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Some Extensions of the Black-Litterman Model

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Abstract

In 1959, Markowitz developed the mean-variance model. With this model, investors can input expected returns of the assets and generate portfolio weights. In 1960s, capital asset pricing model (CAPM) was established. It is the equilibrium version of mean-variance theory. However, the resulting portfolio via these models sometimes could not make sense to general investors, especially when the investors have certain opinions about the future performance of assets. To fix up this problem, in 1992, Black and Litterman established an outstanding technique that is called the Black-Litterman model. It starts from neutral weights by using CAPM, then it sets up a way to combine the investor's views. Finally, a posterior distribution can be obtained.

By assuming that all the assets are normally distributed, the Black-Litterman model takes advantage of this property and uses Bayes' formula to obtain the posterior distribution. However, the existence of many non-normally distributed markets leads to a lot of criticism. Therefore, the CAPM and the mean-variance model in the more general case lack the strength and volatilities alone cannot represent the risk, more generalized distributions involving skewness and kurtosis are considered to take the place of the normal distribution. Researchers tried to use more generalized distributions to model the markets, such as the t-distribution and the stable distribution. So far, several models have been developed in order to compensate the deficiency of the Black-Litterman model. On the other hand, value-at-risk and conditional value-at-risk are two popular risk measures, and are also introduced to generalize the Black-Litterman model. These risk measures emphasize on the probability of loss in a certain time horizon. In particular CVaR, a coherent risk measure, can be used as an objective function of linear optimization to obtain the optimal portfolio.

Guided by the works of Blasi [10] and Meucci [25], in this dissertation, we extend the Black-Litterman by using the skew normal distribution to characterize the data, and by applying CVaR as an alternative risk measure to obtain the optimal portfolio. In order to illustrate the extended Black-Litterman model (EBL model), we present two examples, one with the data of eight Brazilian stocks and the other one with seven country indices. Finally, we find it very efficient to use CVaR portfolio optimization to allocate assets. In particular, in the case of skewed data, which is an important part of the EBL model, we use the skew normal distribution to fit the data. This extension benefits from the fact that skew normal distributions are the generalizations of normal distributions. Moreover, by changing the shape parameter, the skew normal densities can change continuously to the normal densities. In addition, we provide a similar method with the results in Blasi [10] to model the location parameter and obtain the posterior distribution. To close the dissertation, more types of pick matrices are also discussed in the last chapter.

Key words: Black-Littman model, value-at-risk, conditional value-at-risk, skew normal distribution.

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Chapter 1

Introduction

1.1 Background

Optimal asset allocation is a topic that has been widely investigated in finance and economics. A number of models have been proposed in order to suit the investor's needs in an uncertain environment. In 1952, Markowitz established the mean-variance framework in [22], which also can be seen as the origin of modern portfolio theory (MPT). He suggested that when choosing an investment, one should care about the expected return, as well as the corresponding risk. In general, assets with higher expected returns are riskier. For a given amount of risk, MPT describes how to select a portfolio with the highest possible expected return. Alternatively, for a given expected return, MPT explains how to select a portfolio with the lowest possible risk.

Based on this model, in the 1960s, the capital asset pricing model (CAPM) was developed by Treynor (see French [14]), Sharpe [35], Lintner [21] and Mossin [26] independently. This model is used to determine a theoretically appropriate required rate of return for any risky asset. It works by taking into account the asset's sensitivity to systematic risk, which is denoted by beta (β), as well as the expected return of the market and the expected return of a theoretical risk-free asset. The CAPM is still playing an important role in asset pricing and portfolio selection due to its simplicity and utility in a variety of situations.

In 1991, Black and Litterman [8] introduced a model to combine the market equilibrium with the views of the investor. It is called Black-Litterman model (hereafter, BL model). Specifically, they used the CAPM equilibrium market portfolio as a starting point and 'reverse optimization' to generate a stable distribution of returns. Then they gave a way to specify investors' views and used Bayes' formula to blend them with the prior information to obtain a posterior distribution of the portfolio. Finally, a new optimal portfolio is obtained by using mean-variance approach.

Based on the mean-variance model and the CAPM, the Black-Litterman model naturally possesses all the advantages of these two models. Using the CAPM equilibrium market portfolio as a starting point, the Black-Litterman model provides an intuitive reference model. Another significant improvement is to use 'reverse optimization'. Besides, using Bayes' theory to combine the views with the prior is very direct and clear. Views, in this model, are allowed to be relative and absolute. That is, a relative view is about comparing two or some of the assets in the portfolio, such as a spread or a butterfly, whereas an absolute view is to compare a certain asset with the benchmark.

However, as in the mean-variance model and the CAPM, the Black-Litterman model is based on many assumptions, such as the assumptions of normality to value risk, etc, and hence, questioned by many practitioners. To quantify the views involves a lot of uncertainties, therefore the model can generate errors. The parameters in this model, such as risk aversion λ, τ, and Ω are still under discussion. Critics also doubt that this model depends too much on the input data and may result in a useless output. Therefore, many researchers extended the model to be more general and suitable for more types of portfolios.

Firstly, one extension is to deal with some non-normally distributed markets in reality. Giacometti, Bertocchi, Rachev and Fabozzi [15] considered to use the t-distribution or the stable distribution to model the market. Xiao and Valdez [38]) considered the case when returns in the market fall within the class of the elliptical distribution. Blasi [11] gave an example of a very simple volatility trading strategy producing skew normal returns and provided the optimization problem to embed the Black-Litterman model in the skew normal market case. These generations are more practical to model scenarios with fat tails or skewness.

Another extension is to use newly defined measures of risk, such as value-at-risk (VaR) and conditional value-at-risk (CVaR or expected shortfall). Accordingly, in the work of Giacometti, Bertocchi, Rachev and Fabozzi [15], they established a frame work by applying the t and the stable distribution for asset returns and by using value-at-risk and conditional value-at-risk. Meucci [23] used copula-opinion pooling (COP) approach to extend the Black-Litterman methodology to non-normally distributed markets and views. Correspondingly, Meucci [23] minimized the CVaR value subject to the constraint of a minimum target expected return.

Finally, many researchers have investigated to embed other models in the Black-Litterman model. Beach and Orlov [6] used GARCH-derived views as an input into the Black-Litterman model. Fabozzi, Focardi and Kolm [13] used cross-sectional momentum strategy as an input view and combine the strategy with the Black-Litterman model. Instead of using Bayes' formula to combine the prior distribution with views of investors, Meucci [23] implemented copula-opinion pooling approach, as well as minimizing CVaR to obtain optimal portfolio.

1.2 Outline

The first chapter covers the background of the classical Black-Litterman model as well as some related extension works. In Chapter 2, we describe all the details of this model. Then the extended Black-Litterman model will be presented in the following chapters. Chapter 3 explains two popular risk measures, VaR and CVaR. For a certain target expected return, we can obtain the optimal portfolio by minimizing CVaR. We give an example to illustrate different efficient frontiers of $\text{CVaR}_{0.9}$, $\text{CVaR}_{0.95}$ and $\text{CVaR}_{0.99}$. In Chapter 4, we move to the skew normal distribution and describe the methods of generalizing the Black-Litterman model to skew normal version. In addition, we provide another method to estimate the location parameter L. Chapter 5 serves two purposes, we will use the data of 8 stocks in Brazilian stock market to illustrate how our model works and compare the results with those using classical Black-Litterman model. Finally, we will continue with the example in Chapter 3, and talk about another type of view.

Chapter 2

The Black-Litterman Model

Prior to the Black-Litterman model, portfolio optimization only takes the expectations and the covariances of a set of assets as inputs and computes from a given reference econometric model. The outstanding technique by Black and Litterman established a new way in which the investors could combine their ideas with the CAPM and the mean-variance model.

In this chapter, we will report the details of the classical Black-Litterman model and explain how it works. For more details, readers are referred to the website http://www.blacklitterman.org/cookbook.html.

2.1 The Mean-Variance Model and CAPM

Let P_t be the price of an asset at time t. Holding the asset for one period from time $t-1$ to time t, the *simple gross return* is defined by

$$R_t + 1 = \frac{P_t}{P_{t-1}}. \tag{2.1.1}$$

The corresponding *simple net return* or *simple return* is

$$R_t = \frac{P_t}{P_{t-1}} - 1. \tag{2.1.2}$$

More generally, if we hold the asset from time $t-k$ to time t, the *multi-period simple return* is

$$R_t(k) = \frac{P_t - P_{t-k}}{P_{t-k}} = \prod_{j=0}^{k-1}(1 + R_{t-j}) - 1. \tag{2.1.3}$$

The natural logarithm of the simple gross return of an asset is called the *continuously compounded return* or *log return*

$$r_t = \ln(1 + R_t) = \ln \frac{P_t}{P_{t-1}}. \tag{2.1.4}$$

For *multi-period compounded return*, it is not difficult to prove that

$$r_t(k) = \ln(1 + R_t(k)) = r_t + r_{t-1} + \cdots + r_{t-k+1}. \tag{2.1.5}$$

For a portfolio with N assets, the simple net return of the portfolio R_P is a weighted average of the simple net return of the assets involved, where the weight on each asset is the percentage of the portfolio's value invested in that asset. Let $w = (w_1, w_2, \ldots, w_N)'$ be the weights of a portfolio with N assets and expected returns $R_1, R_2, \ldots, R_N$. Then the simple return of the portfolio is

$$R_P = \sum_{i=1}^{N} w_i R_i = w'R, \tag{2.1.6}$$

where $R = (R_1, R_2, \ldots, R_N)'$. The continuously compounded returns of a portfolio, however, do not have this property. If the simple returns R_i are all small in magnitude, then we have

$$\sum_{i=1}^{N} w_i r_i \approx \sum_{i=1}^{N} w_i R_i. \tag{2.1.7}$$

If we consider rates of return to be random variables, then N assets with random rates of return R_i, $i = 1, \ldots, N$ have expected returns $E(R_i)$, $i = 1, \ldots, N$. The expected return of portfolio is

$$E(R_P) = \sum_{i=1}^{N} w_i E(R_i) = w'E(R), \tag{2.1.8}$$

where $E(R) = (E(R_1), E(R_2), \ldots, E(R_N))'$. Return is one of important characteristics of a portfolio. Another one is the *risk*, defined as the unexpected variability of asset prices and/or earnings. In the mean-variance framework, Markowitz defined risk as the variance of the return. Since variance measures the deviation around the expected value, in the same way, the variance of return measures the deviation of the return around the expected return. For a portfolio with N assets, and the i-th asset with variance σ_i^2, the variance of a portfolio is

$$\sigma^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j<1} w_i w_j cov(R_i, R_j) = w'\Sigma w. \tag{2.1.9}$$

where $cov(R_i, R_j)$ is the covariance between the i-th and the j-th asset, Σ is the covariance matrix. For more details on returns and variance, see Tsay [37].

Now we present the *mean-variance framework* pioneered by Markowitz in 1952. Under the assumptions of modern portfolio theory, investors are risk a-verse, meaning that given two portfolios that offer the same expected return, investors will prefer the less risky one. When investors take on increased risk, the corresponding returns should be higher, too. Conversely, if investors want higher expected returns, the risk is higher. With a target expected return R_P, the standard mean-variance optimization problem can be formulated as follows:

$$\begin{aligned} \min_{w\in\mathbb{R}^n}\ & \frac{1}{2}w'\Sigma w \\ s.t.\ & w'E(R) = R_P \\ & \sum_{i=1}^{N} w_i = 1, \end{aligned} \tag{2.1.10}$$

The solution of Problem (2.1.10) can be found by using Lagrange method and (2.1.10) becomes:

$$\min_{w\in\mathbb{R}^n} \frac{1}{2}w'\Sigma w - \lambda w'E(R) \quad \text{for any } \lambda \geq 0,\ \sum_{i=1}^{N} w_i = 1, \tag{2.1.11}$$

where λ is often called the *risk-aversion parameter*. By solving the optimization problem, we have the optimal weights for the portfolio:

$$w^* = (\lambda\Sigma)^{-1}E(R). \tag{2.1.12}$$

In Problem (2.1.10), by changing different target expected returns R_P, we can obtain corresponding weights. We plot all the possible combinations of the assets in risk-expected return space, and obtain a hyperbola. The upper edge of this region is the *efficient frontier*.

Built on the modern portfolio theory, the capital asset pricing model (CAPM) was introduced in 1960s. *Risk-free rate* R_f is the theoretical rate of return of an investment with no risk of financial loss. Together with the risk of the market $E(R_M)$, this model derives the theoretical required expected return for an asset. The CAPM is usually expressed as:

$$E(R_i) = R_f + \beta_i[E(R_M) - R_f], \tag{2.1.13}$$

where $E(R_i)$ is the expected return on asset i, R_f is the risk-free rate of interest, $E(R_M)$ is the expected return of the market and β_i is the sensitivity of the expected excess asset returns to the expected excess market returns with

$$\beta_i = \frac{cov(R_i, R_M)}{\sigma_M^2}.$$

Generally, we call $E(R_M) - R_f$ the *market premium* and $E(R_i) - R_f$ the *risk premium* and the above formula can be written as:

$$E(R_i) - R_f = \beta_i[E(R_M) - R_f]. \tag{2.1.14}$$

2.2 The Market Model

Now we start to describe the Black-Litterman model. Consider a market of N securities or asset classes, whose returns $X = (X_1, X_2, \ldots, X_N)$ have a multivariate normal distribution:

$$X \sim N(\mu, \Sigma), \tag{2.2.1}$$

where μ is an N-dimensional vector and represents the expected outcome of returns and Σ is the covariance matrix. Here we define μ as a random variable distributed as

$$\mu \sim N(\mu_0, \Sigma_0). \tag{2.2.2}$$

Σ_0 can be also set as $\tau\Sigma$, with $0 < \tau < 1$, which denotes the uncertainty on μ_0. We can therefore rewrite the reference model X as $X = \mu + Z$, where $Z \sim N(0, \Sigma)$.

The Black-Litterman model uses 'equilibrium' returns, derived from the CAP-M, as a neutral starting point. By this model and from (2.1.14), the expected return on asset i is:

$$E(R_i) - R_f = \beta_i[E(R_M) - R_f], \tag{2.2.3}$$

where $E(R_i)$, $E(R_M)$ and R_f are the expected return on asset i, the expected return on the market portfolio, and the risk-free rate, respectively. Furthermore, let σ_M^2 denote the variance of the market. We have

$$\beta_i = \frac{cov(R_i, R_M)}{\sigma_M^2}. \tag{2.2.4}$$

This representation identifies the appropriate level of risk for which an investor should be compensated. Let us denote by $w_M = (w_1, w_2, \ldots, w_N)'$ the market capitalization or benchmark weights, so that with an asset universe of N securities the return of the market can be written as

$$R_M = \sum_{j=1}^{N} w_j R_j. \tag{2.2.5}$$

Then the expected excess return on asset i becomes

$$\begin{aligned} E(R_i) - R_f &= \beta_i [E(R_M) - R_f] \\ &= \frac{cov(R_i, R_M)}{\sigma_M^2} [E(R_M) - R_f] \\ &= \frac{E(R_M) - R_f}{\sigma_M^2} \sum_{j=1}^{N} cov(R_i, R_j) w_j. \end{aligned}$$

Hence, we have the vector form as

$$\mu_0 = (E(R_1) - R_f, E(R_2) - R_f, \ldots, E(R_N) - R_f) = \lambda \Sigma w_M, \tag{2.2.6}$$

where

$$\lambda = \frac{E(R_M) - R_f}{\sigma_M^2}$$

and

$$\Sigma = \begin{pmatrix} cov(R_1, R_1) & \cdots & cov(R_1, R_N) \\ \vdots & & \vdots \\ cov(R_N, R_1) & \cdots & cov(R_N, R_N) \end{pmatrix}$$

On the other hand, by minimizing the mean-variance utility function, or Equation (2.1.12), we have

$$w^* = (\lambda \Sigma)^{-1} \mu. \tag{2.2.7}$$

Note that if μ does not equal μ_0, w^* will not equal w_M. So if we have additional information of the portfolio and we want to update the expected return, and hence, we will get new weights for the portfolio.

