Short Wave-Long Wave Interactions in Compressible Fluid Dynamics

Daniel Rodriguez Marroquin

Advisor: Hermano Frid

Instituto Nacional de Matemática Pura e Aplicada

This dissertation is submitted for the degree of

Master in Mathematics

IMPA 2014

Acknowledgements

First of all, I am grateful to IMPA for the opportunity of being here. I have done my best to squeeze every drop of this experience. I am also grateful to the CNPq and to the FAPERJ for the financial support during this course.

I would like to acknowledge Hermano for the orientation and guidance on the topics of this dissertation. I would also like to mention professors Leonardo Rendón and Germán Fonseca because it is thanks to them that I could get into IMPA. Special thanks to my friends, because not everything in life is mathematics. Finally, I am more than thankful to my family and Betina, for the unconditional support and for making everything a lot easier.

Thanks!

Abstract

Following the ideas of Dias and Frid in [4] we adapt the calculations in [1] chapter II to show global existence and uniqueness of solutions to the Cauchy problem for a coupling between a Navier Stokes System and a Shrödinger equation, all of this in the one (space) dimensional context. This coupling was proposed by Dias and Frid in [4] and therein, after proving local solvability through a Faedo-Galerkin type method, they used a priori estimates to prove the existence of global solutions. They did this for the case of a non heat conductive fluid. We generalize these results to the heat conductive case.

Contents

Introduction

The present work is meant as an introduction for the author on some techniques used to study certain partial differential equations; in particular, the Navier Stokes Equations from fluid dynamics.

For that purpose we studied the already called (by the experts on the subject) classic theory contained in the book [1], focusing on chapter 2; as well as the paper [4] of Dias and Frid.

In this paper, Dias and Frid propose a coupling between the Navier Stokes system for a non heat conductive fluid with a Schrödinger equation, in the one dimensional case, proving global solvability for the Cauchy problem. In this work we follow the outline presented in [1], where the Cauchy problem for the Navier Stokes system is solved, and adapt the calculations therein in order to generalize the results of Dias and Frid to the heat conductive case (under certain assumptions on the pressure).

It is worth mentioning that these results have already been generalized to the 3 dimensional case, when the initial data is a small perturbation of of an equilibrium state, in the paper [5] by Frid, Pan and Zhang.

Related references and future work include the paper of Chen and Wang [3] where they study nonlinear magneto-hydrodynamics which consists of a coupling between the Navier Stokes system with Maxwell's equation.

Chapter 1

Coupling

1.1 Coupling

Consider the one dimensional Navier Stokes System from fluid dynamics given in Lagrangian coordinates (see appendix A) by

$$\begin{aligned}
&\rho_t + \rho^2 u_x = 0, \\
&u_t + p_x = \mu(\rho u_x)_x + F, \\
&\theta_t + \frac{1}{C_\vartheta}\theta p_\theta u_x = \frac{1}{C_\vartheta}\kappa(\rho\theta_x)_x + \frac{\mu}{C_\vartheta}\rho u_x^2,
\end{aligned}$$

where u, ρ and θ are the fluid's velocity, density and temperature respectively, p is the pressure, F is an external force and μ, κ and C_ϑ are positive constants. In this model we assume that the pressure p is given by.

$$p = p(\rho, \theta) = R\rho\theta, \tag{1.1}$$

where $R > 0$ is a constant and that C_ϑ is a positive constant. This is the case of a perfect polytropic gas (for a wide discussion on this model we refer the reader to [1] chapter I). Also consider the nonlinear Schrödinger equation:

$$iw_t + w_{xx} = |w|^2 w + wG,$$

where i is the imaginary unit and G is an external force. As in [4] we make the coupling by taking F and G in the above equations of the form

$$F = \alpha\left(g'(1/\rho)h(|w|^2)\right)_x \quad , \quad G = -\alpha g(1/\rho)h'(|w|^2),$$

where $\alpha > 0$ is a constant and $g,h : [0,\infty) \to [0,\infty)$ are real smooth functions satisfying that $supp(g')$ is compact in $(0,\infty)$, $supp(h')$ is compact in $[0,\infty)$ and $g(0) = h(0) = 0$. We arrive at the following system of PDE's

$$\rho_t + \rho^2 u_x = 0, \tag{1.2}$$

$$u_t + R(\rho\theta)_x = \mu(\rho u_x)_x + \alpha\left(g'(1/\rho)h(|w|^2)\right)_x, \tag{1.3}$$

$$\theta_t + \frac{1}{C_\vartheta} R\rho\theta u_x = \frac{1}{C_\vartheta}\kappa(\rho\theta_x)_x + \frac{\mu}{C_\vartheta}\rho u_x^2, \tag{1.4}$$

$$iw_t + w_{xx} = |w|^2 w + \alpha g(1/\rho)h'(|w|^2)w. \tag{1.5}$$

In the present work we study this system, our main goal being to prove global existence and uniqueness of solutions to the Cauchy problem.

1.2 Statement of the problem and outline of the proof

Consider the Cauchy problem for the system (1.2)-(1.5) subject to initial data

$$u(x,0) = u_0(x), \qquad \rho(x,0) = \rho_0(x), \qquad \theta(x,0) = \theta_0(x), \qquad w(x,0) = w_0(x). \tag{1.6}$$

Suppose there exist $\rho_*, \theta_*, m, M > 0$ such that

$$\lim_{|x|\to\infty} \rho_0(x) = \rho_*, \qquad \lim_{|x|\to\infty} \theta_0(x) = \theta_* \tag{1.7}$$

and

$$m < \rho_0 < M, \qquad m < \theta_0 < M. \tag{1.8}$$

Then we can state our main theorem as follows:

Theorem 1. *Let the initial data (1.6) satisfy (1.7), (1.8) and*

$$u_0, \rho_0 - \rho_*, \theta_0 - \theta_* \in H^1(\mathbb{R}), \qquad w(x,0) = w_0(x) \in H^1(\mathbb{R},\mathbb{C}). \tag{1.9}$$

Then for every $T > 0$ there are constants M_1 and m_1 and a unique solution of (1.2)-(1.5),(1.6) *satisfying*

$$
\begin{aligned}
&u, \theta-\theta_* \in C([0,T],H^1(\mathbb{R}))\cap L^2([0,T],H^2(\mathbb{R})),\\
&u_t, \theta_t \in L^2(\mathbb{R}\times[0,T]),\\
&\rho-\rho_* \in C([0,T],H^1(\mathbb{R})), \quad \rho_t \in L^2(\mathbb{R}\times[0,T])\\
&m_1<\rho<M_1,\\
&w\in C([0,T],H^1(\mathbb{R})).
\end{aligned}
$$

The proof of the theorem is divided into the following two chapters for a better understanding. All calculations are carefully made with all the details included. The outline is as follows.

We are going to approximate the initial data $(\rho_0,u_0,\theta_0,w_0)$ by suitable functions $(\rho_{0,k}, u_{0,k},\theta_{0,k},w_{0,k})$ on the interval $[-k,k]$, where k is any natural number. We prove a general existence theorem for the case of a (spatial) bounded interval $\Omega=(a,b)$ and apply it taking $a=-k, b=k$ and $(\rho_{0,k},u_{0,k},\theta_{0,k},w_{0,k})$ as initial data, thus finding a sequence of solution functions $(\rho_k,u_k,\theta_k,w_k)$. Such a theorem can be stated as follows:

Theorem 2. *Let $w_0\in H^1(\Omega,\mathbb{C})$ and $\rho_0,u_0,\theta_0\in H^1(\Omega)$ such that $u_0(a)=u_0(b)=\theta_{0x}(a)=\theta_{0x}(b)=w_0(a)=w_0(b)=0$. Then there exists a unique (local) solution to the problem*

$$\rho_t+\rho^2u_x=0, \tag{1.10}$$

$$u_t+R(\rho\theta)_x=\mu(\rho u_x)_x+\alpha\left(g'(1/\rho)h(|w|^2)\right)_x, \tag{1.11}$$

$$\theta_t+\frac{1}{C_\vartheta}R\rho\theta u_x=\frac{1}{C_\vartheta}\kappa(\rho\theta_x)_x+\frac{\mu}{C_\vartheta}\rho u_x^2, \tag{1.12}$$

$$iw_t+w_{xx}=|w|^2w+\alpha g(1/\rho)h'(|w|^2)w, \tag{1.13}$$

$$u=\theta_x=w=0 \quad \text{at } x=a,b, \tag{1.14}$$

$$u=u_0,\rho=\rho_0,\theta=\theta_0,w=w_0 \quad \text{at } t=0, \tag{1.15}$$

where the boundary values are taken in the sense of traces and (for a small enough $t_0>0$)

$$
\begin{aligned}
&u, \theta-\theta_* \in C([0,t_0],H^1(\Omega))\cap L^2([0,t_0],H^2(\Omega)),\\
&u_t, \theta_t \in L^2(\Omega\times[0,t_0]),\\
&\rho-\rho_* \in C([0,t_0],H^1(\Omega)), \quad \rho_t \in L^2(\Omega\times[0,t_0])\\
&m_1<\rho<M_1,\\
&w\in C([0,t_0],H^1(\Omega)).
\end{aligned}
$$

We will refer to this as the bounded domain problem. Let us point out that this theorem can be stated in a more general way. In fact, the calculations in chapter three can be adapted (or replaced by simpler ones) to show global existence in the bounded domain case. Here, however, this theorem is an intermediate step in the proof of theorem 1.

For the proof of theorem 2, which is the purpose of chapter 2, we apply a Faedo-Galerkin tipe method as in [4] which in turn follows the ideas in [1]. All of this is accomplished in chapter 2.

