Stochastic Models of Urban Traffic

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Abstract

In this text we investigate the Biham-Middleton-Levine Traffic model. In the first chapter we introduce the model for $\mathbb{Z}^2$ lattice and show that the system will be globally blocked for p close to 1. In this chapter we use [1, 3, 4, 6, 7, 9]. In the second chapter we survey the Biham-Middleton-Levine model for a single junction of size N and we introduce the time-normalized model. For this chapter we use [5, 7]. Finally, in the third chapter we focus our study for the Biham-Middleton-Levine model on a finite lattice of size $N \times N$. Also, in this chapter we respond some open questions of [8], and give a MATLAB code for finding the configuration of the system at time t. For this chapter we use [2, 7, 8].

Chapter 1

The Jammed Phase of the Biham-Middleton-Levine Traffic Model

1.1 Introduction

In this chapter we will define The Biham-Middleton-Levine traffic model and see some properties and theorems of this model. The Biham-Middleton-Levine traffic model is a self-organizing cellular automaton traffic flow model. It is consists of a number of cars represented by points on a lattice with a random starting position, which each car may be one of two type: those that only move upwards (shown as blue in this work), and those that only move towards the right (shown as red in this work). The two types of cars take turns to move. During each turn, all the cars for the corresponding type advance by one step if they are not blocked by another car. Our goal in this chapter is to prove that there is $p_1 < 1$ such that for all $p \geq p_1$ the BML traffic model on $\mathbb{Z}^2$ will be block almost surely.

First we give some definitions and then we define the system more rigorously.

Definition 1. (Cellular Automaton) *A cellular automaton consists of a regular grid of cells, each in one of a finite number of states, such as on and off.*

The grid can be in any finite number of dimensions. For each cell, a set of cells called its neighborhood is defined relative to the specified cell. An initial state (time $t = 0$) is selected by assigning a state for each cell. A new generation is created advancing t by 1, according to some fixed rule that determines the new state of each cell in terms of the current state of the cell and the states of the cells in its neighborhood. Typically, the rule for updating the state of cells is the same for each cell and does not change over time, and is applied to the whole grid simultaneously.

Definition 2. (Self-organization) *Ability of a system to spontaneously arrange its components or elements in a purposeful (non-random) manner, under appropriate conditions but without the help of an external agency.*

For constructing The Biham-Middleton-Levine traffic model we present three variants of a simple cellular automaton model that describes traffic flow in two dimensions. The first two variants are three-state cellular automaton models on a square lattice. Each site contains either an arrow (blue particle) pointing upwards, an arrow (red particles) pointing to the right, or is empty. In the first variant the dynamics is controlled by a traffic light, such that the right arrows move only in even time steps and the up arrows move in odd time steps. On even time steps, each right arrow moves one step to the right unless the site on its right-hand side is occupied by another arrow (which can be either an up or a right arrow). If it is blocked by another arrow it does not move, even if during the same time step the blocking arrow moves out of that site. Similar rules apply to the up arrows, which move upwards. Note that this is a fully deterministic model; randomness enters only through the initial conditions.

Note. In this text, "right arrow", "red car", and "red particle" refer to the same concept. Also, "up arrow", "blue car", and "blue particle" are the same concept.

The model is defined on a square lattice of $N \times N$ sites with periodic boundary conditions. Due to the periodic boundary conditions the total number of arrows of each type is conserved. Moreover, the total number of up arrows in each column and the total number of right arrows in each row are conserved, giving rise to $2N$ conservation rules.

The density of right (up) arrows is given by $p_{\rightarrow} = \frac{n_{\rightarrow}}{N^2}$ ($p_{\uparrow} = \frac{n_{\uparrow}}{N^2}$), where

$n_{\rightarrow}$ ($n_{\uparrow}$) is the number of right (up) arrows in the system. The (average) velocity v of an arrow in a time interval τ is defined to be the number of successful moves it makes in τ divided by the number of attempted moves in τ. It has maximal value $v = 1$, indicating that the arrow was never blocked, while $v = 0$ means that the arrow was stopped for the entire duration τ, and never moved at all. The average velocity $\bar{v}$ for the system is then obtained by averaging v over all the arrows in the system.

In this part suppose initially a car is placed with probability p at each site of the two dimensional integer lattice (we consider the BML model on $\mathbb{Z}^2$ instead of a square lattice of $N \times N$ sites with periodic boundary conditions). Each car is equally likely to be red or blue, and different sites receive independent assignments. We prove that when p is sufficiently close to 1 traffic is jammed, in the sense that no car moves infinitely many times.

Let $\mathbb{Z}^2 = \{\mathbf{z} = (z_1, z_2) : z_1, z_2 \in \mathbb{Z}\}$ be two dimensional integer lattice. At each time step $t = 0, 1, \ldots$, each site of $\mathbb{Z}^2$ contains either a red car ($\rightarrow$), a blue car ($\uparrow$) or an empty space (0). Let $p \in [0, 1]$. The initial configuration is given by a random element σ of $\{0, \rightarrow, \uparrow\}^{\mathbb{Z}^2}$ under a probability measure $\mathbb{P}_p$ in which

$$\mathbb{P}_p(\sigma(\mathbf{z}) = \rightarrow) = \mathbb{P}_p(\sigma(\mathbf{z}) = \uparrow) = \frac{p}{2} \quad \textit{and} \quad \mathbb{P}_p(\sigma(\mathbf{z}) = 0) = 1 - p$$

for each site $\mathbf{z} \in \mathbb{Z}^2$, and the initial states of different sites are independent. The configuration evolves in discrete time according to the following deterministic dynamics. On each odd time step, every $\uparrow$ which currently has a 0 immediately to its top (i.e. in direction $(0, 1)$) moves into this space. On the even time step, each $\rightarrow$ which currently has a 0 immediately to its right (i.e. in direction $(1, 0)$) moves into this space. The configuration remains otherwise unchanged.

Our goal here is to prove the following theorem.

Theorem 3. *There exists $p_1 < 1$ such that for all $p \geq p_1$, almost surely no car moves infinitely often and the state of each site is eventually constant.*

1.2 Proof of Main Result (Theorem 3)

Definition 4. *A finite or infinite sequence of sites* $\mathbf{z}^0, \mathbf{z}^1, \mathbf{z}^2, ...[, \mathbf{z}^n]$ *is called a* **blocking path** *if, for each* $m \geq 0$*, one of the following holds:*
(i) $\sigma(\mathbf{z}^m) = \rightarrow$ *and* $\mathbf{z}^{m+1} = \mathbf{z}^m + (1,0)$*;*
(ii) $\sigma(\mathbf{z}^m) = \uparrow$ *and* $\mathbf{z}^{m+1} = \mathbf{z}^m + (0,1)$*;*
(iii) $\sigma(\mathbf{z}^m) = \sigma(\mathbf{z}^m + (1,0)) = \rightarrow$*,* $\sigma(\mathbf{z}^m + (1,-1)) = \uparrow$*, and* $\mathbf{z}^{m+1} = \mathbf{z}^m + (1,1)$*;*
or (iv) $\sigma(\mathbf{z}^m) = \sigma(\mathbf{z}^m + (0,1)) = \uparrow$*,* $\sigma(\mathbf{z}^m + (-1,1)) = \rightarrow$*, and* $\mathbf{z}^{m+1} = \mathbf{z}^m + (1,1)$*.*

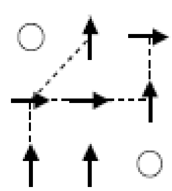

Figure 1.1: There are blocking paths from $(0,0)$ (bottom left) to $(2,2)$ and from $(0,0)$ to $(1,2)$. The former uses just steps (i) and(ii) but the latter uses a step of type (iii) between $(0,1)$ and $(1,2)$.

See Figure 1.1 for an illustration. Note that if $\mathbf{z}^0, ..., \mathbf{z}^n$ and $\mathbf{z}^n, \mathbf{z}^{n+1}, ...$ are blocking paths then so is $\mathbf{z}^0, ..., \mathbf{z}^n, \mathbf{z}^{n+1}, ...$. Cases (i) and (ii) correspond to the naive chains of cars. Cases (iii) and (iv) will provide the key to our argument by allowing for additional types of blocking path.

Lemma 5. *No car on an infinite blocking path ever moves.*

Proof. We claim that the car at $\mathbf{z}^m$ can only move strictly after that at $\mathbf{z}^{m+1}$ has moved. This implies the result, by induction on the time step. The claim is immediate in cases (i) and (ii) above. In case (iii), we note that the car at $\mathbf{z}^m$ can only move after that $\mathbf{z}^m + (1,0)$. If the latter car ever moves then it does so at an even step, and it is replaced immediately at the next step by the car initially at $\mathbf{z}^m + (1,-1)$. But this car now cannot move again until after that at $\mathbf{z}^{m+1}$. An analogous argument applies in case (iv). For all sites we have one of these cases, hence, no car ever moves. □

Now we present a renormalized lattice with the structure of $\mathbb{Z}^2$. Suppose that M and k are positive integers such that $M > 2k > 0$, later we will fix them. Each site in the renormalized lattice covers $2k+1$ sites on a diagonal from higher left hand to the lower right hand. We introduce $D_k = \{(s,-s) : |s| \leq k\}$. For each site $\mathbf{u} = (u_1, u_2) \in \mathbb{Z}^2$ introduce the renormalized site

$$V_{\mathbf{u}} = u_1(10M, 9M) + u_2(9M, 10M) + D_k.$$

By having renormalized sites we have to give a definition for renormalized edge. We call $(\mathbf{u}, \mathbf{v})$ a renormalized edge if $\mathbf{v} - \mathbf{u}$ is equal $(1,0)$ or $(0,1)$.

For example if $k = 3$ and $M = 7$ for $\mathbf{u} = (0,0)$ we have $V_{\mathbf{u}} = D_3 = \{(-3,3), (-2,2), ..., (0,0), ...(2,-2), (3,-3)\}$; that is, the sites on the line that connects $(-3,3)$ to $(3,-3)$ are considered as a one renormalized site. Also, by definition of renormalized edge for $\mathbf{v} = (1,0)$ and $\mathbf{v}' = (0,1)$, $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{u}, \mathbf{v}')$ are renormalized edge. By knowing renormalized sites and edges, we call the edge $(\mathbf{u}, \mathbf{v})$ a "good edge" if for all $\mathbf{x} \in V_{\mathbf{u}}$ there exists a blocking path to some $\mathbf{y} \in V_{\mathbf{v}}$. In fact, renormalization is another way for looking to the configuration in lattice without changing the configuration and good edges are defined in terms of the initial configuration σ. The issue of renormalization is illustrated in Figure 1.2.

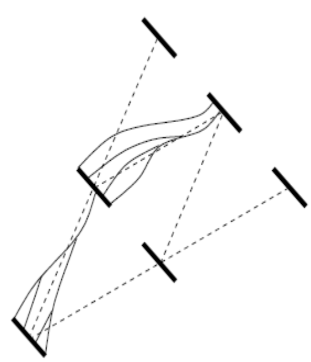

Figure 1.2: Part of renormalized lattice. Renormalized sites are indicated by bold lines, renormalized edges by dashed lines and blocking paths by curved lines. Here $((0,0),(0,1))$ and $((0,1),(1,1))$ are good edges.

Definition 6. (graph-theoretic distance) *The graph-theoretic distance $\delta(\mathbf{u}, \mathbf{v})$ from $\mathbf{u} = (u_1, u_2)$ to $\mathbf{v} = (v_1, v_2)$ is defined by $\delta(\mathbf{u}, \mathbf{v}) =$*

$|u_1 - v_1| + |u_2 - v_2|$.

Definition 7. *We say that the process of good edges is k-dependent for $k \in \mathbb{Z}$, if for any subsets A and B of edges with graph-theoretic distance at least k from each other in the renormalized lattice, the states of the edges in A are independent of the states of the edges in B.*

Lemma 8. *Let $M > 2k > 0$. The process of good edges is 30-dependent.*

Proof. By our definitions of blocking paths and good edges, the event that the edges $(\mathbf{u}, \mathbf{v})$ is good depends only on the initial states $\sigma(\mathbf{x})$ of sites $\mathbf{x}$ in some box containing $V_{\mathbf{u}}$ and $V_{\mathbf{v}}$. Now for the edge $(\mathbf{u}, \mathbf{v})$, $\mathbf{v} - \mathbf{u} = (1, 0)$ or $(0, 1)$, and a rectangle box of dimensions $(9M + 2k)$ and $(10M + 2k)$ can contain $V_{\mathbf{u}}$ and $V_{\mathbf{v}}$. See Figure 1.3. When two good edges at graph theoretic distance their distance is at least 30 in the renormalized lattice, in the $\mathbb{Z}^2$ lattice their graph theoretic distance is $570M = 30 \times 19M$. since $M > 2k$, one may control the dimensions of the isolating boxes such that the boxes separate the two good edges at graph-theoretic distance at least 30 and do not intersect each other. $\square$

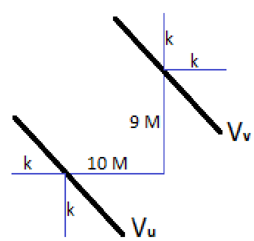

Figure 1.3: In this picture $\mathbf{v} - \mathbf{u} = (1, 0)$.

Note that in the previous lemma the number 30 is not a specific number and may be a number less than 30 satisfies too, but we choose 30 as a big enough number for our purpose.

For stating the proof of theorem 1 we are going to prove that an arbitrary edge is good with probability close to 1. First we show this for the case $p = 1$. Figure 1.4 shows all blocking paths with the initial site at the origin for a random initial configuration with $p = 1$.

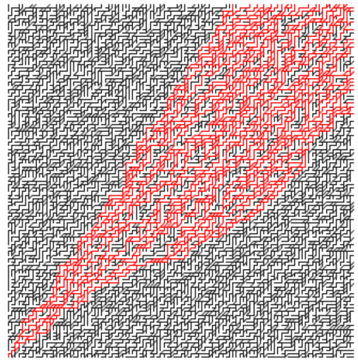

Figure 1.4: Blocking paths for a random initial configuration with $p = 1$. Blocking paths from the origin are highlighted.

In our proof a basic stage is the following lemma which says that these kind of paths are likely to come close to any site in a certain cone.

Lemma 9. *Suppose that $p = 1$ (at any site in the lattice $\mathbb{Z}^2$ there is a car). Assume that $E(\mathbf{y}, k)$ is the event that there exists a blocking path from $(0, 0)$ to $\mathbf{y} + (s, -s)$ for some $s \in [-k, k]$. There exists $c > 0$ such that for any site $\mathbf{y} = (y_1, y_2) \in \mathbb{Z}^2$ which $y_1, y_2 > 0$ and $\frac{y_1}{y_2} \in [\frac{8}{9}, \frac{9}{8}]$ we have*

$$\mathbb{P}_1[E(\mathbf{y}, k)] > 1 - e^{-ck}.$$

It can be interesting to the reader when $p = 1$ why the probability of having a blocking path from $(0, 0)$ to the mentioned $\mathbf{y}$ in the lemma is not 1. It happens because of our definition of a blocking path. See Figure 1.5 for being more clear.

First, we present and prove two propositions and then we present the proof of the lemma 9.

Proposition 10. *Consider the case $p = 1$. For each $\beta < 1$, there is M and k with $M > 2k$ such that for each renormalized edge $(\mathbf{u}, \mathbf{v})$,*

$$\mathbb{P}_1(\textit{edge } (\mathbf{u}, \mathbf{v}) \textit{ is good}) \geq \beta.$$

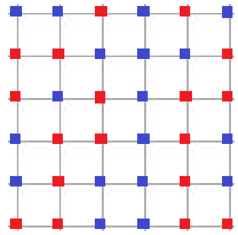

Figure 1.5: There is not any blocking path from $(0,0)$ (bottom left) to $(5,5)$.

