots, 4n-4$. Define $f_0 : Q \to \mathbb{R}$ satisfying:

- $f_0(x, y) = x + y + c_i$, for every $(x, y) \in C_0^i$, where c_i is a constant to be chosen latter;
- $f_0(x, y) > 2\max\{f_0(x, y) : (x, y) \in C_0^0 \cup C_0^4 \cup \cdots \cup C_0^{4n-4}\}$, for every $(x, y) \in R_k \cap \psi_n(R_l)$, where $R_k \cap \psi_n(R_l) \subset Q \setminus (C_0^0 \cup C_0^4 \cup \cdots \cup C_0^{4n-4})$.

We take neighborhoods $\mathcal{U}_n \subset \mathrm{Diff}^2(\mathbb{S}^2)$ neighbourhood of ψ_n and $\mathcal{V}_n \subset C^1(S^2; \mathbb{R})$ of f_0 such that for the pair $(\psi, f) \in \mathcal{U}_n \times \mathcal{V}_n$, we have:

i) By the proof of Proposition 2, for every $i = 0, 4, \cdots, 4n-4$: $L(f, \Lambda_\varphi^i) = M(f, \Lambda_\varphi^i)$ and $\{r_i(f, \psi)\} := M(f, \Lambda_\varphi^i) \cap (-\infty, \inf M'(f, \Lambda_\varphi^i))$, where Λ_φ^i is the hyperbolic continuation of Λ_0^i;

ii) $f(x, y) > \max\{f(x, y) : (x, y) \in C_\psi^0 \cup C_\psi^4 \cup \cdots \cup C_\psi^{4n-4}\}$, for every $(x, y) \in R_k^\psi \cap \psi(R_l^\psi)$ with $R_k^\psi \cap \psi(R_l^\psi) \subset Q \setminus (C_\psi^0 \cup C_\psi^4 \cup \cdots \cup C_\psi^{4n-4})$, where $\{R_0^\varphi, R_2^\varphi, \cdots R_{4n-2}^\varphi\}$ is a Markov partition for Λ_φ and C_ψ^i is analogously defined, given by the hyperbolic continuation of Λ.

Since $r_i(f_0, \psi_n) - \inf M'(f_0, \Lambda_0^i) = \dfrac{2}{4n-1}$, we finally take

$$c_i := r_0(f_0, \psi_n) - r_i(f_0, \psi_n) + \frac{2i}{4n(4n-1)}, \text{ for i} = 0, 4, \cdots 4\text{n} - 4.$$

By *ii*), we get that

$$M(f,\Lambda_\varphi)\cap(-\infty,\inf M'(f,\Lambda_\varphi))\subset M(f,\Lambda_\varphi^0)\cup M(f,\Lambda_\varphi^4)\cup\cdots\cup M(f,\Lambda_\varphi^{4n-4}).$$

Hence, possibly reducing $\mathcal{U}_n$ and $\mathcal{V}_n$, we have:

$$\begin{aligned} M(f,\Lambda_\varphi)\cap(-\infty,\inf M'(f,\Lambda_\varphi)) &= L(f,\Lambda_\varphi)\cap(-\infty,\inf L'(f,\Lambda_\varphi)) = \\ &= \{r_0(f,\psi)<\cdots<r_{4n-4}(f,\psi)\}, \end{aligned}$$

for every $(\psi,f)\in\mathcal{U}_n\times\mathcal{V}_n$. This concludes the proof of the proposition. □

3.3.2 Infinite beginning in Markov spectrum

In this subsection, we build an open set in the pair (dynamics, function) such that the beginning of the dynamical Markov spectrum associate with elements in this neighbourhood is an infinite countable set. Moreover, we also answer negatively a question asked after the Theorem 3, i.e., if $M'(f,\Lambda)=M''(f,\Lambda)$ in some generic context. More specifically, we have the following:

Proposition 4. *There are open neighborhoods $\hat{\mathcal{U}}\subset \mathrm{Diff}^2(\mathbb{S}^2)$ and $\hat{\mathcal{V}}\subset C^1(\mathbb{S}^2;\mathbb{R})$, such that $M(f,\Lambda_\varphi)$ has an infinite beginning, for every $(\varphi,f)\in(\hat{\mathcal{U}},\hat{\mathcal{V}})$. Moreover, $M'(f,\Lambda_\varphi)\neq M''(f,\Lambda_\varphi)$ and $L(f,\Lambda_\varphi)$ has a finite beginning, for every $(\varphi,f)\in(\hat{\mathcal{U}},\hat{\mathcal{V}})$.*

Proof. As in the previous subsection, using g_5 and $h_{5,j}$ for $j=0,2,4$, we define a map $\varphi_0\in\mathrm{Diff}^2(\mathbb{S}^2)$ with an associated horseshoe Λ_0 which has the symbolic representation a full shift in $\Sigma_3=\{1,2,3\}^{\mathbb{Z}}$. Now, take a C^2–neighborhood $\hat{\mathcal{U}}$ of φ, where we have hyperbolic continuation of Λ_0 and we have the same symbolic representation (gave by an associated Markov partition).

Define f_0 using symbolic representation, as each rectangle $R_{a_0}^{\varphi_0}\cap\varphi_0^{-1}(R_{a_1}^{\varphi_0})$ is associated in symbolic language to the cylinder $(a_0^*,a_1)_\varphi:=R_{\varphi_0}(a_0^*,a_1)$, where $*$ indicates the zero position in the kneading sequence. We put:

$$f_0(3^*,3)_{\varphi_0}<f_0(1^*,1)_{\varphi_0}<f_0(1^*,2)_{\varphi_0}<f_0(2^*,2)_{\varphi_0}<f_0(2^*,3)_{\varphi_0}<c,$$

for a fixed constant c, otherwise $f_0(a_0^*,a_1)_{\varphi_0}>c_1>c$. Moreover, we can define f_0 in such a way that $\langle\nabla f_0(p_0),e^s_{\varphi_0}\rangle>a>0$, where $p_0\in\Lambda_0$ has the kneading sequence $(\bar{2};\bar{3})_{\varphi_0}$ and $e^s_{\varphi_0}\in T^1W^s(p_0)$ is orientated from down to up. Note that we can extend f_0 to a C^1 function on $\mathbb{S}^2$. Now, possibly shrinking $\hat{\mathcal{U}}$, we consider a C^1 neighborhood $\hat{\mathcal{V}}$ of f_0, where all these above inequalities hold for every $(\varphi, f) \in (\hat{\mathcal{U}}, \hat{\mathcal{V}})$, with respect to their hyperbolic continuations to φ. Thus,

$$\begin{aligned}\Lambda_c(f, \Lambda_\varphi) &= \bigcap_{n\in\mathbb{Z}} \varphi^n(\{x \in \Lambda_\varphi : f(x) \le c\}) \\ &= \Pi\left(\{\bar{1}, \bar{2}, \bar{3}, \mathcal{O}(\bar{1}; \bar{2}), \mathcal{O}(\bar{2}; \bar{3})\} \cup \{\mathcal{O}(\bar{1}; 2_n, \bar{3}) : n \in \mathbb{N}\}\right),\end{aligned}$$

where $\mathcal{O}(\underline{\theta}) = \{\sigma^n(\underline{\theta}) : n \in \mathbb{Z}\}$ indicates the orbit of $\underline{\theta} \in \Sigma_3$. Note that $m_{f,\Lambda_\varphi}((\bar{2}; \bar{3})_\varphi) = f(\bar{2}; \bar{3})_\varphi$ and $m_{f,\Lambda_\varphi}((\bar{1}; 2_n, \bar{3})_\varphi) = f(\bar{1}2_n; \bar{3})_\varphi$. For $n > n_0$, $(\bar{1}, 2_n; \bar{3})_\varphi$ belongs to the monotonicity region of f in the neighborhood of $(\bar{2}; \bar{3})_\varphi$ in $W^s_\varphi(p_\varphi)$, where $p_\varphi = \Pi((\bar{2}; \bar{3})_\varphi)$. Since the Cantor stable is defined by an increasing map g^s_φ, we have $m_{f,\Lambda_\varphi}((\bar{2}; \bar{3})_\varphi) > m_{f,\Lambda_\varphi}((\bar{1}; 2_n, \bar{3})_\varphi)$. Therefore, $\inf M'(f, \Lambda_\varphi) = m_{f,\Lambda_\varphi}((\bar{2}; \bar{3})_\varphi)$ and

$$M(f, \Lambda_\varphi) \cap (-\infty, \inf M'(f, \Lambda_\varphi)) \supset \{m_{f,\Lambda_\varphi}((\bar{1}; 2_n, \bar{3})_\varphi) : n > n_0\}.$$

Moreover, for $(\varphi, f) \in \hat{\mathcal{U}} \times \hat{\mathcal{V}}$ we have $M'(f, \Lambda_\varphi) \cap (-\infty, c) = m_{f,\Lambda_\varphi}((\bar{2}; \bar{3})_\varphi)$, and any other different accumulation must be bigger than c_1. Therefore, $M'(f, \Lambda_\varphi) \neq M''(f, \Lambda_\varphi)$. Finally, by Theorem 1, since $Per_\varphi(\Lambda_c(f, \Lambda_\varphi)) = \Pi(\{\bar{1}, \bar{2}, \bar{3}\})$, we have that $L(f, \Lambda_\varphi) \cap (-\infty, c) = \{f(\bar{1}), f(\bar{2}), f(\bar{3})\}$. Hence, in these neighbourhoods the dynamical spectra have different beginnings. □

3.3.3 Infinite beginning in a conservative Lagrange spectrum

In [7], Davenport and Schmidt have shown an extension of Dirichlet's Theorem, and for this reason associated with an irrational number $\alpha = [a_0; a_1, \cdots]$ they defined the value $\gamma(\alpha) = \liminf_{n\to\infty} [0; a_{n+1}, a_{n+2}, \cdots] \cdot [0; a_n, a_{n-1}, \cdots, a_1]$. In a convenient way, related with these values we define the *Dirichlet Spectrum* as the set

$$D = \{\tilde{\gamma}(\underline{\theta}) : \underline{\theta} = (a_n)_{n\in\mathbb{Z}} \in \Sigma := (\mathbb{N}^*)^{\mathbb{Z}}\},$$

where $\tilde{\gamma}(\underline{\theta}) := \limsup_{n\to\infty} [a_{n+1}; a_{n+2}, \cdots] \cdot [a_n; a_{n-1}, \cdots]$.

Kopetzky found in [19] the first accumulation point of this spectrum, namely $\chi := [2; \bar{1}][1; \bar{1}]$. He also showed that before this number, at beginning of this set, there are infinitely many points. More specifically, $\tilde{\gamma}(\overline{2, 1_k}) < \chi$, for every k odd. In this section, we see this set as a dynamical Lagrange spectrum, and we show that this property of having a countably infinite beginning is robust in the pair dynamics/function, in a way precisely described in the following proposition.

Let $\varphi_0 : (0,1)^2 \to (0,1)^2$ be a natural extension of Gauss Map on the interval $(0,1)$, $g : (0,1) \to (0,1)$ given by $g(x) = \{1/x\}$, defined by

$$\varphi_0(x,y) = \left(\left\{ \frac{1}{x} \right\}, \frac{1}{\lfloor 1/x \rfloor + y} \right). \tag{3.13}$$

Recall that $C(2) := \{x = [0; a_1, a_2, \cdots] : 1 \leq a_n \leq 2\}$ and $\Lambda_2 = C(2) \times C(2)$ is a horseshoe associated with φ_0 defined in (3.13). Define the following rectangles $R_1 = \{(x,y) : [0;1,\overline{1,2}] \leq x \leq [0;1,\overline{2,1}],\ [0;2,\overline{1,2}] \leq y \leq [0;1,\overline{2,1}]\}$ and $R_2 = \{(x,y) : [0;2,\overline{1,2}] \leq x \leq [0;2,\overline{2,1}],\ [0;2,\overline{1,2}] \leq y \leq [0;1,\overline{2,1}]\}$. Note that $\{R_1, R_2\}$ is a Markov Partition for Λ_2, which induces a coding with $\Sigma_2 := \{1,2\}^{\mathbb{Z}}$. More specifically, there exists a homeomorphism $\Pi : \Sigma_2 \to \Lambda_2$, given by $\Pi(\cdots, a_{-1}; a_0, a_1, \cdots)_{\varphi_0} = (x,y)$, where $x = [0; a_0, a_1, \cdots]$ and $y = [0; a_{-1}, a_{-2}, \cdots]$, that conjugates $\varphi_0 : \Lambda_2 \to \Lambda_2$ with the shift map $\sigma : \Sigma_2 \to \Sigma_2$.

Given an admissible string $\theta^m_{k=-l} = (a_{-l}, \cdots, a_{-1}; a_0, a_1, \cdots a_m)$, we define the rectangle $R_{\varphi_0}(\theta^m_{k=-l}) := \bigcap\limits_{k=-l}^{m} \varphi_0^{-k}(R_{a_k})$ and

$$\begin{aligned} R_{\Lambda_2}(\theta^m_{k=-l}) &:= \{\Pi(\cdots, b_{-1}, b_0, b_1, \cdots)_{\varphi_0} \in \Lambda_2 |\, b_j = a_j,\ -l \leq j \leq m\} = \\ &= \Lambda_2 \cap \bigcap_{k=-l}^{m} \varphi_0^{-k}(R_{a_k}). \end{aligned}$$

In order to see Dirichlet spectrum as a Lagrange dynamical spectrum, we define $f_0 : (0,1)^2 \to \mathbb{R}$ defined by $f_0(x,y) = 1/xy$ and recall the following lemmas from [19]:

Lemma ([19], Lemma 2)**.** *Let $\underline{\theta} = (a_k)_{k \in Z} \in \Sigma$, if $a_k \geq 3$ for infinitely many $k \in \mathbb{N}$ or $a_k = a_{k+1} = 2$ for infinitely many k, then*

$$\tilde{\gamma}(\underline{\theta}) \geq c_{31} := \tilde{\gamma}(\overline{3,1}) > \chi := [2;\bar{1}][1;\bar{1}].$$

Moreover, if $\tilde{f}_0 := f_0 \circ \Pi$, then $\tilde{f}_0((\cdots, 2; 2, \cdots)_{\varphi_0}) > c_{31}$, where the 0th position is on the right of ;.

Lemma ([19], Lemma 3)**.** *Given $\underline{\theta} = (\cdots, B_1; B_0, B_1, \cdots)$, where $B_k = (2, 1_{m_k})$ with $B_k \geq 1$. If $m_k \neq m_{k+1}$ for infinitely many $k \in \mathbb{N}$, then*

$$\tilde{\gamma}(\underline{\theta}) \geq \chi = \tilde{f}_0((\bar{1}2; \bar{1})_{\varphi_0}) = \tilde{f}_0((\bar{1}; 2\bar{1})_{\varphi_0}).$$

More precisely, we have the cases:

i) *Case* $m_k \not\equiv m_{k+1}(mod\ 2)$.
If m_k *is odd, then* $f_0\big|R_{\varphi_0}(1,2,1_{m_k},2;1_{m_{k+1}},2,1) > \tilde{f}_0((\bar{1}2;\bar{1})_{\varphi_0})$.
If m_k *is even, then* $f_0\big|R_{\varphi_0}(1,2,1_{m_k};2,1_{m_{k+1}},2,1) > \tilde{f}_0((\bar{1};2\bar{1})_{\varphi_0})$.

ii) *Case* $m_k \equiv m_{k+1} \equiv 1(mod\ 2)$.
If $m_k < m_{k+1}$*, then* $f_0\big|R_{\varphi_0}(1,2,1_{m_k},2;1_{m_{k+1}},2,1) > \tilde{f}_0((\bar{1}2;\bar{1})_{\varphi_0})$.
If $m_k > m_{k+1}$*, then* $f_0\big|R_{\varphi_0}(1,2,1_{m_k};2,1_{m_{k+1}},2,1) > \tilde{f}_0((\bar{1};2\bar{1})_{\varphi_0})$.

iii) *Case* $m_k \equiv m_{k+1} \equiv 0(mod\ 2)$.
If $m_k < m_{k+1}$*, then* $f_0\big|R_{\varphi_0}(1,2,1_{m_k};2,1_{m_{k+1}},2,1) > \tilde{f}_0((\bar{1}2;\bar{1})_{\varphi_0})$.
If $m_k > m_{k+1}$*, then* $f_0\big|R_{\varphi_0}(1,2,1_{m_k},2;1_{m_{k+1}},2,1) > \tilde{f}_0((\bar{1};2\bar{1})_{\varphi_0})$.

By Lemma 2 [19], we have that:

$$D \cap (-\infty, c_{31}) = L(f_0, \varphi_0\big|_{\Lambda_2}) \cap (-\infty, c_{31}),$$

and the first accumulation point is $\chi = m_{f_0,\Lambda_2}((\bar{1}2;\bar{1})_{\varphi_0}) = f_0((\bar{1}2;\bar{1})_{\varphi_0}) = f_0((\bar{1};2\bar{1})_{\varphi_0})$.

It is known that φ_0 in (3.13) is a smooth conservative diffeomorphism with respect to an area form ω_0, which could be found in [1, 18]. As commented before in the previous sections, it is possible to think of $\varphi_0\big|_{\Lambda_2}$ as a horseshoe of a diffeomorphism $\varphi_0 : \mathbb{S}^2 \to \mathbb{S}^2$. There is an open C^2-neighborhood of φ_0 in $\mathrm{Diff}^2(\mathbb{S}^2)$, such that Λ_2 admits a hyperbolic continuation Λ_φ, for every φ in this neighborhood. Moreover, we have a nearby Markov partition $\{R_1^\varphi, R_2^\varphi\}$ for the corresponding Λ_φ, that induces a coding $\Pi : \Sigma_2 \to \Lambda_\varphi$, as given in to Section 2.1. For each point $p \in \Lambda_\varphi$ denote its kneading sequence by $\theta_\varphi = (\cdots, a_{-1}; a_0, a_1, \cdots)_\varphi$, whenever $\Pi(\theta_\varphi) = p$. Thus, we are able to state the main proposition in this subsection:

Theorem 5. *There are open neighborhoods* $\mathcal{U} \subset \mathrm{Diff}^2_{\omega_0}(\mathbb{S}^2)$ *of* φ_0 *and* $\mathcal{V} \subset C^1(\mathbb{S}^2;\mathbb{R})$ *of* f_0*, such that the beginning of* $L(f,\Lambda_\varphi)$ *is an infinite set, for every* $(\varphi, f) \in \mathcal{U} \times \mathcal{V}$*, where* Λ_φ *is the hyperbolic continuation of* Λ_0*.*

In order to prove the theorem, let us first impose the following restrictions on the pair. Let $\varepsilon_0 > 0$ sufficiently small (to be decided later), then there are a $N \geq 4$ and a neighborhood $\mathcal{U}_1 \times \mathcal{V}_1$ of (φ_0, f_0) in $\mathrm{Diff}^2_{\omega_0}(\mathbb{S}^2) \times \mathrm{C}^1(\mathbb{S}^2;\mathbb{R})$, such that for every $(\varphi, f) \in \mathcal{U}_1 \times \mathcal{V}_1$:

a) Since $\partial_x f_0((\bar{1}2;\bar{1})_{\varphi_0})/\partial_y f_0((\bar{1}2;\bar{1})_{\varphi_0}) = \partial_y f_0((\bar{1};2\bar{1})_{\varphi_0})/\partial_x f_0((\bar{1};2\bar{1})_{\varphi_0}) = [0;2,\bar{1}]/[0;\bar{1}] = 0.61803...$ and $W^u_{loc}(z,\varphi_0)$ and $W^s_{loc}(z,\varphi_0)$ are horizontal and vertical segments respectively for every $z \in \Lambda_0$, we have that:

$$\frac{[0;2,\bar{1}]}{[0;\bar{1}]} - \varepsilon_0 < \frac{\langle \nabla f(p_1), e^u_{p_1} \rangle}{\langle \nabla f(p_2), e^s_{p_2} \rangle} < \frac{[0;2,\bar{1}]}{[0;\bar{1}]} + \varepsilon_0, \tag{3.14}$$

for all $p_1 \in W^u_{loc}(z_1,\varphi) \cap R_\varphi(1_N 2; 1_N)$, $p_2 \in W^s_{loc}(z_2,\varphi) \cap R_\varphi(1_N 2; 1_N)$, with $z_1, z_2 \in R_{\Lambda_\varphi}(1_N 2; 1_N)$, where $e^u_{p_1}$ is the unit tangent vector to $W^u_{loc}(z_1,\varphi)$ at p_1(orientated from left to right) and $e^s_{p_2}$ is the unit tangent vector to $W^s_{loc}(z_2,\varphi)$ at p_2(orientated from down to up), see Figure 3.3. Here we are using the fact that f is C^1-close to f_0 and that the foliations $\mathcal{F}^u_{\Lambda_\varphi}(x)$ and $\mathcal{F}^u_{\Lambda_\varphi}(x)$ defined in a neighborhood of Λ_φ vary C^1-differentiably on the parameters (x,φ), according to Theorem 2.1. We have an analogous inequality as in (3.14) in the neighborhood of the point of which the kneading sequence is $(\bar{1};2\bar{1})_{\varphi_0}$.

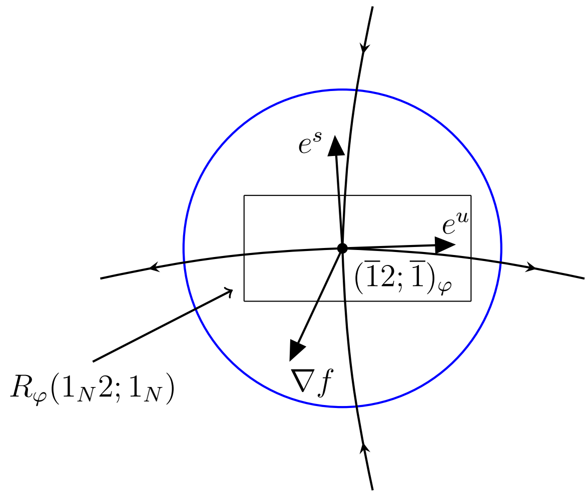

Figure 3.3: Behavior of f in a neighborhood of $(\bar{1}2;\bar{1})_\varphi$.

b) Since the projection given by unstable and stable foliations associates with Λ_2 as horseshoe with respect to φ_0 in (3.13) are respectively $\pi^u = \pi_2$ and $\pi^s = \pi_1$, then the expanding maps of definitions of the $K^u_{\varphi_0}$ and $K^s_{\varphi_0}$ are

$$g^{\varphi_0}_u\big|_{I^u(a_0,a_1)} = g \text{ and } g^{\varphi_0}_s\big|_{I^s(a_1,a_0)} = g,$$

where $(a_0,a_1) \in \mathcal{T}$ and g is the Gauss map. Thus, taking $\mathcal{U}_1$ sufficiently small by the same argument as before, we have $(g^{\varphi}_{u,s})' < 0$, for every $\varphi \in \mathcal{U}_1$.

The proof of the theorem also requires the following two lemmas. The first is a reformulation of the Lemmas 2 and 3[19], and its proof follows directly from the facts that f is C^1-close to f_0 and that the Markov partitions $\{R_1^{\varphi}, R_1^{\varphi}\}$ associated with Λ_{φ} vary continuously with φ.

Lemma 3.5. *For every $(\varphi, f) \in \mathcal{U}_1 \times \mathcal{V}_1$, for possibly reduced $\mathcal{U}_1 \times \mathcal{V}_1$, we get that:*

i) $f(R_{\varphi}(1;1)) < f(R_{\varphi}(1;2))$ & $f(R_{\varphi}(2;1)) < f(R_{\varphi}(2;2))$;

ii) There exists $M = M(N) > N$, such that:

- *If $m_k \in \{1, \cdots, N-1\}$ is odd and $m_{k+1} > M$ is even, then* $f\big|R_{\varphi}(1,2,1_{m_k},2;1_{m_{k+1}}) > \tilde{f}((\bar{1}2;\bar{1})_{\varphi})$;
- *If $m_{k+1} \in \{1, \cdots, N-1\}$ is even and $m_k > M$ is odd, then* $f\big|R_{\varphi}(1_{m_k},2;1_{m_{k+1}},2,1) > \tilde{f}((\bar{1}2;\bar{1})_{\varphi})$;
- *If $m_k \in \{1, \cdots, N-1\}$ is odd and $m_{k+1} > M$ is odd, then* $f\big|R_{\varphi}(1,2,1_{m_k},2;1_{m_{k+1}}) > \tilde{f}((\bar{1}2;\bar{1})_{\varphi})$;
- *If $m_{k+1} \in \{1, \cdots, N-1\}$ is even and $m_k > M$ is even, then* $f\big|R_{\varphi}(1_{m_k},2;1_{m_{k+1}},2,1) > \tilde{f}((\bar{1}2;\bar{1})_{\varphi})$;

and the analogous inequalities related to $\tilde{f}((\bar{1};2,\bar{1})_{\varphi})$, for every $(f,\varphi) \in \mathcal{U}_1 \times \mathcal{V}_1$.

Moreover, for distinct $m_k, m_{k+1} \in \{1, \ldots, M\}$ we have the same inequality as in Lemma 3[19]. Depending on the pair (m_k, m_{k+1}) either

$$f\big|R_{\varphi}(1,2,1_{m_k},2;1_{m_{k+1}},2,1) > \tilde{f}((\bar{1}2;\bar{1})_{\varphi}) \text{ or}$$

$$f\big|R_{\varphi}(1,2,1_{m_k};2,1_{m_{k+1}},2,1) > \tilde{f}((\bar{1};2,\bar{1})_{\varphi}).$$

Note that for every pair (φ, f) in $\mathcal{U}_1 \times \mathcal{V}_1$, $m_{f,\Lambda_{\varphi}}((\bar{1}2;\bar{1})_{\varphi})$ is an accumulation point of $L(f, \Lambda_{\varphi})$. Indeed, by Lemma 3.5*i*), we have that

$$m_{f,\Lambda_{\varphi}}((\bar{1}2;\bar{1})_{\varphi}) = \tilde{f}((\bar{1}2;\bar{1})_{\varphi}) \text{ or } \tilde{f}((\bar{1};2\bar{1})_{\varphi}), \tag{3.15}$$

and $l_{f,\Lambda_{\varphi}}((\overline{2,1_k})_{\varphi}) \to m_{f,\Lambda_{\varphi}}((\bar{1}2;\bar{1})_{\varphi})$. Moreover, again by Lemma 3.5*i*), for the remainder of the theorem's proof, we only are concerned with point of kneading sequence of the form $\underline{\theta}_{\varphi} = (\cdots, B_{-1}; B_0, B_2, \cdots)_{\varphi}$, where $B_k = (2, 1_{m_k})$, $m_k \geq 1$.

We adopt the following notation: given a finite string $(a_1, \cdots, a_l) \in (\mathbb{N}^*)^l$, we write:

$$[0; a_1, \cdots, a_l] = \frac{p(a_1, \cdots, a_l)}{q(a_1, \cdots, a_l)}.$$

Lemma 3.6. *There exists a neighborhood $\mathcal{U}_2 \times \mathcal{V}_2$ such that for every (φ, f) in $\mathcal{U}_2 \times \mathcal{V}_2$ and given $\underline{\theta}_\varphi$ as above, if $m_k \neq m_{k+1}$ for infinitely many k, then $l_{f,\Lambda_\varphi}(\underline{\theta}_\varphi) \geq m_{f,\Lambda_\varphi}((\bar{1}2;\bar{1})_\varphi)$.*

Proof. Let assume that in (3.15) we have that $m_{f,\Lambda_\varphi}((\bar{1}2;\bar{1})_\varphi) = \tilde{f}((\bar{1}2;\bar{1})_\varphi)$. Otherwise, we have to do analogous calculations and possibly reduce the neighborhood.

In order to prove this claim, we give a proof of the inequalities that appear in the Lemma 3[19] for (f_0, φ_0) and for every $m_k, m_{k+1} \geq N$, where we are able to see that those inequalities have uniform gaps, that give us space to ensure that the same inequalities holds for small perturbations of the pair. To this end, in the rest of this subsection we use the following convenient notation:

$$I^u_\varphi(a_0, \cdots, a_r) := \{\Pi((\cdots, b_{-2}, b_{-1}; c_0, c_1, \cdots)_\varphi) \in K^u_{\text{loc}}(p) |\ c_j = a_j,\ 0 \leq j \leq r\},$$

$$I^s_\varphi(a_{-1}, \cdots, a_{-s}) := \{\Pi((\cdots, c_{-2}, c_{-1}; b_0, b_1, \cdots)_\varphi) \in K^s_{\text{loc}}(p) |\ c_j = a_j,\ -s \leq j \leq -1\},$$

where $p \sim (b_n)_{n\in\mathbb{Z}}$. We denote by $\mathcal{U}_2 \times \mathcal{V}_2$ a neighborhood shrunken from $\mathcal{U}_1 \times \mathcal{V}_1$, where the inequalities given bellow in the cases $I)$, $II)$ and $III)$ are true for every pair (φ, f). Consider $(\cdots 121_{m_k}2;1_{m_{k+1}}21\cdots)_\varphi =: (\theta_k^-;\theta_k^+)_\varphi$ a iterated of $\underline{\theta}_\varphi$, by $b)$ we have $(g^\varphi_{u,s})' < 0$ and then, we can locate the quadrant to which this point belongs in the cases:
$I)[m_k$ is odd and m_{k+1} is even$]$ $R_\varphi(2, 1_{m_k}, 2; 1_{m_{k+1}}, 2)$ belongs to the quadrant in the direction of gradient of $\nabla f(\Pi(\bar{1}2;\bar{1})_\varphi)$, in the region given by $a)$, then

$$f\big|R_\varphi(2, 1_{m_k}, 2; 1_{m_{k+1}}, 2) > \tilde{f}((\bar{1}2;\bar{1})_\varphi).$$

$II)[m_k,\ m_{k+1}$ are odd numbers and $m_k < m_{k+1}]$ The point $(\theta_k^-;\theta_k^+)_\varphi$ belongs to the fourth quadrant, see Figure 3.4, and we have:

$$\tilde{f}((\bar{1}2;\theta_k^+)_\varphi) - \tilde{f}((\bar{1}2;\bar{1})_\varphi) = \langle \nabla f(p_1), e^u_{p_1}\rangle \cdot \delta^\varphi_x \text{ and} \tag{3.16}$$

$$\tilde{f}((\bar{1}2;\theta_k^+)_\varphi) - \tilde{f}((\theta_k^-;\theta_k^+)_\varphi) = \langle \nabla f(p_2), e^s_{p_2}\rangle \cdot \delta^\varphi_y, \tag{3.17}$$

where $\delta^\varphi_{x,y} > 0$, $p_1 \in W^u_{loc}(\Pi(\bar{1}2;\bar{1})_\varphi) \cap R_\varphi(1_N2;1_N)$, $p_2 \in W^s_{loc}(\Pi(\bar{1}2;\theta_k^+)_\varphi) \cap R_\varphi(1_N2;1_N)$, $e^u_{p_1}$ is the unit tangent vector to $W^u_{loc}(\Pi(\bar{1}2;\bar{1})_\varphi)$ at p_1(orientated from left to right) and $e^s_{p_2}$ is the unit tangent vector to $W^s_{loc}\Pi(\bar{1}2;\theta_k^+)_\varphi)$ at p_2(orientated from down to up).

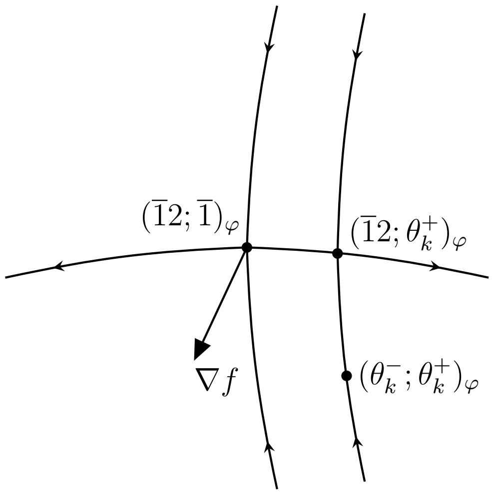

Figure 3.4: The point $(\theta_k^-;\theta_k^+)_\varphi$ in case m_k, m_{k+1} are odd numbers.

We estimate the distances $\delta_{x,y}^{\varphi}$ analyzing the relations between the lengths of intervals and gaps in distinct phases by definition of the regular Cantor set $K_\varphi^{u,s}$. In order to do that, we estimate $\delta_x^{\varphi_0}$ and $\delta_y^{\varphi_0}$ respectively in terms of $|I_{\varphi_0}^u(1_{m_{k+1}}2)|$ and $|I_{\varphi_0}^s(21_{m_k})|$. Note that, according to Figure 3.5:

$$\begin{aligned}\delta_x^{\varphi_0} \le \Delta_x^{\varphi_0} \quad &:= \quad [0;1_{m_{k+1}},\overline{2,1}] - [0;1_{m_{k+1}},1_2,\overline{1,2}] = \\ &= \quad \frac{[2;\overline{1,2}]-[1;1,\overline{1,2}]}{q_{m_{k+1}}^2([2;\overline{1,2}]+\beta_{m_{k+1}})([1;1,\overline{1,2}]+\beta_{m_{k+1}})},\end{aligned}$$

$$\begin{aligned}|I_{\varphi_0}^u(1_{m_{k+1}}2)| \quad &:= \quad [0;1_{m_{k+1}},2,\overline{1,2}] - [0;1_{m_{k+1}},2,\overline{2,1}] = \\ &= \quad \frac{[2;\overline{1,2}]-[2;\overline{2,1}]}{q_{m_{k+1}}^2([2;\overline{1,2}]+\beta_{m_{k+1}})([2;\overline{2,1}]+\beta_{m_{k+1}})},\end{aligned}$$

where $q_{m_{k+1}} = q(1_{m_{k+1}})$ and $\beta_{m_{k+1}} = [0;1_{m_{k+1}}]$.

Thus,

$$\frac{\Delta_x^{\varphi_0}}{|I_{\varphi_0}^u(1_{m_{k+1}}2)|} = \frac{[2;\overline{1,2}]-[1;1,\overline{1,2}]}{[2;\overline{1,2}]-[2;\overline{2,1}]}\cdot\frac{([2;\overline{2,1}]+\beta_{m_{k+1}})}{([1;1,\overline{1,2}]+\beta_{m_{k+1}})}.$$

Since $[0;\overline{1}] \le \beta_{m_{k+1}} \le [0;1,1,1]$, we have:

$$\frac{\delta_x^{\varphi_0}}{|I_{\varphi_0}^u(1_{m_{k+1}}2)|} \le \frac{\Delta_x^{\varphi_0}}{|I_{\varphi_0}^u(1_{m_{k+1}}2)|} \le 4.358. \tag{3.18}$$

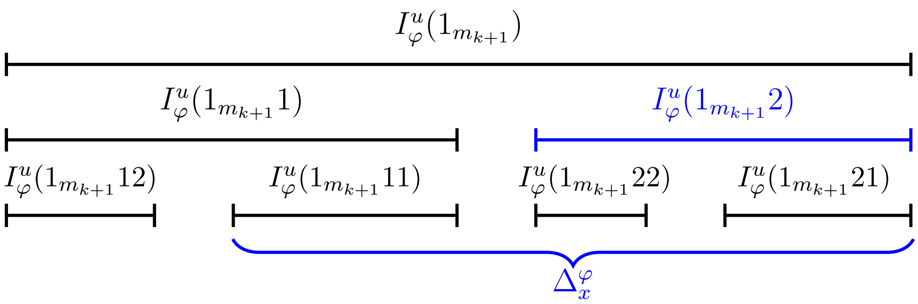

Figure 3.5: Relations between the lengths $|I^u_\varphi(1_{m_{k+1}}2)|$ and Δ^φ_x.

Note that (see Figure 3.6):

$$\begin{aligned}\delta^{\varphi_0}_y \geq \Delta^{\varphi_0}_y &:= [0;2,1_{m_k},1,1,\overline{2,1}] - [0;2,1_{m_k},2,1,\overline{1,2}] = \\ &= \frac{[2;1,\overline{1,2}] - [1;1,\overline{2,1}]}{\tilde{q}^2_{m_k+1}([2;1,\overline{1,2}] + \tilde{\beta}_{m_k+1})([1;1,\overline{2,1}] + \tilde{\beta}_{m_k+1})},\end{aligned}$$

$$\begin{aligned}|I^s_{\varphi_0}(21_{m_k})| &:= [0;21_{m_k},\overline{1,2}] - [0;2,1_{m_k},\overline{2,1}] = \\ &= \frac{[2;\overline{1,2}] - [1;\overline{2,1}]}{\tilde{q}^2_{m_k+1}([2;\overline{1,2}] + \tilde{\beta}_{m_k+1})([1;\overline{2,1}] + \tilde{\beta}_{m_k+1})},\end{aligned}$$

where $\tilde{q}_{m_k+1} = q(21_{m_k})$ and $\tilde{\beta}_{m_{k+1}} = [0;1_{m_k}2]$.

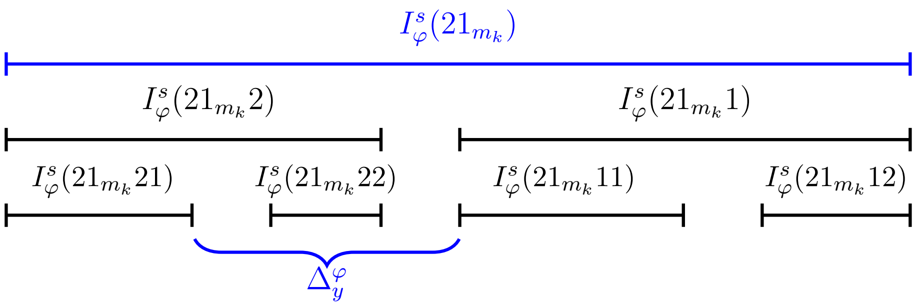

Figure 3.6: Relations between the lengths $|I^s_\varphi(21_{m_k})|$ and Δ^φ_y.

Therefore,

$$\frac{\Delta^{\varphi_0}_y}{|I^s_{\varphi_0}(21_{m_k})|} = \frac{[2;1,\overline{1,2}] - [1;1,\overline{1,2}]}{[2;\overline{1,2}] - [1;\overline{2,1}]} \cdot \frac{([2;\overline{1,2}] + \tilde{\beta}_{m_k+1})([1;\overline{2,1}] + \tilde{\beta}_{m_k+1})}{([2;1,\overline{1,2}] + \tilde{\beta}_{m_k+1})([1;1,\overline{2,1}] + \tilde{\beta}_{m_k+1})}.$$

Since $[0;\overline{1}] \leq \tilde{\beta}_{m_{k+1}} \leq [0;1,1,1,2]$, we have:

$$\frac{\delta^{\varphi_0}_y}{|I^s_{\varphi_0}(21_{m_k})|} \geq \frac{\Delta^{\varphi_0}_y}{|I^s_{\varphi_0}(21_{m_k})|} \geq 0.544. \tag{3.19}$$

Now, for $m \geq 3$ odd, we relate $|I^u_{\varphi_0}(1_m 2)|$ with $|I^s_{\varphi_0}(21_m)|$:

$$\begin{aligned} \frac{|I^s_{\varphi_0}(21_m)|}{|I^u_{\varphi_0}(1_m 2)|} &= \frac{[0;2,1_m,\overline{2,1}]-[0;2,1_m,\overline{1,2}]}{[0;1_m,2,\overline{2,1}]-[0;1_m,2,\overline{1,2}]} = \\ &= \frac{([2;\overline{1,2}]+[0;2,1_m])([1;\overline{2,1}]+[0;2,1_m])}{([2;\overline{1,2}]+[0;1_m,2])([1;\overline{2,1}]+[0;1_m,2])}, \end{aligned}$$

where $[0;2,1,1,1] \leq [0;2,1_m] \leq [0;2,\bar{1}]$ and $[0;\bar{1}] \leq [0;1_m,2] \leq [0;1,1,1,2]$. Thus,

$$0.809 < \frac{|I^s_{\varphi_0}(21_m)|}{|I^u_{\varphi_0}(1_m 2)|} < 0.819. \tag{3.20}$$

We also have, for $m \geq 3$ odd:

$$\begin{aligned} \frac{|I^s_{\varphi_0}(21_{m+2})|}{|I^s_{\varphi_0}(21_m)|} &= \frac{[0;2,1_{m+2},\overline{1,2}]-[0;2,1_{m+2},\overline{2,1}]}{[0;2,1_m,\overline{1,2}]-[0;2,1_m,\overline{2,1}]} \\ &= \frac{[1;1,\overline{2,1}]-[1;1,\overline{1,2}]}{[2;\overline{1,2}]-[1;\overline{2,1}]} \cdot \frac{([2;\overline{1,2}]+\beta_{m+1})([1;\overline{2,1}]+\beta_{m+1})}{([1;1,\overline{1,2}]+\beta_{m+1})([1;1,\overline{2,1}]+\beta_{m+1})}, \end{aligned}$$

where $[0;\bar{1}] < \beta_{m+1} := [0;1_m,2] < [0;1,2]$. Thus,

$$\frac{|I^s_{\varphi_0}(21_{m+2})|}{|I^s_{\varphi_0}(21_m)|} < 0.152. \tag{3.21}$$

By (3.16) and (3.17), we have $\tilde{f}((\theta_k^-,\theta_k^+)_\varphi) \geq \tilde{f}((\bar{1}2;\bar{1})_\varphi)$ if and only if

$$\frac{\delta_y^\varphi}{\delta_x^\varphi} \geq \frac{\langle \nabla f(p_1), e^u_{p_1}\rangle}{\langle \nabla f(p_2), e^s_{p_2}\rangle}.$$

In order to prove this inequality, we take $\varepsilon_0 > 0$ sufficiently small and shrinking $\mathcal{U}_1 \subset \mathrm{Diff}^2_{\omega_0}(\mathbb{S}^2)$ to $\mathcal{U}_2$, such that for φ in $\mathcal{U}_2$ we have almost the same inequalities as (3.18), (3.19), (3.20) and (3.21). More specifically, we guarantee for φ the inequalities (3.18), (3.19) and (3.21) with errors ε_2, ε_1 and ε_4 respectively, because the constant of bounded distortion property varies continuously with the diffeomorphism. We obtained for φ the inequality (3.20) with error ε_3, since we have $\mathcal{U}_2$ sufficiently small in the space of conservative C^2-diffeomorphisms. Thus we get, using that $m_{k+1} \geq m_k + 2$ and *a)*:

$$\begin{aligned} \frac{\delta_y^\varphi}{\delta_x^\varphi} &\geq \frac{\Delta_y^\varphi}{\Delta_x^\varphi} > \frac{0.544-\varepsilon_1}{4.358+\varepsilon_2}\cdot\frac{|I^s_\varphi(21_{m_k})|}{|I^u_\varphi(1_{m_{k+1}}2)|} > \frac{(0.544-\varepsilon_1)(0.809-\varepsilon_3)}{4.358+\varepsilon_2}\cdot\frac{|I^s_\varphi(21_{m_k})|}{|I^s_\varphi(21_{m_k+2})|} \\ &> \frac{(0.544-\varepsilon_1)(0.809-\varepsilon_3)}{(4.358+\varepsilon_2)(0.152+\varepsilon_4)} = 0.639+\varepsilon' > \frac{[0;2,\bar{1}]}{[0;\bar{1}]}+\varepsilon_0 \geq \frac{\langle \nabla f(p_1), e^u_{p_1}\rangle}{\langle \nabla f(p_2), e^s_{p_2}\rangle}, \end{aligned}$$

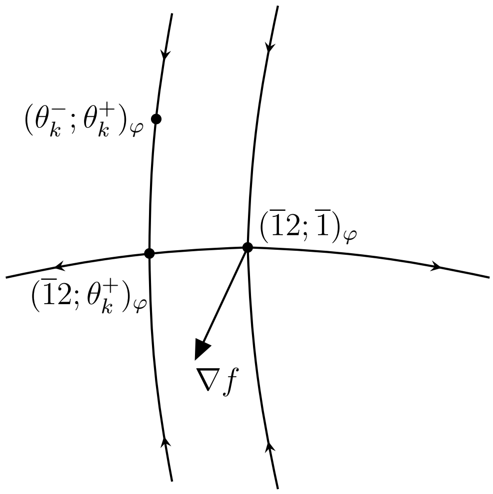

Figure 3.7: The point $(\theta_k^-;\theta_k^+)_\varphi$ in case m_k, m_{k+1} are even numbers.

for $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3,\ \varepsilon_4$ and so, ε' sufficiently small.
$III)[m_k, m_{k+1}$ are even numbers and $m_k > m_{k+1}]$The point $(\theta_k^-;\theta_k^+)_\varphi$ belongs to the second quadrant (see Figure 3.7), and we have:

$$\tilde{f}((\bar{1}2;\bar{1})_\varphi) - \tilde{f}((\bar{1}2;\theta_k^+)_\varphi) = \langle \nabla f(p_3), e^u_{p_3}\rangle \cdot \delta^\varphi_x \text{ and} \tag{3.22}$$

$$\tilde{f}((\theta_k^-;\theta_k^+)_\varphi) - \tilde{f}((\bar{1}2;\theta_k^+)_\varphi) = \langle \nabla f(p_4), e^s_{p_4}\rangle \cdot \delta^\varphi_y, \tag{3.23}$$

where $\delta^\varphi_{x,y} > 0$, $p_3 \in W^u_{loc}(\Pi(\bar{1}2;\bar{1})_\varphi) \cap R_\varphi(1_N2;1_N)$, $p_4 \in W^s_{loc}(\Pi(\bar{1}2;\theta_k^+)_\varphi) \cap R_\varphi(1_N2;1_N)$, $e^u_{p_3}$ is the unit tangent vector to $W^u_{loc}(\Pi(\bar{1}2;\bar{1})_\varphi)$ at p_3(orientated from left to right) and $e^s_{p_4}$ is the unit tangent vector to $W^s_{loc}\Pi(\bar{1}2;\theta_k^+)_\varphi)$ at p_4(orientated from down to up).

By (3.22) and (3.23), we have $\tilde{f}((\theta_k^-,\theta_k^+)_\varphi) \geq \tilde{f}((\bar{1}2;\bar{1})_\varphi)$ if and only if

$$\frac{\delta^\varphi_y}{\delta^\varphi_x} \leq \frac{\langle \nabla f(p_3), e^u_{p_3}\rangle}{\langle \nabla f(p_4), e^s_{p_4}\rangle}. \tag{3.24}$$

We can follow exactly in the same lines as in $II)$ to prove (3.24), for a possible small $\varepsilon_0 > 0$ and possibly shrinked $\mathcal{V}_2 \subset \mathrm{Diff}^2_{\omega_0}(\mathbb{S}^2)$.

Finally we prove the lemma. If there exist infinitely many k such that m_k and m_{k+1} have different parities, then there exist infinitely many k such that m_k is odd and m_{k+1} is even. Thus, by $I)$ we have that

$$f\big|R_\varphi(1,2,1_{m_k},2;1_{m_{k+1}},2,1) > \tilde{f}((\bar{1}2;\bar{1})_\varphi)$$

for $m_k, m_{k+1} > N$ and by Lemma 3.5 $ii)$ we get the same inequality for the other cases. Therefore, in this situation, $l_{f,\Lambda_\varphi}(\underline{\theta}_\varphi) \geq \tilde{f}((\bar{1}2;\bar{1})_\varphi)$. Now, we have only to study $\underline{\theta}_\varphi$ with m_k of same parity for all k sufficiently large. If there exists a subsequence of positive indices $(k_n)_n$, such that m_{k_n}, $m_{k_{n+1}} \to \infty$ as $n \to \infty$, then $(\theta^-_{k_n};\theta^+_{k_n})_\varphi := (\cdots 121_{m_{k_n}}2;1_{m_{k_{n+1}}}21\cdots)_\varphi$ goes to $(\bar{1}2;\bar{1})_\varphi$ and by continuity we have $\tilde{f}((\theta^-_{k_n};\theta^+_{k_n})_\varphi) \to \tilde{f}((\bar{1}2;\bar{1})_\varphi)$. Thus, $l_{f,\Lambda_\varphi}(\underline{\theta}_\varphi) \geq \tilde{f}((\bar{1}2;\bar{1})_\varphi)$. Otherwise, there are subsequences $(k_l)_l$ such that $m_{k_l+1} > m_{k_l}$ and $(k_j)_j$ such that $m_{k_j} > m_{k_j+1}$. Thus, in this case, when m_k is odd (resp. even) for all k large, by II) we get $\tilde{f}((\theta^-_{k_l};\theta^+_{k_l})_\varphi) > \tilde{f}((\bar{1}2;\bar{1})_\varphi)$ for $m_{k_l}, m_{k_l+1} > N$ (resp. by III), we get $\tilde{f}((\theta^-_{k_j};\theta^+_{k_j})_\varphi) > \tilde{f}((\bar{1}2;\bar{1})_\varphi)$ for $m_{k_j}, m_{k_j+1} > N$). And by Lemma 3.5 *ii*) we have the same inequalities for the other cases. Therefore, in these cases, $l_{f,\Lambda_\varphi}(\underline{\theta}_\varphi) \geq \tilde{f}((\bar{1}2;\bar{1})_\varphi)$. This concludes the proof of the claim. □

Finally, we are able to prove the theorem.

Proof of Theorem 5. In view of Lemma 3.5 and Lemma 3.6 we are only concerned with points q in $W^s_\varphi(\alpha_k)$ or $W^s_\varphi(\alpha^*)$, where $\alpha_k = \Pi(\overline{2,1_k},2;1_k,\overline{2,1_k})_\varphi$ and $\alpha^* = \Pi(\bar{1})_\varphi$. Note that $l_{f,\Lambda_\varphi}(q)$ is equal to $l_{f,\Lambda_\varphi}(\alpha_k)$ or $l_{f,\Lambda_\varphi}(\alpha^*)$.

In order to prove the theorem it is sufficient to show that for every k odd, we get $l_{f,\Lambda_\varphi}(\alpha_k) = m_{f,\Lambda_\varphi}(\alpha_k) = f(\alpha_k) < m_{f,\Lambda_\varphi}((\bar{1}2;\bar{1})_\varphi)$. The point α_k belongs to the fourth quadrant, as in Lemma 3.6 II), and we have:

$$\tilde{f}((\bar{1}2;\overline{1_k2})_\varphi) - \tilde{f}((\bar{1}2;\bar{1})_\varphi) = \langle \nabla f(p_5), e^u_{p_5}\rangle \cdot \tilde{\delta}^\varphi_x \text{ and} \tag{3.25}$$

$$\tilde{f}((\bar{1}2;\overline{1_k2})_\varphi) - \tilde{f}((\theta^-_k;\theta^+_k)_\varphi) = \langle \nabla f(p_6), e^s_{p_6}\rangle \cdot \tilde{\delta}^\varphi_y, \tag{3.26}$$

where $\tilde{\delta}^\varphi_{x,y} > 0$, $p_5 \in W^u_{loc}(\Pi(\bar{1}2;\bar{1})_\varphi) \cap R_\varphi(1_N2;1_N)$, $p_6 \in W^s_{loc}(\Pi(\bar{1}2;\overline{1_k2})_\varphi) \cap R_\varphi(1_N2;1_N)$, $e^u_{p_5}$ is the unit tangent vector to $W^u_{loc}(\Pi(\bar{1}2;\bar{1})_\varphi)$ at p_5(orientated from left to right) and $e^s_{p_6}$ is the unit tangent vector to $W^s_{loc}\Pi(\bar{1}2;\overline{1_k2})_\varphi)$ at p_6(orientated from down to up).

Analogously, in order to estimate the distances $\tilde{\delta}^\varphi_{x,y}$, we estimate $\delta^{\varphi_0}_x$ and $\delta^{\varphi_0}_y$ respectively in terms of $|I^u_{\varphi_0}(1_{m_{k+1}}2)|$ and $|I^s_{\varphi_0}(21_{m_k})|$. Note that(see Figure 3.8):

$$\delta^{\varphi_0}_x \geq \Delta^{\varphi_0}_x := [0;1_k,2,1,\overline{1,2}] - [0;1_k,1_2,\overline{2,1}] = \frac{[2;1,\overline{1,2}] - [1;1,\overline{2,1}]}{q_k^2([2;1,\overline{1,2}]+\beta_k)([1;1,\overline{2,1}]+\beta_k)},$$

$$|I^u_{\varphi_0}(1_k2)| := [0;1_k,2,\overline{1,2}] - [0;1_k,2,\overline{2,1}] = \frac{[2;\overline{1,2}] - [2;\overline{2,1}]}{q_k^2([2;\overline{1,2}]+\beta_k)([2;\overline{2,1}]+\beta_k)},$$

where $q_k = q(1_k)$ and $\beta_{k+1} = [0;1_k]$.

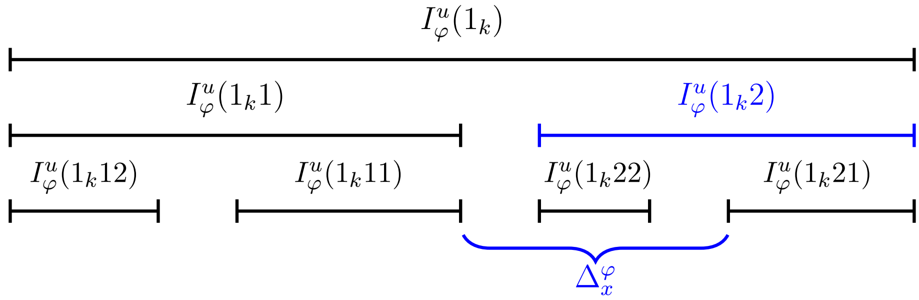

Figure 3.8: Relations between the lengths $|I_\varphi^u(1_k2)|$ and Δ_x^φ.

Thus,

$$\frac{\Delta_x^{\varphi_0}}{|I_{\varphi_0}^u(1_k2)|} = \frac{[2;1,\overline{1,2}]-[1;1,\overline{2,1}]}{[2;\overline{1,2}]-[2;\overline{2,1}]}\cdot\frac{([2;\overline{1,2}]+\beta_k)([2;\overline{2,1}]+\beta_k)}{([2;1,\overline{1,2}]+\beta_k)([1;1,\overline{2,1}]+\beta_k)}.$$

Since $[0;\bar{1}]\leq\beta_k\leq[0;1]$, we have:

$$\frac{\delta_x^{\varphi_0}}{|I_{\varphi_0}^u(1_k2)|}\geq\frac{\Delta_x^{\varphi_0}}{|I_{\varphi_0}^u(1_k2)|}\geq 2.362. \tag{3.27}$$

Note that(the Figure 3.9):

$$\begin{aligned}\delta_y^{\varphi_0}\leq\Delta_y^{\varphi_0} &:= [0;2,1_k,1_2,\overline{1,2}]-[0;2,1_k,\overline{2,1}] = \\ &= \frac{[2;\overline{1,2}]-[1;1,\overline{1,2}]}{\tilde{q}_{k+1}^2([2;\overline{1,2}]+\tilde{\beta}_{k+1})([1;1,\overline{1,2}]+\tilde{\beta}_{k+1})},\end{aligned}$$

$$\begin{aligned}|I_{\varphi_0}^s(21_k)| &:= [0;21_k,\overline{1,2}]-[0;2,1_k,\overline{2,1}] = \\ &= \frac{[2;\overline{1,2}]-[1;\overline{2,1}]}{\tilde{q}_{k+1}^2([2;\overline{1,2}]+\tilde{\beta}_{k+1})([1;\overline{2,1}]+\tilde{\beta}_{k+1})},\end{aligned}$$

where $\tilde{q}_{k+1}=q(21_k)$ and $\tilde{\beta}_{k+1}=[0;1_k2]$.