2.3 The Views

A *view* is a statement on the market that can potentially clash with the reference market model. In the BL model, we only consider views on expectations. A *relative view* means comparing with another asset, one asset will profit or not. An *absolute view* is comparing with the benchmark or the whole portfolio, the asset will profit or not. The sum of the weights will be 0 or 1, respectively. For instance, the portfolio manager might say that the third asset will outperform the second, in which case the view is $r_3 - r_2 \geq 0$, where r_2 and r_3 denote the returns of asset 2 and 3. Note that, we do not require a view on all assets and it is possible for the views to conflict.

Suppose we have K views on the assets. Let $P_{K \times N}$ be a pick matrix to represent the investors views on assets. For every row of P, the sum of the weights will be 0 if we have a relative view, whereas the sum of the weights will be 1 if we have an absolute view. Denote v, a $K \times 1$ matrix, as the returns for each view. Let Ω be a $K \times K$ matrix, which represents the covariance of the views. By construction we can require each view to be unique and uncorrelated with the other views, and Ω is generally diagonal. In order to associate uncertainty with the view, we suppose the views follow a normal model:

$$P\mu \sim N(v, \Omega). \tag{2.3.1}$$

As for (2.3.1), we write it as

$$v = P\mu + Z \tag{2.3.2}$$

where $Z \sim N(0, \Omega)$. Therefore, we can model v as a random variable V whose distribution, conditioned on the realization of μ is:

$$V|\mu \sim N(P\mu, \Omega). \tag{2.3.3}$$

For example, we have 4 assets and 2 views. Our first view is a relative view in which the investor believes that asset 1 will outperform asset 3 and 4 by 2% with confidence ω_{11}. The second view is an absolute view in which the investor believes that asset 2 will have return 3% with confidence ω_{22}. These views are specified as follows:

$$P = \begin{pmatrix} 1 & 0 & -1/2 & -1/2 \\ 0 & 1 & 0 & 0 \end{pmatrix}, V = \begin{pmatrix} 2\% \\ 3\% \end{pmatrix}, \Omega = \begin{pmatrix} \omega_{11} & 0 \\ 0 & \omega_{22} \end{pmatrix}.$$

In the work of Fabozzi, Focardi and Kolm [13], they suggested a way to determine the confidence level matrix Ω. Alternatively, in Meucci [23], it is also convenient to set

$$\Omega = \frac{1}{c} P \Sigma P'$$

where $c \in (0, \infty)$ represents the level of confidence in the views. On one extreme, $c \to 0$ means no confidence. On the other extreme $c \to \infty$, i.e., $\Omega \to 0$, the confidence in the views V is full.

2.4 The Posterior

To avoid the confusion of symbols, we use the symbols in the Black-Litterman model to illustrate the prior and posterior distributions. Let μ and V be two events, in probability theory, we have

$$f_{\mu|V}(\mu) = \frac{f_{\mu,V}(\mu, v)}{f_V(v)} \text{ and } f_{V|\mu}(v) = \frac{f_{\mu,V}(\mu, v)}{f_\mu(\mu)},$$

hence,

$$f_{\mu,V}(\mu,v) = f_{\mu|V}(\mu)f_V(v) = f_{V|\mu}(v)f_\mu(\mu)$$

and

$$f_{\mu|V}(\mu) = \frac{f_{V|\mu}(v)f_\mu(\mu)}{f_V(v)}. \tag{2.4.1}$$

In Bayesian statistics, V is the observed data and μ are the parameters, $f_\mu(\mu)$ is the *prior distribution* of the parameters, $f_{\mu|V}(\mu|v)$ is the *posterior distribution* and $f_{V|\mu}(v|\mu)$ is called *likelihood function*. More intuitively, the posterior probability of a random event is the conditional probability that is assigned after the relevant evidence is taken into account. If we begin with a prior probability of an event, and we want to revise it in the light of new data, the posterior probability is the probability of the event computed following the collection of new data. It can be written in the form as

$$\text{posterior probability} \propto \text{prior probability} \times \text{likelihood}.$$

where $\propto$ means the 'posterior probability' is proportional to the 'prior probability' multiplying 'likelihood'.

Back to the Black-Litterman model, from (2.2.2), (2.3.3) and (2.4.1), we can show that the distribution of μ given the views using Bayes' formula is:

$$\mu|V;\Omega \sim N(\mu_{BL}, \Sigma^\mu_{BL}), \tag{2.4.2}$$

where

$$\begin{aligned}
\mu_{BL} &= [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\mu_0 + P'\Omega^{-1}V] && (2.4.3)\\
&= \mu_0 + \tau\Sigma P'(\tau P\Sigma P' + \Omega)^{-1}(V - P\mu_0) && (2.4.4)\\
\Sigma^\mu_{BL} &= [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}. && (2.4.5)
\end{aligned}$$

By the relation of X and μ as we assumed in Section 2.2, we have

$$X|V;\Omega \sim N(\mu_{BL}, \Sigma_{BL}), \tag{2.4.6}$$

where

$$\begin{aligned}
\mu_{BL} &= [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\mu_0 + P'\Omega^{-1}V] && (2.4.7)\\
&= \mu_0 + \tau\Sigma P'(\tau P\Sigma P' + \Omega)^{-1}(V - P\mu_0) && (2.4.8)\\
\Sigma_{BL} &= \Sigma + \Sigma^\mu_{BL} && (2.4.9)\\
&= (1+\tau)\Sigma - \tau^2\Sigma P'(\tau P\Sigma P' + \Omega)^{-1}P\Sigma. && (2.4.10)
\end{aligned}$$

Fabozzi, Focardi and Kolm [13] gave another way to interpret the result as a 'confidence weighted' linear combination of market equilibrium μ_0 and the expected return $\hat{\mu}$, which is given by $P'(P'P)^{-1}V$. Then μ_{BL} can be rewritten as

$$\begin{aligned} w_{\mu_0} &= [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}(\tau\Sigma)^{-1} \\ w_v &= [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}P'\Omega^{-1}P \end{aligned}$$

Note that $w_{\mu_0} + w_v = I$. In particular, $(\tau\Sigma)^{-1}$ and $P'\Omega^{-1}P$ represent the confidence we have in our estimates of the market equilibrium and the views, respectively. In particular, if our confidence is low in the views, the resulting expected returns will be close to the ones implied by market equilibrium. Conversely, we can also make the resulting expected return close to our views if we have full confidence.

2.5 The Allocation

After blending our views with the prior, we have the posterior distribution (2.4.6), with corresponding new expected return μ_{BL} and covariance matrix Σ_{BL}. Using the same technique as in Section 2.1, Equations (2.1.10) and (2.1.11), the mean-variance optimization problem with regard to μ_{BL} and Σ_{BL} can be written as

$$\min_{w_{BL}} \frac{1}{2} w'_{BL}\Sigma_{BL}w_{BL} - \lambda w'_{BL}\mu_{BL}. \tag{2.5.1}$$

$$s.t. \sum w_{BL} = 1. \tag{2.5.2}$$

The weights of the new optimal portfolio are the solution of this problem. Namely,

$$\begin{aligned} w^*_{BL} &= \arg\min_{w_{BL}}\{(1/2)w'_{BL}\Sigma_{BL}w_{BL} - \lambda w'_{BL}\mu_{BL}\} & (2.5.3) \\ &= (\lambda\Sigma_{BL})^{-1}\mu_{BL}. & (2.5.4) \end{aligned}$$

Chapter 3

Value-at-Risk and Conditional Value-at-Risk

In the classical Black-Litterman model, risk is measured by the standard deviation of unexpected outcomes, also called volatility. However, the use of the standard deviation as a risk measurement is not appropriate for non-normal distributions. Since the shape of the underlying return density function is not symmetrical, the standard deviation does not capture the appropriate probability of obtaining undesirable return outcomes. Whereas value-at-risk (VaR) captures the combined effect of underlying volatility and exposure to financial risks. Conditional value-at-risk (CVaR), introduced by Rockafellar and Uryasev [30], possesses the same goal and much better properties whereby it is used to generalize the BL model for non-normally distributed market. Sarykalin, Serraino and Uryasev [32] presented some experience working with VaR and CVaR. In addition, they explained strong and weak features of these risk measures and gave several examples. Pflug [29] studied the structure of the portfolio optimization problem using the VaR and C-VaR as objective functions. In the work of Krokhmal, Palmquist and Uryasev [20], they extended to use CVaR as constraints to maximize expected returns. In this chapter, we firstly focus on some properties of VaR, then we turn to conditional value at risk, CVaR. At last, we will describe some methods of estimating CVaR and use CVaR as an objective function in our portfolio optimization problem. In the end of this chapter, we will use an example with 7 country funds to illustrate the efficient frontier in return-CVaR space.

3.1 Value-at-Risk

Value-at-Risk (or *VaR* in short) is defined as the maximum loss expected on a portfolio of assets over a certain holding period at a given confidence level. It is currently the most popular measure for financial risk management. In particular, VaR is important in back-testing and stress testing. See Jorion [19] and Sarykalin, Serraino and Uryasev [32] for more references. Conventionally, VaR is reported as a positive number to represent a loss, whereas a negative VaR would imply a probability of making a profit. In Section 3.1 and 3.2, we temporarily use X to represent the loss, the negative of return, and this would impact the sign of functions in the definition of VaR and CVaR. Formally, the VaR with confidence $\beta \in (0,1)$ of a random variable X is:

$$\mathrm{VaR}_\beta(X) = \inf\{x \in \mathbb{R} : F_X(x) \geq \beta\} \tag{3.1.1}$$

where $F_X(x) = P(X \leq x)$ is the cumulative distribution function of X. In addition, suppose $F^{-1}(\beta)$ is the left continuous inverse of $F(x)$, such that $F^{-1}(\beta) = \min\{x : F(x) \geq \beta\}$. So VaR_β is also the β-quantile, i.e.,

$$\mathrm{VaR}_\beta(X) = F^{-1}(\beta). \tag{3.1.2}$$

For example, if the VaR on a portfolio of stocks is \$1 million at one-day, 95% confidence level, there is a 5% probability that the portfolio will fall in value by more than \$1 million over a one day period if there is no trading.

By the definition of VaR, using VaR as a risk constraint is equivalent to the chance constraint on probabilities of losses. However, VaR is not a good measure of risks in some extreme cases.

Suppose we have two assets X_1 and X_2, which have identical and independent distributions of losses as follows:

$$\begin{aligned} P(100000) &= 0.03 \\ P(10) &= 0.03 \\ P(0) &= 0.94 \end{aligned}$$

We can see that $\mathrm{VaR}_{0.95}(X_1) = \mathrm{VaR}_{0.95}(X_2) = 10$. However, the loss of the portfolio $Y = X_1 + X_2$ has the probability distribution:

$$\begin{aligned} P(200000) &= 0.0009 \\ P(100010) &= 0.0018 \\ P(100000) &= 0.0564 \\ P(20) &= 0.0009 \\ P(10) &= 0.0564 \\ P(0) &= 0.8836 \end{aligned}$$

With the same confidence level, $\mathrm{VaR}_{0.95}(Y) = 100000$. Therefore, we have $\mathrm{VaR}_{0.95}(X_1) + \mathrm{VaR}_{0.95}(X_2) < \mathrm{VaR}_{0.95}(Y)$. This is somewhat opposite with our intuition, because we always try to estimate the risk of a portfolio by summing up all the risk value of its assets. The problem is that VaR violates the sub-additive property, which is a very important part in the definition of coherent risk measure:

A function ρ of a bounded random variable X is a *coherent risk measure* if it satisfies the following conditions:

- $\rho(C) = C$ for all constant C;
- monotonicity: If $Y \geq X$, then $\rho(Y) \leq \rho(X)$;
- translation invariance: if $c \in \mathbb{R}$, then $\rho(X + c) = \rho(X) + c$;
- sub-additivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$;
- positive homogeneity: if $\lambda \geq 0$, then $\rho(\lambda X) = \lambda\rho(X)$.

In the case of VaR, it does not satisfy the sub-additive property, neither it is a continuous or convex function. Therefore, when we use VaR as a constraint to find the optimal portfolio, the feasible region of the optimization problem may be non-convex, hence, there would be computational difficulties.

On the other hand, VaR ignores the lost beyond the value at risk level. For example, an asset may significantly increase the largest loss exceeding VaR, with the VaR not changing. This property can be useful if we also want to disregard some outliers and large losses, however, it may be un undesirable property when we do care about some rare losses.

3.2 Conditional Value-at-Risk

In order to deal with the conceptual problems causes by VaR, Rockafellar and Uryasev [30] and [31] introduced a new measure of financial risk referred to as the *conditional value-at-risk (CVaR)*. It can be also referred to as expected shortfall, tail conditional expectation, conditional loss, or expected tail loss. For random variables with continuous distribution functions, $\text{CVaR}_\beta(X)$ equals the conditional expectations of X with constraints $X \geq \text{VaR}_\beta(X)$. That is,

$$\text{CVaR}_\beta(X) = E[X | X \geq \text{VaR}_\beta(X)]. \tag{3.2.1}$$

In Figure 2.1, suppose the red dot represents VaR_β for some β, then CVaR_β must lie on the right side of VaR_β. We use the blue dot in the figure to approximately represent CVaR_β. For general definition of CVaR with confidence level $\beta \in (0, 1)$,

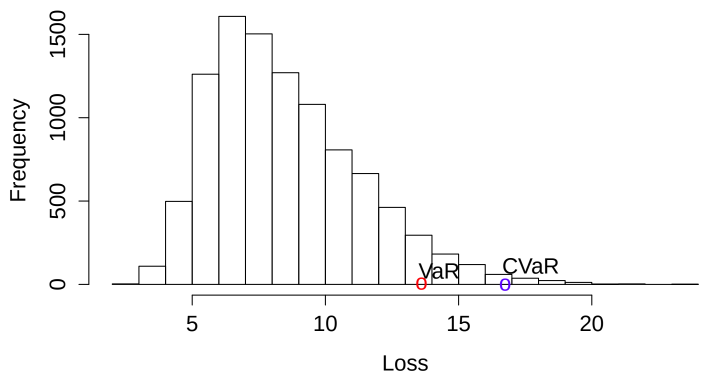

Figure 2.1: Graphical representation of VaR and CVaR

see Rockafellar and Uryasev [31], it is the mean of the generalized distribution,

$$\mathrm{CVaR}_\beta(X) = \int_{-\infty}^{\infty} z dF_X^\beta(z), \tag{3.2.2}$$

where

$$F_X^\beta(z) = \begin{cases} 0, & \text{when } z < \mathrm{VaR}_\beta(X); \\ \frac{F_X(z)-\beta}{1-\beta}, & \text{when } z \geq \mathrm{VaR}_\beta(X). \end{cases}$$

To illustrate this general definition, we borrow the example in Sarykalin, Serrain and Uryasev [32]. Suppose we have six equally likely scenarios with losses $f_1 < f_2 < \cdots < f_6$ and $P(f_1) = P(f_2) = \cdots = P(f_6) = \frac{1}{6}$. Let $\beta = \frac{2}{3}$. Then $\mathrm{VaR}_{\frac{2}{3}}(X) = f_4$ and $\mathrm{CVaR}_{\frac{2}{3}} = \frac{1}{2}f_5 + \frac{1}{2}f_6$. Now, let $\beta = \frac{7}{12}$. In this case, we have $\mathrm{VaR}_{\frac{7}{12}}(X) = f_4$. and

$$F_X^{\frac{7}{12}}(z) = \begin{cases} 0, & z < f_4; \\ \frac{1}{5}, & z = f_4; \\ \frac{3}{5}, & z = f_5; \\ 1, & z = f_6. \end{cases}$$

So $\mathrm{CVaR}_{\frac{7}{12}}(X) = \frac{1}{5}f_4 + \frac{2}{5}f_5 + \frac{2}{5}f_6$.

Acerbi [1] showed that CVaR can also be defined in an equivalent way as expected shortfall:

$$\mathrm{CVaR}_\beta(X) = \frac{1}{\beta}\int_0^\beta \mathrm{VaR}_p(X)dp. \tag{3.2.3}$$

Comparing with VaR, CVaR is a coherent risk measure. It is also a continuous convex function. Furthermore, from different types of definitions, CVaR approximately equals the average of a certain percentage of the worst-case loss scenarios. Obviously, with the same confidence level, CVaR is larger than VaR. However, only when the model on the tails are correct, CVaR can provide more information of risks reflected in extreme tails.