Proof. Since $\frac{2k+1}{e^{ck}} \to 0 < 1-\beta$ as $k \to \infty$, we can take k large enough that $(2k+1)e^{-ck} < 1-\beta$. Also, since $\frac{10M}{9M} \to \frac{10}{9} < \frac{9}{8}$ and $\frac{9M}{10M} \to \frac{9}{10} > \frac{8}{9}$ as $M \to \infty$, we can take $M > 2k$ large enough that $\frac{10M}{9M} \leq \frac{9}{8}$ and $\frac{9M}{10M} \geq \frac{8}{9}$. $\mathbf{v}-\mathbf{u} = (1,0)$ or $(0,1)$, by translation invariance we can consider $\mathbf{u} = (0,0)$ and $\mathbf{v} = (1,0)$ or $(0,1)$ respectively, then for any site $\mathbf{y} \in V_{\mathbf{v}}$, $\frac{y_1}{y_2} \in [\frac{8}{9}, \frac{9}{8}]$ and by lemma 9 we have

$$\mathbb{P}_1(\textit{edge } (\mathbf{u},\mathbf{v}) \textit{ is not good}) \leq \sum_{\mathbf{x}\in V_{\mathbf{u}}} \mathbb{P}_1(\nexists \textit{ a blocking path from } \mathbf{x} \textit{ to } V_{\mathbf{v}})$$

$$\leq \sum_{\mathbf{x}\in V_{\mathbf{u}}} e^{-ck} = (2k+1)e^{-ck} < 1-\beta.$$

Therefore $\mathbb{P}_1(\textit{edge } (\mathbf{u},\mathbf{v}) \textit{ is good}) \geq \beta$. □

Proposition 11. *Let $\alpha < 1$. There exist M and k with $M > 2k$ such that for all p sufficiently close to 1, for every edge $(\mathbf{u},\mathbf{v})$*

$$\mathbb{P}_p(\textit{edge } (\mathbf{u},\mathbf{v}) \textit{ is good}) \geq \alpha. \qquad (1)$$

Proof. Take $\beta \in (\alpha, 1)$, and fix M, k according to Proposition 10. Since the event that an edge is good depends only on the initial states in a finite box, it is a polynomial in p and therefore continuous. By Proposition 10 $\mathbb{P}_1(\textit{edge } (\mathbf{u},\mathbf{v}) \textit{ is good}) \geq \beta > \alpha$. By continuity of $\mathbb{P}$ we have for all p sufficiently close to 1, $\mathbb{P}_p(\textit{edge } (\mathbf{u},\mathbf{v}) \textit{ is good}) \geq \alpha$. □

Definition 12. *Let S be a countable set, and let $Y = \{Y_x; x \in S\}$, $Z = \{Z_x; x \in S\}$ be families of random variables taking values in the set $\{0,1\}$ and indexed by S. We say that Y dominates (stochastically) Z, written $Y \geq_{st} Z$ if $E(f(Y)) \geq E(f(Z))$ for all bounded, increasing, measurable* *function* $f : \{0,1\}^S \to \mathbb{R}$ *(Here,* E *denotes the expectation operator).*

Proof of Theorem 3. By [3], section 10 page 1026, the critical probability for oriented percolation on $\mathbb{Z}^2$ is strictly less than 1 ($p_c \leq \frac{8}{9}$). Now by the results of [9] (Theorem 0.0), a percolation on $\mathbb{Z}^d$, $d \geq 1$, (or any transitive graph) which is k-dependent is stochastically dominated from below by Bernoulli percolation with parameter less than 1, with some control on the parameter. Therefore, if α is sufficiently close to 1 then any 30-dependent bond percolation process on $\mathbb{Z}^2$ satisfying (1) stochastically dominates a Bernoulli percolation process which is super-critical for oriented percolation on $\mathbb{Z}^2$.

Hence by Proposition 11 and Lemma 8, we may choose M, k such that if p is sufficiently close to 1, the event that there is an infinite path of good renormalized edges starting from $V_{(0,0)}$, oriented in the positive directions of both coordinates, occurs with positive probability. On this event, there is an infinite blocking path starting at $(0,0)$, so by Lemma 5 we have

$$\mathbb{P}_p(\textit{there is a car which never moves at } (0,0)) > 0.$$

Now consider any site $\mathbf{z}$. By translation invariance and ergodicity, it follows from the above that almost surely there are cars which never move at $\mathbf{z} + (r, 0)$ and $\mathbf{z} + (0, s)$ for some (random) $r, s \geq 0$. This implies that any car initially at $\mathbf{z}$ moves at most $\max\{r, s\}$ times, which the state at $\mathbf{z}$ changes at most $2(r+s)$ times. □

Proof of Lemma 9. We start by giving an outline of the proof. Given a "target" $\mathbf{y}$, we will algorithmically construct a blocking path $\mathbf{z}^0, \mathbf{z}^1, ...$ starting at $\mathbf{z}^0 = (0,0)$. If we use only steps of types (i) and (ii) in the definition of a blocking path, we obtain a unique random path with asymptotic direction $(1,1)$. If we also allow steps of types (iii) and (iv) then at a positive proportion of steps we have a choice of which direction to move. By always choosing the direction which moves closer to the target we are exponentially unlikely to miss the target by much, provided that the target is within a cone determined by the typical slopes that would result from choosing to go always up or always down.

We now present the details. Let $\mathbf{z}^0 = (0,0)$. Suppose that a blocking path $\mathbf{z}^0, ..., \mathbf{z}^m$ has been constructed, and suppose that $\mathbf{z}^m$ lies on the diagonal line $z_1 + z_2 = 2n$. We will extend the blocking path by one or two sites to some site on the line $z_1 + z_2 = 2n + 2$.

If $\sigma(\mathbf{z}^m) = \rightarrow$, consider the following cases:

1) If $\sigma(\mathbf{z}^m + (1,0)) = \uparrow$ we set $\mathbf{z}^{m+1} = \mathbf{z}^m + (1,0)$ and $\mathbf{z}^{m+2} = \mathbf{z}^m + (1,1)$.

2) If $\sigma(\mathbf{z}^m + (1,0)) = \sigma(\mathbf{z}^m + (1,-1)) = \rightarrow$ we set $\mathbf{z}^{m+1} = \mathbf{z}^m + (1,0)$ and $\mathbf{z}^{m+2} = \mathbf{z}^m + (2,0)$.

3) If $\sigma(\mathbf{z}^m + (1,0)) = \rightarrow$ and $\sigma(\mathbf{z}^m + (1,-1)) = \uparrow$ we have a choice: we can set either

   a) $\mathbf{z}^{m+1} = \mathbf{z}^m + (1,0)$ and $\mathbf{z}^{m+2} = \mathbf{z}^m + (2,0)$, or

   b) $\mathbf{z}^{m+1} = \mathbf{z}^m + (1,1)$ (using a blocking path step pf type (iii)).

   We choose (a) if $z_1^m - z_2^m < y_1 - y_2$, otherwise (b).

Thus we take the naive path (using steps of types (i) and (ii)) unless a step of type (iii) is possible and it moves us closer to $\mathbf{y}$ than the alternative.

On the other hand if $\sigma(\mathbf{z}^m) = \uparrow$, then:

1') If $\sigma(\mathbf{z}^m + (0,1)) = \rightarrow$ we set $\mathbf{z}^{m+1} = \mathbf{z}^m + (0,1)$ and $\mathbf{z}^{m+2} = \mathbf{z}^m + (1,1)$.

2') If $\sigma(\mathbf{z}^m + (0,1)) = \sigma(\mathbf{z}^m + (-1,1)) = \uparrow$ we set $\mathbf{z}^{m+1} = \mathbf{z}^m + (0,1)$ and $\mathbf{z}^{m+2} = \mathbf{z}^m + (0,2)$.

3') If $\sigma(\mathbf{z}^m + (0,1)) = \uparrow$ and $\sigma(\mathbf{z}^m + (-1,1)) = \rightarrow$ we have a choice: we can set either

   a) $\mathbf{z}^{m+1} = \mathbf{z}^m + (0,1)$ and $\mathbf{z}^{m+2} = \mathbf{z}^m + (0,2)$, or

   b) $\mathbf{z}^{m+1} = \mathbf{z}^m + (1,1)$ (using a blocking path step of type (iv)).

   We choose (a) if $z_1^m - z_2^m > y_1 - y_2$, otherwise (b).

It is trivial the above construction yields a blocking path $\mathbf{z}^0, \mathbf{z}^1, ...$ Suppose for the moment that $y_1 + y_2$ is even. For each n, let $\mathbf{z}^{r(n)}$ be the site at which the blocking path intersects the line $z_1 + z_2 = 2n$, and let $W_n = \frac{|(z_1^{r(n)} - z_2^{r(n)}) - (y_1 - y_2)|}{2}$. Now we claim that $(W_n)_{n \geq 0}$ is a Markov chain with transition probabilities

$$P_{j,j-1} = \frac{1}{4}, \qquad P_{j,j} = \frac{5}{8}, \qquad P_{j,j+1} = \frac{1}{8} \qquad for\ j \geq 1;$$

$$P_{0,0} = \frac{3}{4}, \qquad P_{0,1} = \frac{1}{4}.$$

For finding these data, first we show that $P_{0,0} = \frac{3}{4}$. We have $W_n = 0$ thus $z_1^{r(n)} - z_2^{r(n)} = y_1 - y_2$ and we want $W_{n+1} = 0$. It is the case if $\sigma(\mathbf{z}^{r(n)}) =\rightarrow$ and $\sigma(\mathbf{z}^{r(n)} + (1,0)) =\uparrow$, or $\sigma(\mathbf{z}^{r(n)}) =\uparrow$ and $\sigma(\mathbf{z}^{r(n)} + (0,1)) =\rightarrow$, or $\sigma(\mathbf{z}^{r(n)}) = \sigma(\mathbf{z}^{r(n)} + (1,0)) =\rightarrow$ and $\sigma(\mathbf{z}^{r(n)}(1,-1)) =\uparrow$, or $\sigma(\mathbf{z}^{r(n)}) = \sigma(\mathbf{z}^{r(n)} + (0,1)) =\uparrow$ and $\sigma(\mathbf{z}^{r(n)} + (-1,1)) =\rightarrow$.
Hence $P_{0,0} = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{3}{4}$.
Also, $P_{0,1} = 1 - P_{0,0} = \frac{1}{4}$.

Now for proving $P_{j,j} = \frac{5}{8}$, we have $W_n = j$ and we want $W_{n+1} = j$. We have $z_1^{r(n)} - z_2^{r(n)} > y_1 - y_2$ or $z_1^{r(n)} - z_2^{r(n)} < y_1 - y_2$. Without loose of generality suppose $z_1^{r(n)} - z_2^{r(n)} > y_1 - y_2$, then $W_{n+1} = j$ if one of the following cases happens: $\sigma(\mathbf{z}^{r(n)}) =\rightarrow$ and $\sigma(\mathbf{z}^{r(n)} + (1,0)) =\uparrow$, or $\sigma(\mathbf{z}^{r(n)}) =\uparrow$ and $\sigma(\mathbf{z}^{r(n)} + (0,1)) =\rightarrow$, or $\sigma(\mathbf{z}^{r(n)}) = \sigma(\mathbf{z}^{r(n)} + (0,1)) =\uparrow$ and $\sigma(\mathbf{z}^{r(n)} + (-1,1)) =\rightarrow$.
Hence $P_{j,j} = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} + \frac{1}{4} + \frac{1}{8} = \frac{5}{8}$.

For proving $P_{j,j-1} = \frac{1}{4}$, we have $W_n = j$ and we want $W_{n+1} = j - 1$. We have $y_1 - y_2 < 0$ or $= 0$ or > 0. If $y_1 - y_2 = 0$ and $z_1 - z_2 > 0$ then for having $W_{n+1} = j - 1$, $\mathbf{z}_2^{r(n+1)}$ must be $\mathbf{z}_2^{r(n)} + 2$. This will happen in cases 2' and 3'(a). Therefore $P_{j,j-1} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. If $y_1 - y_2 = 0$ and $z_1 - z_2 < 0$ then for having $W_{n+1} = j - 1$, $\mathbf{z}_1^{r(n+1)}$ must be $\mathbf{z}_1^{r(n)} + 2$. This will happen in cases 2 and 3(a). Therefore $P_{j,j-1} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.
If $y_1 - y_2 < 0$ and $z_1^{r(n)} - z_2^{r(n)} > 0$ then for having $W_{n+1} = j - 1$, $z_2^{r(n+1)}$ must be $z_2^{r(n)} + 2$. This will happen in cases 2' and 3'(a). Therefore $P_{j,j-1} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.
If $y_1 - y_2 < 0$ and $z_1^{r(n)} - z_2^{r(n)} < 0$ and $z_1^{r(n)} - z_2^{r(n)} > y_1 - y_2$ then for having $W_{n+1} = j - 1$, $z_2^{r(n+1)}$ must be $z_2^{r(n)} + 2$. This will happen in cases 2' and 3'(a). Therefore $P_{j,j-1} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.
If $y_1 - y_2 < 0$ and $z_1^{r(n)} - z_2^{r(n)} < 0$ and $z_1^{r(n)} - z_2^{r(n)} < y_1 - y_2$ then for having $W_{n+1} = j - 1$, $z_1^{r(n+1)}$ must be $z_1^{r(n)} + 2$. This will happen in cases 2 and 3(a). Therefore $P_{j,j-1} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.
If $y_1 - y_2 > 0$ we have a similar argument. Hence by the above discussion in any case we have $P_{j,j-1} = \frac{1}{4}$. Now $P_{j,j+1} = 1 - P_{j,j} - P_{j,j-1} = 1 - \frac{5}{8} - \frac{1}{4} = \frac{1}{8}$. Since $z_1^{r(n)} + z_2^{r(n)}$ and $y_1 + y_2$ are even, thus $(W_n)_{n \geq 0}$ is a random walk on the natural numbers with drift $-\frac{1}{8}$ ($\frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$). To conclude we use the following claim.

Claim 13. *For the above Markov chain* (W_n)*, there exists* $c_1 > 0$ *such that for any* $N > 9r$ *and any* k*,*

$$\mathbb{P}(W_N > k | W_0 = r) \leq e^{-c_1 k}.$$

Assuming the claim we argue Lemma 9 as follows. If $y_1 + y_2$ is even, then the lemma follows from the claim immediately. In fact,

$$\mathbb{P}_1(W_N > k | W_0 = r) \leq e^{-c_1 k} \Rightarrow \mathbb{P}_1(W_N \leq k | W_0 = r) > 1 - e^{-c_1 k}$$

$$\Rightarrow \mathbb{P}_1(E(\mathbf{y}, k)) > 1 - e^{-c_1 k}.$$

If $y_1 + y_2$ is odd, then we apply the lemma first to $\mathbf{y} - (1, 0)$ or $\mathbf{y} - (0, 1)$ and $k - 1$, and note that any finite blocking path may always be extended by one site in direction (1,0) or (0,1). $\square$

Before proving Claim 13 we recall some definitions and facts which will use in the proof. Suppose $(X_0, X_1, ...)$ is a Markov chain on a state space Ω. A **stopping time** τ for (X_t) is a $\{0, 1, ...\} \cup \{\infty\}$-valued random variable such that, for each t, the event $\{\tau = t\}$ is determined by $X_0, ..., X_t$. In other words, a random time τ is a stopping time if and only if the indicator function $1_{\{\tau = t\}}$ is a function of the vector $(X_0, X_1, ..., X_t)$.