Therefore,

$$\frac{\Delta_y^{\varphi_0}}{|I_{\varphi_0}^s(21_k)|}=\frac{[2;\overline{1,2}]-[1;1,\overline{1,2}]}{[2;\overline{1,2}]-[1;\overline{2,1}]}\cdot\frac{([1;\overline{2,1}]+\tilde{\beta}_{k+1})}{([1;1,\overline{1,2}]+\tilde{\beta}_{k+1})}.$$

Since $[0;\bar{1}]\leq\tilde{\beta}_{k+1}\leq[0;1,2]$, we have:

$$\frac{\delta_y^{\varphi_0}}{|I_{\varphi_0}^s(21_{m_k})|}\leq\frac{\Delta_y^{\varphi_0}}{|I_{\varphi_0}^s(21_{m_k})|}\leq 0.783. \tag{3.28}$$

By (3.25) and (3.26), we have $f(\alpha_k)<\tilde{f}((\bar{1}2;\bar{1})_\varphi)$ if and only if

$$\frac{\tilde{\delta}_y^\varphi}{\tilde{\delta}_x^\varphi}<\frac{\langle\nabla f(p_5),e_{p_5}^u\rangle}{\langle\nabla f(p_6),e_{p_6}^s\rangle}.$$

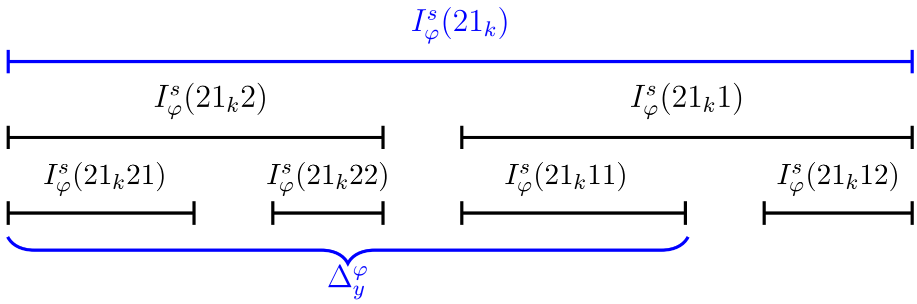

Figure 3.9: Relations between the lengths $|I^s_\varphi(21_k)|$ and Δ^φ_y.

In order to prove this last inequality, we can take $\varepsilon_0 > 0$ sufficiently small and shrinking $\mathcal{U}_2 \times \mathcal{V}_2$ to $\mathcal{U} \times \mathcal{V}$, such that for φ in $\mathcal{U}$ we have the inequalities (3.27) and (3.28) with errors ε_6 and ε_5 respectively, because the constant of bounded distortion property varies continuously with the diffeomorphism. Therefore we get, using Lemma 3.6 that for $\varphi \in \mathcal{U}_2$ we have the inequality as in (3.20) with an error ε_3 and *a)*:

$$\begin{aligned}\frac{\tilde{\delta}^\varphi_y}{\tilde{\delta}^\varphi_x} &< \frac{\Delta^\varphi_y}{\Delta^\varphi_x} < \frac{0.783+\varepsilon_5}{2.362-\varepsilon_6}\cdot\frac{|I^s_\varphi(21_k)|}{|I^u_\varphi(1_k2)|} < \frac{(0.783+\varepsilon_5)(0.819+\varepsilon_3)}{2.362-\varepsilon_6} = \\ &= 0.271+\tilde{\varepsilon} < \frac{[0;2,\bar{1}]}{[0;\bar{1}]} - \varepsilon_0 \leq \frac{\langle \nabla f(p_5), e^u_{p_5}\rangle}{\langle \nabla f(p_6), e^s_{p_6}\rangle},\end{aligned}$$

for ε_5, ε_6, ε_3 and so, $\tilde{\varepsilon}$ sufficiently small. This concludes the proof of the theorem. □

Remark 3.2. It follows from the proof that for a possibly small neighborhood $\mathcal{U} \times \mathcal{V}$, we can use the same ideas to prove that $f(\alpha_k) > m_{f,\Lambda_\varphi}((\bar{1}2;\bar{1})_\varphi))$ for k even, and thus we get:

$$L(f,\Lambda_\varphi) \cap (-\infty, m_{f,\Lambda_\varphi}((\bar{1}2;\bar{1})_\varphi)) = \{f(\alpha_k) : \text{ k odd}\} \cup \tilde{f}((\bar{1})_\varphi),$$

for every $(\varphi, f) \in \mathcal{U} \times \mathcal{V}$, where $m_{f,\Lambda_\varphi}((\bar{1}2;\bar{1})_\varphi) = inf L'(f,\Lambda_\varphi)$.

3.3.4 Infinite beginning in Lagrange spectrum

In this subsection, we build an open set in the pair (dynamics, function) where the Lagrange spectrum for each pair in this set has infinitely many points before the first accumulation point.

Let $\varphi : [0,2]^2 \to \varphi([0,2]^2)$ be a diffeomorphism with a associated linear piecewise horseshoe $\Lambda = \bigcap_{n\in\mathbb{Z}} \varphi([0,2]^2)$, whose the local unstable (resp. stable) manifold are given by horizontal (resp. vertical) lines, and the stable and unstable Cantor sets $K^{s,u}$ are defined by $g_{s,u} : I^{s,u}(0) \cup I^{s,u}(1) \subset [0,1] \to [0,1]$ whose graphics are given in the Figure 3.10, where $|g'_u|_{I^s(0)}| \equiv \mu_0 = \tilde{\mu}_0^2$, $|g'_u|_{I^u(1)}| \equiv \tilde{\mu}_0$ and $|g'_s|_{I^s(0)}| \equiv \lambda_0 = \tilde{\lambda}_0^2$, $|g'_s|_{I^s(1)}| \equiv \tilde{\lambda}_0$. Thus, $|I^u(0)| = \mu_0^{-1}$, $|I^u(1)| = \tilde{\mu}_0^{-3}$, $|I^s(0)| = \lambda_0^{-1}$ and $|I^s(1)| = \tilde{\lambda}_0^{-3}$. We also choose $\lambda_0 = \mu_0^{7.4}$, for $\tilde{\mu}_0 = \mu_0^{1/2} \geq 4$ big enough to be picked a posteriori. Again, we can extended the map $\varphi : [0,2]^2 \to \varphi([0,2]^2)$ to a C^2–diffeomorphism on $\mathbb{S}^2$, i.e., $\varphi \in \mathrm{Diff}^2(\mathbb{S}^2)$.

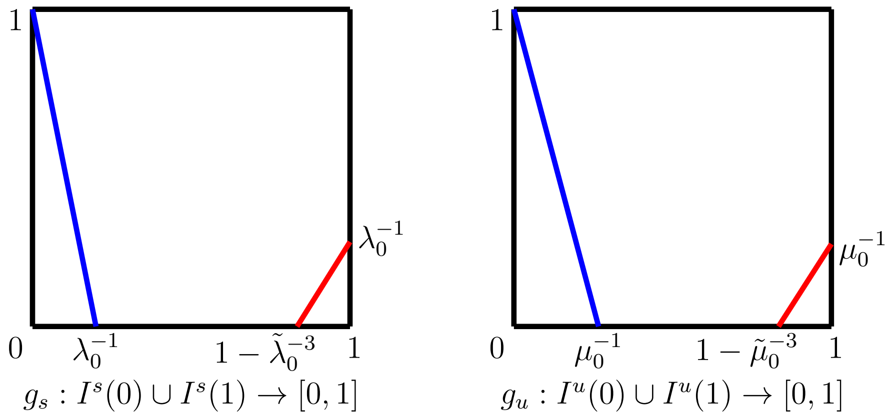

Figure 3.10: The expanding maps of stable and unstable Cantor sets

Moreover, we choose Λ conjugated to the subshift of finite type $\sigma_B : \Sigma_B \to \Sigma_B$, where $\Sigma_B \subset \Sigma_2 := \{0,1\}^{\mathbb{Z}}$ with transition matrix B given by $b_{00} = b_{01} = b_{10} = 1$ and $b_{11} = 0$. In a such way that the dynamics $g_{s,u} : K^{s,u} \to K^{s,u}$ are conjugate to the forward subshift $\sigma^+ : \Sigma_B^+ \to \Sigma_B^+$, given by $\sigma^+((a_n)_{n\geq 0}) = (a_{n+1})_{n\geq 0}$, where $\Sigma_B^+ = \Sigma_B \cap \{0,1\}^{\mathbb{N}}$. Moreover, the branch in $g_{s,u}$ associate to the symbol 0 is decreasing and the branch associate to the symbol 1 is increasing.

Let $f : [0,1]^2 \to \mathbb{R}$ given by $f(x,y) = -x - y$. In the next discussion, we will analyse the beginning of the dynamical Lagrange spectrum $L(f,\Lambda) = \{\limsup_{n\to\infty} f(\varphi^n(x)) : x \in \Lambda\}$.

In order to to that, we will take a real number t_0 such that the set of the Lagrange values $L(f,\Lambda_{t_0})$ of points in $\Lambda_{t_0} := \bigcap_{n=-\infty}^{\infty} \varphi^n(\{x \in \Lambda : f(x) \leq t_0\})$ is essentially the set of Lagrange values computed in another horseshoe, contained in Λ_{t_0}, quite of similar to Λ, as we show precisely in the following.

Remind that given an admissible string $\theta_{k=-l}^{m} = (a_{-l}, \cdots, a_{-1}; a_0, a_1, \cdots a_m)$, we define the rectangle$R_\varphi(\theta_{k=-l}^{m}) := \displaystyle\bigcap_{k=-l}^{m} \varphi^{-k}(R_{a_k})$.

First, we use the symbolic dynamics to explain how we take a such t_0. For now, we may assume that we can take t_0 such that the level curve $[f = t_0] := \{(x, y) : x+y = -t_0\}$ cross $[0, 1]\times 0$ between stages $I^u(010100000)$ and $I^u(010100001)$ of the Cantor set K^u, such that

$$(R_\varphi(0; 0100)\cup R_\varphi(0; 010101)\cup R_\varphi(0; 01010001)\cup R_\varphi(0; 010100000))\cap\Lambda = [f > t_0]\cap\Lambda,$$

where $[f > t_0] = \{(x, y) \in [0, 1]^2 : f(x, y) > t_0\}$. See Figures 3.12 and 3.11.

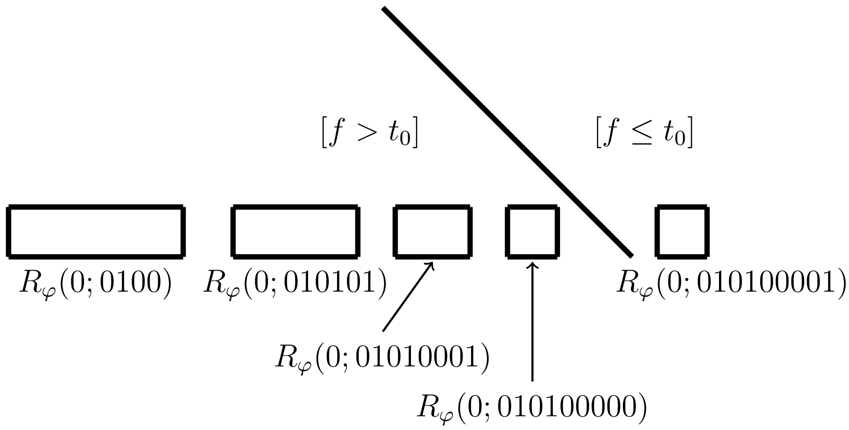

Figure 3.11: Geometrical representation of the cut by $[f = t_0]$.

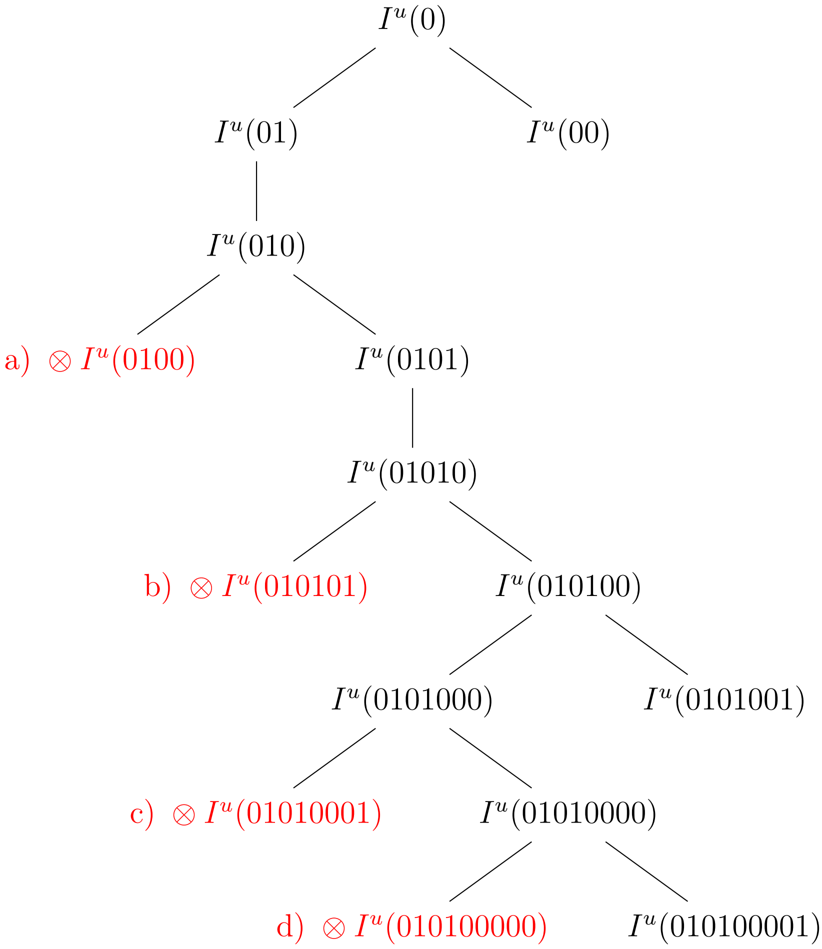

Figure 3.12: Symbolic representation of the cut by $[f = t_0]$ on the unstable Cantor set (where $a_{-1} = 0$).

In order to follow, we may introduce the next notations. Given a set $A \subset \Sigma_B$ we define the set $S_\sigma(A) := \{\sigma^n(x) : x \in A, n \in \mathbb{Z}\}$ of all orbits by σ of elements in A. We define $[01010, 00/00 \nrightarrow 00]$ as the set

$$\{(..., w_{-1}; w_0, w_1, ...) \in \Sigma_B : w_i \in \{01010, 00\} \text{ and } (w_i, w_{i+1}) \neq (00, 00), \forall i \in \mathbb{Z}\},$$

and $[01010, 00/00 \nrightarrow 00]^+ := [01010, 00/00 \nrightarrow 00] \cap \Sigma_B^+$.

We check that Λ_{t_0} is the subset of point in Λ associated to the set $S_\sigma(A_1)$, where A_1 is the set

$$A_1 = \{(\overline{1010}; \overline{1010}), (\overline{1010}; \overline{0}), (\overline{00}; \overline{00})\} \cup \bigcup_{r=0}^{\infty} (\overline{1010}; 0_r 01010[01010, 00/00 \nrightarrow 00]^+)$$
$$\cup\, (\overline{0}; 01010[01010, 00/00 \nrightarrow 00]^+) \cup [01010, 00/00 \nrightarrow 00]\}.$$

Hence, $L(f,\varphi|_{\Lambda_{t_0}}) = L(\tilde{f},\sigma|_{[01010,00/00\nrightarrow 00]})\cup\{l_{f,\Lambda}(\overline{00};\overline{00})\}\cup\{l_{f,\Lambda}(\overline{1010};\overline{1010})\}$, where $\tilde{f} = f|_\Lambda\circ\Pi$.

By the definition of the expanding maps of the stable and unstable Cantor set (where the branch in $g_{s,u}$ associate to the symbol 0 is decreasing and the branch associate to the symbol 1 is increasing) and the fact that $\nabla f\equiv(-1,-1)$, we have:

$$f(R_\varphi(0;0)) > f(R_\varphi(1;0)), f(R_\varphi(0;1)) > f(R_\varphi(1;1)), \tag{3.29}$$

$$f(R_\varphi(10;01)) > f(R_\varphi(00;01)), f(R_\varphi(10;00)) > f(R_\varphi(00;00)). \tag{3.30}$$

Let $\Pi:\Sigma_B\to\Lambda$ the conjugation map, by inequalities (3.29) and (3.30), if $x\in\Lambda_1 := \Pi([01010,00/00\nrightarrow 00])$ with $m_{f,\varphi|_\Lambda}(x) = \sup_{n\in\mathbb{Z}} f(\varphi^n(x)) = f(x)$, then $x\in R_0^1\cap R_1^1$, where $R_0^1 := R_\varphi(0;01010)$ and $R_1^1 := R_\varphi(0;00)$.

Define $\varphi_1 : R_0^1\cup R_1^1\to\varphi_1(R_0^1\cup R_1^1)$ given by $\varphi_1(x) := \varphi^{\tau_1(x)}(x)$, where $\tau_1(x) = \min\{n>0 : \varphi^n(x)\in R_0^1\cup R_1^1\}$. Thus, Λ_1 is a horseshoe to φ_1. Note that, $\varphi_1:\Lambda_1\to\Lambda_1$ is conjugated to $\sigma_1:\Sigma_B\to\Sigma_B$, where $\sigma_1(\theta) := \sigma^{\tilde{\tau}_1(\theta)}(\theta)$ and $\tilde{\tau}_1(\theta) := \min\{n>0 : \sigma^n(\theta)\in(C[0;01010]\cup C[0;00])\subset\Sigma_B\}$. By doing the identification $01010\to 0$ and $00\to 1$, the last subshift is exactly the subshift $\sigma:\Sigma_B\to\Sigma_B$. Thus, $\varphi_1:\Lambda_1\to\Lambda_1$ is also conjugated to $\sigma:\Sigma_B\to\Sigma_B$. Moreover, the definition maps of the unstable and stable Cantor of Λ_1 are respectively $g_{u,s}^{(1)} : I^{u,s}(01010)\cup I^{u,s}(00)\to[0,1]$, where the branch associated to $I^{u,s}(01010)$ is decreasing and the branch associated to $I^{u,s}(00)$ is increasing, and $I^{u,s}(01010)$ is in the left of $I^{u,s}(00)$.

Thus, by previous paragraphs we have that:

$$L(f,\varphi|_{\Lambda_{t_0}}) = L(f,\varphi_1|_{\Lambda_1})\cup\{l_{f,\Lambda}(\overline{00};\overline{00})\}\cup\{l_{f,\Lambda}(\overline{10};\overline{10})\}.$$

Again by (3.29) and (3.30), we get $l_{f,\Lambda}(\overline{00};\overline{00}) = f(\overline{00};\overline{00})$, $l_{f,\Lambda}(\overline{10};\overline{10}) = f(\overline{10};\overline{10})$ or $f(\overline{01};\overline{01})$ and:

$$l_{f,\Lambda}(\overline{10};\overline{10}) < l_{f,\Lambda}(\overline{00};\overline{00}) < y < t_0,\quad \forall y\in L(f,\varphi_1|_{\Lambda_1}).$$

By the above process, we have a renormalization mechanism given by the cut in $[f=t_0]$, where from the $\varphi:\Lambda\to\Lambda$ we get $\varphi_1:\Lambda_1\to\Lambda_1$, where the last system is quite similar to the first. In terms of symbolic dynamic, the renormalization process is given by: $A_0 := 0\to A_1 := 01010$ and $B_0 := 1\to B_1 := 00$, with the symbolic representation given by $\sigma_1 := \sigma^{\tilde{\tau}_1} : [A_1, B_1/B_1\nrightarrow B_1]\to[A_1,B_1/B_1\nrightarrow B_1]$ instead of $\sigma:\Sigma_B\to\Sigma_B$.

We assume inductively that in the stage n of the renormalization process we have $\varphi_n : R_0^n \cup R_1^n \to \varphi_n(R_0^n \cup R_1^n)$ and a horseshoe $\Lambda_n := \Pi[A_n, B_n/B_n \nrightarrow B_n]$ associated with φ_n, such that the stable and unstable Cantor sets $K_s^{(n)}$ and $K_u^{(n)}$ of Λ_n are given by $g_{u,s}^{(n)} : I^{u,s}(A_n) \cup I^{u,s}(B_n) \to [0,1]$, where the branch associated to $I^{u,s}(A_n)$ is decreasing and the branch associated to $I^{u,s}(B_n)$ is increasing, and $I^{u,s}(A_n)$ is in the left of $I^{u,s}(B_n)$, where A_n and B_n are symmetric words with the common begin $A_{n-1}A_{n-2}\cdots A_0$.

For now, we may assume that we can take t_n for the dynamics $\varphi_n : \Lambda_n \to \Lambda_n$ as we take t_0 for the dynamic $\varphi : \Lambda \to \Lambda$, in order to iterate the renormalization process, see Figures 3.11 and 3.12 (we will prove the existence of t_n latter). More specifically, recursively given $\varphi_n : \Lambda_n \to \Lambda_n$ we chose $t_n < t_0$ that induce the renormalization given in symbolic language by $A_n \to A_{n+1} := A_nB_nA_nB_nA_n$ and $B_n \to B_{n+1} := A_nA_n$. First note that, we have that $A_{n+1}^t = A_{n+1}$, $B_{n+1}^t = B_{n+1}$, and A_{n+1} and B_{n+1} has the same begin $A_nA_{n-1}\cdots A_0$. Let $R_0^{n+1} := R_\varphi(A_n; A_{n+1})$ and $R_1^{n+1} := R_\varphi(A_n; B_{n+1})$. Define $\varphi_{n+1} : R_0^{n+1} \cup R_1^{n+1} \to \varphi_{n+1}(R_0^{n+1} \cup R_1^{n+1})$ given by $\varphi_{n+1}(x) := \varphi_n^{\tau_{n+1}(x)}(x)$, where $\tau_{n+1}(x) = \min\{k > 0 : \varphi_n^k(x) \in R_0^{n+1} \cup R_1^{n+1}\}$. Thus, $\Lambda_{n+1} := \Pi[A_{n+1}, B_{n+1}/B_{n+1} \nrightarrow B_{n+1}]$ is a horseshoe associated to φ_{n+1}, such that the stable and unstable Cantor sets $K_s^{(n+1)}$ and $K_u^{(n+1)}$ of Λ_{n+1} are given by $g_{u,s}^{(n+1)} : I^{u,s}(A_{n+1}) \cup I^{u,s}(B_{n+1}) \to [0,1]$, where the branch associated to $I^{u,s}(A_{n+1})$ is decreasing and the branch associated to $I^{u,s}(B_{n+1})$ is increasing, and $I^{u,s}(A_{n+1})$ is in the left of $I^{u,s}(B_{n+1})$. Moreover, we have:

$$f(R_\varphi(A_n; A_n)) > f(R_\varphi(B_n; A_n)), f(R_\varphi(A_n; B_n)) > f(R_\varphi(B_n; B_n)) \quad (3.31)$$

$$\begin{aligned} f(R_\varphi(B_nA_n; A_nB_n)) > f(R_\varphi(A_nA_n; A_nB_n)), f(R_\varphi(B_nA_n; A_nA_n)) > \\ > f(R_\varphi(A_nA_n; A_nA_n)) \quad (3.32) \end{aligned}$$

By (3.31) and (3.32), we get:

$$L(f, \varphi|_{(\Lambda_n)_{t_n}}) = L(f, \varphi_{n+1}|_{\Lambda_{n+1}}) \cup \{l_{f,\Lambda_n}(\overline{A_nA_n}; \overline{A_nA_n})\} \cup \{l_{f,\Lambda}(\overline{B_nA_n}; \overline{B_nA_n})\},$$

with

$$l_{f,\Lambda}(\overline{B_nA_n}; \overline{B_nA_n}) = f(\overline{B_nA_n}; \overline{B_nA_n}) \text{ or } f(\overline{A_nB_n}; \overline{A_nB_n}),$$
$$l_{f,\Lambda}(\overline{A_nA_n}; \overline{A_nA_n}) = f(\overline{A_nA_n}; \overline{A_nA_n})$$

and

$$l_{f,\Lambda}(\overline{B_nA_n}; \overline{B_nA_n}) < l_{f,\Lambda}(\overline{A_nA_n}; \overline{A_nA_n}) < y < t_n < t_0, \ \forall y \in L(f, \varphi_{n+1}|_{\Lambda_{n+1}}).$$

Since the strings A_{n+1} and B_{n+1} have the common begin equal to $A_nA_{n-1}\dots A_1A_0$. Thus, there exists a string $\alpha^* \in \Sigma_B$ such that $\alpha_n := (\overline{A_n};\overline{A_n})$, $\beta_n := (\overline{A_nB_n};\overline{A_nB_n})$ and $\tilde{\beta}_n := (\overline{B_nA_n};\overline{B_nA_n})$ converge to α^* when $n \to \infty$. Since $\alpha_n \to \alpha^*$ and $l_{f,\Lambda}(\alpha_n) = m_{f,\Lambda}(\alpha_n) = f(\alpha_n) \to f(\alpha^*) = m_{f,\Lambda}(\alpha^*)$, by Theorem 1, we have that $m_{f,\Lambda}(\alpha^*) \in L(f,\Lambda)$. Moreover,

$$\begin{aligned} l_{f,\Lambda}(\beta_0) <& l_{f,\Lambda}(\alpha_0) < \dots < l_{f,\Lambda}(\beta_n) < l_{f,\Lambda}(\alpha_n) < \\ &< l_{f,\Lambda}(\beta_{n+1}) < l_{f,\Lambda}(\alpha_{n+1}) < \dots < m_{f,\Lambda}(\alpha^*) \end{aligned}$$

Therefore, $m_{f,\Lambda}(\alpha^*)$ is the first accumulation point of $L(f,\Lambda)$ and

$$L(f,\Lambda) \cap (-\infty, m_{f,\Lambda}(\alpha^*)) = \{l_{f,\Lambda}(\beta_n),\ l_{f,\Lambda}(\alpha_n) :\ n \geq 0\}.$$

In the following we justify how the above renormalization process works for every (ψ, h) in a neighborhood of (φ, f) in $\mathrm{Diff}^2(\mathbb{S}^2) \times \mathrm{C}^1(\mathbb{S}^2;\mathbb{R})$, by justifying that for each of these pair is possible to take the required sequence $(t_n)_n$.

In the initial linear horseshoe Λ associated with φ, we have that the derivatives of the maps $g_s^{(n)}$ are constant in each branch, i.e., $|(g_s^{(n)})'|_{I^s(A_n)}| \equiv \lambda_n$ and $|(g_s^{(n)})'|_{I^s(B_n)}| \equiv \tilde{\lambda}_n$. Moreover, since $A_{n+1} = A_nB_nA_nB_nA_n$ and $B_{n+1} = A_nA_n$, we have $\lambda_{n+1} = \lambda_n^3\tilde{\lambda}_n^2$ and $\tilde{\lambda}_{n+1} = \lambda_n^2$. By the fact that the constant of bounded distortion vary continuously with the Cantor set, we can take an open neighborhood of hyperbolic continuation $\tilde{\mathcal{U}} \subset \mathrm{Diff}^2(\mathbb{S}^2)$ of φ, such that for every $\psi \in \tilde{\mathcal{U}}$, let $g_{s,\psi}$ and $g_{u,\psi}$ be the maps of definitions of the stable and unstable Cantor set associated with Λ_ψ, then there are constants $\lambda_n(\psi), \tilde{\lambda}_n(\psi)$ such that $d_n\lambda_n(\psi) \leq |(g_{s,\psi}^{(n)})'(x)| \leq e_n\lambda_n(\psi)$ and $\tilde{d}_n\tilde{\lambda}_n(\psi) \leq |(g_{s,\psi}^{(n)})'(y)| \leq \tilde{e}_n\tilde{\lambda}_n(\psi)$, for every $x \in I_\psi^s(A_n)$ and $y \in I_\psi^s(B_n)$, where $d_n^{\pm1}, e_n^{\pm1}, \tilde{d}_n^{\pm1}, \tilde{e}^{\pm1} \in (0.9999, 1.0001)$, for every $n \geq 1$. Thus, we have that $\lambda_{n+1}(\psi) = c_n\lambda_n^3(\psi)\tilde{\lambda}_n^2(\psi)$ and $\tilde{\lambda}_{n+1}(\psi) = \tilde{c}_n\lambda_n^2(\psi)$, for $c_n, \tilde{c}_n \in (0.999, 1.001)$.

Fix $\psi \in \tilde{\mathcal{U}}$, for simplicity let $\lambda_n = \lambda_n(\psi)$ and $\tilde{\lambda}_n = \tilde{\lambda}_n(\psi)$. Now, we define $r_n := \log\lambda_n/\log\tilde{\lambda}_n$. Thus,

$$r_{n+1} = \frac{\log\lambda_{n+1}}{\log\tilde{\lambda}_{n+1}} = \frac{3\log\lambda_n + 2\log\tilde{\lambda}_n + \log c_n}{2\log\lambda_n + \log\tilde{c}_n} = \frac{3r_n + 2 + \log c_n/\log\tilde{\lambda}_n}{2r_n + \log\tilde{c}_n/\log\tilde{\lambda}_n}.$$

Hence,

$$|r_{n+1} - 2| = \frac{|r_n - 2| + o(1)}{2r_n + o(1)} < \frac{|r_n - 2| + o(1)}{2}.$$

By shrinking $\tilde{\mathcal{U}}$ if necessarily, we have that $r_0 = r_0(\psi) = \dfrac{\log \lambda_0}{\log \tilde{\lambda}_0} \in (1.9999, 2.0001)$. Thus, by induction using the previous inequality, we have that $r_n = 2 + o(1)$, where $-0.001 < o(1) < 0.001$. Thus, we have, for all $n \geq 0$,

$$\tilde{\lambda}_n^{1.999} < \lambda_n < \tilde{\lambda}_n^{2.001}. \tag{3.33}$$

Since $\lambda_n, \tilde{\lambda}_n \geq 4$, we have $\tilde{\lambda}_n^{-0.001} < 0.999 < 1.001 < \tilde{\lambda}_n^{0.001}$. Thus, for all $n \geq 0$,

$$\tilde{\lambda}_n^{7.996} < \lambda_{n+1} < \tilde{\lambda}_n^{8.004} \text{ and } \tilde{\lambda}_n^{3.997} < \tilde{\lambda}_{n+1} < \tilde{\lambda}_n^{4.003}. \tag{3.34}$$

Similarly we have the same analyses for the unstable Cantor set, there are constants $\mu_n = \mu_n(\psi), \tilde{\mu}_n = \tilde{\lambda}_n(\psi)$ such that $D_n\mu_n \leq |(g_{u,\psi}^{(n)})'(x)| \leq E_n\mu_n$ and $\tilde{D}_n\tilde{\mu}_n \leq |(g_{u,\psi}^{(n)})'(y)| \leq \tilde{E}_n\tilde{\mu}_n$, for every $x \in I_\psi^u(A_n)$ and $y \in I_\psi^u(B_n)$, where $D_n^{\pm 1}, E_n^{\pm 1}, \tilde{D}_n^{\pm 1}, \tilde{E}^{\pm 1} \in (0.9999, 1.0001)$, for every $n \geq 0$. Thus, we have that $\mu_{n+1} = C_n\mu_n^3\tilde{\mu}_n^2$ and $\tilde{\mu}_{n+1} = \tilde{C}_n\mu_n^2$, for $C_n, \tilde{C}_n \in (0.999, 1.001)$. We also have $\tilde{\mu}_n^{1.999} < \mu_n < \tilde{\mu}_n^{2.001}$ and analogous inequalities as in (3.34).

By shrinking again $\tilde{\mathcal{U}}$ if necessarily, we have that $7.3999 < \dfrac{\log \lambda_0(\psi)}{\log \mu_0(\psi)}, \dfrac{\log \tilde{\lambda}_0(\psi)}{\log \tilde{\mu}_0(\psi)} < 7.4001$. Since,

$$\frac{\log \lambda_{n+1}}{\log \mu_{n+1}} = \frac{3\log \lambda_n + 2\log \tilde{\lambda}_n + \log c_n}{3\log \mu_n + 2\log \tilde{\mu}_n + \log C_n},$$

we have, for all $n \geq 0$:

$$7.39 < \frac{\log \lambda_n}{\log \mu_n} < 7.41 \tag{3.35}$$

Let Δ_n (resp. $\tilde{\Delta}_n$) be the length of the support interval of $K_s^{(n)}(\psi)$ (resp. $K_u^{(n)}(\psi)$) of $\Lambda_\psi^{(n)}$ associated to ψ. Since the common begin between A_{n+1} and B_{n+1} is A_n follows by the common begin between A_n and B_n, we have that $\Delta_{n+1} = |(g_{s,\psi}^{(n)})'(x)|^{-1}\Delta_n$, for some $x \in I_\psi^s(A_n)$. Thus,

$$\lambda_n^{-1.001}\Delta_n < e_n^{-1}\lambda_n^{-1}\Delta_n \leq \Delta_{n+1} \leq d_n^{-1}\lambda_n^{-1}\Delta_n < \lambda_n^{-0.999}\Delta_n. \tag{3.36}$$

It follows by induction that

$$\lambda_n^{-1/2.99} < \Delta_n \leq 1 \text{ (resp. } \mu_n^{-1/2.99} < \tilde{\Delta}_n \leq 1). \tag{3.37}$$

Indeed, by continuity we have $\lambda_0^{-1/2.99} < \Delta_0 \leq 1$. By (3.33), (3.34) and (3.36), we have

$$\lambda_{n+1}^{-1/2.99} < \lambda_{n+1}^{\frac{2.001}{7.996}(-1.001-1/2.99)} < \lambda_n^{-1.001}\lambda_n^{-1/2.99} < \Delta_{n+1} < \lambda_n^{-0.999} < 1.$$

In the step n, in order to get the renormalization process (following the cut d) in Figure 3.12), we need to take t_n such that the curve level on t_n satisfies that (see Figure 3.11):

$$(R_\varphi(A_n;A_nB_nA_nA_n)\cup R_\varphi(A_n;A_nB_nA_nB_nA_nB_n)\cup R_\varphi(A_n;A_nB_nA_nB_nA_n^3B_n) \cup R_\varphi(A_n;A_nB_nA_nB_nA_n^4))\cap\Lambda_n=[f>t_n]\cap\Lambda_n. \tag{3.38}$$

Thus, we need to take t_n such that the level curve by t_n crosses the gap J_n between $I_n^1 = I_\psi^u(A_nB_nA_nB_nA_n^5)$ and $I_n^2 = I_\psi^u(A_nB_nA_nB_nA_n^4B_n)$, see Figure 3.13. Since the strings A_n and B_n have the common begin equal to $A_{n-1}\dots A_1A_0$, we have that the smallest stage of the unstable Cantor set containing the two previous stages is $D_n^u := I^u(A_nB_nA_nB_nA_n^4A_{n-1}\cdots A_1A_0)$. Hence, $|J_n| = |D_n^u| - |I_n^1| - |I_n^2|$, by (3.33), (3.34) and (3.37), we have:

$$\begin{aligned}|J_n| &> (\mu_n^{-1.001})^6(\tilde\mu_n^{-1.001})^2\tilde\Delta_n - (\mu_n^{-0.999})^7(\tilde\mu_n^{-0.999})^2\tilde\Delta_n - (\mu_n^{-0.999})^7(\tilde\mu_n^{-0.999})^3\tilde\Delta_n\\ &= (\mu_n^{-1.001})^6(\tilde\mu_n^{-1.001})^2\tilde\Delta_n[1-\mu_n^{-0.987}\tilde\mu_n^{-0.004}-\mu_n^{-0.987}\tilde\mu_n^{-0.995}]\\ &> \mu_n^{-7.346}(1-\mu_n^{-0.989}-\mu_n^{-1.484})\end{aligned}$$

Analogously, let K_n be the smallest stage of the unstable Cantor set containing the two stages $I_\psi^u(A_nB_nA_nA_n)$ and $I_\psi^u(A_nB_nA_nB_n)$, see the cut a) in Figure 3.12, then we have

$$|K_n| < (\mu_n^{-0.999})^2\tilde\mu_n^{-0.999}\tilde\Delta_n < \mu_n^{-2\cdot 0.999-0.999/2.001} < \mu_n^{-2.497}.$$

Let L_n be the interval get from Δ_n minus the stage $I_\psi^s(B_n)$. Thus, $|L_n| = |\Delta_n| - |I_\psi^s(B_n)|$, by (3.33), (3.34) and (3.37), we have:

$$|L_n| \geq \lambda_n^{-1/2.99}-\lambda_n^{-0.999}\tilde\lambda_n^{-0.999}\Delta_n > \lambda_n^{-0.335}-\lambda_n^{-0.999-0.999/2.001} > \lambda_n^{-0.335}-\lambda_n^{-1.496}.$$

We also have $|I_\psi^s(A_n)| < \Delta_n\lambda_n^{-0.999} \leq \lambda_n^{-0.999}$. In order to take t_n such that the renormalization process works we need to have $|J_n| \gg |I_\psi^s(A_n)|$ and $|L_n| \gg |K_n|$, see Figure 3.13. By (3.35), we get $\mu_n^{7.39} < \lambda_n < \mu_n^{7.41}$, for all n. Therefore, using the previous estimates, we get:

$$\frac{|J_n|}{|I_\psi^s(A_n)|} > \frac{\mu_n^{-7.346}(1-\mu_n^{-0.989}-\mu_n^{-1.484})}{\lambda_n^{-0.999}} > \frac{\mu_n^{-7.346}(1-\mu_n^{-0.989}-\mu_n^{-1.484})}{\mu_n^{-7.38261}} \tag{3.39}$$

$$\frac{|L_n|}{|K_n|} > \frac{\lambda_n^{-0.335}-\lambda_n^{-1.496}}{\mu_n^{-2.497}} > \frac{\lambda_n^{-0.335}-\lambda_n^{-1.496}}{\lambda_n^{-2.497/7.41}} > \frac{\lambda_n^{-0.335}-\lambda_n^{-1.496}}{\lambda_n^{-0.336}} \tag{3.40}$$

Thus, if we choose initially μ_0 and λ_0 big enough, then μ_n and λ_n also are big enough (because they grown exponentially fast), we have that the inequalities (3.39) and (3.40) are both uniformly bigger then 1, for all $n \geq 0$.

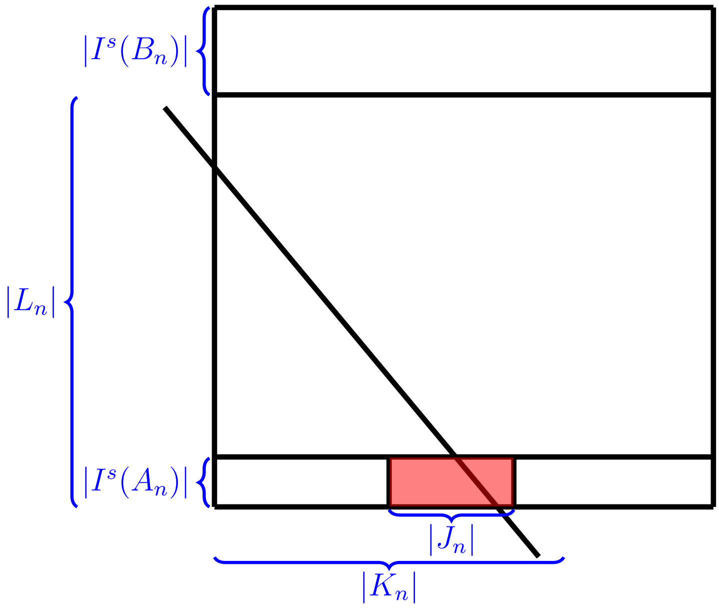

Figure 3.13: The renormalization cut in the stage n.

Finally, we take this neighborhood $\tilde{\mathcal{U}} \subset \mathrm{Diff}^2(\mathbb{S}^2)$ of φ and a neighborhood $\tilde{\mathcal{V}} \subset C^1(\mathbb{S}^2;\mathbb{R})$ of f, such that, for any $n \geq 0$, the inequalities (3.31) and (3.32) hold for every $(\psi, g) \in \tilde{\mathcal{U}} \times \tilde{\mathcal{V}}$. Moreover, we also requires that $\tilde{\mathcal{V}}$ is sufficiently small in C^1 topology such that the level curve $[g = t_n]$ is uniformly close to $[f = t_n]$, in a such way that also allows us to get the renormalization process (this uniformity is given by the inequalities (3.39) and (3.40)).

Therefore, this entire discussion in this subsection can be summarized in the following:

Theorem 6. *There are open neighborhoods $\tilde{\mathcal{U}} \subset \mathrm{Diff}^2(\mathbb{S}^2)$ of φ and $\tilde{\mathcal{V}} \subset C^1(\mathbb{S}^2;\mathbb{R})$ of f, such that the beginning of $L(g, \Lambda_\psi)$ has infinitely many points, for every $(\psi, g) \in \tilde{\mathcal{U}} \times \tilde{\mathcal{V}}$, where Λ_ψ is the hyperbolic continuation of Λ.*

CHAPTER 4

$M \backslash L$ near 3

In this chapter, we provide some evidence in favor of the possibility that $M \setminus L$ is not closed, so that the answer to T. Bousch's question about the closedness of $M \setminus L$ might be negative. We construct four new elements $m_4 < m_3 < m_2 < m_1 < 3.11$ of $M \backslash L$ lying in distinct connected components of $\mathbb{R} \setminus L$.

These elements are part of a decreasing sequence $(m_k)_{k \in \mathbb{N}}$ of elements in M converging to 3 and we give some evidence towards the possibility that $m_k \in M \setminus L$ for all $k \geq 1$. In particular, this indicates that 3 might belong to the closure of $M \setminus L$.

4.1 Main result

For each $k \in \mathbb{N}^*$, consider the finite string $\underline{\omega}_k := (2_{2k}, 1_2, 2_{2k+1}, 1_2, 2_{2k+2}, 1_2)$ and the bi-infinite word $\gamma_k^1 := (\overline{\underline{\omega}_k}\underline{\omega}_k^*\underline{\omega}_k\overline{2})$ where the asterisk indicates that the $(2k+2)$-th position occurs in the first 2 in substring 2_{2k+1} of $\underline{\omega}_k$. In this context, the main result in this chapter is the next theorem:

Theorem 7. *The Markov values $m_k = m(\gamma_k^1)$ form a decreasing sequence converging to 3 whose first four elements belong to $M \setminus L$. Moreover, these four elements belong to distinct connected components of $\mathbb{R} \setminus L$.*

Remark 4.1. Even though we will not pursue this direction here, the technique used in [26], [27], [25] suggests that it might be possible to extend our discussion below to show that, for each $k \in \{1,2,3,4\}$, the connected component of $\mathbb{R} \setminus L$ containing m_k intersects $M \setminus L$ in a Cantor set of positive Hausdorff dimension.

Remark 4.2. The smallest known numbers in $M \setminus L$ was nearby 3.1181, but we have that m_1, m_2, m_3 and m_4 are approximately $3.005, 3.0001, 3.000004$ and 3.0000001, respectively.

4.2 Ideas to construct points in $M \setminus L$

Our construction of elements in $M \setminus L$ follows the ideas of Freiman [10], [11], Flahive [12] and posteriorly of Moreira e Mathues [26], [27], [25]. The approach here is based on some qualitative dynamical insights leading to a series of quantitative estimates with continued fractions, as we can see explained in [25] and it presents in this section.

In [12], Flahive introduced the following notion of *semi-symmetric* words:

Definition 4.1. Let $\alpha = (c_1, c_2, \cdots, c_s)$ be a word of positive integer. We call α a *semi-symmetric* word if $(c_1, c_2, \cdots, c_s) = (c_s, c_{s-1}, \cdots, c_1)$ or there exists a integer $1 \le i \le s-1$, with $(c_1, \cdots, c_i, c_{i+1}, \cdots c_s) = (c_i, \cdots, c_1, c_s, \cdots, c_{i+1})$.

Flahive proved that an element of $M \setminus L$ is usually associated to non semi-symmetric words. In particular, it is not surprising that Freimans construction of elements in $M \setminus L$ is related to the non semi-symmetric words of *odd lengths*, and the construction in [26], [27] and [25] of new elements in $M \setminus L$ is also based on the non semi-symmetric words of odd lengths.

Let α given a word non semi-symmetric of odd length, which the Markov value of the periodic sequence associated $\bar{\alpha} = \cdots \alpha\alpha \cdots$ is $l = m(\bar{\alpha})$, we select a complete subshift Σ_α of sequences whose Markov values are $< l$.

We choose a word of odd length because any modification of the associated infinite periodic sequence will force a definite increasing of the Markov value in one of two consecutive periods.

Since α is not semi-symmetric, the problems of gluing sequences in Σ_α on the left or on the right of $\bar{\alpha} = \cdots \alpha\alpha \cdots$ in such a way that the Markov value of the resulting sequence doesn't increase too much might have distinct answers. In fact, let $\Sigma_\alpha^{+,-} := \Sigma_\alpha \cap (\mathbb{N}^*)^{\mathbb{Z}_{\ge 0}, \mathbb{Z}_{<0}}$ the projections in the non-negatives(≥ 0) and negatives(< 0) positions, and if $\alpha = a_1 a_2 \cdots a_s$ then the smaller Markov values μ of $\bar{\alpha} a_1 a_2 \cdots a_m z$, with $z \in \Sigma_\alpha^+$ and $1 \le m \le s$ is *systematically* smaller than the Markov values ν of smaller Markov values of

$wa_n \cdots a_{s-1}a_s\bar{\alpha}$, with $w \in \Sigma_\alpha^-$ and $1 \le n \le s$, because the gluings of $a_1a_2\cdots a_m$ and z is a different problem from the gluings of w and $a_n \cdots a_{s-1}a_s$.

In other words, the cheapest cost of gluing $z' \in \Sigma_\alpha^+$ on the right of $\bar{\alpha}\alpha\cdots$ as $\bar{\alpha}a_1a_2\cdots a_{m'}z'$ is always smallest than the cost of gluing any $w \in \Sigma_\alpha^-$ on the left of $\cdots\alpha\bar{\alpha}$. Hence, the Markov value μ of $\bar{\alpha}a_1a_2\cdots a_{m'}z'$ is likely to belong to $M \setminus L$, because any attempt to modify the left side of $\bar{\alpha}a_1a_2\cdots a_{m'}z'$ to reproduce big chunks of this sequence (in order to show that $\mu \in L$) would fail since it ends up producing a subword close to the sequence $w\alpha\alpha\cdots\alpha\alpha a_1a_2\cdots a_{m'}z'$ whose Markov value would be $\nu > \mu$.

The previous discussion can be qualitatively rephrased in dynamical terms as follows. Remember that in the Section 3.1 we recover the classical Markov and Lagrange spectra from a dynamical approach. In order to do that, let $\varphi : (0,1)^2 \to (0,1)^2$ defined by $\varphi(x,y) = (\{1/x\}, 1/(\lfloor 1/x \rfloor + y))$ and $f : (0,1)^2 \to \mathbb{R}$ defined by $f(x,y) = x + y$. Given $(x,y) \in (0,1)^2$ a pair of irrational numbers, we associate the sequence $\underline{\theta} = (a_n)_{n\in\mathbb{Z}} \in \Sigma := (\mathbb{N}^*)^{\mathbb{Z}}$, where $x = [0; a_0, a_1, \cdots]$ and $y = [0; a_{-1}, a_{-2}, \cdots]$.

The periodic sequence $\bar{\alpha} \in \Sigma$ provides a periodic point $p_\alpha \in (0,1)^2$ such that $l = f(p_\alpha) = \max_{n\in\mathbb{Z}} f(\varphi^n(p_\alpha))$. The problems of gluing sequences in Σ_α on the left and right of $\bar{\alpha} = \cdots\alpha\alpha\cdots$ have a dynamical meaning: it amounts to study the intersections $W^u_{\text{loc}}(\Lambda_\alpha) \cap W^s_{\text{loc}}(p_\alpha)$ and $W^s_{\text{loc}}(\Lambda_\alpha) \cap W^u_{\text{loc}}(p_\alpha)$.

Geometrically, the fact that p_α comes from a non semi-symmetric word α of odd length suggests that the local stable and unstable manifolds of p_α intersect the invariant manifolds of the hyperbolic subset $\Lambda_\alpha \subset (0,1)^2$ related to Σ_α at distinct heights with respect to $f(x,y) = x + y$. In fact, one can show that the smallest height μ of a point $q_\alpha := W^u_{\text{loc}}(p_\alpha) \cap W^s_{\text{loc}}(\tilde{p})$ for some $\tilde{p} \in \Lambda_\alpha$ is strictly smaller than the minimal height ν of any point $r \in W^s_{\text{loc}}(p_\alpha) \cap W^u_{\text{loc}}(\Sigma_\alpha)$: this is called *self- replication mechanism* and is depicted in Figure 4.1.

Moreover, the φ-orbit of q_α is locally unique in the sense that some portion of the φ-orbit of any point $z \in (0,1)^2$ with $\sup_{n\in\mathbb{Z}} f(\varphi^n(z))$ close to μ must stay close to the first few φ-iterates of q_α: this is called *local uniqueness*.

By using this two previous parts of the argument, we can show that the Markov value μ doesn't belong to the Lagrange spectrum L. More specifically, if $\mu = \limsup_{n\to\infty} f(\varphi^n(z)) \in L$, for some $z \in (0,1)^2$, then the local uniqueness property would say that some portion $\{\varphi^{n_0}(z), \cdots, \varphi^{n_0+m_0}(z)\}$ of the φ-orbit of z is close to the first few φ-iterates $\{\varphi(q_\alpha), \cdots, \varphi^{m_0}(q_\alpha)\}$,

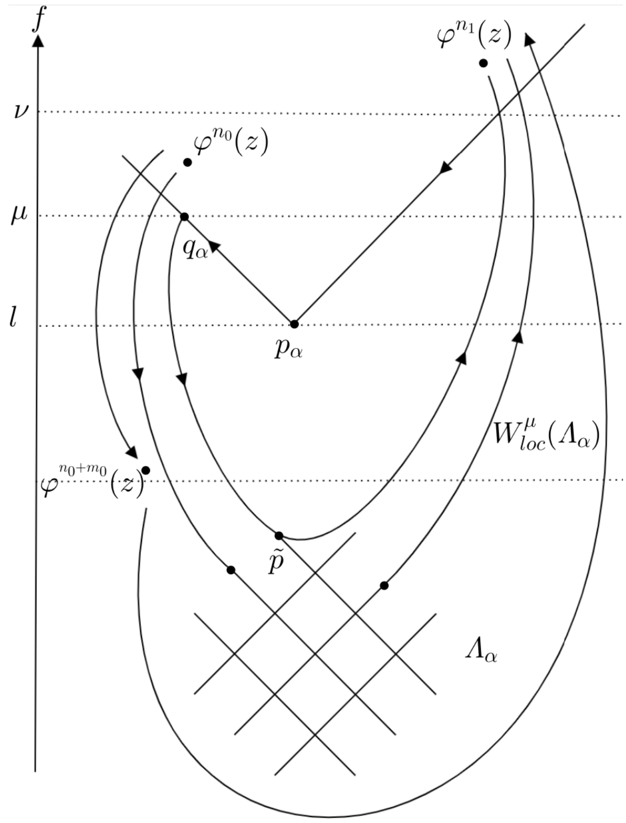

Figure 4.1: Ideas behind a point in $M \setminus L$.

so that $\varphi^{n_0+m_0}(z)$ is close to Λ_α. On the other hand, the assumption that $\mu = \limsup_{n\to\infty} f(\varphi^n(z))$ and the local uniqueness property say that there exists a $n_1 > n_0 + m_0$ such that $\varphi^{n_1}(z)$ is again close to q_α. However, this is impossible because the iterates of $\varphi^{n_0+m_0}(z)$ would follow $W^u_{\text{loc}}(\Lambda_\alpha)$ in their way to reach $\varphi^{n_1}(z)$ and we know that the smallest height of the intersection between $W^s_{\text{loc}}(q_\alpha)$ and $W^u_{\text{loc}}(\Lambda_\alpha)$ is $\nu > \mu$: see Figure 4.1.

Here, we study exclusively the portion of M below $\sqrt{12}$ and, for this reason, we assume that all sequences appearing in the sequel consist of 1 and 2 (i.e., all sequences in this paper belong to $\{1,2\}^{\mathbb{Z}}$ by default).

In this chapter, for the selected non-semi-symmetric word $\underline{\omega}_k$ of odd lengths the *local uniqueness* and *self-replication* properties are quantitatively described as:

- the local uniqueness asks that any word $\theta \in \{1,2\}^{\mathbb{Z}}$ with Markov value $m(\theta) = \lambda_0(\theta)$ sufficiently close to m_k has the form

$$\theta = \dots 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1 \dots$$

(up to transposition)

- the self-replication requires that any word $\theta \in \{1,2\}^{\mathbb{Z}}$ of the form $\theta = \dots 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1 \dots$ whose Markov value $m(\theta)$ is sufficiently close to m_k extends as

$$\theta = \overline{2_{2k}1_2 2_{2k+1}1_2 2_{2k+2}1_2} 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2_{2k+1}1_2 2_{2k+2}1_2 2_{2k} \dots$$

It is not hard to see that these properties imply that $m_k \in M \setminus L$ because they would say that a periodic word θ with Markov value $m(\theta)$ sufficiently close to m_k must coincide with the periodic word $\theta(\underline{\omega}_k)$ determined by $\underline{\omega}_k$, a contradiction with the fact that $m_k \neq m(\theta(\underline{\omega}_k))$.

We establish in Section 5.5 below that the self-replication property holds for every $k \in \mathbb{N}$. Since that the combinatorics of the words in $\{1,2\}^{\mathbb{Z}}$ with Markov value 3 is quite intricate, fact explained in Bombieri's survey [2], we could not find a systematic argument allowing to obtain the local uniqueness property for every $k \in \mathbb{N}$. For this reason, in Section 4.6, we prove the local uniqueness property for $k \in \{1,2,3,4\}$ and an "almost uniqueness" property for all $k \in \mathbb{N}$ in Section 4.8.

Nevertheless, there is still some hope to get the local uniqueness property for m_k because Proposition 1 in [30] seems to indicate that the function $\{1,2\}^{\mathbb{Z}} \ni \theta \mapsto m(\theta) \in \mathbb{R}$ could be injective on $m^{-1}((3, 3.0056))$, and this give some support to the possibility that $m_k \in M \setminus L$ for every $k \in \mathbb{N}$.