In chapter 3 we prove estimates which allow us to take a convergent subsequence of $(\rho_k, u_k, \theta_k, w_k)$. Then we prove that the limit functions are in fact a local solution to our original Cauchy problem. After proving uniqueness, we extend our solution to the time interval $[0,T]$ where $T > 0$ is an arbitrarily large finite time. We achieve this by means of a priori estimates.

As mentioned before, all calculations and ideas in this work are based on the analogues contained in [4] and [1].

Chapter 2

The case of a bounded interval

As stated before, we first consider the case when our space domain is a bounded interval $\Omega = (a,b)$. The present chapter will be devoted to the proof of theorem 2.

2.1 Approximate Problem

Theorem 2 is stated as a general theorem so, throughout all of chapter 2, we are going to forget that, for our purposes, it is a mere tool that will help us prove theorem 1. In other words, for the time being, the functions ρ, u, θ and w are not related to their homonyms considered previously. Keeping that in mind let us begin with the proof.

Note that up to a translation of the involved functions we can assume that our spatial variable takes values in the domain $\Omega = (0,L)$. Consider the subspace X_n of $L^2(\Omega,\mathbb{C})$ given for each $n \in \mathbb{N}$ by

$$X_n = span_{\mathbb{C}}\{cos\frac{\pi jx}{L}, sin\frac{\pi jx}{L} : j = 0,1,...,n\}.$$

Note that $dimX_n < \infty$ and that we can write

$$X_n = X_n^s + X_n^c,$$

where

$$X_n^s = span_{\mathbb{C}}\{sin\frac{\pi jx}{L} : j = 1,...,n\}$$

and

$$X_n^c = span_{\mathbb{C}}\{cos\frac{\pi jx}{L} : j = 0,1,...,n\}.$$

Let $P_n^s : L^2(\Omega,\mathbb{C}) \to X_n^s$ and $P_n^c : L^2(\Omega,\mathbb{C}) \to X_n^c$ be the respective projections. We are going to construct a solution to the problem as a limit of functions $(\rho_n, u_n, \theta_n, w_n)$ where $u_n \in X_n^s$, $\theta_n \in X_n^c$ and $w_n \in X_n^s$. We are going to do so by posing the following approximated problem in X_n. First, approximate the initial data u_0, θ_0, w_0 respectively by the projections $u_{0n} = P_n^s u_0$, $\theta_{0n} = P_n^c \theta_0$ and $w_{0n} = P_n^s w_0$. From (1.10) we can find ρ_n in terms of u_n by the formula

$$\rho_n(x,t) = \frac{\rho_0(x)}{1 + \rho_0(x) \int_0^t u_{nx}(x,s)ds}. \tag{2.1}$$

By defining

$$z_n(x,t) = \int_0^t u_n(x,s)ds$$

we arrive to the following system of ordinary differential equations in the finite dimensional linear space X_n:

$$u_{nt} = \Phi(u_n, \theta_n, w_n, z_n), \tag{2.2}$$

$$\theta_{nt} = \Psi(u_n, \theta_n, w_n, z_n), \tag{2.3}$$

$$iw_{nt} = \Gamma(u_n, \theta_n, w_n, z_n), \tag{2.4}$$

$$z_{nt} = u_n. \tag{2.5}$$

where

$$\begin{aligned}
\Phi(u_n, \theta_n, w_n, z_n) &= P_n^s \left(-R \left(\rho_n \theta_n - \mu \rho u_{nx} - \alpha g'(1/\rho_n) h(|w_n|^2) \right)_x \right), \\
\Psi(u_n, \theta_n, w_n, z_n) &= P_n^c \left(-\frac{1}{C_\vartheta} \left(R\rho_n \theta_n u_{nx} - \kappa(\rho_n \theta_{nx})_x - \mu \rho_n u_{nx}^2 \right) \right), \\
\Gamma(u_n, \theta_n, w_n, z_n) &= P_n^s \left(- \left(w_{nxx} - |w_n|^2 w_n - \alpha g(1/\rho_n) h'(|w_n|^2) w_n \right) \right),
\end{aligned}$$

and the initial data is given by

$$(u_n, \theta_n, w_n, z_n)|_{t=0} = ((u_{0n}, \theta_{0n}, w_{0n}, 0)).$$

We are going to refer to this problem as the *approximate problem*. Local existence and uniqueness of solutions to the approximate problem is given by the well known result on the theory of ordinary differential equations.

2.2 Estimates for the approximate problem

In this section we prove some uniform (over n) estimates on the approximate problem which will allow us to take convergent subsequences to a solution of the bounded domain problem.

Observe that the solution to the approximate problem is defined on a time interval $[0,t_n]$. So, we not only have to bound properly the norms of the involved functions but have to guarantee that they are all defined on a uniform small enough interval $[0,t_0]$. That is, we have to guarantee that t_n is bounded from below by a uniform positive bound t_0.

As is to be seen below, the estimates proven in this section will depend on the length L of the space interval. We will make this dependence explicit.

Proposition 1. *The following equalities hold for all $n \in \mathbb{N}$*

(i) $\frac{d}{dt}||w_n(t)||^2_{L^2(\Omega)} = 0.$

(ii) $\frac{d}{dt}\left(\frac{1}{2}||u_n||^2_{L^2(\Omega)} + C_\vartheta \int_\Omega \theta_n dx\right) = -\int_\Omega \alpha g'(1/\rho_n)h(|w_n|^2)u_{nx}dx.$

Proof. The first equality is easily obtained by multiplying (2.4) by $\overline{w_n}$, taking imaginary part and integrating over Ω. For the second equality, begin by multiplying (2.3) by C_ϑ and integrating over Ω to obtain:

$$C_\vartheta \frac{d}{dt}\int_\Omega \theta_n dx + \int_\Omega R\rho_n\theta_n u_{nx} - \kappa(\rho_n\theta_{nx})_x - \mu\rho_n u_{nx}^2 dx = 0.$$

Since $\theta_n \in X_n^c$ we have

$$\int_\Omega (\rho_n\theta_{nx})_x dx = 0.$$

Thus,

$$C_\vartheta \frac{d}{dt}\int_\Omega \theta_n dx = -\int_\Omega (R\rho_n\theta_n u_{nx} - \mu\rho_n u_{nx}^2)dx. \tag{2.6}$$

To evaluate the right side of this equality, multiply (2.2) by u_n and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}||u_n||^2_{L^2(\Omega)} + \int_\Omega (R\rho_n\theta_n - \mu\rho_n u_{nx} - \alpha g'(1/\rho_n)h(|w_n|^2))_x u_n dx.$$

Since $u_n \in X_n^s$ we can integrate by parts to get

$$\frac{1}{2}\frac{d}{dt}||u_n||^2_{L^2(\Omega)} = \int_\Omega R\rho_n\theta_n u_{nx} - \mu\rho_n u_{nx}^2 - \alpha g'(1/\rho_n)h(|w_n|^2)u_{nx}dx.$$

Adding this last equality to (2.6) we complete the proof. □

From this proposition we can derive two corollaries. Namely:

Corollary 1. *There is a constant $C > 0$ independent of n (however, dependent on L) such that*

$$\max_{x\in\Omega}|\theta_n| \leq C\left(1+||u_{nx}||^2_{L^2(\Omega)}+\int_0^t ||u_{nxx}||^2_{L^2(\Omega)}ds+||\theta_{nx}||_{L^2(\Omega)}\right) \tag{2.7}$$

for all $0 \leq t \leq \min\{t_n, 1\}$. Here, and all through this work, ds denotes integration with respect to the time variable.

Proof. Since θ_n is continuous (with respect to the space variable), there exists a point $a = a(t) \in \Omega$ such that

$$\theta_n(a,t) = \frac{1}{L}\int_\Omega \theta_n dx.$$

By equality (ii) in proposition 1, we have that

$$|LC_\vartheta \theta_n(a,t)| \leq \frac{1}{2}||u_n||^2_{L^2(\Omega)} + \int_0^t\int_\Omega \alpha|g'(1/\rho_n)h(|w_n|^2))_x u_n|dxds + C_1,$$

where C_1 depends on the initial data. Since $u_n \in X_n^s$ we have the inequality

$$||u_n||_{L^2(\Omega)} \leq L^{1/2}||u_{nx}||_{L^2(\Omega)},$$

which is obtained by writing u_n in the form

$$u_n(x,t) = \int_0^x u_{nx}(\xi,t)d\xi.$$

Similarly, for u_{nx}, since $u_n(0) = u_n(L) = 0$, then there exists a point $b = b(t) \in \Omega$ such that $u_{nx}(b) = 0$. So writing $u_{nx}(x,t) = \int_0^x u_{nxx}(\xi,t)d\xi$, we have the inequality

$$||u_{nx}||_{L^2(\Omega)} \leq L||u_{nxx}||_{L^2(\Omega)}.$$

Using this and Young's inequality we get

$$\begin{aligned}\int_\Omega \alpha|g'(1/\rho_n)h(|w_n|^2))_x u_n|dx &\leq C_2L^{1/2}||u_{nx}||_{L^2(\Omega)}\\ &\leq C_2L^{3/2}||u_{nxx}||_{L^2(\Omega)}\\ &\leq \frac{1}{2}\left(C_2^2L^3+||u_{nxx}||^2_{L^2(\Omega)}\right),\end{aligned}$$

where $C_2 = \alpha\max_{s\in\mathbb{R}}|g'(s)|\max_{s\in\mathbb{R}}|h(s)|$. Putting all of this together we conclude that for all $0 \leq t \leq \min\{t_n, 1\}$

$$|\theta_n(a,t)| \leq C\left(1+||u_{nx}||^2_{L^2(\Omega)}+\int_0^t ||u_{nxx}||^2_{L^2(\Omega)}ds\right),$$

for some constant $C > 0$ which depens on L but is uniform over n. In order to finish the proof we write

$$\begin{aligned} |\theta_n(x,t)| &= \left| \int_a^x \theta_{nx}(\xi,t)d\xi + \theta_n(a,t) \right| \\ &\leq L^{1/2}||\theta_{nx}||_{L^2(\Omega)} + C\left(1 + ||u_{nx}||^2_{L^2(\Omega)} + \int_0^t ||u_{nxx}||^2_{L^2(\Omega)} ds\right). \end{aligned}$$