If τ is a stopping time, then an immediate consequence of the definition and the Markov property is

$$\mathbb{P}_{x_0}\{(X_{\tau+1}, X_{\tau+2}, ..., X_l) \in A | \tau = k \textit{ and } (X_1, ..., X_k) = (x_1, ..., x_k)\}$$

$$= \mathbb{P}_{x_k}\{(X_1, ..., X_l) \in A\}.$$

For any $A \in \Omega^l$. This is referred to as the **strong Markov property**. Informally, we say that the chain "starts afresh" at a stopping time.

Let $S_n = X_1 + \cdots + X_n$. For $a > \mu = \mathbb{E}X_i$ if the **moment-generating function** $\varphi(\theta) = \mathbb{E}\exp(\theta X_i) < \infty$ for some $\theta > 0$, $\mathbb{P}(S_n \geq an) \to 0$ exponentially rapidly by **large deviation** argument in [3]. In fact, by large deviation argument in [3], if $\varphi(\theta) < \infty$ for some $\theta > 0$, $\mathbb{P}(S_n \geq na) \leq$

$\exp(-n\{a\theta - \kappa(\theta)\})$ where $\kappa(\theta) = \log \varphi(\theta)$. Also, by Lemma 2.6.2 of [3] if $a > \mu$ and $\theta > 0$ is small, then $a\theta - \kappa(\theta) > 0$.

For a Markov chain $(X_0, X_1, ...)$ on the state space $\Omega = \{0, 1, 2, ...\}$ let

- p_j is the probability of moving from j to $j+1$ when $j \geq 0$,
- q_j is the probability of moving from j to $j-1$ when $j \geq 1$,
- r_j is the probability of remaining at j when $j \geq 0$,
- $q_0 = 0$.

By Section 2.5 of [1] we have a function ω on $\Omega = \{0, 1, 2, ...\}$ given by $\omega_0 = 1$ and

$$\omega_j = \prod_{i=1}^{j} \frac{p_{i-1}}{q_i} \qquad for\ j \geq 1.$$

Normalizing so that the sum is unity yields $\pi_j = \frac{\omega_j}{\sum_{i=0}^{\infty} \omega_i}$ for $j \geq 0$. Then $\pi_j \to \pi$ as $j \to \infty$ and by Proposition 2.8 of [1] π is the stationary distribution.

We define a **coupling of Markov chains** with transition matrix P to be a process $(X_t, Y_t)_{t=0}^{\infty}$ with the property that both (X_t) and (Y_t) are Markov chains with transition matrix P, although the two chains may possibly have different starting distributions. Any coupling of Markov chains with transition matrix P can be modified so that the two chains stay together at all times after their first simultaneous visit to a single state– more precisely, so that

$$if\ \ X_s = Y_s,\ then\ X_t = Y_t\ for\ t \geq s. \qquad (*)$$

To construct a coupling satisfying $(*)$, simply run the chains according to the original coupling until they meet; then run them together.

Proof of Claim 13. Since the chain has increments at most 1, we have $W_N \leq r + N < \frac{N}{9} + N < 2N$. Hence the probability in question is zero when $k > 2N$, so we may assume $k \leq 2N$.

Let T be the first time (W_n) hits 0. Before T, the increments are i.i.d with mean $\mu = \mathbb{E}X_i = -\frac{1}{8}$. We have the moment-generating function $\varphi(\theta) =$

$\mathbb{E}\exp(\theta X_i) \le \frac{1}{4}e^{\theta.1} + \frac{1}{4}e^{\theta.(-1)} + \frac{3}{4}e^{\theta.0}$, therefore $\varphi(\theta) < \infty$ for some $\theta > 0$. Now

$$\mathbb{P}(T > N) \le \mathbb{P}(W_N > 0) \le \mathbb{P}(\sum_{i=1}^{N} X_i > -r) \le \mathbb{P}(\sum_{i=1}^{N} X_i > -\frac{1}{9}N)$$

$$\le_{(large\ deviation)} \exp(-N\{-\frac{1}{9}\theta - \kappa(\theta)\}).$$

Since $-\frac{1}{9} > -\frac{1}{8}$ we have $-\frac{1}{9}\theta - \kappa(\theta) > 0$ by choosing $\theta > 0$ small. Therefore, put $c_2 = -\frac{1}{9}\theta - \kappa(\theta)$ we have $\mathbb{P}(T > N) \le e^{-c_2 N} \le e^{-c_2 k/2}$. Therefore, applying the strong Markov property at T, the claim will follow if we can establish for fixed $c_3 > 0$ and all $n \ge 0$ that $\mathbb{P}(W_n > k | W_0 = 0) \le e^{-c_3 k}$. To check this, note that $\omega_j = (\frac{1/8}{1/4})^j = (\frac{1}{2})^j$ and $\pi_j = \frac{\omega_j}{\sum_{i=0}^{\infty}\omega_i} = \frac{(\frac{1}{2})^j}{2} = (\frac{1}{2})^{j-1}$, then $\pi = \lim_{j\to\infty}\pi_j$ is the stationary distribution, and observe that we may couple (W_n) with a stationary copy $(\widetilde{W_n})$ in such a way that $W_n \le \widetilde{W_n}$ for all n, then note that the stationary distribution has exponentially decaying tail. $\square$

In Figure 1.6 we can see for $p = 0.1$ and $p = 0.3$ the system organized itself and has free-flowing, for $p = 0.32$ the system organized itself and has free-flowing while it has local jammed. For $p = 0.34$ and $p = 0.8$ the system is globally jammed in both cases but the former is organized while the latter one is not. In fact, for $p = 0.34$ by having a part of the lattice we can guess the structure of the system for rest of the lattice but for $p = 0.8$ we do not have this possibility.

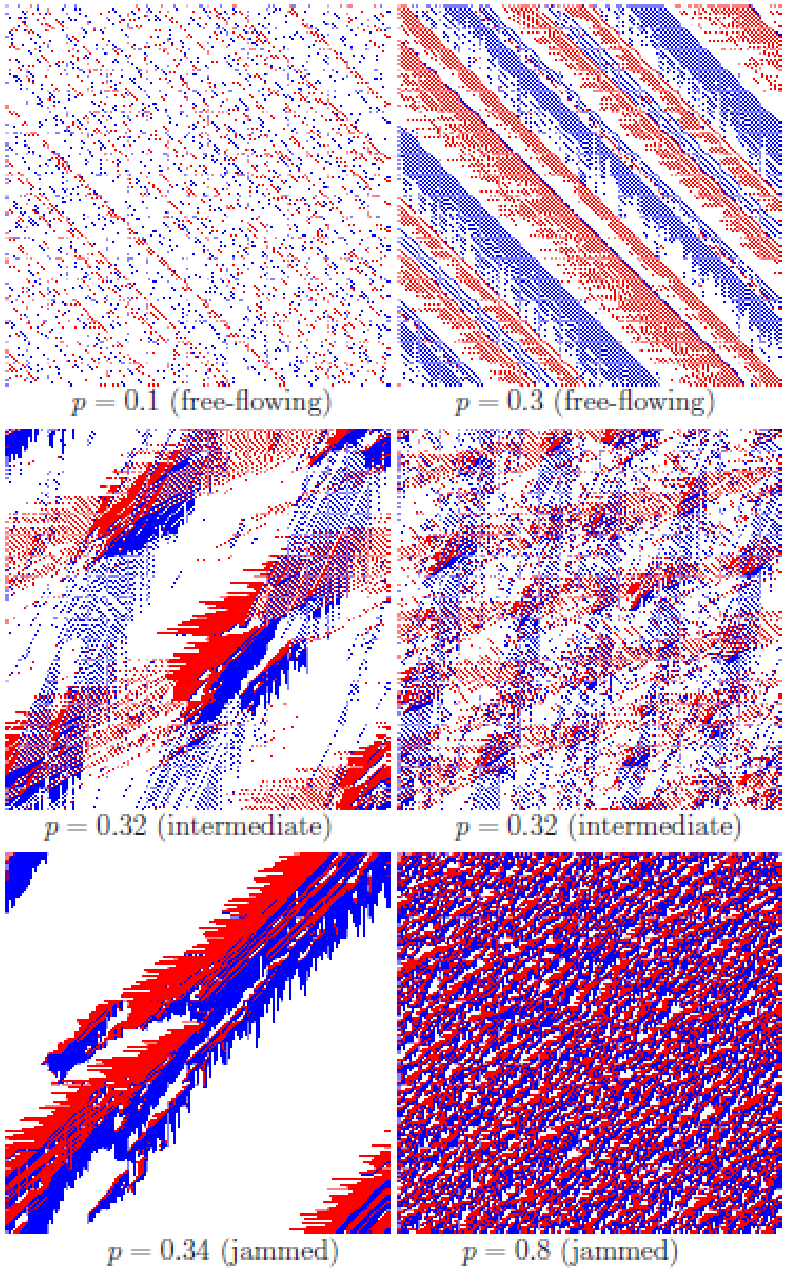

Figure 1.6: Example of the model after 20,000 steps on a 200-by-200 torus. East-facing and North-facing are shown in red and blue respectively.

Chapter 2

The Biham-Middleton-Levine Traffic Model for a Single Junction

2.1 The Junction Model

In this chapter we introduce a slight variant of the original BML model by permitting a car (say, red) to move not only if there is vacant place right next to it but also if there is a red car next to it that moves. Thus, for sequence of red cars placed in a row with a single vacant place to its right - all cars will move together (as oppose to only rightmost car in the sequence for the original BML model). Not only does this new variant exhibits the same phenomena of self-organization and phase transition, they even seem to appear more quickly (i.e. it takes less time for the system to reach a stable state). Note that the result of the first chapter appear to apply equally well to the variant model.

In the following, we will analyze a simplified version of the BML model: BML on a single junction. Meaning, we place red cars in some density p on a single row of the torus, and blue cars are placed in density p on a single column. For $p < 0.5$ we will show the system reaches velocity 1, while for $p > 0.5$ the velocity cannot be 1, but the system will reach the same velocity, regardless of the initial configuration. Moreover, at $p = 0.5$ the system's behavior undergoes a phase transition: we will prove that while for $p < 0.5$ the stable configuration will have linearly many sequences of cars, for $p > 0.5$ we will have only $O(1)$ different sequences after some time. We will also examine what happens at a small window around $p = 0.5$.

Pay attention in the variant BML model car sequences are never split. Therefore, the simplified version of the variant BML model can be viewed as some kind of 1-dimensional coalescent process.

We start with the exact definition of the simplified model. On a cross shape, containing a single horizontal segment and a single vertical segment, both of identical length N, red cars are placed in exactly pN randomly (and uniformly) chosen locations along the row, and blue cars are similarly placed in pN locations along the column. For simplicity we may assume that the junction is left unoccupied. Also, the segments are cyclic (periodic boundary condition).

At each turn, all the red cars move one step to the right, except that the red car that is just left of the junction will not move if a blue car is in the junction (i.e. blocking it), in which case also all red cars immediately following it will stay still. Afterwards, the blue cars move similarly, with red and blue roles switched.

2.2 Time-normalized model

Though less natural, it will be sometimes useful to consider the equivalent description, in which two rows of cars - red and blue - are placed one beneath the other, with "special phase" -the junction- where at most one car can be present at any time. In every step first the red line shifts one to right (except cars immediately to the left of the junction, if it contains a blue car) and then the blue line does the same. Furthermore, instead of having the cars move to the right, we can have the junction moves to the left, and when a blue car is in the junction, the (possibly empty) sequence of red cars immediately to the left of the junction moves to the left, and vice verse. Figure 2.1 clarifies the correspondence between these models.

From the discussion above we get the following equivalent system, which

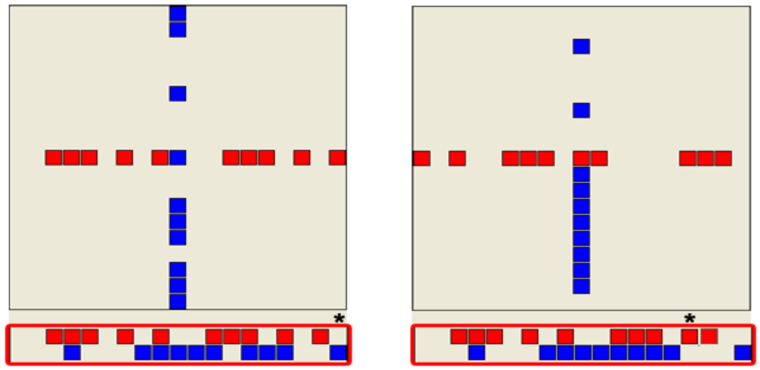

Figure 2.1: On the left hand side is a junction configuration and analogous configuration under it. The junction is showed by star. On the right hand side there is the same configuration after 3 turns in both views.

we will call the time-normalized junction:

1. Fix $S = N - 1 \in \mathbb{Z}_N$ and fix some initial configuration $\{R_i\}_{i=0}^{N-1}$, $\{B_i\}_{i=0}^{N-1} \in \{0,1\}^N$ representing the red and blue cars respectively, i.e. $R_i = 1$ if and only if there is a red car at the i-th place. We require that $\sum R_i = \sum B_i = p.N$, and at place $S(= N - 1)$ there is at most one car in both rows which means that the junction itself can contain only one car at the beginning.

2. In each turn:

   - If place S contains a blue car, and place $S - 1$ contains a red car (if $B_S = R_{S-1} = 1$), push this car one step to the left. By pushing the car we mean also moving all red cars that immediately followed it one step to the left, i.e. set $R_{S-1} = 0$, $R_{S-i} = 1$ for $i = min_{j\geq 1}[R_{S-j} = 0]$.
   - If place S does not contain a blue car and place $S - 1$ contains both a red and a blue car (if $B_S = 0$ and $R_{S-1} = B_{S-1} = 1$), push the blue car at $S - 1$ one step to the left (set $B_{S-1} = 0$ and $B_{S-i} = 1$ for $i = min_{j\geq 1}[B_{S-j} = 0]$).
   - set $S = S - 1$.

Note that the time-normalized model guarantee that always there is at most one car at the junction. Also, when the system flows freely the time-normalized system configuration does not change except the position of the junction, and cars in the time-normalized system move only when some cars are waiting for the junction to clear in the non-time-normalized system.

2.3 Analysis of an (N, p) junction

By investigating the junction we will show that for any p, regardless of the initial configuration, the system will approach to an optimal velocity that depends only on p. First we clarify what is an optimal velocity.

Theorem 1. *For a junction with density p, the maximum of velocity for any initial configuration is* $\min(1, \frac{1}{2p})$.

Proof. It is straightforward that the velocity cannot exceed 1. Consider the system when it is in its stable state, and denote the velocity by s. Therefore, at time t, a car has gone forward ts steps (on average); it means that it has passed the junction $\frac{ts}{N}$ times because we have N sites on each line. But just one car can occupy the junction at any time, all number of cars passing the junction until time t is bounded by t. Now on both lines we have $2pN$ cars, therefore, we have $2pN \times \frac{ts}{N} \leq t$. This implies that $s \leq \frac{1}{2p}$. $\square$

We will demonstrate the system with any initial configuration approaches to these velocities.

2.4 The case $p < 0.5$

In this section we want to prove that for $p < 0.5$ the junction eventually reachs velocity very close to 1.