4.3 Prohibited and avoided strings

In this section, we introduce the notions of prohibited and avoided strings. Before, recall that $\underline{\omega}_k := (2_{2k}, 1_2, 2_{2k+1}, 1_2, 2_{2k+2}, 1_2)$ is a finite string determining a periodic word $\theta(\underline{\omega}_k)$ and a bi-infinite word $\gamma_k^1 := (\overline{\omega}_k \underline{\omega}_k^* \underline{\omega}_k \bar{2})$ where the asterisk indicates the $(2k+2)$-position occurs at the first 2 in 2_{2k+1} in $\underline{\omega}_k$. In the following, we analyse the Markov value of $\theta(\underline{\omega}_k)$ and γ_k^1.

Lemma 4.1. *If $\theta = (a_n)_{n\in\mathbb{Z}}$ contains $(a_n)_{i-1\le n\le i+1} = (222)$, then $\lambda_i(\theta) < 2.85$.*

Proof. In fact, $\lambda_i(\theta) = [2;2,...] + [0;2,...] \le 2 + 2[0;2,\overline{2,1}] < 2.85$. □

Lemma 4.2. *Let $Y = \dfrac{(n_1+x_1+x_2)(n_2+x_3+x_4)}{(n_3+x_5+x_6)(n_4+x_7+x_8)}$, where $n_i \in \{1,2\}$ and $x_i \in [0,1]$ has at least two digits in its continued fraction expansion and they are only 1 or 2, then $0.226 < 100/441 \le Y \le 441/100 = 4.41$.*

Proof. We minimize the numerator with $(1+[0;2,1]+[0;2,1])(1+[0;2,1]+[0;2,1]) = (1+2[0;2,1])^2 = 25/9$. And the denominator is upper bound by $(2+2[0;1,2,1])^2 = 49/4$. Then, $100/441 \leq Y \leq 441/100$. □

Lemma 4.3. *The Markov value of $\theta(\underline{\omega}_k)$ is attained at the position $2k+2$. In particular, $m(\theta(\underline{\omega}_k))$ is a decreasing sequence converging to 3.*

Proof. First, by Lemma 4.1, $\lambda_i(\theta(\underline{\omega}_k)) < 2.85$ for $i \in \{1,\dots,2k-2, 2k+3,\dots,4k+1, 4k+6,\dots,6k+5\}$. Moreover, if $\alpha_k := [2_{2k}, 1_2, 2_{2k+2}, ...]$ and $\beta_k := [2_{2k-1}, 1_2, ...]$, then $\beta_k > \alpha_k$. Thus, (2.2) implies that

$$\lambda_{2k+2}(\theta(\underline{\omega}_k)) = [2, 1_2, \alpha_k] + [0; 2, \beta_k] > 3.$$

Therefore, $m(\theta(\underline{\omega}_k)) = \lambda_i(\theta(\underline{\omega}_k))$ for some $i \in \{0, 2k-1, 2k+2, 4k+2, 4k+5, 6k+6\}$. Since we also have that $\lambda_{2k+2}(\theta(\underline{\omega}_k)) > \lambda_{2k+4}(\theta(\underline{\omega}_{k+1}))$ and $\lim_{k\to+\infty} \lambda_{2k+2}(\theta(\underline{\omega}_k)) = 3$ (because $\lim_{k\to+\infty}\alpha_k = \lim_{k\to+\infty}\beta_k = [2;\overline{2}]$), our task is reduced to show that

$$\lambda_i(\theta(\underline{\omega}_k)) \leq \lambda_{2k+2}(\theta(\underline{\omega}_k))$$

for each $i \in \{0, 2k-1, 4k+2, 4k+5, 6k+6\}$.

In this direction, note that

$$\lambda_0(\theta(\underline{\omega}_k)) = [2; 2_{2k-1}, 1_2, 2_{2k+1}, 1_2, 2_{2k+2}, 1_2, 2_{2k}...] + [0; 1_2, 2_{2k+2}, 1_2, 2_{2k+1}, 1_2, 2_{2k}, ...],$$

$$\lambda_{2k-1}(\theta(\underline{\omega}_k)) = [2; 1_2, 2_{2k+1}, 1_2, 2_{2k+2}, 1_2, 2_{2k}, ...] + [0; 2_{2k-1}, 1_2, 2_{2k+2}, 1_2, 2_{2k+1}, 1_2, ...],$$

$$\lambda_{2k+2}(\theta(\underline{\omega}_k)) = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k}, 1_2, ...] + [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k+1}, ...],$$

$$\lambda_{4k+2}(\theta(\underline{\omega}_k)) = [2; 1_2, 2_{2k+2}, 1_2, 2_{2k}, 1_2, 2_{2k+1}, 1_2, ...] + [0; 2_{2k}, 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, ...],$$

$$\lambda_{4k+5}(\theta(\underline{\omega}_k)) = [2; 2_{2k+1}, 1_2, 2_{2k}, 1_2, 2_{2k+1}, 1_2, ...] + [0; 1_2, 2_{2k+1}, 1_2, 2_{2k}, 1_2, ...],$$

$$\lambda_{6k+6}(\theta(\underline{\omega}_k)) = [2; 1_2, 2_{2k}, 1_2, 2_{2k+1}, 1_2, 2_{2k+2}, 1_2, ...] + [0; 2_{2k+1}, 1_2, 2_{2k+1}, 1_2, 2_{2k}, ...]$$

A direct inspection of these formulas reveals that $\lambda_{2k+2}(\theta(\underline{\omega}_k)) > \lambda_i(\theta(\underline{\omega}_k))$ for each $i \in \{0, 2k-1, 4k+5, 6k+6\}$. Thus, it suffices to prove that

$$\lambda_{2k+2}(\theta(\underline{\omega}_k)) > \lambda_{4k+2}(\theta(\underline{\omega}_k)).$$

For this sake, let us write

$$\lambda_{2k+2}(\theta(\underline{\omega}_k)) - \lambda_{4k+2}(\theta(\underline{\omega}_k)) = A_k - D_k + B_k - C_k,$$

where

$$A_k = [0; 2_{2k}, 1_2, 2_{2k+2}, 1_2, \overline{\underline{\omega}_k}], \quad D_k = [0; 2_{2k}, 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k+1}, 1_2, 2_{2k}, \overline{\underline{\omega}_k^t}],$$

$$B_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k+1}, 1_2, 2_{2k}, \overline{\underline{\omega}_k^t}], \quad C_k = [0; 1_2, 2_{2k+2}, 1_2, \overline{\underline{\omega}_k}],$$

and $\underline{\omega}_k^t$ is the transpose of $\underline{\omega}_k$.

Observe that $g^{4k+2}(A_k) = g^{2k+2}(C_k) := 1/x$ and $g^{4k+2}(D_k) = g^{2k+2}(B_k) := 1/y$, where g is the Gauss map acts. Then,

$$D_k - A_k = \frac{(y-x)}{q_{4k+2}^2(x+\beta)(y+\beta)} \text{ and } B_k - C_k = \frac{y-x}{\tilde{\beta}_{2k+2}^2(x+\tilde{\beta})(y+\tilde{\beta})},$$

where $q_{4k+2} = q(2_{2k}1_2 2_{2k})$, $\tilde{q}_{2k+2} = q(1_2 2_{2k})$, $\beta = [0; 2_{2k}, 1_2, 2_{2k}]$ and $\tilde{\beta} = [0; 2_{2k}, 1_2]$. Note that

$$A_k - D_k + B_k - C_k > 0 \Leftrightarrow \frac{B_k - C_k}{D_k - A_k} > 1.$$

We have that

$$\frac{B_k - C_k}{D_k - A_k} = \frac{q_{4k+2}^2}{\tilde{q}_{2k+2}^2} \cdot \frac{(x+\beta)(y+\beta)}{(x+\tilde{\beta})(y+\tilde{\beta})},$$

since $q_{4k+2} > q(2_2)\tilde{q}_{2k+2} = 5\tilde{q}_{2k+2}$, by Lemma 4.2, we get the result. $\square$

Lemma 4.4. *The Markov value of γ_k^1 is attained at the position $2k+2$. In particular, $m(\theta(\underline{\omega}_k)) < m(\gamma_k^1) < m(\theta(\underline{\omega}_{k-1}))$, $k > 1$.*

Proof. First, let i be the position such that

$$\lambda_i(\gamma_k^1) = [2; \overline{2}] + [0; 1_2, 2_{2k+2}, 1_2, ...] = [2; \overline{2}] + [0; \overline{\underline{\omega}_k^t}].$$

Since $[2; \overline{2}] < [2; 2_{2k}, 1_2, ...]$ and $[0; 1_2, 2_{2k+2}, 1_2, ...] < [0; 1_2, 2_{2k}, 1_2, ...]$ we have that

$$\lambda_i(\gamma_k^1) < \lambda_{2k+2}(\gamma_k^1).$$

Then, like above, it suffices to prove that $\lambda_{2k+2}(\gamma_k^1) > \lambda_{4k+2}(\gamma_k^1)$. For this sake remember that

$$\lambda_{2k+2}(\gamma_k^1) = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, \underline{\omega}_k, \overline{2}] + [0, 1_2, 2_{2k}, \overline{\underline{\omega}_k^t}],$$

while

$$\lambda_{4k+2}(\gamma_k^1) = [2; 2_{2k}, 1_2, 2_{2k}, \overline{\omega_k^t}] + [0; 1_2, 2_{2k+2}, 1_2, \underline{\omega}_k, \overline{2}].$$

Then, $\lambda_{2k+2}(\gamma_k^1) - \lambda_{4k+2}(\gamma_k^1) = A_k - C_k + B_k - D_k$, where

$A_k = [0; 2_{2k}, 1_2, 2_{2k+2}, 1_2, \underline{\omega}_k, \overline{2}]$, $\quad B_k = [0, 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k+1}, 1_2, 2_{2k}, \underline{\overline{\omega_k^t}}]$

and

$$C_k = [0; 2_{2k}, 1_2, 2_{2k}, \overline{\omega_k^t}], \quad D_k = [0; 1_2, 2_{2k+2}, 1_2, \underline{\omega}_k, \overline{2}].$$

Observe that $g^{4k+2}(A_k) = g^{2k+2}(D_k) := 1/x$ and $g^{2k+2}(B_k) = g^{4k+2}(C_k) := 1/y$.

Thus,

$$C_k - A_k = \frac{y-x}{q_{4k+2}^2(x+\beta)(y+\beta)} \text{ and } B_k - D_k = \frac{y-x}{\tilde{q}_{2k+2}^2(x+\tilde{\beta})(y+\tilde{\beta})},$$

where $q_{4k+2} = q(2_{2k}1_2 2_{2k})$, $\tilde{q}_{2k+2} = q(1_2 2_{2k})$, $\beta = [0; 2_{2k}, 1_2, 2_{2k}]$ and $\tilde{\beta} = [0; 2_{2k}, 1_2]$. Note that

$$A_k - C_k + B_k - D_k > 0 \Leftrightarrow \frac{B_k - D_k}{C_k - A_k} > 1.$$

We have that

$$\frac{B_k - D_k}{C_k - A_k} = \frac{q_{4k+2}^2}{\tilde{q}_{2k+2}^2} \cdot \frac{(x+\beta)(y+\beta)}{(x+\tilde{\beta})(y+\tilde{\beta})},$$

since $q_{4k+2} > q(2_2)\tilde{q}_{2k+2} = 5\tilde{q}_{2k+2}$, by Lemma 4.2, we get that $\lambda_{2k+2}(\gamma_k^1) > \lambda_{4k+2}(\gamma_k^1)$.

Finally, note that

$$\lambda_{2k+2}(\gamma_k^1) > \lambda_{2k+2}(\theta(\underline{\omega}_k)). \tag{4.1}$$

In fact, since $|\underline{\omega}_k|$ is odd, we have

$$[2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, \underline{\omega}_k, \overline{2}] > [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, \overline{\omega}_k, 2_{2k}, 1...] \tag{4.2}$$

and

$$[0; 1_2, 2_{2k}, \underline{\overline{\omega^t}}_k] = [0; 1_2, 2_{2k}, \underline{\overline{\omega^t}}_k]. \tag{4.3}$$

By (4.2) and (4.3), we have (4.1). It is easy to see that $m(\theta(\underline{\omega}_{k-1})) > m(\gamma_k^1)$, $k > 1$. $\square$

Given a finite string $\underline{u} = (a_i)_{i=-m}^n$, we define

$$\lambda_i^-(\underline{u}) := \min\{[a_i; a_{i+1}, ..., a_n, \theta_1] + [0; a_{i-1}, ..., a_{-m}, \theta_2] : \theta_1, \theta_2 \in \{1,2\}^{\mathbb{N}}\},$$

and

$$\lambda_i^+(\underline{u}) := \max\{[a_i; a_{i+1}, ..., a_n, \theta_1] + [0; a_{i-1}, ..., a_{-m}, \theta_2]; \theta_1, \theta_2 \in \{1,2\}^{\mathbb{N}}\}.$$

Definition 4.2. A finite string $\underline{u} = (a_i)_{i=-m}^{n}$ is:

- k-prohibited if $\lambda_i^-(\underline{u}) > m(\gamma_k^1)$, for some $-m \leq i \leq n$.
- k-avoided if $\lambda_0^+(\underline{u}) < m(\theta(\underline{\omega}_k))$.

A word $\theta \in \{1,2\}^{\mathbb{Z}}$ is (k,λ)-admissible whenever $m(\theta(\underline{\omega}_k)) < m(\theta) = \lambda_0(\theta) < \lambda$.

These notions are crucial in the study of the self-replication and local uniqueness properties. Indeed, the self-replication is based on the construction of an appropriate finite set of prohibited strings, the local uniqueness relies on the identification of an adequate finite set of prohibited and avoided strings, and the self-replication and local uniqueness properties imply that the Markov value of any (k,λ_k)-admissible word belongs to $M \setminus L$ whenever λ_k is close to $m_k = m(\gamma_k^1)$.

Remark 4.3. By Lemmas 4.3 and 4.4, if $\underline{u}$ is $(k-1)$-prohibited, resp. $(k+1)$-avoided, then it is also k-prohibited, resp. k-avoided. Also, by definition, a k-avoided string can not appear in the center of a (k,λ)-admissible word.

In the sequel, we give basic examples of prohibited, avoided and admissible words.

Lemma 4.5. *The strings* (12^*1), (2^*12), (1112^*22), (21_32^*211) *(and their transpositions) are k-prohibited for all $k \in \mathbb{N}$.*

Proof. In fact, we have

(1) $\lambda_0^-(12^*1) = [2;1,\overline{1,2}] + [0;1,\overline{1,2}] > 3.15$;

(2) $\lambda_0^-(2^*12) = [2;1,2,\overline{2,1}] + [0;\overline{2,1}] > 3.06$;

(3) $\lambda_0^-(1_32^*2_2) = [2;2_2,\overline{2,1}] + [0;1_3,\overline{1,2}] > 3.02$;

(4) $\lambda_0^-(21_32^*21_2) = [2;2,1,1,\overline{1,2}] + [0;1,1,1,2,\overline{2,1}] > 3.009$.

Since $m(\gamma_1^1) = 3.00558731248699779947\ldots$, it follows from Remark 4.3 that the proof of the lemma is complete. $\square$

Remark 4.4. Let θ be a $(k, 3.009)$-admissible word. It follows from the proof of Lemma 4.5 that:

- if $\theta = ...12...$, then $\theta = ...12_2...$;
- if $\theta = ...21..$, then $\theta = ...21_2...$;

- if $\theta = ...1_3 2_2...$, then $\theta = ...1_3 2_2 1_2...$, and
- if $\theta = ...1_3 2_2 1_2...$, then $\theta = ...1_4 2_2 1_2.....$

We use this remark systematically in what follows.

Corollary 4.1. *Given $k \geq 1$, if θ is $(k, 3.009)$-admissible, then, up to transposition, $\theta = (...1_2 2^* 2 1_2...)$ or $(...1_2 2^* 2_2...)$.*

Proof. Note that 222 is k-avoided (cf. Lemma 4.1). Thus, by Remark 4.4, it follows that, up to transposition, a $(k, 3.009)$-admissible word θ is $\theta = (...1_2 2^* 2 1_2...)$ or $\theta = (...1_2 2^* 2_2...)$. □

Lemma 4.6. *The string $2 1_2 2^* 2 1_2$ is k-avoided for any $k \in \mathbb{N}$.*

Proof. In fact, $\lambda_0^+(\theta) = [2; 2, 1_2, \overline{2, 1}] + [0; 1_2, 2, \overline{2, 1}] < 2.98$. □

Corollary 4.2. *Given $k \geq 1$, if θ is $(k, 3.009)$-admissible, then, up to transposition, either $\theta = (...1_4 2^* 2 1_2...)$ or $(...2_2 1_2 2^* 2_2...)$.*

Proof. By Corollary 4.1, $\theta = (...1_2 2^* 2 1_2...)$ or $\theta = (...1_2 2^* 2_2...)$. In the first case, by Lemma 4.6, θ extends as $\theta = (...1_3 2^* 2 1_2...)$. So, by Remark 4.4, it follows that θ extends as $(...1_4 2^* 2 1_2...)$ or $(...2_2 1_2 2^* 2_2...)$. □

4.4 Replication mechanism for γ_k^1

In this section, we investigate the extensions of a word θ containing the string

$$\theta_k^0 := 2_{2k} 1_2 2_{2k+2} 1_2 2_{2k} 1_2 2^* 2_{2k} 1_2 2_{2k+2} 1_2 2_{2k} 1 \tag{4.4}$$

4.4.1 Extension from θ_k^0 to $2_{2k} 1_2 2 \theta_k^0 1 2_2$

Lemma 4.7. *A $(k, 3.0055873128)$-admissible word θ containing (4.4) extends as*

$$\theta = ...\theta_k^0 1 2_2... = ...2_{2k} 1_2 2_{2k+2} 1_2 2_{2k} 1_2 2^* 2_{2k} 1_2 2_{2k+2} 1_2 2_{2k} 1_2 2_2...$$

Proof. If $k = 1$, the desired result follows from Remark 4.4 and the fact that

$$\lambda_0^-(2_2 1_2 2_4 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_2 1_4) > 3.0055873128 > m(\gamma_1^1).$$

If $k \geq 2$, this is an immediate consequence of Remark 4.4. □

Lemma 4.8. *If $0 \leq j < k$, then $\lambda_0^-(1_2 2_{2j} 1_2 2^* 2_{2k}) > m(\gamma_k^1)$.*

Proof. We remember that $m(\gamma_k^1) = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, \underline{\omega}_k, \overline{2}] + [0, 1_2, 2_{2k}, \overline{\underline{\omega}_k^t}]$. Let $C_k := [2; 2_{2k}, \overline{1,2}]$ and $D_k := [0, 1_2, 2_{2k}, \overline{1,2}]$, thus $m(\gamma_k^1) < C_k + D_k$.

Note that $\lambda_0^-(1_2 2_{2j} 1_2 2^* 2_{2k}) \geq \lambda_0^-(1_2 2_{2k-2} 1_2 2^* 2_{2k}) := A + B$, where $A = [2; 2_{2k}, \overline{2,1}]$ and $B = [0; 1_2, 2_{2k-2}, 1_2, \overline{2,1}]$, for each $j < k$. Thus, our task is reduced to prove that $B_k - D_k > C_k - A_k$.

In order to establish this estimate, we observe that

$$C_k - A_k = \frac{[2; \overline{1,2}] - [1; \overline{2,1}]}{q_{2k}^2([2; \overline{1,2}] + \beta_{2k})([1; \overline{2,1}] + \beta_{2k})}$$

and

$$B_k - D_k = \frac{[2; \overline{2,1}] - [1; \overline{1,2}]}{\tilde{q}_{2k}^2([2; \overline{2,1}] + \tilde{\beta}_{2k})([1; \overline{1,2}] + \tilde{\beta}_{2k})},$$

where $q_{2k} = q(2_{2k})$, $\tilde{q}_{2k} = q(1_2 2_{2k-2})$, $\beta_{2k} = [0; 2_{2k}]$ and $\tilde{\beta}_{2k} = [0; 2_{2k-2}, 1_2]$. Note that, $\tilde{\beta}_{2k} = [0; 2_{2k-1}] =: \beta_{2k-1}$ and $\tilde{q}_{2k} = q(2_{2k-2} 1_2) = q(2_{2k-2} 1) + q(2_{2k-2}) = 2q(2_{2k-2}) + q(2_{2k-3}) = q(2_{2k-1}) =: q_{2k-1}$

Thus,

$$\frac{B_k - D_k}{C_k - A_k} = \frac{q_{2k}^2}{\tilde{q}_{2k}^2} \cdot X \cdot Y$$

where

$$X = \frac{[2; \overline{2,1}] - [1; \overline{1,2}]}{[2; \overline{1,2}] - [1; \overline{2,1}]} > 0.464,$$

$$\frac{q_{2k}}{\tilde{q}_{2k}} > \frac{2q_{2k-1}}{q_{2k-1}} = 2$$

and

$$Y = \frac{([2; \overline{1,2}] + \beta_{2k})([1; \overline{2,1}] + \beta_{2k})}{([2; \overline{2,1}] + \beta_{2k-1})([1; \overline{1,2}] + \beta_{2k-1})} \geq \frac{([2; \overline{1,2}] + 0.4)([1; \overline{2,1}] + 0.4)}{([2; \overline{2,1}] + 0.5)([1; \overline{1,2}] + 0.5)} > 0.864,$$

because $\beta_{2k} \geq \beta_2 = 0.4$ and $\beta_{2k-1} \leq \beta_1 = 0.5$. Thus,

$$\frac{B_k - D_k}{C_k - A_k} > 4 \cdot 0.464 \cdot 0.864 > 1.$$

□

Lemma 4.9. *If* $0 \leq j < k$ *then* $\lambda_0^-(1_2 2_{2j} 1_2 2^* 2_{2k+1}) > m(\gamma_k^1)$.

Proof. Since $\lambda_0^-(1_2 2_{2j} 1_2 2^* 2_{2k+1}) = \lambda_0^-(1_2 2_{2j} 1_2 2^* 2_{2k})$, the desired result follows from Lemma 4.8. □

Lemma 4.10. *If* $0 \leq m < k$, *then* $\lambda_0^-(2_{2k} 1_2 2^* 2_{2m} 1_2 2_2) > m(\gamma_k^1)$.

Proof. We note that it is suffices to show the case $m = k-1$, because $\lambda_0^-(2_{2k}1_2 2^* 2_{2m}1_2 2_2)$ increases when m decreases. For this sake, let us write

$$m(\gamma_k^1) < [2; 2_{2k}, 1_2, 2_2, \overline{1,2}] + [0; 1_2, 2_{2k}, \overline{1,2}] = C_k + D_k.$$

Then, we shall show that

$$\lambda_0^-(2_{2k}1_2 2^* 2_{2k-2}1_2 2_2) = [2; 2_{2k-2}, 1_2, 2_2, \overline{2,1}] + [0; 1_2, 2_{2k}, \overline{2,1}] := A_k + B_k > C_k + D_k.$$

In fact, $A_k - C_k = [0; 2_{2k-2}, 1_2, 2_2, \overline{2,1}] - [0; 2_{2k}, 1_2, 2_2, \overline{1,2}]$. That is,

$$A_k - C_k = \frac{[2; 2, 1_2, 2_2, \overline{1,2}] - [1; 1, 2_2, \overline{2,1}]}{q_{2k-2}^2([2; 2, 1_2, 2_2, \overline{1,2}] + \beta_{2k-2})([1; 1, 2_2, \overline{2,1}] + \beta_{2k-2})},$$

where $q_{2k-2} = q(2_{2k-2})$ and $\beta_{2k-2} = [0; 2_{2k-2}]$ (in case $k = 1$, we have $q(2_{2k-2}) = q(0) := 1$ and $\beta_{2k-2} := 0$.). Moreover, we also have $D_k - B_k = [0; 1_2, 2_{2k}, \overline{1,2}] - [0; 1_2, 2_{2k}, \overline{2,1}]$, thus

$$D_k - B_k = \frac{[2; \overline{1,2}] - [1; \overline{2,1}]}{\tilde{q}_{2k+2}^2([2; \overline{1,2}] + \tilde{\beta}_{2k+2})([1; \overline{2,1}] + \tilde{\beta}_{2k+2})},$$

where $\tilde{q}_{2k+2} = q(2_{2k}1_2) = 2q(2_{2k}) + q(2_{2k-1}) = q(2_{2k+1}) =: q_{2k+1}$, and $\tilde{\beta}_{2k+2} = [0; 2_{2k}, 1_2] = [0; 2_{2k+1}] =: \beta_{2k+1}$.

Thus,

$$\frac{A_k - C_k}{D_k - B_k} = \frac{q_{2k+1}^2}{q_{2k-2}^2} \cdot X \cdot Y,$$

where

$$X = \frac{[2; 2, 1_2, 2_2, \overline{1,2}] - [1; 1, 2_2, \overline{2,1}]}{[2; \overline{1,2}] - [1; \overline{2,1}]} > 0.26,$$

$$Y = \frac{([2; \overline{1,2}] + \beta_{2k+1})([1; \overline{2,1}] + \beta_{2k+1})}{([2; 2, 1_2, 2_2, \overline{1,2}] + \beta_{2k-2})([1; 1, 2_2, \overline{2,1}] + \beta_{2k-2})}.$$

Therefore, by Lemma 4.2 and since $q_{2k+1} > 12 q_{2k-2}$, we have:

$$\frac{A_k - C_k}{D_k - B_k} => 144 \cdot 0.26 \cdot 0.22 > 1.$$

Then, $A_k + B_k > C_k + D_k$. □

Lemma 4.11. *If $0 \le m < k$, then $\lambda_0^-(2_2 1_2 2_{2k+1} 1_2 2^* 2_{2m} 1_2 2_2) > m(\gamma_k^1)$.*

Proof. Since $\lambda_0^-(2_2 1_2 2_{2k+1} 1_2 2^* 2_{2m} 1_2 2_2) > \lambda_0^-(2_{2k} 1_2 2^* 2_{2m} 1_2 2_2)$, the desired result follows from Lemma 4.10. □

Lemma 4.12. *If $0 \le m < k$, $\lambda_0^-(1_2 2_{2k+2} 1_2 2^* 2_{2m} 1_2 2_2) > m(\gamma_k^1)$.*

Proof. Since $\lambda_0^-(1_2 2_{2k+2} 1_2 2^* 2_{2m} 1_2 2_2) > \lambda_0^-(2_{2k} 1_2 2^* 2_{2m} 1_2 2_2)$, the desired result follows from Lemma 4.11. □

Lemma 4.13. *If $k \geq 2$, then $\lambda_0^-(1_2\theta_k^0 12_2) > \lambda_0^-(2_2\theta_k^0 12_2) > m(\gamma_k^1)$, where θ_k^0 is the string in* (4.4). *Also,*

$$\lambda_0^-(1_2\theta_1^0 12_2) > 3.005587313 > m(\gamma_1^1),$$

and

$$\lambda_0^-(2_2\theta_1^0 12_2 1_2) > \lambda_0^-(2_2\theta_1^0 12_3) > 3.0055873125 > m(\gamma_1^1).$$

Proof. The inequalities

$$\lambda_0^-(1_2\theta_k^0 12_2) > \lambda_0^-(2_2\theta_k^0 12_2), \quad \lambda_0^-(1_2\theta_1^0 12_2) > 3.005587313 > m(\gamma_1^1), \quad \text{and}$$

$$\lambda_0^-(2_2\theta_1^0 12_2 1_2) > \lambda_0^-(2_2\theta_1^0 12_3) > 3.0055873125 > m(\gamma_1^1)$$

are clear. Hence, it remains only to prove that $\lambda_0^-(2_2\theta_k^0 12_2) > m(\gamma_k^1)$ for all $k \geq 2$.

For this sake, let us show that $A_k + B_k > C_k + D_k$, where $\lambda_0^-(2_2\theta_k^0 12_2) := A_k + B_k$ and $m(\gamma_k^1) \leq C_k + B_k$, with $A_k = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k}, 1_2, 2_2, \overline{2,1}]$, $B_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k+2}, \overline{2,1}]$, $C_k = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k}, 1_2, 2_{2k+1}, 1_2, 2_2, \overline{2,1}]$ and $D_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k+1}, 1_2, 2_2, \overline{2,1}]$. Note that,

$$C_k - A_k = \frac{[2; 2_{2k-3}, 1_2, 2_2, \overline{2,1}] - [1; \overline{2,1}]}{q_{6k+11}^2([2; 2_{2k-3}, 1_2, 2_2, \overline{2,1}] + \beta_{6k+11})([1; \overline{2,1}] + \beta_{6k+11})}$$

and

$$B_k - D_k = \frac{[2; 1_2, 2_2, \overline{2,1}] - [2; 2, \overline{2,1}]}{\tilde{q}_{6k+8}^2([2; 2, \overline{2,1}] + \tilde{\beta}_{6k+8})([2; 1_2, 2_2, \overline{2,1}] + \tilde{\beta}_{6k+8})},$$

where $q_{6k+11} = q(2_{2k}1_2 2_{2k+2} 1_2 2_{2k} 1_2 2_3)$, $\tilde{q}_{6k+8} = q(1_2 2_{2k} 1_2 2_{2k+2} 1_2 2_{2k})$.

Thus,

$$\frac{C_k - A_k}{B_k - D_k} = \frac{[2; 2_{2k-3}, 1_2, 2_2, \overline{2,1}] - [1; \overline{2,1}]}{[2; 1_2, 2_2, \overline{2,1}] - [2; 2, \overline{2,1}]} \cdot X \cdot \frac{\tilde{q}_{6k+8}^2}{q_{6k+11}^2},$$

where

$$X = \frac{([2; 2, \overline{2,1}] + \tilde{\beta}_{6k+8})([2; 1_2, 2_2, \overline{2,1}] + \tilde{\beta}_{6k+8})}{([2; 2_{2k-3}, 1_2, 2_2, \overline{2,1}] + \beta_{6k+11})([1; \overline{2,1}] + \beta_{6k+11})}.$$

We have

$$\frac{[2; 2_{2k-3}, 1_2, 2_2, \overline{2,1}] - [1; \overline{2,1}]}{[2; 1_2, 2_2, \overline{2,1}] - [1; 1, \overline{2,1}]} \leq \frac{[2; \overline{2}] - [1; \overline{2,1}]}{[2; 1_2, 2_2, \overline{2,1}] - [2; 2, \overline{2,1}]} < 6.44.$$

Furthermore, by Euler's rule, $q_{6k+11} > q(2_{2k}1_2 2_{2k+2}1_2 2_{2k})q(1_2 2_3) =: 29q_{6k+6}$ and $\tilde{q}_{6k+8} = q(1_2 2_{2k}1_2 2_{2k+2}1_2 2_{2k}) = p(2_{2k}1_2 2_{2k+2}1_2 2_{2k}) + 2q_{6k+6}$. Thus,

$$\frac{\tilde{q}_{6k+8}}{q_{6k+11}} < \frac{1}{29} \cdot \frac{p(2_{2k}1_2 2_{2k+2}1_2 2_{2k})}{q_{6k+6}} + \frac{2}{29} < \frac{3}{29}.$$

By Lemma 4.2, $X < 4$ and therefore,

$$\frac{C_k - A_k}{B_k - D_k} \leq 6.44 \cdot 4.41 \cdot \left(\frac{3}{29}\right)^2 < 1.$$

□

Corollary 4.3. *Consider the following parameters , for $k \geq 2$ let*

$$\lambda_k^{(1)} := \min\{\lambda_0^-(1_2 2_{2k-2}1_2 2^* 2_{2k}1_2 2_2), \lambda_0^-(2_2 1_2 2_{2k+1}1_2 2^* 2_{2k-2}1_2 2_2), \lambda_0^-(2_2\theta_k^0 12_2)\},$$

and

$$\lambda_1^{(1)} := \min\{\lambda_0^-(1_2 2_{2k-2}1_2 2^* 2_{2k}1_2 2_2), \lambda_0^-(2_2 1_2 2_{2k+1}1_2 2^* 2_{2k-2}1_2 2_2), 3.0055873125\}.$$

Then, $\lambda_k^{(1)} > m(\gamma_k^1)$ and any $(k, \lambda_k^{(1)})$-admissible word θ containing the string θ_k^0 from (4.4) *extends as*

$$\theta = ...2_{2k}1_2 2\theta_k^0 12_2... = ...2_{2k}1_2 2_{2k+1}1_2 2_{2k+2}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2_2...$$

Proof. The fact that $\lambda_k^{(1)} > m(\gamma_k^1)$ follows from Lemmas 4.8, 4.11 and 4.13.

By Lemma 4.7, a $(k, \lambda_k^{(1)})$-admissible word θ containing θ_k^0 extends as $...\theta_k^0 12_2....$ By (Remark 4.4 and) Lemma 4.13, θ must keep extending as

$$\theta = ...2_2 1_2 2\theta_k^0 12_2...$$

Finally, by Lemma 4.8 and 4.11 (together with Remark 4.4), θ must keep extending as $\theta = ...2_{2k}1_2 2\theta_k^0 12_2....$ □

4.4.2 Extension from $2_{2k}1_2 2\theta_k^0 12_2$ to $2_{2k}1_2 2\theta_k^0 12_{2k+1}1_2 2_2$

Lemma 4.14. *If $1 \leq j \leq k$, then $\lambda_0^-(2_{2k}1_2 2\theta_k^0 12_{2j}1_2) > \lambda_0^-(2_{2k}1_2 2\theta_k^0 12_{2k+2}) > m(\gamma_k^1)$.*

Proof. By definition, $m(\gamma_k^1) \leq C_k + D_k$, where

$$C_k = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k}, 1_2, 2_{2k+1}, 1_2, 2_2, \overline{2, 1}], \quad \text{and}$$
$$D_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k+1}, 1_2, 2_{2k}, 1_2, 2_2, \overline{2, 1}].$$

Note that $\lambda_0^-(2_{2k}1_22\theta_k^012_{2j}1_2) > \lambda_0^-(2_{2k}1_22\theta_k^012_{2k+2}) = A_k + B_k$, where

$$A_k = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k}, 1_2, 2_{2k+2}, \overline{2,1}] \text{ and}$$
$$B_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k+1}, 1_2, 2_{2k}\overline{1,2}].$$

Hence, our work is reduced to prove that $A_k - C_k > D_k - B_k$.

In order to prove this inequality, we observe that

$$A_k - C_k = \frac{[2;\overline{2,1}] - [1;1,2_2,\overline{2,1}]}{q_{8k+9}^2([2;\overline{2,1}] + \beta_{8k+9})([1;1,2_2,\overline{2,1}] + \beta_{8k+9})},$$

and

$$D_k - B_k = \frac{[1;1,2_2,\overline{2,1}] - [1;\overline{2,1}]}{\tilde{q}_{8k+11}^2([1;1,2_2,\overline{2,1}] + \tilde{\beta}_{8k+11})([1;\overline{2,1}] + \tilde{\beta}_{8k+11})},$$

where $q_{8k+9} = q(2_{2k}1_22_{2k+2}1_22_{2k}1_22_{2k+1})$ and $\tilde{q}_{8k+11} = q(1_22_{2k}1_22_{2k+2}1_22_{2k+1}1_22_{2k})$. Thus,

$$\frac{A_k - C_k}{D_k - B_k} = \frac{[2;\overline{2,1}] - [1;1,2_2,\overline{2,1}]}{[1;1,2_2,\overline{2,1}] - [1;\overline{2,1}]} \cdot Y \cdot \frac{\tilde{q}_{8k+11}^2}{q_{8k+9}^2},$$

where

$$Y = \frac{([1;1,2_2,\overline{2,1}] + \tilde{\beta}_{8k+11})([1;\overline{2,1}] + \tilde{\beta}_{8k+11})}{([2;\overline{2,1}] + \beta_{8k+9})([1;1,2_2,\overline{2,1}] + \beta_{8k+9})}.$$

We have

$$Y \geq \frac{([1;1,2_2,\overline{2,1}] + [0;\overline{2}])([1;\overline{2,1}] + [0;\overline{2}])}{([2;\overline{2,1}] + [0;\overline{2}])([1;1,2_2,\overline{2,1}] + [0;\overline{2}])} > 0.64.$$

Let $\alpha = 2_{2k}1_22_{2k+2}1_22_{2k}$ and $\tilde{\alpha} = 1_2\alpha$, by Euler's rule, $\tilde{q}_{8k+11} \geq q(\tilde{\alpha})q(21_22_{2k}) = (p(\alpha)+2q(\alpha))(p(1_22_{2k})+2q(1_22_{2k}))$ and $q_{8k+9} \leq 2q(\alpha)q(1_22_{2k+1}) \leq 2q(\alpha)3q(1_22_{2k})$. Thus,

$$\frac{\tilde{q}_{8k+11}}{q_{8k+9}} \geq \left(1 + \frac{1}{2}[0;\alpha]\right)\left(\frac{2}{3} + \frac{1}{3}[0;1_22_{2k}]\right) \geq \left(1 + \frac{1}{2}[0;\overline{2}]\right)\left(\frac{2}{3} + \frac{1}{3}[0;1_22_2]\right) > 1.03$$

Therefore,

$$\frac{A_k - C_k}{D_k - B_k} > 1.925 \cdot 0.64 \cdot (1.03)^2 > 1.$$

□

Corollary 4.4. *Consider the parameter*

$$\lambda_k^{(2)} := \min\{\lambda_0^-(2_{2k}1_22^*2_{2k-2}1_22_2), \lambda_0^-(2_{2k}1_22\theta_k^012_{2k+2})\}.$$

Then, $\lambda_k^{(2)} > m(\gamma_k^1)$ *and any* $(k, \lambda_k^{(2)})$*-admissible word* θ *containing* $2_{2k}1_22\theta_k^012_2$ *extends as*

$$\begin{aligned}\theta &= \ldots 2_{2k}1_22\theta_k^012_{2k+1}1_22_2\ldots \\ &= \ldots 2_{2k}1_22_{2k+1}1_22_{2k+2}1_22_{2k}1_22^*2_{2k}1_22_{2k+2}1_22_{2k}1_22_{2k+1}1_22_2\ldots.\end{aligned}$$

Proof. The fact that $\lambda_k^{(2)} > m(\gamma_k^1)$ follows from Lemmas 4.10 and 4.14. Moreover, these lemmas (and Remark 4.4) imply that any $(k, \lambda_k^{(2)})$-admissible word θ containing $2_{2k}1_2 2\theta_k^0 1 2_2$ extends as $\theta = ...2_{2k}1_2 2\theta_k^0 1 2_{2k+1}1_2 2_2...$ □

4.4.3 Extension from $2_{2k}1_2 2\theta_k^0 1 2_{2k+1}1_2 2_2$ to $2_{2k}1_2 2\theta_k^0 1 2_{2k+1}1_2 2_{2k+1}$

Lemma 4.15. *If $0 \leq m < k$, then $\lambda_0^-(2_2 1_2 2_{2k+2} 1_2 2_{2k} 1_2 2^* 2_{2k} 1_2 2_{2m+2} 1_2) > m(\gamma_k^1)$.*

Proof. We write $\lambda_0^-(2_2 1_2 2_{2k+2} 1_2 2_{2k} 1_2 2^* 2_{2k} 1_2 2_{2m+2} 1_2) := A_k + B_k$, where $A_k = [2; 2_{2k}, 1_2, 2_{2m+2}, 1_2, \overline{2, 1}]$ and $B_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_2, \overline{2, 1}]$. Remember that

$$m(\gamma_k^1) < [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_2, \overline{1, 2}] + [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_2, \overline{1, 2}] := C_k + D_k.$$

It is suffices take $m = k - 1$. We have

$$A_k - C_k = [0; 2_{2k}, 1_2, 2_{2k}, 1_2, \overline{2, 1}] - [0; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_2, \overline{1, 2}]$$

then

$$A_k - C_k = \frac{[2; 2, 1_2, 2_2, \overline{1, 2}] - [1; 1, \overline{2, 1}]}{q_{4k+2}^2([2; 2, 1_2, 2_2, \overline{1, 2}] + \beta_{4k+2})([1; 1, \overline{2, 1}] + \beta_{4k+2})},$$

where $q_{4k+2} = q(2_{2k}1_2 2_{2k})$ and $\beta_{4k+2} = [0; 2_{2k}, 1_2, 2_{2k}]$. Moreover,

$$D_k - B_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_2, \overline{1, 2}] - [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_2, \overline{2, 1}]$$

then

$$D_k - B_k = \frac{[1; 1, 2_2, \overline{2, 1}] - [1; 1, 2_2, \overline{1, 2}]}{\tilde{q}_{4k+8}^2([1; 1, 2_2, \overline{2, 1}] + \tilde{\beta}_{4k+8})([1; 1, 2_2, \overline{1, 2}] + \tilde{\beta}_{4k+8})},$$

where $\tilde{q}_{4k+8} = q(1_2 2_{2k} 1_2 2_{2k+2} 1_2)$ and $\tilde{\beta}_{4k+8} = [0; 1_2, 2_{2k+2}, 1_2, 2_{2k}, 1_2]$. Thus,

$$D_k - B_k = \frac{[1; 1, 2_2, \overline{2, 1}] - [1; 1, 2_2, \overline{1, 2}]}{\tilde{q}_{4k+8}^2([1; 1, 2_2, \overline{2, 1}] + \tilde{\beta}_{4k+8})([1; 1, 2_2, \overline{1, 2}] + \tilde{\beta}_{4k+8})},$$

where $\tilde{q}_{4k+8} = q(1_2 2_{2k} 1_2 2_{2k+2} 1_2)$. Thus,

$$\frac{A_k - C_k}{D_k - B_k} = \frac{[2; 2, 1_2, 2_2, \overline{1, 2}] - [1; 1, \overline{2, 1}]}{[1; 1, 2_2, \overline{2, 1}] - [1; 1, 2_2, \overline{1, 2}]} \cdot X_2 \cdot \frac{\tilde{q}_{4k+8}^2}{q_{4k+2}^2},$$

where

$$X_2 = \frac{([1; 1, 2_2, \overline{2, 1}] + \tilde{\beta}_{4k+8})([1; 1, 2_2, \overline{1, 2}] + \tilde{\beta}_{4k+8})}{([2; 2, 1_2, 2_2, \overline{1, 2}] + \beta_{4k+2})([1; 1, \overline{2, 1}] + \beta_{4k+2})}.$$

Therefore, by Lemma 4.2 and since $\tilde{q}_{4k+8} > q(1_2)q_{4k+2}q(2_21_2) = 24q_{4k+2}$, we have:

$$\frac{A_k - C_k}{D_k - B_k} > 133 \cdot 0.22 \cdot (24)^2 > 1.$$

□

Corollary 4.5. *Consider the parameter*

$$\lambda_k^{(3)} := \min\{\lambda_0^-(2_21_22_{2k+1}1_22^*2_{2k-2}1_22_2), \lambda_0^-(2_21_22_{2k+2}1_22_{2k}1_22^*2_{2k}1_22_{2k}1_2)\}.$$

Then, $\lambda_k^{(3)} > m(\gamma_k^1)$ *and any* $(k, \lambda_k^{(3)})$*-admissible word* θ *containing* $2_{2k}1_22\theta_k^012_{2k+1}1_22_2$ *extends as*

$$\begin{aligned} \theta &= ...2_{2k}1_22\theta_k^012_{2k+1}1_22_{2k+1}... \\ &= ...2_{2k}1_22_{2k+1}1_22_{2k+2}1_22_{2k}1_22^*2_{2k}1_22_{2k+2}1_22_{2k}1_22_{2k+1}1_22_{2k+1}... \end{aligned}$$

Proof. The fact that $\lambda_k^{(3)} > m(\gamma_k^1)$ follows from Lemmas 4.11 and 4.15. Moreover, these lemmas (and Remark 4.4) imply that any $(k, \lambda_k^{(3)})$-admissible word θ containing $2_{2k}1_22\theta_k^012_2$ extends as $\theta = ...2_{2k}1_22\theta_k^012_{2k+1}1_22_{2k+1}...$ □

4.4.4 Extension from $2_{2k}1_22\theta_k^012_{2k+1}1_22_{2k+1}$ to $2_{2k+1}1_22_{2k}1_22\theta_k^012_{2k+1}1_22_{2k+1}$

Lemma 4.16. *If* $0 \le j < k$*, then*

$$\lambda_0^-(1_22_{2j+2}1_22_{2k}1_22^*2_{2k}1_22_{2k+2}1_22_2) > m(\gamma_k^1).$$

Proof. Let $u = 1_22_{2j+2}1_22_{2k}1_22^*2_{2k}1_22_{2k+2}1_22_2$. We can suppose $j = k-1$. Note that

$$\lambda_0^-(u) = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_2, \overline{2,1}] + [0; 1_2, 2_{2k}, 1_2, 2_{2k}, 1_2, \overline{2,1}] = A_k + B_k$$

In the same way as before

$$m(\gamma_k^1) < [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_2, \overline{1,2}] + [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_2, \overline{1,2}] := C_k + D_k.$$

Therefore, we have

$$\frac{B_k - D_k}{C_k - A_k} = \frac{q_{4k+4}^2}{\tilde{q}_{4k+4}^2} \cdot \frac{[2; 2, 1_2, 2_2, \overline{1,2}] - [1; 1, \overline{2,1}]}{[1; 1, 2_2, \overline{2,1}] - [1; 1, 2_2, \overline{1,2}]} \cdot Q$$

where $q_{4k+4} = q(2_{2k}1_22_{2k+2})$, $\tilde{q}_{4k+4} = q(1_22_{2k}1_22_{2k})$ and

$$Q = \frac{([1; 1, 2_2, \overline{2,1}] + \beta_{4k+4})([1; 1, \overline{1,2}] + \beta_{4k+4})}{([2; 2, 1_2, 2_2, \overline{1,2}] + \tilde{\beta}_{4k+4})([1; 1, \overline{2,1}] + \tilde{\beta}_{4k+4})}.$$

Since $q_{4k+4} > 2q(2_{2k}1_2 2_{2k+1}) = 2\tilde{q}_{4k+4}$ and

$$\frac{[2;2,1_2,2_2,\overline{1,2}] - [1;1,\overline{2,1}]}{[1;1,2_2,\overline{2,1}] - [1;1,2_2,\overline{1,2}]} > 133,$$

we have, using the Lemma 4.2, that:

$$\frac{B_k - D_k}{C_k - A_k} > 4 \cdot 133 \cdot 0.22 > 1.$$

□

Lemma 4.17. *If* $u_4 = 2_{2k+1}1_2 2_{2k+1}1_2 2_{2k+2}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2_{2k+1}1_2 2_{2k+1}$, *then* $\lambda_0^-(u_4) > m(\gamma_k^1)$.

Proof. By definition, $\lambda_0^-(u_4) = A_k + B_k$, where

$$A_k = [2;2_{2k},1_2,2_{2k+2},1_2,2_{2k},1_2,2_{2k+1},1_2,2_{2k+1},\overline{2,1}] \ \text{ and}$$

$$B_k = [0;1_2,2_{2k},1_2,2_{2k+2},1_2,2_{2k+1},1_2,2_{2k+1},\overline{2,1}].$$

Moreover, $m(\gamma_k^1) \le C_k + D_k$, where

$$C_k = [2;2_{2k},1_2,2_{2k+2},1_2,2_{2k},1_2,2_{2k+1},1_2,2_{2k+2},1_2,\overline{2}] \text{ and}$$
$$D_k = [0;1_2,2_{2k},1_2,2_{2k+2},1_2,2_{2k+1},1_2,2_{2k},1_2,2_2,\overline{2,1}].$$

We shall show that $A_k + B_k > C_k + D_k$. In order to establish this inequality, we observe that

$$C_k - A_k = \frac{[2;\overline{1,2}] - [1;\overline{2}]}{q_{10k+14}^2([1;\overline{2}] + \beta_{10k+14})([2;\overline{1,2}] + \beta_{10k+14})}$$

and

$$B_k - D_k = \frac{[2;\overline{2,1}] - [1;1,2_2,\overline{2,1}]}{\tilde{q}_{8k+11}^2([2;\overline{2,1}] + \tilde{\beta}_{8k+11})([1;1,2_2,\overline{2,1}] + \tilde{\beta}_{8k+11})},$$

where

$$q_{10k+14} = q(2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2_{2k+1}1_2 2_{2k+2}1),\ \tilde{q}_{8k+11} = q(1_2 2_{2k}1_2 2_{2k+2}1_2 2_{2k+1}1_2 2_{2k}).$$

Thus,

$$\frac{C_k - A_k}{B_k - D_k} = \frac{[2;\overline{1,2}] - [1;\overline{2}]}{[2;\overline{2,1}] - [1;1,2_2,\overline{2,1}]} \cdot X_4 \cdot \frac{\tilde{q}_{8k+11}^2}{q_{10k+14}^2},$$

where

$$X_4 = \frac{([2;\overline{2,1}] + \tilde{\beta}_{8k+11})([1;1,2_2,\overline{2,1}] + \tilde{\beta}_{8k+11})}{([1;\overline{2}] + \beta_{10k+14})([2;\overline{1,2}] + \beta_{10k+14})}.$$

Let $\alpha = 2_{2k}1_22_{2k+2}1_22_{2k}$ and $\tilde{\alpha} = 1_2\alpha$, by Euler's rule,

$$\tilde{q}_{8k+11} < 2q(\tilde{\alpha})q(21_22_{2k}) \leq 2q(\tilde{\alpha})3q(1_22_{2k}),$$

and

$$\begin{aligned} q_{10k+14} &> q(\alpha)q(1_22_{2k+1})q(1_22_{2k+2}1) \geq q(\alpha)2q(2_{2k+1})2q(12_{2k+2}) \\ &\geq 4q(\alpha)q(1_22_{2k})q(12_4). \end{aligned}$$

Thus,

$$\frac{\tilde{q}_{8k+11}}{q_{10k+14}} < \frac{3}{2q(12_4)}\left(\frac{p(\alpha)+2q(\alpha)}{q(\alpha)}\right) = \frac{3}{2\cdot 41}\left([0;\alpha]+2\right) < \frac{9}{2\cdot 41}.$$

Therefore, since that $X_4 \leq 4.41$, by Lemma 4.2, we obtain

$$\frac{C_k - A_k}{B_k - D_k} < 2.003 \cdot 4.41 \cdot \left(\frac{9}{2\cdot 41}\right)^2 < 1.$$

$\square$

Corollary 4.6. *Consider the parameter*

$$\begin{aligned} \lambda_k^{(4)} := \min\{&\lambda_0^-(2_{2k}1_22^*2_{2k-2}1_22_2), \lambda_0^-(1_22_{2k}1_22_{2k}1_22^*2_{2k}1_22_{2k+2}1_22_2), \\ &\lambda_0^-(2_{2k+1}1_22\theta_k^012_{2k+1}1_22_{2k+1})\}. \end{aligned}$$

Then, $\lambda_k^{(4)} > m(\gamma_k^1)$ *and any* $(k, \lambda_k^{(4)})$*-admissible word* θ *containing* $2_{2k}1_22\theta_k^012_{2k+1}1_22_{2k+1}$ *extends as*

$$\begin{aligned} \theta &= ...2_{2k+1}1_22_{2k}1_22\theta_k^012_{2k+1}1_22_{2k+1}... \\ &= ...2_{2k+1}1_22_{2k}1_22_{2k+1}1_22_{2k+2}1_22_{2k}1_22^*2_{2k}1_22_{2k+2}1_22_{2k}1_22_{2k+1}1_22_{2k+1}... \end{aligned}$$

Proof. The fact that $\lambda_k^{(4)} > m(\gamma_k^1)$ follows from Lemmas 4.10, 4.16 and 4.17. Moreover, these lemmas (and Remark 4.4) imply that any $(k, \lambda_k^{(4)})$-admissible word θ containing $2_{2k}1_22\theta_k^012_{2k+1}1_22_{2k+1}$ extends as

$$\theta = ...2_{2k+1}1_22_{2k}1_22\theta_k^012_{2k+1}1_22_{2k+1}....$$

$\square$

4.4.5 Extension from $2_{2k+1}1_22_{2k}1_22\theta_k^0 12_{2k+1}1_22_{2k+1}$ to $2_{2k+1}1_22_{2k}1_22\theta_k^0 12_{2k+1}1_22_{2k+2}1_22_{2k}$

Lemma 4.18. *One has $\lambda_0^-(u_5) > \lambda_0^-(u_6) > m(\gamma_k^1)$, where*

$$u_5 = 2_{2k+1}1_22_{2k}1_22_{2k+1}1_22_{2k+2}1_22_{2k}1_22^*2_{2k}1_22_{2k+2}1_22_{2k}1_22_{2k+1}1_22_{2k+1}1_22_2$$

and

$$u_6 = 2_{2k+1}1_22_{2k}1_22_{2k+1}1_22_{2k+2}1_22_{2k}1_22^*2_{2k}1_22_{2k+2}1_22_{2k}1_22_{2k+1}1_22_{2k+3}$$

Proof. Let $\lambda_0^-(u_6) = A_k + B_k$, where

$$A_k = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k}, 1_2, 2_{2k+1}, 1_2, 2_{2k+3}, \overline{2,1}] \text{ and}$$

$$B_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k+1}, 1_2, 2_{2k}, 1_2, 2_{2k+1}\overline{2,1}].$$

Moreover, by definition, $m(\gamma_k^1) \leq C_k + D_k$, where

$$C_k = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k}, 1_2, 2_{2k+1}, 1_2, 2_{2k+2}, 1_2, \overline{2}] \text{ and}$$
$$D_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k+1}, 1_2, 2_{2k}, 1_2, 2_{2k+2}1_22_2, \overline{2,1}].$$

Let us show that $A_k + B_k > C_k + D_k$. For this sake, we observe that

$$A_k - C_k = \frac{[2;\overline{2,1}] - [1;1,\overline{2}]}{q_{10k+13}^2([2;\overline{2,1}] + \beta_{10k+13})([1;1,\overline{2}] + \beta_{10k+13})}$$

and

$$D_k - B_k = \frac{[2;\overline{1,2}] - [1;2_2,\overline{2,1}]}{\tilde{q}_{10k+16}^2([1;2_2,\overline{2,1}] + \tilde{\beta}_{10k+16})([2;\overline{1,2}] + \tilde{\beta}_{10k+16})},$$

where $\tilde{q}_{10k+16} = q(1_22_{2k}1_22_{2k+2}1_22_{2k+1}1_22_{2k}1_22_{2k+2}1)$ and $q_{10k+13} = q(2_{2k}1_22_{2k+2}1_22_{2k}1_22_{2k+1}1_22_{2k+2})$. Thus,

$$\frac{A_k - C_k}{D_k - B_k} = \frac{[2;\overline{2,1}] - [1;1,\overline{2}]}{[2;\overline{1,2}] - [1;2_2,\overline{2,1}]} \cdot X_6 \cdot \frac{\tilde{q}_{10k+16}^2}{q_{10k+13}^2},$$

where

$$X_6 = \frac{([1;2_2,\overline{2,1}] + \tilde{\beta}_{10k+16})([2;\overline{1,2}] + \tilde{\beta}_{10k+16})}{([2;\overline{2,1}] + \beta_{10k+13})([1;1,\overline{2}] + \beta_{10k+13})}.$$