For a normal distribution $X \sim N(\mu, \sigma^2)$, a CVaR deviation is proportional to the standard deviation, see Rockafellar and Uryasev [30]. Namely,

$$\mathrm{CVaR}_\beta(X) = \mu + k(\beta)\sigma, \tag{3.2.4}$$

where $k(\beta) = (\sqrt{2\pi}\exp(g^{-1}(2\beta - 1))^2(1-\beta))^{-1}$ and $g(z) = (2/\sqrt{\pi})\int_0^z e^{-t^2}dt$.

For a skew normal distribution with the following density function,

$$f_X(x) = \frac{2}{\sigma}\phi(\frac{x-\mu}{\sigma})\Phi(\alpha\frac{x-\mu}{\sigma}), \tag{3.2.5}$$

for $x \in \mathbb{R}$, $\alpha \in \mathbb{R}$ and $\sigma > 0$, where $\phi(x)$ is the standard normal pdf and $\Phi(x)$ is the standard normal cdf. The cdf for this skew normal distribution is

$$F_X(x) = \Phi(\frac{x-\mu}{\sigma}) - 2T(\frac{x-\mu}{\sigma}, \alpha), \tag{3.2.6}$$

where

$$T(h, a) = \frac{1}{2\pi}\int_0^a \frac{\exp\{-h^2(1+x^2)/2\}}{1+x^2}dx \tag{3.2.7}$$

is Owen's T function, see Owen [28]. Bernardi [7] proved that the $\mathrm{CVaR}_\beta(X)$ for the skew normal distribution is given by

$$\mathrm{CVaR}_\beta(X) = \mu + \frac{\sigma\sqrt{2}}{\alpha\sqrt{\pi}}[\tilde{\alpha}\Phi(z_\beta) - \sqrt{2\pi}\varphi(y_\beta)\Phi(\alpha y_\beta)], \tag{3.2.8}$$

where $\tilde{\alpha} = \alpha/\sqrt{1+\alpha}$, $z_\beta = \sqrt{1+\alpha^2}y_\beta$, $y_\beta = (x_\beta - \mu)/\sigma$ and x_β satisfies $F_X(x_\beta) = \beta$.

However, it is not efficient to minimize Equation (3.2.8) and get the optimal portfolio. Hence, we still try to use some nonparametric methods.

Pflug [29] and Rockafellar and Uryasev [30] defines CVaR via an optimization problem, as

$$\mathrm{CVaR}_\beta(X) = \min_C\{C + \frac{1}{1-\beta}E[\max\{X - C, 0\}]\}. \tag{3.2.9}$$

We will report the proof of the equivalence of Equation (3.2.1) and Equation (3.2.9) in the Appendix.

3.3 Portfolio Optimization and CVaR

Suppose we have a portfolio with N assets and weight vector $w = (w_1, w_2, \ldots, w_N)'$. Let $X = (X_1, \ldots, X_N)'$ be the return of the assets of the portfolio. Then we have the return $R(w, X)$ of the portfolio as a function of w and X. Based on the discussions in Section 3.1 and 3.2, let $\mathrm{VaR}_\beta(w, -X)$ and $\mathrm{CVaR}_\beta(w, -X)$ be the value-at-risk and conditional value-at-risk of the portfolio. In the paper by Rockafellar and Uryasev [30], they considered minimizing CVaR, with a given expected return. In the work of Krokhmal, Palmquist and Uryasev [20], they proved that the following three formulations of the optimization problem are equivalent in a general setting:

$$\min_w \quad \mathrm{CVaR}_\beta(w, -X) - aR(w, X) \ s.t. \ w \in W, a \geq 0 \tag{3.3.1}$$

$$\min_w \quad \mathrm{CVaR}_\beta(w, -X) \ s.t. \ R(w, X) \geq r, w \in W \tag{3.3.2}$$

$$\min_w \quad -R(w.X) \ s.t. \ \mathrm{CVaR}_\beta(w, -X) \leq C, w \in W \tag{3.3.3}$$

where the set W is constraints of w. For example, if we restrict the portfolio to be full-investment and long-only, $W = \{w : w \geq 0, \sum_i w_i = 1\}$, whereas if short selling is allowed, and the long-short positions offset, $W = \{w : \sum_i w_i = 0\}$. They are equivalent in the sense that they produce the same efficient frontier.

In this work, however, we only consider the optimization problem as (3.3.2), long-only and the sum of the weights is 1. Let $R_{S\times N} = (R_1, R_2, \ldots, R_S)$ be the panel data with S simulated joint scenarios of returns and R_i, $i = 1, 2, \ldots, S$ be vectors of N-dimension. $\hat{R}_{1\times N} = (\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_N)'$ is a vector of mean values of all the assets, or more generally, the expected returns of the assets. Denote $d_i = \max\{-w'R_i - \mathrm{VaR}_\beta(w, -X), 0\}$, $i = 1, 2, \ldots, S$ and $R_{\min}$ is the target return. By Equation (3.2.9), we report the portfolio optimization problem here with more details:

$$\begin{aligned}
&\min_w \ \mathrm{VaR}_\beta(w, -X) + \frac{1}{S(1-\beta)} \sum_{n=1}^{S} \max\{-w'R_n - \mathrm{VaR}_\beta(w, -X), 0\} \\
&s.t. \ d_i \geq -w'R_i - \mathrm{VaR}_\beta(w, -X), i = 1, 2, \ldots, S \\
&\quad\ \ w'\hat{R}_{1\times N} \geq R_{\min} \\
&\quad\ \ \sum_{j}^{N} w_j = 1 \\
&\quad\ \ w_j \geq 0, j = 1, 2, \ldots, N. \\
&\quad\ \ d_i \geq 0, i = 1, 2, \ldots, S
\end{aligned} \tag{3.3.4}$$

Let $d = \mathrm{VaR}_\beta(w, -X)$. We can solve the optimization problem (3.3.4) by means of linear programming as follows:

$$\begin{aligned} \min_x : \quad & c'x \\ s.t. : \quad & Ax \geq b \\ & w_i \geq 0, i = 1, 2, \dots, N \\ & d_j \geq 0, j = 1, 2, \dots, S \end{aligned} \tag{3.3.5}$$

where

$$c' = (0, 0, \dots, 0, \frac{1}{(1-\beta)S}, \dots, \frac{1}{(1-\beta)S}, 1)$$

$$x' = (w_1, w_2, \dots, w_N, d_1, d_2, \dots, d_S, d)$$

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ \hat{r_1} & \hat{r_2} & \cdots & \hat{r_N} & 0 & \cdots & 0 & 0 \\ r_{11} & r_{12} & \cdots & r_{1N} & 1 & 0 & \cdots & 1 \\ r_{21} & r_{22} & \cdots & r_{2N} & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{S1} & r_{S2} & \cdots & r_{SN} & 0 & \cdots & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ R_{\min} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

By solving this problem, we can get the optimal vector x^*, and the corresponding VaR_β and CVaR_β are d and $c'x^*$, respectively. The optimal vector of weights is the first N entries of x^*. By changing different minimum returns, and solving the corresponding optimal CVaR_β, we can draw a return-CVaR efficient frontier of the portfolio.

3.4 An Example

In this section we will give an example of return-CVaR efficient frontier and also compare with the return-variance efficient frontier.

Suppose our portfolio contains 7 single country funds in international equity market. Specifically, they are EWA (Australia), EWC (Canada), EWQ (France), EWG (Germany), EWJ (Japan), EWU (U.K.), SPY (U.S.A.). This example of portfolio is widely used in illustrating the Black-Litterman model and can be found in He and Litterman [16] and a website http://www.r-bloggers.com/black-litterman-model/. We use the historical prices series from 1996-2014 and calculate simple monthly returns. The data can be found from Yahoo Finance.

We set $\beta = 0.9$, 0.95 and 0.99 and plot the efficient frontiers, respectively. Figure 4.2 shows the CVaR-return efficient frontiers. Especially, Figure 4.3 shows

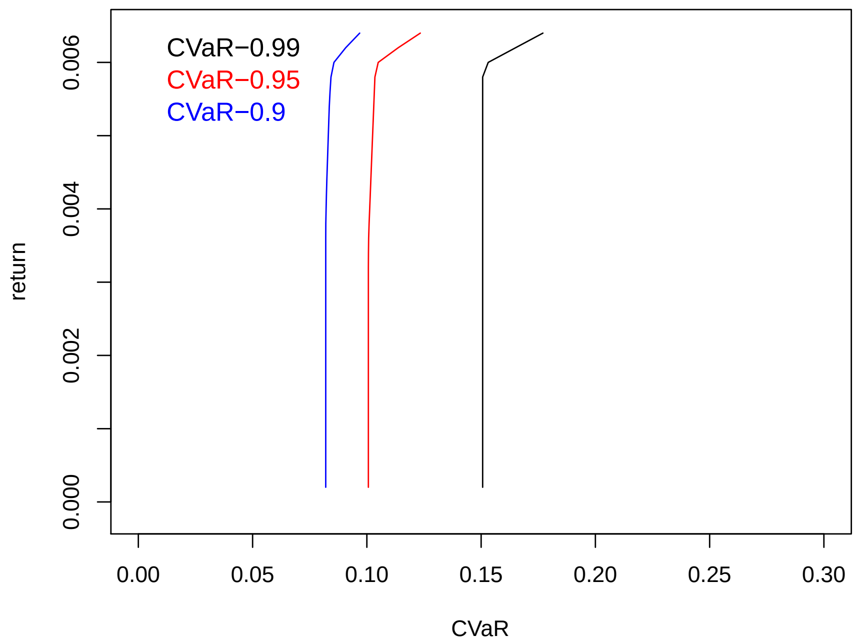

Figure 4.2: The CVaR-return efficient frontier with $\beta = 0.9$, 0.95 and 0.99.

the case of $\beta = 0.95$. It can be seen that with different β, the positions of efficient frontiers vary a lot. When β is bigger, the target CVaR value is bigger and the efficient frontiers moves to the right. Some parts we want to underlines are when the target returns are small, CVaR value changes only a little with the target returns. We also compute the return-variance efficient frontier, as is shown in Figure 4.4. Comparing with Figure 4.4, different with the relation of mean and variance, the CVaR efficient frontier always increases when CVaR increases. Finally, as will be seen in the next chapter, we will fit a skew normal distribution to the data. In Figure 4.5, we restrict the returns larger than 0.004. We find that using the normal and the skew normal distributions to fit the data can generate different CVaR. Since the historical data are quite limited, when expected return is larger, the historical CVaR is smaller than the two simulated models.

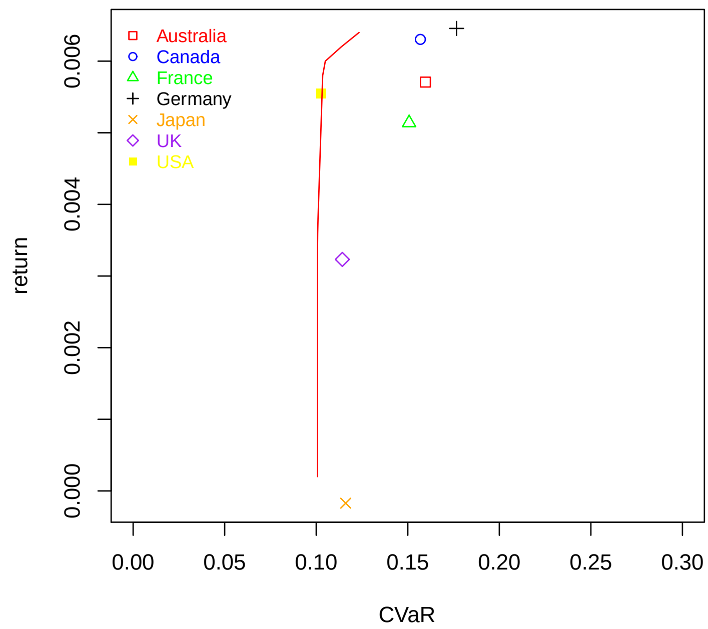

Figure 4.3: The CVaR-return efficient frontier with $\beta = 0.95$.

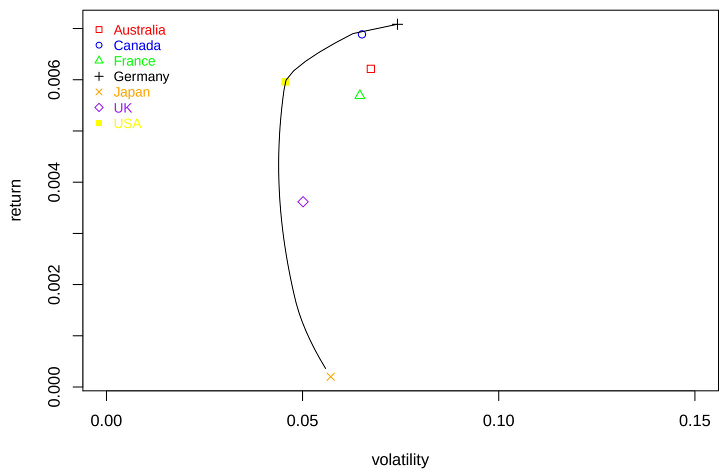

Figure 4.4: The return-variance efficient frontier.

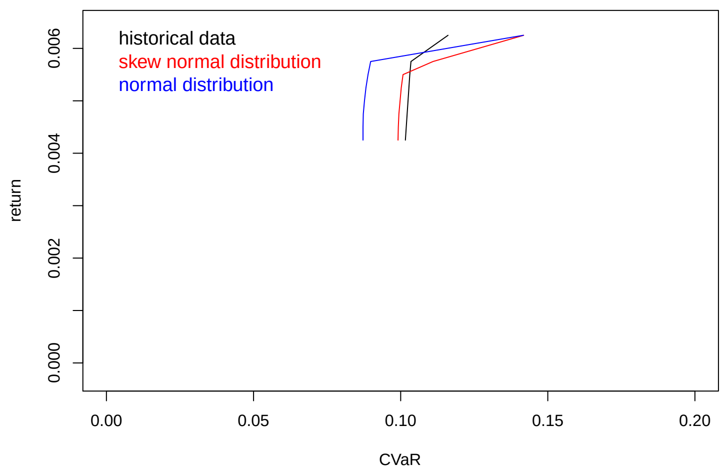

Figure 4.5: The CVaR-return efficient frontier with different methods.

Chapter 4

Skew Normal Distribution

In probability theory and statistics, skewness is a measure of the extent to which a probability distribution of a real values random variable 'leans' to one side of the mean. For a positive skewness (negative skewness), the tail on the right side (left side) is longer or fatter than the other side. Hill and Dixon [17] discussed the presence of skewness in real data. Data of financial returns always exhibits asymmetry and fat tails. When we adopt the normal distribution to model it, it is very difficult to precisely capture all the information of the assets. Because of the existence of skewness and kurtosis, in this case, the model risk can be significant. We may either underestimate or overestimate the value at risk and cause losses. To fix it up in some degree, we consider to use the skew normal distribution to model the financial returns. The skew normal distribution has been widely researched in the last few years. It can be noted that this class of distributions is a generalization of the normal distribution. As is concerned by investors when considering financial risks, the skew normal distribution is very useful in modeling returns with non-zero skewness or fat tails. For an introductory overview of the skew normal family of distributions, see Azzalini [3] and Azzalini and Dalla Valle [5]. For statistical issues of the skew normal distribution, see Azzalini and Capitanio [4]. With regards to risk measures of the skew normal distribution, Ngoussou [27] computed some VaR estimates for the skew normal and the skew t distributions. Bernardi [7] showed some results of computing VaR and CVaR for finite mixtures of skew normal distributions. Soltyk and Gupta [36] used expectation maximization algorithm to estimate VaR in the multivariate skew normal mixture model. As for results more related to this dissertation, Blasi [10], [11] and Scarlatti and Blasi [34] extended the Black-Litterman model to a market where the asset returns follow a multivariate skew normal distribution. Following these ideas, we try to extend the BL model by using skew normal distributions to model the asset returns and use CVaR to measure risks. Section 4.1 outlines some properties of the multivariate skew normal distribution. After that, using some modified methods different with Blasi [10], we will describe how to use the skew normal distribution in the BL model to obtain the posterior distribution.