Lemma 2. *A junction is free-flowing (no car is ever waiting to enter the junction) if and only if the time-normalized junction satisfies the following:*
1) For all $0 \leq i \leq N-1$ there exists only one car in site i in both rows.
2) For all $0 \leq i \leq N-1$ if site i contains a blue car, site $(i-1)$ mod N does not contain a red car.

Proof. $\Rightarrow$) Suppose that we have a free-flowing system. We have (1) otherwise suppose that without loose of generality at site $N-2$, exactly before the junction, there are a red and a blue car. Now
i) if the next movement is for red cars then when the blue car wants to move to the junction, it cannot because the junction is occupied by the red car.
ii) if the next movement is for blue cars, we have the same argument.
This is contradiction with the free-flowing system (This argument is followed by time-normalized model, step 2, condition 2).
For proving (2), without loose of generality suppose that at site $N-1$ we have a blue car and at site $N-2$ we have a red car then by time-normalized model step 2 condition 1 we do not have a free-flowing system and is a contradiction.

$\Leftarrow$) Suppose that we have condition (1) and (2), therefore, in time-normalized model, step 2 conditions (1) and (2) never happen and in all time we have the order: "set $S = S - 1$;" which means that the system is free-flowing. $\square$

Now we will show that for $p < 0.5$ the system of time-normalized junction necessarily reaches to a free-flowing state or at worst situation to almost free-flowing state, it means that the velocity of the system will be arbitrary close to 1 for large enough N.

For doing this we consider a configuration and look at the sets of violations of the previous lemma. In fact, we consider the places that have both red and blue cars and the places that has a blue car and exactly one site left to it contains a red car.

Let For a configuration R and B we define two disjoint sets for two types of violations:

$$V_R = \{0 \leq i \leq N-1 : R_{i-1} = B_i = 1\}$$

$$V_B = \{0 \leq i \leq N-1 : R_{i-1} = B_{i-1} = 1, B_i = 0\}$$

Let $V = V_R \cup V_B$, and consider the indicators $X = \{X(i)\}_{i=0}^{N}$ where

$$X(i) = \begin{cases} 1 & i \in V \\ 0 & i \notin V \end{cases} .$$

For a junction with an initial configuration R and B, let R^t and B^t be the system configuration at time t, and $V_t = V_{R^t} \cup V_{B^t}$ be the set of violations for this configuration, and X^t be the corresponding indicator vector, and S^t denote the junction's position at time t.

The following lemma will show that the size of the set of violations is strongly related to the system velocity, and has an important property. In fact, it is non-increasing in time.

Lemma 3.
1) $|V_{t+1}| \leq |V_t|$.
2) For any t*, the system velocity is at least* $(1 + \frac{|V_t|}{N})^{-1} \geq 1 - \frac{|V_t|}{N}$.

Proof. For the first part of the lemma we examine the three possible cases which can happen at time t:

1. If at time t, S^t does not contain to a violation then in the time-normalized model the configurations do not change for $t+1$, and $R^{t+1} = R^t$, $B^{t+1} = B^t$. Hence $|V_{t+1}| = |V_t|$.

2. If $S^t \in V_{B^t}$, then the configuration B^{t+1} changes in two places:

   a) $B^t_{S^t-1}$ is changed from 1 to 0, $B^{t+1}_{S^t-1} = 0$. By this change S^t is no longer in V_{B^t} and $X^t(S^t)$ changes from 1 to 0.

   b) $B^t_{S^t-i}$ is changed from 0 to 1 for $i = \min_{j\geq 1}(B^t_{S^t-j} = 0)$. This change may affect $X^{t+1}(S^t - i)$ and $X^{t+1}(S^t - i + 1)$. But by changing $B^t_{S^t-i}$ from 0 to 1 no violation can be created for place $S^t - i + 1$ because by definition of i in time-normalized model $B^t_{S^t-i+1} = 1$ and $B^{t+1}_{S^t-i+1} = 1$ so $X^{t+1}(S^t - i + 1) = 1$ if and only if $R^t_{S^t-i} = R^{t+1}_{S^t-i} = 1$ regardless of $B^{t+1}_{S^t-i}$.
   For other indices because R and B do not change $X^{t+1}(i) = X^t(i)$. Therefore, from time t to $t+1$ we have $X^t(S^t)$ changes from 1 to 0, and at worst situation only $X^t(S^t - i)$ changes from 0 to 1. Hence $|V_{t+1}| = \sum_{i=0}^{N-1} X^{t+1}(i) \leq \sum_{i=0}^{N-1} X^t(i) = |V_t|$.

3. Similarly, if place $S^t \in V_{R^t}$, then the configuration R^{t+1} changes in two places:

   a) $R^t_{S^t-1}$ is changed from 1 to 0, $R^{t+1}_{S^t-1} = 0$. By this change S^t is no longer in the set of violations and $X^t(S^t)$ changes from 1 to 0.

   b) $R^t_{S^t-i}$ is changed from 0 to 1 for $i = \min_{j\geq 1}(R^t_{S^t-j} = 0)$. It may affect $X^{t+1}(S^t - i)$ and $X^{t+1}(S^t - i + 1)$. However, for place $S^t - i$ changing $R^t_{S^t-i}$ does not affect whether this place is a violation or

not, because we do not change $R^t_{S^t-i-1}$ and B^t. Hence at worst case $X^t(S^t-i+1)$ changes from 0 to 1.
Now by the same argument we get $|V_{t+1}| \leq |V_t|$.

Finally, by these three possibilities we have $|V_{t+1}| \leq |V_t|$.

For the second part of the lemma we can see in the time-normalized system, considering a specific car in the system, its velocity is $\frac{N}{N+k} = (1+\frac{k}{N})^{-1}$ where k is the number of times the car pushed to the left during the last N system turns (in fact, k is the number of attempts for movement which fails to move). We note that if a car at place j is pushed to the left at some time t by some violation at place S^t, this violation can reappear only to the left of j, therefore, it can push the car again only after S^t passes j. Thus any violation can push a car to the left only one time in a car's cycle (car's cycle is N moves). Because by part (1) of the lemma at any time t onwards the number of violations in the system is at most $|V_t|$, therefore, each car can push left only $|V_t|$ times in N turns and its velocity from time t onwards is at least $(1+\frac{|V_t|}{N})^{-1}$ and we know $N^2 > N^2 - |V_t|^2 = (N+|V_t|)(N-|V_t|)$. Therefore, $(1+\frac{|V_t|}{N})^{-1} > 1 - \frac{|V_t|}{N}$ as asserted. □

By having this lemma we are ready to prove system self organization for $p < 0.5$. We are going to show that for $p < 0.5$, after $2N$ system turns $|V_t| = O(1)$, therefore by second part of the previous lemma the system approaches to velocity $1 - \frac{O(1)}{N} \to 1$ as $N \to \infty$.

As the junction goes to the left it pushes some cars, hence affecting the configuration to its left. The next lemma will prove that when $p < 0.5$, for some $T < N$, the number of cars affected to the left of the junction is only a constant, independent of N.

Lemma 4. *Consider a junction with density $p < 0.5$. There exists a constant $C = C(p) = \frac{p}{1-2p}$, independent of N such that:*
From any configuration R, B with junction at site S there exist some $0 < T < N$ such that after T turns:
1) For $i \in \{S-T, ..., S\}$, $X^T(i) = 0$ (there are no violations there).
2) For $i \in \{S+1, ..., N-1, 0, ..., S-T-C\}$, $R^T_i = R^0_i$ and $B^T_i = B^0_i$ (R, B are unchanged there).

Proof. First suppose that $T = 1$. If $T = 1$ does not satisfy in the results of the lemma, we need to have a car sequence red or blue of length more than C, which is pushed left by junction as it moves. Following this process, if for some $T < N - C$, the sequence currently pushed by the junction has length less than C, then this is the desired T. Hence, for the results not to satisfy, the length of the car sequence pushed by the junction must be more than C for all $0 < T < N$. If this happens, then leaving the junction we see alternating red and blue sequences, all of lengths more than C and one vacant place after any blue sequence an before any red one.

However, if this happens for all $0 < T < T'$ then the average number of cars per site in $\{S-T', ..., S\}$ at time T' must be at least $\frac{2C}{2C+1}$ (at least $2C$ cars between vacant places, because between vacant places we have a sequence of red cars and a sequence of blue cars, and each of them has length more than C). Therefore, all number of cars in $\{S - T', ..., S\}$ at time T' is more than $\frac{2C}{2C+1}T' = \frac{(2p)/(1-2p)}{(2p+1-2p)/(1-2p)}T' = 2pT'$.

But we have only $2pN$ cars in the system, and this cannot hold for all T up to N and is contradiction. Therefore, there is some $0 < T < N$ for which the results of the lemma is satisfied. $\square$

Now we can demonstrate the main theorem of this section.

Theorem 5. *Consider a junction of size N with density $p < 0.5$. For any initial configuration the system reaches the velocity $1 - \frac{C(p)}{N}$.*

Proof. Let R, B be an initial configuration with the junction $S = N - 1$ and suppose that V is the corresponding set of violations and X its indicators vector. By the previous lemma there exists $T_0 > 0$ such that $X^{T_0}(i) = 0$ for $i \in [N - 1 - T_0, N - 1]$. Now starting at R^{T_0}, B^{T_0} and $S = N - 1 - T_0$ and using the lemma again, there exists $T_1 > 0$ such that $X^{T_0+T_1}(i) = 0$ for $i \in [N - 1 - T_0 - T_1, N - 1 - T_0]$, and also, as long as $N - 1 - T_0 - T_1 > C(p) = \frac{p}{1-2p}$, $X^{T_0+T_1}(i) = X^{T_0}(i) = 0$ for $i \in [N - 1 - T_0 - T_1, N - 1]$ as well.
Continuing in this way until $T = \sum T_i \leq N$ we have after T turns, $X^T(i) = 0$ for all but at most $C(p)$ places. Therefore, by lemma 3 the system velocity after this time is at least $1 - \frac{C(p)}{N}$. $\square$

We note no one can show that the exact velocity is 1 for the case $p < 0.5$, because by Figure 2.2 you see that this is not the case and in this figure the junction has velocity $1 - \frac{1}{N}$ for all N.

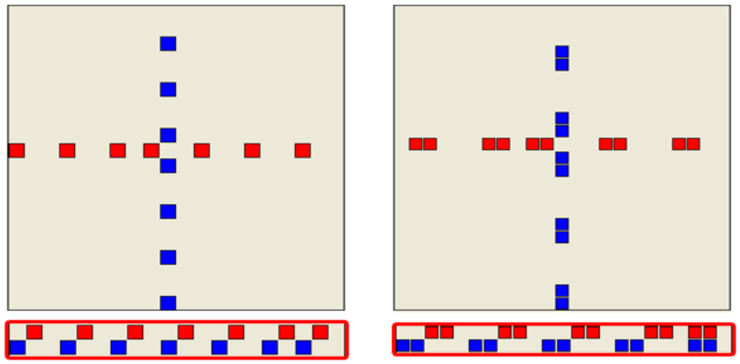

Figure 2.2: On the left hand side there is a junction configuration with density $p = \frac{1}{3}$ ($\frac{p}{1-2p} = 1$) which never reaches velocity 1, and its velocity is $1 - \frac{1}{N}$. On the right hand side there is a similar construction for $p = 0.4$ ($\frac{p}{1-2p} = 2$) and approaching velocity $1 - \frac{2}{N}$.

2.5 Number of segments for $p < 0.5$

Here we want to state and prove a theorem which gives us information about the number of segments of cars for $p < 0.5$. There are configurations such that the number of different segments of cars in each row is $\Theta(N)$. Trivially, if the cars are arranged in a single red sequence and a single blue sequence in the initial configuration, we are going to have only one sequence of each color of cars at any time.

Nevertheless, in the following theorem we go to prove that for a random initial configuration, the system has linearly many different segments of cars with high probability.

Theorem 6. *Consider a junction of size N with density $p < 0.5$, and with a random initial configuration. It has $\Theta(N)$ different segments of cars at all times with high probability.*

Proof. In the proof of Lemma 4 we see that, as the system completes a full round, at each place we have at most a single car, therefore, there must be $2pN$ places with car and $(1-2p)N$ places in which no car is present. Each two empty places which are not adjacent must correspond to a segment of cars in the configuration.

By time-normalized model it is clear that the number of places for which $R_i = B_i = R_{i-1} = B_{i-1} = 0$ is non-increasing. In other words, only places for which $R_i = B_i = R_{i-1} = B_{i-1} = 0$ in the initial configuration remain like this in the future and two empty places do not connect to each other. In a random initial configuration with density p, the initial number of these places is expected to be $(1-p)^4 N$ (each site is empty with probability $1-p$ and we want to R_i, B_i, R_{i-1}, and B_{i-1} be empty), and by standard Central Limit Theorem, we obtain that with high probability this number is at most $((1-p)^4 + \varepsilon)N$, for arbitrary $\varepsilon > 0$. Two empty places cannot decrease the number of segments of cars because they cannot connect to each other in the future. Therefore, the number of different segments of cars in the system configuration at any time is at least $((1-2p) - (1-p)^4 - \varepsilon)N$. But for p very close to 0.5 this number may be negative, hence does not suffice.

To take out this problem, note that also for any fixed K, we have the number of consecutive K empty places in a configuration is non-increasing by the system dynamics, and with high probability is at most $((1-p)^{2K} + \varepsilon)N$ for a random initial configuration. This guarantees we have at least $\frac{(1-2p)-(1-p)^{2K}-\varepsilon}{K-1}N$ different segments of cars in the system. By choosing $K(p)$ sufficiently large such that $(1-p)^{2K} < (1-2p)$ we obtain a linear lower bound for the number of segments of cars from a random initial configuration (note that K is independent of N). $\square$

2.6 The case $p > 0.5$

In this part we give a definition and state two facts about the system, then we continue to survey the system by stating and proving some lemmas and theorems.

Definition (*stable configuration*). A *stable configuration* for the system is a configuration that reappears after running the system for some M turns.

The proof of having optimal velocity and segment structure for $p > 0.5$ depend strongly on combinatorial properties and a stable configuration. First, we note that the number of possible configurations for the system is finite for fixed N, therefore, the system necessarily will reach to a stable configuration regardless of the initial configuration.

Now we state two simple facts that ought to hold after a system reached a stable configuration:

(a) $|V_t|$ cannot change. It means that no violation can disappear. This fact is clear by Lemma 3.

(b) Two disjoint segments of car cannot merge to one.

For proving (b) suppose that in a stable configuration we have n segments of car. If after some time two segments of car merge to one, it means that we have $n-1$ segments of car in new time, but again when the system comes back to the stable configuration we ought to have n segments of car and this is contradiction because we know that the number of segments in the system is non-increasing in time.

Note. In a stable configuration R, B, when $S=0$, $B_0=0$, and there is a sequence of exactly s_R consecutive red cars at places $[N-s_R, N-1]$ and s_B blue cars at places $[N-s_B, N-1]$, then $\min(s_R, s_B)$ is equal s_R. By contradiction, suppose $s_B < s_R$, then we have s_B violations for overlapping places of the segments s_R, s_B and one violation for the place $S=0$ ($B_{N-1}=R_{N-1}=1$ and $B_0=0$), therefore we have s_B+1 violations in the stable configuration. Now run the system for s_R-s_B turns. Note that at places $[N-s_R, N-s_B]$ there is not any blue cars otherwise two segments of blue cars merge to one which is impossible by (b). At time $t=s_R-s_B$, s_R and s_B segments have the same endpoint and $R_{N-s_R-1}=0$. Hence, the endpoint of the segments is not a violation and at this time we have s_B violations in the stable configuration. But this is contradiction, since by (a) the number of violations cannot change. Therefore, $s_R \leq s_B$. See Figure 2.3.