Note that

$$X_6 \geq \frac{([1;2_2,\overline{2,1}] + [0,1,2_4,1])([2;\overline{1,2}] + [0,1,2_4,1])}{([2;\overline{2,1}] + [0,2_4,1])([1;1,\overline{2}] + [0,2_4,1])} > 1.23.$$

Let $\theta = 2_{2k+2}1_2 2_{2k}1_2 2_{2k+1}1_2 2_{2k+2}$, since $q(2_{2k+1}1_2 2_{2k}1_2 2_{2k+1}1_2 2_{2k+2}) < (1/2)q(\theta)$ and $q(2_{2k}1) < q(2_{2k}1_2)$, by Euler's rule, we have:

$$q_{10k+13} = q(2_{2k}1_2)q(\theta) + q(2_{2k}1)q(2_{2k+1}1_2 2_{2k}1_2 2_{2k+1}1_2 2_{2k+2}) < (3/2)q(2_{2k}1_2)q(\theta).$$

Analogously, since $q(1_2 2_{2k}1) > (1/2)q(1_2 2_{2k}1_2)$ and $q(2_{2k+1}1_2 2_{2k+1}1_2 2_{2k}1_2 2_{2k+2}1) > (1/3)q(\theta^t 1)$, we obtain:

$$\begin{aligned}\tilde{q}_{10k+16} &= q(1_2 2_{2k}1_2\theta^t 1) = q(1_2 2_{2k}1_2)q(\theta^t 1) + q(1_2 2_{2k}1)q(2_{2k+1}1_2 2_{2k+1}1_2 2_{2k}1_2 2_{2k+2}1)\\ &> (7/6)q(1_2 2_{2k}1_2)q(\theta^t 1) > 2(7/6)q(1_2 2_{2k})q(\theta^t).\end{aligned}$$

Thus,

$$\frac{\tilde{q}_{10k+16}}{q_{10k+13}} > \frac{7}{3}\cdot\frac{2}{3} = \frac{14}{9}.$$

Therefore,

$$\frac{A_k - C_k}{D_k - B_k} > 0.49 \cdot 1.23 \cdot \left(\frac{14}{9}\right)^2 > 1.$$

□

Corollary 4.7. *Consider the parameter*

$$\lambda_k^{(5)} := \min\{\lambda_0^-(1_2 2_{2k+2}1_2 2^* 2_{2k-2}1_2 2_2), \lambda_0^-(1_2 2_{2k-2}1_2 2^* 2_{2k+1}1_2 2_2), \lambda_0^-(u_6)\}.$$

Then, $\lambda_k^{(5)} > m(\gamma_k^1)$ *and any* $(k, \lambda_k^{(5)})$*-admissible word* θ *containing* $2_{2k+1}1_2 2_{2k}1_2 2\theta_k^0 1 2_{2k+1}1_2 2_{2k+1}$ *extends as*

$$\begin{aligned}\theta &= ...2_{2k+1}1_2 2_{2k}1_2 2\theta_k^0 1 2_{2k+1}1_2 2_{2k+2}1_2 2_{2k}...\\ &= ...2_{2k+1}1_2 2_{2k}1_2 2_{2k+1}1_2 2_{2k+2}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2_{2k+1}1_2 2_{2k+2}1_2 2_{2k}...\end{aligned}$$

Proof. The fact that $\lambda_k^{(5)} > m(\gamma_k^1)$ follows from Lemmas 4.12, 4.9 and 4.18. Moreover, these lemmas (and Remark 4.4) imply that any $(k, \lambda_k^{(5)})$-admissible word θ containing $2_{2k+1}1_2 2_{2k}1_2 2\theta_k^0 1 2_{2k+1}1_2 2_{2k+1}$ extends as

$$\theta = ...2_{2k+1}1_2 2_{2k}1_2 2\theta_k^0 1 2_{2k+1}1_2 2_{2k+2}1_2 2_{2k}....$$

□

4.4.6 Replication lemma

Lemma 4.19. *One has* $\lambda_0^-(u_7) > \lambda_0^-(u_8) > m(\gamma_k^1)$*, where*

$$u_7 = 2_2 1_2 2_{2k+1} 1_2 2_{2k} 1_2 2_{2k+1} 1_2 2_{2k+2} 1_2 2_{2k} 1_2 2^* 2_{2k} 1_2 2_{2k+2} 1_2 2_{2k} 1_2 2_{2k+1} 1_2 2_{2k+2} 1_2 2_{2k}$$

and

$$u_8 = 2_{2k+3} 1_2 2_{2k} 1_2 2_{2k+1} 1_2 2_{2k+2} 1_2 2_{2k} 1_2 2^* 2_{2k} 1_2 2_{2k+2} 1_2 2_{2k} 1_2 2_{2k+1} 1_2 2_{2k+2} 1_2 2_{2k}.$$

Proof. Let $\lambda_0^-(u_8) = A_k + B_k$, where

$$A_k = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k}, 1_2, 2_{2k+1}, 1_2, 2_{2k+2}, 1_2, 2_{2k}, \overline{1,2}] \text{ and}$$
$$B_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k+1}, 1_2, 2_{2k}, 1_2, 2_{2k+3}, \overline{2,1}].$$

Furthermore, by definition, $m(\gamma_k^1) \le C_k + D_k$, where

$$C_k = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k}, 1_2, 2_{2k+1}, 1_2, 2_{2k+2}, 1_2, \overline{2}] \text{ and}$$
$$D_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k+1}, 1_2, 2_{2k}, 1_2, 2_{2k+2} 1_2 2_2, \overline{2,1}].$$

Thus, our task is prove that $B_k - D_k > C_k - A_k$. In order to establish this estimative, we observe that

$$C_k - A_k = \frac{[2;\overline{2}] - [1;\overline{2,1}]}{q_{12k+15}^2([2;\overline{2}] + \beta_{12k+15})([1;\overline{2,1}] + \beta_{12k+15})}$$

and

$$B_k - D_k = \frac{[2;\overline{2,1}] - [1;1,2_2,\overline{2,1}]}{\tilde{q}_{10k+15}^2([2;\overline{2,1}] + \tilde{\beta}_{10k+15})([1;1,2_2,\overline{2,1}] + \tilde{\beta}_{10k+15})},$$

where $q_{12k+15} = q(2_{2k} 1_2 2_{2k+2} 1_2 2_{2k} 1_2 2_{2k+1} 1_2 2_{2k+2} 1_2 2_{2k})$ and $\tilde{q}_{10k+15} =$ $= q(1_2 2_{2k} 1_2 2_{2k+2} 1_2 2_{2k+1} 1_2 2_{2k} 1_2 2_{2k+2})$. Thus,

$$\frac{C_k - A_k}{B_k - D_k} = \frac{[2;\overline{2,1}] - [1;1,\overline{2}]}{[2;\overline{2,1}] - [1;1,2_2,\overline{2,1}]} \cdot X_8 \cdot \frac{\tilde{q}_{10k+15}^2}{q_{12k+15}^2},$$

where

$$X_8 = \frac{([2;\overline{2,1}] + \tilde{\beta}_{10k+15})([1;1,2_2,\overline{2,1}] + \tilde{\beta}_{10k+15})}{([2;\overline{2}] + \beta_{12k+15})([1;\overline{2,1}] + \beta_{12k+15})}.$$

Note that

$$X_8 < \frac{([2;\overline{2,1}] + [0,2_4,1])([1;1,2,2,\overline{2,1}] + [0,2_4,1])}{([2;\overline{2}] + [0,\overline{2}])([1;\overline{2,1}] + [0,\overline{2}])} < 1.72.$$

Let $\omega = 2_{2k}1_2 2_{2k+1}1_2 2_{2k+2}1_2 2_{2k}$, by Euler's rule, we have:

$$q_{12k+15} > q(2_{2k}1_2)q(2_{2k+2}1_2)q(\omega) \geq q(2_2 1_2)q(2_{2k+2}1_2)q(\omega) = 12q(2_{2k+2}1_2)q(\omega)$$

and

$$\tilde{q}_{10k+15} < 2q(1_2\omega^t)q(1_2 2_{2k+2}) < 2 \cdot 3q(\omega)q(1_2 2_{2k+2}).$$

Thus,

$$\frac{\tilde{q}_{10k+15}}{q_{12k+15}} < \frac{1}{2}.$$

Therefore,

$$\frac{C_k - A_k}{B_k - D_k} < 1.6 \cdot 1.72 \cdot \left(\frac{1}{2}\right)^2 < 1.$$

□

Corollary 4.8. *Consider the parameter*

$$\lambda_k^{(6)} := \min\{\lambda_0^-(1_2 2_{2k+2}1_2 2^* 2_{2k-2}1_2 2_2), \lambda_0^-(1_2 2_{2k-2}1_2 2^* 2_{2k+1}1_2 2_2), \lambda_0^-(u_8)\}.$$

Then, $\lambda_k^{(6)} > m(\gamma_k^1)$ *and any* $(k, \lambda_k^{(6)})$*-admissible word* θ *containing* $2_{2k+1}1_2 2_{2k}1_2 2\theta_k^0 1 2_{2k+1}1_2 2_{2k+2}1_2 2_{2k}$ *extends as*

$$\begin{aligned}\theta &= \ldots 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2\theta_k^0 1 2_{2k+1}1_2 2_{2k+2}1_2 2_{2k}\ldots \\ &= \ldots 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2_{2k+1}1_2 2_{2k+2}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2_{2k+1}1_2 2_{2k+2}1_2 2_{2k}\ldots\end{aligned}$$

Proof. The fact that $\lambda_k^{(6)} > m(\gamma_k^1)$ follows from Lemmas 4.12, 4.9 and 4.19. Moreover, these lemmas (and Remark 4.4) imply that any $(k, \lambda_k^{(6)})$-admissible word θ containing $2_{2k+1}1_2 2_{2k}1_2 2\theta_k^0 1 2_{2k+1}1_2 2_{2k+2}1_2 2_{2k}$ extends as

$$\theta = \ldots 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2\theta_k^0 1 2_{2k+1}1_2 2_{2k+2}1_2 2_{2k}\ldots.$$

□

The entire discussion of this section can be summarized into the following key lemma establishing the self-replication property of γ_k^1 for all $k \in \mathbb{N}$:

Lemma 4.20 (Replication Lemma)**.** *For each* $k \in \mathbb{N}$*, there exists an explicit constant* $\lambda_k > m(\gamma_k^1)$ *such that any* (k, λ_k)*-admissible word* θ *containing* $\theta_k^0 = 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1$ *must extend as*

$$\theta = \ldots 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2_{2k+1}1_2 2_{2k+2}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2_{2k+1}1_2 2_{2k+2}1_2 2_{2k}\ldots,$$

and the neighbourhood of the position $-(6k+9)$ *is*

$$...2_{2k}1_22_{2k+2}1_22_{2k}1_22^*2_{2k}1_22_{2k+2}1_22_{2k}1....$$

In particular, any (k,λ_k)*-admissible word* θ *containing* θ_k^0 *has the form*

$$\overline{2_{2k}1_22_{2k+1}1_22_{2k+2}1_2}2_{2k}1_22^*2_{2k}1_22_{2k+2}1_22_{2k}1_22_{2k+1}1_22_{2k+2}1_22_{2k}...$$

Proof. This result for $\lambda_k := \min\{\lambda_k^{(i)} : i = 1,\dots,6\}$ is an immediate consequence of Corollaries 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7 and 4.8. □

4.5 Going to the Replication (Extensions of $2_21_22_{2k}1_22^*2_{2k}1_22_2$)

In this section, we investigate for every $k \geq 2$ the extensions of a word θ containing the string

$$\theta_k^1 := 2_21_22_{2k}1_22^*2_{2k}1_22_2. \tag{4.5}$$

Let $\tilde{\lambda}_k^{(1)} = \min\{\lambda_0^-(1_22_{2k-2}1_22^*2_{2k}), \lambda_0^-(2_21_22_{2k+1}1_22^*2_{2k-2}1_22_2)\}$. By Lemmas 4.8 and 4.11, $\tilde{\lambda}_k^{(1)} > m(\gamma_k^1)$ and a $(k,\tilde{\lambda}_k^{(1)})$-admissible word θ containing θ_k^1 must extend as

$$...\theta_k^12_{2k-2}....$$

Lemma 4.21. *Let* θ_k^1 *be the string in* (4.5)*, then* $\lambda_0^-(1_22_{2k-4}\theta_k^12_{2k-2}) > m(\gamma_k^1)$*. In particular,* $\lambda_0^-(1_22_{2j}1_22_{2k}1_22^*2_{2k}1_22_{2k}) > m(\gamma_k^1)$*, for each* $1 \leq j \leq k-1$*.*

Proof. The inequality $\lambda_0^-(1_22_{2j}1_22_{2k}1_22^*2_{2k}1_22_{2k}) > \lambda_0^-(1_22_{2k-4}\theta_k^12_{2k-2})$ is clear, for each $1 \leq j \leq k-1$. Hence, it remains only to prove that $\lambda_0^-(1_22_{2k-4}\theta_k^12_{2k-2}) > m(\gamma_k^1)$. For this sake, let $\lambda_0^-(u) = A + B$, where $A = [2;2_{2k},1_2,2_{2k},\overline{2,1}]$ and $B = [0;1_2,2_{2k},1_2,2_{2k-2},1_2,\overline{2,1}]$. By definition, $m(\gamma_k^1) \leq C_k + D_k$, where $C_k = [2;2_{2k},1_2,2_{2k+2},1_2,2_2,\overline{1,2}]$ and $D_k = [0;1_2,2_{2k},1_2,2_{2k+2},1_2,2_2,\overline{1,2}]$.

Thus, our work is reduced to prove that $A + B > C_k + D_k$. Note that

$$C_k - A = \frac{[2;1_2,2_2,\overline{1,2}] - [1;\overline{2,1}]}{q_{4k+3}^2([2;1_2,2_2,\overline{1,2}] + \beta_{4k+3})([1;\overline{2,1}] + \beta_{4k+3})},$$

while

$$B - D_k = \frac{[2;2_3,1_2,2_2,\overline{1,2}] - [1;1,\overline{2,1}]}{\tilde{q}_{4k+2}^2([1;1,\overline{2,1}] + \tilde{\beta}_{4k+2})([2;2_3,1_2,2_2,\overline{1,2}] + \tilde{\beta}_{4k+2})},$$

where $q_{4k+3} = q(2_{2k}1_2 2_{2k+1})$ and $\tilde{q}_{4k+2} = q(1_2 2_{2k}1_2 2_{2k-2})$. Thus,

$$\frac{C_k - A}{B - D_k} = \frac{[2; 1_2, 2_2, \overline{1, 2}] - [1; \overline{2, 1}]}{[2; 2_3, 1_2, 2_2, \overline{1, 2}] - [1; 1, \overline{2, 1}]} \cdot X \cdot \frac{\tilde{q}^2_{4k+2}}{q^2_{4k+3}},$$

where

$$X = \frac{([1; 1, \overline{2, 1}] + \tilde{\beta}_{4k+2})([2; 2_3, 1_2, 2_2, \overline{1, 2}] + \tilde{\beta}_{4k+2})}{([2; 1_2, 2_2, \overline{1, 2}] + \beta_{4k+3})([1; \overline{2, 1}] + \beta_{4k+3})}.$$

Let $\alpha = 2_{2k}1_2 2_{2k-2}$, then $\tilde{q}_{4k+2} = p(\alpha) + 2q(\alpha)$ and $q_{4k+2} = q(\alpha 2_3) > 12q(\alpha)$. Thus,

$$\frac{\tilde{q}_{4k+2}}{q_{4k+2}} < \frac{p(\alpha) + 2q(\alpha)}{12q(\alpha)} < \frac{1}{4}.$$

By Lemma 4.2, $X \leq 4.41$ and therefore,

$$\frac{C_k - A}{B - D_k} \leq 1.8 \cdot 4.41 \cdot \left(\frac{1}{4}\right)^2 < 1.$$

□

Let $\tilde{\lambda}^{(2)}_k = \min\{\lambda^-_0(1_2 2_{2k-4}\theta^1_k 2_{2k-2}), \lambda^-_0(2_{2k}1_2 2^* 2_{2k-2}1_2 2_2)\}$. By Lemmas 4.21 and Lemma 4.10, $\tilde{\lambda}^{(2)}_k > m(\gamma^1_k)$ and a $(k, \tilde{\lambda}^{(2)}_k)$-admissible word θ containing $\theta^1_k 2_{2k-2}$ must extend as

$$...2_{2k-2}\theta^1_k 2_{2k-2}... = ...2_{2k}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k}....$$

Lemma 4.22. *If* $\tilde{\lambda}^{(3)}_k := \lambda^-_0(2_{2k-2}\theta^1_k 2_{2k-2}1_2) = \lambda^-_0(2_{2k}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k}1_2)$, *then* $\tilde{\lambda}^{(3)}_k > m(\gamma^1_k)$.

Proof. By definition, $m(\gamma^1_k) \leq C_k + D_k$, where $C_k = [2; 2_{2k}, 1_2, 2_{2k+2}, \overline{1, 2}]$ and $D_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, \overline{1, 2}]$.

Note that $\lambda^-_0(2_{2k-2}\theta^1_k 2_{2k-2}1_2) = A_k + B_k$, where $A_k = [2; 2_{2k}, 1_2, 2_{2k}, 1_2, \overline{2, 1}]$ and $B_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k}, \overline{2, 1}]$. Hence, our work is reduced to prove that $A_k - C_k > D_k - B_k$.

In order to prove this inequality, we observe that

$$A_k - C_k = \frac{[2; 2, \overline{2, 1}] - [1; 1, \overline{2, 1}]}{q^2_{4k+2}([1; 1, \overline{2, 1}] + \beta_{4k+2})([2; 2, \overline{1, 2}] + \beta_{4k+2})},$$

and

$$D_k - B_k = \frac{[2; \overline{1, 2}] - [1; \overline{2, 1}]}{\tilde{q}^2_{4k+5}([2; \overline{1, 2}] + \tilde{\beta}_{4k+5})([1; \overline{2, 1}] + \tilde{\beta}_{4k+5})},$$

where $q_{4k+2} = q(2_{2k}1_2 2_{2k})$ and $\tilde{q}_{4k+5} = q(1_2 2_{2k}1_2 2_{2k+1})$. Thus,

$$\frac{A_k - C_k}{D_k - B_k} = \frac{[2; 2, \overline{2, 1}] - [1; 1, \overline{2, 1}]}{[2; \overline{1, 2}] - [1; \overline{2, 1}]} \cdot Y \cdot \frac{\tilde{q}^2_{4k+5}}{q^2_{4k+2}},$$

where

$$Y = \frac{([2;\overline{1,2}] + \tilde{\beta}_{4k+5})([1;\overline{2,1}] + \tilde{\beta}_{4k+5})}{([1;1,\overline{2,1}] + \beta_{4k+2})([2;2,\overline{1,2}] + \beta_{4k+2})}.$$

Let $\alpha = 2_{2k}1_2 2_{2k}$, then $\tilde{q}_{4k+5} = q(1_2\alpha 2) > 2q(1_2\alpha) = 2(p(\alpha) + 2q(\alpha))$. Thus,

$$\frac{\tilde{q}_{4k+5}}{q_{4k+2}} > 2 \cdot \frac{p(\alpha) + 2q(\alpha)}{q(\alpha)} > 4.$$

By Lemma 4.2, $Y \geq 0.22$ and therefore,

$$\frac{A_k - C_k}{D_k - B_k} > 0.46 \cdot 0.22 \cdot (4)^2 > 1.$$

$\square$

By Lemma 4.22 and Remark 4.4 any $(k, \tilde{\lambda}_k^{(3)})$-admissible word θ containing $2_{2k-2}\theta_k^1 2_{2k-2}$ must to extend to right as

$$...2_{2k-2}\theta_k^1 2_{2k-1} = ...2_{2k}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k+1}....$$

Lemma 4.23. *If* $\tilde{\lambda}_k^{(4)} := \lambda_0^-(2_2 1_2 2_{2k-2}\theta_k^1 2_{2k}) = \lambda_0^-(2_2 1_2 2_{2k} 1_2 2_{2k} 1_2 2^* 2_{2k} 1_2 2_{2k+2})$, *then* $\tilde{\lambda}_k^{(4)} > m(\gamma_k^1)$

Proof. By definition, $\lambda_0^-(2_2 1_2 2_{2k-2}\theta_k^1 2_{2k}) = A_k + B_k$, where $A_k = [2; 2_{2k}, 1_2, 2_{2k+2}, \overline{2,1}]$ and $B_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k}, 1_2, 2_2, \overline{2,1}]$. Moreover, $m(\gamma_k^1) \leq C_k + D_k$, where $C_k = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, \overline{1,2}]$ and $D_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, \overline{1,2}]$.

We shall show that $A_k + B_k > C_k + D_k$. In order to establish this inequality, we observe that

$$C_k - A_k = \frac{[2;\overline{1,2}] - [1;1,\overline{1,2}]}{q_{4k+4}^2([1;1,\overline{1,2}] + \beta_{4k+4})([2;\overline{1,2}] + \beta_{4k+4})}$$

and

$$B_k - D_k = \frac{[2;2,\overline{1,2}] - [1;1,2_2,\overline{2,1}]}{\tilde{q}_{4k+4}^2([1;1,2_2,\overline{2,1}] + \tilde{\beta}_{4k+4})([2;2,\overline{1,2}] + \tilde{\beta}_{4k+4})},$$

where $q_{4k+4} = q(2_{2k}1_2 2_{2k+2})$ and $\tilde{q}_{4k+4} = q(1_2 2_{2k} 1_2 2_{2k})$. Thus,

$$\frac{C_k - A_k}{B_k - D_k} = \frac{[2;\overline{1,2}] - [1;1,\overline{1,2}]}{[2;2,\overline{1,2}] - [1;1,2_2,\overline{2,1}]} \cdot X \cdot \frac{\tilde{q}_{4k+4}^2}{q_{4k+4}^2},$$

where

$$X = \frac{([1;1,2_2,\overline{2,1}] + \tilde{\beta}_{4k+4})([2;2,\overline{1,2}] + \tilde{\beta}_{4k+4})}{([1;1,\overline{1,2}] + \beta_{4k+4})([2;\overline{1,2}] + \beta_{4k+4})}.$$

We have

$$X < \frac{([1;1,2_2,\overline{2,1}]+[0;2])([2;2,\overline{1,2}]+[0;2])}{([1;1,\overline{1,2}]+[0;\overline{2}]([2;\overline{1,2}]+[0;\overline{2}])} < 1.1.$$

Let $\alpha = 2_{2k}1_2 2_{2k}$, then $q_{4k+4} = q(\alpha 2_2) > 5q(\alpha)$ and $\tilde{q}_{4k+4} = q(1_2\alpha) = p(\alpha)+2q(\alpha)$. Thus,

$$\frac{\tilde{q}_{4k+4}}{q_{4k+4}} < \frac{p(\alpha)+2q(\alpha)}{5q(\alpha)} < \frac{3}{5}.$$

Therefore,

$$\frac{C_k - A_k}{B_k - D_k} < 1.76 \cdot 1.1 \cdot \left(\frac{3}{5}\right)^2 < 1.$$

□

Lemma 4.24. *Let θ_k^1 be the string in* (4.5)*, then*

$$\lambda_0^+(2_{2k-1}\theta_k^1 2_{2k-1}1_2 2_2) < \lambda_0^+(2_2 1_2 2_{2k-2}\theta_k^1 2_{2k-1}1_2 2_2) < m(\theta(\underline{\omega}_k)).$$

Proof. Let $\lambda_0^+(2_2 1_2 2_{2k-2}\theta_k^1 2_{2k-1}1_2 2_2) = A_k + B_k$, where

$$A_k = [2;2_{2k},1_2,2_{2k+1},1_2,2_2,\overline{2,1}] \text{ and } B_k = [0;1_2,2_{2k},1_2,2_{2k},1_2,2_2,\overline{1,2}].$$

Furthermore, by definition, $m(\theta(\underline{\omega})^k) \geq C_k + D_k$, where

$$C_k = [2;2_{2k},1_2,2_{2k+2},1_2,\overline{2,1}] \text{ and } D_k = [0;1_2,2_{2k},1_2,2_{2k+2},1_2,\overline{2,1}].$$

Thus, our task is prove that $C_k + D_k > A_k + B_k$. In order to establish this estimative, we observe that

$$C_k - A_k = \frac{[2;1_2,\overline{2,1}] - [1;1,2_2,\overline{2,1}]}{q_{4k+3}^2([2;1_2,\overline{2,1}]+\beta_{4k+3})([1;1,2_2,\overline{2,1}]+\beta_{4k+3})}$$

and

$$B_k - D_k = \frac{[2;2,1_2,\overline{2,1}] - [1;1,2_2,\overline{1,2}]}{\tilde{q}_{4k+4}^2([1;1,2_2,\overline{1,2}]+\tilde{\beta}_{4k+4})([2;2,1_2,\overline{2,1}]+\tilde{\beta}_{4k+4})},$$

where $q_{4k+3} = q(2_{2k}1_2 2_{2k+1})$ and $\tilde{q}_{4k+4} = q(1_2 2_{2k}1_2 2_{2k})$. Thus,

$$\frac{C_k - A_k}{B_k - D_k} = \frac{[2;1_2,\overline{2,1}] - [1;1,2_2,\overline{2,1}]}{[2;2,1_2,\overline{2,1}] - [1;1,2_2,\overline{1,2}]} \cdot Y \cdot \frac{\tilde{q}_{4k+4}^2}{q_{4k+3}^2},$$

where

$$Y = \frac{([1;1,2_2,\overline{1,2}]+\tilde{\beta}_{4k+4})([2;2,1_2,\overline{2,1}]+\tilde{\beta}_{4k+4})}{([2;1_2,\overline{2,1}]+\beta_{4k+3})([1;1,2_2,\overline{2,1}]+\beta_{4k+3})}.$$

Note that

$$Y \geq \frac{([1;1,2_2,\overline{1,2}]+[0;\overline{2}])([2;2,1_2,\overline{2,1}]+[0;\overline{2}])}{([2;1_2,\overline{2,1}]+[0;\overline{2}])([1;1,2_2,\overline{2,1}]+[0;\overline{2}])} > 0.93.$$

Let $\alpha = 2_{2k}1_2 2_{2k}$, then $q_{4k+3} = q(\alpha 2) = 2q(\alpha)+q(2_{2k}1_2 2_{2k-1}) < (2+1/2)q(\alpha)$ and $\tilde{q}_{4k+4} = p(\alpha)+2q(\alpha)$. Thus,

$$\frac{\tilde{q}_{4k+4}}{q_{4k+3}} > \frac{2}{5}\cdot\frac{p(\alpha)+2q(\alpha)}{q(\alpha)} = \frac{2}{5}\cdot([0,\alpha]+2) = \frac{2}{5}\cdot[2,\overline{2}] > 0.96.$$

Therefore,

$$\frac{C_k - A_k}{B_k - D_k} > 1.26\cdot 0.93\cdot(0.96)^2 > 1.$$

□

By Lemmas 4.23, 4.24 and Remark 4.4 any $(k,\tilde{\lambda}_k^{(4)})$-admissible word θ containing $2_{2k-2}\theta_k^1 2_{2k-1}$ must to extend as

$$...2_{2k-1}\theta_k^1 2_{2k}... = ...2_{2k+1}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k+2}....$$

Lemma 4.25. $\lambda_0^+(2_2 1_2 2_{2k-1}\theta_k^1 2_{2k}) = \lambda_0^+(2_2 1_2 2_{2k+1}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k+2}) < m(\theta(\underline{\omega}_k))$.

Proof. Let $\lambda_0^+(2_2 1_2 2_{2k-1}\theta_k^1 2_{2k}) = A_k + B_k$, where $A_k = [2;2_{2k},1_2,2_{2k+2},\overline{1,2}]$ and $B_k = [0;1_2,2_{2k},1_2,2_{2k+1},1_2,2_2,\overline{2,1}]$. Moreover, by definition, $m(\theta(\underline{\omega}_k)) \geq C_k + D_k$, where $C_k = [2;2_{2k},1_2,2_{2k+2},1_2,2_2,\overline{2,1}]$ and $D_k = [0;1_2,2_{2k},1_2,2_{2k+2},1_2,\overline{2,1}]$.

Let us show that $A_k + B_k < C_k + D_k$. For this sake, we observe that

$$A_k - C_k = \frac{[2;\overline{1,2}]-[1;2_2,\overline{2,1}]}{q_{4k+5}^2([2;\overline{1,2}]+\beta_{4k+5})([1;2_2,\overline{2,1}]+\beta_{4k+5})}$$

and

$$D_k - B_k = \frac{[2;1_2,\overline{2,1}]-[1;1,2_2,\overline{2,1}]}{\tilde{q}_{4k+5}^2([2;1_2,\overline{2,1}]+\tilde{\beta}_{4k+5})([1;1,2_2,\overline{2,1}]+\tilde{\beta}_{4k+5})},$$

where $\tilde{q}_{4k+5} = q(1_2 2_{2k}1_2 2_{2k+1})$ and $q_{4k+5} = q(2_{2k}1_2 2_{2k+2}1)$. Thus,

$$\frac{A_k - C_k}{D_k - B_k} = \frac{[2;\overline{1,2}]-[1;2_2,\overline{2,1}]}{[2;1_2,\overline{2,1}]-[1;1,2_2,\overline{2,1}]}\cdot X\cdot\frac{\tilde{q}_{4k+5}^2}{q_{4k+5}^2},$$

where

$$X = \frac{([2;1_2,\overline{2,1}]+\tilde{\beta}_{4k+5})([1;1,2_2,\overline{2,1}]+\tilde{\beta}_{4k+5})}{([2;\overline{1,2}]+\beta_{4k+5})([1;2_2,\overline{2,1}]+\beta_{4k+5})}.$$

Note that

$$X < \frac{([2;1_2,\overline{2,1}]+[0;\overline{2}])([1;1,2_2,\overline{2,1}]+[0;\overline{2}])}{([2;\overline{1,2}]+[0;1,2])([1;2_2,\overline{2,1}]+[0;1,2])} < 0.9.$$

Let $\alpha = 2_{2k}1_2 2_{2k+1}$, since $q(2_{2k}1_2 2_{2k}) > (1/3)q(\alpha)$, we have:

$$q_{4k+5} = q(\alpha 21) = q(\alpha 2) + q(\alpha) = 3q(\alpha) + q(2_{2k}1_2 2_{2k}) > (10/3)q(\alpha).$$

Thus,

$$\frac{\tilde{q}_{4k+5}}{q_{4k+5}} < \frac{3}{10}\cdot\frac{p(\alpha)+2q(\alpha)}{q(\alpha)} < \frac{3}{10}([0;\alpha]+2) < \frac{3}{10}\cdot[2;2] = 0.75.$$

Therefore,

$$\frac{A_k - C_k}{D_k - B_k} < 1.52\cdot 0.9\cdot (0.75)^2 < 1.$$

$\square$

Lemma 4.26. $\lambda_0^+(2_{2k}\theta_k^1 2_{2k+1}) = \lambda_0^+(2_{2k+2}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k+3}) < m(\theta(\underline{\omega}_k))$.

Proof. By definition, $\lambda_0^+(2_{2k}\theta_k^1 2_{2k+1}) = A_k + B_k$, where $A_k = [2;2_{2k},1_2,2_{2k+3},\overline{2,1}]$ and $B_k = [0;1_2,2_{2k},1_2,2_{2k+2},\overline{1,2}]$. Moreover, $m(\theta(\underline{\omega}_k)) \geq C_k + D_k$, where $C_k = [2;2_{2k},1_2,2_{2k+2},1_2,2_2,\overline{2,1}]$ and $D_k = [0;1_2,2_{2k},1_2,2_{2k+2},1_2,2_2,\overline{2,1}]$.

We shall show that $A_k + B_k < C_k + D_k$. In order to establish this inequality, we observe that

$$C_k - A_k = \frac{[2;\overline{2,1}] - [1;1,2_2,\overline{2,1}]}{q_{4k+4}^2([1;1,2_2,\overline{2,1}]+\beta_{4k+4})([2;\overline{2,1}]+\beta_{4k+4})}$$

and

$$B_k - D_k = \frac{[2;\overline{1,2}] - [1;2_2,\overline{2,1}]}{\tilde{q}_{4k+7}^2([2;\overline{2,1}]+\tilde{\beta}_{4k+7})([1;2_2,\overline{2,1}]+\tilde{\beta}_{4k+7})},$$

where $q_{4k+4} = q(2_{2k}1_2 2_{2k+2})$ and $\tilde{q}_{4k+7} = q(1_2 2_{2k}1_2 2_{2k+2}1)$.

Thus,

$$\frac{C_k - A_k}{B_k - D_k} = \frac{[2;\overline{2,1}] - [1;1,2_2,\overline{2,1}]}{[2;\overline{1,2}] - [1;2_2,\overline{2,1}]}\cdot Y\cdot\frac{\tilde{q}_{4k+7}^2}{q_{4k+4}^2},$$

where

$$Y = \frac{([2;\overline{2,1}]+\tilde{\beta}_{4k+7})([1;2_2,\overline{2,1}]+\tilde{\beta}_{4k+7})}{([1;1,2_2,\overline{2,1}]+\beta_{4k+4})([2;\overline{2,1}]+\beta_{4k+4})}.$$

Let $\alpha = 2_{2k}1_2 2_{2k+2}$ and $\tilde{\alpha} = 1_2\alpha$, since that $q(1_2 2_{2k}1_2 2_{2k+1}) > (1/3)q(\tilde{\alpha})$, we have

$$\tilde{q}_{4k+7} = q(\tilde{\alpha}1) = q(\tilde{\alpha}) + q(1_2 2_{2k}1_2 2_{2k+1}) > (4/3)q(\tilde{\alpha}).$$

Thus,

$$\frac{\tilde{q}_{4k+7}}{q_{4k+4}} > \frac{4}{3}\cdot\frac{p(\alpha)+2q(\alpha)}{q(\alpha)} = \frac{4}{3}\left([0;\alpha]+2\right) > \frac{4}{3}\cdot[2;\overline{2}] > 3.2.$$

Therefore, since that $Y \geq 0.226$, by Lemma 4.2, we obtain

$$\frac{C_k - A_k}{B_k - D_k} > 0.49\cdot 0.226\cdot(3.2)^2 > 1.$$

□

Let $\tilde{\lambda}_k^{(5)} = \min\{\lambda_0^-(1_2 2_{2k+2} 1_2 2^* 2_{2k-2} 1_2 2_2), \lambda_0^-(1_2 2_{2k-2} 1_2 2^* 2_{2k+1})\}$. We have $\tilde{\lambda}_k^{(5)} > m(\gamma_k^1)$ from Lemmas 4.9 and 4.12. Moreover, these lemmas, Lemmas 4.25 and 4.26 (and Remark 4.4) imply that any $(k, \tilde{\lambda}_k^{(5)})$-admissible θ containing $2_{2k-1}\theta_k^1 2_{2k}$ must extend as

$$...2_{2k}\theta_k^1 2_{2k} 1_2 2_{2k}... = ...2_{2k+2} 1_2 2_{2k} 1_2 2^* 2_{2k} 1_2 2_{2k+2} 1_2 2_{2k}....$$

Lemma 4.27.

$$\lambda_0^+(2_{2k+1}\theta_k^1 2_{2k} 1_2 2_{2k}) = \lambda_0^+(2_{2k+3} 1_2 2_{2k} 1_2 2^* 2_{2k} 1_2 2_{2k+2} 1_2 2_{2k}) < m(\theta(\underline{\omega}_k))$$

Proof. By definition, $m(\theta(\underline{\omega}_k)) \geq C_k + D_k$, where

$C_k = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k}, 1_2, 2_2, \overline{2,1}]$ and $D_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_2, \overline{2,1}]$.

Note that $\lambda_0^+(2_{2k+1}\theta_k^1 2_{2k} 1_2 2_{2k}) = A_k + B_k$, where $A_k = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k}, \overline{1,2}]$ and $B_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+3}, \overline{2,1}]$.

Hence, our work is reduced to prove that $A_k + B_k < C_k + D_k$. In order to prove this inequality, we observe that

$$A_k - C_k = \frac{[2;\overline{1,2}] - [1; 2_2, \overline{2,1}]}{q_{6k+7}^2([2;\overline{1,2}] + \beta_{6k+7})([1; 2_2, \overline{2,1}] + \beta_{6k+7})}$$

and

$$D_k - B_k = \frac{[2;\overline{2,1}] - [1; 1, 2_2, \overline{2,1}]}{\tilde{q}_{4k+6}^2([1; 1, 2_2, \overline{2,1}] + \tilde{\beta}_{4k+6})([2;\overline{2,1}] + \tilde{\beta}_{4k+6})},$$

where $q_{6k+7} = q(2_{2k} 1_2 2_{2k+2} 1_2 2_{2k} 1)$ and $\tilde{q}_{4k+6} = q(1_2 2_{2k} 1_2 2_{2k+2})$.

Thus,

$$\frac{A_k - C_k}{D_k - B_k} = \frac{[2;\overline{1,2}] - [1; 2_2, \overline{2,1}]}{[2;\overline{2,1}] - [1; 1, 2_2, \overline{2,1}]}\cdot X\cdot\frac{\tilde{q}_{4k+6}^2}{q_{6k+7}^2},$$

where

$$X = \frac{([1; 1, 2_2, \overline{2,1}] + \tilde{\beta}_{4k+6})([2;\overline{2,1}] + \tilde{\beta}_{4k+6})}{([2;\overline{1,2}] + \beta_{6k+7})([1; 2_2, \overline{2,1}] + \beta_{6k+7})}.$$

Let $\alpha = 2_{2k}1_2 2_{2k+2}$, since that $2k \geq 4$, we have $q_{6k+7} = q(\alpha 1_2 2_{2k} 1) > 2^4 q(\alpha)$. Thus,

$$\frac{\tilde{q}_{4k+6}}{q_{6k+7}} < \frac{p(\alpha) + 2q(\alpha)}{2^4 q(\alpha)} < \frac{3}{16}.$$

By Lemma 4.2, we have $X < 4.41$ and therefore,

$$\frac{A_k - C_k}{D_k - B_k} < 2.1 \cdot 4.41 \cdot \left(\frac{3}{16}\right)^2 < 1.$$

□

Lemma 4.28.

$\lambda_0^+(2_{2k}1_2 2_{2k}\theta_k^1 2_{2k}1_2 2_{2k+1}) = \lambda_0^+(2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k+2}1_2 2_{2k+1}) < m(\theta(\underline{\omega}_k))$.

Proof. Let $\lambda_0^+(2_{2k}1_2 2_{2k}\theta_k^1 2_{2k}1_2 2_{2k+1}) = A_k + B_k$, where

$A_k = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k+1}, 1_2, \overline{2, 1}]$ *and* $B_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k}, \overline{1, 2}]$.

Moreover, $m(\theta(\omega_k)) \geq C_k + D_k$, where

$$C_k = [2; 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k}, 1_2, 2_2, \overline{2, 1}] \text{ and}$$
$$D_k = [0; 1_2, 2_{2k}, 1_2, 2_{2k+2}, 1_2, 2_{2k+1}, 1_2, 2_2, \overline{1, 2}].$$

Let us show that $A_k + B_k < C_k + D_k$. For this sake, we observe that

$$C_k - A_k = \frac{[2; 1_2, \overline{2, 1}] - [1; 1, 2_2, \overline{2, 1}]}{q_{6k+6}^2([1; 1, 2_2, \overline{2, 1}] + \beta_{6k+6})([2; 1_2, \overline{2, 1}] + \beta_{6k+6})},$$

while

$$B_k - D_k = \frac{[2; 1_2, 2_2, \overline{1, 2}] - [1; \overline{2, 1}]}{\tilde{q}_{6k+8}^2([1; \overline{2, 1}] + \tilde{\beta}_{6k+8})([2; 1_2, 2_2, \overline{1, 2}] + \tilde{\beta}_{6k+8})},$$

where $q_{6k+6} = q(2_{2k}1_2 2_{2k+2}1_2 2_{2k})$ and $\tilde{q}_{6k+8} = q(1_2 2_{2k}1_2 2_{2k+2}1_2 2_{2k})$.

Thus,

$$\frac{C_k - A_k}{B_k - D_k} = \frac{[2; 1_2, \overline{2, 1}] - [1; 1, 2_2, \overline{2, 1}]}{[2; 1_2, 2_2, \overline{1, 2}] - [1; \overline{2, 1}]} \cdot Y \cdot \frac{\tilde{q}_{6k+8}^2}{q_{6k+6}^2},$$

where

$$Y = \frac{([1; \overline{2, 1}] + \tilde{\beta}_{6k+8})([2; 1_2, 2_2, \overline{1, 2}] + \tilde{\beta}_{6k+8})}{([1; 1, 2_2, \overline{2, 1}] + \beta_{6k+6})([2; 1_2, \overline{2, 1}] + \beta_{6k+6})}.$$

Note that,

$$Y > \frac{([1; \overline{2, 1}] + [0; \overline{2}])([2; 1_2, 2_2, \overline{1, 2}] + [0; \overline{2}])}{([1; 1, 2_2, \overline{2, 1}] + [0; 2, 2, 1])([2; 1_2, \overline{2, 1}] + [0; 2, 2, 1])} > 0.78.$$

Let $\alpha = 2_{2k}1_2 2_{2k+2}1_2 2_{2k}$, then

$$\frac{\tilde{q}_{6k+8}}{q_{6k+6}} = \frac{p(\alpha) + 2q(\alpha)}{q(\alpha)} = 2 + [0; \alpha] > 2 + [0; \overline{2}] > 2.41.$$

Therefore,

$$\frac{C_k - A_k}{B_k - D_k} > 0.7 \cdot 0.78 \cdot (2,41)^2 > 1.$$

□

Let $\tilde{\lambda}_k^{(5)} > m(\gamma_k^1)$ be as before. By Lemma 4.27 and Remark 4.4, a $(k, \tilde{\lambda}_k^{(5)})$-admissible word θ containing $2_{2k}\theta_k^1 2_{2k}1_2 2_{2k}$ extend as $2_2 1_2 2_{2k}\theta_k^1 2_{2k}1_2 2_{2k}$. By Lemmas 4.9 and 4.12, θ must keeping extending as $2_{2k}1_2 2_{2k}\theta_k^1 2_{2k}1_2 2_{2k}$. Finally, by Lemma 4.28, θ must keeping extending as

$$2_{2k}1_2 2_{2k}\theta_k^1 2_{2k}1_2 2_{2k}1 = ...2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1....$$

The full discussion of this section can be compiled into the following lemma establishing that a word θ containing the right string $2_2 1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_2$ must extend until the beginning of replication mechanism:

Lemma 4.29 (Going to the Replication)**.** *For every $k \geq 2$, there exists a explicit constant $\tilde{\lambda}_k > m(\gamma_k^1)$ such that any $(k, \tilde{\lambda}_k)$-admissible word θ containing $\theta_k^1 := 2_2 1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_2$ must extend as*

$$...2_{2k}1_2 2_{2k+2}1_2 2_{2k}1_2 2^* 2_{2k}1_2 2_{2k+2}1_2 2_{2k}1....$$

Proof. This result for $\tilde{\lambda}_k := \min\{\tilde{\lambda}_k^{(i)} : i = 1, ..., 5\}$ is a consequence of this subsection. □

4.6 Local uniqueness

In this section, we proved the local uniqueness for γ_k^1, $k \in \{1, 2, 3, 4\}$.

4.6.1 Local uniqueness for γ_1^1

Note that

$$m(\theta(\underline{\omega}_1)) = \lambda_0(\overline{2_3 1_2 2_4 1_2 2_2 1_2} 2^* 221_2 2_4 1_2 2_2 1_2 \overline{2_3 1_2 2_4 1_2 2_2 1_2})$$
$$= 3.00558731248699779818\ldots$$

and

$$
\begin{aligned}
m(\gamma_1^1) &= \lambda_0(\overline{2_3 1_2 2_4 1_2 2_2 1_2} 2^* 2 2 1_2 2_4 1_2 2_2 1_2 2_3 1_2 2_4 1_2 \overline{2}) \\
&= 3.00558731248699779947\ldots
\end{aligned}
$$

By Corollary 4.2, up to transposition, a $(1, 3.009)$-admissible word θ is

- $\theta = \ldots 1_4 2^* 2 1_2 \ldots$ or $\theta = \ldots 2_2 1_2 2^* 2_2 \ldots$.

Lemma 4.30. $\lambda_0^+(1_4 2^* 2 1_3 \ldots) < 3.0032$.

Lemma 4.31. $\lambda_0^+(2 1_2 2^* 2_3) < 3.0017$ *and* $\lambda_0^+(2_3 1_2 2^* 2_2 1_2 2_2) < 3.00486$.

By Lemmas 4.30, 4.31 and Remark 4.4, it follows that a $(1, 3.009)$-admissible word θ is

- $\theta = \ldots 1_4 2^* 2 1_2 2_2 \ldots$ or $\theta = \ldots 1_2 2_2 1_2 2^* 2_2 1_2 2_2 \ldots$.

By applying Remark 4.4 once again, we have that

- $\theta = \ldots 1_4 2^* 2 1_2 2_2 1_2 \ldots$, or $\theta = \ldots 1_4 2^* 2 1_2 2_3 \ldots$, or
- $\theta = \ldots 1_2 2_2 1_2 2^* 2_2 1_2 2_2 1_2 \ldots$, or $\theta = \ldots 1_2 2_2 1_2 2^* 2_2 1_2 2_3 \ldots$,

whenever θ is $(1, 3.009)$-admissible.

Lemma 4.32. *(i)* $\lambda_0^+(2 1_4 2^* 2 1_2 2_2 1_2) < \lambda_0^+(2 1_4 2^* 2 1_2 2_3 \ldots) < 3.00026$

(ii) $\lambda_0^-(1_3 2_2 1_2 2^* 2_2 1_2 2_2 1_2) > \lambda_0^-(2_2 1_2 2_2 1_2 2^* 2_2 1_2 2_2 1_2) > 3.0056$

(iii) $\lambda_0^-(1_3 2_2 1_2 2^* 2_2 1_2 2_3) > 3.0056$

By Lemma 4.32, if θ is $(1, 3.0056)$-admissible, then

- $\theta = \ldots 1_5 2^* 2 1_2 2_2 1_2 \ldots$, or $\theta = \ldots 1_5 2^* 2 1_2 2_3 \ldots$, or $\theta = \ldots 2_2 1_2 2_2 1_2 2^* 2_2 1_2 2_3 \ldots$.

By Remark 4.4, it follows that

- $\theta = \ldots 1_5 2^* 2 1_2 2_2 1_4 \ldots$, or $\theta = \ldots 1_5 2^* 2 1_2 2_2 1_2 2_2 \ldots$, or
- $\theta = \ldots 1_5 2^* 2 1_2 2_3 1_2 2_2 \ldots$, or $\theta = \ldots 1_5 2^* 2 1_2 2_4 \ldots$, or
- $\theta = \ldots 2_2 1_2 2_2 1_2 2^* 2_2 1_2 2_3 1_2 2_2 \ldots$, or $\theta = \ldots 2_2 1_2 2_2 1_2 2^* 2_2 1_2 2_4 \ldots$,

whenever θ is $(1, 3.0056)$-admissible.

Lemma 4.33. *(i)* $\lambda_0^+(1_6 2^* 21_2 2_2 1_4) < \lambda_0^+(1_6 2^* 21_2 2_2 1_2 2_2) < \lambda_0^+(1_6 2^* 21_2 2_4) < \lambda_0^+(1_6 2^* 21_2 2_3 1_2 2_2) < 3.00513$

(ii) $\lambda_0^-(21_5 2^* 21_2 2_3 1_2 2_2) > \lambda_0^-(21_5 2^* 21_2 2_4) > \lambda_0^-(21_5 2^* 21_2 2_2 1_2 2_2) > \lambda_0^-(21_5 2^* 21_2 2_2 1_4) > 3.0063$

(iii) $\lambda_0^+(2_3 1_2 2_2 1_2 2^* 2_2 1_2 2_3 1_2 2_2) < 3.005584$

(iv) $\lambda_0^-(12_2 1_2 2_2 1_2 2^* 2_2 1_2 2_4) > 3.005589$

By Lemma 4.33, if θ is $(1, 3.005589)$-admissible, then

- $\theta = \dots 1_2 2_2 1_2 2_2 1_2 2^* 2_2 1_2 2_3 1_2 2_2 \dots$, or $\theta = \dots 2_3 1_2 2_2 1_2 2^* 2_2 1_2 2_4 \dots$.

Lemma 4.34. $\lambda_0^+(2_3 1_2 2_2 1_2 2^* 2_2 1_2 2_5) < 3.0055868$

By Lemma 4.34 and Remark 4.4, if θ is $(1, 3.005589)$-admissible, then

- $\theta = \dots 1_2 2_2 1_2 2_2 1_2 2^* 2_2 1_2 2_3 1_2 2_2 1_2 \dots$, or $\theta = \dots 1_2 2_2 1_2 2_2 1_2 2^* 2_2 1_2 2_3 1_2 2_3 \dots$,
- $\theta = \dots 2_3 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_2 \dots$.

Lemma 4.35. *(i)* $\lambda_0^+(21_2 2_2 1_2 2_2 1_2 2^* 2_2 1_2 2_3 1_2 2_2 1_2) < \lambda_0^+(21_2 2_2 1_2 2_2 1_2 2^* 2_2 1_2 2_3 1_2 2_3) < \lambda_0^+(1_4 2_2 1_2 2_2 1_2 2^* 2_2 1_2 2_3 1_2 2_3) < 3.00558725$

(ii) $\lambda_0^+(12_3 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_2) < 3.0055867$

By Lemma 4.35 and Remark 4.4, if θ is $(1, 3.005589)$-admissible, then

- $\theta = \dots 1_4 2_2 1_2 2_2 1_2 2^* 2_2 1_2 2_3 1_2 2_2 1_2 \dots$, or $\theta = \dots 2_4 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_2 \dots$.

By applying Remark 4.4 once more, we get that

- $\theta = \dots 1_4 2_2 1_2 2_2 1_2 2^* 2_2 1_2 2_3 1_2 2_2 1_4 \dots$, or $\theta = \dots 1_4 2_2 1_2 2_2 1_2 2^* 2_2 1_2 2_3 1_2 2_2 1_2 2_2 \dots$,
- $\theta = \dots 2_4 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_2 1_2 \dots$, or $\theta = \dots 2_4 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_3 \dots$,

whenever θ is $(1, 3.005589)$-admissible.

Lemma 4.36. *One has*

(i) $\lambda_0^+(1_4 2_2 1_2 2_2 1_2 2^* 2_2 1_2 2_3 1_2 2_2 1_4) < \lambda_0^+(1_4 2_2 1_2 2_2 1_2 2^* 2_2 1_2 2_3 1_2 2_2 1_2 2_2) < 3.0055872244$

(ii) $\lambda_0^+(2_5 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_3) < \lambda_0^+(2_2 1_2 2_4 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_3) < 3.0055873108$

(iii) $\lambda_0^+(2_5 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_2 1_2) < 3.005587211$

By Lemma 4.36(i), if θ is $(1, 3.005589)$-admissible, then

- $\theta = \dots 2_4 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_2 1_2 \dots$, or $\theta = \dots 2_4 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_3 \dots$.

By Remark 4.4, it follows that

- $\theta = \dots 2_2 1_2 2_4 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_2 1_2 \dots$, or $\theta = \dots 2_5 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_2 1_2 \dots$,
- $\theta = \dots 2_2 1_2 2_4 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_3 \dots$, or $\theta = \dots 2_5 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_3 \dots$,

whenever θ is $(1, 3.005589)$-admissible.

Hence, Lemma 4.36 implies the desired local uniqueness result for γ_1^1:

Lemma 4.37 (Local uniqueness of γ_1^1)**.** *A $(1, 3.005589)$-admissible word θ has the form*

$$\theta = \dots 2_2 1_2 2_4 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_2 1_2 \dots$$

In particular, it contains the string $\theta_1^0 = 2_2 1_2 2_4 1_2 2_2 1_2 2^* 2_2 1_2 2_4 1_2 2_2 1$.

4.6.2 Local uniqueness for γ_2^1

Observe that

$$\begin{aligned} m(\theta(\underline{\omega}_2)) &= \lambda_0(\overline{2_5 1_2 2_6 1_2 2_4 1_2} 2^* 2_4 1_2 2_6 1_2 2_4 1_2 \overline{2_5 1_2 2_6 1_2 2_4 1_2}) \\ &= 3.000164231218189413925594268 22\dots \end{aligned}$$

and

$$\begin{aligned} m(\gamma_2^1) &= \lambda_0(\overline{2_5 1_2 2_6 1_2 2_4 1_2} 2^* 2_4 1_2 2_6 1_2 2_4 1_2 2_5 1_2 2_6 1_2 \overline{2}) \\ &= 3.00016423121818941392559426906\dots \end{aligned}$$

By Corollary 4.2, up to transposition, a $(2, 3.009)$-admissible word x is

- $x = \dots 1_4 2^* 2 1_2 \dots$ or $x = \dots 2_2 1_2 2^* 2_2 \dots$.

Lemma 4.38. $\lambda_0^-(2_2 1_2 2^* 2_2 1) > 3.0043$.

By Lemma 4.38, if x is $(2, 3.009)$-admissible, then

- $x = \dots 1_4 2^* 2 1_3 \dots$ or $x = \dots 1_4 2^* 2 1_2 2_2 \dots$ or
- $x = \dots 1 2_2 1_2 2^* 2_3 \dots$ or $x = \dots 2_3 1_2 2^* 2_3 \dots$.

Lemma 4.39. *(i)* $\lambda_0^-(1 2_2 1_2 2^* 2_4) > 3.00073$.

(ii) $\lambda_0^+(2_3 1_2 2^* 2_3 1) < 3$.

By Lemma 4.39, Lemma 4.5 and Lemma 4.38, if x is $(2, 3.00073)$-admissible, then

- $x = \dots 1_4 2^* 2 1_4 \dots$ or $x = \dots 1_4 2^* 2 1_2 2_2 1_2 \dots$ or
- $x = \dots 1_4 2^* 2 1_2 2_4 \dots$ or $x = \dots 1_2 2_2 1_2 2^* 2_3 1_2 2_2 \dots$ or
- $x = \dots 2_4 1_2 2^* 2_4 \dots$

Lemma 4.40. *(i)* $\lambda_0^+(2 1_2 2_2 1_2 2^* 2_3 1_2 2_2) < 3.00000758$.

(ii) $\lambda_0^+(1_2 2_2 1_4 2^* 2 1_2 2_2 1_2) < \lambda_0^+(1_2 2_2 1_4 2^* 2 1_2 2_4) < 3.0001551$.

(iii) $\lambda_0^-(1_5 2^* 2 1_2 2_4) > \lambda_0(1_5 2^* 2 1_2 2_2 1_2) > 3.003$.