4.1 Properties of the Skew Normal Distribution

In statistics, studies of generalizing the normal distribution have been developed for decades. Researchers want to find classes of distributions strictly including normal distributions, mathematically tractable and with wide range of the indices of skewness and kurtosis. In this circumstance, Azzalini [2] proposed the skew normal distribution and studied their properties. This class of densities includes the normal density; it has a clear representation in mathematics and a wide range of the skewness and kurtosis to some extent. Besides, as we will see in the following, the shape parameter allows the skew normal density function to change 'continuously' from non-normality to normality. Therefore, the skew normal distribution is very useful to model the real data.

Let $\phi(x)$ denote the standard normal probability density function for variable X, then it can be written as

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \ x \in \mathbb{R}. \tag{4.1.1}$$

The cumulative distribution function is given by

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)dt, \ x \in \mathbb{R}. \tag{4.1.2}$$

Then the probability density function of *a univariate skew normal distribution* with *shape parameter* α is

$$f_X(x) = 2\phi(x)\Phi(\alpha x), \ x \in \mathbb{R}. \tag{4.1.3}$$

In other words, for a random variable X having density function as (4.1.3), we say X follows a *skew normal distribution* and is denoted by

$$X \sim SN(\alpha). \tag{4.1.4}$$

Furthermore, if $Y = \mu + \sigma X$, with $\mu, \sigma \in \mathbb{R}$ and $\sigma > 0$, then we have

$$Y \sim SN(\mu, \sigma^2, \alpha). \tag{4.1.5}$$

Its probability density function is

$$\begin{aligned} f_Y(y) &= 2\phi(\frac{y-\mu}{\sigma})\Phi(\alpha\frac{y-\mu}{\sigma}), \quad y \in \mathbb{R} \qquad (4.1.6) \\ &= 2\phi(y;\mu,\sigma)\Phi(\alpha\sigma^{-1}(y-\mu)), \quad y \in \mathbb{R}. \qquad (4.1.7) \end{aligned}$$

The following properties come directly:

1. If $\alpha = 0$, then $X \sim N(0,1)$ and $Y \sim N(\mu, \sigma^2)$.

2. If $X \sim SN(\alpha)$, then $-X \sim SN(-\alpha)$.

Now we are ready to generalize the univariate skew normal distribution (4.1.6) into an n-dimensional case. Denote Σ as an $n \times n$ positive definite matrix and σ the diagonal matrix of standard deviations of Σ. Besides, μ and α are the n-dimensional location and shape vector, respectively. Then the multivariate density function is

$$f_Y(y) = 2\phi(y;\mu,\Sigma)\Phi(\alpha'\sigma^{-1}(y-\mu)), y \in \mathbb{R}^n. \qquad (4.1.8)$$

We call that the n-dimensional random variable Y has a *multivariate skew normal distribution* and $Y \sim SN(\mu, \Sigma, \alpha)$. To be different with the normal distribution, Σ is not always the covariance matrix of Y. The expectation and covariance matrix are given as follows:

$$\begin{aligned} E(Y) &= \mu + (\sigma\tilde{\alpha})\sqrt{\frac{2}{\pi}} \qquad (4.1.9) \\ Cov(Y) &= \Sigma - \frac{2}{\pi}(\sigma\tilde{\alpha})(\sigma\tilde{\alpha})', \qquad (4.1.10) \end{aligned}$$

where

$$\tilde{\alpha} = \frac{\overline{\Sigma}\alpha}{\sqrt{1+\alpha'\overline{\Sigma}\alpha}}, \ \overline{\Sigma} = \sigma^{-1}\Sigma\sigma^{-1}$$

To make it complete, we present a general form of the skew normal density function, which will be applied in the following section. An n-dimensional random variable W is distributed according to the extended skew normal distribution, or $W \sim SN(\mu, \Sigma, \alpha, \theta)$ if its density function is

$$f_W(w) = \phi(w;\mu,\Sigma)\Phi(\alpha_0 + \alpha'\sigma^{-1}(w-\mu))/\Phi(\theta), \qquad (4.1.11)$$

where $\alpha_0 = \theta(1+\alpha'\overline{\Sigma}\alpha)^{1/2}$.

4.2 The Black-Litterman Model in Skew Normal Markets

In this section, we will follow the similar idea in the works of Blasi [10] and [11] to obtain the posterior distribution for a market that is a multivariate skew normal distribution. To make all the formulas and representations consistent, we rewrite the procedure of the Black-Litterman model here.

4.2.1 The Market

Consider a portfolio of N securities or asset classes, whose returns have a multivariate skew normal distribution:

$$X \sim SN(\mu, \Sigma, \alpha), \tag{4.2.1}$$

where μ is the location parameter, and is considered to be random and normally distributed with $\mu \sim N(\mu_0, \tau\Sigma)$. Note that Σ is a positive definite matrix and α is the shape parameter. Similar with the standard Black-Litterman model, τ is a small number, meaning the uncertainty on the expectation of μ. There are some methods to calibrate τ. One empirical way is to estimate it close to zero. For example in He and Litterman [16], Black and Litterman [9] and Idzorek [18], they suggested to set it between 0.025 to 0.05. However, there are also supporters for τ being close to 1, see Satchell and Scowcroft [33] and Meucci [24].

From the work of Blasi [10], we have the following lemma:

Lemma 4.2.1. *(Blasi [10]) If the returns of assets follow a multivariate skew normal distribution:*

$$\begin{aligned} X|_{L=\mu} &\sim SN(\mu, \Sigma, \alpha_L) &(4.2.2)\\ L &\sim N(\mu_0, \Sigma_0). &(4.2.3)\end{aligned}$$

Then the marginal density function of X is:

$$f_X(x) = 2\varphi(x; \mu_0, \Sigma + \Sigma_0)\Phi(\alpha'\sigma_1^{-1}(x - \mu_0)), \tag{4.2.4}$$

where

$$\begin{aligned} \alpha &= \alpha_L'\sigma^{-1}\Sigma(\Sigma + \Sigma_0)^{-1}(1 + \alpha_1'\overline{\Delta}\alpha_1)^{-1/2}\sigma_1 &(4.2.5)\\ \Delta &= (\Sigma^{-1} + \Sigma_0^{-1})^{-1} &(4.2.6)\\ \overline{\Delta} &= d^{-1}\Delta d^{-1} &(4.2.7)\\ \alpha_1 &= -\alpha_L\sigma^{-1}d, &(4.2.8)\end{aligned}$$

and d is the diagonal matrix of standard deviations of Δ, and σ_1 is the diagonal matrix of standard deviations of $\Sigma + \Sigma_0$.

In particular, if $\Sigma_0 = \tau\Sigma$, we can simplify (4.2.4) and yield:

$$X \sim SN(\mu_0, (1+\tau)\Sigma, \alpha), \tag{4.2.9}$$

where $\alpha = \frac{\frac{\alpha_L}{\sqrt{1+\tau}}}{\sqrt{1+\frac{\tau}{1+\tau}\alpha_L'\overline{\Sigma}\alpha_L}}$.

Note that the positive definite matrix Σ is not the covariance matrix of X. Therefore we can alternatively model the covariance matrix of location as $\tau \cdot Cov(X)$, where

$$Cov(X) = \Sigma - \frac{2}{\pi}(\sigma\tilde{\alpha})(\sigma\tilde{\alpha})', \tag{4.2.10}$$

here σ and $\tilde{\alpha}$ are defined as before. Therefore,

$$L \sim N(\mu_0, \tau Cov(X)) \tag{4.2.11}$$

Substituting the assumption (4.2.11) to Lemma 4.2.1, the marginal density function of X is

$$f_X(x) = 2\varphi(x; \mu_0, \Sigma + \tau Cov(X))\Phi(\alpha'\sigma_1^{-1}(x-\mu_0)), \tag{4.2.12}$$

where

$$\alpha = \alpha_L'\sigma^{-1}\Sigma[\Sigma + \tau Cov(X)]^{-1}(1 + \alpha_1'\overline{\Delta}\alpha_1)^{-1/2}\sigma_1 \tag{4.2.13}$$
$$\Delta = [\Sigma^{-1} + [\tau Cov(X)]^{-1}]^{-1} \tag{4.2.14}$$
$$\overline{\Delta} = d^{-1}\Delta d^{-1} \tag{4.2.15}$$
$$\alpha_1 = -\alpha_L\sigma^{-1}d, \tag{4.2.16}$$

and d is the diagonal matrix of standard deviations of Δ, and σ_1 is the diagonal matrix of standard deviations of $\Sigma + \tau Cov(X)$.

4.2.2 The Views

In order to put the views in the prior distribution of asset returns, we use the same technique with the standard BL model. That is, given the expected returns of the assets, the views expressed on the expected returns are normally distributed:

$$V|_{E(X)} \sim N(v, \Omega). \tag{4.2.17}$$

But in this model, the distribution of X conditioned on the location parameter L is described in Section 4.2.1. We, therefore, try to 'move' our views from the expectations of X to the location parameter. This is an alternative way to connect the market and the views. Note we have $E(X) = L + \sigma\tilde{\alpha}\sqrt{\frac{2}{\pi}}$. Suppose P is the $K \times N$ pick matrix: the k-th row of the pick matrix determines the weights of the k-th view. So (4.2.17) is equivalent with

$$V|_{L=\mu} \sim N(P(\mu + \sigma\tilde{\alpha}\sqrt{\frac{2}{\pi}}), \Omega). \tag{4.2.18}$$

Now we are ready to apply Bayes' rule

$$\begin{aligned} f_{L|V}(\mu|v) &\propto f_{V|\mu}(v|\mu) f_L(\mu) \\ &\propto |\tau\Sigma|^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \cdot \\ &\quad e^{-\frac{1}{2}[(\mu-\mu_0)'(\tau\Sigma)^{-1}(\mu-\mu_0)+(v-P(\mu+\sigma\tilde{\alpha}\sqrt{\frac{2}{\pi}}))'\Omega^{-1}(v-P(\mu+\sigma\tilde{\alpha}\sqrt{\frac{2}{\pi}}))]} \\ &\propto |(\tau\Sigma)^{-1} + P'\Omega^{-1}P|^{\frac{1}{2}} \cdot \\ &\quad e^{-\frac{1}{2}(\mu-\mu_{BL}^L)'((\tau\Sigma)^{-1}+P'\Omega^{-1}P)(\mu-\mu_{BL}^L)}, \end{aligned} \tag{4.2.19}$$

where

$$\begin{aligned} \mu_{BL}^L &= [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\mu_0 + P'\Omega^{-1}(v - P(\sigma\tilde{\alpha}\sqrt{\frac{2}{\pi}}))] \\ &= \mu_0 + (\tau\Sigma)P'[P(\tau\Sigma)P' + \Omega]^{-1}[v - P(\sigma\tilde{\alpha}\sqrt{\frac{2}{\pi}}) - P\mu_0], \end{aligned} \tag{4.2.20}$$

with the covariance matrix

$$\Sigma_{BL}^L = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}. \tag{4.2.21}$$

Eventually, the posterior distribution of locations given the views is a normal distribution similar with the standard BL model:

$$L|V \sim N(\mu_{BL}^L, \Sigma_{BL}^L) \tag{4.2.22}$$

As is discussed in Section 4.2.1, an alternative way to model the location parameter L is to use the covariance matrix $Cov(X)$ of X as in (4.2.11). The mean and covariance matrix of the posterior distribution is

$$\mu_{BL}^L = \mu_0 + [\tau Cov(X)]P'[P(\tau Cov(X))P' + \Omega]^{-1}[v - P(\sigma\tilde{\alpha}\sqrt{\frac{2}{\pi}}) - P\mu_0], \tag{4.2.23}$$

as well as the covariance matrix

$$\Sigma_{BL}^L = [(\tau Cov(X))^{-1} + P'\Omega^{-1}P]^{-1}. \tag{4.2.24}$$

4.2.3 The Posterior

In Bayesian statistics, when no data is available, a prior distribution is used to describe the parameter. Once the data is available, we can update the prior distribution using the conditional distribution of parameters, which is our posterior distribution. That is to say, the posterior probability of a random event or an uncertain proposition is the conditional probability that is assigned after the relevant evidence is taken into account.

Namely, we have a prior distribution with probability distribution function $f_1(x)$. By Bayes' theorem, the posterior distribution by accounting for the data y is

$$f_1(x|y) = \frac{f_2(y|x)f_1(x)}{\int_x f_2(y|x)f_1(x)dx}. \tag{4.2.25}$$

Furthermore, we can also make use of the posterior distribution of the parameter given the observed data to yield a probability distribution over an interval rather than simply a point estimate. In detail, we can obtain the posterior predictive distribution, represented as

$$f_1(x|y) = \int_\theta f_1(x|\theta)f_2(\theta|y)d\theta \tag{4.2.26}$$

Continuing to our model of asset allocation, we have

$$f_{X|V}(x|v) = \int f_{X|L}(x|\mu) \cdot f_{L|V}(\mu|v)d\mu. \tag{4.2.27}$$

We substitute Equations (4.2.2) and (4.2.19) to (4.2.27), and by Lemma 4.2.1 we get:

$$f_{X|V}(x|v) = 2\phi(x; \mu_{BL}^L, \Sigma + \Sigma_{BL}^L)\Phi(\alpha_{BL}'\sigma_{BL}^{-1}(x - \mu_{BL}^L)) \tag{4.2.28}$$

where σ_{BL} is the diagonal matrix of the standard deviations of $\Sigma+\Sigma_{BL}^L$. To make it satisfy the form of a multivariate skew normal distribution, the parameter α_{BL} is given by:

$$\begin{aligned} \alpha_{BL}'\sigma_{BL}^{-1} &= \alpha'\sigma^{-1}\Sigma(\Sigma + \Sigma_{BL}^L)^{-1}(1 + \alpha_\Delta'\overline{\Delta}_{BL}\alpha_\Delta)^{-1/2} && (4.2.29) \\ \alpha_\Delta' &= -\alpha'\sigma^{-1}d_{BL} && (4.2.30) \\ \Delta_{BL} &= [\Sigma^{-1} + (\Sigma_{BL}^L)^{-1}]^{-1}, && (4.2.31) \end{aligned}$$

where $\overline{\Delta}_{BL}$ is the correlation matrix of Δ_{BL} and d_{BL} is the diagonal matrix of standard deviations of Δ_{BL}. Obviously, we have $d_{BL}\overline{\Delta}_{BL}d_{BL} = \Delta_{BL}$.

Therefore, (4.2.28) is a multivariate skew normal distribution density function. We have

$$X|V \sim SN(\mu_{BL}^L, \Sigma + \Sigma_{BL}^L, \alpha_{BL}). \tag{4.2.32}$$

4.2.4 The Allocation

Since the posterior distribution is also a multivariate skew normal distribution, we cannot use the mean-variance optimization or CAPM freely. However, what we concern is the riskiness, especially the loss in some extreme scenarios, such as economic recession or financial crisis; therefore we choose to use VaR or CVaR to measure the risk. Based on the argument in Chapter 3, we choose CVaR as the risk measure. Given a target expected return, we will minimize the CVaR to obtain the optimal portfolio.

To finish this chapter, we repeat the optimization problem here again:

$$
\begin{aligned}
&\min_{w}\ \mathrm{VaR}_{\beta}(w,-X)+\frac{1}{S(1-\beta)}\sum_{n=1}^{S}\max\{-w'R_n-\mathrm{VaR}_{\beta}(w,-X),0\}\\
&s.t.\ d_i \geq -w'R_i-\mathrm{VaR}_{\beta}(w,-X), i=1,2,\ldots,S\\
&\qquad w'\hat{R}_{1\times N}\geq R_{\min}\\
&\qquad \sum_{j}^{N} w_j = 1\\
&\qquad w_j \geq 0, j=1,2,\ldots,N.\\
&\qquad d_i \geq 0, i=1,2,\ldots,S
\end{aligned}
\tag{4.2.33}
$$

Chapter 5

Implementation

Based on the analysis in previous chapters, we call the Black-Litterman model using the skew normal distribution to fit the data and CVaR portfolio optimization to choose the optimal portfolio as an *extended Black-Litterman model* (EBL model). The goal of this chapter is to test the extended Black-Litterman model and compare the results with those of the standard Black-Litterman model. The role of the comparison, then, is not to decide which procedure outperform the other, but rather to find out the different parts. For the first example, the data of eight stocks in BM&F Bovespa will be used. These stocks are contained in the Ibovespa index and are mid-large cap. We will test the distribution of the data and use the normal and the skew normal distributions to model the returns of assets, respectively. Then, we express the same views on returns and proceed with the study of the two models. Finally, we conduct the stability analysis of the EBL model. We adjust the data by 1% and 2%, then we implement the EBL model again. To end this chapter, we will continue with the seven country indices example in Section 3.4 and try two different types of views to implement the extended Black-Litterman model.