Figure 2.3: This figure illustrates the contradiction. In the left-hand side there are 4 violations and after 3 turns there are 3 violations.

These two facts give us a lot of information about the stable configuration for $p > 0.5$. Now we give twin lemmas on the stable configuration.

Lemma 7. *Let R, B be a stable configuration with junction $S = 0$ and $B_0 = 0$. Suppose that there exists a sequence of exactly s_R consecutive red cars at places $[N - s_R, N - 1]$ and s_B blue cars at places $[N - s_B, N - 1]$, for some $s_R, s_B \geq 1$. Then:*

1) $B_i = 0$ for $i \in [N - s_R - s_B - 1, N - s_B - 1]$.

2) $R_i = 1$ for $i \in [N - s_R - s_B - 1, N - \max(s_R, s_B) - 2]$.

3) $R_i = 0$ for $i \in [N - \max(s_R, s_B) - 1, N - s_R - 1]$.

Proof. By assumption:

- $B_i = 1$ for $i \in [N - s_B, N - 1]$, $B_{N-s_B-1} = 0$.

- $R_i = 1$ for $i \in [N - s_R, N - 1]$, $R_{N-s_R-1} = 0$.

For proving item (1), pay attention that $B_0 = 0$, $B_{N-1} = R_{N-1} = 1$, hence the blue sequence will be pushed to the left in the next s_R turns. Now by fact (b) because two disjoint blue segments do not merge, we have $B_i = 0$ for $i \in [N - s_B - s_R - 1, N - s_B - 1]$. Therefore, following the system after s_R turns we have: $B_i = 1$ for $i \in [N - s_R - s_B, N - s_R - 1]$ (the blue segment pushed to the left s_R times), and $B_i = 0$ for $i \in [N - s_R, N - 1]$ and B_i not changed left to the $N - s_R - s_B$, and R is unchanged.

For proving item (2), note that $s_R = \min(s_R, s_B)$ then R, B contained s_R consecutive violations in places $[N - s_R + 1, 0]$ which all vanished after s_R turns, and possible violations at places $[N - s_B, N - s_R]$ remained as they were. But by the fact (a) no violation can disappear, therefore, we get that we must have $R_i = 1$ for s_R places within $[N - s_R - s_B - 1, N - s_B - 1]$. But R_{N-s_B-1} cannot be 1 because if $R_{N-s_B-1} = 1$ then exactly one place within $[N - s_R - s_B - 1, N - s_B - 2]$ is empty in the red row, after s_R turns there is a violation in $N - s_B$ and $B_{N-s_B} = 1$, hence the segment of red car which contains $N - s_B - 1$ pushed one place to the left and two red segments merge to one. This is contradiction with (b). Hence, $R_{N-s_B-1} = 0$ and we ought to have $R_i = 1$ for $i \in [N - s_R - s_B - 1, N - s_B - 2]$. In other words, $R_i = 1$ for $i \in [N - s_R - s_B - 1, N - \max(s_R, s_B) - 2]$.

For proving item (3), note that $s_B = \max(s_R, s_B)$ then follow the system for s_B turns we note that any red car in $[N - s_B - 1, N - s_R - 1]$ will be pushed left until eventually hitting the red car already proven to be present at place $N - \max(s_R, s_B) - 2$ and this is contradiction by (b). Thus $R_i = 0$ for $i \in [N - s_B - 1, N - s_R - 1]$. Therefore, $R_i = 0$ for $i \in [N - \max(s_R, s_B) - 1, N - s_R - 1]$. $\square$

The next lemma is exactly similar when we are reversing the roles of R, B, and can be proven the same way.

Lemma 8. *Let R, B be a stable configuration with junction at $S = 0$ and $B_0 = 1$. Assume that there is a sequence of exactly s_R consecutive red cars at places $[N - s_R, N - 1]$ and s_B blue cars at places $[N - s_B + 1, 0]$, $s_R, s_B \geq 1$. Then:*

1) $R_i = 0$ *for* $i \in [N - s_B - s_R - 1, N - s_R - 1]$.

2) $B_i = 1$ *for* $i \in [N - s_B - s_R, N - \max(s_B, s_R) - 1]$.

3) $B_i = 0$ *for* $i \in [N - \max(s_B, s_R), N - s_B]$.

By Lemmas 7 and 8 together we get the following characterization for stable configuration:

Lemma 9. *Let R, B be a stable configuration with junction at $S = 0$ and $B_0 = 0$. Suppose that there is a sequence of exactly s_R consecutive red cars at places $[N - s_R, N - 1]$ and s_B consecutive blue cars at places $[N - s_B, N - 1]$. Denote $M = \max(s_R, s_B)$. Then:*

a) There are no additional cars at $[N - M, N - 1]$.

b) Place $i = N - M - 1$ *is empty. That is* $R_i = B_i = 0$.

c) Starting at $N - M - 2$ *there is a sequence of* $K_1 \geq \min(s_R, s_B)$ *places for which* $R_i = 1$, $B_i = 0$, $i \in [N - M - K_1 - 1, N - M - 2]$.

d) Starting at $N - M - K_1 - 2$ *(right after the red sequence) there is a sequence of* $K_2 \geq \min(s_R, s_B)$ *places for which* $B_i = 1$, $R_i = 0$, $i \in [N - M - K_1 - K_2 - 1, N - M - K_1 - 2]$.

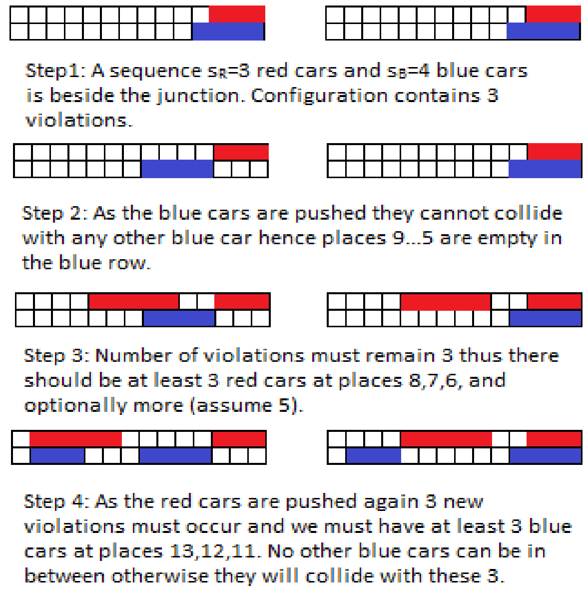

Figure 2.4: Sketch of proof ideas for Lemma 7.

Proof. First of all by Lemma 7 we get:
1) $B_i = 0$ for $i \in [N - s_R - s_B - 1, N - s_B - 1]$.
2) $R_i = 1$ for $i \in [N - s_R - s_B - 1, N - \max(s_R, s_B) - 2]$.
3) $R_i = 0$ for $i \in [N - \max(s_R, s_B) - 1, N - s_R - 1]$.

From (1) we get $B_i = 0$ for $i \in [N - \max(s_R, s_B) - 1, N - s_B - 1] \subseteq [N - s_R - s_B - 1, N - s_B - 1]$. From (3) we get $R_i = 0$ for $i \in [N - \max(s_R, s_B) - 1, N - s_R - 1]$. Therefore, no additional cars are at $[N - M, N - 1]$, and place $N - M - 1$ is empty, proving claims (a) and (b) in the lemma.

From (1) we get $B_i = 0$ for $i \in [N - s_R - s_B - 1, N - \max(s_R, s_B) - 2] \subseteq [N - s_R - s_B - 1, N - s_B - 1]$, and from (2) $R_i = 1$ for $i \in [N - s_R - s_B - 1, N - \max(s_R, s_B) - 2]$, hence places $[N - s_R - s_B - 1, N - \max(s_R, s_B) - 2]$ contain a sequence of length $\min(s_R, s_B)$ of red cars with no blue cars in parallel to it. This sequence is possibly a part of a larger sequence of length $s'_R \geq \min(s_R, s_B)$, located at $[N - \max(s_R, s_B) - s'_R - 1, N - \max(s_R, s_B) - 2]$.

Now running the system for $\max(s_R, s_B) + 1$ turns, we are going to have the junction at place $S = N - \max(s_R, s_B) - 1$, when we are running the system for $\max(s_R, s_B) + 1$ the blue sequence is pushing s_R turns to left, then $B_S = 1$, followed by sequences of s'_R red cars and $s'_B = \min(s_R, s_B) (\leq s'_R)$ blue cars (note that $s_B = \max(s_R, s_B)$ when we run the system for $s_B + 1$

turns the blue sequence pushed left s_R times and $S = N - s_B - 1$ and the blue sequence is from $N - s_R - 1$ until $N - s_R - s_B - 1$, therefore, from $S = N - s_B - 1$ we have a blue sequence of length $s_R = \min(s_R, s_B)$). We apply Lemma 8 for the system for $N' = N - \max(s_R, s_B) - 1$:

1') $R_i = 0$ for $i \in [N' - s'_B - s'_R - 1, N' - s'_R - 1] = [N - \max(s_R, s_B) - \min(s_R, s_B) - s'_R - 2, N - \max(s_R, s_B) - s'_R - 2] = [N - s_R - s_B - s'_R - 2, N - \max(s_R, s_B) - s'_R - 2]$.

2') $B_i = 1$ for $i \in [N' - s'_B - s'_R, N' - \max(s'_R, s'_B) - 1] = [N - \max(s_R, s_B) - \min(s_R, s_B) - s'_R - 1, N - \max(s_R, s_B) - \max(s'_R, s'_B) - 2] = [N - s_R - s_B - s'_R - 1, N - \max(s_R, s_B) - s'_R - 2]$.

3') $B_i = 0$ for $i \in [N' - \max(s'_R, s'_B), N' - s'_B] = [N - \max(s_R, s_B) - s'_R - 1, N - \max(s_R, s_B) - \min(s_R, s_B) - 1] = [N - \max(s_R, s_B) - s'_R - 1, N - s_R - s_B - 1]$.

From (1) we have $B_i = 0$ for $i \in [N - s_R - s_B - 1, N - s_B - 1]$ and $N - \max(s_R, s_B) - 2 \leq N - s_B - 1$, therefore $B_i = 0$ for $i \in [N - s_R - s_B - 1, N - \max(s_R, s_B) - 2]$, by $(3')$ we have $B_i = 0$ for $i \in [N - \max(s_R, s_B) - s'_R - 1, N - s_R - s_B - 1]$. Hence, we have no blue cars are in parallel to the entire red segment in $[N - \max(s_R, s_B) - s'_R - 1, N - \max(s_R, s_B) - 2]$.

Moreover, by $(2')$ we have a sequence of blue cars in $[N - s_R - s_B - s'_R - 1, N - \max(s_R, s_B) - s'_R - 2]$ with no red cars parallel to it in $[N - s_R - s_B - s'_R - 2, N - \max(s_R, s_B) - s'_R - 2]$ by $(1')$. Note that $N - \max(s_R, s_B) - s'_R - 2$ is exactly to the left of $N - \max(s_R, s_B) - s'_R - 1$ where the red sequence ended. Now by choosing $K_1 = s'_R$ and $K_2 = s'_B = \min(s_R, s_B)$ we get claims (c) and (d) in the lemma. □

Theorem 10. *Let R, B be a stable configuration with junction at $S = 0$ and $B_0 = 0$. Suppose that there is a sequence of exactly s_R consecutive red cars at places $[N - s_R, N - 1]$ and s_B blue cars at places $[N - s_B, N - 1]$. Denote $M = \max(s_R, s_B)$. Then no additional cars are at $[N - M, N - 1]$, and at places $[0, N - M - 1]$ the configurations R, B satisfies:*

1) Each place contains at most one type of car, red or blue.

2) Place $N - M - 1$ is empty. Each empty place, is followed by a sequence of places containing red cars immediately left to it, which is followed by a sequence of places containing blue cars immediately left to it.

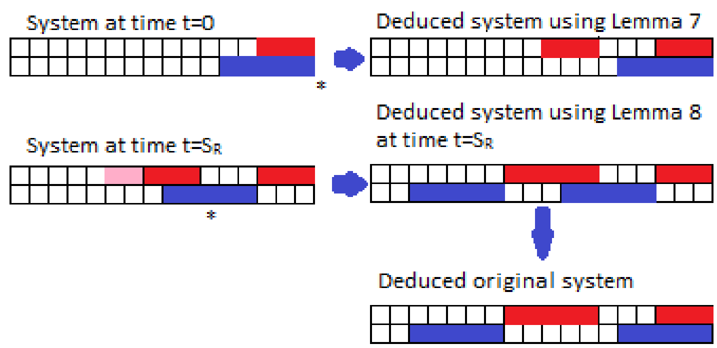

Figure 2.5: Lemmas 7 and 8 combined together result Lemma 9.

3) Any sequence of red or blue cars is of length at least $\min(s_R, s_B)$.

Proof. The proof is obtained by using Lemma 9 repeatedly. By using Lemma 9 we know that there exist $K_1, K_2 \geq \min(s_R, s_B)$ such that: The place $N - M - 1$ is empty, followed by K_1 consecutive places with only red cars and K_2 consecutive places with only blue cars left to it. Therefore, the assertion of the theorem holds for the segment $[T, N - M - 1]$ for $T = N - M - 1 - K_1$.

Now we know R, B completely in $[N - M - 1 - K_1, N - M - 1]$, and we run the system for $M + K_1 + 1$ turns. We have first s_B blue segment is pushed left s_R places, then the K_1 red sequence is pushing $\min(s_R, s_B)$ places to the left, therefore, its last $\min(s_R, s_B)$ cars now overlap with the K_2 blue sequence.

So after $M + K_1 + 1$ turns the system evolves to a state where $S = N - K_1 - M - 1$, $B_S = 0$ and left to S there are K_2 consecutive blue cars and exactly $\min(s_R, s_B)$ consecutive red cars. Noting that this time $M' = \max(K_2, \min(s_R, s_B)) = K_2$, once again we can result from Lemma 9 that: there are no additional cars in $[N - M - K_1 - K_2 - 1, N - M - K_1 - 2]$, place $N - M - K_1 - K_2 - 2$ is empty, followed by some K_3 consecutive places with only red cars and K_4 consecutive places with only blue cars left to it, for $K_3, K_4 \geq \min(\min(s_R, s_B), K_2) = \min(s_R, s_B)$ thus the assertion holds for the segment $[T', N - M - 1]$ for $T' = N - M - K_1 - K_2 - K_3 - 1 < T$.

We are applying Lemma 9 repeatedly as long as $T > 0$, and repeatedly we find that the claim holds for some $[T, N - M - 1]$ for T strictly decreasing, therefore, the claim holds in $[0, N - M - 1]$. $\square$

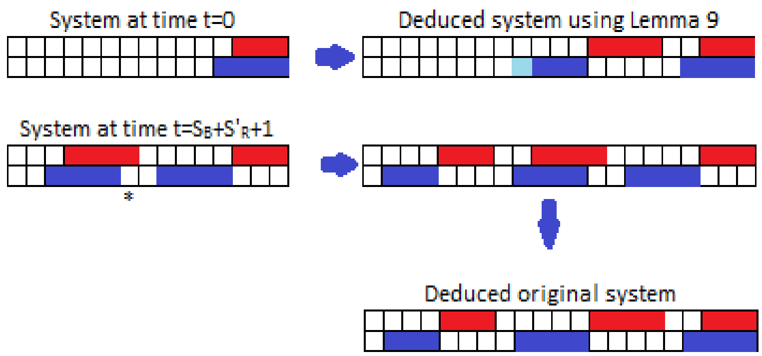

Figure 2.6: We are applying repetitively Lemma 9, and reveal a longer segment in the configuration $[T, N-1]$, for which properties of Theorem 10 hold.