□

Corollary 2. *For all $n \in \mathbb{N}$ we have*

$$\frac{d}{dt}\left(\frac{1}{2}||u_n||^2_{L^2(\Omega)} + C_\vartheta \int_\Omega \theta_n dx + \alpha||g(1/\rho_n)h(|w_n|^2)||_{L^1(\Omega)} + ||w_{nx}||^2_{L^2(\Omega)} + || \, |w_n|^4||_{L^1(\Omega)}\right) = 0. \tag{2.8}$$

Proof. From (2.1) we can write

$$\begin{aligned} -\int_\Omega \alpha g'(1/\rho_n)h(|w_n|^2)u_{nx}dx &= -\int_\Omega \alpha g(1/\rho_n)_t h(|w_n|^2)dx \\ &= \frac{d}{dt}\left(-\int_\Omega \alpha g(1/\rho_n)h(|w_n|^2)dx\right) \\ &\quad + \int_\Omega \alpha g(1/\rho_n)h'(|w_n|^2)2Re(w_n\overline{w_{nt}})dx. \end{aligned}$$

Multiplying (2.4) by $2\overline{w_{nt}}$, taking real part and integrating over Ω we get

$$\begin{aligned} \int_\Omega \alpha g(1/\rho_n)h'(|w_n|^2)2Re(w_n\overline{w_{nt}})dx &= 2Re\left(\int_\Omega w_{nxx}\overline{w_{nt}} - |w_n|^2 w_n\overline{w_{nt}}dx\right) \\ &= -\frac{d}{dt}\left(\int_\Omega |w_{nx}|^2 + \frac{1}{2}|w_n|^4 dx\right). \end{aligned}$$

Note that in this last equality we used the fact that $w_n \in X_n^s$ in order to integrate by parts. Putting all of this together with (ii) in proposition 1 we arrive at the desired result. □

Proposition 2. *There exists $t_0 > 0$ such that*

$$\max_{0\leq t\leq t_0} ||u_{nx}||^2_{L^2(\Omega)} + \int_0^{t_0} ||u_{nxx}||^2_{L^2(\Omega)} ds \leq C, \tag{2.9}$$

$$\max_{0\leq t\leq t_0} ||\theta_{nx}||^2_{L^2(\Omega)} + \int_0^{t_0} ||\theta_{nxx}||^2_{L^2(\Omega)} ds \leq C, \tag{2.10}$$

$$\frac{1}{2}m \leq \rho_n(x,t) \leq 2M, \quad x \in \Omega, 0 \leq t \leq t_0 \tag{2.11}$$

for some constant $C > 0$ independent of n.

The proof of this proposition is long and will rely on a series of lemmas. We begin assuming that the estimates (2.11) hold. This is certainly true, for each n, on a suficiently small interval which we can assume to be $[0,t_n]$. We will later on prove that this interval can be chosen to be uniform over n.

Define $y_n = y_n(t)$ by

$$y_n(t) = ||u_{nx}||^2_{L^2(\Omega)} + ||\theta_{nx}||^2_{L^2(\Omega)} + \int_0^t ||u_{nxx}||^2_{L^2(\Omega)} ds + \int_0^t ||\theta_{nxx}||^2_{L^2(\Omega)} ds.$$

We are going to prove the following inequality:

$$y_n' \leq C_1(1+y_n^4) \tag{2.12}$$

for a constant $C_1 > 0$ independent of n. In this way, since all $y_n(0)$ are bounded by a constant C_2, which depends only on (u_0,θ_0), if $y = y(t)$ is a solution to the ODE

$$y' = C_1(1+y^4), \quad y(0) = C_2,$$

then we have the estimate

$$y_n(t) \leq y(t)$$

on a sufficiently small interval $[0,t_0]$ on which y is defined. After that, by (2.1) we see that t_0 can be chosen in such a way that (2.11) is satisfied and the proof will be completed.

Let us begin proving (2.12). Multiply (2.2) by u_{nxx} and integrate over Ω to obtain

$$0 = \int_\Omega u_{nt}u_{nxx} + \big(R\rho_n\theta_n - \mu\rho_n u_{nx} - \alpha g'(1/\rho_n)h(|w_n|^2)\big)_x u_{nxx} dx.$$

This equality implies

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||u_{nx}||^2_{L^2(\Omega)} + \mu\int_\Omega \rho_n u_{nxx}^2 dx \\ &\quad = \int_\Omega (R\rho_{nx}\theta_n + R\rho_n\theta_{nx} - \mu\rho_{nx}u_{nx} - \alpha(g'(1/\rho_n)h(|w_n|^2))_x)u_{nxx} dx.\end{aligned}$$

Similarly, multiplying (2.3) by $C_\vartheta\theta_{nxx}$ and integrating gives

$$\begin{aligned}&C_\vartheta\frac{1}{2}\frac{d}{dt}||\theta_{nx}||^2_{L^2(\Omega)} + \kappa\int_\Omega \rho_n\theta_{nxx}^2 dx \\ &\quad = \int_\Omega (R\rho_n\theta_n u_{nx} - \kappa\rho_{nx}\theta_{nx} - \mu\rho_n u_{nx}^2)\theta_{nxx} dx.\end{aligned}$$

Adding these two equations we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(||u_{nx}||^2_{L^2(\Omega)} + C_\vartheta||\theta_{nx}||^2_{L^2(\Omega)}\right) + \int_\Omega \rho_n(\mu u_{nxx}^2 + \kappa\theta_{nxx}^2)dx \\
&\quad = \int_\Omega (R\rho_{nx}\theta_n + R\rho_n\theta_{nx} - \mu\rho_{nx}u_{nx} - \alpha(g'(1/\rho_n)h(|w_n|^2))_x)u_{nxx}dx \\
&\quad + \int_\Omega (R\rho_n\theta_n u_{nx} - \kappa\rho_{nx}\theta_{nx} - \mu\rho_n u_{nx}^2)\theta_{nxx}dx. \qquad (2.13)
\end{aligned}$$

The rest of this section will be devoted to bound appropriately each one of the terms in the right side of this last equality. For this we need the following two lemmas, as well as the corollaries proven before.

Lemma 1. *There exists a constant $C > 0$ independent of n such that for all $0 \le t \le \min\{t_n, 1\}$*

$$||\rho_{nx}||_{L^2(\Omega)} \le C\left(1 + \left[\int_0^t ||u_{nxx}||^2_{L^2(\Omega)}\right]^{1/2}\right). \qquad (2.14)$$

Proof. Direct calculation of $\frac{\partial}{\partial x}\rho_n$ in (2.1) gives

$$\begin{aligned}
\rho_{nx} &= \frac{\rho_{0x} - \rho_0^2\int_0^t u_{nxx}ds}{\left(1+\rho_0\int_0^t u_{nx}ds\right)^2} \\
&= \left(\rho_{0x} - \rho_0^2\int_0^t u_{nxx}ds\right)\left(\frac{\rho_n}{\rho_0}\right)^2.
\end{aligned}$$

By (2.11) and the triangle inequality we have

$$||\rho_{nx}||_{L^2(\Omega)} \le \left(\frac{2M}{m}\right)^2\left(||\rho_{0x}||_{L^2(\Omega)} + M^2\left[\int_\Omega\left(\int_0^t u_{nxx}ds\right)^2 dx\right]^{1/2}\right).$$

Using Jensen's inequality we see that for $t < 1$ we have

$$\left[\int_\Omega\left(\int_0^t u_{nxx}ds\right)^2 dx\right]^{1/2} \le \left[\int_0^t ||u_{nxx}||^2_{L^2(\Omega)}ds\right]^{1/2}.$$

This, together with our assumptions on the initial data, implies the result. □

Lemma 2. *The following inequalities hold for all n*

$$\max_{x\in\Omega}|u_{nx}| \le \sqrt{2}||u_{nx}||^{1/2}_{L^2(\Omega)}||u_{nxx}||^{1/2}_{L^2(\Omega)}, \qquad (2.15)$$

$$\max_{x\in\Omega}|\theta_{nx}| \le \sqrt{2}||\theta_{nx}||^{1/2}_{L^2(\Omega)}||\theta_{nxx}||^{1/2}_{L^2(\Omega)}. \qquad (2.16)$$

Proof. Since $u_n(0) = u_n(L)$ (for $u_n \in X_n^s$) there exists a point $b = b_n(t) \in \Omega$ such that $u_{nx}(b) = 0$. Therefore, we can write

$$\begin{aligned} u_{nx}^2(x,t) &= \int_b^x (u_{nx}(\xi,t)^2)_x d\xi \\ &= \int_b^x 2u_{nx}(\xi,t)u_{nxx}(\xi,t)d\xi \\ &\le 2||u_{nx}||_{L^2(\Omega)}||u_{nxx}||_{L^2(\Omega)} \end{aligned}$$

from which inequality (2.15) follows. The proof of inequality (2.16) is identical once we observe that $\theta_{nx}(0) = 0$, for $\theta_n \in X_n^c$. □

We now have the necessary to bound the terms in (2.13). We begin by the term

$$I_1 := \int_\Omega R\rho_{nx}\theta_n u_{nxx}dx.$$

Applying first Cauchy's inequality, lemma 1 and corollary 1 we have

$$\begin{aligned} |I_1| &\le R||\rho_{nx}||_{L^2(\Omega)}||u_{nxx}||_{L^2(\Omega)} \max_{x\in\Omega}|\theta_n| \\ &\le C\left(1+\left[\int_0^t ||u_{nxx}||^2_{L^2(\Omega)}ds\right]^{1/2}\right)||u_{nxx}||_{L^2(\Omega)}\left(1+||u_{nx}||^2_{L^2(\Omega)}+\right. \\ &\left.\int_0^t ||u_{nxx}||^2_{L^2(\Omega)}ds+||\theta_{nx}||_{L^2(\Omega)}\right). \end{aligned}$$