5.1 The Data

To implement our model, we select eight stocks in BM&F Bovespa, ITUB4, PETR4, VALE5, BRFS3, ITSA4, BBAS3, GGBR4, EMBR3. These stocks are contained in the Ibovespa index and all together make up a great percentage. We summarize some information of the stocks in the table below.

Code	Sector	Part. %
ITUB4	Financial / Financial Intermediaries	7.036%
PETR4	Oil, Gas and Biofuels	7.820%
VALE5	Basic Materials Mining	8.278%
BRFS3	Consumer Non Cyclical / Food Processors	2.292%
ITSA4	Financial / Financial Intermediaries	2.869%
BBAS3	Financial / Financial Intermediaries	2.599%
GGBR4	Basic Materials / Steel and Metalurgy	1.994%
EMBR3	Capital Goods and Services / Transportation Equipment and Co	1.355%

Since we will calculate the CVaR of the assets, we need as much as possible historical data. The data series starts from 2004 to 2014, with weekly observations. This time period includes the 2008 finance crisis, the European debt crisis and also other events that may affect the Brazilian market. Short-term trading is always conducted in stock market or futures market. We calculate the compound 5-day returns of every stock. The statistic characteristics of the data are shown in the table below. We also present the box-plot for the 8 stocks. In the following table and Figure 1.1, we can see that BRFS3 has the biggest mean value and positive skewness. This means BRFS3 has more extreme values to the right, and its historical performance would be the best. On the contrary, GGBR4 has the lowest mean, the biggest negative skewness and the largest kurtosis. Using R 3.0.2, we obtain a correlation plot of the 8 stocks, see Figure 1.2. It can be seen that ITUB4 and ITSA4 are strongly correlated. Others have moderate correlations or even no correlations. The prices and returns of index of ITUB4 are shown in Figure 1.3.

Variable	Mean $\times 10^{-3}$	Std Dev	Skewness	Kurtosis	Min & Max
ITUB4	2.2	0.052	-0.00	5.37	-0.27 & 0.29
PETR4	0.0	0.050	-0.37	2.71	-0.03 & 0.03
VALE5	2.2	0.047	-0.12	1.30	-0.17 & 0.16
BRFS3	3.8	0.087	0.28	9.14	-0.41 & 0.57
ITSA4	2.0	0.052	0.11	7.91	-0.29 & 0.34
BBAS3	2.6	0.056	-0.24	3.10	-0.31 & 0.23
GGBR4	-1.5	0.076	-3.03	26.10	-0.74 & 0.26
EMBR3	0.8	0.050	-0.42	3.78	-0.29 & 0.17

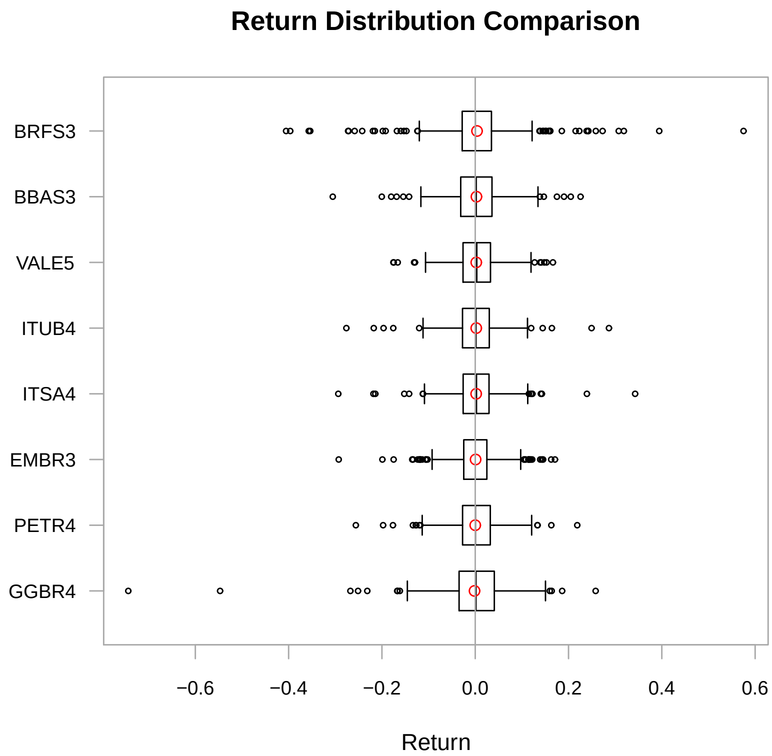

Figure 1.1: The box plot for returns of 8 stocks

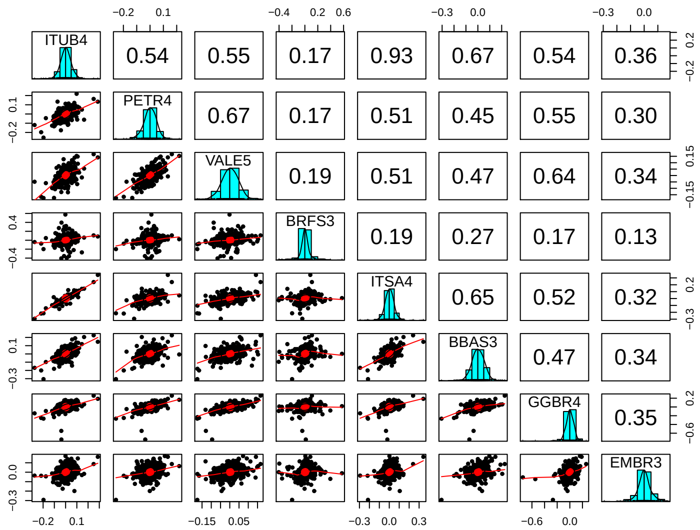

Figure 1.2: Scatter plot of all the returns

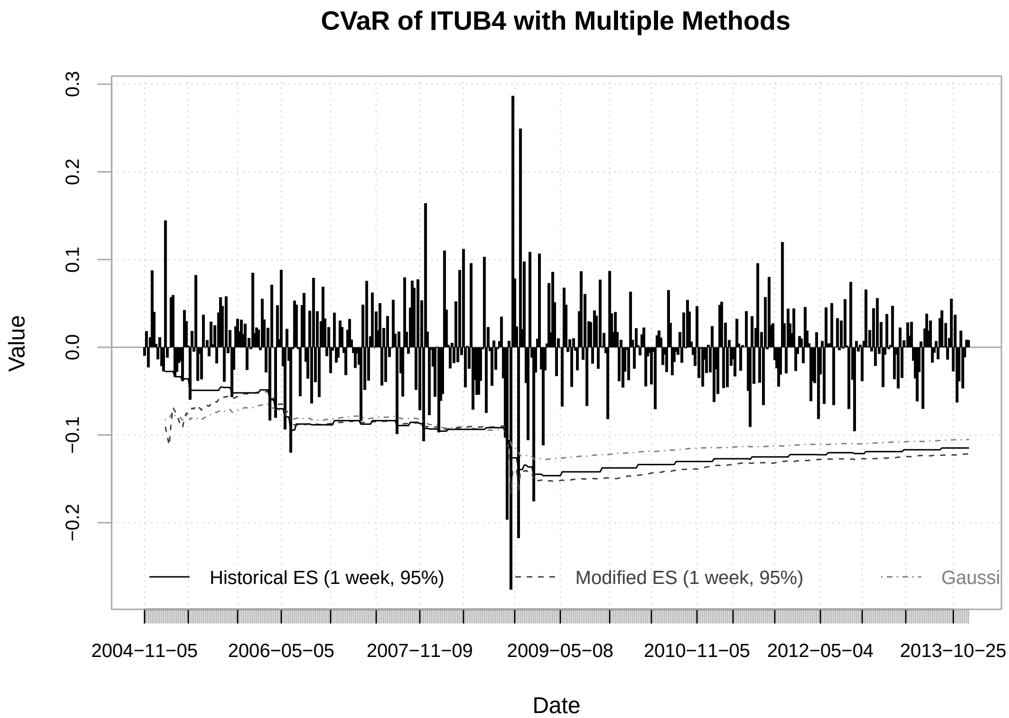

Figure 1.3: Weekly returns of ITUB4.SA from 2004 to 2014

5.2 The Models

In the EBL model, we will use CVaR portfolio optimization to obtain the optimal portfolios. We use historical method and Gaussian method to calculate CVaR for 8 stocks. In Figure 2.4, we can see that for all the 8 stocks, the historical CVaR are larger than gaussian CVaR in magnitude.

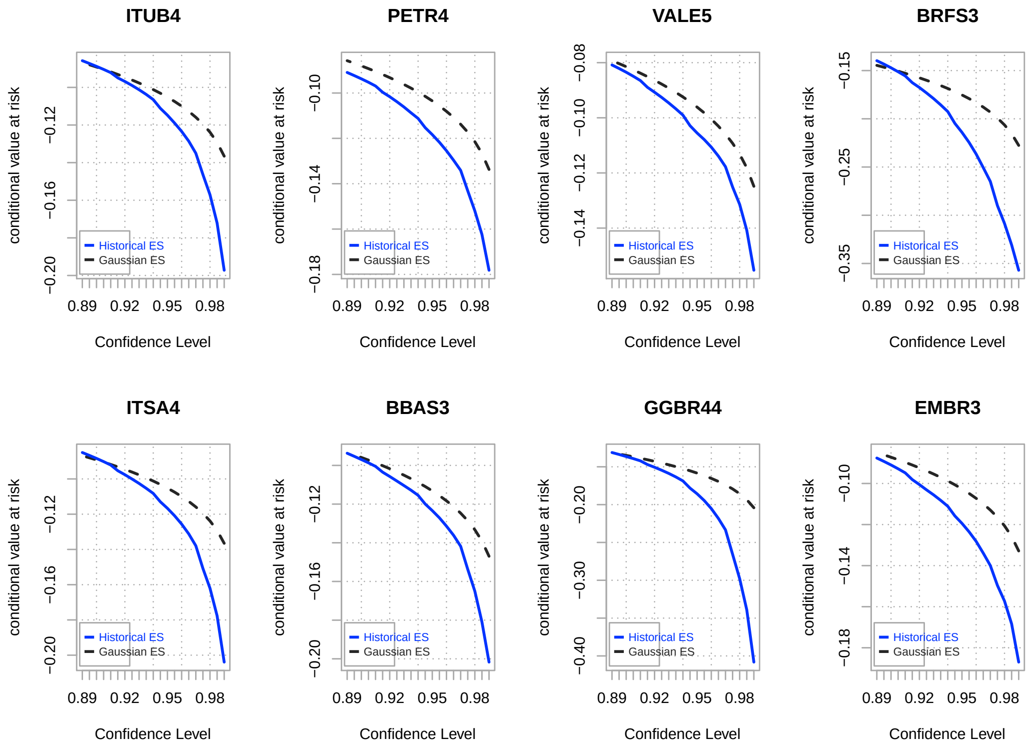

Figure 2.4: Risk confidence sensitivity of 8 stocks

From the statistics of the returns, we make an insight that the distribution of the data is skewed and has a fat tail. Take GGBR4 as an example, the histogram of the returns of GGBR4, Figure 2.5, has a left fat tail. We fit the normal and the skew normal distributions to the data, respectively. Note that in Figure 2.5 the solid line represents the skew normal density function, whereas the dotted line to plot the fitted normal density function. Figure 2.6 is the Mahalanobis distances QQ-plot. Almost all the points in the plot for the skew normal distribution are on a straight line. Comparing with the plot for the normal distribution, the one for the skew normal distribution is better. The same situation is in PP-plot, Figure 2.7. These plots make us conclude that GGBR4 is skewed and use the skew normal distribution is more proper. For the other indices, we fail to reject that ITUB4, VALE5, ITSA4 and BBAS3 are normally distributed. However, for PETR4, BRFS3, GGBR4 and EMBR3, we reject that they are normally distributed. As a conclusion, for the portfolio, we reject the portfolio is multivariate normally distributed.

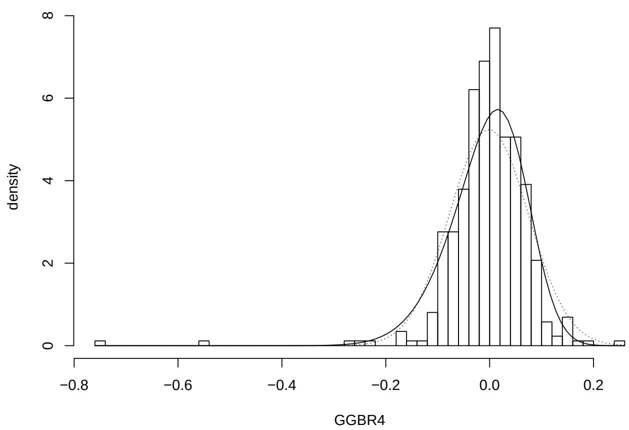

Figure 2.5: Fit the normal and the skew normal distribution to the compound returns of GGBR4.

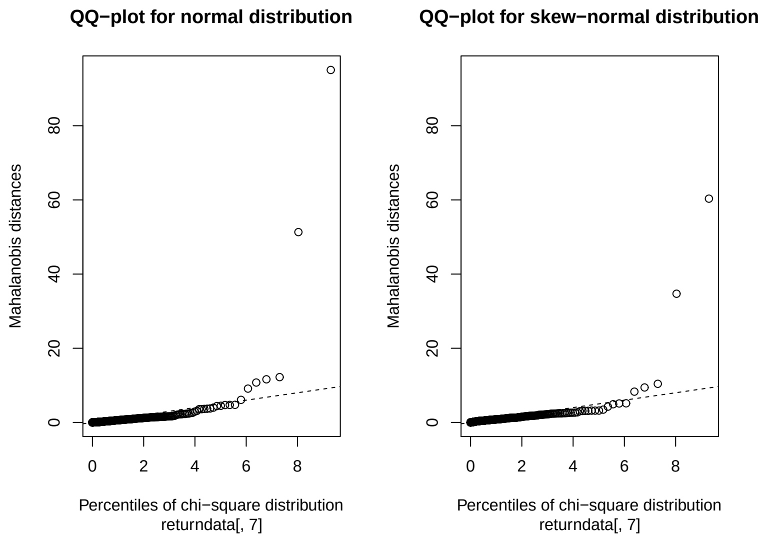

Figure 2.6: QQ-plot

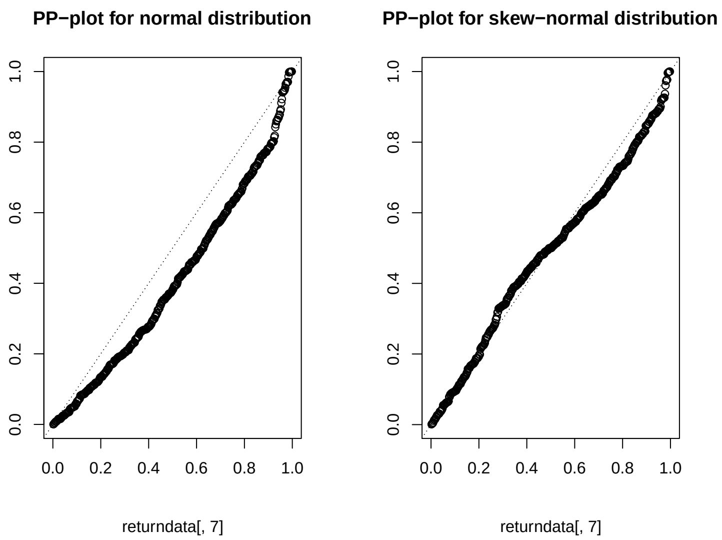

Figure 2.7: PP-plot

Finally, based on the results of MLE, we can conclude that GGBR4 is skew normally distributed.

	likelihood ratio test (test.normality)
LRT	63
$p-$ value	0

The MLE estimations for 8 stocks are listed below.