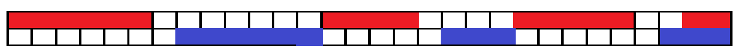

Figure 2.7: A typical stable configuration.

For reaching to this theorem we worked hard and stating some lemmas, but it is worthwhile, now we state and prove a useful corollary which uses Theorem 10.

Corollary 11. *Let R, B be a stable configuration with junction at $S = 0$ and $B_0 = 0$. Suppose that there is a sequence of exactly s_R consecutive red cars at places $[N - s_R, N - 1]$ and s_B consecutive blue cars at places $[N - s_B, N - 1]$. Denote $m = \min(s_R, s_B), (m = s_R)$. Then:*
1) The number of blue segments and red segments are equal in the system.
2) System velocity is at least $(1 + \frac{m}{N})^{-1}$.
3) All number of cars in the system is at most $N + m$.
4) All number of cars in the system is at least $\frac{2m}{2m+1}N + m$.
5) All number of segments in the system is at most $\frac{N}{m} + 1$.

Proof. By looking to the structure described in Theorem 10, there is exactly one red and one blue segment in $[N-M, N-1]$ where $M = \max(s_R, s_B)$

and in $[0, N-M-1]$ because any red sequence is immediately proceeded by a blue sequence, we have an equal number of red and blue segments. This proves (1).

For proving (2), by Theorem 10 the configuration R, B has exactly m violations. In fact, the m overlapping places of the segments s_R, s_B in $[N-M, N-1]$ produce the only violations in the configuration because any places in $[0, N-M-1]$ contains at most a single car, and no red car can appear immediately to the left of a blue car. Therefore, by Lemma 3 the system velocity is at least $(1+\frac{m}{N})^{-1}$.

For proving (3), by Theorem 10 any place in $[0, N-m-1]$ has at most one car or it is empty, and places $[N-m, N-1]$ has both red and blue cars. Hence, all number of cars in the system is at most $N-m+2m = N+m$.

For proving (4), we know that any sequence of a red car or a blue car is of length at least m, and an empty place in $[0, N-m]$ can happen only one time in each $2m+1$ places, and other places has one car, and places $[N-m, N-1]$ has both a red car and a blue car. Therefore, the number of cars has the lower bound

$$\frac{2m}{2m+1}(N-m)+2m = \frac{2m}{2m+1}N + \frac{2m^2+2m}{2m+1} \geq \frac{2m}{2m+1}N + m.$$

For proving (5), we know that any sequence of cars has length at least m, and by item (3) of this corollary all number of cars is at most $N+m$, therefore, the number of different sequences is at most $\frac{N+m}{m} = \frac{N}{m}+1$ □

Theorem 12. *All cars in the system have the same asymptotic velocity.*

Proof. As we can see, when the system reaches a stable configuration, it contains of alternating red and blue sequences of cars. This is trivial that the order of the sequences cannot change. Hence, the difference between the number of steps two different cars have taken cannot be more than the length of the longest sequence, which is less than N. In other words, for two cars when one of them passes the junction, until it passes the junction again the other car cannot pass the junction more than one time (for arbitrary two cars they pass the junction alternatingly). Thus, the asymptotic velocity with respect to t is the same for all cars. $\square$

With these statements which we have in hand until now, we can completely characterize the stable configuration of a junction with $p > 0.5$. Since the number of cars in the system is greater than N, at all times, including after reaching stable configuration, there are violations in the system. We are going to look at some time when the junction reaches a violation when the system is in stable configuration. Now at this point conditions of Theorem 10 are satisfied, therefore, we have the following theorem.

Theorem 13. *A junction of size N and density $p > 0.5$ reaches velocity of $\frac{1}{2p} - O(\frac{1}{N})$ (i.e. arbitrarily close to the optimal velocity of $\frac{1}{2p}$, for large enough N), and has at most a bounded number (depending only on p) of car sequences.*

Proof. We consider the system after it reached a stable configuration. Since $2pN > N$ at some time after that conditions of Theorem 10 are satisfied for some $s_R, s_B \geq 1$. Put $m = \min(s_R, s_B)$ at this time. The number of cars is $2pN$ and by using claims (3) and (4) in Corollary 11 we obtain:

$$\frac{2m}{2m+1}N + m \leq 2pN \leq N + m$$

From this inequality we get

$$(2p-1)N \leq m \leq (2p - \frac{2m}{2m+1})N$$
$$= (2p - 1 + 1 - \frac{2m}{2m+1})N = (2p-1)N + \frac{N}{2m+1}.$$

By using $(2p-1)N \leq m$ on the left hand side of the above inequality for its right hand side we get:

$$(2p-1)N \leq m \leq (2p-1)N + \frac{N}{2(2p-1)N+1} \leq (2p-1)N + \frac{1}{4p-2}$$

For $C = \frac{1}{4p-2}$ a constant independent of N ($C = O(1)$). Therefore, $m = (2p-1)N + K$ for some $K \leq C$. Now by applying claim (2) in Corollary 11, system velocity is at least

$$(1 + \frac{m}{N})^{-1} = (1 + \frac{(2p-1)N + K}{N})^{-1} = (2p + \frac{K}{N})^{-1} \geq \frac{1}{2p} - \frac{K}{N}.$$

For proving the last inequality we know that $N^2 \geq N^2 - K^2$, it concludes that $\frac{N}{N+K} \geq 1 - \frac{K}{N}$ $(*)$. Now

$$(2p + \frac{K}{N})^{-1} \geq_{(2p>1)} (2p + \frac{2pK}{N})^{-1} = \frac{1}{2p}(\frac{N}{N+K})$$

$$\geq_{(*)} \frac{1}{2p}(1 - \frac{K}{N}) = \frac{1}{2p} - \frac{K}{2pN} \geq_{(2p>1)} \frac{1}{2p} - \frac{K}{N}.$$

But by theorem 1 system velocity is at most $\frac{1}{2p}$, therefore, the system velocity is exactly $\frac{1}{2p} - \frac{K'}{N}$ for some $0 \leq K' \leq K \leq C = O(1)$, hence, $K' = O(1)$ proving the first part of the theorem.

By claim (5) in Corollary 11 we have all number of segments in the system is at most $\frac{N}{m} + 1$, using $m \geq (2p-1)N$ we obtain the number of segments is bounded by

$$\frac{N}{m} + 1 \leq \frac{1}{2p-1} + 1 = \frac{2p}{2p-1} = O(1).$$

Therefore, we conclude the second part of the theorem. □

2.7 $p < 0.5$ revisited

The characterization in Theorem 10 is useful to survey $p < 0.5$. In fact, the main result for $p < 0.5$, Theorem 5, can be proved by a similar technique, and even sharpened.

Corollary 14. *For $p < 0.5$ the junction reaches velocity of at least $1 - \frac{C(p)}{N}$, for $C(p) = \lfloor \frac{p}{1-2p} \rfloor$. In particular, for $p < \frac{1}{3}$ the junction reaches velocity 1, for any initial configuration.*

Proof. Let R, B be any initial configuration. Looking at the configuration after it reached the stable configuration, if the system reached velocity 1 we have nothing to prove. Suppose that the velocity is less than 1. Since in this case violations still occur, at some time the stable configuration will satisfy conditions of Theorem 10. As before, letting $m = \min(s_R, s_B)$ at this time, by claim (4) in Corollary 11 we have:

$$\frac{2m}{2m+1}N + m \leq 2pN$$

$$\Rightarrow m \leq (2p - \frac{2m}{2m+1})N = (2p - 1 + 1 - \frac{2m}{2m+1})N = (2p - 1 + \frac{1}{2m+1})N.$$

Sine $m \geq 1$ we conclude that $2p - 1 + \frac{1}{2m+1} > 0$, by rearranging we have $m < \frac{p}{1-2p}$, and since $m \in \mathbb{Z}$, $m \leq \lfloor \frac{p}{1-2p} \rfloor = C(p)$. Now by claim (2) of Corollary 11 the system velocity is at least

$$(1 + \frac{m}{N})^{-1} \geq 1 - \frac{m}{N} \geq 1 - \frac{C(p)}{N}.$$

Also, for $p < \frac{1}{3}$, $m \leq C(p) = \lfloor \frac{p}{1-2p} \rfloor = 0$, but m must be positive, we get a contradiction, thus the assumption velocity less than 1 cannot hold for $p < \frac{1}{3}$, and for $p < \frac{1}{3}$ the system velocity reaches 1 for any initial configuration. $\square$

2.8 The critical $p = 0.5$

Until now for junction of size N and density p we have the following description of the behavior:

- If $p < 0.5$ the junction will reach velocity $1 - o(1)$ (asymptotically optimal), and contain linearly many different segments in the stable configuration.
- If $p > 0.5$ the junction will reach velocity $\frac{1}{2p} - o(1)$ (asymptotically optimal), and contain constant many segments in the stable configuration.

Therefore, we can see that the junction system undergoes a sharp phase transition at $p = 0.5$, as the number of segments of cars as the system stabilizes drops from being linear to merely constant. Again by using the powerful theorem 10 we obtain the following theorem for the case $p = 0.5$.

Theorem 15. *A junction of size N with $p = 0.5$ reaches velocity of at least $1 - \frac{1}{\sqrt{N}}$ and contains at most $\sqrt{N}$ different segments.*

Proof. First of all since $p = 0.5$ we have exactly N cars in the system. When we reach stable configuration since we have N places and N cars in the system, at least one red car is immediately left to a blue car, hence, violations must still occur. Therefore, at some time conditions of Theorem 10 is satisfied. For $m = \min(s_R, s_B)$ at this time, by claim (4) in Corollary 11 we have:

$$\frac{2m}{2m+1}N + m \leq N \Rightarrow m(2m+1) \leq N,$$

Thus, $m \leq \sqrt{N}$. From here by claim (2) in Corollary 11 the system velocity is at least

$$(1 + \frac{m}{N})^{-1} \geq 1 - \frac{m}{N} \geq 1 - \frac{1}{\sqrt{N}},$$

it proves the first part of the theorem.

For the second part of the theorem, if r is the number of segments, then by Theorem 10 we can deduce the following about the number of cars: $2m$ cars are in places $[N-m, N-1]$, and since each place in $[0, N-m-1]$ contains one car, except transitions between segments that are empty, we have $N-m-r$ cars in places $[0, N-m-1]$. Therefore,

$$N = (N-m-r) + 2m = N + m - r \Rightarrow r = m \leq \sqrt{N}.$$

Hence, the configuration has at most $\sqrt{N}$ segments. □

2.9 Simulation results

In the following we give some simulation results for the junction for p near to 0.5 and $p = 0.5$. The columns of the tables consist of N, the average asymptotic velocity, the average number of car segments in the stable configuration, and the average longest segment in the stable configuration.

$\mathbf{p = 0.48}$

For large N, system reaches velocity 1 and the average number of segments is linear.

N	Velocity	No. segs	Longest
1000	0.99970	38.7	6.8
5000	1.00000	186.4	8.5
10000	1.00000	369.8	7.5
50000	1.00000	1850.6	8.2

$\mathbf{p = 0.52}$

For large N, system reaches velocity $0.961 = \frac{1}{2p}$ and the average number of segments ia about constant.

N	Velocity	No. segs	Longest
1000	0.95703	5.7	76.7
5000	0.96041	6.9	330.0
10000	0.96091	7.3	416.1
50000	0.96142	7.2	3207.1

$\mathbf{p = 0.5}$

At critical point, the velocity is tending 1 like $1 - \frac{C}{\sqrt{N}}$.

N	Velocity	No. segs	Longest
1000	0.98741	13.4	38.4
5000	0.99414	30.0	82.8
10000	0.99570	43.8	142.1
50000	0.99812	95.0	248.4

In this chapter we investigated a very simplified version of Biham-Middleton-Levine Traffic Model, which, despite its relative simplicity, had very analogous phenomena of phase transition at some critical density and of self-organization, which in single junction model both can be proven and well understood.

Chapter 3

The Biham-Middleton-Levine Traffic Model for a finite lattice of size $N \times N$

3.1 Introduction

In this chapter we focus on a lattice of finite size with periodic boundary conditions. By periodic boundary condition we mean that when a red car left the lattice on the right-hand side, then it enters the lattice on the left-hand side of the same row, and we have the similar rule for blue cars. First, we state an easy theorem for a system which has some red cars and just one blue car. Then we continue our discussion to find the answer of the following question: For what number of cars must self organization occur in the BML model from any possible initial configuration? Finally, we give a MATLAB code for BML model in a finite lattice with periodic boundary conditions.

First, we are going to introduce a function, with the number of cars as a variable, which for a finite lattice with periodic boundary conditions of size $N \times N$ gives the probability of happening fully jammed configuration.

Theorem 1. *Suppose that we have a finite lattice of size $N \times N$ with periodic boundary conditions and we have i red cars. Then the function $f_N(i)$, which is defined in the following, gives us the probability of having fully jammed configuration if we add a blue car to the lattice.*

$$f_N(i) = \begin{cases} 0 & i < N \\ N\frac{1}{N^2}\frac{1}{N^2-1}\cdots\frac{1}{N^2-(N-1)}\begin{pmatrix} N^2-N \\ i-N \end{pmatrix} & N \leq i \leq N^2-N \\ 1 & N^2-N < i \leq N^2 \end{cases}.$$

Proof. We have a fully jammed configuration if at least one row is fulfilled by red cars. Let the number of red cars be strictly less that N, $i < N$. Clearly no row can be fulfilled, hence, with probability zero we have a fully jammed configuration. If $N \leq i < N^2 - N$, with probability $\frac{1}{N^2}$ the first car is on the lowest row and the first site in the left-hand side, and with probability $\frac{1}{N^2-1}$ the second car is on the lowest row and second site, and by continuing this process with probability $\frac{1}{N^2-(N-1)}$, N^{th} car is on the lowest row and last site in the right-hand side, and for the rest of the cars we have $\binom{N^2-N}{i-N}$ choices. Therefore, with probability $\frac{1}{N^2}\frac{1}{N^2-1}\cdots\frac{1}{N^2-(N-1)}\binom{N^2-N}{i-N}$ the lowest row is fulfilled. Now we have N rows, thus, we multiply the last expression by N. Finally, assume that $N^2 - N \leq i \leq N^2$, in the worst case each row has $N - 1$ occupied sites and no row is not fulfilled; that is, the number of occupied sites is $N^2 - N$. Now by adding another red car with probability 1 we have a fulfilled row. It means that we have a fully jammed configuration. □

By this theorem, we know that for less than N cars we never have a fully jammed configuration. Now it is interesting we discover what will happen to the system if we have less than N cars. The next theorem says that for any arbitrary initial configuration the system will organize itself, and by passing a finite time it behaves periodically.