We use Young's inequality with ε to show that for every $\varepsilon_1 > 0$ there exists C_{ε_1} such that

$$\begin{aligned} |I_1| &\le \varepsilon_1||u_{nxx}||^2_{L^2(\Omega)}+C_{\varepsilon_1}\left(1+\left[\int_0^t ||u_{nxx}||^2_{L^2(\Omega)}ds\right]^{1/2}\right)\left(1+||u_{nx}||^2_{L^2(\Omega)}+\right. \\ &\left.\int_0^t ||u_{nxx}||^2_{L^2(\Omega)}ds+||\theta_{nx}||_{L^2(\Omega)}\right). \end{aligned}$$

Applying Young's inequality and redefining C_{ε_1}

$$\begin{aligned} |I_1| &\le \varepsilon_1||u_{nxx}||^2_{L^2(\Omega)}+C_{\varepsilon_1}\left(1+\int_0^t ||u_{nxx}||^2_{L^2(\Omega)}ds+||u_{nx}||^4_{L^2(\Omega)}+\right. \\ &\left.\left[\int_0^t ||u_{nxx}||^2_{L^2(\Omega)}ds\right]^2+||\theta_{nx}||^2_{L^2(\Omega)}\right). \end{aligned} \tag{2.17}$$

We now pass to the term

$$I_2 := \int_\Omega R\rho_n\theta_{nx}u_{nxx}dx.$$

Using (2.11), Cauchy-Schwarz inequality and Young's inequality with ε we have

$$\begin{aligned} |I_2| &\leq 2MR||\theta_{nx}||_{L^2(\Omega)}||u_{nxx}||_{L^2(\Omega)} \\ &\leq \varepsilon_2||u_{nxx}||^2_{L^2(\Omega)} + C_{\varepsilon_2}||\theta nx||^2_{L^2(\Omega)}. \end{aligned} \tag{2.18}$$

For

$$I_3 := \int_\Omega \mu \rho_{nx} u_{nx} u_{nxx} dx,$$

we use, yet again, the Cauchy-Schwarz inequality, followed by lemma 1 and (2.15) to obtain

$$\begin{aligned} |I_3| &\leq \mu||\rho_{nx}||_{L^2(\Omega)}||u_{nxx}||_{L^2(\Omega)} \max_{x\in\Omega}|u_{nx}| \\ &\leq C\left(1+\left[\int_0^t ||u_{nxx}||^2_{L^2(\Omega)}ds\right]^{1/2}\right)||u_{nxx}||^{3/2}_{L^2(\Omega)}||u_{nx}||^{1/2}_{L^2(\Omega)}. \end{aligned}$$

Now, we apply Young's inequality with ε and Young inequality again resulting in the estimates

$$\begin{aligned} |I_3| &\leq \varepsilon_3||u_{nxx}||^2_{L^2(\Omega)} + C_{\varepsilon_3}\left(1+\left[\int_0^t ||u_{nxx}||^2_{L^2(\Omega)}ds\right]^{1/2}\right)^4 ||u_{nx}||^2_{L^2(\Omega)} \\ &\leq \varepsilon_3||u_{nxx}||^2_{L^2(\Omega)} + C_{\varepsilon_3}\left(1+\left[\int_0^t ||u_{nxx}||^2_{L^2(\Omega)}ds\right]^4 + ||u_{nx}||^4_{L^2(\Omega)}\right). \end{aligned} \tag{2.19}$$

We write the fourth term in the form

$$\begin{aligned} I_4 &= \int_\Omega \alpha(g'(1/\rho_n)h(|w_n|^2))_x u_{nxx} dx \\ &= -\int_\Omega \alpha g''(1/\rho_n)\frac{\rho_{nx}}{\rho^2}h(|w_n|^2)u_{nxx}dx \\ &\quad + \int_\Omega \alpha g'(1/\rho_n)h'(|w_n|^2)2Re(w_n\overline{w}_{nx})u_{nxx}dx \\ &= I_4^{(1)} + I_4^{(2)}. \end{aligned}$$

To estimate $I_4^{(1)}$ we use (2.11), Cauchy-Schwarz inequality and lemma 1, followed by Young's inequality with ε obtaining

$$\begin{aligned} |I_4^{(1)}| &\leq \frac{2}{m}\alpha \max|g''| \max|h| \, ||\rho_{nx}||_{L^2(\Omega)}||u_{nxx}||_{L^2(\Omega)} \\ &\leq \varepsilon_4||u_{nxx}||^2_{L^2(\Omega)} + C_{\varepsilon_4}\left(1+\int_0^t ||u_{nxx}||^2_{L^2(\Omega)}ds\right). \end{aligned} \tag{2.20}$$

For $I_4^{(2)}$, we note that since $supp(h')$ is compact in $[0,\infty)$, there is a constant $A>0$ such that if $|w_n|\geq A$, $h(|w_n|^2)=0$. So, using this fact, Cauchy-Schwarz inequality and Young's inequality with ε, we have

$$\begin{aligned}|I_4^{(2)}| &\leq \alpha A \max|g'|\max|h'|\ ||w_{nx}||_{L^2(\Omega)}||u_{nxx}||_{L^2(\Omega)}\\ &\leq \varepsilon_4||u_{nxx}||^2_{L^2(\Omega)}+C_{\varepsilon_4}||w_{nx}||^2_{L^2(\Omega)}.\end{aligned}$$

By (2.8) we know that $||w_{nx}||^2_{L^2(\Omega)}$ is uniformly bounded over n by a constant which depends only on the initial data so we complete the estimate for $I_4^{(2)}$ as

$$|I_4^{(2)}|\leq \varepsilon_4||u_{nxx}||^2_{L^2(\Omega)}+C. \tag{2.21}$$

We continue with

$$I_5 := \int_\Omega R\rho_n\theta_n u_{nx}\theta_{nxx}dx$$

By (2.11), corollary 1 and Cauchy-Schwarz inequality

$$|I_5|\leq 2MRC\left(1+||u_{nx}||^2_{L^2(\Omega)}+||\theta_{nx}||_{L^2(\Omega)}+\int_0^t||u_{nxx}||_{L^2(\Omega)}ds\right)||u_{nx}||_{L^2(\Omega)}||\theta_{nxx}||^2_{L^2(\Omega)}.$$

Applying Young's inequality wth ε and Young's inequality again

$$\begin{aligned}|I_5| &\leq \varepsilon_5||\theta_{nxx}||^2_{L^2(\Omega)}+C_{\varepsilon_5}\Big(1+||u_{nx}||^2_{L^2(\Omega)}+||\theta_{nx}||_{L^2(\Omega)}+\\ &\quad \int_0^t||u_{nxx}||_{L^2(\Omega)}ds\Big)^2||u_{nx}||^2_{L^2(\Omega)}\\ &\leq \varepsilon_5||\theta_{nxx}||^2_{L^2(\Omega)}+C_{\varepsilon_5}\Big(1+||u_{nx}||^8_{L^2(\Omega)}+||\theta_{nx}||^4_{L^2(\Omega)}+\\ &\quad \left[\int_0^t||u_{nxx}||_{L^2(\Omega)}ds\right]^4+||u_{nx}||^4_{L^2(\Omega)}\Big),\end{aligned}$$

possibly redefining the constant C_{ε_5}. Finally, applying Jensen inequality we see that for $t<1$ we have

$$\begin{aligned}|I_5| &\leq \varepsilon_5||\theta_{nxx}||^2_{L^2(\Omega)}+C_{\varepsilon_5}\Big(1+||u_{nx}||^8_{L^2(\Omega)}+||\theta_{nx}||^4_{L^2(\Omega)}+\\ &\quad \left[\int_0^t||u_{nxx}||^2_{L^2(\Omega)}ds\right]^2+||u_{nx}||^4_{L^2(\Omega)}\Big), \end{aligned}\tag{2.22}$$

(2.23)

The second to last term is

$$I_6 = \int_\Omega \kappa \rho_{nx} \theta_{nx} \theta_{nxx} dx$$

We use lemma 1, (2.16) and Cauchy-Schwarz inequality to obtain

$$\begin{aligned} |I_6| &\leq \kappa ||\rho_{nx}||_{L^2(\Omega)} ||\theta_{nxx}||_{L^2(\Omega)} \max_{x\in\Omega} |\theta_{nx}| \\ &\leq C\left(1+\left[\int_0^t ||u_{nxx}||^2_{L^2(\Omega)} ds\right]^{1/2}\right) ||\theta_{nxx}||^{3/2}_{L^2(\Omega)} ||\theta_{nx}||^{1/2}_{L^2(\Omega)}. \end{aligned}$$

Young's inequality with ε gives

$$|I_6| \leq \varepsilon_6 ||\theta_{nxx}||^2_{L^2(\Omega)} + C_{\varepsilon_6}\left(1+\left[\int_0^t ||u_{nxx}||^2_{L^2(\Omega)} ds\right]^{1/2}\right)^4 ||\theta_{nx}||^2_{L^2(\Omega)}.$$

And using Young's inequality again

$$|I_6| \leq \varepsilon_6 ||\theta_{nxx}||^2_{L^2(\Omega)} + C_{\varepsilon_6}\left(1+\left[\int_0^t ||u_{nxx}||^2_{L^2(\Omega)} ds\right]^4 + ||\theta_{nx}||^4_{L^2(\Omega)}\right). \tag{2.24}$$