	ITUB4	PETR4	VALE5	BRFS3	ITSA4	BBAS3	GGBR4	EMBR3
$\mu_0 \times 10^{-3}$	9.0	5.1	14.7	-1.6	7.2	9.9	59.7	12.8
α	-0.06	0.58	0.34	0.24	0.57	0.09	-2.99	-0.15

The matrix of Σ is

$$1000 \times \begin{pmatrix} 2.8 & 1.4 & 1.4 & 0.7 & 2.5 & 2.0 & 2.5 & 1.0 \\ & 2.5 & 1.6 & 0.7 & 1.3 & 1.3 & 2.4 & 0.8 \\ & & 2.4 & 0.7 & 1.3 & 1.3 & 3.1 & 0.9 \\ & & & 7.5 & 0.8 & 1.3 & 0.8 & 0.5 \\ & & & & 2.7 & 1.9 & 2.4 & 0.9 \\ & & & & & 3.2 & 2.4 & 1.0 \\ & & & & & & 9.5 & 2.1 \\ & & & & & & & 2.7 \end{pmatrix}$$

Since we also want to implement the classical Black-Litterman model, the market weights w_M and the equilibrium expected returns μ_0 are as follows. The correlation values can be found in Figure 1.2 and the standard deviations can be found in Section 5.1.

	ITUB4	PETR4	VALE5	BRFS3	ITSA4	BBAS3	GGBR4	EMBR3
w_M	21%	23%	24%	6%	8%	8%	6%	4%
$\mu_0 \times 10^{-3}$	3.9	3.7	3.6	2.8	3.8	3.6	4.9	2.0

Hereafter, we denote the results of the two models by the following symbols:

- μ_{BL}: mean value of the posterior distribution of the Black-Litterman model;
- Σ_{BL}: the covariance matrix of the posterior distribution of the Black-Litterman model;
- μ^1_{EBL}: location parameter of the posterior distribution of the EBL model;
- α^1_{EBL}: the shape parameter of the posterior distribution of the EBL model;
- Σ^1_{EBL}: the definite matrix of the posterior distribution of the EBL model;
- μ^2_{EBL}: location parameter of the posterior distribution of the EBL model obtained by the alternative approach;
- α^2_{EBL}: the shape parameter of the posterior distribution of the EBL model obtained by the alternative approach;
- Σ^2_{EBL}: the definite matrix of the posterior distribution of the EBL model obtained by the alternative approach.

5.3 Methodology

For this particular problem, we will process the BL model and EBL model, respectively. Firstly, as is described in Chapter 2, we use a normal distribution to model the returns of the portfolio. For convenience, we set $R_f = 0$. We use the Ibovespa index as the market index and calculate the risk aversion parameter $\lambda \approx 1.1$ and use CAPM to obtain the expected return of the assets. For EBL model, we set the risk confidence $\beta = 0.95$ and calculate the efficient frontier of the optimal portfolio using the CVaR frame work. That is, we obtain the weights of the portfolio by minimizing the $\text{CVaR}_{0.95}$ value. This model can be converted to a linear optimization problem, as is presented in Chapter 3. We report it here:

$$
\begin{aligned}
&\min_{w}\ \mathrm{VaR}_{\beta}(w,-X)+\frac{1}{S(1-\beta)}\sum_{n=1}^{S}\max\{-w'R_n-\mathrm{VaR}_{\beta}(w,-X),0\}\\
&s.t.\ d_i \geq -w'R_i-\mathrm{VaR}_{\beta}(w,-X), i=1,2,\ldots,S\\
&\qquad w'\hat{R}_{1\times N}\geq R_{\min}\\
&\qquad \sum_{j}^{N} w_j = 1\\
&\qquad w_j \geq 0, j=1,2,\ldots,N.\\
&\qquad d_i \geq 0, i=1,2,\ldots,S
\end{aligned}
$$

By solving the linear optimization, we can plot an efficient frontier as in Figure 3.8.

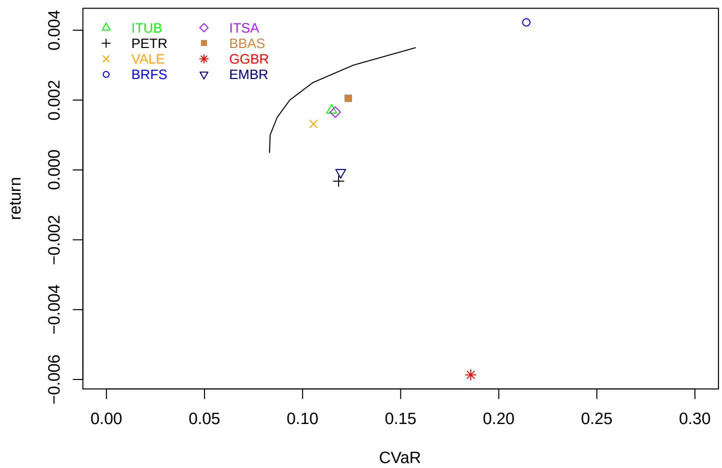

Figure 3.8: The efficient frontier

Then, we blend our views with the two models respectively. Assume that our views are:

'VALE5 will have a weekly return of 0.6%'

and

'BBAS4 will outperform ITUB4 by 0.4%'.

Therefore the pick matrix reads

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Accordingly, the views vector becomes $V = (0.6\%, 0.4\%)'$, and the confidence matrix of the views is

$$\Omega = \begin{pmatrix} 0.02^2 & 0 \\ 0 & 0.05^2 \end{pmatrix}.$$

For the BL model, see Chapter 2, we can obtain the posterior distribution and calculate the expected return and the covariance matrix:

$$\begin{aligned} \mu_{BL} &= \mu_0 + \tau\Sigma P'(\tau P\Sigma P' + \Omega)^{-1}(V - P\mu_0), \\ \Sigma_{BL} &= (1+\tau)\Sigma - \tau^2\Sigma P'(\tau P\Sigma P' + \Omega)^{-1}P\Sigma. \end{aligned}$$

In the case of the EBL model, we need to calculate the location parameter $\mu_{EBL}^{1,2}$, the matrix $\Sigma_{EBL}^{1,2}$ and also the shape parameter α_{EBL}.

5.4 The Results of the Black-Litterman Model

Following the formulas in Black-Litterman model, μ_{BL} and Σ_{BL} are given below:

	ITUB4	PETR4	VALE5	BRFS3	ITSA4	BBAS3	GGBR4	EMBR3
$\mu_{BL} \times 10^{-3}$	3.9	3.9	3.8	3.1	3.8	4.1	5.3	2.1
$diag(\Sigma_{BL}) \times 10^{-3}$	2.8	2.8	2.3	7.7	2.8	3.2	6.0	2.6

The weights are given in Figure 4.9. Comparing the differences of the two plots, we found that ITUB4 has less proportions in the portfolio, whereas VALE5 and BBAS3 have bigger proportions as we expected.

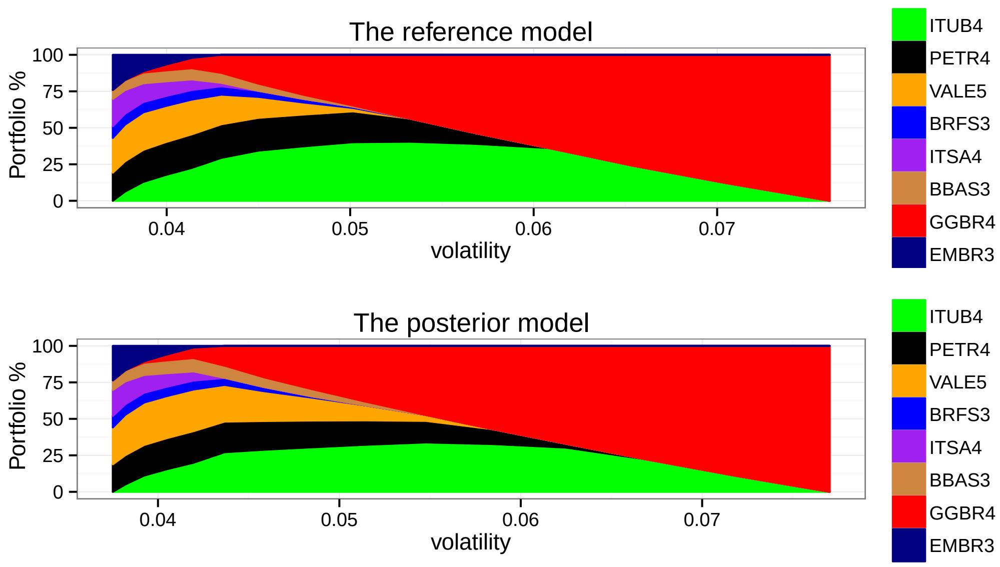

Figure 4.9: The transition maps for the Black-Litterman model

5.5 The Results of the Extended Black-Litterman Model

Now we fit a multivariate skew normal distribution to the data and calculate the posterior distribution and the optimal portfolio.

By the results of Chapter 4, we have the posterior distribution with the following parameters.

	ITUB4	PETR4	VALE5	BRFS3	ITSA4	BBAS3	GGBR4	EMBR3
$\mu^1_{EBL}(\times 10^{-3})$	9.3	5.6	15.3	-1.3	7.5	10.3	60.4	13.1
α^1_{BL}	-0.06	0.53	0.31	0.22	0.52	0.08	-2.75	-0.14
$diag(\Sigma^1_{EBL}) \cdot 10^{-3}$	2.8	2.6	2.5	7.8	2.8	3.2	9.7	2.7

The purpose of using CVaR is to control and manage the risk. We discard the portfolios when their CVaR is larger than 15% in one week. Figure 5.10 shows the relation between the CVaR and the resulting portfolio. It can be seen that, after blending our views, the weights of VALE3 and BBAS3 increase a little in the new portfolio. On the other hand, the weights of ITUB4 decrease.

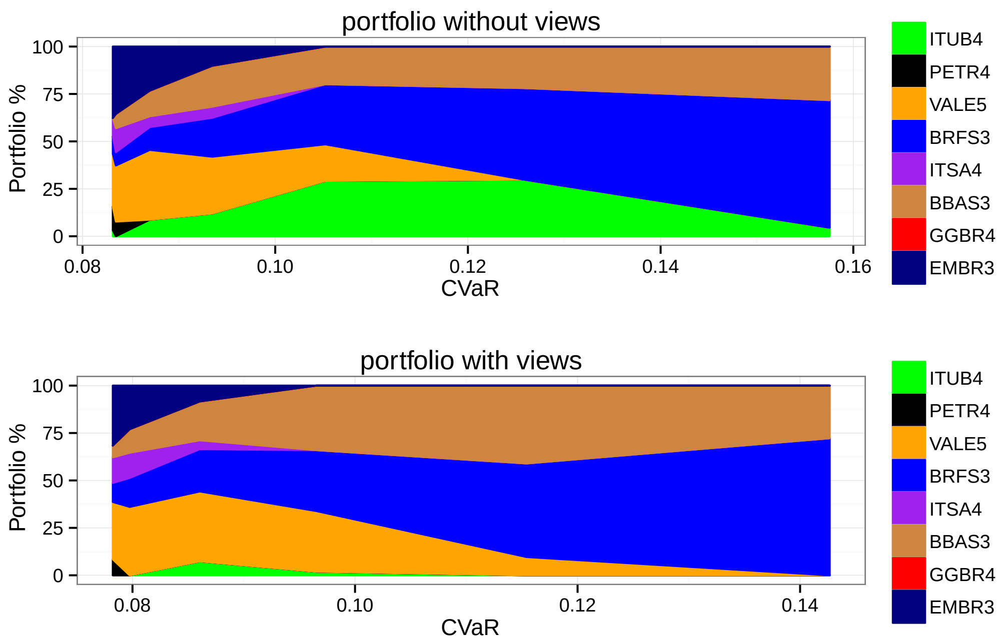

Figure 5.10: The transition maps for EBL model

5.6 An Alternative Approach

As is discussed in Chapter 4, in extended Black-Litterman model Σ is a semi-definite matrix, though it is not the covariance matrix of the distribution. Therefore, it is natural to model the distribution of the location parameter L using the covariance matrix

$$L \sim N(\mu_0, \tau Cov(X)),$$

where

$$Cov(X) = \Sigma - \frac{2}{\pi}(\sigma\tilde{\alpha})(\sigma\tilde{\alpha})',$$

and

$$\tilde{\alpha} = \frac{\overline{\Sigma}\alpha}{\sqrt{1 + \alpha'\overline{\Sigma}\alpha}}.$$

We list the location parameter μ_{BL} and shape parameter α_{BL} below:

	ITUB4	PETR4	VALE5	BRFS3	ITSA4	BBAS3	GGBR4	EMBR3
$\mu^2_{EBL} \cdot 10^{-3}$	9.3	5.6	15.3	-1.3	7.5	10.2	60.2	13.0
α^2_{EBL}	-0.06	0.56	0.33	0.23	0.55	0.08	-2.88	-0.15
$diag(\Sigma^2_{EBL}) \cdot 10^{-3}$	2.8	2.6	2.5	7.8	2.8	3.3	9.6	2.7

We can obtain similar results shown in Figure 6.11.

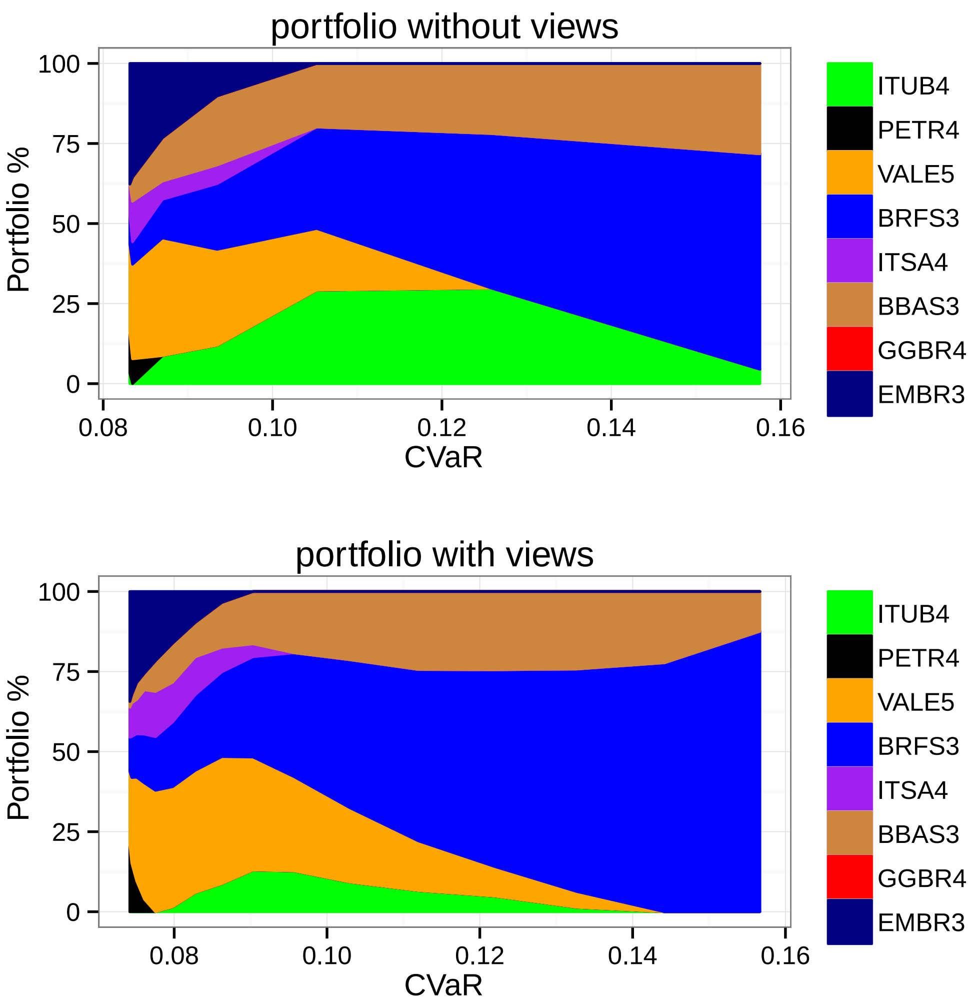

Figure 6.11: The transition maps of the alternative approach

5.7 Discussion of the Stability of the EBL Model

In this section, we will adjust the data by 1% and 2%, respectively, and test stability of the EBL model.