Theorem 2. *Assume that we have a finite lattice of size $N \times N$ with periodic boundary conditions and we have less than N cars with an arbitrary initial configuration. Then by passing a finite time the movement of cars becomes periodic.*

Proof. Suppose that on the even time steps each red car moves and on the odd time steps blue ones. In the lattice of size $N \times N$ we have N^2 sites, therefore, the number of all configurations are finite, by considering the time which is produced (it is produced in odd or even time; more clearly, we also pay attention to the time to know that the configuration is produced by movement of red cars or blue ones). But we do not have a fully jammed configuration, hence, we have an infinite number of movements and the result of each of them is chosen from the finite set of all configurations. Therefore, at least two configurations are equal so that the next movement in both is by cars of the same color. In other words, let F_1 be the initial configuration, and the configurations F_t and F_{t+T} are equal so that the next movement in both of them is by cars of the same color. Hence, $F_{t+T+1} = F_{t+1}$ and so on. Therefore, by passing time t the movement of cars becomes periodic. Note that this theorem is like stable configuration in chapter 2. $\square$

Note that in the previous theorem it is not necessary to have less than N cars. In fact, it is enough the system never has a fully jammed configuration.

Here we give two examples:
$i)$

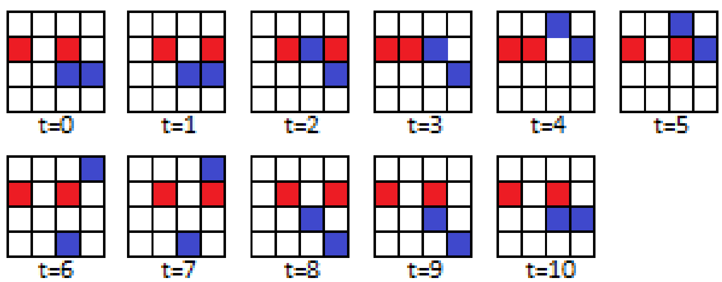

Figure 3.1: $N = 4$, and there are 4 cars in the system.

ii)

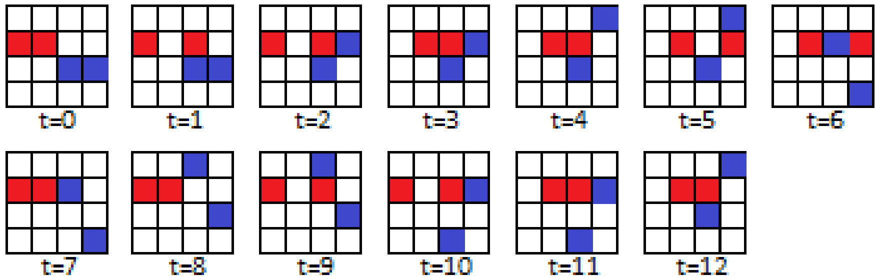

Figure 3.2: $N = 4$, and there are 4 cars in the system.

In the example (i), configurations at time $t = 0$ and $t = 10$ are equal and they have the same next steps. In the example (ii), configurations at time $t = 4$ and $t = 12$ are equal and they have the same next steps.

3.2 For what number of cars must self organization occur from any initial configuration?

One may see in the example (i) the period has length 10 while in the example (ii) is 8. In fact, in the second example the number of collisions when time tends to infinity is finite, but in the first one we have an infinite number of collisions. Therefore, it is interesting if we find a condition such that under it the number of collisions is finite as time tends to infinity.

For the square lattice of size $N \times N$, consider when the time comes to move first red cars move and then the blue ones; i.e. in any given time step, first all red cars try to move, then all the blue cars. Here we prove that when the number of cars is less that $\frac{N}{2}$, $m < \frac{N}{2}$, where m is the number of cars, the system must attain velocity one, which means the number of collisions is finite.

Write $\mathbb{Z}_N = \frac{\mathbb{Z}}{N\mathbb{Z}}$, and consider the $N \times N$ discrete torus $\mathbb{Z}_N \times \mathbb{Z}_N$. Consider the locations of the cars on the N North-West to South-East diagonals of the torus, and define $D_1, D_2, ..., D_N$ by

$$D_k = \{(i, j) \in \mathbb{Z}_N^2 : i + j = k \quad mod\ N\}$$

see Figure 3.

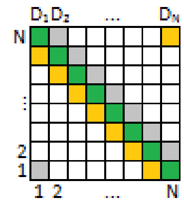

Figure 3.3: We can see the positions of $D_1, ..., D_N$ here.

If a car moves during a time step, then it moves up one diagonal otherwise it stays without movement. For each $t \geq 0$ let $\phi_t : \mathbb{Z}_N^2 \longrightarrow \mathbb{Z}_N$ be the associated 'time-corrected diagonal map' defines as $\phi_t(i, j) = i + j - t \ \ mod\ N$.

Suppose $X^1, X^2, ..., X^m$ are the initial positions of the cars, and write the X_t^i for the position of car i at time $t \geq 0$ ($X_0^i = X^i$) and $Y_t^i = \phi_t(X_t^i)$. Thus, knowing the Y_t^i at a given time t tells us something about the configuration $(X_t^i)_{i \leq m}$, but far from specifies it uniquely. The proof here will use constrains on the behavior of $Y_t^1, Y_t^2, ..., Y_t^m$ as t increases.

At a given time t, the points Y_t^i are distributed within $\mathbb{Z}_N$, some points of $\mathbb{Z}_N$ may be occupied by many such Y_t^i, while others will be empty, note that several cars may occupy different points on the same diagonal. We will partition the set $\mathbb{Z}_N \backslash \{Y_t^i : i \leq m\}$ of empty points at time t into a union of arcs in $\mathbb{Z}_N$ say $A_t^1 \cup A_t^2 \cup ... \cup A_t^{r(t)}$, where we label the arcs in order and choose A_t^1 to be the arc containing the first non-occupied point when $\mathbb{Z}_N$ is written as $\{1, 2, ..., N\}$.

Lemma 3. *Suppose that at time t some arc $A_t^s = \{y, y+1, ..., y+l\}$ has length at least 2. Then at time $t+1$ the point immediately to its left, $y-1$, is still occupied by some Y_{t+1}^i, and the set $A_t^s \backslash \{y+l\} = \{y, y+1, ..., y+l-1\}$ is still an arc of unoccupied points.*

Proof. This is a direct observation from the dynamics of the cars. The lower boundary point cannot move, since given a number of cars all in the same diagonal in the discrete torus and there is not any car in the diagonal above because the diagonal above is in the arc, therefore, at least one of those cars on the diagonal of the lower boundary will not be blocked during the next time step. Hence, when it moves $i \to i+1$ or $j \to j+1$ and under the time-corrected diagonal map we have: $\phi_{t+1}(\textit{the car that moved at time } t+1) = i+1+j-(t+1) \textit{ or } i+j+1-(t+1) = i+j-t \textit{ mod } N$, which is equal the lower boundary point of A_t^s. Therefore, the lower boundary point cannot move. Also, images of cars under the map ϕ can only either stay still or move one step to the left in one time step, since if a car moves we have $+1$ in the i or j term and $+1$ in the t term of ϕ and they cancel each other, thus the image of the car under the map ϕ stay still, and if cars do not move, then in the next time step i and j do not change and the t term in ϕ changes to $t+1$ and the image of cars moves one step to the left. Hence, it is clear that the points $\{y, y+1, ..., y+l-1\}$ cannot become occupied during the next time step. $\square$

Lemma 4. *The dynamics cannot create new arcs A_t^s of length greater than 1; the number of such long arcs is non-increasing in t.*

Proof. Suppose that at time $t+1$ we have an arc A_{t+1}^s of length at least 2. During the time step from t to $t+1$, the images of those cars that are now in diagonals immediately above and below A_{t+1}^s either stayed still or moved one step to the left. Thus, by previous lemma, at time t there must have been an empty arc at least as long as A_{t+1}^s and with the same lower end-point. Thus to each empty arc of length at least 2 at time $t+1$ we can associate such an arc at time t; since this association is also clearly unique, the number of such arcs cannot increase. $\square$

Lemma 5. *If the system never attains velocity one, then there must come a time when no arcs A_t^s have length greater than 1.*

Proof. Because the system never attains velocity 1 we know there are infinitely many times at which cars are blocked, then in particular some car must be blocked infinitely often. Suppose it is car $i \leq m$. We know that when the car blocks Y_t^i moves one point backward because in time-corrected diagonal map t changes to $t+1$. This means that as t increases Y_t^i describes infinitely many circuits around the discrete circle $\mathbb{Z}_N$.

But after completing one such circuit (say at time T), there can be no arcs of length greater than 1 remaining. By contradiction, if there is still an arc A_T^s of length at least 2 at time T, then, arguing as in the proof of the previous lemma, there must be a sequence of arcs $A_t^{s(t)}$ for $t = 0, 1, ..., T$, all of length at least 2 because arc greater than 1 cannot create, and all with same lower end-point, say y. Thus we deduce that $y \in \mathbb{Z}_N$ must remain occupied, and have an arc of length at least 2 immediately to its right, for all time $t \leq T$. This contradicts the fact that Y^i passes through all points of $\mathbb{Z}_N$ by time T. $\square$

Theorem 6. *If $m < \frac{N}{2}$, the system must attain velocity one.*

Proof. This is now immediate: if $m < \frac{N}{2}$ then, however the images $Y_t^1, Y_t^2, ..., Y_t^m$ are distributed in $\mathbb{Z}_N$, there will always be some arc of length at least 2, and so, by previous lemma, the system must attain velocity one in finite time; that is, the number of collisions is finite. $\square$

In the following theorem we will make a better bound for the number of cars in the system.

Theorem 7. *If $m \leq [\frac{N}{2}]$, the system must attain velocity one ($[K]$ denotes the integer part of K).*

Proof. Let N be odd. Since $m \in \mathbb{Z}$ we have $m \leq [\frac{N}{2}]$ if and only if $m < \frac{N}{2}$. Therefore by Theorem 6, the system must attain velocity one.

Now suppose N is even and $m \leq [\frac{N}{2}] = \frac{N}{2}$. For $m < \frac{N}{2}$ we have the result by using Theorem 6. Assume that $m = \frac{N}{2}$. If for all times there exists an arc of length at least 2 then ,by Lemma 5, the system must attain velocity one in finite time. Suppose for time t^* there is not any arc of length greater than 1. Hence, $Y_{t^*}^1, Y_{t^*}^2, \cdots, Y_{t^*}^m$ are distributed in $\mathbb{Z}_{N=2m}$ so that none of them are not adjacent. In fact, between each $Y_{t^*}^i$ and $Y_{t^*}^{i+1}$ there is an arc of length 1, for $1 \leq i \leq N \bmod N$. Therefore, for each diagonal which contains a car we have two facts: 1) That diagonal contains exactly one car. 2) The diagonal above that diagonal is empty. Clearly, for this configuration in the next move we do not have any collision and all cars move freely. Thus the configuration of $Y_{t^*}^1, Y_{t^*}^2, \cdots, Y_{t^*}^m$ does not change in $\mathbb{Z}_N$ from t^* to $t^* + 1$. By the same argument for all $t \geq t^*$, we do not have any change in the configuration of $Y_t^1, \cdots, Y_t^m$ in $\mathbb{Z}_N$ and any collision in the system. It means that the velocity is one. ☐

Example. In this example $N = 8$ and we have 4 cars.

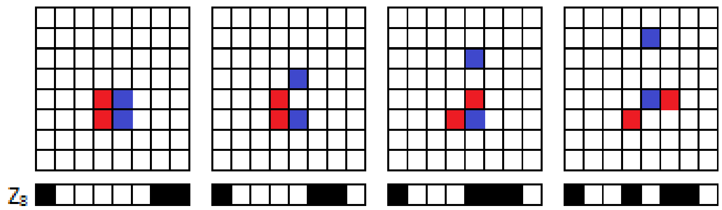

Figure 3.4: As you can see, all of the times there is an empty arc of length at least 2.

Now in the following proposition we are going to prove that it is possible the configuration stuck when we have at least $2N$ cars.

Proposition 8. *There is a configuration with m cars which is stuck if and only if $m \geq 2N$.*

Proof. $\Rightarrow$) First we note that no row can contain some red but no blue cars and no column can contain some blue but no red cars, since in this case those cars would be able to move freely. Now assume that the system is stuck. Then we conclude that in every column there must be at least one red car otherwise for there cannot be only blue cars, and if there were no cars in that column, then there would be red cars in some column to the left of it which are not blocked, and we get a contradiction. By the same argument, in every row there must be at least one blue car. Therefore, there are at least N red cars and at least N blue cars, so $m \geq 2N$.

$\Leftarrow$) We need only find a configuration of $2N$ stuck cars. We choose two adjacent southwest-northeast diagonals, and occupy whole of the lower diagonal with blue cars and whole of the upper one with red cars. Now for any $m \geq 2N$, we can add more cars randomly to this configuration. After a finite time they reach to the blocked diagonals, so the system becomes blocked. ☐

Here we present a question of [7] then answer it.

Question. *Does the system necessarily self-organize to attain velocity 1 for any $m > \frac{1}{2}N$? Put differently, for which $\frac{1}{2}N < m < 2N$ can the system stay forever below velocity 1, even though (by proposition 7) it can never get stuck?*

Answer. We claim that for $\frac{1}{2}N < m < 2N$ the velocity is not forever below 1. For $N = 4$, and $m = 4$ we give an initial configuration and run it in Figure 5 such that the velocity is 1.

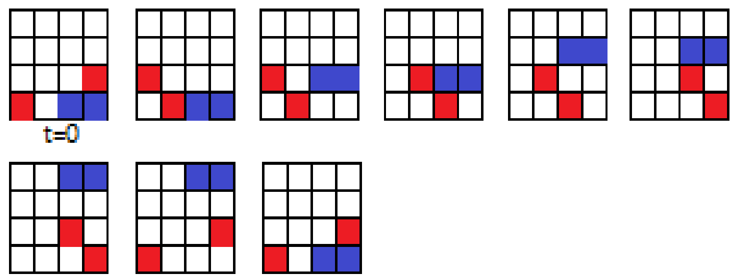

Figure 3.5: As you can see by having $N = m = 4$ and the initial configuration at time $t = 0$, the velocity is exactly 1.

3.3 How many collisions will occur before attaining velocity one?

In this section, we are trying to answer the following open question in [8].

Question. *Suppose we place a configuration of $m \leq [\frac{N}{2}]$ cars on $\mathbb{Z}_N \times \mathbb{Z}_N$ uniformly at random. By Theorem 7 the system will self organize to attain velocity one, but how many collisions will occur before it does so?*

In this section our system is of size $N \times N$ with $m \leq [\frac{N}{2}]$ cars. First we state two facts.

a) Suppose there exists a diagonal having some cars which do not collide with other cars (out of the diagonal). Assume that the image of these cars is $Y \in \mathbb{Z}_N$. If some collisions between these cars happen, under the time-corrected diagonal map, the image of cars which could not move (because a collision affected them) will take one step to the left; i.e. their new image is $Y-1 \in \mathbb{Z}_N$. Now cars whose images are pushed to $Y-1$ never will collide with the cars whose images are Y.

b) Suppose we have some cars on two consecutive diagonals with images $Y-1, Y \in \mathbb{Z}_N$. If some cars whose images are $Y-1$ collide with some cars whose images are Y, then the image of them will take one step to the left (to $Y-2 \in \mathbb{Z}_N$) and they do not have collision any more with the cars whose images forever are Y.