Finally the last term is given by

$$I_7 = \int_\Omega \mu \rho_n u^2_{nx} \theta_{nxx} dx.$$

By (2.11), (2.15) and Cauchy-Schwarz inequality

$$\begin{aligned} |I_7| &\leq 2M\mu ||u_{nx}||_{L^2(\Omega)} ||\theta_{nxx}||_{L^2(\Omega)} \max_{x\in\Omega} |u_{nx}| \\ &\leq C ||\theta_{nxx}||_{L^2(\Omega)} ||u_{nx}||^{3/2}_{L^2(\Omega)} ||u_{nxx}||^{1/2}_{L^2(\Omega)}. \end{aligned}$$

Using Young's inequality with ε

$$|I_7| \leq \varepsilon_7 ||\theta_{nxx}||^2_{L^2(\Omega)} + C_{\varepsilon_7} ||u_{nx}||^3_{L^2(\Omega)} ||u_{nxx}||_{L^2(\Omega)}.$$

And Young's inequality with ε for the last time gives

$$|I_7| \leq \varepsilon_7 (||\theta_{nxx}||^2_{L^2(\Omega)} + ||u_{nxx}||^2_{L^2(\Omega)}) + C_{\varepsilon_7} ||u_{nx}||^6_{L^2(\Omega)}. \tag{2.25}$$

Gathering all these estimates in (2.13) we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(||u_{nx}||^2_{L^2(\Omega)}+C_\vartheta||\theta_{nx}||^2_{L^2(\Omega)}\right)+\frac{m}{2}\left(\mu||u_{nxx}||^2_{L^2(\Omega)}+\kappa||\theta_{nxx}||^2_{L^2(\Omega)}\right)\\
&\leq\varepsilon(||u_{nxx}||^2_{L^2(\Omega)}+||\theta_{nxx}||^2_{L^2(\Omega)})+C_\varepsilon\left\{1+||u_{nx}||^4_{L^2(\Omega)}+||u_{nx}||^6_{L^2(\Omega)}\right.\\
&+||\theta_{nx}||^4_{L^2(\Omega)}+\int_0^t||u_{nxx}||^2_{L^2(\Omega)}ds+\left[\int_0^t||u_{nxx}||^2_{L^2(\Omega)}ds\right]^2\\
&\left.+\left[\int_0^t||u_{nxx}||^2_{L^2(\Omega)}ds\right]^4\right\}.
\end{aligned}$$

Note that this inequality is valid for all $0\leq t\leq\min\{1,t_n\}$, where t_n is the largest time of definition of the solution $(u_n(t),w_n(t),\rho_n(t),\theta_n(t))$ to the approximate problem such that (2.11) is valid, and for arbitrary $\varepsilon>0$. Choosing ε small enough and redefining the constant C_ε we may write this last inequality in the form

$$\begin{aligned}
&\frac{d}{dt}\left(||u_{nx}||^2_{L^2(\Omega)}+||\theta_{nx}||^2_{L^2(\Omega)}\right)+\left(||u_{nxx}||^2_{L^2(\Omega)}+||\theta_{nxx}||^2_{L^2(\Omega)}\right)\\
&\leq C\left\{1+||u_{nx}||^4_{L^2(\Omega)}+||u_{nx}||^6_{L^2(\Omega)}+||\theta_{nx}||^4_{L^2(\Omega)}+\int_0^t||u_{nxx}||^2_{L^2(\Omega)}ds\right.\\
&\left.+\left[\int_0^t||u_{nxx}||^2_{L^2(\Omega)}ds\right]^2+\left[\int_0^t||u_{nxx}||^2_{L^2(\Omega)}ds\right]^4\right\}. \qquad (2.26)
\end{aligned}$$

Finally, observe that this last inequality implies the following one

$$\begin{aligned}
&\frac{d}{dt}\left(||u_{nx}||^2_{L^2(\Omega)}+||\theta_{nx}||^2_{L^2(\Omega)}\right)+\left(||u_{nxx}||^2_{L^2(\Omega)}+||\theta_{nxx}||^2_{L^2(\Omega)}\right)\\
&\leq 6C\left\{1+||u_{nx}||^8_{L^2(\Omega)}+||\theta_{nx}||^8_{L^2(\Omega)}+\left[\int_0^t||u_{nxx}||^2_{L^2(\Omega)}ds\right]^4\right\}. \qquad (2.27)
\end{aligned}$$

(It is easy to see it by considering cases when each one of the terms in the right side of (2.26) is greater or lower than 1). This inequality implies directly (2.12) and, as was mentioned before, this is enough to prove proposition 2.

2.3 Local existence and uniqueness

In this section we are going to show how the estimates from the previous section help us prove theorem 2.

As a corollary to proposition 2 we have the estimates

$$\max_{t\in[0,t_0]} ||\theta_n||^2_{L^2(\Omega)} \le C \tag{2.28}$$

$$\max_{t\in[0,t_0]} ||u_{nt}||^2_{L^2(\Omega)} + \int_0^t ||u_{nxt}||^2_{L^2(\Omega)} ds \le C, \qquad t\in[0,t_0] \tag{2.29}$$

$$\int_0^{t_0} ||\rho_{nt}||^2_{L^2(\Omega)} ds \le C \tag{2.30}$$

$$\max_{t\in[0,t_0]} ||\theta_{nt}||^2_{L^2(\Omega)} + \int_0^t ||\theta_{nxt}||^2_{L^2(\Omega)} ds \le C, \qquad t\in[0,t_0] \tag{2.31}$$

$$\max_{t\in[0,t_0]} ||w_{nt}||_{H^{-1}(\Omega)} \le C. \tag{2.32}$$

where $C>0$ is independent of n. Let us show how (2.28) is obtained. Estimates (2.29) trhu (2.32) will be similarly deduced. Multiplying (2.3) by θ, integrating over Ω and using Young's inequality appropriately we arrive at the following inequality

$$C_\vartheta \frac{d}{dt}||\theta_n||^2_{L^2(\Omega)} \le C_1||\theta_n||^2_{L^2(\Omega)} + 2MR\max_{x\in\Omega}|\theta|\ ||u_x||^2_{L^2(\Omega)} + \kappa\max_{x\in\Omega}|\theta_x|\ ||\rho_x||^2_{L^2(\Omega)}$$
$$+2M\kappa||\theta_{xx}||^2_{L^2(\Omega)} + 2M\mu\max_{x\in\Omega}|u_x|\ ||u_x||^2_{L^2(\Omega)}.$$

Using corollary 1, proposition 2 and Gronwall's inequality we deduce (2.28).

Let us take a moment to analyze some consequences of all the calculations we have done so far. First, note that from (2.8) and (2.9), Morrey's inequality tells us that not only are all u_n Hölder continuous, but they are all equicontinuous (with respect to the space variable x). So, from the Arzelà-Ascoli theorem, for all fixed $t\in[0,t_0]$, there is a subsequence $\{u_{n_{m(t)}}\}$, which depends on t, such that $u_{n_{m(t)}}(\cdot,t)$ converges uniformly to a function $u^{(t)}(\cdot)$. In fact, applying a diagonal argument, we can take a subsequence $\{u_{n_m}\}$ that, for all $t\in[0,t_0]\cap\mathbb{Q}$ fixed, converges uniformly to a function $u^{(t)}$. We would like that this convergence held for all $t\in[0,t_0]$ (or at least for almost all t) and for that we need some kind of continuity in t. Fix $t_1,t_2\in[0,t_0]$. Using Jensen's inequality and (2.29) we have the following

$$\begin{aligned} ||u_n(\cdot,t_1)-u_n(\cdot,t_2)||^2_{L^2(\Omega)} &= \int_\Omega \left|\int_{t_1}^{t_2} u_t(x,s)ds\right|^2 dx \\ &\le |t_1-t_2|\int_\Omega\int_0^{t_0} u_t^2 ds dx \\ &= |t_1-t_2|\int_0^{t_0} ||u_t||^2_{L^2(\Omega)} ds \\ &\le C|t_1-t_2|. \end{aligned}$$

Thus the functions u_n, viewed as functions of $t \in [0,t_0]$ taking values in $L^2(\Omega)$, are equicontinuous. This justifies that the sequence $\{u_{n_m}\}$ converges a.e. and in $L^2(\Omega)$ to a function u, for almost all $t \in [0,t_0]$. Finally, observe that $\{||u_{n_m}(\cdot,t) - u(\cdot,t)||^2_{L^2(\Omega)}\}_{m\in\mathbb{N}}$ is a sequence of measurable functions in t which converges a.e. to zero. Since (2.8) holds, from the dominated convergence theorem we have that

$$\int_0^{t_0} ||u_{n_m}(\cdot,t) - u(\cdot,t)||^2_{L^2(\Omega)} ds \to 0, \tag{2.33}$$

when $m \to \infty$. A similar argument shows that there is a subsequence $(\rho_{n_m}, u_{n_m}, \theta_{n_m}, w_{n_m})$ of $(\rho_n, u_n, \theta_n, w_n)$ such that $\rho_{n_m} \to \rho$, $u_{n_m x} \to u_x$, $\theta_{n_m} \to \theta$, $\theta_{n_m x} \to \theta_x$ and $w_{n_m} \to w$ in $L^2(\Omega \times [0,t_0])$ and a.e.

We affirm that (ρ, u, θ, w) is a solution to the bounded domain problem. The approximations were made in such a way that the values at $t = 0$ and the boundary values are achieved. Also, for all $n \in \mathbb{N}$, we have that ρ_n satisfies the equation

$$\rho_{nt} + \rho_n^2 u_{nx} = 0. \tag{2.34}$$

So, by the continuity of the inner product in L^2, if $\varphi \in C_0^\infty(\Omega \times (0,t_0))$ we have that

$$0 = \int_0^{t_0}\int_\Omega -\rho_{n_m}\varphi_t + \rho_{n_m}^2 u_{n_m x}\varphi dxds \longrightarrow \int_0^{t_0}\int_\Omega -\rho\varphi_t + \rho^2 u_x \varphi dxds. \tag{2.35}$$

This shows that equation (1.10) is satisfied.