The period of data covers 10 years, including the 2008 financial crisis. Furthermore, we have 435 weekly returns. Some of the stocks have large volatilities during the financial crisis, such as ITUB4 and PETR4. Some, however, have large volatilities all the time before 2006 or 2008, such as BRFS3 and VALE5. These extreme losses, as well as some extreme gains make the distributions skewed and have fat tails, then affect fitting a normal distribution to the data. These severe losses also affect the process of CVaR portfolio optimization. Therefore, in this part, we will treat these large losses happened years ago as outliers and adjust them. For this, we modify the procedure in Boudt, Peterson and Croux [12]. The general idea is that assuming the data follows a normal distribution, hence, we find out some outliers until the number of the negative outliers reaches the target. During the process, we ignore the positive outliers because we concern more about the losses. For the negative outliers, we adjust them. In this way, we adjust, or we say clean the worst 1% and 2% of the losses for every stock, respectively. For every stock, suppose we want to clean 1% of the data, that is $1\%\times 435 \approx 4$ negative outliers. In the case of 2% adjustment, we use the same method, but only have to find 9 outliers. The algorithm has the following steps:

1. *Ranking the observations in function of their extremeness.* Denote μ and Σ the mean and covariance matrix of the data. We calculate the extremeness for every return r_n, $n = 1, 2, \ldots, S$. That is, we calculate the squared Mahalanobis distance

$$d_n^2 = (r_n - \mu)'\Sigma^{-1}(r_n - \mu).$$

   We sort these results as

$$d_{(1)}^2 \leq d_{(2)}^2 \leq \cdots \leq d_{(S)}^2.$$

2. *Outlier identification.* For every Mahalanobis distance d_n^2, if d_n^2 is one of the four largest values and $r_n < 0$, we denote r_n as an outlier. Besides, we set $\Delta = \max\{\delta/1000 : d_n^2 \geq \chi^2_{\delta/1000}, d_n^2 \text{ is an outlier}, \delta \in \mathbb{Z}^+\}$

3. *Data cleaning.* For the outlying return r_n, we replace r_n by

$$r_n\sqrt{\chi^2_{1,\Delta}/d_n^2}.$$

In step 2, the reason of choosing Δ in this way is that we want to adjust the extreme value into a reasonable loss. Since in our case, when Δ gets bigger, $\chi^2_{1,\Delta}$ is bigger but no more than d_n^2, then the adjusted value is $r_n\sqrt{\chi^2_{1,\Delta}/d_n^2} < r_n$ and is not too small in magnitude.

Figure 7.12 and Figure 7.13 illustrate the original data and the adjusted data for all the stocks. As is shown in the previous sections, we use CVaR linear optimization to obtain the optimal portfolio. Later, with the same views, we obtain the posterior distribution and the corresponding optimal portfolio. In Figure 7.14 and Figure 7.15, the results of the reference model and the posterior model using adjusted data are presented. Comparing with the results in the previous sections, for both 1% adjustment and 2% adjustment, the reference model using adjusted data does not change very much. The main parts constituting the portfolio are still ITUB4 and BRFS3, whereas the small parts are PETR4, VALE5, ITSA4, BBAS3 and EMBR3. After blending our views with the reference model, the posterior model becomes closer with the one using original data. From these results, we conclude that the EBL model is stable with respect of slightly adjusted data.

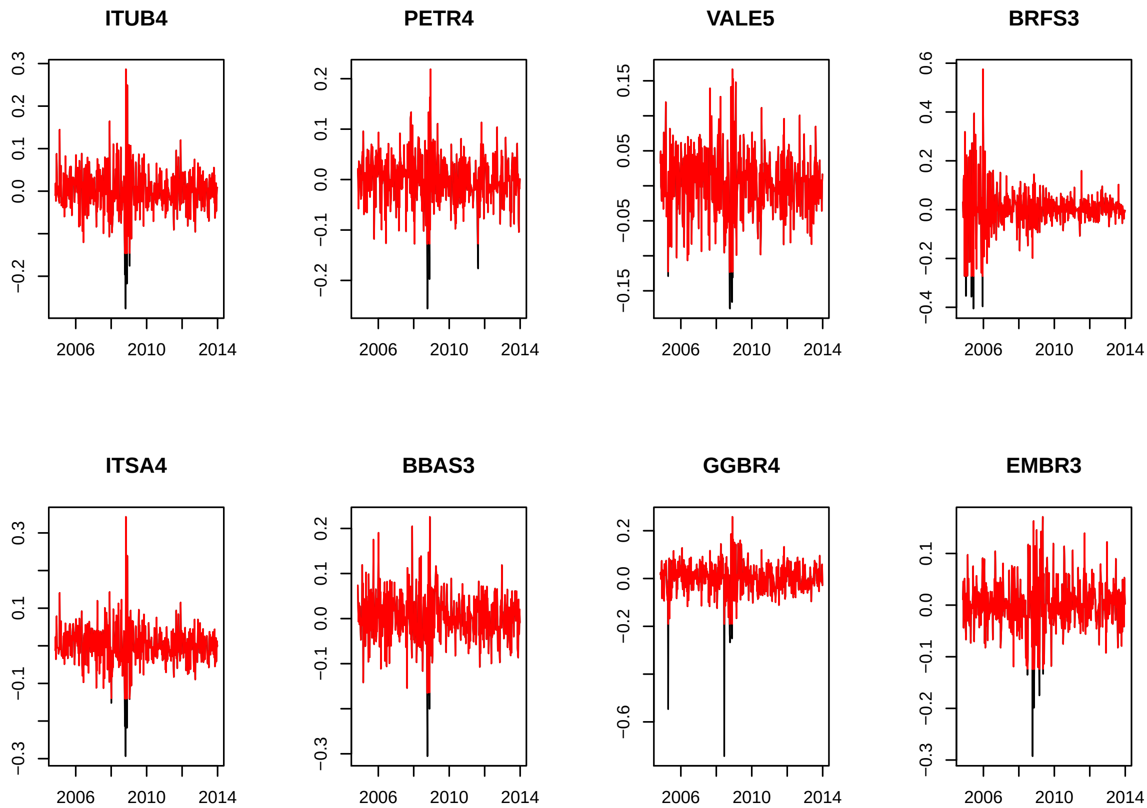

Figure 7.12: The original (black) and the 1% adjusted data (red).

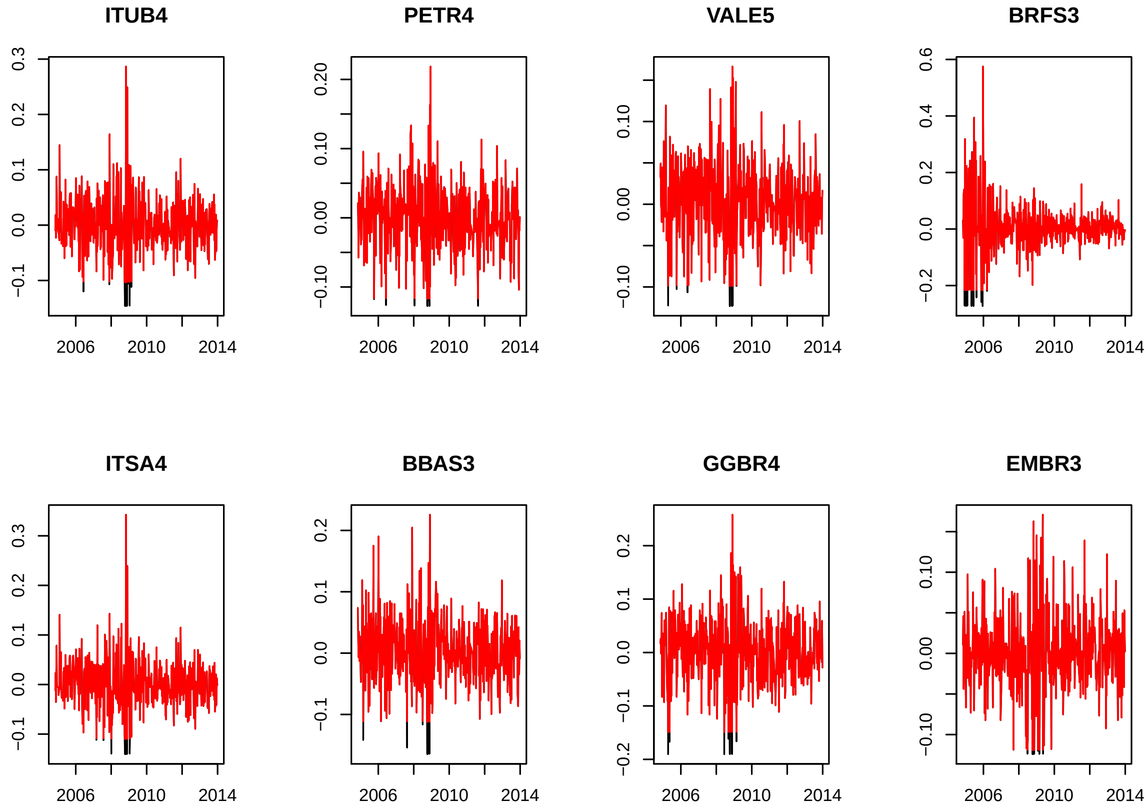

Figure 7.13: The 1% adjusted data (black) and the 2% adjusted data (red).

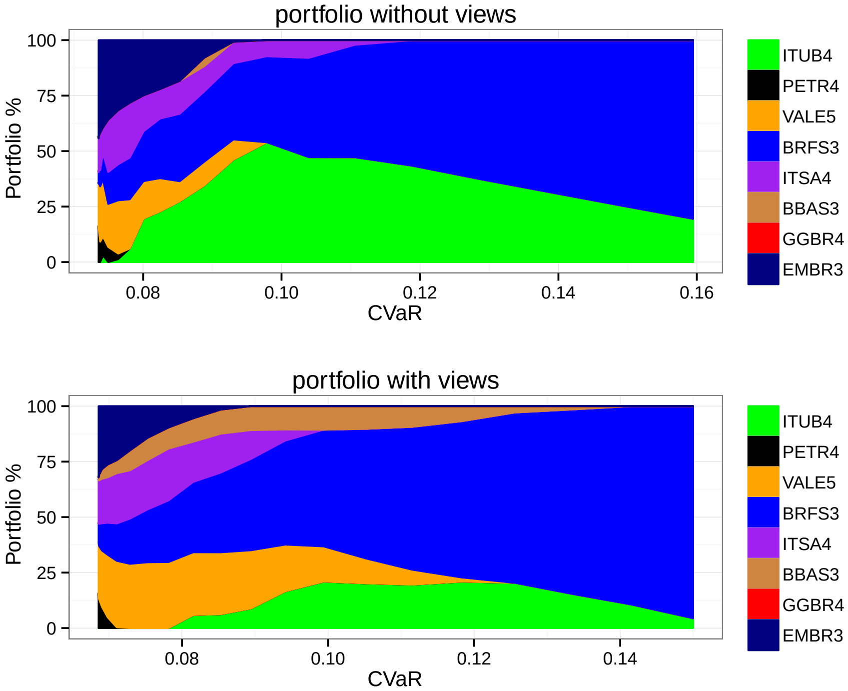

Figure 7.14: The transition maps for adjusted data (1%)

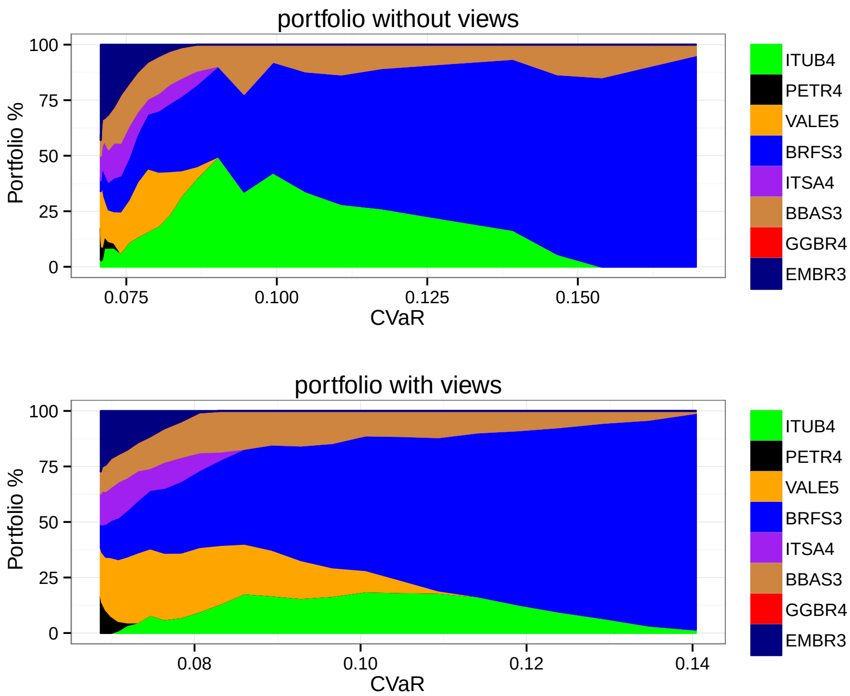

Figure 7.15: The transition maps for adjusted data (2%)

5.8 Continuation of the Country Index Example

In this section, we want to investigate if the EBL model is also suitable for other types of views. We continue with the discussion of the example in Section 3.4. Besides, we will deal with CVaR with one-month time horizon.

After observing the histograms of all the indices and testing the normality, we conclude that some returns of the indices are skewed. We, therefore, fit a multivariate skew normal distribution to the data and fix $\beta = 0.95$.

From the efficient frontier in Section 3.4, Figure 4.3, we can see that the positions of the indices of three European countries (France, Germany and UK) and Australia are in the inside of the efficient frontier. Suppose our views are,

'EWQ (France) will outperform EWG (Germany) and EWU (U.K.)'

and

'EWC (Canada) will outperform SPY (USA)'.

The pick matrix is

$$P = \begin{pmatrix} 0 & 0 & 1 & -1/2 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

We set the views vector as $V = (1\%, 1\%)'$, and the confidence matrix of the views is

$$\Omega = \begin{pmatrix} 0.01^2 & 0 \\ 0 & 0.03^2 \end{pmatrix}.$$

By the results of Chapter 3 and 4, we have the posterior distribution with the following parameters:

	EWA	EWC	EWQ	EWG	EWJ	EWU	SPY
$\mu_{EBL} \times 10^{-2}$	6.4	7.3	7.0	7.8	4.2	4.7	5.2
α_{EBL}	-0.04	-1.07	-0.70	-0.24	-0.26	0.91	-1.09
$diag(\Sigma_{EBL}) \times 10^{-3}$	7.9	8.6	8.1	10.6	5.1	4.4	4.2

The results are shown in the Figure 8.16. As expected, the EBL model has more allocations to EWQ (France) and EWC (Canada). EWU (U.K.) and SPY (U. S. A.), however, have less allocations. In contrast with the former portfolios, the new portfolios are well diversified.

Sometimes, however, we do not have a specific view on a certain index. We may have a certain view on the European countries, or the North American countries. For instance, the Eurozone crisis makes us have less confidence on EWG (Germany), EWU (U.K.) and EWQ (France) in our example. Therefore, in the following, we will try another view. In the prior result, EWC (Canada) and SPY (U.S.A.) together take a major part. The portfolio is not diversified. Hence, if we have a bullish view on EWQ (France), EWG (Germany) and EWU (U.K.), and a bearish view on EWC (Canada) and SPY (U.S.A.). So the pick matrix reads

$$P = \begin{pmatrix} 0 & 0 & 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix}.$$

We set the views vector as $V = (3\%, -3\%)'$, and the confidence matrix of the views remains the same

$$\Omega = \begin{pmatrix} 0.01^2 & 0 \\ 0 & 0.03^2 \end{pmatrix}.$$

The transition maps are shown in the Figure 8.16. From this figure, we notice that Germany, U.K. and France are in the new portfolio, whereas Canada is not.