Examples:

i) Assume that $N = 4$ and $m = 2$. It is straightforward the number of collisions is at most 1.

ii) Suppose $N = 6$ and $m = 3$. By Theorem 7, the system must attain velocity 1. If the image of initial configuration in $\mathbb{Z}_6$ is alternately empty and occupied, by the argument in the proof of Theorem 7 there is not any collision in the system and it starts with velocity 1, Figure 3.6(a). In Figure 3.6 we discuss other possibilities. In Figure 3.6(b) we have 3 cars in one diagonal. Definitely, one of them moves without any stop, thus we have at most 2 collisions. Therefore, their image moves one step to the left and by fact (a) they do not have collision with the car in the diagonal above anymore, and between these two cars there is at most one collision, then one of them moves one step to the left and there is no collision between them anymore. Hence, there are at most 3 collisions. In Figure 3.6(c), on the diagonal with 2 cars we have at most one collision, because one of them moves freely and the car on the lower diagonal has at most one collision. Then between the two last cars can exist at most one collision. Hence, we have at most 3 collisions. In Figure 3.6(d), the car on the upper diagonal moves freely and the two other cars on the lower diagonal can have at most 2 collisions, hence their image moves one step to the left. By fact (b), they cannot have more collision with the car on the upper diagonal, and it can exist at most one collision between these two cars. Therefore, we have at most 3 collisions. We can argue Figure 3.6(e) similarly, and all other possible initial configurations. Then we find that there exist at most 3 collisions in the system.

iii) Suppose $N = 8$ and $m = 4$. We have at most $6 = 3 + 2 + 1$ collisions. In Figure 3.7 we check the images of four initial configurations. The reader

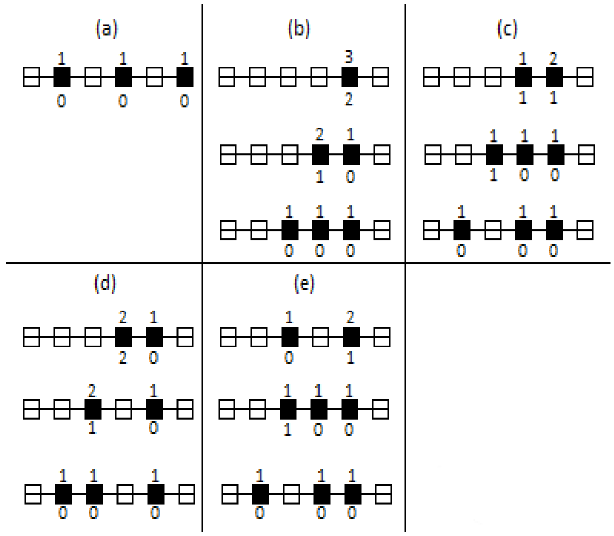

Figure 3.6: The number above each occupied place shows the number of cars, and the number under each occupied place shows the number of possible collisions.

can check all possible initial configurations.

Lemma 9. *Consider a system of size $N \times N$ with $m \leq [\frac{N}{2}]$ cars. There exists at least a car which never stops.*

Proof. Assume that $m = [\frac{N}{2}]$. By contradiction suppose all cars in the system have some stops. Let car i be the first car which has a stop and whose image is $Y \in \mathbb{Z}_N$; the image of car i moves to $Y-1$ after its stop. For the cars whose images are in $Y-1$ and $Y-2$, after a stop their image move to $Y-2$ and $Y-3$, respectively. Note that for cars which immediately right to their images exist an empty arc, it is necessary some image appear immediately right to their image so that they can have their stop; otherwise there is a car which never stops. Now by Lemma 4, we know that the dynamics cannot create new arcs of length greater than 1. Therefore, since all cars must have a stop, we conclude that there is no empty arc of length greater than 1 after passing enough time. Since car i with image Y had a stop (now its image is in $Y-1$) necessarily we had a car with image in Y or $Y+1$. Now:
i) If this car had image Y, after a stop its image is $Y-1$, and since we do not have any empty arc of length greater than 1 we have m occupied places in $\mathbb{Z}_N$, which each of them contains at least one car, and two cars have image

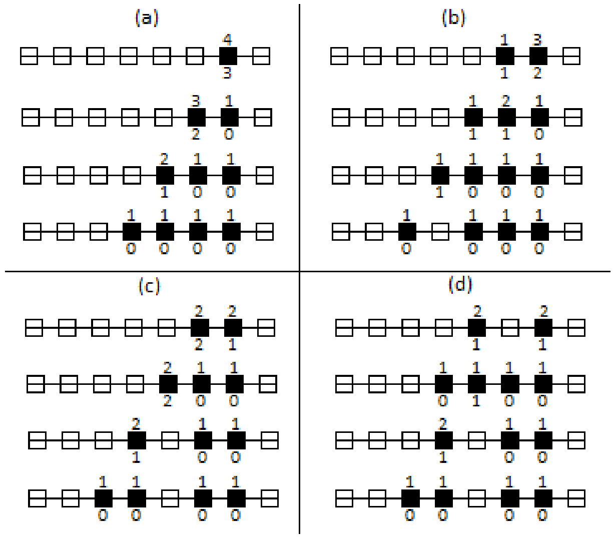

Figure 3.7: The number above each occupied place shows the number of cars, and the number under each occupied place shows the number of possible collisions.

$Y-1$. Therefore we have at least $m+1$ cars, which is a contradiction.
ii) If this car had image $Y+1$, after a stop its image is Y (immediately next to $Y-1$), and since we do not have any empty arc of length greater than 1, we have at least m cars, and places Y and $Y-1$ next to each other are occupied, therefore we have at least $m+1$ cars, which is a contradiction.
Hence, there is a car which never stops. $\square$

In Figure 3.8 we give some initial configurations such that for the images with red color never appear another image immediately to its right. In fact, by the argument in the proof of Lemma 9, we see there is an image which never appear another image immediately to its right. In other words, if for all images appear another image immediately to its right (even temporarily) we reach to a contradiction with the number of cars in the system.

Figure 3.8: No image will appear immediately right to the red images.

Theorem 10. *Consider a system of size $N \times N$ with $m \leq [\frac{N}{2}]$ cars. Then* *we have at most $\sum_{i=1}^{m-1}(m-i)$ collisions in this system.*

Proof. Assume that $0 \leq Y^1 \leq \cdots \leq Y^l \leq N$, $l \leq m$, are all occupied places (images) in $\mathbb{Z}_N$. Let Y^k, $1 \leq k \leq l$, be an image which ,immediately to its right, does not appear another image. Consider the new order $Y^{k+1}, Y^{k+2}, \cdots, Y^l, Y^1, \cdots, Y^k$, and denote them without changing the order $Y^{1,1}, Y^{2,1}, \cdots, Y^{r_1,1} = Y^k$ where $r_1 = l$. Suppose $Y^{i,1}$ is the image of $y^{i,1}$ cars for $1 \leq i \leq r_1$, and $\sum_{i=1}^{r_1} y^{i,1} = m$. At least there exists one car with image $Y^{r_1,1} = Y^k$ so that it moves without any stop. Thus, for cars with the image $Y^{r_1,1}$ we have at most $y^{r_1,1} - 1$ collisions and for cars with the image $Y^{i,1}$ $i \neq r_1$, we have at most $y^{i,1}$ collisions. Therefore, we have $y^{1,1} + \cdots + y^{r_1-1,1} + y^{r_1,1} - 1 = m-1$ collisions. Now for the cars which had a collision their images move one step to the left and the new configuration of the images is $Y^{1,2}, Y^{2,2}, \cdots, Y^{r_2,2}, Y^{r_1,1}$, where by having $y^{r_1,1} - 1$ collisions in $Y^{r_1,1}$, we know $Y^{r_1,1}$ is the image of just one car (the car which moves without any collision) and $y^{i,2}$ is the number of cars which their image is $Y^{i,2}$ for $i = 1, \cdots, r_2$. Now by facts (a) and (b), cars with the image $Y^{r_2,2}$ do not have any collision anymore with the car whose image is $Y^{r_1,1}$, and at least there exists one car with image $Y^{r_2,2}$ so that it does not stop anymore. Thus, for cars with the image $Y^{r_2,2}$ we have at most $y^{r_2,2} - 1$ collisions and for cars with the image $Y^{i,2}$, $i \neq r_2$, we have at most $y^{i,2}$, $i \neq r_2$, collisions. Therefore, we have at most $y^{1,2} + \cdots + y^{r_2-1,2} + y^{r_2,2} - 1 = m-2$ collisions. For the cars which had a collision their images move one step to the left and the new configuration of the images is $Y^{1,3}, Y^{2,3}, \cdots, Y^{r_3,3}, Y^{r_2,2}, Y^{r_1,1}$. By continuing this process, we reach to the configuration

$$Y^{1,m-1}, Y^{2,m-1}, \cdots, Y^{r_{m-1},m-1}, Y^{r_{m-2},m-2}, Y^{r_{m-3},m-3}, \cdots, Y^{r_2,2}, Y^{r_1,1}$$

where just one collision, (m-[m-1]), can happen and this collision is between cars with the image $Y^{r_{m-1},m-1}$, and since after all these collisions $Y^{1,m-1}$ cannot reach immediately right to $Y^{r_1,1}$, we cannot have more collision in the system. Now we sum the number of all possible collisions in the system, we have $(m-1) + (m-2) + (m-3) + \cdots + 1 = \sum_{i=1}^{m-1}(m-i)$. $\square$

Example. Here we give an example of an initial configuration with $N = 8$ and $m = 4$ so that the number of collisions reaches to its maximum.

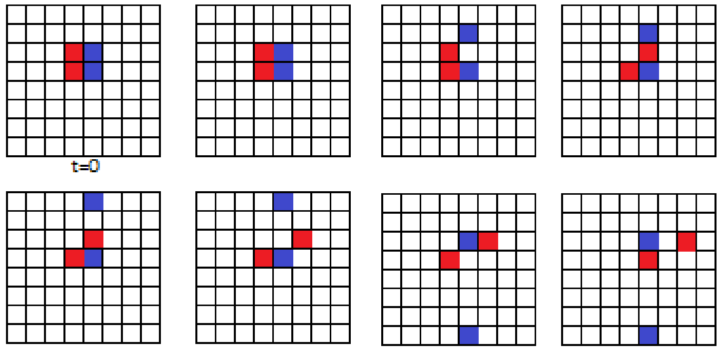

Figure 3.9: There exactly 6 collisions in this system.

3.4 Simulation Code

Now we present a MATLAB code function, biham(n), where the lattice is of size $n \times n$. We show red cars by 1 and blue cars by -1.

```
function biham(n)
% Create a matrix that have the information of the position of each car and
its color
val=cell(1,3,n);
for i=1:n
      val(:,:,i)=input('');
end
val=cell2mat(val);
A=cell(n,n,(n^2)+n);
for i=1:(n^2)+n
      for j=1:n
         for k=1:n
            A(j,k,i)=0;
         end
      end
end
A=cell2mat(A);
%Creat the first configuration
for i=1:n
      A(val(1,1,i),val(1,2,i),1)=val(1,3,i);
```

```
end
i=1;
%first configuration
t=1;
disp(A(:,:,i))
for j=2:(n^3)
     if mod(j,2)==0
        for co=1:n
           for ro=n:-1:1
              if ro==1
                 if A(ro,co,i)==1
                    A(ro,co,j)=1;
                 end
                 if A(ro,co,i)==-1
                    if A(n,co,i)==0
                       A(n,co,j)=-1;
                    else
                       A(ro,co,j)=-1;
                    end
                 end
              end
              if (1<ro)&(ro<=n)
                 if A(ro,co,i)==1
                    A(ro,co,j)=1;
                 end
                 if A(ro,co,i)==-1
                    if A(ro-1,co,i)==0
                       A(ro-1,co,j)=-1;
                    else
                       A(ro,co,j)=-1;
                    end
                 end
              end
           end
        end
        disp(A(:,:,j))
        for pr=2:2:(j-(2*n)+1)
           if (A(:,:,pr)==A(:,:,j))
```

```
                disp(pr)
                disp(j)
                disp(A(:,:,pr))
                disp(A(:,:,j))
                return
            end
        end
    end
    if mod(j,2)==1
        for ro=1:n
            for co=1:n
                if A(ro,co,i)==-1
                    A(ro,co,j)=-1;
                end
                if A(ro,co,i)==1
                    if A(ro,mod(co,n)+1,i)==0
                        A(ro,mod(co,n)+1,j)=1;
                    else
                        A(ro,co,j)=1;
                    end
                end
            end
        end
        disp(A(:,:,j))
        for pr=1:2:(j-(2*n)+1)
            if (A(:,:,pr)==A(:,:,j))
                disp(pr) disp(j)
                disp(A(:,:,pr))
                disp(A(:,:,j))
                return
            end
        end
    end
    i=j;
end
end
```

Appendix

Here we are going to give a new definition which may help us to prove the existence of $p_c > 0$.

In fact, instead of considering the asymptotic velocity of a car, we consider a rate of visit by cars per site. The benefit of this method is that the site is fixed and does not move as time increases. First, we define $\Omega_0 = \{e, r, b\}^{\mathbb{Z}^2} = \{\omega | \omega(x) = e, r,\ or\ b;\ x \in \mathbb{Z}^2\}$ where e means that site x is empty, b and r means that site x is occupied by a blue car and red car respectively. $\Omega_T = \underbrace{\Omega_0 \times ... \times \Omega_0}_{T-times}$, $\Omega = \Omega_0^Z$ where $Z = \{0, 1, 2, ...\}$.

Now we define $\alpha_T(x) = \frac{\Sigma_{t=0}^{T-1} 1_{\{\omega_t(x)=r\ or\ b\}}}{T}$, $x \in \mathbb{Z}^2$. We define asymptotic rate of visit by $\alpha(x) = \overline{\lim}_{T\to\infty} \alpha_T(x),\ x \in \mathbb{Z}^2$

Lemma. $v = 0$ (asymptotic velocity) if and only if there exists an $x \in \mathbb{Z}^2$ such that $\alpha(x) = 1$, ($v > 0$ if and only if for all $x \in \mathbb{Z}^2,\ \alpha(x) < 1$).

Proof. $\Rightarrow$) $v = 0$ means that by choosing a car randomly, its velocity is zero. By lemma 5 of Chapter 1 it implies for some $x \in \mathbb{Z}^2$, $\alpha(x) = 1$.

$\Leftarrow$) $\alpha(x) = 1$ for some $x \in \mathbb{Z}^2$, then x is in an infinite blocking path. Therefore, $v = 0$ because we have an infinite blocking path. $\square$

Theorem. $\mathbb{E}_p[\alpha(x)] = p$, where $x \in \mathbb{Z}^2$, p is the density of cars, and $\mathbb{E}[.]$ is expectation.

Proof. We have

$$\mathbb{E}_p[\alpha_T(x)] = \frac{\sum_{t=0}^{T-1} \mathbb{E}_p[1_{\{\omega_t(x)=r\ or\ b\}}]}{T} = \frac{\sum_{t=0}^{T-1} \mathbb{P}_p[\omega_t(x)\ is\ occupied]}{T}$$

Now for $t = 0$ the site x with probability p is occupied because the density of cars is p. When $t = 1$, in two dimensional lattice some cars move without changing the number of cars, thus the density of cars does not change and we can consider this stage as an initial configuration with the same density, therefore again the probability of existence a car at site x is p. We have the same argument for $t = 2, 3, ..., T$. Hence, we obtain

$$\mathbb{E}_p[\alpha_T(x)] = \frac{pT}{T} = p$$

Now for each T we have the above statement which it means that $\mathbb{E}_p[\alpha_T(x)]$ is independent of T. Therefore

$$\mathbb{E}_p[\alpha(x)] = \mathbb{E}_p[\overline{\lim}_{T\to\infty}\alpha_T(x)] = \overline{\lim}_{T\to\infty}\mathbb{E}_p[\alpha_T(x)] = \overline{\lim}_{T\to\infty}p = p$$

and we are done. $\square$

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