Concerning equation (1.11), we have that

$$\int_0^{t_0}\int_\Omega u_{n_m}\varphi_t + \left(R\rho_{n_m}\theta_{n_m} - \mu\rho_{n_m}u_{n_m x} - \alpha g'(1/\rho_{n_m})h(|w_{n_m}|^2)\right)\varphi_x dxds$$
$$\longrightarrow \int_0^{t_0}\int_\Omega u\varphi_t + \left(R\rho\theta - \mu\rho u_x - \alpha g'(1/\rho)h(|w|^2)\right)\varphi_x dxds. \tag{2.36}$$

On the other hand, given $N \in \mathbb{N}$, for all $n_m \geq N$ we have that $P_N^s\varphi = P_n^s\varphi$ and therefore, from

(2.2), we have

$$\begin{aligned}
&\left|\int_0^{t_0}\int_\Omega u_{n_m}\varphi_t + \left(R\rho_{n_m}\theta_{n_m} - \mu\rho_{n_m}u_{n_mx} - \alpha g'(1/\rho_{n_m})h(|w_{n_m}|^2)\right)\varphi_x dxds\right| \\
&= \left|\int_0^{t_0}\int_\Omega \left[u_{n_mt} + \left(R\rho_{n_m}\theta_{n_m} - \mu\rho_{n_m}u_{n_mx} - \alpha g'(1/\rho_{n_m})h(|w_{n_m}|^2)\right)_x\right](\varphi - P_N^s\varphi)dxds\right| \\
&\leq \int_0^{t_0}\Bigg(||u_{n_mt}||_{L^2(\Omega)} + R\max_{x\in\Omega}|\theta_{n_m}|\,||\rho_{n_mx}||_{L^2(\Omega)} + 2MR||\theta_{n_mx}||_{L^2(\Omega)} + \mu\max_{x\in\Omega}|u_{n_mx}|\,||\rho_{n_mx}||_{L^2(\Omega)} \\
&\quad + 2M\mu||u_{n_mxx}||_{L^2(\Omega)} - \frac{4\tilde{A}}{m^2}||\rho_{n_mx}||_{L^2(\Omega)} + 2\tilde{\tilde{A}}\max_{x\in\Omega}|w_{n_m}|\,||w_{n_mx}||_{L^2(\Omega)}\Bigg)||\varphi - P_N^s\varphi||_{L^2(\Omega)}ds.
\end{aligned} \tag{2.37}$$

The estimates found before, being uniform in n, allow us to take the limit when $N \to \infty$ and, by uniqueness of the limit, from (2.36) and (2.37), we conclude that

$$\int_0^{t_0}\int_\Omega u\varphi_t + \left(R\rho\theta - \mu\rho u_x - \alpha g'(1/\rho)h(|w|^2)\right)\varphi_x dxds = 0. \tag{2.38}$$

Since this holds for all $\varphi \in C_0^\infty(\Omega\times(0,t_0))$, equation (1.11) is satisfied. A similar argument can be made in order to show that equations (1.12) and (1.13) are satisfied as well. Finally, the estimates continue to hold for the limit functions and they prove the smoothness properties asserted in the theorem. Thus, the existence part of theorem 2 is proved.

The only thing left is the uniqueness part and it goes as follows. Suppose that $(\rho_1, u_1, \theta_1, w_1)$ and $(\rho_2, u_2, \theta_2, w_2)$ are solutions to the bounded domain problem. Then a straightforward calculation shows that the difference $(\rho, u, \theta, w) := (\rho_1 - \rho_2, u_1 - u_2, \theta_1 - \theta_2, w_1 - w_2)$ satisfies the following system

$$\rho_t+\rho_1^2 u_x+\rho(\rho_1+\rho_2)u_{2x}=0, \tag{2.39}$$
$$u_t=\mu(\rho_1 u_x+\rho u_{2x})_x-R(\rho_1\theta+\rho\theta_2)_x+\alpha[g'(1/\rho_1)C(|w_1|^2)(\overline{w_1}w+w_2\overline{w})$$
$$-h(|w_2|^2)D(1/\rho_1)\frac{1}{\rho_1\rho_2}\rho]_x, \tag{2.40}$$
$$C_\vartheta\theta_t=\kappa(\rho_1\theta_x+\rho\theta_{2x})_x+\mu\rho_1(u_{1x}+u_{2x})u_x+\mu\rho u_{2x}^2-R\rho_1\theta_1 u_x$$
$$-R(\rho_1\theta+\rho\theta_2)u_{2x}, \tag{2.41}$$
$$iw_t+w_{xx}=\overline{w_1}(w_1+w_2)w+w_2^2\overline{w}+g(1/\rho)[(A(|w_1|^2)\overline{w_1}+h'(|w_1|^2))w$$
$$+A(|w_1|^2)w_2^2\overline{w}]-h'(|w_2|)w_2B(1/\rho_1)\frac{1}{\rho_1\rho_2}\rho. \tag{2.42}$$
$$u=\theta_x=w=0 \quad \text{at x}=a,b, \tag{2.43}$$
$$u=\rho=\theta=w=0 \quad \text{at } t=0, \tag{2.44}$$

where, according to the Taylor theorem, we can write $h'(x)=h'(|w_2|^2)+A(x)(x-|w_2|^2)$, $g(x)=g(1/\rho_2)+B(x)(x-1/\rho_2)$, $h(x)=h(|w_2|^2)+C(x)(x-|w_2|^2)$, $g'(x)=g'(1/\rho_2)+D(x)(x-1/\rho_2)$ for certain functions $A(x)$, $B(x)$, $C(x)$ and $D(x)$.

Multiplying (2.39) by ρ, (2.40) by u and (2.41) by θ, integrating by parts and using Young's inequality with ε we get

$$\begin{aligned}
||\rho||^2_{L^2(\Omega)} &\le C\int_0^t\left(||u_x||^2_{L^2(\Omega)}+||\rho||^2_{L^2(\Omega)}\right)ds\\
||u||^2_{L^2(\Omega)}+\int_0^t||u_x||^2_{L^2(\Omega)}ds &\le C\int_0^t\left(||\rho||^2_{L^2(\Omega)}+||\theta||^2_{L^2(\Omega)}+||w||^2_{L^2(\Omega)}\right)ds\\
||\theta||^2_{L^2(\Omega)}+\int_0^t||\theta_x||^2_{L^2(\Omega)} &\le C\int_0^t\left(||\rho||^2_{L^2(\Omega)}+||u_x||^2_{L^2(\Omega)}+||\theta||^2_{L^2(\Omega)}\right)ds.
\end{aligned}$$

Finally, multiplying (2.42) by $\overline{w}$, taking imaginary part and integrating by parts we get

$$||w||^2_{L^2(\Omega)} \le C\int_0^t\left(||\rho||^2_{L^2(\Omega)}+||w||^2_{L^2(\Omega)}\right)ds.$$

From these last four inequalities we get

$$||\rho||^2_{L^2(\Omega)}+||u||^2_{L^2(\Omega)}+||\theta||^2_{L^2(\Omega)}+||w||^2_{L^2(\Omega)}\le C\int_0^t\left(||\rho||^2_{L^2(\Omega)}+||\theta||^2_{L^2(\Omega)}+||w||^2_{L^2(\Omega)}\right)ds. \tag{2.45}$$

Therefore, using Gronwall's inequality, we conclude that $||\rho||^2_{L^2(\Omega)}=||u||^2_{L^2(\Omega)}=||\theta||^2_{L^2(\Omega)}=||w||^2_{L^2(\Omega)}=0$ for $t\in[0,t_0]$. Thus proving the uniqueness of the solution, which completes the proof of theorem 2.

Chapter 3

Extension to the whole $\mathbb{R}$

In this chapter we are going to prove theorem 1 using the results from the previous chapter.

3.1 Local solutions

Let us assume the hypotheses of theorem 1. For $k \in \mathbb{N}$ define $\Omega_k := (-k,k)$ and let $\eta_k \in C_0^\infty(\mathbb{R})$ be such that

(i) $\eta_k(x) = 1$ for all $x \in (-k+1, k-1)$,

(ii) $0 \leq \eta_k \leq 1$,

(iii) $supp(\eta_k) \subseteq [-k,k]$,

(iv) All the derivatives of η_k are uniformly bounded over k.

Define the functions $\rho_{0,k}, u_{0,k}, \theta_{0,k} : \mathbb{R} \to \mathbb{R}$ and $w_{0,k} : \mathbb{R} \to \mathbb{C}$ by

$$\rho_{0,k} = \eta_k \rho_0 + (1-\eta_k)\rho_* \tag{3.1}$$

$$u_{0,k} = \eta_k u_0 \tag{3.2}$$

$$\theta_{0,k} = \eta_k \theta_0 + (1-\eta_k)\theta_* \tag{3.3}$$

$$w_{0,k} = \eta_k w_0 \tag{3.4}$$

Clearly $\rho_{0,k} - \rho_* \to \rho_0 - \rho_*$, $u_{0,k} \to u_0$, $\theta_{0,k} - \theta_* \to \theta_0 - \theta_*$ in $H^1(\mathbb{R})$ and $w_{0,k} \to w_0$ in $H^1(\mathbb{R},\mathbb{C})$. Moreover, $m \leq \rho_{0,k}, \theta_{0,k} \leq M$ and $u_{0,k}(-k) = u_{0,k}(k) = \theta'_{0,k}(-k) = \theta'_{0,k}(k) = w_{0,k}(-k) = w_{0,k}(k) = 0$. Thus we can apply theorem 2 with inicial data $(\rho_{0,k}, u_{0,k}, \theta_{0,k}, w_{0,k})|_{\Omega_k}$ to find a solution $(\rho_k, u_k, \theta_k, w_k)$ to equations (1.10)-(1.13), defined on a time interval $[0,t_k]$.