(iv) $\lambda_0^+(2 1_4 2^* 2 1_4) < 3$.

By Lemma 4.40 and Remark 4.4, if x is $(2, 3.00073)$-admissible, then

- $x = \dots 1_5 2^* 2 1_4 \dots$ or $x = \dots 1_4 2_2 1_2 2^* 2_3 1_2 2_2 \dots$ or
- $x = \dots 2_2 1_2 2_4 1_2 2^* 2_4 \dots$ or $x = \dots 2_5 1_2 2^* 2_4 \dots$.

Lemma 4.41. *(i)* $\lambda_0^+(2_4 1_2 2^* 2_5) < 3.00005$.

(ii) $\lambda_0^+(2_5 1_2 2^* 2_4 1_2 2) < 3.0001426$.

(iii) $\lambda_0^+(1_4 2_2 1_2 2^* 2_3 1_2 2_2 1) < \lambda_0^+(1_4 2_2 1_2 2^* 2_3 1_2 2_3) < 3.0001544$.

(iv) $\lambda_0^-(2 1_5 2^* 2 1_4) > 3.0014$.

By Lemma 4.41, if x is $(2, 3.00073)$-admissible, then

(a) $x = \dots 1_6 2^* 2 1_5 \dots$ or $x = \dots 1_6 2^* 2 1_4 2_2 1_2 \dots$;

(b) $x = \dots 2_2 1_2 2_4 1_2 2^* 2_4 1_2 2_2 \dots$

First, we start proving that there is no possible continuations of x with central combinatorics in the branch (a).

Lemma 4.42. $\lambda_0^+(1_6 2^* 2 1_5 2 \dots) < 3.000083$.

By Lemma 4.42 and Lemma 4.38, if x in the branch (a) is $(2, 3.00073)$-admissible, then

- $x = \dots 1_6 2^* 21_6 \dots$ or
- $x = \dots 1_6 2^* 21_4 2_2 1_4 \dots$ or $x = \dots 1_6 2^* 21_4 2_2 1_2 2_2 \dots$

Lemma 4.43. *(i)* $\lambda_0^-(1_7 2^* 21_4 2_2 1_2) > 3.000545$.

(ii) $\lambda_0^+(21_6 2^* 21_6) < \lambda_0^+(21_6 2^* 21_4 2_2 1_4) < \lambda_0^+(21_6 2^* 21_4 2_2 1_2 2_2) < 3.000014$.

By Lemma 4.43 and Lemma 4.38, if x in the branch (a) is $(2, 3.000545)$-admissible, then $x = \dots 1_7 2^* 21_6 \dots$. And this one must to extend by Remark 4.4 as

- $x = \dots 1_7 2^* 21_7 \dots$ or $x = \dots 1_7 2^* 21_6 2_2 1_2 \dots$.

Lemma 4.44. *(i)* $\lambda_0^+(1_8 2^* 21_7) < \lambda_0^+(1_8 2^* 21_6 2_2) < 3.0001516$

(ii) $\lambda_0^-(21_7 2^* 21_6 \dots) > 3.0002048$.

By Lemma 4.44, there is no x in the branch (a) which is $(2, 3.000248)$-admissible. Thus, it remains just the branc (b). More specifically:

Corollary 4.9. *Any* $(2, 3.000248)$*-admissible word x has the form*

$$x = \dots 2_2 1_2 2_4 1_2 2^* 2_4 1_2 2_2 \dots.$$

Finally, we follow the script in the Section 4.5, which is condensed in Lemma 4.29, by this lemma, there is a explicit constant $\tilde{\lambda}_2 > m(\gamma_2^1)$, for which one we get the desired local uniqueness result for γ_2^1:

Lemma 4.45 (Local uniqueness of γ_2^1)**.** *A* $(2, 3.000164233)$*-admissible word θ has the form*

$$\theta = \dots 2_4 1_2 2_6 1_2 2_4 1_2 2^* 2_4 1_2 2_6 1_2 2_4 1_2 2_2 \dots$$

In particular, it contains the string $\theta_2^0 = 2_4 1_2 2_6 1_2 2_4 1_2 2^* 2_4 1_2 2_6 1_2 2_4 1$.

4.6.3 Local uniqueness for γ_3^1

Note that

$$m(\theta(\underline{\omega}_3)) = 3.00000483430477638242797442234744984 23...$$

and

$$m(\gamma_3^1) = 3.0000048343047763824279744223474498428...$$

By Corollary 4.2, up to transposition, a $(3, 3.009)$-admissible word has the form

(a) $\theta = \ldots 1_4 2^* 2 1_2 \ldots$ or

(b) $\theta = \ldots 2_2 1_2 2^* 2_2 \ldots$

First, we start studying the possible continuations of θ with central combinatorics in the branch (a), let us now show that $1_4 2^* 2 1_2$ in (a) can not extend into a $(3, 3.0000075)$-admissible word. By Lemma 4.5, θ extends as either $\theta = ...1_4 2^* 2 1_4 \ldots$ or $\theta = ...1_4 2^* 2 1_2 2_2 ....$ Again, by Lemma 4.5, $2_3 1_3$ and 212 are prohibited, the possible continuation of these words on the left hand are

- $\theta = ...1_5 2^* 2 1_4$ or $\theta = ...1_2 2_2 1_4 2^* 2 1_4 ...$;
- $\theta = ...1_5 2^* 2 1_2 2_2 ...$ or $\theta = ...1_2 2_2 1_4 2^* 2 1_2 2_2 ....$

Recall from Lemma 4.5 and the case $k = 2$ (i.e., Subsection 4.6.2) the following 2-prohibited strings: $v_1 = 2 1_3 2^* 2 1_2$, $v_2 = 2 1_3 2^* 2 1_3$, $v_3 = 1_5 2^* 2 1_2 2_2 1_2$, $v_4 = 1_5 2^* 2 1_2 2_4$, $v_5 = 2 1_5 2^* 2 1_4$, $v_6 = 1_7 2^* 2 1_4 2_2 1_2$ and $v_7 = 2 1_7 2^* 2 1_6$

Lemma 4.46. $\lambda_0^+(1_2 2_2 1_4 2^* 2 1_4) < 2.997.$

By Lemma 4.46, the possible continuation of these words in the branch (a) on the right hand side are

- $\theta = ...1_5 2^* 2 1_5 ...$ or $\theta = ...1_5 2^* 2 1_4 2_2 1_2 ...$, because $2_3 1_3$ and 212 are prohibited;
- $\theta = ...1_5 2^* 2 1_2 2_3 1_2 2_2 ...$, because v_3 and v_4 are prohibited;
- $\theta = ...1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 ...$ or $\theta = ...1_2 2_2 1_4 2^* 2 1_2 2_3 ....$

Lemma 4.47.

$$\lambda_0^-(1_5 2^* 2 1_2 2_3 1_2 2_2) > \lambda_0^-(1_2 2_2 1_4 2^* 2 1_2 2_3) > 3.0001,\ \lambda_0^-(1_2 2_2 1_5 2^* 2 1_4 2_2 1_2) > 3.002.$$

By Lemma 4.47, if θ in the branch (a) is admissible, then

- $\theta = ...1_6 2^* 2 1_6 ...$, because v_5 is 2-prohibited;
- $\theta = ...1_6 2^* 2 1_4 2_2 1_2 ...$, because by above Lemma $1_2 2_2 1_5 2^* 2 1_4 2_2 1_2$ is prohibited;
- $\theta = ...1_4 2_2 1_4 2^* 2 1_2 2_2 1_2 ...$ or $\theta = ...2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 ...$, because $2 1_3 2_2 1_2$ is prohibited.

Lemma 4.48. $\lambda_0^+(1_2 2_2 1_6 2^* 2 1_6) < 3$

By Lemma 4.48 and Remark 4.4 , the possible continuation of these words in the branch (a) on the left hand side are

- $\theta = ...1_7 2^* 2 1_6$;
- $\theta = ...1_2 2_2 1_6 2^* 2 1_4 2_2 1_2 ...$, because v_6 is 2-prohibited;
- $\theta = ...1_5 2_2 1_4 2^* 2 1_2 2_2 1_2 ...$ or $\theta = ...1_2 2_2 1_4 2_2 1_4 2^* 2 1_2 2_2 1_2 ...$;
- $\theta = ...1_2 2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 ...$ or $\theta = ...2_3 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 ....$

By Remark 4.4, the possible continuation of these words in the branch (a) on the right (sometimes also on the left) hand side are

- $\theta = ...1_8 2^* 2 1_8 ...$ or $\theta = ...1_8 2^* 2 1_6 2_2 1_2 ...$, because $v_7 = 2 1_7 2^* 2 1_6$ is 2-prohibited;
- $\theta = ...1_2 2_2 1_6 2^* 2 1_4 2_2 1_4 ...$ or $\theta = ...1_2 2_2 1_6 2^* 2 1_4 2_2 1_2 2_2 ...$, because v_1 and $v_2 = 2 1_3 2_2 1_2$ are 2-prohibited;
- $\theta = ...1_5 2_2 1_4 2^* 2 1_2 2_2 1_3 ...$ or $\theta = ...1_5 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2 ...$;
- $\theta = ...1_2 2_2 1_4 2_2 1_4 2^* 2 1_2 2_2 1_3 ...$ or $\theta = ...1_2 2_2 1_4 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2 ...$;
- $\theta = ...1_2 2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_4 ...$ or $\theta = ...1_2 2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2 ...$;
- $\theta = ...2_3 1_2 2_2 1_4 2^* 2 1_2 2_2 1_4 ...$ or $\theta = ...2_3 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2 ...$;

Lemma 4.49. *i)* $\lambda_0^+(1_2 2_2 1_6 2^* 2 1_4 2_2 1_4) < 3.00000211$, $\lambda_0^+(1_2 2_2 1_6 2^* 2 1_4 2_2 1_2 2_2) < 3.00000469$*;*

ii) $\lambda_0^+(1_5 2_2 1_4 2^* 2 1_2 2_2 1_4) < 3$, $\lambda_0^+(1_5 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2) < 3$*;*

iii) $\lambda_0^+(1_2 2_2 1_4 2_2 1_4 2^* 2 1_2 2_2 1_4) < 3.0000009352$ *and* $\lambda_0^-(1_2 2_2 1_4 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2) > 3.00001$*;*

iv) $\lambda_0^+(1_2 2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_4) < 3$ *and* $\lambda_0^+(1_2 2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2) < 3.000001133$*;*

v) $\lambda_0^+(2_3 1_2 2_2 1_4 2^* 2 1_2 2_2 1_4) < 3$ *and* $\lambda_0^+(2_3 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2) < 3.00000019457.$

By Lemma 4.49, if θ in the branch (a) is $(3, 3.00001)$-admissible, then either $\theta = ...1_8 2^* 2 1_8 ...$ or $\theta = ...1_8 2^* 2 1_6 2_2 1_2 ....$ And by Remark 4.4, their left hand side continuations are

- $\theta = ...1_9 2^* 21_8...$ or $\theta = ...1_2 2_2 1_8 2^* 21_8...$
- $\theta = ...1_9 2^* 21_6 2_2 1_2...$ or $\theta = ...1_2 2_2 1_8 2^* 21_6 2_2 1_2....$

Lemma 4.50. $\lambda_0^+(1_2 2_2 1_8 2^* 21_8) < 3$, $\lambda_0^-(1_9 2^* 21_6 2_2 1_2) > 3.00007$ *and* $\lambda_0^+(1_2 2_2 1_8 2^* 21_6 2_2 1_2) < 3.00000080093$

By Lemma 4.50, if θ in the branch (a) is $(3, 3.00001)$-admissible, then $\theta = ...1_9 2^* 21_8....$ By Remark 4.4, this word must extend to the right as $\theta = ...1_9 2^* 21_9...$ or $\theta = ...1_9 2^* 21_8 2_2 1_2...$, ant then must to extend to the left as:

- $\theta = ...1_{10} 2^* 21_9...$ or $\theta = ...1_2 2_2 1_9 2^* 21_9...$;
- $\theta = ...1_{10} 2^* 21_8 2_2 1_2...$ or $\theta = ...1_2 2_2 1_9 2^* 21_8 2_2 1_2....$

Lemma 4.51. $\lambda_0^-(1_2 2_2 1_9 2^* 21_9) > 3.00003$ *and* $\lambda_0^-(1_2 2_2 1_9 2^* 21_8 2_2 1_2) > 3.00005.$

By Lemma 4.51, if θ in the branch (a) is $(3, 3.00001)$-admissible, then $\theta = ...1_{10} 2^* 21_9...$ or $\theta = ...1_{10} 2^* 21_8 2_2 1_2....$ By Remark 4.4, the first word must extend to the right as $\theta = ...1_{10} 2^* 21_{10}...$, because $\theta = ...1_{10} 2^* 21_9 2_2 1_2...$ contains $1_9 2^* 21_9 2_2 1_2$ (string is 3-prohibited). The second word must extend to the right as $\theta = ...1_{10} 2^* 21_8 2_2 1_4...$ or $\theta = ...1_{10} 2^* 21_8 2_2 1_2 2_2....$ Again by Remark 4.4, the continuations on the left hand side are

- $\theta = ...1_{11} 2^* 21_{10}...$ or $\theta = ...1_2 2_2 1_{10} 2^* 21_{10}...$;
- $\theta = ...1_{11} 2^* 21_8 2_2 1_4...$ or $\theta = ...1_2 2_2 1_{10} 2^* 21_8 2_2 1_4...$;
- $\theta = ...1_{11} 2^* 21_8 2_2 1_2 2_2...$ or $\theta = ...1_2 2_2 1_{10} 2^* 21_8 2_2 1_2 2_2....$

Lemma 4.52. *(i)* $\lambda_0^+(1_2 2_2 1_{10} 2^* 21_{10}) < 3$*;*

(ii) $\lambda_0^-(1_{11} 2^* 21_8 2_2 1_4) > 3.00001$*,* $\lambda_0^+(1_2 2_2 1_{10} 2^* 21_8 2_2 1_4) < 3.000000044$*;*

(iii) $\lambda_0^-(1_{11} 2^* 21_8 2_2 1_2 2_2) > 3.00001$*,* $\lambda_0^+(1_2 2_2 1_{10} 2^* 21_8 2_2 1_2 2_2) < 3.000000099.$

By Lemma 4.52, if θ in the branch (a) is $(3, 3.00001)$-admissible, then $\theta = ...1_{11} 2^* 21_{10}....$ By Remark 4.4, this word must extend to the right as $\theta = ...1_{11} 2^* 21_{11}$ or $\theta = ...1_{11} 2^* 21_{10} 2_2 1_2....$ Again by Remark 4.4, the continuations on the left hand side are

- $\theta = ...1_{12} 2^* 21_{11}...$ or $\theta = ...1_2 2_2 1_{11} 2^* 21_{11}...$;
- $\theta = ...1_{12} 2^* 21_{10} 2_2 1_2...$ or $\theta = ...1_2 2_2 1_{11} 2^* 21_{10} 2_2 1_2....$

Lemma 4.53. $\lambda_0^+(1_{12}2^*21_9) < 3.000003786$*,* $\lambda_0^-(1_22_21_{11}2^*21_{10}2_21_2) > 3.0000075$.

By Lemma 4.53, if θ in the branch (a) is $(3, 3.0000075)$-admissible, then $\theta = ...1_22_21_{11}2^*21_{11}....$ Again, by Lemma 4.53, this word must extend to the right as $\theta = ...1_22_21_{11}2^*21_{11}2_21_2....$

Lemma 4.54. $\lambda_0^+(1_22_21_{11}2^*21_{11}2_21_2) < 3.00000473$.

By Lemma 4.54, there is no θ in the branch (a) which is $(3, 3.0000075)$-admissible. Therefore:

Corollary 4.10. *Any* $(3, 3.0000075)$*-admissible word* θ *has the form* $\theta = ...2_21_22^*2_2....$

Second, we study the possible continuations of θ with central combinatorics in the branch (b) from Corollary 4.2. By Lemma 4.38, we need to continue as $\theta = ...2_21_22^*2_3...$ which continue as

- $\theta = ...1_22_21_22^*2_3...$ or $\theta = ...2_31_22^*2_3....$

By Lemmas 4.39 and 4.5, $\theta = ...1_22_21_22^*2_3...$ must to continue as $\theta = ...1_22_21_22^*2_31_22_2...$ and $\theta = ...2_31_22^*2_3...$ must to continue as $\theta = ...2_31_22^*2_4....$

Lemma 4.55. $\lambda_0^-(1_32_21_22^*2_31_22_2) > 3.0001$.

By Remark 4.4, Lemmas 4.38 and 4.55, if θ in the branch (b) is $(3, 3.0001)$-admissible, then

- $\theta = ...2_21_22_21_22^*2_31_22_2...$ or $\theta = ...2_41_22^*2_4....$

Lemma 4.56. *(i)* $\lambda_0^-(2_41_22^*2_41_22_2) > 3.0001$.

(ii) $\lambda_0^+(2_21_22_21_22^*2_31_22_21_2) < 3.000003$.

By Lemmas 4.5, 4.56 and 4.38 and Remark 4.4, if θ in the branch (b) is $(3, 3.0001)$-admissible, then

- $\theta = ...1_22_21_22_21_22^*2_31_22_4...$ or $\theta = ...2_31_22_21_22^*2_31_22_4...$ or
- $\theta = ...2_21_22_41_22^*2_5...$ or $\theta = ...2_51_22^*2_5...$

Lemma 4.57. *(i)* $\lambda_0^+(2_31_22_21_22^*2_31_22_4) < \lambda_0^+(1_22_21_22_21_22^*2_31_22_4) < 3.0000047$*;*

(ii) $\lambda_0^+(2_21_22_41_22^*2_51_22_2) < 3.00000023$*,* $\lambda_0^-(1_22_41_22^*2_6) > 3.00002$*;*

(iii) $\lambda_0^+(2_5 1_2 2^* 2_5 1_2 2_2) < 3$.

By Lemmas 4.5 and 4.57, if θ in the branch (b) is $(3, 3.00002)$-admissible, then

- $\theta = ...2_2 1_2 2_5 1_2 2^* 2_6...$ or $\theta = ...2_6 1_2 2^* 2_6....$

Lemma 4.58. $\lambda_0^+(2_2 1_2 2_5 1_2 2^* 2_6) < 3.0000032$.

By Lemmas 4.5 and 4.58, if θ in the branch (b) is $(3, 3.00002)$-admissible, then $\theta = ...2_6 1_2 2^* 2_6 1_2 2_2...$ or $\theta = ...2_6 1_2 2^* 2_7....$

Lemma 4.59.

$$\lambda_0^+(2_7 1_2 2^* 2_7) < \lambda_0^+(2_2 1_2 2_6 1_2 2^* 2_7) < \lambda_0^+(2_7 1_2 2^* 2_6 1_2 2_2) < 3.000004196.$$

By Lemmas 4.5 and 4.59, if θ in the branch (b) is $(3, 3.00002)$-admissible, then

$$\theta = ...2_2 1_2 2_6 1_2 2^* 2_6 1_2 2_2....$$

Thus, in summary this discussion over the two branches (a) and (b), from Corollary 4.2, give to us that if θ is $(3, 3.0000075)$-admissible, then

$$\theta = ...2_2 1_2 2_6 1_2 2^* 2_6 1_2 2_2....$$

Finally, we follow the script in the Section 4.5, which is condensed in Lemma 4.29, by this lemma, there is a explicit constant $\tilde{\lambda}_3 > m(\gamma_3^1)$, for which one we get the desired local uniqueness result for γ_3^1:

Lemma 4.60 (Local uniqueness of γ_3^1)**.** *A $(3, \tilde{\lambda}_3)$-admissible word θ has the form*

$$\theta = \ldots 2_6 1_2 2_8 1_2 2_6 1_2 2^* 2_6 1_2 2_8 1_2 2_6 1_2 2_2 \ldots$$

In particular, it contains the string $\theta_3^0 = 2_6 1_2 2_8 1_2 2_6 1_2 2^* 2_6 1_2 2_8 1_2 2_6 1$.

4.6.4 Local uniqueness for γ_4^1

Note that:

$$m(\theta(\underline{\omega}_4)) = \lambda_0(\overline{2_9 1_2 2_{10} 1_2 2_8 1_2} 2^* 2_8 1_2 2_{10} 1_2 \overline{2_8 1_2 2_9 1_2 2_{10} 1_2})$$
$$= 3.00000014230846289515772187541301530809498052633\ldots$$

and

$$\begin{aligned} m(\gamma_4^1) &= \lambda_0(\overline{2_9 1_2 2_{10} 1_2 2_8 1_2} 2^* 2_8 1_2 2_{10} 1_2 2_8 1_2 2_9 1_2 2_{10} 1_2 \overline{2}) \\ &= 3.000000142308462895157721875413015308094980526 69\ldots. \end{aligned}$$

By Corollary 4.2, up to transposition, a $(4, 3.009)$-admissible word θ is

(a) $\theta = \ldots 1_4 2^* 2 1_2 \ldots$ or

(b) $\theta = \ldots 2_2 1_2 2^* 2_2 \ldots$.

First, we start studying the possible continuations of θ with central combinatorics in the branch (a). By previous sections, after the Lemma 4.49 v), if θ in the branch (a) is $(4, 3.0001)$-admissible, then

- $\theta = ...1_8 2^* 2 1_8...$ or $\theta = ...1_8 2^* 2 1_6 2_2 1_2...$;
- $\theta = ...1_2 2_2 1_6 2^* 2 1_4 2_2 1_4...$ or $\theta = ...1_2 2_2 1_6 2^* 2 1_4 2_2 1_2 2_2...$;
- $\theta = ...1_2 2_2 1_4 2_2 1_4 2^* 2 1_2 2_2 1_4...$, because $2 1_3 2_2 1_2$ is prohibited;
- $\theta = ...2_3 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2...$ or $\theta = ...1_2 2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2....$

Lemma 4.61. *(i)* $\lambda_0^+(2_2 1_2 2_2 1_6 2^* 2 1_4 2_2 1_4) < 3$

(ii) $\lambda_0^-(1_4 2_2 1_6 2^* 2 1_4 2_2 1_2 2_2) > 3.0000023$

(iii) $\lambda_0^+(1_2 2_3 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2) < 3.00000008$

By Remark 4.4, Lemmas 4.50 and 4.61, if θ in the branch (a) is $(4, 3.0000023)$-admissible, then

- $\theta = ...1_9 2^* 2 1_8...$,
- $\theta = ...1_2 2_2 1_8 2^* 2 1_6 2_2 1_2...$,
- $\theta = ...1_4 2_2 1_6 2^* 2 1_4 2_2 1_4...$,
- $\theta = ...2_2 1_2 2_2 1_6 2^* 2 1_4 2_2 1_2 2_2...$,
- $\theta = ...1_4 2_2 1_4 2_2 1_4 2^* 2 1_2 2_2 1_4...$ or $\theta = ...2_2 1_2 2_2 1_4 2_2 1_4 2^* 2 1_2 2_2 1_4...$,
- $\theta = ...2_4 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2...$,
- $\theta = ...1_4 2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2...$ or $\theta = ...2_2 1_2 2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2...$,

where $3.00000008 < m(\theta) = \lambda_0(\theta) < 3.0000023$.

By Remark 4.4, the possible continuation of these words on the right hand side are

- $\theta = ...1_9 2^* 2 1_9 ...$ or $\theta = ...1_9 2^* 2 1_8 2_2 1_2 ...$,
- $\theta = ...1_2 2_2 1_8 2^* 2 1_6 2_2 1_4 ...$ or $\theta = ...1_2 2_2 1_8 2^* 2 1_6 2_2 1_2 2_2 ...$,
- $\theta = ...1_4 2_2 1_6 2^* 2 1_4 2_2 1_6 ...$ or $\theta = ...1_4 2_2 1_6 2^* 2 1_4 2_2 1_4 2_2 ...$, because v_5 is 2-prohibited,
- $\theta = ...2_2 1_2 2_2 1_6 2^* 2 1_4 2_2 1_2 2_2 1_2 ...$ or $\theta = ...2_2 1_2 2_2 1_6 2^* 2 1_4 2_2 1_2 2_3 ...$,
- $\theta = ...1_4 2_2 1_4 2_2 1_4 2^* 2 1_2 2_2 1_4 2_2 1_2 ...$, because v_3 is 2-prohibited,
- $\theta = ...2_2 1_2 2_2 1_4 2_2 1_4 2^* 2 1_2 2_2 1_4 2_2 1_2 ...$, , because v_3 is 2-prohibited,
- $\theta = ...2_4 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2 1_2 ...$ or $\theta = ...2_4 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_3 ...$,
- $\theta = ...1_4 2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2 1_2 ...$ or $\theta = ...1_4 2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_3 ...$,
- $\theta = ...2_2 1_2 2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2 1_2 ...$ or $\theta = ...2_2 1_2 2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_3 ....$

Lemma 4.62. *(i)* $\lambda_0^+(2_2 1_2 2_2 1_6 2^* 2 1_4 2_2 1_2 2_2 1_2) < 3.000000066$.

(ii) $\lambda_0^+(2_2 1_2 2_2 1_4 2_2 1_4 2^* 2 1_2 2_2 1_4 2_2 1_2) < \lambda_0^+(1_4 2_2 1_4 2_2 1_4 2^* 2 1_2 2_2 1_4 2_2 1_2) < 3.000000019$.

(iii) $\lambda_0^+(2_4 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2 1_2) < 3$.

(iv) $\lambda_0^-(1_4 2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_3) > \lambda_0^-(2_2 1_2 2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_3) > 3.00000051$.

(v) $\lambda_0^+(2_2 1_2 2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2 1_2) < \lambda_0^+(1_4 2_2 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_2 1_2) < 3.000000129$.

By Lemma 4.62 and Remark 4.4, if θ in the branch (a) is $(4, 3.00000051)$-admissible, then

- $\theta = ...1_9 2^* 2 1_9 ...$ or $\theta = ...1_9 2^* 2 1_8 2_2 1_2 ...$,
- $\theta = ...1_2 2_2 1_8 2^* 2 1_6 2_2 1_4 ...$ or $\theta = ...1_2 2_2 1_8 2^* 2 1_6 2_2 1_2 2_2 ...$,
- $\theta = ...1_4 2_2 1_6 2^* 2 1_4 2_2 1_6 ...$ or $\theta = ...1_4 2_2 1_6 2^* 2 1_4 2_2 1_4 2_2 ...$, because v_5 is prohibited,
- $\theta = ...2_2 1_2 2_2 1_6 2^* 2 1_4 2_2 1_2 2_3 ...$,
- $\theta = ...2_4 1_2 2_2 1_4 2^* 2 1_2 2_2 1_2 2_3 ....$

Lemma 4.63. *(i)* $\lambda_0^+(2_21_22_21_82^*21_62_21_4) < \lambda_0^+(1_42_21_82^*21_62_21_4) < 3.000000118$.

(ii) $\lambda_0^-(1_42_21_82^*21_62_21_22_2) > 3.00000035$.

(iii) $\lambda_0^+(2_21_22_21_82^*21_62_21_22_2) < 3.000000025$.

(iv) $\lambda_0^+(2_21_42_21_62^*21_42_21_6) < \lambda_0^+(1_62_21_62^*21_42_21_6) < 3.000000118$.

(v) $\lambda_0^-(1_62_21_62^*21_42_21_42_2) > 3.00000035$.

(vi) $\lambda_0^+(2_21_42_21_62^*21_42_21_42_2) < 3.000000025$.

(vii) $\lambda_0^+(2_31_22_21_62^*21_42_21_22_3) < \lambda_0^+(1_22_21_22_21_62^*21_42_21_22_3) < 3.000000126$.

By Lemmas 4.51 and 4.62, Remark 4.4 and since that v_5 is prohibited, if θ in the branch (a) is $(4, 3.00000035)$-admissible, then

- $\theta = ...1_{10}2^*21_9...$,
- $\theta = ...1_{10}2^*21_82_21_2...$,
- $\theta = ...1_22_41_22_21_42^*21_22_21_22_3...$ or $\theta = ...2_51_22_21_42^*21_22_21_22_3....$

Lemma 4.64. *(i)* $\lambda_0^+(1_22_41_22_21_42^*21_22_21_22_4) < \lambda_0^+(1_22_41_22_21_42^*21_22_21_22_31_2) < 3.000000139$.

(ii) $\lambda_0^+(2_51_22_21_42^*21_22_21_22_4) < \lambda_0^+(2_51_22_21_42^*21_22_21_22_31_2) < 3.00000012$.

By Lemmas 4.51 and 4.64 (and Remark 4.4), if θ in the branch (a) is $(4, 3.00000035)$-admissible, then

- $\theta = ...1_{10}2^*21_{10}...$,
- $\theta = ...1_{10}2^*21_82_21_4...$, or $\theta = ...1_{10}2^*21_82_21_22_2...$,

By Lemma 4.52, if θ in the branch (a) is $(4, 3.00000035)$-admissible, then $\theta = ...1_{11}2^*21_{10}...$, and by Remark 4.4, we must extend as $\theta = ...1_{11}2^*21_{11}...$ or $\theta = ...1_{11}2^*21_{10}2_21_2....$

Lemma 4.65. $\lambda_0^-(1_22_21_{11}2^*21_{11}) > 3.0000044$.

By Lemmas 4.53 and 4.65, if θ in the branch (a) is $(4, 3.00000035)$-admissible, then

- $\theta = ...1_{12}2^*21_{11}...$ or $\theta = ...1_{12}2^*21_{10}2_21_2....$

By Remark 4.4, the possible continuation of these words on the right hand side are

- $\theta = ...1_{12}2^*21_{12}...$ or $\theta = ...1_{12}2^*21_{11}2_21_2...$,
- $\theta = ...1_{12}2^*21_{10}2_21_4...$ or $\theta = ...1_{12}2^*21_{10}2_21_22_2....$

Lemma 4.66. *(i)* $\lambda_0^+(1_22_21_{12}2^*21_{11}) < 3$.

(ii) $\lambda_0^+(1_22_21_{12}2^*21_{10}2_21_2) < 3.0000000171.$

(iii) $\lambda_0^-(1_{13}2^*21_{10}2_21_2) > 3.00000169.$

By Lemma 4.66, if θ in the branch (a) is $(4, 3.00000035)$-admissible, then

- $\theta = ...1_{13}2^*21_{11}2_21_2...$ or $\theta = ...1_{13}2^*21_{12}....$

Lemma 4.67. *(i)* $\lambda_0^+(1_{14}2^*21_{11}2_21_2) < 3.$

(ii) $\lambda_0^+(1_22_21_{13}2^*21_{11}2_21_2) < 3.0000000066.$

(iii) $\lambda_0^-(1_22_21_{13}2^*21_{12}) > 3.00000064.$

By Lemma 4.67, if θ in the branch (a) is $(4, 3.00000035)$-admissible, then $\theta = ...1_{14}2^*21_{12}....$ And by Remark 4.4, this word must extend as $\theta = ...1_{14}2^*21_{13}...$ or $\theta = ...1_{14}2^*21_{12}2_21_2....$

Lemma 4.68. *(i)* $\lambda_0^+(1_22_21_{14}2^*21_{13}) < 3.$

(ii) $\lambda_0^-(1_{15}2^*21_{12}2_21_2) > 3.00000024.$

(iii) $\lambda_0^+(1_22_21_{14}2^*21_{12}2_21_2) < 3.0000000025.$

By Lemma 4.68, if θ in the branch (a) is $(4, 3.00000024)$-admissible, then

$$\theta = ...1_{15}2^*21_{13}....$$

Lemma 4.69. $\lambda_0^+(1_{15}2^*21_{13}2_21_2) < 3.000000037$

By Lemma 4.69,if θ in the branch (a) is $(4, 3.00000024)$-admissible, then

$$\theta = ...1_{15}2^*21_{14}....$$

Lemma 4.70. $\lambda_0^+(1_{16}2^*21_{14}) < 3.000000081$

By Lemma 4.70, if θ in the branch (a) is $(4, 3.00000024)$-admissible, then

$$\theta = ...1_2 2_2 1_{15} 2^* 21_{14}....$$

Lemma 4.71. *(i)* $\lambda_0^+(1_2 2_2 1_{15} 2^* 21_{15}) < 3.000000127$

(ii) $\lambda_0^-(1_2 2_2 1_{15} 2^* 21_{14} 2_2 1_2) > 3.000000161$

By Lemma 4.71, there is no θ in the branch (a) which is $(4, 3.000000161)$-admissible. Therefore,

Corollary 4.11. *Any* $(4, 3.000000161)$*-admissible word* θ *has the form* $\theta = ...2_2 1_2 2^* 2_2....$

Second, we study the possible continuations of θ with central combinatorics in the branch (b) from Corollary 4.2. By previous subsections, after the Lemma 4.55, if θ in the branch (b) is $(4, 3.0001)$-admissible, then

- $\theta = ...2_2 1_2 2_2 1_2 2^* 2_3 1_2 2_2...$ or
- $\theta = ...2_4 1_2 2^* 2_4....$

By Lemma 4.56(i) and Remark 4.4, if θ in the branch (b) is $(4, 3.0001)$-admissible, then

- $\theta = ...2_2 1_2 2_2 1_2 2^* 2_3 1_2 2_2 1_2...$ or $\theta = ...2_2 1_2 2_2 1_2 2^* 2_3 1_2 2_4...$,
- $\theta = ...2_4 1_2 2^* 2_5....$

Lemma 4.72. *(i)* $\lambda_0^+(2_4 1_2 2_2 1_2 2^* 2_3 1_2 2_2 1_2) < 3.$

(ii) $\lambda_0^-(1_2 2_2 1_2 2_2 1_2 2^* 2_3 1_2 2_4) > 3.000003.$

By Lemma 4.72 and Remark 4.4, if θ in the branch (b) is $(4, 3.000003)$-admissible, then

- $\theta = ...1_2 2_2 1_2 2_2 1_2 2^* 2_3 1_2 2_2 1_2...$ or $\theta = ...2_4 1_2 2_2 1_2 2^* 2_3 1_2 2_4...$,
- $\theta = ...2_2 1_2 2_4 1_2 2^* 2_5...$ or $\theta = ...2_5 1_2 2^* 2_5....$

Lemma 4.73. $\lambda_0^+(2_4 1_2 2_2 1_2 2^* 2_3 1_2 2_4 1_2) < 3.000000088.$

By Lemmas 4.73, 4.57*(ii)*-*(iii)* and Remark 4.4, if θ in the branch (b) is $(4, 3.000003)$-admissible, then

- $\theta = ...1_2 2_2 1_2 2_2 1_2 2^* 2_3 1_2 2_2 1_4...$ or $\theta = ...1_2 2_2 1_2 2_2 1_2 2^* 2_3 1_2 2_2 1_2 2_2...$,

- $\theta = ...2_4 1_2 2_2 1_2 2^* 2_3 1_2 2_5...$,
- $\theta = ...2_2 1_2 2_4 1_2 2^* 2_5 1_2 2_2...$ or $\theta = ...2_5 1_2 2^* 2_6....$

Lemma 4.74. *(i)* $\lambda_0^+(2_2 1_2 2_2 1_2 2_2 1_2 2^* 2_3 1_2 2_2 1_4) < 3$.

(ii) $\lambda_0^-(1_4 2_2 1_2 2_2 1_2 2^* 2_3 1_2 2_2 1_2 2_2) > 3.00000049$.

(iii) $\lambda_0^+(2_2 1_2 2_2 1_2 2_2 1_2 2^* 2_3 1_2 2_2 1_2 2_2) < 3.000000034$.

(iv) $\lambda_0^+(2_2 1_2 2_4 1_2 2_2 1_2 2^* 2_3 1_2 2_5) < 3.000000142$.

(v) $\lambda_0^+(2_3 1_2 2_4 1_2 2^* 2_5 1_2 2_2) < 3.00000004$.

By Lemmas 4.74 and 4.39 (and Remark 4.4), if θ in the branch (b) is $(4, 3.00000049)$-admissible, then

- $\theta = ...1_4 2_2 1_2 2_2 1_2 2^* 2_3 1_2 2_2 1_4...$,
- $\theta = ...2_3 1_2 2_4 1_2 2^* 2_5 1_2 2_2...$,
- $\theta = ...2_2 1_2 2_5 1_2 2^* 2_6...$ or $\theta = ...2_6 1_2 2^* 2_6....$

Lemma 4.75. *(i)* $\lambda_0^+(1_4 2_2 1_2 2_2 1_2 2^* 2_3 1_2 2_2 1_5) < 3.000000063$.

(ii) $\lambda_0^+(1_4 2_2 1_2 2_2 1_2 2^* 2_3 1_2 2_2 1_4 2_2 1_2) < 3.000000138$.

(iii) $\lambda_0^+(1_2 2_2 1_2 2_4 1_2 2^* 2_5 1_2 2_4) < 3.000000137$.

(iv) $\lambda_0^+(2_2 1_2 2_5 1_2 2^* 2_7) < \lambda_0^+(2_2 1_2 2_5 1_2 2^* 2_6 1_2 2_2) < 3.00000004$.

(v) $\lambda_0^-(2_6 1_2 2^* 2_6 1_2 2_2) > 3.000003$.

By Lemmas 4.75, 4.38 and 4.39(and Remark 4.4), if θ in the branch (b) is $(4, 3.00000049)$-admissible, then $\theta = ...2_6 1_2 2^* 2_7....$ And by Remark 4.4 this word must extend as $\theta = ...2_2 1_2 2_6 1_2 2^* 2_7...$ or $\theta = ...2_7 1_2 2^* 2_7....$

Lemma 4.76. *(i)* $\lambda_0^+(2_2 1_2 2_6 1_2 2^* 2_7 1_2 2_2) < 3.00000007$.

(ii) $\lambda_0^-(2_2 1_2 2_6 1_2 2^* 2_8) > 3.0000006$.

(iii) $\lambda_0^+(2_7 1_2 2^* 2_7 1_2 2_2) < 3$.

By Lemmas 4.76 and Remark 4.4, if θ in the branch (b) is $(4, 3.00000049)$-admissible, then

$$\theta = ...2_7 1_2 2^* 2_8....$$

Lemma 4.77. $\lambda_0^+(2_2 1_2 2_7 1_2 2^* 2_8) < 3.000000094$.

By Lemmas 4.77 and Remark 4.4, if θ in the branch (b) is $(4, 3.00000049)$-admissible, then

$$\theta = ...2_8 1_2 2^* 2_8....$$

Lemma 4.78. $\lambda_0^+(2_8 1_2 2^* 2_9) < 3.00000005$.

By Lemmas 4.78 and Remark 4.4, if θ in the branch (b) is $(4, 3.00000049)$-admissible, then

$$\theta = ...2_8 1_2 2^* 2_8 1_2 2_2....$$

Lemma 4.79. $\lambda_0^+(2_9 1_2 2^* 2_8 1_2 2_2) < 3.00000013$.

By Lemmas 4.79 and Remark 4.4, if θ in the branch (b) is $(4, 3.00000049)$-admissible, then

$$\theta = ...2_2 1_2 2_8 1_2 2^* 2_8 1_2 2_2....$$

Thus, in summary this discussion over the two branches (a) and (b), from Corollary 4.2, give to us that if θ is $(4, 3.000000161)$-admissible, then

$$\theta = ...2_2 1_2 2_8 1_2 2^* 2_8 1_2 2_2....$$

Finally, we follow the script in the Section 4.5, which is condensed in Lemma 4.29. Let

$$\tilde{\lambda}_4 := \lambda_0^-(2_2 1_2 2_8 1_2 2_8 1_2 2^* 2_8 1_2 2_{10}) > 3.000000142308464 > m(\gamma_4^1)$$

be as in this lemma. Thus, we get the desired local uniqueness result for γ_4^1:

Lemma 4.80 (Local uniqueness of γ_4^1)**.** *A $(4, \tilde{\lambda}_4)$-admissible word θ has the form*

$$\theta = \dots 2_8 1_2 2_{10} 1_2 2_8 1_2 2^* 2_8 1_2 2_{10} 1_2 2_8 1_2 2_2 \dots$$

In particular, it contains the string $\theta_4^0 = 2_8 1_2 2_{10} 1_2 2_8 1_2 2^* 2_8 1_2 2_{10} 1_2 2_8 1$.

4.7 Proof of Theorem 7

The fact that $m_k = m(\gamma_k^1)$ is a decreasing sequence converging to 3 is an immediate consequence of Lemmas 4.3 and 4.4.

Next, let us show that $m_j \in M \setminus L$ for each $j \in \{1, 2, 3, 4\}$. For this sake, assume that $m_j \in L$ for some $1 \le j \le 4$: this would mean that m_j is the limit of the Markov values $m(\theta_n)$ of certain periodic words $\theta_n \in \{1,2\}^{\mathbb{Z}}$. By combining the local uniqueness for γ_j^1, i.e., Lemma 4.37, 4.45, 4.60, 4.80 resp. when $j = 1, 2, 3, 4$ resp., with the replication property in Lemma 4.20, we get that $\theta_n = \theta(\underline{\omega}_j)$ for all n sufficiently large. Therefore, $m_j = m(\gamma_j^1) = \lim\limits_{n\to\infty} m(\theta_n) = m(\theta(\underline{\omega}_j))$, a contradiction.

Finally, the quantities m_j, $j \in \{1,2,3,4\}$, belong to distinct connected components of L because for any $k \in \mathbb{N}$ one has that $m(\theta(\underline{\omega}_k)) \in L$ and Lemma 4.4 ensures that

$$m(\theta(\underline{\omega}_k)) < m_k < m(\theta(\underline{\omega}_{k-1})).$$

4.8 Local almost uniqueness for γ_k^1

We know from Corollary 4.2 that any $(k, 3.009)$-admissible word θ has the form $\theta = \dots 1_4 2^* 2 1_2 \dots$ or $\dots 2_2 1_2 2^* 2_2 \dots$ (up to transposition).

In this section, we will establish the following local almost uniqueness property for γ_k^1 with $k \geq 4$ with respect the branch $\dots 2_2 1_2 2^* 2_2 \dots$:

- there exists an explicit constant $\mu_k > m(\gamma_k^1)$ such that any (k, μ_k)-admissible word $\theta = \dots 2_2 1_2 2^* 2_2 \dots$ has the form
  - $\theta = \dots 2_{2k} 1_2 2_{2k+2} 1_2 2_{2k} 1_2 2^* 2_{2k} 1_2 2_{2k+2} 1_2 2_{2k} 1 \dots$ or
  - $\theta = \dots 1_2 2_{2m} 1_2 2^* 2_{2m+1} 1_2 2_2 \dots$ with $m < k$ or
  - $\theta = \dots 2_2 1_2 2_{2m-1} 1_2 2^* 2_{2m} 1_2 2_2 \dots$ with $1 < m < k-1$.

Remark 4.5. In view of the statements above, the local uniqueness property for γ_k^1 is equivalent to the existence of $\nu_k > m(\gamma_k^1)$ such that no (k, ν_k)-admissible word has the form

- $\dots 1_4 2^* 2 1_2 \dots$ or
- $\dots 1_2 2_{2m} 1_2 2^* 2_{2m+1} 1_2 2_2 \dots$ with $m < k$ or
- $\dots 2_2 1_2 2_{2m-1} 1_2 2^* 2_{2m} 1_2 2_2 \dots$ with $1 < m < k-1$

In the rest of this section we use the next notations. We write $p_j = p(2_j)$ and $q_j = q(2_j)$. Moreover, $\tilde{p}_{j+2} = p(1_2 2_j)$ and $\tilde{q}_{j+2} = q(1_2 2_j)$. Note that:

$$\frac{\tilde{p}_{s+2}}{\tilde{q}_{s+2}} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{p_s}{q_s}}} = \frac{p_s + q_s}{p_s + 2q_s}.$$

Since $\gcd(p_s+q_s, p_s+2q_s)=1$, we have $\tilde{q}_{s+2}=p_s+2q_s$. On the other hand,

$$\frac{p_s}{q_s}=\frac{1}{2+\dfrac{p_{s-1}}{q_{s-1}}}=\frac{q_{s-1}}{2q_{s-1}+p_{s-1}}.$$

Since $\gcd(q_{s-1}, 2q_{s-1}+p_{s-1})=1$, we have $p_s=q_{s-1}$. Therefore

$$\tilde{q}_{s+2}=2q_s+q_{s-1}=q_{s+1}.$$

If we write $\tilde{\beta}_{s+2}=[0;2_s,1_2]$ then $\tilde{\beta}_{s+2}=\dfrac{\tilde{q}_{s+1}}{\tilde{q}_{s+2}}=\dfrac{q_s}{q_{s+1}}=\beta_{s+1}:=[0,2_{s+1}]$.

Lemma 4.81. *If $s,t>2k$, then $\lambda_0^+(2_s1_22^*2_t)<m(\theta(\underline{\omega}_k))$.*

Proof. Since $[2;2_t,...]<[2;2_{2k},1,...]$ and $[0;1_2,2_s,...]<[0;1_2,2_{2k},1,...]$, we have that $\lambda_0^+(2_s1_22^*2_t)<m(\theta(\underline{\omega}_k))$. $\square$

This lemma says that any $(k,3.009)$-admissible word of the form $\theta=...2_21_22^*2_2...$ extends as

$(\mathrm{A})_{a,b}$ $\theta=\dots 1_22_a1_22^*2_b1_22_2\dots$ with $2\le a,b<2k+1$ or

$(\mathrm{B})_a$ $\theta=\dots 1_22_a1_22^*2_{2k+1}\dots$ with $2\le a<2k+1$ or

$(\mathrm{C})_b$ $\theta=\dots 2_{2k+1}1_22^*2_b1_22_2\dots$ with $2\le b<2k+1$.

In the rest of this section we analyse this cases above and ruling out case that can not appear.

Let us start **ruling out $(\mathbf{B})_a$ with a odd**. This situation never occurs:

Lemma 4.82. *If $1\le j<k$, then $\lambda_0^+(1_22_{2j+1}1_22^*2_{2k+1})<m(\theta(\underline{\omega}_k))$.*

Proof. $\lambda_0^+(1_22_{2j+1}1_22^*2_{2k+1})=[2;2_{2k+1},...]+[0;1_2,2_{2j+1},1...]<[2;2_{2k},1,...]+[0;1_2,2_{2k},1...]$. $\square$

We **rule out $(\mathbf{B})_a$ with a even**. This case never occurs. Indeed, by Lemma 4.9, a word $\theta=\dots 1_22_{2j}1_22^*2_{2k+1}\dots$ with $0\le j<k$ is not $(k,\lambda_k^{(1)})$-admissible. Moreover, a word $\theta=\dots 1_22_{2j}1_22^*2_{2k+1}\dots$ with $j=k$ is also not $(k,\lambda_k^{(1)})$-admissible:

Lemma 4.83. *If $0\le m<k$, then*

$$\lambda_0^+(1_22_{2k}1_22^*2_{2k+1})<\lambda_0^+(1_22_{2k}1_22^*2_{2m+1})<m(\theta(\underline{\omega}_k)).$$

Proof. In fact, as before,

$$\lambda_0^+(1_2 2_{2k} 1_2 2^* 2_{2m+1}) = [2; 2_{2m+1}, \overline{2,1}] + [0; 1_2, 2_{2k}, 1_2, \overline{1,2}] := A_k + B_k.$$

Moreover, $m(\theta(\underline{\omega}_k)) > [2; 2_{2k}, \overline{2,1}] + [0; 1_2, 2_{2k}, 1_2, \overline{2,1}] := C_k + D_k$. We have

$$C_k - A_k = \frac{[2; \overline{1,2}] - [2; 2_{2k-2m-2}, \overline{2,1}]}{q_{2m+1}^2([2; 2_{2k-2m-2}, \overline{2,1}] + \beta_{2m+1})([2; \overline{1,2}] + \beta_{2m+1})}.$$

and

$$B_k - D_k = \frac{[1; \overline{1,2}] - [1; 1, \overline{1,2}]}{\tilde{q}_{2k+2}^2([1; \overline{1,2}] + \tilde{\beta}_{2k+2})([1; 1, \overline{1,2}] + \tilde{\beta}_{2k+2})}.$$

Thus,

$$\frac{B_k - D_k}{C_k - A_k} = \frac{q_{2k+1}^2}{q_{2m+1}^2} \cdot X \cdot Y$$

where

$$X = \frac{[2; \overline{1,2}] - [2; 2_{2k-2m-2}, \overline{2,1}]}{[1; \overline{1,2}] - [1; 1, \overline{1,2}]} > 1$$

and

$$Y = \frac{([2; 2_{2k-2m-2}, \overline{2,1}] + \beta_{2m+1})([2; \overline{1,2}] + \beta_{2m+1})}{([1; \overline{1,2}] + \tilde{\beta}_{2k+2})([1; 1, \overline{1,2}] + \tilde{\beta}_{2k+2})}.$$

Since $m \leq k-1$, we have $\dfrac{q_{2k+1}}{q_{2m+1}} \geq 5 + 2\beta_{2m+1}$. Then, by Lemma 4.2

$$\frac{C_k - A_k}{B_k - D_k} = 25 \cdot 1 \cdot 0.226 > 1.$$

Therefore, we have that

$$\lambda_0^+(1_2 2_{2k} 1_2 2^* 2_{2m+1}) = A_k + B_k < C_k + D_k < m(\theta(\underline{\omega}_k)).$$

Finally, since $[2; 2_{2k+2}, \overline{1,2}] < [2; 2_{2m+2}, \overline{1,2}]$ when $m < k$, we have

$$\lambda_0^+(1_2 2_{2k} 1_2 2^* 2_{2k+1}) < \lambda_0^+(1_2 2_{2k} 1_2 2^* 2_{2m+1})$$

□

We **rule out** $\mathbf{(C)}_b$ **with** b **odd**. This situation never occurs:

Lemma 4.84. *If* $0 \leq m < k$, *then* $\lambda_0^+(2_{2k+1} 1_2 2^* 2_{2m+1} 1_2) < m(\theta(\underline{\omega}_k))$.

Proof.

$$\begin{aligned}\lambda_0^+(2_{2k+1} 1_2 2^* 2_{2m+1} 1_2) &= [2; 2_{2m+1}, 1...] + [0; 1_2, 2_{2k+1}, ...] \\ &< [2; 2_{2k}, 1, ...] + [0; 1_2, 2_{2k}, 1...].\end{aligned}$$

□

In the following, we **rule out $(\mathbf{C})_b$ with b even**. This case never occurs. Indeed, by Lemma 4.10, a word $\theta = \dots 2_{2k+1}1_2 2^* 2_{2m} 1_2 2_2 \dots$ with $0 \le m < k$ is not $(k, \lambda_k^{(1)})$-admissible. Moreover, a word $\theta = \dots 2_{2k+1}1_2 2^* 2_{2m} 1_2 2_2 \dots$ with $m = k$ is also not $(k, \lambda_k^{(1)})$-admissible:

Lemma 4.85. *If $1 \le j < k-1$, then*

$$\lambda_0^+(2_{2k+1}1_2 2^* 2_{2k} 1_2) < \lambda_0^+(1_2 2_{2j+1} 1_2 2^* 2_{2k} 1_2) < m(\theta(\underline{\omega}_k)).$$

Proof. $\lambda_0^+(1_2 2_{2j+1} 1_2 2^* 2_{2k} 1_2) = [0; 2_{2k}, 1_2, \overline{1,2}] + [2, 1_2, 2_{2j+1}, 1_2, \overline{2,1}] = [0; 2, \beta] + [2; 1_2, \alpha]$, where $\alpha = [2; 2_{2j}, 1_2, \overline{2,1}]$ and $\beta = [2; 2_{2k-2}, 1_2, \overline{1,2}]$. If $j < k-1$, then $\beta < \alpha$. By (2.2), we get that $\lambda_0^+(1_2 2_{2j+1} 1_2 2^* 2_{2k} 1_2) < 3$. □

In the next, we **rule out $(\mathbf{A})_{a,b}$ with a, b odd**. This situation never occurs:

Lemma 4.86. *If $1 \le j, m < k$, then $\lambda_0^+(1_2 2_{2j+1} 1_2 2^* 2_{2m+1} 1_2) < m(\theta(\omega_k))$.*

Proof.

$$\begin{aligned}\lambda_0^+(1_2 2_{2j+1} 1_2 2^* 2_{2m+1} 1_2) &= [2; 2_{2m+1}, 1...] + [0; 1_2, 2_{2j+1}, 1...] \\ &< [2; 2_{2k}, 1, ...] + [0; 1_2, 2_{2k}, 1...].\end{aligned}$$

□

Now, we **rule out $(\mathbf{A})_{a,b}$ with a, b even, $a < 2k$**. This case never happens: Lemma 4.8 implies that $\theta = \dots 1_2 2_{2j} 1_2 2^* 2_{2m} 1_2 2_2 \dots$ is not $(k, \lambda_k^{(1)})$-admissible when $1 \le j < k$ and $1 \le m \le k$.

We also **rule out $(\mathbf{A})_{2k,b}$ with $b < 2k$ even**. This situation never occurs. Indeed, by Lemma 4.10, a word $\theta = \dots 2_{2k} 1_2 2^* 2_{2m} 1_2 2_2 \dots$ with $1 \le m < k$ is not $(k, \lambda_k^{(2)})$-admissible.

The **case $(\mathbf{A})_{2k,2k}$** corresponds to a word $\theta = \dots 1_2 2_{2k} 1_2 2^* 2_{2k} 1_2 2_2 \dots$.

Now, we analyse **the case $(\mathbf{A})_{a,b}$ with a odd, b even**. This situation can not occur except possibly when $b = a+1 < 2k-2$. Indeed, let us establish this fact by analysing the subcases $1 < a < 2k-1$ and $a = 2k-1$. Remember that 121 is k-prohibited.

Note that Lemma 4.85 implies that a $(k, \lambda_k^{(1)})$-admissible word

$$\theta = \dots 2_2 1_2 2_a 1_2 2^* 2_b 1_2 2_2 \dots$$

with $a < 2k-1$ odd and b even satisfies $b < 2k$.

Lemma 4.87. *We have:*

(i) If $k \geq j+1 > m \geq 1$ *then* $\lambda_0^-(2_{2j+1}1_2 2^* 2_{2m}1_2) > m(\gamma_k^1)$.

(ii) If $1 \leq j+1 < m < k$ *then* $\lambda_0^+(1_2 2_{2j+1}1_2 2^* 2_{2m}1_2) < m(\theta(\underline{\omega}_k))$.