Australia is also in the new portfolios. The posterior distribution is with the following parameters:

	EWA	EWC	EWQ	EWG	EWJ	EWU	SPY
$\mu_{EBL} \times 10^{-2}$	7.4	8.4	8.1	9.3	4.9	5.6	6.0
α_{EBL}	-0.04	-1.10	-0.73	-0.26	-0.28	0.93	-1.12
$diag(\Sigma_{EBL}) \times 10^{-3}$	7.7	8.5	8.0	10.4	5.0	4.4	4.2

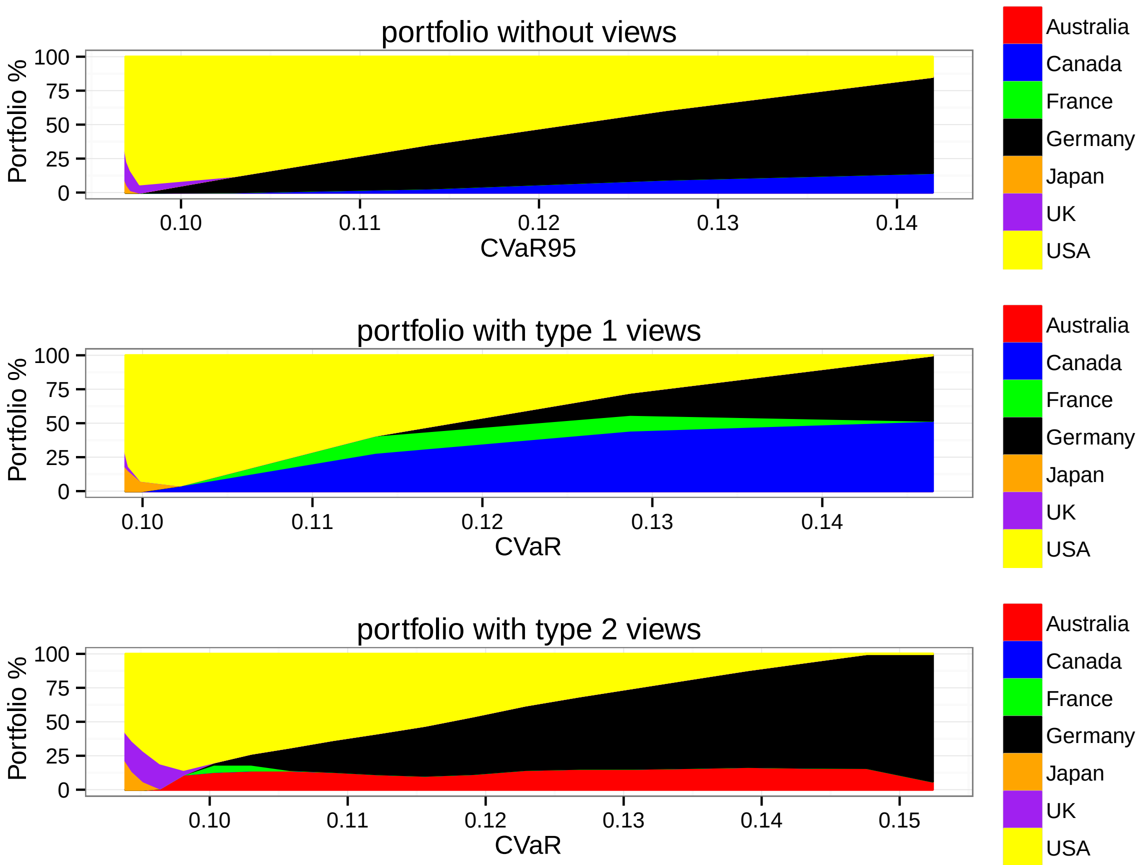

Figure 8.16: The CVaR-weights graph, the first one is the portfolio without views, the second one is the extended Black-Litterman model with the first view and the last one is with the second view.

5.9 Conclusion

In this chapter, we implement the standard Black-Litterman model and the extended Black-Litterman model. In the EBL model, we try to use CVaR portfolio optimization to obtain the optimal portfolio. To calculate CVaR we need either a large size of samples or a precise formula of the distribution. In the first example, we have more than 400 weekly returns, and we can use the historical data to estimate CVaR and obtain the optimal portfolio for the reference model. For the second example, we plot several efficient frontiers using historical, normal and skew normal methods, respectively. When β is close to 1, using historical method may underestimate the CVaR, because we only have 200 observations. Fitting a normal distribution to the data is a common method. However, in our cases, the data is significantly skewed. In conclusion, for the two examples, fitting a skew normal distribution to the data is an important part. Nevertheless, the weakness of fitting a skew normal distribution to the data is that we cannot estimate the annual returns from the monthly returns for the assumption of non-normally distributed and, therefore, we are unable to calculate the one year time horizon CVaR. For the posterior distribution of two models, we calculate the CVaR from the simulated data from the posterior distribution. In order to make the result stable, we recommend to simulate the samples as many as possible.

From the results in this chapter, the Black-Litterman and the extended Black-Litterman models yield quite different results in the following two aspects.

Firstly, the extended Black-Litterman model is affected by the tail behavior of the assets, since it uses CVaR portfolio optimization instead of mean-variance approach. By comparing the first transition maps of Figure 4.9 and Figure 5.10, we find that GGBR4 makes up the greatest percentage in the portfolios in the BL reference model, whereas in EBL reference model, it is not in any portfolio. We observe that the histogram of the returns of GGBR4 in Figure 2.5 has a fat tail on the left. BRFS3 has the opposite situation; it makes up a small percentage in the reference and posterior model in the standard BL model, whereas in EBL model, it gives a great part.

Secondly, as far as the weights of BBAS3, ITUB4 and VALE5 to be concerned, they change in the same direction in both of the models. The percentage of VALE5 increases more than that of BBAS3 does, because our expectation and the confidence level are higher. ITUB4 is another difference in the two models. In BL model, it decreases a little, while in EBL model, it decreases significantly.

For EBL model, we also provide an alternative approach. As is shown in Figure 6.11, as well as result tables, there are only slight differences compared with Figure 5.10.

Finally, we also discuss the stability of the EBL model. After adjusting the data by 1% and 2%, respectively, the reference models change a little. The percentage of every stock has a small change, but the relative major parts and minor parts remain the same. Despite the changes in both of the reference models, after blending the views, the results of posterior models are similar. Hence, we believe that using CVaR portfolio optimization is stable, especially when the size of the samples are large, several extreme data will not determine the optimal portfolio.

Based on the two examples, we make some comments on the common aspects of the Black-Litterman, as well as the extended Black-Litterman models. The two models are very sensitive to the input, a large change of the historical data, the parameters and the views. In EBL model, CVaR as a risk criterion is affected a lot by the tail behavior of the asset. When we blend the views in the models, the weights of all the assets will change. The parameters we are talking about are τ, risk aversion λ, and β. If we fix $\tau = 0.025$ and choose Ω having the same order of magnitude with Σ, based on the discussion in Chapter 2, our views are not fully confident. When the elements of Σ tends to zero, the confidence is stronger.

Due to these properties, both of the two models are very useful tools in diversifying the portfolios and asset allocation. In particular, for skewed data, skew normal distributions can be fitted better than normal distributions. The EBL model can perform better. To make the model completed, sometimes, back testing and stress testing are also needed as a complement part. For the future work related to the Black-Litterman model, there are still many topics to investigate. One of them is to develop a new kind of view on the volatilities.

Chapter 6

Appendix

In this appendix we discuss some technical results that can be skipped at first reading.

6.1 The Equivalence of Different Definitions of CVaR

Proof of the two definitions of $\mathrm{CVaR}_\beta(X)$,

$$\begin{aligned} \mathrm{CVaR}_\beta(X) &= E[X|X \geq \mathrm{VaR}_\beta(X)] & (6.1.1)\\ \mathrm{CVaR}_\beta(X) &= \min_C\{C + \frac{1}{1-\beta}E[\max\{X - C, 0\}]\}, & (6.1.2) \end{aligned}$$

are equivalent. The proof is borrowed from the paper of Rockafellar and Uryasev [30].

We also assume that X is a random variable representing the loss and $F_X(x)$ is continuous with respect to x, i.e.,

$$F_X(x) = P(X \leq x) = P(X < x).$$

We will now derive (6.1.2) is CVaR. Let

$$G_\beta(y) = y + \frac{1}{1-\beta}E[\max\{X - y, 0\}].$$

Note that $G_\beta(y)$ is convex and continuously differentiable with respect to y, and

$$\frac{d}{dy}G_\beta(y) = 1 + (1-\beta)^{-1}[F(y) - 1] = (1-\beta)^{-1}[F(y) - \beta].$$

Therefore, the minimum of $G_\beta(y)$ can be achieved if and only if there exists $\tilde{y}$ such that $F(\tilde{y}) - \beta = 0$. Since $F_X(x)$ is continuous and nondecreasing and by the definition of $\mathrm{VaR}_\beta(X)$, we can choose $\tilde{y} = \mathrm{VaR}_\beta(X)$ and

$$\min_{y\in\mathbb{R}} G_\beta(y) = G_\beta(\mathrm{VaR}_\beta(X)) = \mathrm{VaR}_\beta(X) + \frac{1}{1-\beta} E[\max\{X - \mathrm{VaR}_\beta(X), 0\}].$$

Let

$$[t]^+ = \begin{cases} t, & t > 0; \\ 0, & t \leq 0. \end{cases}$$

The expectation here equals

$$\begin{aligned} E[[X - \mathrm{VaR}_\beta(X)]^+] = (E[X|X \geq \mathrm{VaR}_\beta(X)] - \\ E[\mathrm{VaR}_\beta(X)|X \geq \mathrm{VaR}_\beta(X)]) \cdot P(X \geq \mathrm{VaR}_\beta(X)) \quad (6.1.3) \\ = (1-\beta)[\mathrm{CVaR}_\beta(X) - \mathrm{VaR}_\beta(X)]. \quad (6.1.4) \end{aligned}$$

Equation (6.1.4) is achieved from the definition of CVaR (6.1.1). Thus,

$$\min_{y\in\mathbb{R}} G_\beta(y) = \mathrm{VaR}_\beta(X) + (1-\beta)^{-1}(1-\beta)[\mathrm{CVaR}_\beta(X) - \mathrm{VaR}_\beta(X)] = \mathrm{CVaR}_\beta(X).$$

Furthermore, $E[\max\{X - \mathrm{VaR}_\beta(X), 0\}]$ can be obtained approximately by sampling the probability distribution of X. If the sampling generates a collection of $X_1, X_2, \ldots, X_S$, then the corresponding approximation to the second definition of $\mathrm{CVaR}_\beta(X)$ is

$$\mathrm{CVaR}_\beta(X) = \mathrm{VaR}_\beta(X) + \frac{1}{S(1-\beta)} \sum_{n=1}^{S} \max\{X_n - \mathrm{VaR}_\beta(X), 0\}$$

∎

6.2 Proof of Lemma 4.2.1

We revise the proof the Lemma 4.2.1 here, which can be found in Blasi [11].

Lemma 6.2.1. *If the returns of assets follow a multivariate skew normal distribution:*

$$X|_{L=\mu} \sim SN(\mu, \Sigma, \alpha_L) \quad (6.2.1)$$

$$L \sim N(\mu_0, \Sigma_0). \quad (6.2.2)$$

Then the marginal density function of X is:

$$f_X(x) = 2\phi(x; \mu_0, \Sigma + \Sigma_0)\Phi(\alpha' \sigma_1^{-1}(x - \mu_0)), \quad (6.2.3)$$

where

$$\alpha = \alpha_L' \sigma^{-1}\Sigma(\Sigma+\Sigma_0)^{-1}(1+\alpha_1'\overline{\Delta}\alpha_1)^{-1/2}\sigma_1 \tag{6.2.4}$$
$$\Delta = (\Sigma^{-1}+\Sigma_0^{-1})^{-1} \tag{6.2.5}$$
$$\overline{\Delta} = d^{-1}\Delta d^{-1} \tag{6.2.6}$$
$$\alpha_1 = -\alpha_L\sigma^{-1}d, \tag{6.2.7}$$

and d *is the diagonal matrix of standard deviations of* Δ*,* σ_1 *is the diagonal matrix of standard deviations of* $\Sigma+\Sigma_0$*.*

In particular, if $\Sigma_0 = \tau\Sigma$*, we can simplify (6.2.3) and get:*

$$X \sim SN(\mu_0, (1+\tau)\Sigma, \alpha), \tag{6.2.8}$$

where $\alpha = \frac{\frac{\alpha_L}{\sqrt{1+\tau}}}{\sqrt{1+\frac{\tau}{1+\tau}\alpha_L'\overline{\Sigma}\alpha_L}}$.

Proof. From the definitions of posterior distribution and posterior predict distribution, the marginal distribution of X is given by:

$$\begin{aligned} f_X(x) &= \int f_{X|L}(x|\mu) f_L(\mu) d\mu \\ &= \int 2\phi(x;\mu,\Sigma)\Phi(\alpha_L'\sigma^{-1}(x-\mu)) \cdot \phi(\mu;\mu_0,\Sigma_0)d\mu \end{aligned}$$

Similar with the standard BL model, we have:

$$\begin{aligned} &\phi(x;\mu,\Sigma)\cdot\phi(\mu;\mu_0,\Sigma_0) \\ =&\phi(x;\mu_0,\Sigma+\Sigma_0)\cdot\phi(\mu;z(x,\mu_0),\Delta), \end{aligned}$$

where

$$z(x,\mu_0) = \Delta\cdot(\Sigma^{-1}x + \Sigma_0^{-1}\mu_0)$$
$$\Delta = (\Sigma^{-1}+\Sigma_0^{-1})^{-1}.$$

So it becomes:

$$\begin{aligned} f_X(x) &= \int 2\phi(x;\mu_0,\Sigma+\Sigma_0)\cdot\phi(\mu;z(x,\mu_0),\Delta)\Phi(\alpha_L'\sigma^{-1}(x-\mu))d\mu \\ &= \int 2\phi(x;\mu_0,\Sigma+\Sigma_0)\cdot\phi(\mu;z(x,\mu_0),\Delta) \\ &\cdot\Phi(\alpha_L'\sigma^{-1}(x-z(x,\mu_0)) - \alpha_L'\sigma^{-1}(\mu-z(x,\mu_0)))d\mu \\ &= \int 2\phi(x;\mu_0,\Sigma+\Sigma_0)\cdot\phi(\mu;z(x,\mu_0),\Delta) \\ &\cdot\Phi(\rho_0+\alpha_1'\delta^{-1}(\mu-z(x,\mu_0)))d\mu \end{aligned}$$

where δ is the diagonal matrix of standard deviations of Δ and $\overline{\Delta} = \delta^{-1}\Delta\delta^{-1}$, and

$$\begin{aligned} \alpha_1' &= -\alpha_L'\sigma^{-1}\delta \\ \rho_0 &= \rho\sqrt{1+\alpha_1'\overline{\Delta}\alpha_1} \\ \rho &= \alpha_L'\sigma^{-1}(1+\alpha_1'\overline{\Delta}\alpha_1)^{-1/2}(x-z(x,\mu_0)) \\ &= \alpha_L'\sigma^{-1}\Sigma(\Sigma+\Sigma_0)^{-1}(1+\alpha_1'\overline{\Delta}\alpha_1)^{-1/2}(x-\mu_0). \end{aligned}$$

Continuing with the proof, we have

$$\begin{aligned} f_X(x) = &\int 2\phi(x;\mu_0,\Sigma+\Sigma_0)\Phi(\rho)\cdot\frac{1}{\Phi(\rho)}\phi(\mu;z(x,\mu_0),\Delta) \\ &\cdot\Phi(\rho_0+\alpha_1'\delta^{-1}(\mu-z(x,\mu_0)))d\mu \\ &= 2\phi(x;\mu_0,\Sigma+\Sigma_0)\Phi(\rho) \\ &\cdot\int\frac{1}{\Phi(\rho)}\phi(\mu;z(x,\mu_0),\Delta)\Phi(\rho_0+\alpha_1'\delta^{-1}(\mu-z(x,\mu_0)))d\mu. \end{aligned}$$

By the definition of the generalized the skew normal distribution (4.1.11), we have the value of the integral is 1. Hence, the expression is:

$$\begin{aligned} f_X(x) &= 2\phi(x;\mu_0,\Sigma+\Sigma_0)\Phi(\alpha_L'\sigma^{-1}\Sigma(\Sigma+\Sigma_0)^{-1}(1+\alpha_1'\overline{\Delta}\alpha_1)^{-1/2}(x-\mu_0)) \\ &= 2\phi(x;\mu_0,\Sigma+\Sigma_0)\Phi(\alpha'\sigma_1^{-1}(x-\mu_0)), \end{aligned}$$

where α and σ_1 are defined as in the Lemma 4.2.1. ∎

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