In a similar way as in the previous chapter when dealing with the approximate problem, for any fixed $T > 0$ we are going to prove uniform over k estimates which will serve two purposes: extend all solutions $(\rho_k, u_k, \theta_k, w_k)$ to the same time interval $[0,T]$ and guarantee the existence of a subsequence that converges to a solution to the problem.

In order to take such convergent subsequence, we also have to guarantee that all solutions are defined in a uniform (over k) time interval $[0,t_0]$. The estimates proven below are not only uniform in k, but also in t_k so that we can assume that such t_0 actually exists.

3.2 Estimates on the density

We first need to guarantee that the density ρ does not approach zero, for otherwise the Lagrangian transformation becomes singular (see appendix A). Also, high positiveness and boundedness of ρ are essential in the subsequent calculations.

With theorem 2 at hand we can assume that $\rho_k > 0$ and $\theta_k > 0$. Let us drop the subscripts k for simplicity and begin with the estimates.

Multiply (1.2) by $R\frac{1}{\rho}(1-\frac{\rho_*}{\rho})$ and integrate over Ω to obtain

$$R\int_\Omega \frac{1}{\rho}\rho_t - \frac{\rho_*}{\rho^2}\rho_t + \rho u_x - \rho_* u_x dx = 0.$$

This implies

$$R\frac{d}{dt}\int_\Omega \frac{1}{\rho}\left(\rho ln\frac{\rho}{\rho_*} + \rho_* - \rho\right)dx + \int_\Omega R\rho u_x dx = 0. \tag{3.5}$$

Now, multiply (1.3) by $\frac{1}{\theta_*}u$ and integrate over Ω to obtain

$$\frac{1}{\theta_*}\frac{d}{dt}\int_\Omega \frac{u^2}{2}dx - \int_\Omega\left(\frac{R}{\theta_*}\rho\theta u_x - \frac{\mu}{\theta_*}\rho u_x^2\right)dx + \frac{1}{\theta_*}\int_\Omega \alpha g'(1/\rho)h(|w|^2)u_x dx = 0.$$

The proof of corollary 2 can be easily adapted to show

$$\int_\Omega \alpha g'(1/\rho)h(|w|^2)u_x dx = \frac{d}{dt}\left(||g(1/\rho)h(|w|^2)||_{L^1(\Omega)} + ||w_x||_{L^2(\Omega)} + ||\,|w|^4||_{L^1(\Omega)}\right).$$

So,

$$\frac{1}{\theta_*}\frac{d}{dt}\int_\Omega \frac{u^2}{2}dx - \int_\Omega\left(\frac{R}{\theta_*}\rho\theta u_x - \frac{\mu}{\theta_*}\rho u_x^2\right)dx + \frac{1}{\theta_*}\frac{d}{dt}\left(||g(1/\rho)h(|w|^2)||_{L^1(\Omega)} + ||w_x||_{L^2(\Omega)} + ||\,|w|^4||_{L^1(\Omega)}\right) = 0. \tag{3.6}$$

Finally, multiplying (1.4) by $C_\vartheta\left(\frac{1}{\theta_*}-\frac{1}{\theta}\right)$ and integrating over Ω we get

$$C_\vartheta \frac{d}{dt}\int_\Omega \left(\frac{\theta}{\theta_*}-1-ln\frac{\theta}{\theta_*}\right)dx \;+\; \int_\Omega \left(\frac{R}{\theta_*}\rho\theta u_x-\frac{\mu}{\theta_*}\rho u_x^2\right)dx$$
$$-\;\int_\Omega R\rho u_x dx+\int_\Omega \left(\kappa\frac{\rho}{\theta^2}\theta_x^2+\mu\frac{\rho}{\theta}u_x^2\right)dx=0. \quad (3.7)$$

Adding (3.5), (3.6) and (3.7) yields

$$\frac{d}{dt}\int_\Omega \left[\frac{1}{2\theta_*}u^2+\frac{1}{\rho}\left(\rho ln\frac{\rho}{\rho_*}+\rho_*-\rho\right)+\left(\frac{\theta}{\theta_*}-1-ln\frac{\theta}{\theta_*}\right)\right]dx$$
$$+\frac{d}{dt}\int_\Omega \left[g(1/\rho)h(|w|^2)+|w_x|^2+|w|^4\right]dx+\int_\Omega \left(\kappa\frac{\rho}{\theta^2}\theta_x^2+\mu\frac{\rho}{\theta}u_x^2\right)dx=0. \quad (3.8)$$

Let E_k be given by

$$E_k=\frac{1}{2\theta_*}||u_{0,k}||^2_{L^2(\Omega)}+||g(1/\rho_{0,k})h(|w_{0,k}|^2)||_{L^1(\Omega)}+||w_{0,kx}||^2_{L^2(\Omega)}+||\;|w_{0,k}|^4||_{L^1(\Omega)}$$
$$+\int_\Omega \left[\frac{1}{\rho_{0,k}}\left(\rho_{0,k}ln\frac{\rho_{0,k}}{\rho_*}+\rho_*-\rho_{0,k}\right)+\left(\frac{\theta_{0,k}}{\theta_*}-1-ln\frac{\theta_{0,k}}{\theta_*}\right)\right]dx. \quad (3.9)$$

We assert that E_k is finite. In fact, there is a constant $E_0>0$ independent of k such that $E_k\le E_0$. Because of our assumptions on the initial data, we only have to show that the last integral is finite. Since $\theta_{0,k}$ is given by (3.3) we have that $\theta_{0,k}|_{(-k+1,k-1)}=\theta_0|_{(-k+1,k-1)}$. Moreover, $|\theta_*-\theta_{0,k}(x)|\le|\theta_*-\theta_0(x)|$ for all $x\in\mathbb{R}$. Now, Taylor series expansion of the function $y\to ln y$ about $y_0=1$ gives $ln y=(y-1)-(y-1)^2+o(|y-1|^3)$. Thus, since $\lim_{|x|\to\infty}\theta_0(x)/\theta_*=1$, we have that

$$\int_\Omega \left(\frac{\theta_{0,k}}{\theta_*}-1-ln\frac{\theta_{0,k}}{\theta_*}\right)dx\le C(1+||\theta_0-\theta_*||^2_{L^2(\Omega)}),$$

where $C>0$ is independent of k. Here we used (1.8). A similar argument applied to the function $v=1/\rho$ can be made in order to show that

$$\int_\Omega \frac{1}{\rho_{0,k}}\left(\rho_{0,k}ln\frac{\rho_{0,k}}{\rho_*}+\rho_*-\rho_{0,k}\right)dx=\int_\Omega \left(\frac{v_{0,k}}{v_*}-1-ln\frac{v_{0,k}}{v_*}\right)dx\le C(1+||v_0-v_*||^2_{L^2(\Omega)}),$$

where $v_*=1/\rho_*$ and $v_{0,k}=1/\rho_{0,k}$. Note that the fact that $\rho_0-\rho_*\in L^2(\Omega)$ and (1.8) imply that $v_0-v_*\in L^2(\Omega)$. This proves our assertion.

Observe that the function $y \to y-1-ln\,y$ is non-negative (it attains its minimum at y=1). Coming back to (3.8) we have that

$$\frac{1}{2\theta_*}||u||^2_{L^2(\Omega)} + ||g(1/\rho)h(|w|^2)||_{L^1(\Omega)} + ||w_x||^2_{L^2(\Omega)} + ||\,|w|^4||_{L^1(\Omega)} + \int_\Omega \left[\frac{1}{\rho}\left(\rho ln\frac{\rho}{\rho_*} + \rho_* - \rho\right) + \left(\frac{\theta}{\theta_*} - 1 - ln\frac{\theta}{\theta_*}\right)\right] dx \leq E_0 \tag{3.10}$$

and

$$\int_0^t \int_\Omega \left(\kappa\frac{\rho}{\theta^2}\theta_x^2 + \mu\frac{\rho}{\theta}u_x^2\right) dx \leq E_0. \tag{3.11}$$

By Morrey's inequality we have that ρ and θ are continuous. So, for all $N = -k, -k+1, ..., k-1$ there is a point $a = a_N(t) \in [N, N+1]$ such that

$$\left[\frac{1}{\rho}\left(\rho ln\frac{\rho}{\rho_*} + \rho_* - \rho\right) + \left(\frac{\theta}{\theta_*} - 1 - ln\frac{\theta}{\theta_*}\right)\right]\Bigg|_{x=a} \leq E_0.$$

In particular,

$$\left[\frac{1}{\rho}\left(\rho ln\frac{\rho}{\rho_*} + \rho_* - \rho\right)\right]\Bigg|_{x=a} \leq E_0, \quad \left[\frac{\theta}{\theta_*} - 1 - ln\frac{\theta}{\theta_*}\right]\Bigg|_{x=a} \leq E_0. \tag{3.12}$$

Since the function $y \to y - ln\,y - 1$ is decreasing in $(0,1)$, increasing in $(1,\infty)$ and $\lim_{y\to 0^+}(y - ln\,y - 1) = \lim_{y\to\infty}(y - ln\,y - 1) = \infty$, if ν_1, ν_2 are roots of the equation

$$y - ln\,y - 1 = E_0,$$

such that $\nu_1 \leq 1$ and $\nu_2 \geq 1$, then $\nu_1 \leq 1/\rho(a,t) \leq \nu_2$ and $\nu_1 \leq \theta(a,t) \leq \nu_2$.