Proof. To prove (i), we write $\lambda_0^-(2_{2j+1}1_2 2^* 2_{2m}1_2) = A_k + B_k$, where $A_k = [2; 2_{2m}, 1_2, \overline{2,1}]$ and$B = [0; 1_2, 2_{2j+1}, \overline{1,2}]$. Moreover,

$$m(\gamma_k^1) < [2; 2_{2k}, 1_2, \overline{1,2}] + [0; 1_2, 2_{2k}, \overline{1,2}] := C_k + D_k.$$

Then,

$$A_k - C_k = \frac{[2; 2_{2k-2m-1}, 1_2, \overline{2,1}] - [1; \overline{2,1}]}{q_{2m}^2([2; 2_{2k-2m-1}, 1_2, \overline{2,1}] + \beta_{2m})([1; \overline{2,1}] + \beta_{2m})}$$

while

$$D_k - B_k = \frac{[2; 2_{2k-2j-2}, \overline{1,2}] - [1; \overline{2,1}]}{\tilde{q}_{2j+3}([2; 2_{2k-2j-2}, \overline{1,2}] + \tilde{\beta}_{2j+3})([1; \overline{2,1}] + \tilde{\beta}_{2j+3})}.$$

Thus,

$$\frac{A_k - C_k}{D_k - B_k} = \frac{q_{2j+2}^2}{q_{2m}^2} \cdot X \cdot Y$$

where

$$X = \frac{[2; 2_{2k-2m-1}, 1_2, \overline{2,1}] - [1; \overline{2,1}]}{[2; 2_{2k-2j-2}, \overline{1,2}] - [1; \overline{2,1}]} > \frac{[2; 2, 1_2, \overline{2,1}] - [1; \overline{2,1}]}{[2; \overline{1,2}] - [1; \overline{2,1}]} > 0.7481.$$

and

$$Y = \frac{([2; 2_{2k-2j-2}, \overline{1,2}] + \tilde{\beta}_{2j+3})([1; \overline{2,1}] + \tilde{\beta}_{2j+3})}{([2; 2_{2k-2m-1}, 1_2, \overline{2,1}] + \beta_{2m})([1; \overline{2,1}] + \beta_{2m})}.$$

By Lemma 4.2, we have

$$\frac{A_k - C_k}{D_k - B_k} > 25 \cdot 0.74 \cdot 0.22 > 1,$$

because $j+1 > m$ implies $q_{2j+2} \geq q_{2m+2} = 5q_{2m} + q_{2m-1} > 5q_{2m}$.

To prove (ii), note that if $j+1 < m$, then writing $\alpha = [2; 2_{2j}, 1_2, \overline{2,1}]$ and $\beta = [2; 2_{2m-2}, 1_2, \overline{1,2}]$, we have that $\lambda_0^+(1_2 2_{2j+1}1_2 2^* 2_{2m}1_2) = [2, 1_2, \alpha] + [0; 2, \beta]$. But, $\beta < \alpha$ and by (2.2), we get $\lambda_0^+(1_2 2_{2j+1}1_2 2^* 2_{2m}1_2) < 3$. $\square$

Let $\mu_k^{(1)} := \min\{\lambda_k^{(1)}, \lambda_0^-(2_{2j+1}1_2 2^* 2_{2m}1_2) : m < j+1 \leq k\}$. By Lemma 4.87, a $(k, \lambda_k^{(1)})$-admissible word

$$\theta = \dots 2_2 1_2 2_a 1_2 2^* 2_b 1_2 2_2 \dots$$

with $a < 2k-1$ odd and b even satisfies $b = a+1$.

The next lemma allows to rule out case $a = 2k-3$:

Lemma 4.88. *If $k > 2$ then $\lambda_0^+(2_2 1_2 2_{2k-3} 1_2 2^* 2_{2k-2} 1_2 2_2) < m(\theta(\underline{\omega}_k))$.*

Proof. In this case

$\lambda_0^+(2_2 1_2 2_{2k-3} 1_2 2^* 2_{2k-2} 1_2 2_2) = [2; 2_{2k-2}, 1_2, 2_2, \overline{1,2}] + [0; 1_2, 2_{2k-3}, 1_2, 2_2, \overline{2,1}] := A_k + B_k$

and

$$m(\theta(\underline{\omega}_k)) > [2; 2_{2k}, 1_2, 2_2, \overline{2,1}] + [0; 1_2, 2_{2k}, 1_2, 2_2, \overline{2,1}] := C_k + D_k.$$

Hence,

$$A_k - C_k = \frac{[2; 2, 1_2, 2_2, \overline{2,1}] - [1; 1, 2_2, \overline{1,2}]}{q_{2k-2}^2([2; 2, 1_2, 2_2, \overline{2,1}] + \beta_{2k-2})([1; 1, 2_2, \overline{1,2}] + \beta_{2k-2})}$$

and

$$D_k - B_k = \frac{[2; 2_2, 1_2, 2_2, \overline{2,1}] - [1; 1, 2_2, \overline{2,1}]}{q_{2k-2}^2([2; 2_2, 1_2, 2_2, \overline{2,1}] + \beta_{2k-2})([1; 1, 2_2, \overline{2,1}] + \beta_{2k-2}}.$$

Thus

$$\frac{D_k - B_k}{A_k - C_k} = X \cdot Y$$

where

$$X = \frac{[2; 2_2, 1_2, 2_2, \overline{2,1}] - [1; 1, 2_2, \overline{2,1}]}{[2; 2, 1_2, 2_2, \overline{2,1}] - [1; 1, 2_2, \overline{1,2}]} > 1.03$$

and

$$Y = \frac{([2; 2, 1_2, 2_2, \overline{2,1}] + \beta_{2k-2})([1; 1, 2_2, \overline{1,2}] + \beta_{2k-2})}{([2; 2_2, 1_2, 2_2, \overline{2,1}] + \beta_{2k-2})([1; 1, 2_2, \overline{2,1}] + \beta_{2k-2})} > 0.986.$$

Then,

$$\frac{D_k - B_k}{A_k - C_k} > 1.03 \cdot 0.986 > 1.01.$$

Therefore, $C_k + D_k > A_k + B_k$. □

So far, we showed that a $(k, \lambda_k^{(1)})$-admissible word

$$\theta = \dots 2_2 1_2 2_a 1_2 2^* 2_b 1_2 2_2 \dots$$

with $a < 2k-1$ odd and b even satisfies $b = a + 1 < 2k - 2$.

Closing our discussion of the case $(\mathrm{A})_{a,b}$ with a odd, b even, let us now show that the case $a = 2k - 1$ can not occur:

Lemma 4.89. *If $j = k - 1$, then $\lambda_0^+(1_2 2_{2j+1} 1_2 2^* 2_{2k} 1_2) < m(\theta(\underline{\omega}_k))$. Moreover, if $m < k$ then $\lambda_0^-(1_2 2_{2k-1} 1_2 2^* 2_{2m} 1_2) > m(\gamma_k^1)$.*

Proof. Note that

$$\lambda_0^+(1_2 2_{2k-1} 1_2 2^* 2_{2k} 1_2) = [2; 2_{2k}, 1_2, \overline{1,2}] + [0; 1_2, 2_{2k-1}, 1_2, \overline{2,1}] := A_k + B_k$$

while

$$m(\theta(\underline{\omega}_k)) > [2; 2_{2k}, 1_2, \overline{1,2}] + [0; 1_2, 2_{2k}, 1_2, \overline{2,1}] = C_k + D_k.$$

Hence,

$$A_k - C_k = \frac{[1; \overline{1,2}] - [1; 1, \overline{1,2}]}{q_{2k}^2([1; \overline{1,2}] + \beta_{2k})([1; 1, \overline{1,2}] + \beta_{2k})}$$

and

$$D_k - B_k = \frac{[2; 1_2, \overline{2,1}] - [1; \overline{1,2}]}{\tilde{q}_{2k+1}^2([2; 1_2, \overline{2,1}] + \tilde{\beta}_{2k+1})([1; \overline{1,2}] + \tilde{\beta}_{2k+1})}.$$

Thus,

$$\frac{D_k - B_k}{A_k - C_k} = X \cdot Y$$

where

$$X = \frac{[2; 1_2, \overline{2,1}] - [1; \overline{1,2}]}{[1; \overline{1,2}] - [1; 1, \overline{1,2}]} > 5.46$$

and

$$Y = \frac{([1; \overline{1,2}] + \beta_{2k})([1; 1, \overline{1,2}] + \beta_{2k})}{([2; 1_2, \overline{2,1}] + \tilde{\beta}_{2k+1})([1; \overline{1,2}] + \tilde{\beta}_{2k+1})}.$$

By Lemma 4.2, we have

$$\frac{D_k - B_k}{A_k - C_k} > 5.46 \cdot 0.22 > 1$$

and this implies that

$$m(\theta(\underline{\omega}_k)) > C_k + D_k > A_k + B_k = \lambda_0^+(1_2 2_{2k-1} 1_2 2^* 2_{2k} 1_2).$$

By Lemma 4.87 we have that if $m < k$ then $\lambda_0^-(1_2 2_{2k-1} 1_2 2^* 2_{2m} 1_2) > m(\gamma_k^1)$, because $k = j + 1 > m$. □

In summary, we showed that

Corollary 4.12. *If $\theta = \ldots 2_2 1_2 2_a 1_2 2^* 2_b 1_2 2_2 \ldots$ is $(k, \mu_k^{(1)})$-admissible word with a odd and b even, then $b = a + 1$ and $3 < b < 2k - 2$.*

In the following, we analyse **the case $(\mathbf{A})_{a,b}$ with a even, b odd**. This case can not occur except possibly when $b = a + 1 < 2k + 1$. Indeed, by Lemma 4.83, a $(k, \lambda_1^{(k)})$-admissible word

$$\theta = \ldots 2_2 1_2 2_a 1_2 2^* 2_b 1_2 2_2 \ldots$$

with a even and b odd satisfies $a < 2k$.

Lemma 4.90. *We have:*

(i) if $1 \le j < m < k$*, then* $\lambda_0^-(1_2 2_{2j} 1_2 2^* 2_{2m+1}) > m(\gamma_k^1)$.

(ii) if $k > j > m \ge 1$ *then* $\lambda_0^+(2_{2j} 1_2 2^* 2_{2m+1} 1_2) < m(\theta(\underline{\omega}_k))$.

Proof. To prove (i), let

$$\lambda_0^-(1_2 2_{2j} 1_2 2^* 2_{2m+1}) = [2; 2_{2m+1}, \overline{1,2}] + [0; 1_2, 2_{2j}, 1_2, \overline{2,1}] := A_k + B_k.$$

We know that $m(\gamma_k^1) < [2; 2_{2k}, \overline{1,2}] + [0; 1_2, 2_{2k}, 1_2, \overline{1,2}] := C_k + D_k$. Hence,

$$C_k - A_k = \frac{[2; 2_{2k-2m-2}, \overline{1,2}] - [1; \overline{2,1}]}{q_{2m+1}^2([2; 2_{2k-2m-2}, \overline{1,2}] + \beta_{2m+1})([1; \overline{2,1}] + \beta_{2m+1})}$$

and

$$B_k - D_k = \frac{[2; 2_{2k-2j-1}, 1_2, \overline{1,2}] - [1; \overline{1,2}]}{\tilde{q}_{2j+2}^2([2; 2_{2k-2j-1}, 1_2, \overline{1,2}] + \tilde{\beta}_{2j+2})([1; \overline{1,2}] + \tilde{\beta}_{2j+2})}.$$

Thus,

$$\frac{B_k - D_k}{C_k - A_k} = \frac{q_{2m+1}^2}{q_{2j+1}^2} \cdot X \cdot Y$$

where

$$X = \frac{[2; 2_{2k-2j-1}, 1_2, \overline{1,2}] - [1; \overline{1,2}]}{[2; 2_{2k-2m-2}, \overline{1,2}] - [1; \overline{2,1}]} > \frac{[2; 2_3, 1_2, \overline{1,2}] - [1; \overline{1,2}]}{[2; \overline{1,2}] - [1; \overline{2,1}]} > 0.75$$

and

$$Y = \frac{([2; 2_{2k-2m-2}, \overline{1,2}] + \beta_{2m+1})([1; \overline{2,1}] + \beta_{2m+1})}{([2; 2_{2k-2j-1}, 1_2, \overline{1,2}] + \tilde{\beta}_{2j+2})([1; \overline{1,2}] + \tilde{\beta}_{2j+2})}.$$

By Lemma 4.2, we have $Y > 0.22$ and it follows that

$$\frac{B_k - D_k}{C_k - A_k} > 25 \cdot 0.75 \cdot 0.22,$$

because $m \ge j+1$ implies $q_{2m+1} \ge q_{2j+3} = 2q_{2j+2} + q_{2j+1} = 5q_{2j+1} + 2q_{2j}$ and then

$$\frac{q_{2m+1}}{q_{2j+1}} \ge 5 + 2\beta_{2j+1} > 5.$$

If $j > m$, put $\alpha = [2; 2_{2j-1}, \overline{1,2}] > \beta = [2; 2_{2m-1}, 1_2, \overline{2,1}]$. Hence,

$$\lambda_0^+(2_{2j} 1_2 2^* 2_{2m+1} 1_2) = [2; 1_2, \alpha] + [0; 2, \beta].$$

By (2.2), we have

$$\lambda_0^+(2_{2j} 1_2 2^* 2_{2m+1} 1_2) = [2; 1_2, \alpha] + [0; 2, \beta] < 3.$$

□

Let $\mu_k^{(2)} := \min\{\lambda_k^{(1)}, \lambda_0^-(1_2 2_{2j} 1_2 2^* 2_{2m+1}) : j < m < k\}$. A direct consequence of Lemma 4.90 is the following result:

Corollary 4.13. *If $\theta = ...2_2 1_2 2_a 1_2 2^* 2_b 1_2 2_2...$ with a even, b odd, and $2 \leq a, b < 2k+1$ is $(k, \mu_k^{(2)})$-admissible, then $3 \leq b = a+1 < 2k+1$.*

In particular, we established the following statement:

Corollary 4.14. *If $\theta = ...2_2 1_2 2^* 2_2...$ is $(k, \mu_k^{(2)})$-admissible for $k \geq 4$, then*

(a) $\theta = ...2_2 1_2 2_{2k} 1_2 2^* 2_{2k} 1_2 2_2...$

(b) $\theta = ...1_2 2_{2m} 1_2 2^* 2_{2m+1} 1_2 2_2...$*, with* $1 \leq m < k$

(c) $\theta = ...2_2 1_2 2_{2m-1} 1_2 2^* 2_{2m} 1_2 2_2...$ *with* $1 < m < k-1$

As it was announced in the beginning of this section, Corollary 4.14 and Lemma 4.29 give us the following local almost uniqueness property for γ_k^1:

Theorem 4.1. *There exists an explicit constant $\mu_k := \min\{\mu_k^{(2)}, \tilde{\lambda}_k\} > m(\gamma_k^1)$ for $k \geq 4$, such that any (k, μ_k)-admissible word has the form*

- $\theta = \ldots 1_4 2^* 2 1_2 \ldots$ *or*
- $\theta = \ldots 2_{2k} 1_2 2_{2k+2} 1_2 2_{2k} 1_2 2^* 2_{2k} 1_2 2_{2k+2} 1_2 2_{2k} 1 \ldots$ *or*
- $\theta = \ldots 1_2 2_{2m} 1_2 2^* 2_{2m+1} 1_2 2_2 \ldots$ *with* $1 \leq m < k$ *or*
- $\theta = \ldots 2_2 1_2 2_{2m-1} 1_2 2^* 2_{2m} 1_2 2_2 \ldots$ *with* $1 < m < k-1$.

CHAPTER 5

$M \backslash L$ is not closed

In this chapter, we show that $1+3/\sqrt{2}$ is a point of the Lagrange spectrum L which is accumulated by a sequence of elements of the complement $M \setminus L$ of the Lagrange spectrum in the Markov spectrum, i.e., $1+3/\sqrt{2} \in L \cap \overline{(M \setminus L)}$. In particular, $M \setminus L$ is not a closed subset of $\mathbb{R}$, so that a question by T. Bousch about the closedness of $M \setminus L$ has a negative answer.

5.1 Main result

For each $k \in \mathbb{N}$, consider the periodic word $\theta(\underline{\eta}_k) = \overline{\underline{\eta}_k} \in \{1,2\}^{\mathbb{Z}}$ associated to the finite string

$$\underline{\eta}_k = (2_{2k-1}, 1, 2_{2k}, 1, 2_{2k+1}, 1)$$

and define $\zeta_k^1 \in \{1,2\}^{\mathbb{Z}}$,

$$\zeta_k^1 := \overline{2_{2k-1}, 1, 2_{2k}, 1, 2_{2k+1}, 1}2^*2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2_{2k-1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 1, \overline{2}.$$

The main theorem of this chapter is:

Theorem 8. *$M \setminus L$ is not a closed subset of $\mathbb{R}$.*

In order to do that, we proved that the Markov values of $\theta(\underline{\eta}_k)$ and ζ_k^1 satisfy:

- $m(\theta(\underline{\eta}_k)) < m(\zeta_k^1) < m(\theta(\underline{\eta}_{k-1}))$ for all $k \geq 3$;

- $\lim_{k\to\infty} m(\theta(\underline{\eta}_k)) = 1 + \frac{3}{\sqrt{2}}$;
- $m(\zeta_k^1) \in M \setminus L$ for all $k \geq 4$.

In particular, $1 + \frac{3}{\sqrt{2}} \in L \cap \overline{(M \setminus L)}$ and $M \setminus L$ is not a closed subset of $\mathbb{R}$.

Remark 5.1. An interesting by-product of our arguments is the fact that $m(\theta(\underline{\eta}_k))$ is an isolated point of L for all $k \geq 4$: cf. Remark 5.2 below.

5.2 The strategy of the proof

The general strategy for the proof of Theorem C is construct a sequence of elements of $M \setminus L$ accumulating at $1+3/\sqrt{2} \in L$. In order to do that, we use the arguments from the Section 4.2 of the previous chapter. More specifically, we prove a local *uniqueness property* and a *replication mechanism* for a given sequence.

In Section 5.3, we prove the fundamental local uniqueness property in Theorem 5.1 saying that a Markov value sufficiently close to $m(\zeta_k^1)$ must come from a sequence of the form $\dots 12_{2k+1}12^*2_{2k-2}1\dots$. The main novelty here in comparison with the previous chapter is the fact that we could establish Theorem 5.1 below ensuring the local uniqueness property near $1+3/\sqrt{2}$.

In Sections 5.4 and 5.5, we prove a replication mechanism saying that any sequence $\theta \in \{1,2\}^{\mathbb{Z}}$ of the form $\theta = \dots 12_{2k+1}12^*2_{2k-2}1\dots$ whose Markov value $m(\theta)$ is sufficiently close to $m(\zeta_k^1)$ must come from a sequence of the form $\overline{2_{2k-1}12_{2k}12_{2k+1}1}2^*2_{2k-2}12_{2k}12_{2k+1}12_{2k-1}12_{2k}12_4\dots$.

Finally, we put together these ingredients to conclude the proof of Theorem C in Section 5.6.

In this chapter, we also deal exclusively with Markov values below $\sqrt{12}$ and, for this reason, we can and do assume that all sequences appearing below belong to $\{1,2\}^{\mathbb{Z}}$.

In order to follow, recall that $\underline{\eta}_k := (2_{2k-1}, 1, 2_{2k}, 1, 2_{2k+1}, 1)$ is a finite string determining a periodic word $\theta(\underline{\eta}_k) = \dots\underline{\eta}_k\underline{\eta}_k^*\underline{\eta}_k\dots$, where the asterisk indicates the 0-th position which occurs at the first 2 in $\underline{\eta}_k$ from the left to the right. Also, recall that ζ_k^1 is the bi-infinite word given by:

$$\zeta_k^1 := \overline{2_{2k-1}, 1, 2_{2k}, 1, 2_{2k+1}, 1}2^*2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2_{2k-1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 1, \overline{2},$$

where $*$ indicates the 0-position. Thus, we have the next lemma relating two important sequences converging to $1+3/\sqrt{2}$.

Lemma 5.1. *For all $k \geq 2$, one has $\lambda_0(\theta(\underline{\eta}_k)) < \lambda_0(\zeta_k^1) < \lambda_0(\theta(\underline{\eta}_{k-1}))$. In particular, $(\lambda_0(\theta(\underline{\eta}_k)))_{k\geq 2}$ and $(\lambda_0(\zeta_k^1))_{k\geq 2}$ are decreasing sequences converging to $[2;\overline{2}] + [0;1,\overline{2}] = 1 + 3/\sqrt{2} = 3.12132034....$*

Now, we recall from the previous chapter the next important definition. Given a finite string $\underline{u} = (a_i)_{i=-m}^n$, let

$$\lambda_i^-(\underline{u}) := \min\{[a_i; a_{i+1}, ..., a_n, \theta_1] + [0; a_{i-1}, ..., a_{-m}, \theta_2] : \theta_1, \theta_2 \in \{1,2\}^{\mathbb{N}}\},$$

and

$$\lambda_i^+(\underline{u}) := \max\{[a_i; a_{i+1}, ..., a_n, \theta_1] + [0; a_{i-1}, ..., a_{-m}, \theta_2]; \theta_1, \theta_2 \in \{1,2\}^{\mathbb{N}}\}.$$

Definition 5.1. We say that $\underline{u} = (a_i)_{i=-m}^n$ is:

- *k-prohibited* whenever $\lambda_i^-(\underline{u}) > \lambda_0(\zeta_k^1)$, for some $-m \leq i \leq n$.
- *k-avoided* if $\lambda_0^+(\underline{u}) < \lambda_0(\theta(\underline{\eta}_k))$.

A word $\theta \in \{1,2\}^{\mathbb{Z}}$ is (k,λ)-*admissible* when $\lambda_0(\theta(\underline{\eta}_k)) < m(\theta) = \lambda_0(\theta) < \lambda$.

These notions are the key to obtain local uniqueness and self-replication properties: in a nutshell, the local uniqueness is based on the construction of a finite set of prohibited and avoided strings and the self-replication relies on a finite set of prohibited strings. In this setting, our main goal is to setup local uniqueness and self-replication properties in such a way that the Markov value of any (k,λ_k)-admissible word belongs to $M \setminus L$ whenever λ_k is close to $m_k = m(\zeta_k^1)$.

5.3 Local uniqueness

We begin this section by the following lemma:

Lemma 5.2. *i)* $\lambda_0^-(12^*1) > 3.154$

ii) $\lambda_0^+(22^*2) < \lambda_0^+(112^*2) < 3.057$

In particular, up transposition, if θ is $(k, 3.154)$-admissible, then $\theta = ...2212^*2....$

On the other hand, if $\theta = ...2_a 12^* 2_b...$ with $a > 2k+1$ and $b > 2k-2$, then $\lambda_0^+(\theta) < \lambda_0(\theta(\underline{\eta}_k))$, because

$$[2; 2_{b-1}, 2, ...] < [2; 2_{2k-2}, 1, ...] \quad \text{and} \quad [0; 1, 2_{a-1}, 2, ...] < [0; 1, 2_{2k+1}, 1, ...].$$

Thus, a $(k, 3.154)$-admissible word θ falls into one of the following categories:

$A_{a,b}$: $\theta = ...12_a12^*2_b1...$ with $a \leq 2k+1$ and $b \leq 2k-2$,

B_a: $\theta = ...12_a12^*2_{2k-1}...$, with $a \leq 2k+1$.

C_b: $\theta = ...2_{2k+2}12^*2_b1...$ with $b \leq 2k-2$.

The main theorem of this section is the following, that describe precisely the *local uniqueness* in this case:

Theorem 5.1. *For each $k \geq 3$, there is a constant $\lambda_k^{(1)} > \lambda_0(\zeta_k^1)$ such that any $(k, \lambda_k^{(1)})$-admissible word θ falls into the category $A_{2k+1,2k-2}$, i.e., has the form*

$$\theta = ...12_{2k+1}12^*2_{2k-2}1...$$

The proof of this result consists into excluding all other categories B_a, C_b and $A_{a,b}$ and it occupies the remainder of this section.

5.3.1 Ruling out B_a with a even

Lemma 5.3. *If $u = 12_{2j}12^*2_{2k-1}, 0 \leq j \leq k$, then $\lambda_0^+(u) < m(\theta(\underline{\eta}_k))$.*

Proof. Note that

$$[2; 2_{2k-2}, 2, ...] < [2; 2_{2k-2}, 1] \quad \text{and} \quad [0; 1, 2_{2j}, 1, ...] < [0; 1, 2_{2j}, 2_{2k-2j}, 2, ...]$$

□

5.3.2 Ruling out B_a with a odd

Lemma 5.4. *Let $u_j = 12_{2j+1}12^*2_{2k-1}$, with $0 \leq j \leq k$. Then,*

$$\lambda_0^+(u_k) < \lambda_0^+(u_{k-1}) < \lambda_0(\theta(\underline{\eta}_k)) \quad \textit{and} \quad \lambda_0(\zeta_k^1) < \lambda_0^-(u_{k-2}) \leq \lambda_0^-(u_j)\, \forall\, j \leq k-2.$$

Proof. Write $\lambda_0^+(u_{k-1}) = [2; 2_{2k-1}, \overline{2,1}] + [0; 1, 2_{2k-1}, 1, \overline{2,1}] := A + B$ and

$$\lambda_0(\theta(\underline{\eta}_k)) > [2; 2_{2k-2}, 1, \overline{1,2}] + [0; 1, 2_{2k+1}, 1, \overline{1,2}] := C + D.$$

Note that $C - A = [0; 2_{2k-2}, 1, \overline{1,2}] - [0; 2_{2k-1}, \overline{2,1}]$, so that

$$C - A = \frac{[2; \overline{2,1}] - [1; \overline{1,2}]}{q^2(2_{2k-2})([2; \overline{2,1}] + \beta(2_{2k-2}))([1; \overline{1,2}] + \beta(2_{2k-2}))}.$$

Moreover, $D - B = [0; 1, 2_{2k+1}, 1, \overline{1,2}] - [0; 1, 2_{2k-1}, 1, \overline{2,1}]$, so that

$$B - D = \frac{[2; 2, 1, \overline{1,2}] - [1; \overline{2,1}]}{q^2(12_{2k-1})([2; 2, 1, \overline{1,2}] + \beta(12_{2k-1}))([1; \overline{2,1}] + \beta(12_{2k-1}))}.$$

This implies that
$$\frac{C-A}{B-D} = \frac{q^2(12_{2k-1})}{q^2(2_{2k-2})} \cdot X \cdot Y,$$
where
$$X = \frac{[2;\overline{2,1}] - [1;\overline{1,2}]}{[2;2,1,\overline{1,2}] - [1;\overline{2,1}]} > 0.62$$
and
$$Y = \frac{([2;2,1,\overline{1,2}] + \beta(12_{2k-1}))([1;\overline{2,1}] + \beta(12_{2k-1}))}{[2;\overline{2,1}] + \beta(2_{2k-2}))([1;\overline{1,2}] + \beta(2_{2k-2}))} > 0.62.$$
Since $q(12_j) = q(2_j1) = q(2_j) + q(2_{j-1})$, we have
$$\frac{C-A}{B-D} = \left(\frac{q(2_{2k-1})}{q(2_{2k-2})} + 1\right)^2 \cdot X \cdot Y = (3 + \beta(2_{2k-2}))^2 \cdot X \cdot Y > 1.$$
In particular, $C - A > B - D$ and
$$\lambda_0^+(u_{k-1}) < \lambda_0(\theta(\underline{\eta}_k)).$$

Next, we write
$$\lambda_0^-(u_{k-2}) = [2;2_{2k-1},\overline{1,2}] + [0;1,2_{2k-3},1,\overline{1,2}] := A' + B'$$
and
$$\lambda_0(\zeta_k^1) < [2;2_{2k-2},1,\overline{2,1}] + [0;1,2_{2k+1},1,\overline{2,1}] := C' + D'.$$
Note that
$$C' - A' = \frac{[2;\overline{1,2}] - [1;\overline{2,1}]}{q^2(2_{2k-2})([2;\overline{1,2}] + \beta(2_{2k-2}))([1;\overline{2,1}] + \beta(2_{2k-2}))}$$
and
$$B' - D' = \frac{[2;2,2,\overline{2,1}] - [1;\overline{1,2}]}{q^2(12_{2k-3})([2;2,2,\overline{2,1}] + \beta(12_{2k-3}))([1;\overline{1,2}] + \beta(12_{2k-3}))}.$$
Therefore,
$$\frac{B'-D'}{C'-A'} = \frac{q^2(2_{2k-2})}{q^2(12_{2k-3})} \cdot X' \cdot Y' = \left(1 + \frac{1}{1+\beta(2_{2k-3})}\right)^2 \cdot X' \cdot Y',$$
where
$$X' = \frac{[2;2,2,\overline{2,1}] - [1;\overline{1,2}]}{[2;\overline{1,2}] - [1;\overline{2,1}]} > 0.4983$$
and
$$Y' = \frac{([2;\overline{1,2}] + \beta(2_{2k-2}))([1;\overline{2,1}] + \beta(2_{2k-2}))}{([2;2,2,\overline{2,1}] + \beta(12_{2k-3}))([1;\overline{1,2}] + \beta(12_{2k-3}))} > 0.91.$$

Since $\left(1+\dfrac{1}{1+\beta(2_{2k-3})}\right)^2 > 2.9$ (because $\beta(2_{2k-3}) \leq [0;2,2,2]$ for $k \geq 3$), we get

$$\frac{B'-D'}{C'-A'} > 2.9 \cdot 0.49 \cdot 0.91 > 1.$$

In particular, $\lambda_0^-(u_{k-2}) > \lambda_0(\zeta_k^1)$. This completes the proof of the lemma. $\square$

5.3.3 Ruling out C_b with b odd

Lemma 5.5. *If $u = 2_{2k+2}12^*2_{2m-1}1$ with $m < k$, then $\lambda_0^+(u) < \lambda_0(\theta(\underline{\eta}_k))$.*

Proof. Note that

$$[2;2_{2m-1},1,...] < [2;2_{2m-1},2_{2k-2m-1},...] \text{ and } [0;1,2_{2k+1},2,...] < [0;1,2_{2k+1},1,...].$$

$\square$

5.3.4 Ruling out C_b with b even

Lemma 5.6. *Let $A = [a_0;\underline{a},\alpha]$, $B = [b_0;\underline{b},\zeta]$, $C = [a_0;\underline{a},\gamma]$ and $D = [b_0;\underline{b},\eta]$ with $\underline{a}$, resp. $\underline{b}$, a finite string of 1 and 2 of length ≥ 2, resp. ≥ 3 and $\alpha,\zeta,\gamma,\eta \in \{1,2\}^{\mathbb{N}}$, $\alpha_1 \neq \gamma_1$, $\zeta_1 \neq \eta_1$. Suppose that $q(\underline{b}) \geq 3q(\underline{a})$. Then,*

$$A+B > C+D \quad \textit{if} \quad A > C \textit{ and } D > B$$

and

$$C+D > A+B \quad \textit{if} \quad C > A \textit{ and } B > D.$$

Moreover, the same statement is also true when the assumptions $\underline{a}$ has length ≥ 2 and/or $\underline{b}$ has length ≥ 3 are replaced by $\underline{a}$ starts with 2 and/or $\underline{b}$ starts with 1.

Proof. If $A > C$ and $D > B$, we have

$$A - C = \frac{|[\gamma]-[\alpha]|}{q^2(\underline{a})([\alpha]+\beta(\underline{a}))([\gamma]+\beta(\underline{a}))}$$

and

$$D - B = \frac{|[\zeta]-[\eta]|}{q^2(\underline{b})([\zeta]+\beta(\underline{b}))([\eta]+\beta(\underline{b}))}.$$

Consider

$$X = \frac{|[\gamma]-[\alpha]|}{|[\zeta]-[\eta]|}$$

and

$$Y = \frac{([\zeta] + \beta(\underline{b}))([\eta] + \beta(\underline{b}))}{([\alpha] + \beta(\underline{a}))([\gamma] + \beta(\underline{a}))}.$$

Therefore,

$$\frac{A - C}{D - B} = \frac{q^2(\underline{b})}{q^2(\underline{a})} \cdot X \cdot Y.$$

Since $\underline{a}$ and $\underline{b}$ are finite strings of 1 and 2 with lengths ≥ 2 and ≥ 3 (resp.) and $\alpha, \zeta, \gamma, \eta \in \{1, 2\}^{\mathbb{N}}$ with $\alpha_1 \neq \gamma_1$, $\zeta_1 \neq \eta_1$, we have that $X \geq \frac{1+[0;\overline{2,1}]-[0;\overline{1,2}]}{1+[0;\overline{1,2}]-[0;\overline{2,1}]}$, $Y \geq \frac{(1+[0;\overline{2,1}]+[0;2,1,2,1])^2}{(2+[0;\overline{1,2}]+[0;1,2,1])^2}$ and $X \cdot Y > \dfrac{1}{9}$. On the other hand, we are assuming that $\dfrac{q^2(\underline{b})}{q^2(\underline{a})} \geq 9$. Thus,

$$\frac{A - C}{D - B} > 1.$$

The other cases are analogous. $\square$

Lemma 5.7. *Let* $u_m = 2_{2k+2}12^*2_{2m}1$. *If* $m \leq k-2$ *and* $k \geq 3$, *then* $\lambda_0^-(u_m) \geq \lambda_0^-(u_{k-2}) > \lambda_0(\zeta_k^1)$.

Proof. Write $\lambda_0^-(u_{k-2}) = [2; 2_{2k-4}, 1, \overline{1,2}] + [0; 1, 2_{2k+2}, \overline{1,2}] := A + B$ and

$$\lambda_0(\zeta_k^1) < [2; 2_{2k-2}, 1, \overline{2,1}] + [0; 1, 2_{2k+1}, \overline{1,2}] := C + D.$$

If we take $\underline{a} = 2_{2k-4}$ and $\underline{b} = 12_{2k+1}$, we have by Euler's rule $q(12_{2k+1}) > 4q(2_{2k-4})$. Since $A > C$ and $D > B$, we deduce from Lemma 5.6 that $A + B > C + D$. $\square$

The next lemma is quite simple, but we use it a lot in the rest of the chapter to estimate certain inequalities.

Lemma 5.8. *Let* α *be a finite string. We have:*

i) $\dfrac{q(\alpha 2)}{3} < q(\alpha) < \dfrac{q(\alpha 2)}{2}$ *and* $\dfrac{4}{3}q(\alpha 2) < q(\alpha 21) < \dfrac{3}{2}(\alpha 2)$;

ii) $\dfrac{7}{17}q(\alpha 2_4) < q(\alpha 2_3) < \dfrac{5}{12}q(\alpha 2_4)$ *and* $\dfrac{24}{17}q(\alpha 2_4) < q(\alpha 2_4 1) < \dfrac{17}{12}q(\alpha 2_4)$.

Lemma 5.9. *Let* $\theta = 2_{2k+2}12^*2_{2k-2}1$ *with* $k \geq 3$. *Then,*

$$\lambda_0^+(\theta 1) < \lambda_0^+(\theta 22) < \lambda_0(\theta(\underline{\eta}_k)).$$

Proof. Note that $\lambda_0^+(\theta 1) < \lambda_0^+(\theta 22)$ because $[0; 2_{2k-2}, 1, 1, ...] < [0; 2_{2k-2}, 1, 2, ...]$. In order to prove that $\lambda_0^+(\theta 22) < \lambda_0(\theta(\underline{\eta}_k))$, let us write

$$\lambda_0^+(\theta 22) = [2; 2_{2k-2}, 1, 2_2, \overline{2, 1}] + [0; 1, 2_{2k+2}, \overline{2, 1}] := C + D$$

and

$$\lambda_0(\theta(\underline{\eta}_k)) > [2; 2_{2k-2}, 1, 2_5, \overline{2, 1}] + [0; 1, 2_{2k+1}, 1, 2_5, \overline{2, 1}] := A + B.$$

Observe that

$$B - D = \frac{[2; \overline{2, 1}] - [1; 2_5, \overline{2, 1}]}{q_{2k+2}^2([2; \overline{2, 1}] + \beta)([1; 2_5, \overline{2, 1}] + \beta)}$$

and

$$C - A = \frac{[2; 2, \overline{2, 1}] - [2; 2_4, \overline{2, 1}]}{\tilde{q}_{2k-1}^2([2; 2_2, \overline{2, 1}] + \tilde{\beta})([2; 2_5, \overline{2, 1}] + \tilde{\beta})},$$

where $q_{2k+2} = q(12_{2k+1})$, $\tilde{q}_{2k-1} = q(2_{2k-2}1)$, $\beta = [0; 2_{2k+1}, 1]$ and $\tilde{\beta} = [0; 1, 2_{2k-2}]$.

Thus,

$$\frac{B - D}{C - A} = X \cdot Y \cdot \frac{\tilde{q}_{2k-1}^2}{q_{2k+2}^2},$$

where

$$X = \frac{[2; \overline{2, 1}] - [1; 2_5, \overline{2, 1}]}{[2; 2, \overline{2, 1}] - [2; 2_4, \overline{2, 1}]} > 112.25,$$

and, since $\beta < [0; \overline{2}]$ and $\tilde{\beta} > [0; 1, 2_3]$,

$$Y = \frac{[2; 2, \overline{2, 1}] + \tilde{\beta})([2; 2_4\overline{2, 1}] + \tilde{\beta})}{([2; \overline{2, 1}] + \beta)([1; 2_5, \overline{2, 1}] + \beta)} > 1.9201.$$

By Lemma 5.8 ii), we have

$$q_{2k+2} = 12q(12_{2k-2}) + 5q(12_{2k-3}) < q(12_{2k-2})\left(12 + 5 \cdot \frac{5}{12}\right).$$

Since $q(12_{2k-2}) = \tilde{q}_{2k-1}$, we get $\dfrac{\tilde{q}_{2k-1}^2}{q_{2k+2}^2} > \left(\dfrac{12}{169}\right)^2$. Therefore,

$$\frac{B - D}{C - A} = 112.25 \cdot 1.92 \cdot \left(\frac{12}{169}\right)^2 > 1.08 > 1.$$

□

5.3.5 Ruling out $A_{a,b}$ with a odd and b even

We want to show that this case essentially never occurs, except when $a = 2k+1$ and $b = 2k-2$. In order to see this fact, we analyse now the following cases:

I) $a < 2k+1$ odd and $b < 2k-2$ even;

II) $a = 2k+1$ and $b < 2k-2$ even;

III) $a < 2k+1$ odd and $b = 2k-2$;

IV) $a = 2k+1$ and $b = 2k-2$.

The next lemma ensures that the case *I*) essentially never occurs:

Lemma 5.10. *If $u = 12_{2j+1}12^*2_{2m}1$ with $m < k-1$, $j < k$, then $\lambda_0^-(u) > \lambda_0(\zeta_k^1)$.*

Proof. Note that

$$[2; 2_{2m}, 1, ...] > [2; 2_{2k-2}, 1, ...] \text{ and } [0; 1, 2_{2j+1}, 1, ...] > [0; 1, 2_{2k+1}, 1, ...],$$

whenever $m < k-1$ and $j < k$. □

The next lemma guarantees that the case *II*) essentially never occurs:

Lemma 5.11. *If $2m \leq 2k-4$, then*

$$\lambda_0^-(2_{2k-2}12^*2_{2m}1) \geq \lambda_0^-(2_{2k-2}12^*2_{2k-4}1) > \lambda_0(\zeta_k^1).$$

Proof. Let $\lambda_0^-(2_{2k-2}12^*2_{2k-4}1) = [2; 2_{2k-4}, 1, \overline{1,2}] + [0; 1, 2_{2k-2}, 1, \overline{1,2}] := A + B$ and $\lambda_0(\zeta_k^1) < [2; 2_{2k-2}, 1, \overline{2,1}] + [0; 1, 2_{2k+1}, 1, \overline{2,1}] := C + D$. In particular, $A > C$ and $D > B$. Take $\underline{a} = 2_{2k-4}$ and $\underline{b} = 12_{2k-1}$. By Euler's rule $q(12_{2k-1}) > 4q(2_{2k-4})$. By Lemma 5.6, we have

$$A + B > C + D.$$

This completes the argument because $[0; 2_{2k-4}, 1, ...] \leq [0; 2_{2m}, 1, ...]$ and, *a fortiori*, $\lambda_0^-(2_{2k-2}12^*2_{2m}1) \geq \lambda_0^-(2_{2k-2}12^*2_{2k-4}1)$ whenever $2m \leq 2k-4$. □

The case *III*) essentially never occurs thanks to Lemma 5.2 i) and the next two lemmas:

Lemma 5.12. *If $2j+1 \leq 2k-3$ and $k \geq 3$, then*

$$\lambda_0^-(12_{2j+1}12^*2_{2k-2}) > \lambda_0^-(12_{2k-3}12^*2_{2k-2}) > \lambda_0(\zeta_k^1).$$

Proof. We begin by noticing that $q(12_{2k-3}) = q(2_{2k-3}) + q(2_{2k-4})$ and $q(2_{2k-2}) = 2q(2_{2k-3}) + q(2_{2k-4})$. Therefore,

$$\frac{q(2_{2k-2})}{q(12_{2k-3})} = 1 + \frac{1}{1+\beta(2_{2k-3})} > 1.6.$$

Next, we write $\lambda_0^-(12_{2k-3}12^*2_{2k-2}) = [2;2_{2k-2},\overline{2,1}] + [0;1,2_{2k-3},1,\overline{1,2}] := A+B$ and $\lambda_0(\zeta_k^1) < [2;2_{2k-2},1,\overline{2,1}] + [0;1,2_{2k+1},1,\overline{2,1} := C+D$. It follows that

$$C - A = \frac{[2;\overline{1,2}] - [1;\overline{2,1}]}{q^2(2_{2k-2})([2;\overline{1,2}] + \beta(2_{2k-2})([1;\overline{2,1}] + \beta(2_{2k-2}))}$$

and

$$B - D = \frac{[2;2,2,2,1,\overline{2,1}] - [1;\overline{1,2}]}{q^2(12_{2k-3})([2;2,2,2,1,\overline{2,1}] + \beta(12_{2k-3}))([1;\overline{1,2}] + \beta(12_{2k-3}))}.$$

Therefore, $\dfrac{B-D}{C-A} = \dfrac{q^2(2_{2k-2})}{q^2(12_{2k-3})} \cdot X \cdot Y$, where

$$X = \frac{[2;2,2,2,1,\overline{2,1}] - [1;\overline{1,2}]}{[2;\overline{1,2}] - [1;\overline{2,1}]} > 0.498$$

and

$$Y = \frac{([2;\overline{1,2}] + \beta(2_{2k-2})([1;\overline{2,1}] + \beta(2_{2k-2}))}{([2;2,2,2,1,\overline{2,1}] + \beta(12_{2k-3}))([1;\overline{1,2}] + \beta(12_{2k-3}))}.$$

Note that

$$Y > \frac{([2;\overline{1,2}] + 0.4)([1;\overline{2,1}] + 0.4)}{([2;2,2,2,1,\overline{2,1}] + 0.5)([1;\overline{1,2}] + 0.5)} > 0.85.$$

Thus

$$\frac{B-D}{C-A} > 2.56 \cdot 0.498 \cdot 0.85 > 1.$$

□

Lemma 5.13. *Let $\theta = 12_{2k-1}12^*2_{2k-2}1$ with $k \geq 3$. We have:*

i) $\lambda_0^-(2\theta 22) > \lambda_0^-(1\theta 22) > \lambda_0(\zeta_k^1)$;

ii) $\lambda_0^+(1\theta 1) < \lambda_0^+(22\theta 1) < \lambda_0(\theta(\underline{\eta}_k))$.

Proof. Let us first establish i). For this sake, we write

$$\lambda_0^-(1\theta 22) = [2; 2_{2k-2}122\overline{12}] + [0; 12_{2k-1}11\overline{21}] := A + B$$

and $\lambda_0(\zeta_k^1) < [2; 2_{2k-2}12_5\overline{12}] + [0; 12_{2k+1}1\overline{21}] := C + D$. Note that

$$C - A = \frac{[2; 2, 2, \overline{1, 2}] - [1; \overline{2, 1}]}{q^2(2_{2k-2}122)([2; 2, 2, \overline{1, 2}] + \beta(2_{2k-2}122)([1; \overline{2, 1}] + \beta(2_{2k-2}122))}$$

and

$$B - D = \frac{[2; 2, 1, \overline{2, 1}] - [1; \overline{1, 2}]}{q^2(12_{2k-1})([2; 2, 1, \overline{2, 1}] + \beta(12_{2k-1}))([1; \overline{1, 2}] + \beta(12_{2k-1}))}.$$

Hence, $\dfrac{B - D}{C - A} = \dfrac{q^2(2_{2k-2}122)}{q^2(12_{2k-1})} \cdot X \cdot Y$, where

$$X = \frac{[2; \overline{2, 1}] - [1; \overline{1, 2}]}{[2; 2, \overline{2, 1}] - [1; \overline{2, 1}]} = 0.6$$

and

$$Y = \frac{([2; 2, 2, \overline{1, 2}] + \beta(2_{2k-2}122)([1; \overline{2, 1}] + \beta(2_{2k-2}122))}{([2; 2, 1, \overline{2, 1}] + \beta(12_{2k-1}))([1; \overline{1, 2}] + \beta(12_{2k-1}))}.$$

Since $[0; 2, 2, 2, 1] < \beta(2_{2k-2}) < [0; 2, 2, 2]$, $\beta(12_{2k-1}) < [0; 2, 2, 2]$ and $\beta(2_{2k-2}122) > [0; 2, 2, 1, 2]$, we have

$$\frac{q(2_{2k-2}122)}{q(12_{2k-1})} = \frac{7 + \beta(2_{2k-2})}{3 + \beta(2_{2k-2})} > 2.1692,$$

$Y > 0.84993$ and, *a fortiori*,

$$\frac{B - D}{C - A} > 2.399 > 1.$$

Let us now prove ii). In this direction, we write

$$\lambda_0^+(22\theta 1) = [2; 2_{2k-2}, 1, 1, \overline{1, 2}] + [0; 1, 2_{2k-1}, 1, 2, 2, \overline{2, 1}] := A' + B'$$

and $\lambda_0(\theta(\underline{\eta}_k)) > [2; 2_{2k-2}, 1, 2, 2, \overline{1, 2}] + [0; 1, 2_{2k+1}, 1, \overline{1, 2}] := C' + D'$. Observe that

$$\frac{C' - A'}{B' - D'} = \frac{q^2(12_{2k-1})}{q^2(2_{2k-2}1)} \cdot X' \cdot Y',$$

where

$$X' = \frac{[2; \overline{2, 1}] - [1; \overline{1, 2}]}{[2; 2, 1, \overline{1, 2}] - [1; 2, 2, \overline{2, 1}]} > 0.65$$

and

$$Y' = \frac{([2;2,1,\overline{1,2}] + \beta(12_{2k-1})([1;2,2,\overline{2,1}] + \beta(12_{2k-1}))}{([2;\overline{2,1}] + \beta(2_{2k-2}1))([1;\overline{1,2}] + \beta(2_{2k-2}1))}.$$

Since $\beta(2_{2k-2}1) < [0;122]$ and $[0;2222] < \beta(12_{2k-1}) < \beta(12_{2k-2}) < [0;222]$, we see that $Y' > 0.67$,

$$\frac{q(12_{2k-1})}{q(2_{2k-2}1)} = 2 + \beta(12_{2k-2}) > 2.41$$

and, *a fortiori*, $(C' - A')/(B' - D') > 2.529 > 1$. $\square$

5.3.6 Ruling out $A_{a,b}$ with a even and b odd

This case essentially never occurs.

Lemma 5.14. *If $u = 12_{2j}12^*2_{2m+1}1$ with $2j \leq 2k+1$ and $2m+1 \leq 2k-2$, then $\lambda_0^+(u) < \lambda_0(\theta(\underline{\eta}_k))$.*

Proof. Note that

$$[2;2_{2m+1},1,...] < [2;2_{2k-2},1,...] \text{and} [0;1,2_{2j},1,...] < [0;1,2_{2k+1},1,...],$$

whenever $2m+1 \leq 2k-2$ and $2j \leq 2k+1$. $\square$

5.3.7 Ruling out $A_{a,b}$ with a, b even

This case essentially never occurs.

Lemma 5.15. *Let $u_{j,m} = 12_{2j}12^*2_{2m}1$ with $j \leq k$ and $m \leq k-1$. We have:*

i) If $k-1 \geq m > j$, then $\lambda_0^+(u_{j,m}) < \lambda_0(\theta(\underline{\eta}_k))$;

ii) If $k-1 > m$ and $j > m$, then $\lambda_0^-(u_{j,m}) > \lambda_0(\zeta_k^1)$;

iii) If $k-1 > m = j$, then $\lambda_0^-(u_{j,m}22) > \lambda_0(\zeta_k^1)$ and

$$\lambda_0^-(1u_{j,m}1) > \lambda_0^-(22u_{j,m}1) > \lambda_0(\zeta_k^1);$$

iv) If $j = m = k-1$, then $\lambda_0^+(u_{k-1,k-1}) < \lambda_0(\theta(\underline{\eta}_k))$;

v) If $m = k-1$ and $j = k$, then $\lambda_0^+(u_{k,k-1}1) < \lambda_0^+(u_{k,k-1}22) < \lambda_0(\theta(\underline{\eta}_k))$.

Proof. Let us prove i). For this sake, write

$$\lambda_0^+(u_{j,m}) = [2; 2_{2m}1\overline{21}] + [0; 12_{2j}1\overline{12}] := B + A$$

and $\lambda_0(\theta(\underline{\eta}_k)) > [2; 2_{2k-2}1\overline{12}] + [0; 12_{2k+1}1\overline{12}] := D + C$. By Lemma 5.6, we get $A + B < C + D$ because $C > A$, $B > D$ and

$$\frac{q(2_{2k-2}1)}{q(12_{2j})} \geq \frac{q(2_{2m})}{q(12_{2j})} \geq \frac{q(2_{2j+2})}{q(12_{2j})} = \frac{5 + 2\beta(2_{2j})}{1 + \beta(2_{2j})} \geq \frac{5 + 2[0; 22]}{1 + [0; 2]} > 3.$$

Let us now establish ii). In this direction, we set $\lambda_0^-(u_{j,m}) = [2; 2_{2m}1\overline{12}] + [0; 12_{2j}1\overline{21}] := A' + B'$ and $\lambda_0(\zeta_k^1) < [2; 2_{2k-2}1\overline{21}] + [0; 12_{2k+1}1\overline{21}] = C' + D'$. Since $A' > C'$, $B' < D'$ and

$$\frac{q(12_{2j})}{q(2_{2m})} = \frac{q(2_{2j}) + q(2_{2j-1})}{q(2_{2m})} \geq \frac{q(2_{2m+2}) + q(2_{2m+1})}{q(2_{2m})} > 3,$$

it follows from Lemma 5.6 that $A' + B' > C' + D'$.

Let us show iii). For this purpose, we denote $\lambda_0^-(u_{j,m}22) = [2; 2_{2m}122\overline{12}] + [0; 12_{2m}1\overline{21}] := A'' + B''$, $\lambda_0^-(22u_{j,m}1) = A''' + B''' := [2; 2_{2m}11\overline{21}] + [0; 12_{2m}122\overline{21}]$ and $\lambda_0(\zeta_k^1) < [2; 2_{2k-2}1\overline{21}] + [0; 12_{2k+1}1\overline{21}] = C' + D'$. Observe that

$$\frac{A'' - C'}{D' - B''} = \frac{q^2(12_{2m})}{q^2(2_{2m})} \cdot X'' \cdot Y'' \quad \text{and} \quad \frac{A''' - C'}{D' - B'''} = \frac{q^2(12_{2m})}{q^2(2_{2m})} \cdot X''' \cdot Y'''$$

where

$$X'' = \frac{[2; 2_{2k-2m-3}1\overline{21}] - [1; 22\overline{12}]}{[2; 2_{2k-2m}1\overline{21}] - [1; \overline{21}]}, \quad X''' = \frac{[2; 2_{2k-2m-3}1\overline{21}] - [1; 1\overline{12}]}{[2; 2_{2k-2m}1\overline{21}] - [1; 22\overline{21}]},$$

$$Y'' = \frac{([2; 2_{2k-2m}1\overline{21}] + \beta(12_{2m}))([1; \overline{21}] + \beta(12_{2m}))}{([2; 2_{2k-2m-3}1\overline{21}] + \beta(2_{2m}))([1; 22\overline{12}] + \beta(2_{2m}))}$$

and

$$Y''' = \frac{([2; 2_{2k-2m}1\overline{21}] + \beta(12_{2m}))([1; 22\overline{21}] + \beta(12_{2m}))}{([2; 2_{2k-2m-3}1\overline{21}] + \beta(2_{2m}))([1; 1\overline{12}] + \beta(2_{2m}))}.$$

Since $\dfrac{q(12_{2m})}{q(2_{2m})} = 1 + \beta(2_{2m}) \geq 1 + [0; 22] = 1.4$,

$$X'' \geq \frac{[2; 21\overline{21}] - [1; 22\overline{12}]}{[2; 2_4 1\overline{21}] - [1; \overline{21}]} > 0.899, \quad X''' \geq \frac{[2; 21\overline{21}] - [1; 1\overline{12}]}{[2; 2_4 1\overline{21}] - [1; 22\overline{21}]} > 0.787,$$

$$Y'' \geq \frac{([2; 2_3 1\overline{21}] + [0; 22])([1; \overline{21}] + [0; 22])}{([2; 2\overline{21}] + [0; 2])([1; 22\overline{12}] + [0; 2])} > 0.884,$$

and

$$Y''' \geq \frac{([2;2_3 1\overline{21}]+[0;22])([1;222\overline{21}]+[0;22])}{([2;2\overline{21}]+[0;2])([1;1\overline{12}]+[0;2])} > 0.839,$$

we see that $\dfrac{A''-C'}{D'-B''} > 1.55$ and $\dfrac{A'''-C'}{D'-B'''} > 1.29$.

Let us now check iv). In order to do this, we put $\lambda_0^+(u_{k-1,k-1}) = [2;2_{2k-2}1\overline{21}] + [0;12_{2k-2}1\overline{12}] := A^* + B^*$ and $\lambda_0(\theta(\underline{\eta}_k)) > C^* + D^* := [2;2_{2k-2}122\overline{21}] + [0;12_{2k+1}12\overline{21}]$. Note that

$$\frac{D^*-B^*}{A^*-C^*} = \frac{q^2(2_{2k-2}12)}{q^2(12_{2k-2})} \cdot X^* \cdot Y^*$$

where

$$X^* = \frac{[2;22122\overline{21}]-[1;\overline{12}]}{[2;\overline{12}]-[1;\overline{21}]}$$

and

$$Y^* = \frac{([2;\overline{12}]+\beta(2_{2k-2}12))([1;\overline{21}]+\beta(2_{2k-2}12))}{([2;22122\overline{21}]+\beta(12_{2k-2}))([1;\overline{12}]+\beta(12_{2k-2}))}.$$

Since $\dfrac{q(2_{2k-2}12)}{q(12_{2k-2})} = 2+\beta(2_{2k-2}1) \geq 2+[0;12] > 2.6$, $X^* > 0.5$ and

$$Y^* \geq \frac{([2;\overline{12}]+[0;2122])([1;\overline{21}]+[0;2122])}{([2;22122\overline{21}]+[0;221])([1;\overline{12}]+[0;221])} > 0.87,$$

we deduce that $(D^*-B^*)/(A^*-C^*) > 2.94 > 1$.

Finally, let us verify v). For this sake, let us define

$$\lambda_0^+(u_{k,k-1}22) = [2;2_{2k-2}1222\overline{21}] + [0;12_{2k}1\overline{12}] := A^{**} + B^{**}$$

and $\lambda_0(\theta(\underline{\eta}_k)) > [2;2_{2k-2}12222\overline{12}] + [0;12_{2k+1}12\overline{21}] := C^{**} + D^{**}$. Observe that

$$\frac{D^{**}-B^{**}}{A^{**}-C^{**}} = \frac{q^2(2_{2k-2}1222)}{q^2(12_{2k})} \cdot X^{**} \cdot Y^{**}$$

where

$$X^{**} = \frac{[2;12\overline{21}]-[1;\overline{12}]}{[2;\overline{12}]-[1;\overline{21}]}$$

and

$$Y^{**} = \frac{([2;\overline{12}]+\beta(2_{2k-2}1222))([1;\overline{21}]+\beta(2_{2k-2}1222))}{([2;12\overline{21}]+\beta(12_{2k}))([1;\overline{12}]+\beta(12_{2k}))}.$$

Since $\dfrac{q(2_{2k-2}1222)}{q(12_{2k})} = \dfrac{17+12\beta(2_{2k-2})}{7+3\beta(2_{2k-2})} \geq \dfrac{17+12[0;2222]}{7+3[0;222]} > 2.6$, $X^{**} > 0.71$ and

$$Y^{**} \geq \frac{([2;\overline{12}]+[0;2221])([1;\overline{21}]+[0;2221])}{([2;12\overline{21}]+[0;221])([1;\overline{12}]+[0;221])} > 0.82,$$

we conclude that $(D^{**}-B^{**})/(A^{**}-C^{**}) > 3.93 > 1$. $\square$

5.3.8 Ruling out $A_{a,b}$ with a, b odd

This case essentially never occurs.

Lemma 5.16. *Let $u = 12_{2j+1}12^*2_{2m+1}1$ with $2m+1 \leq 2k-2$ and $2j+1 \leq 2k+1$. If $m \leq j$, resp. $j < m$, then $\lambda_0^+(u) < \lambda_0(\theta(\underline{\eta}_k))$, resp. $\lambda_0^-(u) > \lambda_0(\zeta_k^1)$.*

Proof. Let us first establish that $\lambda_0^+(u) < \lambda_0(\theta(\underline{\eta}_k))$ whenever $m \leq j$. For this purpose, we write $\lambda_0^+(u) = [2; 2_{2m+1}, 1, \overline{1,2}] + [0; 1, 2_{2j+1}, 1, \overline{2,1}] := A + B$ and $\lambda_0(\theta(\underline{\eta}_k)) > [2; 2_{2k-2}, 1, \overline{1,2}] + [0; 1, 2_{2k+1}, 1, \overline{1,2}] := C + D$. If $j = k$, then we can apply Lemma 5.6 to derive that $C + D > A + B$ because $C > A$, $B > D$ and $q(12_{2k+1}1)/q(2_{2m+1}) > 3$. If $j < k$, then

$$\frac{C-A}{B-D} = \frac{q^2(12_{2j+1})}{q^2(2_{2m+1})} \cdot X \cdot Y$$

where

$$X = \frac{[2; 2_{2k-2m-4}1\overline{12}] - [1; \overline{12}]}{[2; 2_{2k-2j-1}1\overline{12}] - [1; \overline{21}]} \geq \frac{[2; \overline{2}] - [1; \overline{12}]}{[2; \overline{2}] - [1; \overline{21}]} > 0.65$$

and

$$Y = \frac{([2; 2_{2k-2j-1}1\overline{12}] + \beta(12_{2j+1}))([1; \overline{21}] + \beta(12_{2j+1}))}{([2; 2_{2k-2m-4}1\overline{12}] + \beta(2_{2m+1}))([1; \overline{12}] + \beta(2_{2m+1}))}.$$

Since

$$Y \geq \begin{cases} \dfrac{([2; 2, 1, \overline{1,2}] + [0, 2, 2, 2, 1])([1; \overline{2,1}] + [0; 2, 2, 2, 1])}{([2; 1, \overline{1,2}] + [0; 2, 2, 2])([1; \overline{1,2}] + [0; 2, 2, 2])} > 0.773, & \text{if } m > 0 \\ \dfrac{([2; 2, 1, \overline{1,2}] + [0, 2, 1])([1; \overline{2,1}] + [0; 2, 1])}{([2; 2, 2, \overline{1,2}] + [0; 2])([1; \overline{1,2}] + [0; 2])} > 0.7, & \text{if } m = 0 \end{cases}$$

and

$$\frac{q(12_{2j+1})}{q(2_{2m+1})} \geq 1 + \beta(2_{2m+1}) \geq \begin{cases} 1 + [0; \overline{2}], & \text{if } m > 0 \\ 3/2, & \text{if } m = 0 \end{cases}$$

we see that $(C-A)/(D-B) > 1.004$.

Let us now show that $\lambda_0^-(u) > \lambda_0(\zeta_k^1)$ when $j < m$. In order to do this, we write $\lambda_0^-(u) = [2; 2_{2m+1}1\overline{21}] + [0; 12_{2j+1}1\overline{12}] := B' + A'$ and $\lambda_0(\zeta_k^1) < [2; 2_{2k-2}1\overline{21}] + [0; 12_{2k+1}1\overline{21}] := D' + C'$. Since $A' > C'$, $B' < D'$ and

$$\frac{q(2_{2m+1})}{q(12_{2j+1})} \geq \frac{q(2_{2m+1})}{q(12_{2m-1})} = \frac{5 + 2\beta(2_{2m-1})}{1 + \beta(2_{2m-1})} \geq \frac{5 + 2[0; 22]}{1 + [0; 2]} > 3.8,$$

we can use Lemma Lemma 5.6 to conclude that $C' + D' < A' + B'$. $\square$

5.3.9 The Markov values of the two sequences

Let us compute the Markov values of the sequences $\theta(\underline{\eta}_k)$ and ζ_k^1.

Proposition 5.1. *For each $k \geq 3$, the Markov values of $\theta(\underline{\eta}_k)$ and ζ_k^1 are attained at the position* 0.

Proof. The Markov value of $\theta(\underline{\eta}_k)$ can be calculated as follows. Recall that

$$\theta(\underline{\eta}_k) = \ldots 12^*2_{2k-2}12_{2k}12_{2k+1}12_{2k-1}1\ldots$$

By Lemma 5.2, $\lambda_j(\theta(\underline{\eta}_k)) < \lambda_0(\theta(\underline{\eta}_k))$ for all $j \neq 0, 2k-2, 2k, 4k-1, 4k+1, 6k+1$. Moreover, by Lemma 5.15 v), $\lambda_{2k-2}(\theta(\underline{\eta}_k)) < \lambda_0(\theta(\underline{\eta}_k))$. Furthermore, by Lemma 5.4, $\lambda_i(\theta(\underline{\eta}_k)) < \lambda_0(\theta(\underline{\eta}_k))$ for $i = 2k, 4k-1, 6k+1$. Also, by Lemma 5.3, $\lambda_{4k+1}(\theta(\underline{\eta}_k)) < \lambda_0(\theta(\underline{\eta}_k))$. This proves that $m(\theta(\underline{\eta}_k)) = \lambda_0(\theta(\underline{\eta}_k))$.

Similarly, the Markov value of ζ_k^1 can be obtained in the following way. Recall that

$$\zeta_k^1 = \overline{2_{2k-1}12_{2k}12_{2k+1}1}2^*2_{2k-2}12_{2k}12_{2k+1}12_{2k-1}12_{2k}12_{2k-1}11\overline{2}$$

The arguments in the previous paragraph imply that

$$\lambda_j(\zeta_k^1) < \lambda_0(\theta(\underline{\eta}_k)) < \lambda_0(\zeta_k^1),$$

for all $j \notin -(6k+4)\mathbb{N}^* \cup \{6k+3, 8k+1, 10k+4, 12k+2\}$. Also, a direct comparison shows that $\lambda_i(\zeta_k^1) < \lambda_0(\zeta_k^1)$ for each $i \in -(6k+4)\mathbb{N}^* \cup \{6k+3, 8k+1, 10k+4, 12k+2\}$. This completes the proof of the proposition. $\square$

5.3.10 Proof of Theorem 5.1

As it was said right before the statement of Theorem 5.1, a $(k, 3.154)$-admissible word θ necessarily extends in one of the following ways:

$A_{a,b}$: $\theta = ...12_a12^*2_b1...$ with $a \leq 2k+1$ and $b \leq 2k-2$,

B_a: $\theta = ...12_a12^*2_{2k-1}...$, with $a \leq 2k+1$.

C_b: $\theta = ...2_{2k+2}12^*2_b1...$ with $b \leq 2k-2$.

By Lemmas 5.3 and 5.4, there is a constant $\lambda_k^{(1),B} > \lambda_0(\zeta_k^1)$ such that a $(k, \lambda_k^{(1),B})$-admissible word θ can not be of type B_a. Similarly, it follows from Lemmas 5.5, 5.7, 5.9 and Lemma 5.2 that there exists a constant

$\lambda_k^{(1),C} > \lambda_0(\zeta_k^1)$ such that a $(k, \lambda_k^{(1),C})$-admissible word θ can not be of type C_b. Moreover, we have from Lemmas 5.10, 5.11, 5.12, 5.13, 5.14, 5.15, 5.16 (together with Lemma 5.2) that there is a constant $\lambda_k^{(1),A} > \lambda_0(\zeta_k^1)$ such that a $(k, \lambda_k^{(1),A})$-admissible word θ has the form $A_{2k+1,2k-2}$. This shows the validity of Theorem 5.1 for $\lambda_k^{(1)} := \min\{\lambda_k^{(1),A}, \lambda_k^{(1),B}, \lambda_k^{(1),C}\} > \lambda_0(\zeta_k^1)$.

5.4 Going for the replication

In this section, we investigate for every $k \geq 4$ the extensions of a word θ containing the string

$$\alpha_k^1 = 12_{2k+1}12^*2_{2k-2}1.$$

More concretely, the main result of this section is the following statement:

Theorem 5.2. *For each $k \geq 4$, there is an explicit constant $\mu_k^{(1)} > \lambda_0(\zeta_k^1)$ such that any $(k, \mu_k^{(1)})$-admissible word θ containing α_k^1 extends as*

$$\theta = ...2_{2k+1}12_{2k-1}12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_{2k+1}12_{2k-1}....$$

Once again, the proof of this theorem will take this entire section.

5.4.1 Extension from α_k^1 to $2_{2k}\alpha_k^1 2_{2k}$

Lemma 5.17. *Let $\alpha_k^1 = 12_{2k+1}12^*2_{2k-2}1$ with $k \geq 3$. We have:*

i) $\lambda_0^+(\alpha_k^1 1) < \lambda_0^+(\alpha_k^1 2_2 1) < m(\theta(\underline{\eta}_k))$;

ii) $\lambda_0^+(1\alpha_k^1 2222) < m(\theta(\underline{\eta}_k))$;

Proof. Note that $[2; 2_{2k-2}, 1, 1, ...] < [2; 2_{2k-2}, 1, 2, 2, 1, ...]$. In particular, $\lambda_0^+(\alpha_k^1 1) < \lambda_0^+(\alpha_k^1 221)$. To prove that $\lambda_0^+(\alpha_k^1 221) < m(\theta(\underline{\eta}_k))$, we can use Lemma 5.6 with $\underline{a} = 2_{2k-2}12_2$ and $\underline{b} = 12_{2k+1}12$. In fact, observe that

$$\lambda_0^+(\alpha_k^1 221) = [2; 2_{2k-2}, 1, 2_2, 1, \overline{1, 2}] + [0; 1, 2_{2k+1}, 1, \overline{2, 1}] := A + B$$

and

$$m(\theta(\underline{\eta}_k)) > [2; 2_{2k-2}, 1, 2_{2k}, 1, \overline{2, 1}] + [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, \overline{2, 1}] := C + D.$$

We have $C > A$ and $B > D$. Moreover, by Euler's rule,

$$q(\underline{b}) = q(\underline{b}^t) > q(212_3)q(2_{2k-2}1) = 46q(2_{2k-2}1)$$

and

$$q(\underline{a}) = q(\underline{a}^t) = q(2_2)q(2_{2k-2}1) + q(2)q(2_{2k-2}) < 7q(2_{2k-2}1).$$

This implies that

$$q(\underline{b}) > 4q(\underline{a})$$

and, hence, $C + D > A + B$ thanks to Lemma 5.6. This completes the proof of i).

To prove ii) we write $\lambda_0^+(1\alpha_k^1 2_4) := A' + B'$, where $A' = [2; 2_{2k-2}, 1, 2_4, \overline{2, 1}]$ and $B' = [0; 1, 2_{2k+1}, 1, 1, \overline{1, 2}] := A' + B'$, and $m(\theta(\underline{\eta}_k)) > C + D$ as above. By Euler's rule

$$\frac{q(2_{2k-2}12_5)}{q(12_{2k+1}1)} = \frac{99 + 70\beta(2_{2k-2})}{24 + 10\beta(2_{2k-2})} > 4,$$

so that $A' + B' < C + D$ thanks to Lemma 5.6. $\square$

Since the word 12^*1 is k-prohibited, it follows from Lemma 5.17 that α_k^1 must be continued as $\alpha_k^1 2_3$. Furthermore, by Lemma 5.11 and Lemma 5.17 ii), we must continue $\alpha_k^1 2_3$ as $2_2\alpha_k^1 2_4$. In summary, we have:

Corollary 5.1. *Consider the parameter*

$$\lambda_k^{(2)} := \lambda_0^-(2_{2k-2}12^*221).$$

Then, $\lambda_k^{(2)} > m(\zeta_k^1)$ and any $(k, \lambda_k^{(2)})$-admissible word θ containing α_k^1 extends as

$$\theta = ...2_2\alpha_k^1 2_4... = ...2_2 12_{2k+1}12^*2_{2k-2}12_4....$$

In general, the word $\theta = ...2_2\alpha_k^1 2_4$ continues as $\theta = ...2_a\alpha_k^1 2_b...$ with $a \geq 2$ and $b \geq 4$. If $a, b > 2k$, then $\lambda_0^-(\theta) > m(\gamma_k^1)$. Thus, we have four cases:

Ext1A) The string $2_{2k}\alpha_k^1 2_{2k}$.

Ext1B) The string $\gamma_{a,b} = 12_a\alpha_k^1 2_b 1$, with $a, b < 2k$.

Ext1C) The string $\gamma_b = 2_{2k}\alpha_k^1 2_b 1$, with $b < 2k$.

Ext1D) The string $\gamma^a = 12_a\alpha_k^1 2_{2k}$, with $a < 2k$.

5.4.1.1 Ruling out Ext1B)

This case essentially never occurs. In order to see this, let $\gamma_{a,b} = 12_a\alpha_k^1 2_b 1 = 12_a 12_{2k+1}12^* 2_{2k-2}12_b 1$. We have the following subcases:

Ext1B1) b odd and a odd;

Ext1B2) b odd and a even;

Ext1B3) b even and a odd;

Ext1B4) b even and a even.

The next lemma asserts that the case Ext1B1) essentially never occurs:

Lemma 5.18. *If $a = 2j+1 < 2k$ and $b = 2m+1 < 2k$, then*

$$\lambda_0^-(\gamma_{a,b}) \geq \lambda_0^-(\gamma_{2k-1,2k-1}) > m(\zeta_k^1).$$

Proof. For $a = 2j+1 < 2k$ and $b = 2m+1 < 2k$, the inequality $\lambda_0^-(\gamma_{a,b}) \geq \lambda_0^-(\gamma_{2k-1,2k-1})$ is straightforward. Hence, it remains to prove that $\lambda_0^-(\gamma_{2k-1,2k-1}) > m(\zeta_k^1)$. For this sake, note that:

$$A := [2; 2_{2k-2}, 1, 2_{2k-1}, 1, \overline{1,2}] > [2; 2_{2k-2}, 1, 2_{2k}, 1, 2, \overline{2,1}] =: C \text{ and}$$

$$B := [0; 1, 2_{2k+1}, 1, 2_{2k-1}, 1, \overline{1,2}] > [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2, \overline{2,1}] =: D.$$

Therefore, $\lambda_0^-(\gamma_{2k-1,2k-1}) := A + B > C + D > m(\zeta_k^1)$. □

The case Ext1B2) essentially never occurs. Indeed, first note that in this setting ($b = 2m+1 < 2k$ is odd) one actually has $b = 2k-1$ by Lemma 5.11. Also, note that $\lambda_0^-(\gamma_{2j,2k-1})$ and $\lambda_0^+(\gamma_{2j,2k-1})$ are increasing functions of j. In particular, $\lambda_0^-(\gamma_{2k-2,2k-1}) > \lambda_0^-(\gamma_{2k-4,2k-1})$ and $\lambda_0^+(\gamma_{2j,2k-1}) \leq \lambda_0^+(\gamma_{2k-6,2k-1})$ for all $2j \leq 2k-6$. Thus, we can rule out Ext1B2) using the following lemma:

Lemma 5.19. *We have:*

i) $\lambda_0^-(\gamma_{2k-4,2k-1}) > m(\zeta_k^1)$;

ii) $\lambda_0^+(\gamma_{2k-6,2k-1}) < m(\theta(\underline{\eta}_k))$.

Proof. To prove i) we write

$$\lambda_0^-(\gamma_{2k-4,2k-1}) = [2; 2_{2k-2}, 1, 2_{2k-1}, 1, \overline{1,2}] + [0; 1, 2_{2k+1}, 1, 2_{2k-4}, 1, \overline{2,1}] := A+B.$$

and

$$m(\zeta_k^1) < [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_2, \overline{1,2}] + [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_2, \overline{1,2}] := C + D.$$

Therefore,

$$A - C = \frac{[2; 1, 2_2, \overline{1,2}] - [1; \overline{1,2}]}{q^2(2_{2k-2}12_{2k-1})([2; 1, 2_2, \overline{1,2}] + \beta)([1; \overline{1,2}] + \beta)},$$

where $\beta = \beta(2_{2k-2}, 1, 2_{2k-1}) = [0; 2_{2k-1}, 1, 2_{2k-2}] < [0; \overline{2}]$. Moreover, we have

$$D - B = \frac{[2; 2_3, 1, 2_2, \overline{1,2}] - [1; \overline{2,1}]}{q^2(12_{2k+1}12_{2k-4})([2; 2_3, 1, 2_2, \overline{1,2}] + \tilde{\beta})([1; \overline{2,1}] + \tilde{\beta})},$$

where $\tilde{\beta} = \beta(12_{2k+1}12_{2k-4}) = [0; 2_{2k-4}, 1, 2_{2k+1}, 1] > [0; \overline{2}]$. In particular,

$$\frac{A - C}{D - B} = \frac{q^2(12_{2k+1}12_{2k-4})}{q^2(2_{2k-2}12_{2k-1})} \cdot X \cdot Y,$$

where

$$X = \frac{[2; 1, 2_2, \overline{1,2}] - [1; \overline{1,2}]}{[2; 2_3, 1, 2_2, \overline{1,2}] - [1; \overline{2,1}]} > 0.927$$

and

$$Y = \frac{([2; 2_3 12_2 \overline{12}] + \tilde{\beta})([1; \overline{21}] + \tilde{\beta})}{([2; 12_2\overline{12}] + \beta)([1; \overline{12}] + \beta)} > \frac{([2; 2_3 12_2 \overline{12}] + [0; \overline{2}])([1; \overline{21}] + [0; \overline{2}])}{([2; 12_2\overline{12}] + [0; \overline{2}])([1; \overline{12}] + [0; \overline{2}])} > 0.752.$$

Also, by Euler's rule,

$$\begin{aligned}
\frac{q(12_{2k+1}12_{2k-4})}{q(2_{2k-2}12_{2k-1})} &= \frac{7q(2_{2k-4}12_{2k-1}) + 3q(2_{2k-4}12_{2k-2})}{2q(2_{2k-1}12_{2k-3}) + q(2_{2k-1}12_{2k-4})} > \frac{7\beta(2_{2k-1}12_{2k-3})}{2 + \beta(2_{2k-1}12_{2k-3})} \\
&> \frac{7}{\frac{2}{[0;2_4]} + 1} = 1.2.
\end{aligned}$$

Thus,

$$\frac{A - C}{D - B} > (1.2)^2 \cdot 0.927 \cdot 0.752 > 1.003.$$

The proof of ii) follows from Lemma 5.6 because

$$\frac{q(2_{2k-2}12_{2k-1})}{q(12_{2k+1}12_{2k-6})} = \frac{29q(2_{2k-1}12_{2k-6}) + 12q(2_{2k-1}12_{2k-7})}{3q(2_{2k-6}12_{2k}) + q(2_{2k-6}12_{2k-1})} > \frac{29}{\frac{3}{\beta(2_{2k-6}12_{2k})} + 1} > 3.5$$

thanks to Euler's rule. □

The case Ext1B3) essentially never occurs. In fact, note that in this context ($a = 2j + 1 < 2k$ is odd), we can apply Lemma 5.11 to assume that $a = 2k - 1$. The following lemma asserts that this possibility doesn't occur:

Lemma 5.20. *If $b = 2m \leq 2k-2$, then $\lambda_0^+(\gamma_{2k-1,2m}) \leq \lambda_0^+(\gamma_{2k-1,2k-2}) < m(\theta(\underline{\eta}_k))$.*

Proof. First, we have the inequality $\lambda_0^+(\gamma_{2k-1,2m}) \leq \lambda_0^+(\gamma_{2k-1,2k-2})$ for every $b = 2m \leq 2k-2$.

Thus, it remains prove that $\lambda_0^+(\gamma_{2k-1,2k-2}) < m(\theta(\underline{\eta}_k))$. This estimate follows from Lemma 5.6 because

$$\lambda_0^+(\gamma_{2k-1,2k-2}) = [2; 2_{2k-2}, 1, 2_{2k-2}, 1, \overline{1,2}] + [0; 1, 2_{2k+1}, 1, 2_{2k-1}, 1, \overline{2,1}] := C + D,$$

$$m(\theta(\underline{\eta}_k)) > [2; 2_{2k-2}, 1, 2_{2k}, 1, 2, \overline{1,2}] + [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2, \overline{1,2}] := A + B,$$

and

$$\begin{aligned}\frac{q(12_{2k+1}12_{2k-1})}{q(2_{2k-2}12_{2k-2})} &= \frac{2q(12_{2k+1}12_{2k-2}) + q(12_{2k+1}12_{2k-3})}{q(2_{2k-2}12_{2k-2})} \\ &\geq \left(2 + \frac{1}{3}\right)\frac{q(12_{2k+1}12_{2k-2})}{q(2_{2k-2}12_{2k-2})} \geq \frac{7}{3}q(2_3 1) > 3\end{aligned}$$

thanks to Euler's rule. □

Finally, a direct comparison of continued fractions reveals that the case Ext1B4) essentially never occurs.

Lemma 5.21. *If $a = 2j < 2k$ and $b = 2m < 2k$, then*

$$\lambda_0^+(\gamma_{2j,2m}) \leq \lambda_0^+(\gamma_{2k-2,2k-2}) < m(\theta(\underline{\eta}_k)).$$

Proof. Note that

$$[2; 2_{2k-2}, 1, 2_{2m}, 1, ...] \leq [2; 2_{2k-2}, 1, 2_{2k-2}, 1, ...] < [2; 2_{2k-2}, 1, 2_{2k}, 1...]$$

and

$$[0; 1, 2_{2k+1}, 1, 2_{2j}, 1, ..] \leq [2; 2_{2k-2}, 1, 2_{2k-2}, 1, ...] < [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, ...]$$

whenever $j, m < k$. □

5.4.1.2 Ruling out Ext1C)

We begin by excluding Ext1C) with b odd:

Lemma 5.22. *If $0 < m \leq k-1$ and $u_m = 2_{2k}\alpha_k^1 2_{2m+1}1$ then*

$$\lambda_0^-(u_m) \geq \lambda_0^-(u_{k-1}) > m(\zeta_k^1).$$

Proof. We write

$$\lambda_0^-(u_{k-1}) = [2; 2_{2k-2}, 1, 2_{2k-1}, 1, \overline{1, 2}] + [0; 1, 2_{2k+1}, 1, 2_{2k}, \overline{1, 2}] := A + B$$

and

$$m(\zeta_k^1) < [2; 2_{2k-2}, 1, 2_{2k}, 1, \overline{1, 2}] + [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, \overline{1, 2}] := C + D.$$

Then $A > C$ and $D > B$. By Lemma 5.6, it follows that

$$A + B > C + D$$

since $q(12_{2k+1}12_{2k}1) > 4 \cdot q(2_{2k-2}12_{2k-1})$. □

Let us now exclude Ext1C) with b even:

Lemma 5.23. *If $m < k$ and $u_m = 2_{2k}12_{2k+1}12^*2_{2k-2}12_{2m}1$ then*

$$\lambda_0^+(u_m) \le \lambda_0^+(u_{k-1}) < m(\theta(\underline{\eta}_k)).$$

Proof. The proof is similar to the proof of Lemma 5.22. Just note that now $C > A$ and $B > D$ and, by Lemma 5.6, $A + B < C + D$. □

5.4.1.3 Ruling out Ext1D)

Let us first show that Ext1D) with a even essentially never occurs. For this sake, we use the Lemma 5.2 i) and the next two lemmas:

Lemma 5.24. *Let $\gamma^a = 12_a\alpha_k^1 2_{2k} = 12_a 12_{2k+1}12^*2_{2k-2}12_{2k}$. If $a = 2j \le 2k-4$, then $\lambda_0^+(\gamma^{2j}) \le \lambda_0^+(\gamma^{2k-4}) < m(\theta(\underline{\eta}_k))$.*

Proof. First, we have that $\lambda_0^+(\gamma^{2j}) \le \lambda_0^+(\gamma^{2k-4})$, for every $a = 2j \le 2k-4$.

Let $\lambda_0^+(\gamma^{2k-4}) = [2; 2_{2k-2}, 1, 2_{2k}, \overline{2, 1}] + [0; 1, 2_{2k+1}, 1, 2_{2k-4}, 1, \overline{1, 2}] := C + D$ and $m(\theta(\underline{\eta}_k)) > A + B$, where $A = [2; 2_{2k-2}, 1, 2_{2k}, 1, 2, 2, \overline{2, 1}]$ and $B = [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2, 2, \overline{2, 1}]$. Our task is reduced to prove that $B - D > C - A$. In order to establish this inequality, we observe that

$$B - D = \frac{[2; 2_3, 1, 2, 2, \overline{2, 1}] - [1; \overline{1, 2}]}{q_{4k-1}^2([2; 2_3, 1, 2, 2, \overline{2, 1}] + \beta)([1; \overline{1, 2}] + \beta)},$$

and

$$C - A = \frac{[2; \overline{1, 2}] - [1; 2, 2, \overline{2, 1}]}{\tilde{q}_{4k-1}^2([2; \overline{1, 2}] + \tilde{\beta})([1; 2, 2, \overline{2, 1}] + \tilde{\beta})}$$

where $q_{4k-1} = q(12_{2k+1}12_{2k-4})$, $\tilde{q}_{4k-1} = q(2_{2k-2}12_{2k})$, $\beta = [0; 2_{2k-4}, 1, 2_{2k+1}, 1]$ and $\tilde{\beta} = [0; 2_{2k}, 1, 2_{2k-2}]$. Thus,

$$\frac{B-D}{C-A} = \frac{[2; 2_3, 1, 2, 2, \overline{2,1}] - [1; \overline{1,2}]}{[2; \overline{1,2}] - [1; 2, 2, \overline{2,1}]} \cdot Y \cdot \frac{\tilde{q}_{4k-1}^2}{q_{4k-1}^2} > 0.51 \cdot Y \cdot \frac{\tilde{q}_{4k-1}^2}{q_{4k-1}^2},$$

where

$$Y = \frac{([2; \overline{1,2}] + \tilde{\beta})([1; 2, 2, \overline{2,1}] + \tilde{\beta})}{([2; 2_3, 1, 2, 2, \overline{2,1}] + \beta)([1; \overline{1,2}] + \beta)}.$$

Note that

$$Y > \frac{([2; \overline{1,2}] + [0, \overline{2}])([1; 2, 2, \overline{2,1}] + [0, \overline{2}])}{([2; 2_3, 1, 2, 2, \overline{2,1}] + [0, 2_4, 1])([1; \overline{1,2}] + [0, 2_4, 1])} > 0.94.$$

Let $\Gamma = 2_{2k}12_{2k-4}$. By Euler's rule and Lemma 5.8 i), we have:

$$q_{4k-1} = q(2_{2k-4}12_{2k+1}) + q(\Gamma^t) = 3q(\Gamma^t) + q(2_{2k-4}12_{2k-1}) < (3 + 1/2)q(\Gamma^t)$$

and

$$\tilde{q}_{4k-1} = 2q(2_{2k}12_{2k-3}) + q(2_{2k}12_{2k-4}) = 5q(\Gamma) + 2q(2_{2k}12_{2k-5}) > q(\Gamma)(5 + 2/3).$$

Thus,

$$\frac{\tilde{q}_{4k-1}}{q_{4k-1}} > \frac{34}{21}.$$

Therefore, $\dfrac{B-D}{C-A} > 0.51 \cdot 0.94 \cdot \left(\dfrac{34}{21}\right)^2 > 1.25 > 1.$ $\square$

Lemma 5.25. *Let* $\gamma^{2k-2} = 12_{2k-2}\alpha_k^1 2_{2k} = 12_{2k-2}12_{2k+1}12^*2_{2k-2}12_{2k}$. *We have:*

i) $\lambda_0^-(\gamma^{2k-2}2) > \lambda_0^-(\gamma^{2k-2}11) > m(\zeta_k^1)$;

ii) $\lambda_0^+(\gamma^{2k-2}122) < m(\theta(\underline{\eta}_k))$.

Proof. In order to prove i) we first note that $[2; 2_{2k-2}, 1, 2_{2k}, 2, ...] > [2; 2_{2k-2}, 1, 2_{2k}, 1...]$ and, hence, $\lambda_0^-(\gamma^{2k-2}2) > \lambda_0^-(\gamma^{2k-2}11)$. Next, we write

$$\lambda_0^-(\gamma^{2k-2}11) = [2; 2_{2k-2}, 1, 2_{2k}, 1_2, \overline{1,2}] + [0; 1, 2_{2k+1}, 1, 2_{2k-2}, 1, \overline{2,1}] := A + B$$

and

$$m(\zeta_k^1) < [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_2, \overline{1,2}] + [0; 1, 2_{2k+1}, 1, 2_{2k-1}, \overline{1,2}] := C + D.$$

Note that $A > C$, $D > B$ and

$$\frac{A-C}{D-B} = \frac{[2;2,\overline{1,2}]-[1;\overline{1,2}]}{[2;\overline{1,2}]-[1;\overline{1,2}]}\cdot Y\cdot\frac{q^2(12_{2k+1}12_{2k-2})}{q(2_{2k-2}12_{2k}1)} > 0.63\cdot Y\cdot\frac{q^2(12_{2k+1}12_{2k-2})}{q(2_{2k-2}12_{2k}1)}$$

where

$$\begin{aligned} Y &= \frac{([2;\overline{1,2}]+\beta(12_{2k+1}12_{2k-2}))([1;\overline{1,2}]+\beta(12_{2k+1}12_{2k-2}))}{([2;2,\overline{1,2}]+\beta(2_{2k-2}12_{2k}1))([1;\overline{1,2}]+\beta(2_{2k-2}12_{2k}1))} \\ &> \frac{([2;\overline{12}]+[0;\overline{2}])([1;\overline{1,2}]+[0;\overline{2}])}{([2;2,\overline{1,2}]+[0;1,\overline{2}])([1;\overline{1,2}]+[0;1,\overline{2}])} > 0.9 \end{aligned}$$

Since

$$q(12_{2k+1}12_{2k-2}) = 3q(2_{2k-2}12_{2k}) + q(2_{2k-2}12_{2k-1})$$

and

$$q(2_{2k-2}12_{2k}1) = q(2_{2k-2}12_{2k}) + q(2_{2k-2}12_{2k-1}),$$

we also have that

$$\frac{q(12_{2k+1}12_{2k-2})}{q(2_{2k-2}12_{2k}1)} = \frac{3+\beta(2_{2k-2}12_{2k})}{1+\beta(2_{2k-2}12_{2k})} > 2.41.$$

Therefore, $(A-C)/(D-B) > 1$.

To prove ii), it suffices to apply Lemma 5.6. In fact, we can write

$$\lambda_0^+(\gamma^{2k-2}122) = [2;2_{2k-2},1,2_{2k},1,2_2,\overline{1,2}]+[0;1,2_{2k+1},1,2_{2k-2},1,\overline{1,2}] := D'+C'$$

and

$$m(\theta(\underline{\eta}_k)) > [2;2_{2k-2},1,2_{2k},1,2_4,\overline{2,1}] + [0;1,2_{2k+1},1,2_{2k}1,\overline{2,1}] := B'+A',$$

with $B' > D'$, $C' > A'$ and $q(2_{2k-2}12_{2k}12_2) > 4\cdot q(12_{2k+1}12_{2k-2})$. $\square$

Now, let us prove that Ext1D) with a odd essentially never occurs. In this regime ($a = 2j+1 < 2k$ is odd), Lemma 5.11 says that we can assume that $a = 2k-1$. So, we can exclude Ext1D) with a odd thanks to Lemma 5.2 i) and the next lemma:

Lemma 5.26. *Let* $\gamma^{2k-1} = 12_{2k-1}\alpha_k^1 2_{2k} = 12_{2k-1}12_{2k+1}12^*2_{2k-2}12_{2k}$. *Then,* $\lambda_0^-(\gamma^{2k-1}2) > \lambda_0^-(\gamma^{2k-1}11) > \lambda_0^-(\gamma^{2k-1}122) > m(\zeta_k^1)$.

Proof. First, by parity we check that $\lambda_0^-(\gamma^{2k-1}2) > \lambda_0^-(\gamma^{2k-1}11) > \lambda_0^-(\gamma^{2k-1}122)$. It remains to prove that $\lambda_0^-(\gamma^{2k-1}122) > m(\zeta_k^1)$. We write

$$\lambda_0^-(\gamma^{2k-1}122) := C+D := [2;2_{2k-2},1,2_{2k},1,2_2,\overline{2,1}]+[0;1,2_{2k+1},1,2_{2k-1},1,\overline{1,2}]$$

and

$$m(\zeta_k^1) < A + B := [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_4, \overline{1, 2}] + [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_2, \overline{1, 2}],$$

so that our task is reduced to prove that $D - B > A - C$.

Observe that

$$D - B = \frac{[2; 1, 2, 2, \overline{1, 2}] - [1; \overline{1, 2}]}{q_{4k+2}^2([2; 1, 2, 2, \overline{1, 2}] + \beta)([1; \overline{1, 2}] + \beta)},$$

and

$$A - C = \frac{[1; 2_4\overline{1, 2}] - [1; 2, 2, \overline{2, 1}]}{\tilde{q}_{4k-1}^2([1; 2_4, \overline{1, 2}] + \tilde{\beta})([1; 2, 2, \overline{2, 1}] + \tilde{\beta})}$$

where $q_{4k+2} = q(12_{2k+1}12_{2k-1})$, $\tilde{q}_{4k-1} = q(2_{2k-2}12_{2k})$, $\beta = [0; 2_{2k-1}, 1, 2_{2k+1}, 1]$ and $\tilde{\beta} = [0; 2_{2k}, 1, 2_{2k-2}]$. Thus,

$$\frac{D - B}{A - C} = \frac{[2; 1, 2, 2, \overline{1, 2}] - [1; \overline{1, 2}]}{[1; 2_4, \overline{1, 2}] - [1; 2, 2, \overline{2, 1}]} \cdot Y \cdot \frac{\tilde{q}_{4k-1}^2}{q_{4k+2}^2} > 574.47 \cdot Y \cdot \frac{\tilde{q}_{4k-1}^2}{q_{4k+2}^2},$$

where

$$Y = \frac{([1; 2_4, \overline{1, 2}] + \tilde{\beta})([1; 2, 2, \overline{2, 1}] + \tilde{\beta})}{([2; 1, 2, 2, \overline{1, 2}] + \beta)([1; \overline{1, 2}] + \beta)}.$$

Note that

$$Y > \frac{([1; 2_4, \overline{1, 2}] + [0; \bar{2}])([1; 2, 2, \overline{2, 1}] + [0; \bar{2}])}{([2; 1, 2, 2, \overline{1, 2}] + [0; \bar{2}])([1; \overline{1, 2}] + [0; \bar{2}])} > 0.5.$$

Let $\Gamma = 2_{2k-2}12_{2k}$, by Euler's rule and Lemma 5.8i), we have:

$$q_{4k+2} = 2q(12_{2k+1}12_{2k-2}) + q(12_{2k+1}12_{2k-3}) < \left(2 + \frac{1}{2}\right) q(12_{2k+1}12_{2k-2}) =$$

$$= \frac{5}{2}[q(2_{2k-2}12_{2k+1}) + q(\Gamma)] = \frac{5}{2}[3q(\Gamma) + q(2_{2k-2}12_{2k-1})] = \frac{5}{2}\left(3 + \frac{1}{2}\right) q(\Gamma)$$

Thus, $\dfrac{\tilde{q}_{4k-1}}{q_{4k+2}} > \dfrac{4}{35}$ and, therefore, $\dfrac{D - B}{A - C} > 574.47 \cdot 0.5 \cdot \left(\dfrac{4}{35}\right)^2 > 3.75 > 1.$ □

5.4.1.4 Conclusion: Ext1B), Ext1C), Ext1D) are ruled out

Our discussion after Corollary 5.1 until now implies that Ext1A) is essentially the sole possible extension of $\theta = 2_2\alpha_k^1 2_4$: in fact, we have proved that

Corollary 5.2. *There exists an explicit parameter $\lambda_k^{(3)} > m(\zeta_k^1)$ such that any $(k, \lambda_k^{(3)})$-admissible word θ containing $2_2\alpha_k^1 2_4$ extends as*

$$\theta = ...2_{2k}\alpha_k^1 2_{2k} = ...2_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}....$$

5.4.2 Extension from $2_{2k}\alpha_k^1 2_{2k}$ to $2_{2k-1}12_{2k}\alpha_k^1 2_{2k}12_{2k+1}$

Lemma 5.27. $\lambda_0^-(2_{2k}\alpha_k^1 2_{2k}2) > \lambda_0^-(2_{2k}\alpha_k^1 2_{2k}11) > \lambda_0^-(2_{2k}\alpha_k^1 2_{2k}1221) > m(\zeta_k^1)$.

Proof. It is not hard to see that

$$\lambda_0^-(2_{2k}\alpha_k^1 2_{2k}2) > \lambda_0^-(2_{2k}\alpha_k^1 2_{2k}11) > \lambda_0^-(2_{2k}\alpha_k^1 2_{2k}1221),$$

just observe that

$$[0;2_{2k-2},1,2_{2k},2,...] > [0;2_{2k-2},1,2_{2k},1,1,...] > [0;2_{2k-2},1,2_{2k},1,2,2,1,...].$$

In order to prove that $\lambda_0^-(2_{2k}\alpha_k^1 2_{2k}1221) > m(\zeta_k^1)$, we write

$$\lambda_0^-(2_{2k}\alpha_k^1 2_{2k}1221) = [2;2_{2k-2},1,2_{2k},1,2_2,1,\overline{1,2}]+[0;1,2_{2k+1},1,2_{2k},\overline{1,2}] := A+B$$

and

$$m(\zeta_k^1) < [2;2_{2k-2},1,2_{2k},1,2_{2k+1},1,\overline{1,2}]+[0;1,2_{2k+1},1,2_{2k},1,2_{2k-1},1,\overline{1,2}] := C+D$$

Since $q(2_{2k-2}12_{2k}12_2) < 3\cdot q(2_{2k-2}12_{2k}12)$ and

$$q(12_{2k+1}12_{2k}12) > q(12_3)q(2_{2k-2}12_{2k}12) > 17\cdot q(2_{2k-2}12_{2k}12),$$

we have $q(12_{2k+1}12_{2k}12) > 4\cdot q(2_{2k-2}12_{2k}12_2)$. Because $A > C$ and $D > B$, it follows from Lemma 5.6 that $A+B > C+D$. □

Lemma 5.28. $\lambda_0^-(22_{2k}\alpha_k^1 2_{2k}12_4) > \lambda_0^-(112_{2k}\alpha_k^1 2_{2k}12_4) > m(\zeta_k^1)$.

Proof. By direct inspection, we see that

$$\lambda_0^-(22_{2k}\alpha_k^1 2_{2k}12_4) > \lambda_0^-(112_{2k}\alpha_k^1 2_{2k}12_4).$$

It remains to prove that $\lambda_0^-(112_{2k}\alpha_k^1 2_{2k}12_4) > m(\zeta_k^1)$. In order to prove this inequality, let

$$\lambda_0^-(112_{2k}\alpha_k^1 2_{2k}12_4) = [2;2_{2k-2},1,2_{2k},1,2_4,\overline{2,1}]+[0;1,2_{2k+1},1,2_{2k},1,1,\overline{1,2}] := C+D$$

and

$$m(\zeta_k^1) < [2;2_{2k-2},1,2_{2k},1,2_6,\overline{1,2}] + [0;1,2_{2k+1},1,2_{2k},1,2,2,\overline{1,2}] := A+B.$$

Our task is reduced to prove that $D-B > A-C$. We have:

$$D-B = \frac{[2;2,\overline{1,2}]-[1;\overline{1,2}]}{\tilde{q}_{4k+4}^2([2;2,\overline{1,2}]+\tilde{\beta})([1;\overline{1,2}]+\tilde{\beta})},$$

and

$$A - C = \frac{[2;2_3,\overline{2,1}] - [2;2_5,\overline{1,2}]}{q_{4k}^2([2;2_3,\overline{2,1}]+\beta)([2;2_5,\overline{1,2}]+\beta)}$$

where $q_{4k} = q(2_{2k-2}12_{2k}1)$, $\tilde{q}_{4k+4} = q(12_{2k+1}12_{2k}1)$, $\beta = [0;1,2_{2k},1,2_{2k-2}]$ and $\tilde{\beta} = [0;1,2_{2k},1,2_{2k+1},1]$. Thus,

$$\frac{D-B}{A-C} = \frac{[2;2,\overline{1,2}] - [1;\overline{1,2}]}{[2;2_3,\overline{2,1}] - [2;2_5,\overline{1,2}]} \cdot Y \cdot \frac{q_{4k}^2}{\tilde{q}_{4k+4}^2} > 2185.35 \cdot Y \cdot \frac{q_{4k}^2}{\tilde{q}_{4k+4}^2},$$

where

$$Y = \frac{([2;2_3,\overline{2,1}]+\beta)([2;2_5,\overline{1,2}]+\beta)}{([2;2,\overline{1,2}]+\tilde{\beta})([1;\overline{1,2}]+\tilde{\beta})}.$$

Note that

$$Y > \frac{([2;2_3,\overline{2,1}]+[0;1,2_5])([2;2_5,\overline{1,2}]+[0;1,2_5])}{([2;2,\overline{1,2}]+[0;1,\overline{2}])([1;\overline{1,2}]+[0;1,\overline{2}])} > 1.29.$$

Also, by Euler's rule, we have:

$$\tilde{q}_{4k+4} = q(12_{2k}12_{2k+1}1) < 2q(12_{2k}12_{2k-2})q(2_31) = 2 \cdot q_{4k} \cdot 17$$

Therefore,

$$\frac{D-B}{A-C} > 2185.35 \cdot 1.29 \cdot \left(\frac{1}{34}\right)^2 > 2.43 > 1.$$

□

As a direct consequence of the previous two lemmas and Corollary 5.1, we get:

Corollary 5.3. *Consider the parameter*

$\lambda_k^{(4)} := \min\{\lambda_0^-(2_{2k}\alpha_k^1 2_{2k}1221), \lambda_0^-(112_{2k}\alpha_k^1 2_{2k}12_4), \lambda_0^-(2_{2k-2}12^*2_21) := \lambda_k^{(2)}\}$.

Then, $\lambda_k^{(4)} > m(\zeta_k^1)$ *and any* $(k,\lambda_k^{(4)})$*-admissible word* θ *containing* $2_{2k}\alpha_k^1 2_{2k}$ *extends as*

$$\theta = ...2_212_{2k}\alpha_k^1 2_{2k}12_4 = ...2_212_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_4....$$

Let $\alpha_k^2 = 12_{2k}\alpha_k^1 2_{2k}1 = 12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}1$. The word $\theta = ...2_2\alpha_k^2 2_4$ in the conclusion of the previous corollary continues as $\theta = ...2_a\alpha_k^2 2_b...$ with $a \geq 2$, $b \geq 4$. If $a > 2k-1$ and $b > 2k+1$, then $\lambda_0^-(\theta) > m(\zeta_k^1)$. Thus, we have four cases:

Ext2A) The string $2_{2k-1}\alpha_k^2 2_{2k+1}$.

Ext2B) The string $\Delta_{a,b} = 12_a\alpha_k^2 2_b1$, with $a < 2k-1$ and $b < 2k+1$.

Ext2C) The string $\Delta_a = 12_a\alpha_k^2 2_{2k+1}$, with $a < 2k-1$.

Ext2D) The string $\Delta^b = 2_{2k-1}\alpha_k^2 2_b1$, with $b < 2k+1$.

5.4.2.1 Ruling out Ext2B)

This case essentially never occurs. In fact, by the Lemma 5.11, a can not be odd in this regime. It remains the case where $a = 2j < 2k-1$ is even. Again by the Lemma 5.11, $\lambda_0^-(2_{2k-2}12^*2_{2m}1) > m(\zeta_k^1)$, $m \le k-2$, so that if $b < 2k+1$ is odd, then we must have $b = 2k-1$. In particular, we are left with the possibilities that $b = 2k-1$ or $b < 2k+1$ is even. In order to eliminate these cases, we use the next two lemmas:

Lemma 5.29. *Let* $\Delta_{a,b} = 12_a\alpha_k^2 2_b 1 = 12_a 12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_b 1$. *We have:*

i) $\lambda_0^+(\Delta_{2k-2,2k-1}) < \lambda_0^+(\Delta_{2k-4,2k-1}) < m(\theta(\underline{\eta}_k))$;

ii) $\lambda_0^-(\Delta_{2j,2k-1}) \ge \lambda_0^-(\Delta_{2k-6,2k-1}) > m(\zeta_k^1)$ *for* $2j \le 2k-6$.

Proof. It is easy to see that $\lambda_0^+(\Delta_{2k-2,2k-1}) < \lambda_0^+(\Delta_{2k-4,2k-1})$. In order to show that $\lambda_0^+(\Delta_{2k-4,2k-1}) < m(\theta(\underline{\eta}_k))$, we write $\lambda_0^+(\Delta_{2k-4,2k-1}) := A + B$, where

$$A = [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k-1}, 1, \overline{1,2}] \quad \text{and} \quad B := [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-4}, 1, \overline{2,1}].$$

Since $m(\theta(\underline{\eta}_k)) > C + D$ with

$$C := [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, \overline{2,1}] \quad \text{and} \quad D := [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, \overline{2,1}],$$

our task is reduced to prove that $A + B < C + D$.

Note that

$$C - A = \frac{[2;\overline{2,1}] - [1;\overline{1,2}]}{q^2(2_{2k-2}12_{2k}12_{2k-1})([2;\overline{2,1}] + \beta)([1;\overline{1,2}] + \beta)},$$

where $\beta = \beta(2_{2k-2}12_{2k}12_{2k-1}) = [0; 2_{2k-1}, 1, 2_{2k}, 1, 2_{2k-2}] < [0;\overline{2}]$. Moreover,

$$B - D = \frac{[2; 2_2, \overline{1,2}] - [1;\overline{2,1}]}{q^2(12_{2k+1}12_{2k}12_{2k-4})([2; 2_2, \overline{1,2}] + \tilde{\beta})([1;\overline{2,1}] + \tilde{\beta})},$$

where $\tilde{\beta} = \beta(12_{2k+1}12_{2k}12_{2k-4}) = [0; 2_{2k-4}, 1, 2_{2k}, 1, 2_{2k+1}, 1] > [0;\overline{2}]$. Then

$$\frac{C-A}{B-D} = \frac{q^2(12_{2k+1}12_{2k}12_{2k-4})}{q^2(2_{2k-2}12_{2k}12_{2k-1})} \cdot X \cdot Y,$$

where

$$X = \frac{[2;\overline{2,1}] - [1;\overline{1,2}]}{[2; 2_2, \overline{1,2}] - [1;\overline{2,1}]} = 0.6$$

and

$$Y = \frac{([2;2_2,\overline{1,2}]+\tilde{\beta})([1;\overline{2,1}]+\tilde{\beta})}{([2;\overline{2,1}]+\beta)([1;\overline{1,2}]+\beta)} > \frac{([2;2_2,\overline{1,2}]+[0;\overline{2}])([1;\overline{2,1}]+[0;\overline{2}])}{([2;\overline{2,1}]+[0;\overline{2}])([1;\overline{1,2}]+[0;\overline{2}])} > 0.84.$$

On the other hand, by Euler's rule,

$$\begin{aligned} q(12_{2k+1}12_{2k}12_{2k-4}) &= q(12_2)q(2_{2k-1}12_{2k}12_{2k-4}) + q(12)q(2_{2k-2}12_{2k}12_{2k-4}) \\ &= 7q(2_{2k-1}12_{2k}12_{2k-4}) + 3q(2_{2k-2}12_{2k}12_{2k-4}) \end{aligned}$$

and

$$q(2_{2k-2}12_{2k}12_{2k-1}) = 5q(2_{2k-1}12_{2k}12_{2k-4}) + 2q(2_{2k-1}12_{2k}12_{2k-5}).$$

Hence,

$$\frac{q(12_{2k+1}12_{2k}12_{2k-4})}{q(2_{2k-2}12_{2k}12_{2k-1})} = \frac{7+3\beta(2_{2k-4}12_{2k}12_{2k-1})}{5+2\beta(2_{2k-1}12_{2k}12_{2k-4})} > 1.41.$$

In particular,

$$\frac{C-A}{B-D} > (1.41)^2 \cdot 0.6 \cdot 0.84 > 1.001 > 1.$$

To prove ii) we write $\lambda_0^-(\Delta_{2k-6,2k-1}) = A' + B'$ with

$$B' := [2;2_{2k-2},1,2_{2k},1,2_{2k-1},1,\overline{2,1}] \quad \text{and} \quad A' := [0;1,2_{2k+1},1,2_{2k},1,2_{2k-6},1,\overline{1,2}],$$

and $m(\zeta_k^1) < C' + D'$ with

$$D' := [2;2_{2k-2},1,2_{2k},1,2_{2k+1},1,\overline{1,2}] \quad \text{and} \quad C' := [0;1,2_{2k+1},1,2_{2k},1,2_{2k-1},1,\overline{1,2}].$$

Let $\underline{c} = 2_{2k-2}12_{2k}12_{2k-1}$ and $\underline{d} = 12_{2k+1}12_{2k}12_{2k-6}$. By Lemma 5.8 i) and Euler's rule, we have

$$q(\underline{d}) = q(12_3)q(2_{2k-2}12_{2k}12_{2k-6}) + q(12_2)q(2_{2k-3}12_{2k}12_{2k-6}) < \frac{41}{2}q(2_{2k-2}12_{2k}12_{2k-6})$$

and

$$q(\underline{c}) = q(2_4)q(2_{2k-5}12_{2k}12_{2k-2}) + q(2_3)q(2_{2k-6}12_{2k}12_{2k-2}) > 70q(2_{2k-2}12_{2k}12_{2k-6}),$$

so that $q(\underline{c}) > 3 \cdot q(\underline{d})$. Since $A' > C'$ and $D' > B'$, it follows from Lemma 5.6 that $A' + B' > C' + D'$. □

Lemma 5.30. *Let* $\Delta_{a,b} = 12_a\alpha_k^2 2_b 1 = 12_a 12_{2k}12_{2k+1}12^* 2_{2k-2}12_{2k}12_b 1$*, if* $b = 2m < 2k+1$ *and* $a = 2j < 2k-1$*, then*

$$\lambda_0^-(\Delta_{2j,2m}) > m(\zeta_k^1).$$

Proof. If $a = 2j \le 2k-2$ and $b = 2m \le 2k$, then $\lambda_0^-(\Delta_{a,b}) \ge \lambda_0^-(\Delta_{2k-2,2k})$. Hence, it remains to prove that $\lambda_0^-(\Delta_{2k-2,2k}) > m(\zeta_k^1)$. For this sake, note that:

$$C := [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k}, 1, \overline{1,2}] > [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2, \overline{2,1}] =: A \text{ and}$$

$$D := [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-2}, 1, \overline{1,2}] > [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 2, \overline{2,1}] =: B.$$

Therefore, $\lambda_0^-(\Delta_{2k-2,2k}) := C + D > A + B > m(\zeta_k^1)$. □

5.4.2.2 Ruling out Ext2C)

This case essentially never occurs. Again, if $a < 2k-1$, then, by Lemma 5.11, a can not be odd. It remains the case where $a = 2j < 2k-1$ is even, which is eliminated by the next lemma:

Lemma 5.31. *Let $\Delta_a = 12_a 12_{2k} 12_{2k+1} 12^* 2_{2k-2} 12_{2k} 12_{2k+1}$ with $k \ge 4$. If $a = 2j \le 2k-2$, then $\lambda_0^-(\Delta_{2j}) \ge \lambda_0^-(\Delta_{2k-2}) > m(\zeta_k^1)$.*

Proof. As usual, let us write

$$\lambda_0^-(\Delta_{2k-2}) := A + B \quad \text{and} \quad m(\zeta_k^1) < C + D,$$

where $A = [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, \overline{1,2}]$, $B = [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-2}, 1, \overline{1,2}]$,

$$C = [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2_2, \overline{1,2}], \quad D := [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, \overline{1,2}].$$

Then,

$$C - A = \frac{[2; \overline{1,2}] - [1; \overline{2,1}]}{q^2(\underline{c})([2; \overline{1,2}] + \beta(\underline{c}))([1; \overline{2,1}] + \beta(\underline{c}))}$$

and

$$B - D = \frac{[2; 1, \overline{1,2}] - [1; \overline{1,2}]}{q^2(\underline{d})([2; 1, \overline{1,2}] + \beta(\underline{d}))([1; \overline{1,2}] + \beta(\underline{d}))}.$$

where $\underline{c} = 2_{2k-2}12_{2k}12_{2k+1}12$ and $\underline{d} = 12_{2k+1}12_{2k}12_{2k-2}$. It follows that

$$\frac{B-D}{C-A} = \frac{q^2(\underline{c})}{q^2(\underline{d})} \cdot \frac{[2; 1, \overline{1,2}] - [1; \overline{1,2}]}{[2; \overline{1,2}] - [1; \overline{2,1}]} \cdot Y > \frac{q^2(\underline{c})}{q^2(\underline{d})} \cdot 0.61 \cdot Y,$$

where

$$Y = \frac{([2; \overline{1,2}] + \beta(\underline{c}))([1; \overline{2,1}] + \beta(\underline{c}))}{([2; 1, \overline{1,2}] + \beta(\underline{d}))([1; \overline{1,2}] + \beta(\underline{d}))}.$$

Since

$$\beta(\underline{c}) = [0; 2, 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-2}] > [0; 2, 1, 2_9] > 0.369$$

and

$$\beta(\underline{d}) = [0; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1] < [0; 2_6, 1]$$

we have

$$Y > \frac{([2; \overline{1,2}] + 0.369)([1; \overline{2,1}] + 0.369)}{([2; 1, \overline{1,2}] + [0; 2_6, 1])([1; \overline{1,2}] + [0; 2_6, 1])} > 0.83.$$

Because $q(\underline{c}) > 2q(\underline{d})$, we conclude that

$$\frac{B-D}{C-A} > 2^2 \cdot 0.61 \cdot 0.83 > 2,$$

i.e., $A + B > C + D$. □

5.4.2.3 Ruling out Ext2D)

This case essentially never occurs. Indeed, if $b = 2m+1 < 2k+1$ is odd, then Lemma 5.11 forces $b = 2k-1$. This subcase is eliminated by the next lemma:

Lemma 5.32. *Let* $\Delta^b = 2_{2k-1}12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_b1$. *We have*

$$\lambda_0^+(\Delta^{2k-1}) < m(\theta(\underline{\eta}_k)).$$

Proof. By definition, $m(\theta(\underline{\eta}_k)) > A+B$, where $A := [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, \overline{2,1}]$ and $B := [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 2, 2, \overline{2,1}]$. Note that $\lambda_0^+(\Delta^{2k-1}) = C + D$, where $C := [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k-1}, 1, \overline{1,2}]$ and $D := [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, \overline{2,1}]$. Hence, our work is reduced to prove that $A - C > D - B$.

In order to prove this inequality, note that $A > C$, $D > B$, and, by Euler's rule,

$$q(12_{2k+1}12_{2k}12_{2k-1}) > q(2_{2k-1}12_{2k}12_{2k-2})q(2_31) = 17q(2_{2k-2}12_{2k}12_{2k-1}).$$

Therefore, the desired inequality follows from Lemma 5.6. □

It remains the subcase where $b = 2m < 2k+1$ is even, but this possibility does not occur thanks to the next lemma:

Lemma 5.33. *Let* $\Delta^b = 2_{2k-1}12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_b1$. *If* $b = 2m < 2k+1$, *then* $\lambda_0^-(\Delta^{2m}) \geq \lambda_0^-(\Delta^{2k}) > m(\zeta_k^1)$.

Proof. It is not hard to show that $\lambda_0^-(\Delta^{2m}) \geq \lambda_0^-(\Delta^{2k})$ for $2m \leq 2k$. To see that $\lambda_0^-(\Delta^{2k}) > m(\zeta_k^1)$, we write

$$\lambda_0^-(\Delta^{2k}) = [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k}, 1, \overline{1,2}] + [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, \overline{1,2}] := A + B$$

and

$$m(\zeta_k^1) < [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, \overline{1,2}] + [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, \overline{1,2}] := C + D.$$

Note that $A > C$ and $D > B$. Moreover,

$$q(12_{2k+1}12_{2k}12_{2k-1}1) > q(12_3)q(2_{2k-2}12_{2k}12_{2k-1}1) = 17q(2_{2k-2}12_{2k}12_{2k-1}1)$$

and

$$q(2_{2k-2}12_{2k}12_{2k}) < 3q(2_{2k-2}12_{2k}12_{2k-1}).$$

In particular, $q(12_{2k+1}12_{2k}12_{2k-1}1) > 4q(2_{2k-2}12_{2k}12_{2k})$ and, by Lemma 5.6, we have $A + B > C + D$. □

5.4.2.4 Conclusion: Ext2B), Ext2C), Ext2D) are ruled out

Our discussion after Corollary 5.3 until now implies that Ext2A) is essentially the sole possible extension of $\theta = 2_2\alpha_k^2 2_4$: in fact, we have proved that

Corollary 5.4. *There exists an explicit parameter $\lambda_k^{(5)} > m(\zeta_k^1)$ such that any $(k, \lambda_k^{(5)})$-admissible word θ containing $2_2\alpha_k^2 2_4$ extends as*

$$\theta = ...2_{2k-1}\alpha_k^2 2_{2k+1} = ...2_{2k-1}12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_{2k+1}....$$

5.4.3 Extension from $2_{2k-1}\alpha_k^2 2_{2k+1}$ to $2_{2k+1}12_{2k-1}\alpha_k^2 2_{2k+1}12_{2k-1}$

Lemma 5.34. *Let $\alpha_k^2 = 12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}1$. We have:*

i) $\lambda_0^-(2_{2k-1}\alpha_k^2 2_{2k+1}2) > \lambda_0^-(2_{2k-1}\alpha_k^2 2_{2k+1}11) > m(\zeta_k^1)$;

ii) $\lambda_0^-(22_{2k-1}\alpha_k^2 2_{2k+1}122) > \lambda_0^-(112_{2k-1}\alpha_k^2 2_{2k+1}122) > m(\zeta_k^1)$;

Proof. The inequality $\lambda_0^-(2_{2k-1}\alpha_k^2 2_{2k+1}2) > \lambda_0^-(2_{2k-1}\alpha_k^2 2_{2k+1}11)$ is straightforward. Thus, the proof of item i) is reduced to check that $\lambda_0^-(2_{2k-1}\alpha_k^2 2_{2k+1}11) > m(\zeta_k^1)$. In order to do this, we write $m(\zeta_k^1) < A + B$, where $A := [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2, 2, \overline{1,2}]$ and $B := [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 2_4, \overline{1,2}]$. Note that $\lambda_0^-(2_{2k-1}\alpha_k^2 2_{2k+1}11) = C + D := [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 1, \overline{1,2}] + [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, \overline{1,2}]$.

Hence, our work is reduced to prove that $C - A > B - D$. In order to show this inequality, we observe that:

$$C - A = \frac{[2;2,\overline{1,2}] - [1;\overline{1,2}]}{q_{6k+2}^2([2;2,\overline{1,2}]+\beta)([1;\overline{1,2}]+\beta)}$$

and

$$B - D = \frac{[1;2_4,\overline{1,2}] - [1;\overline{2,1}]}{\tilde{q}_{6k+3}^2([1;2_4,\overline{1,2}]+\tilde{\beta})([1;\overline{2,1}]+\tilde{\beta})},$$

where $q_{6k+2} = q(2_{2k-2}12_{2k}12_{2k+1}1)$, $\tilde{q}_{6k+3} = q(12_{2k+1}12_{2k}12_{2k-1})$, $\beta = [0;1,2_{2k+1},1,2_{2k},1,2_{2k-2}]$ and $\tilde{\beta} = [0;2_{2k-1},1,2_{2k},1,2_{2k+1},1]$. Thus,

$$\frac{C-A}{B-D} = \frac{[2;2,\overline{1,2}] - [1;\overline{1,2}]}{[1;2_4,\overline{1,2}] - [1;\overline{2,1}]} \cdot Y \cdot \frac{\tilde{q}_{6k+3}^2}{q_{6k+1}^2} > 13.08 \cdot Y \cdot \frac{\tilde{q}_{6k+3}^2}{q_{6k+1}^2},$$

where

$$Y = \frac{([1;2_4,\overline{1,2}]+\tilde{\beta})([1;\overline{2,1}]+\tilde{\beta})}{([2;2,\overline{1,2}]+\beta)([1;\overline{1,2}]+\beta)} > \frac{([1;2_4,\overline{1,2}]+[0;2_2])([1;\overline{2,1}]+[0;2_2])}{([2;2,\overline{1,2}]+[0;1,2_2])([1;\overline{1,2}]+[0;1,2_2])} > 0.42.$$

By Euler's rule and Lemma 5.8 i), we have:

$$\tilde{q}_{6k+3} = 2q(12_{2k+1}12_{2k}12_{2k-2}) + q(12_{2k+1}12_{2k}12_{2k-3}) > (2+1/3)q_{6k+2}.$$

Therefore,

$$\frac{C-A}{B-D} > 13.08 \cdot 0.42 \cdot \left(\frac{7}{3}\right)^2 > 29.9 > 1.$$

Now, we prove ii). By parity, we can easily check that

$$\lambda_0^-(22_{2k-1}\alpha_k^2 2_{2k+1}122) > \lambda_0^-(112_{2k-1}\alpha_k^2 2_{2k+1}122).$$

It remains to prove that $\lambda_0^-(112_{2k-1}\alpha_k^2 2_{2k+1}122) > m(\zeta_k^1)$. By definition, we have $m(\zeta_k^1) < A' + B'$ with $A' := [2;2_{2k-2},1,2_{2k},1,2_{2k+1},1,2_4,\overline{1,2}]$ and $B' := [0;1,2_{2k+1},1,2_{2k},1,2_{2k-1},1,2_2,\overline{1,2}]$. Note that $\lambda_0^-(112_{2k-1}\alpha_k^2 2_{2k+1}122) = [2;2_{2k-2},1,2_{2k},1,2_{2k+1},1,2_2,\overline{2,1}] + [0;1,2_{2k+1},1,2_{2k},1,2_{2k-1},1_2,\overline{1,2}] := C' + D'$. Our task is reduced to show that $D' - B' > A' - C'$. We have:

$$D' - B' = \frac{[1;1,\overline{1,2}] - [1;2,2,\overline{1,2}]}{\tilde{q}_{6k+3}^2([1;1,\overline{1,2}]+\tilde{\beta})([1;2,2,\overline{1,2}]+\tilde{\beta})}$$

and

$$A' - C' = \frac{[2;2,\overline{2,1}] - [2;2_3,\overline{1,2}]}{q_{6k+2}^2([2;2,\overline{2,1}]+\beta)([2;2_3,\overline{1,2}]+\beta)},$$

where $q_{6k+2} = q(2_{2k-2}12_{2k}12_{2k+1}1)$, $\tilde{q}_{6k+3} = q(12_{2k+1}12_{2k}12_{2k-1})$, $\beta = [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-2}]$ and $\tilde{\beta} = [0; 2_{2k-1}, 1, 2_{2k}, 1, 2_{2k+1}, 1]$. Thus,

$$\frac{D' - B'}{A' - C'} = \frac{[1; 1, \overline{1, 2}] - [1; 2, 2, \overline{1, 2}]}{[2; 2, \overline{2, 1}] - [2; 2_3, \overline{1, 2}]} \cdot Y' \cdot \frac{q_{6k+2}^2}{\tilde{q}_{6k+3}^2} > 15.66 \cdot Y' \cdot \frac{q_{6k+2}^2}{\tilde{q}_{6k+3}^2},$$

where

$$Y' = \frac{([2; 2, \overline{2, 1}] + \beta)([2; 2_3, \overline{1, 2}] + \beta)}{([1; 1, \overline{1, 2}] + \tilde{\beta})([1; 2, 2, \overline{1, 2}] + \tilde{\beta})} > \frac{([2; 2, \overline{2, 1}] + [0; 1\bar{2}])([2; 2_3, \overline{1, 2}] + [0; 1\bar{2}])}{([1; 1, \overline{1, 2}] + [0; \bar{2}])([1; 2, 2, \overline{1, 2}] + [0; \bar{2}])},$$

and so, $Y > 2.66$. By Euler's rule and Lemma 5.8 i), we have:

$$\tilde{q}_{6k+3} = 2q(12_{2k+1}12_{2k}12_{2k-2}) + q(12_{2k+1}12_{2k}12_{2k-3}) < (2 + 1/2)q_{6k+2}.$$

Therefore,

$$\frac{D' - B'}{A' - C'} > 15.66 \cdot 2.66 \cdot \left(\frac{2}{5}\right)^2 > 6.65 > 1.$$

□

Corollary 5.5. *Consider the parameter*

$$\lambda_k^{(6)} := \min\{\lambda_0^-(12^*1), \lambda_0^-(2_{2k-1}\alpha_k^2 2_{2k+1}11), \lambda_0^-(112_{2k-1}\alpha_k^2 2_{2k+1}122)\}.$$

Then, $\lambda_k^{(6)} > m(\zeta_k^1)$ *and any* $(k, \lambda_k^{(6)})$*-admissible word* θ *containing* $2_{2k-1}\alpha_k^2 2_{2k+1}$ *extends as*

$$\theta = ...2_2 12_{2k-1}\alpha_k^2 2_{2k+1}12_2 = ...2_2 12_{2k-1}12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_{2k+1}12_2....$$

Denote $\alpha_k^3 = 12_{2k-1}\alpha_k^2 2_{2k+1}1 = 12_{2k-1}12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_{2k+1}1$. We continue the word $\theta = ...2_2\alpha_k^3 2_2$ as $\theta = ...2_a\alpha_k^3 2_b....$ If $a > 2k+1$ and $b > 2k-1$, then $\lambda_0^-(\theta) > m(\zeta_k^1)$. Thus, we have four cases:

Ext3A) The string $2_{2k+1}\alpha_k^3 2_{2k-1}$.

Ext3B) The string $\eta_{a,b} = 12_a\alpha_k^3 2_b 1$, with $a < 2k+1$ and $b < 2k-1$.

Ext3C) The string $\eta_a = 12_a\alpha_k^3 2_{2k-1}$, with $a < 2k+1$.

Ext3D) The string $\eta^b = 2_{2k+1}\alpha_k^3 2_b 1$, with $b < 2k-1$.

5.4.3.1 Ruling out Ext3B)

This case essentially never occurs. In fact, if $b = 2m+1 < 2k-1$ is odd, then Lemma 5.11 says that this string contains a k-prohibited string. Thus, it remains $b = 2m < 2k-1$ even. Analogously, the case a is odd with $a = 2j+1 < 2k-1$ is also eliminate by Lemma 5.11. In the case $a = 2k-1$, we use the Lemma 5.13 i) to show that the word $\eta_{2k-1,b}$ contains a k-prohibited string. Thus, it remain just the case where both a and b are even. As it turns out, this case is eliminated by the next lemma:

Lemma 5.35. *Let* $\eta_{a,b} = 12_a12_{2k-1}12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_{2k+1}12_b1$. *If* $a = 2j \leq 2k$ *and* $b = 2m \leq 2k-2$, *then* $\lambda_0^-(\eta_{2j,2m}) > m(\zeta_k^1)$.

Proof. This follows from the fact that

$$[0; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2_{2m}, 1, ...] > [0; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2_{2k-1}, 1, ...]$$

and

$$[0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 2_{2j}, 1, ...] > [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 2_{2k+1}, 1, ...],$$

whenever $j \leq k$ and $m \leq k-1$. □

5.4.3.2 Ruling out Ext3C)

This case essentially never occurs. Indeed, by Lemma 5.11, a can not be of the form $a = 2j+1 < 2k-1$. Moreover, the case $a = 2k-1$ is not possible by Lemma 5.13 i). It remains the case $a = 2j < 2k+1$, which is eliminated by the following lemma (together with Lemma 5.2 i)):

Lemma 5.36. *Let* $\eta_a = 12_a12_{2k-1}12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_{2k+1}12_{2k-1}$. *If* $a = 2j < 2k+1$, *then* $\lambda_0^-(\eta_{2j}122) \geq \lambda_0^-(\eta_{2k}122) > m(\zeta_k^1)$. *Moreover, for every* $2j < 2k+1$, *one has* $\lambda_0^-(\eta_{2j}2) > \lambda_0^-(\eta_{2j}11) > \lambda_0^-(\eta_{2j}122)$.

Proof. By parity, the inequalities $\lambda_0^-(\eta_{2j}2) > \lambda_0^-(\eta_{2j}11) > \lambda_0^-(\eta_{2j}12_2) \geq \lambda_0^-(\eta_{2k}12_2)$ for $2j \leq 2k$ are clear. Now, we show that $\lambda_0^-(\eta_{2k}122) > m(\zeta_k^1)$. In order to do this, we write $\lambda_0^-(\eta_{2k}122) = C + D$, where

$$C := [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2_{2k-1}, 1, 2_2, \overline{2, 1}] \quad \text{and}$$

$$D := [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 2_{2k}, 1, \overline{1, 2}]$$

and $m(\zeta_k^1) < A + B$, where

$$A := [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2_{2k-1}, 1, 2_6, \overline{1,2}] \quad \text{and}$$

$$B := [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 2_{2k+1}, 1, 2_2, \overline{1,2}].$$

In this context, our task is reduced to prove that $D-B > A-C$. We observe that:

$$D - B = \frac{[2; 1, \overline{1,2}] - [2; 2, 1, 2_2, \overline{1,2}]}{\tilde{q}_{8k+3}^2([2; 1, \overline{1,2}] + \tilde{\beta})([2; 2, 1, 2_2, \overline{1,2}] + \tilde{\beta})}$$

and

$$A - C = \frac{[2; 2, \overline{2,1}] - [2; 2_5, \overline{1,2}]}{q_{8k+2}^2([2; 2, \overline{2,1}] + \beta)([2; 2_5, \overline{1,2}] + \beta)},$$

where $q_{8k+2} = q(2_{2k-2}12_{2k}12_{2k+1}12_{2k-1}1)$, $\tilde{q}_{8k+3} = q(12_{2k+1}12_{2k}12_{2k-1}12_{2k-1})$, $\beta = [0; 1, 2_{2k-1}, 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-2}]$ and $\tilde{\beta} = [0; 2_{2k-1}, 1, 2_{2k-1}, 1, 2_{2k}, 1, 2_{2k+1}, 1]$. Thus,

$$\frac{D-B}{A-C} = \frac{[2; 1, \overline{1,2}] - [2; 2, 1, 2_2, \overline{1,2}]}{[2; 2, \overline{2,1}] - [2; 2_5, \overline{1,2}]} \cdot Y \cdot \frac{q_{8k+2}^2}{\tilde{q}_{8k+3}^2} > 24.45 \cdot Y \cdot \frac{q_{8k+2}^2}{\tilde{q}_{8k+3}^2},$$

where

$$Y = \frac{([2; 2, \overline{2,1}] + \beta)([2; 2_5, \overline{1,2}] + \beta)}{([2; 1, \overline{1,2}] + \tilde{\beta})([2; 2, 1, 2_2, \overline{1,2}] + \tilde{\beta})} > \frac{([2; 2, \overline{2,1}] + [0; 1, \bar{2}])([2; 2_5, \overline{1,2}] + [0; 1, \bar{2}])}{([2; 1, \overline{1,2}] + [0; \bar{2}])([2; 2, 1, 2_2, \overline{1,2}] + [0; \bar{2}])},$$

and so, $Y > 1.17$. Let $\Gamma = 2_{2k-2}12_{2k}12_{2k+1}1$ and $\Sigma = 2_{2k-1}1$. By Euler's rule and Lemma 5.8 i), we have:

$$q_{8k+2} = q(\Gamma)q(\Sigma) + q(2_{2k-2}12_{2k}12_{2k+1})q(2_{2k-2}1) > q(\Gamma)q(\Sigma)(1 + 2/3 \cdot 1/3),$$

$$\tilde{q}_{8k+3} = q(12_{2k+1}12_{2k}12_{2k-1})q(\Sigma^t) + q(\Gamma^t)q(2_{2k-1}) < q(\Gamma^t)q(\Sigma^t)(3 + 3/4).$$

Thus,

$$\frac{D-B}{A-C} > 24.45 \cdot 1.17 \cdot \left(\frac{44}{135}\right)^2 > 3 > 1.$$

□

5.4.3.3 Ruling out Ext3D)

This case essentially never occurs. Indeed, by Lemma 5.11, b can not be of the form $b = 2m + 1 < 2k - 1$. Thus, it remains the case $b = 2m < 2k - 1$ even. As it turns out, this case is excluded by the following lemma:

Lemma 5.37. *Let* $\eta^b = 2_{2k+1}12_{2k-1}12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_{2k+1}12_b1$. *If* $b = 2m < 2k-1$, *then* $\lambda_0^-(\eta^{2m}) \geq \lambda_0^-(\eta^{2k-2}) > m(\zeta_k^1)$.

Proof. It follows the same ideia of Lemma 5.33. In fact, let $\underline{c} = 2_{2k-2}12_{2k}12_{2k+1}12_{2k-2}$ and $\underline{d} = 12_{2k+1}12_{2k}12_{2k-1}12_{2k+1}$, and denote

$$A = [2; \underline{c}, 1, \overline{1,2}] \quad \text{and} \quad B = [0; \underline{d}, \overline{1,2}]$$

and

$$C = [2; \underline{c}, 2, \overline{2,1}] \quad \text{and} \quad D = [0; \underline{d}, \overline{2,1}].$$

One can check that $\lambda_0^-(\eta^{2k-2}) = A + B$, $m(\zeta_k^1) < C + D$, $A > C$ and $D > B$. Also, Euler's rule implies $q(\underline{d}) > 4q(\underline{c})$, so that $A + B > C + D$ thanks to Lemma 5.6. $\square$

5.4.3.4 Conclusion: Ext3B), Ext3C) and Ext3D) are ruled out

Our discussion after Corollary 5.5 until now implies that Ext3A) is essentially the sole possible extension of $\theta = 2_2\alpha_k^3 2_2$: in fact, we have proved that

Corollary 5.6. *There exists an explicit parameter* $\lambda_k^{(7)} > m(\zeta_k^1)$ *and any* $(k, \lambda_k^{(7)})$*-admissible word* θ *containing* $2_2\alpha_k^3 2_2$ *extends as*

$$\theta = ...2_{2k+1}\alpha_k^3 2_{2k-1} = ...2_{2k+1}12_{2k-1}12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_{2k+1}12_{2k-1}....$$

5.4.4 End of proof of Theorem 5.2

From Corollaries 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, we see that the statement of Theorem 5.2 is true for $\mu_k^{(1)} := \min\{\lambda_k^{(i)} : i = 2, \ldots, 7\}$.

5.5 Replication mechanism for ζ_k^1

In this section, we investigate the extension of a word θ containing the string

$$\alpha_k^4 := 2_{2k+1}12_{2k-1}12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_{2k+1}12_{2k-1}$$

Lemma 5.38. *We have:*

i) $\lambda_0^-(\alpha_k^4 2) > \lambda_0^-(\alpha_k^4 11) > \lambda_0^-(\alpha_k^4 1221) > m(\zeta_k^1)$;

ii) $\lambda_0^-(2\alpha_k^4 12_4) > \lambda_0^-(11\alpha_k^4 12_4) > m(\zeta_k^1)$.

Proof. By parity, we get the inequalities $\lambda_0^-(\alpha_k^4 2) > \lambda_0^-(\alpha_k^4 11) > \lambda_0^-(\alpha_k^4 1221)$. Thus, the proof of i) is reduced to check the inequality $\lambda_0^-(\alpha_k^4 1221) > m(\zeta_k^1)$. In this direction, we write $m(\zeta_k^1) < [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2_{2k-1}, 1, 2_4, \overline{1,2}] + [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 2_{2k+1}, 1, 2_4, \overline{1,2}] := A + B$ and we note that $\lambda_0^-(\alpha_k^4 1221) = C + D$, where

$$C := [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2_{2k-1}, 1, 2, 2, 1, \overline{1,2}]$$

and

$$D := [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 2_{2k+1}, \overline{1,2}].$$

Hence, our work is reduced to prove that $C - A > B - D$. In order to prove this estimate, we observe that:

$$C - A = \frac{[1; 2_2, 1, \overline{1,2}] - [1; 2_4, \overline{1,2}]}{q_{8k+1}^2([1; 2_2, 1, \overline{1,2}] + \beta)([1; 2_4, \overline{1,2}] + \beta)}$$

and

$$B - D = \frac{[2; 2, 1, 2_4, \overline{1,2}] - [2; 2, 1, 2, \overline{1,2}]}{\tilde{q}_{8k+3}^2([2; 2, 1, 2_4, \overline{1,2}] + \tilde{\beta})([2; 2, 1, 2, \overline{1,2}] + \tilde{\beta})},$$

where $q_{8k+1} = q(2_{2k-2}12_{2k}12_{2k+1}12_{2k-1})$, $\tilde{q}_{8k+3} = q(12_{2k+1}12_{2k}12_{2k-1}12_{2k-1})$, $\beta = [0; 2_{2k-1}, 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-2}]$ and $\tilde{\beta} = [0; 2_{2k-1}, 1, 2_{2k-1}, 1, 2_{2k}, 1, 2_{2k+1}, 1]$. Thus,

$$\frac{C - A}{B - D} = \frac{[1; 2_2, 1, \overline{1,2}] - [1; 2_4, \overline{1,2}]}{[2; 2, 1, 2_4, \overline{1,2}] - [2; 2, 1, 2, \overline{1,2}]} \cdot Y \cdot \frac{\tilde{q}_{8k+3}^2}{q_{8k+1}^2} > 1.26 \cdot Y \cdot \frac{\tilde{q}_{8k+3}^2}{q_{8k+1}^2},$$

where

$$Y = \frac{([2; 212_4\overline{12}] + \tilde{\beta})([2; 212\overline{12}] + \tilde{\beta})}{([1; 2_2 1\overline{12}] + \beta)([1; 2_4\overline{12}] + \beta)} > \frac{([2; 212_4\overline{12}] + [0; 2_4])([2; 212\overline{12}] + [0; 2_4])}{([1; 2_2 1\overline{12}] + [0; \overline{2}])([1; 2_4\overline{12}] + [0; \overline{2}])},$$

and so, $Y > 2.3$. Let $\Gamma = 12_{2k+1}12_{2k}12_{2k-2}$ and $\Sigma = 2_{2k-1}$. By Euler's rule and Lemma 5.8 i):

$$\begin{aligned}
\tilde{q}_{8k+3} &= q(12_{2k+1}12_{2k}12_{2k-1})q(12_{2k-1}) + q(\Gamma)q(\Sigma) \\
&> \frac{4}{3}q(12_{2k+1}12_{2k}12_{2k-1})q(\Sigma) + q(\Gamma)q(\Sigma) \\
&= \frac{4}{3}q(\Sigma)\,[2q(12_{2k+1}12_{2k}12_{2k-2}) + q(12_{2k+1}12_{2k}12_{2k-3})] + q(\Gamma)q(\Sigma) \\
&> q(\Gamma)q(\Sigma)\,[4/3(2+1/3)+1] = 37q(\Gamma)q(\Sigma)/9
\end{aligned}$$

and

$$q_{8k+1} = q(\Gamma^T)q(\Sigma) + q(2_{2k-2}12_{2k}12_{2k+1})q(2_{2k-2})$$
$$< q(\Gamma)q(\Sigma)\left(1 + \frac{3}{4}\cdot\frac{1}{2}\right) = \frac{11}{8}q(\Gamma)q(\Sigma).$$

Therefore,

$$\frac{C-A}{B-D} > 1.26\cdot 2.3\cdot\left(\frac{296}{99}\right)^2 > 1.$$

Now, we prove ii). By parity, we can easily check that $\lambda_0^-(2\alpha_k^4 12_4) > \lambda_0^-(11\alpha_k^4 12_4)$. It remains to prove that $\lambda_0^-(11\alpha_k^4 12_4) > m(\zeta_k^1)$. We have $m(\zeta_k^1) < A' + B' := [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2_{2k-1}, 1, 2_8, \overline{1,2}] + [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 2_{2k+1}, 1, 2_4, \overline{1,2}]$. Also, $\lambda_0^-(11\alpha_k^4 12_4) = C' + D'$ with

$$C' := [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2_{2k-1}, 1, 2_4, \overline{2,1}] \quad \text{and}$$
$$D' := [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 2_{2k+1}, 1_2, \overline{1,2}].$$

Hence, our task is reduced to show that $D' - B' > A' - C'$. We have:

$$D' - B' = \frac{[2; 2, 1_2, \overline{1,2}] - [2; 2, 1, 2_4, \overline{1,2}]}{\tilde{q}_{8k+3}^2([2; 2, 1_2, \overline{1,2}] + \tilde{\beta})([2; 2, 1, 2_4, \overline{1,2}] + \tilde{\beta})}$$

and

$$A' - C' = \frac{[2; 2_3, \overline{2,1}] - [2; 2_7, \overline{1,2}]}{q_{8k+2}^2([2; 2_3, \overline{2,1}] + \beta')([2; 2_7, \overline{1,2}] + \beta')},$$

where $q_{8k+2} = q(2_{2k-2}12_{2k}12_{2k+1}12_{2k-1}1)$, $\tilde{q}_{8k+3} = q(12_{2k+1}12_{2k}12_{2k-1}12_{2k-1})$, $\beta' = [0; 1, 2_{2k-1}, 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-2}]$ and $\tilde{\beta} = [0; 2_{2k-1}, 1, 2_{2k-1}, 1, 2_{2k}, 1, 2_{2k+1}, 1]$. Thus,

$$\frac{D' - B'}{A' - C'} = \frac{[2; 2, 1_2, \overline{1,2}] - [2; 2, 1, 2_4, \overline{1,2}]}{[2; 2_3, \overline{2,1}] - [2; 2_7, \overline{1,2}]}\cdot Y\cdot\frac{q_{8k+2}^2}{\tilde{q}_{8k+3}^2} > 41.14\cdot Y'\cdot\frac{q_{8k+2}^2}{\tilde{q}_{8k+3}^2},$$

where

$$Y' = \frac{([2; 2_3, \overline{2,1}] + \beta')([2; 2_7, \overline{1,2}] + \beta')}{([2; 2, 1_2, \overline{1,2}] + \tilde{\beta})([2; 2, 1, 2_4, \overline{1,2}] + \tilde{\beta})} > \frac{([2; 2_3\overline{21}] + [0; \overline{2}])([2; 2_7\overline{12}] + [0; \overline{2}])}{([2; 21_2\overline{12}] + [0; \overline{2}])([2; 212_4\overline{12}] + [0; \overline{2}])},$$

and so, $Y > 1$. Let $\tilde{\Gamma} = 2_{2k-2}12_{2k}12_{2k+1}1$ and $\tilde{\Sigma} = 2_{2k-1}1$. By Euler's rule and Lemma 5.8 ii):

$$q_{8k+2} = q(\tilde{\Gamma})q(\tilde{\Sigma}) + q(2_{2k-2}12_{2k}12_{2k+1})q(2_{2k-2}1) > q(\tilde{\Gamma})q(\tilde{\Sigma})(1 + (12/17)\cdot(7/17)),$$

$$\tilde{q}_{8k+3} = q(\tilde{\Gamma}^T 2)q(\tilde{\Sigma}) + q(\tilde{\Gamma}^T)q(2_{2k-1}) < q(\tilde{\Gamma})q(\tilde{\Sigma})(17/7 + 17/24).$$

Therefore,

$$\frac{D-B}{A-C} > 41.14 \cdot \left(\frac{373 \cdot 168}{289 \cdot 527}\right)^2 > 6.96 > 1.$$

□

A direct consequence of the previous lemma and Lemmas 5.11 and 5.2 i) is:

Corollary 5.7. *Consider the parameter*

$$\lambda_k^{(8)} := \min\{\lambda_0^-(12^*1), \lambda_0^-(2_{2k-2}12^*2_21), \lambda_0^-(\alpha_k^4 1221), \lambda_0^-(11\alpha_k^4 12_4)\}.$$

Then, $\lambda_k^{(8)} > m(\zeta_k^1)$ and the neighbourhood of the string α_k^4 in any $(k, \lambda_k^{(8)})$-admissible word θ has the form

$$\theta = ...2_21\alpha_k^4 12_4 = ...2212_{2k+1}12_{2k-1}12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_{2k+1}12_{2k-1}12_4....$$

5.5.1 Extension from $2_21\alpha_k^4 12_4$ to $2212_{2k}1\alpha_k^4 12_{2k}12_4$

Let $\theta = ...2_21\alpha_k^4 12_4....$ It extends as $\theta = ...2_a1\alpha_k^4 12_b...$ with $a \geq 2$, $b \geq 4$. By Lemma 5.27 and Lemma 5.28, respectively we have that $b \leq 2k$ and $a \leq 2k$. Using Lemma 5.22, we get that b can not be odd. Using Lemmas 5.11 and 5.26, we have that a can not be odd. Thus, it remains the cases where $a = 2j$ and $b = 2m$ are both even. We have four cases:

Rep1) $a = 2k$ and $b = 2k$;

Rep2) $a = 2j < 2k$ and $b = 2m < 2k$;

Rep3) $a = 2k$ and $b = 2m < 2k$;

Rep4) $a = 2j < 2k$ and $b = 2k$;

The case Rep2) essentially never occurs by the next lemma:

Lemma 5.39. *If $a = 2j < 2k$ and $b = 2m < 2k$, then $\lambda_0^-(12_{2j}1\alpha_k^4 12_{2m}1) > m(\zeta_k^1)$.*

Proof. For $a = 2j \leq 2k-2$ and $b = 2m \leq 2k-2$, the inequality $\lambda_0^-(12_{2j}1\alpha_k^4 12_{2m}1) \geq \lambda_0^-(12_{2k-2}1\alpha_k^4 12_{2k-2}1)$ is straightforward. Hence, it remains to prove that

$$\lambda_0^-(12_{2k-2}1\alpha_k^4 12_{2k-2}1) > m(\zeta_k^1).$$

For this sake, note that $C > A$ and $D > B$, where:

$$
\begin{aligned}
C &:= [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2_{2k-1}, 1, 2_{2k-2}, 1, \overline{1, 2}],\\
A &:= [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2_{2k-1}, 1, 2_{2k}, 1, \overline{2, 1}],\\
D &:= [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 2_{2k+1}, 1, 2_{2k-2}, 1, \overline{1, 2}] \text{ and}\\
B &:= [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 2_{2k+1}, 1, 2_{2k}, 1, \overline{2, 1}].
\end{aligned}
$$

Therefore, $\lambda_0^-(12_{2k-2}1\alpha_k^4 12_{2k-2}1) := C + D > A + B > m(\zeta_k^1)$. □

The case Rep3) essentially never occurs by Lemma 5.27 and the next lemma:

Lemma 5.40. *If $a = 2j < 2k$, then*

$$\lambda_0^-(12_{2j}1\alpha_k^4 12_{2k}12_3) \geq \lambda_0^-(12_{2k-2}1\alpha_k^4 12_{2k}12_3) > m(\zeta_k^1).$$

Proof. It is easy to see that $\lambda_0^-(12_{2j}1\alpha_k^4 12_{2k}12_3) \geq \lambda_0^-(12_{2k-2}1\alpha_k^4 12_{2k}12_3)$. In order to show that $\lambda_0^-(12_{2k-2}1\alpha_k^4 12_{2k}12_3) > m(\zeta_k^1)$, let

$$\underline{c} = 2_{2k-2}12_{2k}12_{2k+1}12_{2k-1}12_{2k}12_3 \quad \text{and} \quad \underline{d} = 12_{2k+1}12_{2k}12_{2k-1}12_{2k+1}12_{2k-2}.$$

We have

$$\lambda_0^-(12_{2k-2}1\alpha_k^4 12_{2k}12_3) := A + B = [2; \underline{c}, \overline{2, 1}] + [0; \underline{d}, 1, \overline{1, 2}]$$

and

$$m(\zeta_k^1) < [2; \underline{c}, 2_2, \overline{2, 1}] + [0; \underline{d}, 2, \overline{2, 1}] := C + D.$$

Then,

$$C - A = \frac{[2; \overline{2, 1}] - [1; \overline{2, 1}]}{q^2(\underline{c}2)([2; \overline{2, 1}] + \beta(\underline{c2}))([1; \overline{2, 1}] + \beta(\underline{c2}))}$$

while

$$B - D = \frac{[2; \overline{2, 1}] - [1; \overline{1, 2}]}{q^2(\underline{d})([2; \overline{2, 1}] + \beta(\underline{d}))([1; \overline{1, 2}] + \beta(\underline{d}))}.$$

In particular,

$$\frac{B - D}{C - A} = \frac{q^2(\underline{c}2)}{q^2(\underline{d})} \cdot X \cdot Y,$$

where

$$X = \frac{[2; \overline{2, 1}] - [1; \overline{1, 2}]}{[2; \overline{2, 1}] - [1; \overline{2, 1}]} > 0.6339$$

and

$$Y = \frac{([2; \overline{2, 1}] + \beta(\underline{c}2))([1; \overline{2, 1}] + \beta(\underline{c}2))}{([2; \overline{2, 1}] + \beta(\underline{d}))([1; \overline{1, 2}] + \beta(\underline{d}))} > 0.82$$

By Euler's rule,

$$\begin{aligned} q(\underline{c}2) &> q(2_{2k-2}12_2)q(2_{2k-2}12_{2k+1}12_{2k-1}12_{2k}12_4) \\ &> 8q(2_{2k-3}1)q(2_{2k-2}12_{2k+1}12_{2k-1}12_{2k}12_4) \end{aligned}$$

and

$$q(\underline{d}) < 2q(12_{2k-3})q(2_4 12_{2k}12_{2k-1}12_{2k+1}12_{2k-2}).$$

Thus, $B - D > C - A$, that is, $A + B > C + D$. □

The case Rep4) essentially never occurs by Lemma 5.28, Lemma 5.2 i) and the next lemma:

Lemma 5.41. *If $b = 2m < 2k$, then*

$$\lambda_0^-(2212_{2k}1\alpha_k^4 12_{2m}1) \geq \lambda_0^-(2212_{2k}1\alpha_k^4 12_{2k-2}1) > m(\zeta_k^1).$$

Proof. By parity, it is easy to check that

$$\lambda_0^-(2212_{2k}1\alpha_k^4 12_{2m}1) \geq \lambda_0^-(2212_{2k}1\alpha_k^4 12_{2k-2}1).$$

It remains to prove that $\lambda_0^-(2212_{2k}1\alpha_k^4 12_{2k-2}1) > m(\zeta_k^1)$.

Note that $\lambda_0^-(2212_{2k}1\alpha_k^4 12_{2k-2}1) = C + D$, where

$$\begin{aligned} C &:= [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2_{2k-1}, 1, 2_{2k-2}, 1, \overline{1,2}] \text{ and} \\ D &:= [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_2, \overline{1,2}]. \end{aligned}$$

Moreover, by definition, we have $m(\zeta_k^1) < A + B$, where

$$\begin{aligned} A &:= [2; 2_{2k-2}, 1, 2_{2k}, 1, 2_{2k+1}, 1, 2_{2k-1}, 1, 2_{2k}, 1, 2_3, \overline{1,2}] \text{ and} \\ B &:= [0; 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-1}, 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_3, \overline{1,2}]. \end{aligned}$$

Hence, our work is reduced to prove that $C + D > A + B$. In order to prove this inequality, we observe that:

$$C - A = \frac{[2; 2, 1, 2_3, \overline{1,2}] - [1; \overline{1,2}]}{\tilde{q}_{10k}^2([2; 2, 1, 2_3, \overline{1,2}] + \tilde{\beta})([1; \overline{1,2}] + \tilde{\beta})},$$

and

$$B - D = \frac{[1; 2, 2, \overline{1,2}] - [1; 2_3, \overline{1,2}]}{q_{10k+6}^2([1; 2, 2, \overline{1,2}] + \beta)([1; 2_3, \overline{1,2}] + \beta)}$$

where $\tilde{q}_{10k} = q(2_{2k-2}12_{2k}12_{2k+1}12_{2k-1}12_{2k-2})$, $q_{10k+6} = q(12_{2k+1}12_{2k}12_{2k-1}12_{2k+1}12_{2k})$,

$$\tilde{\beta} = [0; 2_{2k-2}, 1, 2_{2k-1}, 1, 2_{2k+1}, 1, 2_{2k}, 1, 2_{2k-2}] \quad \text{and}$$

$$\beta = [0; 2_{2k}, 1, 2_{2k+1}, 1, 2_{2k-1}, 1, 2_{2k}, 1, 2_{2k+1}, 1].$$

Thus,

$$\frac{C-A}{B-D} = \frac{[2;2,1,2_3,\overline{1,2}] - [1;\overline{1,2}]}{[1;2,2,\overline{1,2}] - [1;2_3,\overline{1,2}]} \cdot Y \cdot \frac{q_{10k+6}^2}{\tilde{q}_{10k}^2} > 64.5 \cdot Y \cdot \frac{q_{10k+6}^2}{\tilde{q}_{10k}^2},$$

where

$$Y = \frac{([1;2,2,\overline{1,2}]+\beta)([1;2_3,\overline{1,2}]+\beta)}{([2;2,1,2_3,\overline{1,2}]+\tilde{\beta})([1;\overline{1,2}]+\tilde{\beta})} > \frac{([1;2_2,\overline{1,2}]+[0;2_4])([1;2_3,\overline{1,2}]+[0;2_4])}{([2;2,1,2_3,\overline{1,2}]+[0;2_3])([1;\overline{1,2}]+[0;2_3])},$$

and so, $Y > 0.56$. Let $\Gamma = 2_{2k-2}12_{2k}12_{2k+1}1$ and $\Sigma = 2_{2k-1}12_{2k-2}$. By Euler's rule, we have:

$$\begin{aligned} q_{10k+6} &> q(\Gamma^t 2)q(12_{2k+1}12_{2k}) > 2q(\Gamma^t)q(12_{2k+1}12_{2k-2})q(2_2) \\ &= 10q(\Gamma^t)q(2_{2k-2}12_{2k+1}1) > 10q(\Gamma^t)q(2_{2k-2}12_{2k-1})q(2_21) = 70q(\Gamma^t)q(\Sigma^t), \end{aligned}$$

and

$$\tilde{q}_{10k} < 2q(\Gamma)q(\Sigma).$$

Thus,

$$\frac{C-A}{B-D} > 64.50 \cdot 0.56 \cdot (35)^2 > 1.$$

□

An immediate consequence of the previous three lemmas is the fact that essentially only the case Rep1) occurs:

Corollary 5.8. *There is an explicit constant $\lambda_k^{(9)} > m(\zeta_k^1)$ such that the neighbourhood of the string $2_21\alpha_k^4 12_4$ in any $(k, \lambda_k^{(9)})$-admissible word θ has the form*

$$\begin{aligned} \theta &= ...2212_{2k}\alpha_k^4 12_{2k}12_4 \\ &= ...2212_{2k}12_{2k+1}12_{2k-1}12_{2k}12_{2k+1}12_{2k-1}12_{2k}12_{2k+1}12_{2k-1}12_{2k}12_4.... \end{aligned}$$

5.5.2 Extension from $2212_{2k}1\alpha_k^4 12_{2k}12_4$ to $2212_{2k-1}12_{2k}1\alpha_k^4 12_{2k}12_4$

Let $\theta = ...2212_{2k}1\alpha_k^4 12_{2k}12_4....$ It extends as $\theta = ...2_a12_{2k}1\alpha_k^4 12_{2k}12_4....$ By Lemma 5.34 ii), we have that $a \le 2k-1$. Using Lemma 5.11, we have that if a is odd, then $a = 2k-1$. Moreover, by Lemma 5.31, we can not have $a = 2j < 2k-1$.

Corollary 5.9. *There exists an explicit constant* $\lambda_k^{(10)} > m(\zeta_k^1)$ *such that the neighbourhood of the string* $2212_{2k}1\alpha_k^4 12_{2k}12_4$ *in any* $(k, \lambda_k^{(10)})$*-admissible word* θ *has the form* $\theta = ...2212_{2k-1}12_{2k}1\alpha_k^4 12_{2k}12_4 =$

$$= ...2212_{2k-1}12_{2k}12_{2k+1}12_{2k-1}12_{2k}12_{2k+1}12_{2k-1}12_{2k}12_{2k+1}12_{2k-1}12_{2k}12_4....$$

5.5.3 Extension from $2212_{2k-1}12_{2k}1\alpha_k^4 12_{2k}12_4$ to $2212_{2k+1}12_{2k-1}12_{2k}1\alpha_k^4 12_{2k}12_4$

Let $\theta = ...2212_{2k-1}12_{2k}1\alpha_k^4 12_{2k}12_4....$ It extends as $\theta = ...2_a12_{2k-1}12_{2k}1\alpha_k^4 12_{2k}12_4....$ By Lemma 5.38 ii), we have that $a \le 2k+1$. By Lemma 5.36, we can not have $a = 2m < 2k+1$. Using Lemma 5.11, we have that if a is odd, then $a \ge 2k-1$. Finally, by Lemma 5.13 i), we can not have $a = 2k-1$. Thus, we have the following corollary:

Corollary 5.10. *Consider the parameter*

$\lambda_k^{11} := \min\{\lambda_0^-(11\alpha_k^4 12_4), \lambda_0^-(\Delta_{2k-2}), \lambda_0^-(2_{2k-2}12^*2_{2k-4}1), \lambda_0^-(112_{2k-1}12^*2_{2k-2}122)\}.$

Then, $\lambda_k^{11} > m(\zeta_k^1)$ *and the neighbourhood of the string* $2212_{2k-1}12_{2k}1\alpha_k^4 12_{2k}12_4$ *in any* $(k, \lambda_k^{(10)})$*-admissible word* θ *has the form*

$$\begin{aligned}\theta &= ...2212_{2k+1}12_{2k-1}12_{2k}1\alpha_k^4 12_{2k}12_4 \\ &= ...2212_{2k+1}12_{2k-1}12_{2k}12_{2k+1}12_{2k-1}12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_{2k+1}12_{2k-1}12_{2k}12_4....\end{aligned}$$

The discussion on this section can be summarised into the following lemma establishing the self-replication property of ζ_k^1 for all $k \ge 4$:

Lemma 5.42 (Replication Lemma)**.** *For each natural number* $k \ge 4$*, there exists an explicit constant* $\nu_k^{(1)} > m(\zeta_k^1)$ *such that any* $(k, \nu_k^{(1)})$*-admissible word* θ *containing* $\alpha_k^4 := 2_{2k+1}12_{2k-1}12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_{2k+1}12_{2k-1}$ *must extend as*

$$\theta = ...2212_{2k+1}12_{2k-1}12_{2k}12_{2k+1}12_{2k-1}12_{2k}12_{2k+1}12_{2k-1}12_{2k}12_{2k+1}12_{2k-1}12_{2k}12_4...$$

and the neighbourhood of the position $-(6k+3)$ *is*

$$...2_212_{2k+1}12_{2k-1}12_{2k}12_{2k+1}12^*2_{2k-2}12_{2k}12_{2k+1}12_{2k-1}....$$

In particular, any $(k, \nu_k^{(1)})$*-admissible word* θ *containing* α_k^4 *has the form*

$$\overline{2_{2k-1}12_{2k}12_{2k+1}1}2^*2_{2k-2}12_{2k}12_{2k+1}12_{2k-1}12_{2k}12_4$$

Proof. This result for $\nu_k^{(1)} := \min\{\lambda_k^{(i)} : i = 8, ..., 11\}$ is a consequence of Corollaries 5.7, 5.8, 5.9 and 5.10. $\square$

5.6 End of the proof of Theorem 8

By Lemma 5.1 and Proposition 5.1, we have that the Markov values $m(\theta(\underline{\eta}_k)) = \lambda_0(\theta(\underline{\eta}_k))$ and $m(\zeta_k^1) = \lambda_0(\zeta_k^1)$ satisfy $m(\theta(\underline{\eta}_k)) < m(\zeta_k^1) < m(\theta(\underline{\eta}_{k-1}))$ for all $k \geq 3$ and $\lim\limits_{k\to\infty} m(\theta(\underline{\eta}_k)) = 1 + 3/\sqrt{2}$.

Moreover, we affirm that $m(\zeta_k^1) \notin L$ for all $k \geq 4$. Indeed, it follows from Theorems 5.1, 5.2 and Lemma 5.42 that if $\lambda_k := \min\{\lambda_k^{(1)}, \mu_k^{(1)}, \nu_k^{(1)}\}$, then any element $\ell \in L$ with $m(\theta(\underline{\eta}_k)) < \ell < \lambda_k$ would necessarily have the form $\ell = m(\overline{2_{2k-1}12_{2k}12_{2k+1}1}) = m(\theta(\underline{\eta}_k))$, a contradiction. This completes the proof of the desired theorem.

Remark 5.2. For each $k \geq 4$, our arguments above were based on the construction of a *finite* set of k-prohibited and k-avoided strings. In particular, we proved that there is also an explicit constant $\rho_k < m(\theta(\underline{\eta}_k))$ such that the statements of Theorems 5.1, 5.2 and Lemma 5.42 are valid for any word θ with $\rho_k < m(\theta) = \lambda_0(\theta) < \lambda_k$. Thus, an element $\ell \in L$ with $\rho_k < \ell < \lambda_k$ has the form $\ell = m(\overline{2_{2k-1}12_{2k}12_{2k+1}1}) = m(\theta(\underline{\eta}_k))$ and, *a fortiori*, $m(\theta(\underline{\eta}_k))$ is an isolated point of L.

Bibliography

[1] P. Arnoux, *Le codage du flot géodésique sur la surface modulaire*, Enseign. Math. (2) **40** (1994), no. 1-2, 29–48. MR 1279059

[2] E. Bombieri, *Continued fractions and the Markoff tree*, Expo. Math. **25** (2007), no. 3, 187–213. MR 2345177

[3] A. Cerqueira, C. Matheus, and C. G. Moreira, *Continuity of Hausdorff dimension across generic dynamical Lagrange and Markov spectra*, J. Mod. Dyn. **12** (2018), 151–174. MR 3808211

[4] A. Cerqueira, C.G. Moreira, and S. Romaña, *Continuity of Hausdorff Dimension Across Generic Dynamical Lagrange and Markov Spectra II*, arXiv e-prints (2017), arXiv:1711.03851.

[5] T. W. Cusick, *The connection between the Lagrange and Markoff spectra*, Duke Math. J. **42** (1975), no. 3, 507–517. MR 374040

[6] T. W. Cusick and M. E. Flahive, *The Markoff and Lagrange spectra*, Mathematical Surveys and Monographs, vol. 30, American Mathematical Society, Providence, RI, 1989. MR 1010419

[7] H. Davenport and W. M. Schmidt, *Dirichlet's theorem on diophantine approximation*, Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69), Academic Press, London, 1970, pp. 113–132. MR 0272722

[8] G. L. Dirichlet, *Recherches sur les formes quadratiques à coëfficients et à indéterminées complexes. Première partie*, J. Reine Angew. Math. **24** (1842), 291–371. MR 1578330

[9] G. A. Freiman, *Diophantine aproximation and the geometry of numbers(markov's problem).russian*, (1975), 144. MR 0485714

[10] G. A. Freĭman, *Non-coincidence of the spectra of Markov and of Lagrange*, Mat. Zametki **3** (1968), 195–200. MR 227110

[11] ———, *Non-coincidence of the Markov and Lagrange spectra*, Number-theoretic studies in the Markov spectrum and in the structural theory of set addition (Russian), 1973, pp. 10–15, 121–125. MR 0429767

[12] M. E. Gbur, *Accumulation points of the Lagrange and Markov spectra*, Monatsh. Math. **84** (1977), no. 2, 91–108. MR 457361

[13] M. Hall Jr., *On the sum and product of continued fractions*, Ann. of Math. (2) **48** (1947), 966–993. MR 22568

[14] A. Hurwitz, *Ueber die angenäherte Darstellung der Irrationalzahlen durch rationale Brüche*, Math. Ann. **39** (1891), no. 2, 279–284. MR 1510702

[15] S. A. Ibarra, *On the Lagrange and Markov Dynamical Spectra for Anosov Flows in dimension 3*, arXiv e-prints (2016), arXiv:1605.01783.

[16] S. A. Ibarra and C. G. Moreira, *On the Lagrange and Markov Dynamical Spectra for Geodesic Flows in Surfaces with Negative Curvature*, arXiv e-prints (2015), arXiv:1505.05178.

[17] ———, *On the Lagrange and Markov dynamical spectra*, Ergodic Theory Dynam. Systems **37** (2017), no. 5, 1570–1591. MR 3668000

[18] S. Ito, *Number theoretic expansions, algorithms and metrical observations*, Seminar on number theory, 1984–1985 (Talence, 1984/1985), Univ. Bordeaux I, Talence, 1985, pp. Exp. No. 3, 27. MR 848365

[19] H. G. Kopetzky, *Some results on Diophantine approximation related to Dirichlet's theorem*, Number theory (Ulm, 1987), Lecture Notes in Math., vol. 1380, Springer, New York, 1989, pp. 137–149. MR 1009798

[20] D. Lima, C. Matheus, C. G. Moreira, and S. Vieira, *$M\backslash L$ is not closed*, arXiv e-prints (2019), arXiv:1912.02698.

[21] ———, *$M \setminus L$ near 3*, arXiv e-prints (2019), arXiv:1904.00269.

[22] D. Lima and C. G. Moreira, *Phase Transitions on the Markov and Lagrange Dynamical Spectra*, ArXiv e-prints (2018).

[23] A. Markoff, *Sur les formes quadratiques binaires indéfinies*, Mathematische Annalen **15** (1879), no. 3, 381–406.

[24] ———, *Sur les formes quadratiques binaires indéfinies.ii*, Math. Ann. **17** (1880), no. 3, 379–399. MR 1510073

[25] C. Matheus and C. G. Moreira, *Fractal geometry of the complement of Lagrange spectrum in Markov spectrum*, arXiv e-prints (2018), arXiv:1803.01230.

[26] ———, *Markov spectrum near Freiman's isolated points in $M \setminus L$*, J. Number Theory **194** (2019), 390–408. MR 3860483

[27] ———, $\mathrm{HD}(M \setminus L) > 0.353$, Acta Arith. **188** (2019), no. 2, 183–208. MR 3925086

[28] C. G. Moreira, *Sums of regular Cantor sets, dynamics and applications to number theory*, vol. 37, 1998, International Conference on Dimension and Dynamics (Miskolc, 1998), pp. 55–63. MR 1719468

[29] ———, *There are no c1 -stable intersections of regular cantor sets*, Acta Math. **206** (2011), no. 2, 311–323.

[30] ———, *On the minima of Markov and Lagrange Dynamical Spectra*, arXiv e-prints (2017), arXiv:1711.01565.

[31] ———, *Geometric properties of the Markov and Lagrange spectra*, Ann. of Math. (2) **188** (2018), no. 1, 145–170. MR 3815461

[32] C. G. Moreira, M. José Pacifico, and S. Romaña, *Hausdorff Dimension, Lagrange and Markov Dynamical Spectra for Geometric Lorenz Attractors*, arXiv e-prints (2016), arXiv:1611.01174.

[33] C. G. Moreira and J. C. Yoccoz, *Corrigendum: "Stable intersections of regular Cantor sets with large Hausdorff dimensions"*, Ann. of Math. (2) **154** (2001), no. 2, 527. MR 1865980

[34] J. Palis and F. Takens, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*, Cambridge Studies in Advanced Mathematics, vol. 35, Cambridge University Press, Cambridge, 1993, Fractal dimensions and infinitely many attractors. MR 1237641

[35] J. Parkkonen and F. Paulin, *On the closedness of approximation spectra*, J. Théor. Nombres Bordeaux **21** (2009), no. 3, 701–710. MR 2605541

[36] O. Perron, *Über die approximation irrationaler zahlen durch rationale*, Sitzungsb. d. Heidelb. Akad. d. Wiss. Math.-naturw. Kl, no. v. 1, Winter, 1921.

[37] M. Shub, *Global stability of dynamical systems*, Springer-Verlag, New York, 1987, With the collaboration of Albert Fathi and Rémi Langevin, Translated from the French by Joseph Christy. MR 869255