Let us cite a lemma found in [1] chapter II, section 6:

Lemma 3. *Let $\gamma(x,t)$ be a non-negative function in $(N, N+1) \times (0,T)$ such that*

$$\int_N^{N+1} [\gamma(x,t) - 1 - ln\gamma(x,t)]dx \leq E, \quad \forall t \in [0,T].$$

Then there exist constants $n(E)$ and $M(E)$ such that

$$0 < n(E) \leq \int_N^{N+1} \gamma(x,t)dx \leq M(E).$$

In fact,

$$n(E) = \frac{1}{2}e^{-2E-1}, \quad M(E) = 2\left(1 + \frac{E}{1 - ln2}\right).$$

With the results above we can derive some information about ρ in the following way. Note that we can write equation (1.2) as

$$\rho u_x = (ln\rho)_t.$$

Putting this together with equation (1.3) we get

$$u_t + R(\rho\theta)_x + \mu(ln\rho)_{tx} = \alpha(g'(1/\rho)h(|w|^2))_x.$$

Integrate this equality with respect to t and get

$$u - u_0 + \int_0^t R(\rho\theta)_x ds + \mu(ln\rho)_x - \mu(ln\rho_0)_x = \alpha\int_0^t \left(g'(1/\rho)h(|w|^2)\right)_x ds.$$

Now integrate with respect to the space variable from a (satisfying (3.12)) to an arbitrary $x \in [N, N+1]$ to obtain

$$\begin{aligned}\frac{1}{\mu}\int_a^x (u(\xi,t) - u_0(\xi))d\xi + ln\rho(x,t) - ln(a,t) + \frac{R}{\mu}\int_0^t (\rho(x,s)\theta(x,s) - \rho(a,s)\theta(a,s))ds \\ = \frac{\alpha}{\mu}\int_0^t (g'(1/\rho(x,s))h(|w(x,s|^2) - g'(1/\rho(a,s))h(|w(a,s)|^2))ds \\ + ln\rho_{0,k}(x) - ln\rho_{0,k}(a). \qquad (3.13)\end{aligned}$$

Potentiate this last equality and after rearranging the terms involved we get

$$\rho(x,t)exp\left\{\int_0^t \rho(x,s)\theta(x,s)ds\right\} = \rho_{0,k}(x)Y(t)B(x,t), \qquad (3.14)$$

where $Y = Y_N(t)$ and $B = B_N(x,t)$ are given by

$$Y(t) = \frac{1}{\rho_{0,k}(a)}exp\left\{\frac{R}{\mu}\int_0^t \rho(a,s)\theta(a,s)ds - \frac{\alpha}{\mu}\int_0^t (g'(1/\rho(a,s))h(|w(a,s)|^2))ds\right\} \quad (3.15)$$

and

$$B(x,t) = \rho(a,t)exp\left\{\frac{1}{\mu}\int_a^x (u_0(\xi) - u(\xi,t))d\xi + \frac{\alpha}{\mu}\int_0^t (g'(1/\rho(x,s))h(|w(x,s)|^2))ds\right\}. \qquad (3.16)$$

Observe that if we multiply equality (3.14) by $\frac{R}{\mu}\theta(x,t)$ we have

$$\frac{d}{dt}exp\left\{\int_0^t \rho(x,s)\theta(x,s)ds\right\} = \frac{R}{\mu}\rho_{0,k}(x)Y(t)B(x,t)\theta(x,t).$$

Thus,

$$exp\left\{\int_0^t \rho(x,s)\theta(x,s)ds\right\} = 1+\frac{R}{\mu}\rho_{0,k}(x)\int_0^t Y(s)B(x,s)\theta(x,s)ds.$$

Substitute in (3.14) and we arrive to the following expression for ρ

$$\rho(x,t) = \frac{\rho_{0,k}(x)Y(t)B(x,t)}{1+\frac{R}{\mu}\rho_{0,k}(x)\int_0^t Y(s)B(x,s)\theta(x,s)ds} \tag{3.17}$$

and this holds for all $x \in [N, N+1]$.

Lemma 4. *There are positive constants C_1 and C_2 independent of N such that*

$$C_1 \le B(x,t) \le C_2, \quad C_1 \le Y(t) \le C_2, \tag{3.18}$$

for all $x \in \Omega$ and $t \in [0,T]$.

The proof of this lemma is very similar to the one of an analogue lemma in [1].

Proof. From Cauchy-Schwarz inequality and (3.10) we have that

$$\int_N^{N+1} u(\xi,t)d\xi \le ||u||_{L^2(\Omega)} \le E_0.$$

Thus,

$$\nu_1 A_1^{-1} \le B(x,t) \le \nu_2 A_1,$$

where $A_1 = exp\left\{\frac{2}{\mu}E_0 + T\frac{\alpha}{\mu}\max|g'|\max|h|\right\}$. Now, let us write (3.17) in the form

$$Y(t)\frac{1}{\rho(x,t)} = \frac{1}{B(x,t)}\left[\frac{1}{\rho_{0,k}(x)} + \frac{R}{\mu}\int_0^t Y(s)B(x,s)\theta(x,s)ds\right].$$

Integrate with respect to x from N to $N+1$ and using upper and lower bounds for B as well as lemma 3 applied to the functions $\frac{1}{\rho}$ and θ we have the following inequalities

$$Y(t) \le A_2\left(1+\int_0^t Y(s)ds\right) \tag{3.19}$$

and

$$Y(t) \geq A_3 \left(1 + \int_0^t Y(s) \int_\Omega \theta(x,t) dx ds\right), \tag{3.20}$$

for some positive constants A_2 and A_3. By using Gronwall's inequality on (3.19) we conclude that $Y(t) \leq A_2 exp(A_2 T)$. Finally, by the positiveness of θ, from (3.20) we see that $Y(t) \geq A_3$. □

For $t \in [0,T]$ define

$$m_\rho(t) = inf_{x\in\Omega}\rho(x,t), \qquad m_\theta(t) = inf_{x\in\Omega}\theta(x,t),$$
$$M_\rho(t) = sup_{x\in\Omega}\rho(x,t), \qquad M_\theta(t) = sup_{x\in\Omega}\theta(x,t).$$

In this notation from (3.17) we have that

$$M_\rho(t) \leq \frac{C_1}{1 + \int_0^t m_\theta(s) ds}, \tag{3.21}$$

$$m_\rho(t) \geq \frac{C_2}{1 + \int_0^t M_\theta(s) ds}. \tag{3.22}$$

From (3.21) and by positiveness of θ we have that

$$M_\rho(t) \leq C_1. \tag{3.23}$$

for all $t \in [0,T]$. In order to estimate m_ρ from below it suffices to show finiteness of $\int_0^T M_\theta(s) ds$.

Lemma 5.

$$\int_0^T M_\theta(s) ds \leq C(T), \tag{3.24}$$

where $C : [0,\infty) \to [0,\infty)$ is an increasing continuous function.

Yet again, we follow the proof of an analogue lemma in [1]

Proof. Let $a = a_1(t) \in [0,1]$ as in (3.12). Let $b = b(t) = \theta(a,t)/\theta_*$. Then $\nu_1/\theta_* \leq b \leq \nu_2/\theta_*$. Define

$$\psi(\theta) = \int_b^{\theta/\theta_*} \frac{(z - lnz - 1)^{1/2}}{z} dz.$$

Note that $\psi(\theta(a,t)) = 0$ and so

$$|\psi(\theta(\cdot,t))| = \left|\int_a^\cdot \frac{\partial}{\partial x}\psi(\theta(x,t))dx\right| \leq \int_\Omega \left|\frac{\partial}{\partial x}\psi(\theta(x,t))\right| dx.$$

Now,

$$\begin{aligned}\frac{\partial}{\partial x}\psi(\theta) &= \frac{(\frac{\theta}{\theta_*}-ln\frac{\theta}{\theta_*}-1)^{1/2}}{\frac{\theta}{\theta_*}}\frac{\theta_x}{\theta_*} \\ &= \frac{(\frac{\theta}{\theta_*}-ln\frac{\theta}{\theta_*}-1)^{1/2}}{\rho^{1/2}}\frac{\rho^{1/2}}{\theta}\theta_x.\end{aligned}$$

From the Cauchy-Schwarz inequality we have

$$|\psi(\theta)| \le \frac{1}{m_\rho^{1/2}}\left(\int_\Omega(\frac{\theta}{\theta_*}-ln\frac{\theta}{\theta_*}-1)dx\right)^{1/2}\left(\int_\Omega\frac{\rho}{\theta^2}\theta_x^2dx\right)^{1/2}. \tag{3.25}$$

Observe that, since $\lim_{z\to\infty}\left(\frac{lnz}{z}+\frac{1}{z}\right)=0$, there exists $N>\frac{\nu_2}{\theta_*}$ such that for $z\ge N$, $\frac{lnz}{z}+\frac{1}{z}<\frac{1}{2}$. In this way <u>if</u> $\frac{\theta}{\theta_*}>N$ then

$$\begin{aligned}\psi(\theta) &= \int_b^{\theta/\theta_*}\left(\frac{1}{z}-\frac{lnz+1}{z^2}\right)^{1/2}dz \\ &\ge \int_N^{\theta/\theta_*}\left(\frac{1}{2z}\right)^{1/2}dz=\frac{2}{\sqrt{2}}\left(\left(\frac{\theta}{\theta_*}\right)^{1/2}-N^{1/2}\right).\end{aligned}$$

Thus, if $\frac{\theta}{\theta_*}>N$

$$\theta^{1/2}\le\theta_*\left(\frac{\sqrt{2}}{2}\psi(\theta)+N^{1/2}\right). \tag{3.26}$$

Define

$$A(t)=\int_\Omega\frac{\rho}{\theta^2}\theta_x^2dx.$$

From (3.25), (3.10) and (3.22) we have that

$$|\psi(\theta)|\le\left(\frac{1+\int_0^t M_\theta(s)ds}{C_2}\right)^{1/2}E_0^{1/2}A(t)^{1/2}.$$

Consequently,

$$M_\theta(t)\le C_1A(t)\left(1+\int_0^t M_\theta(s)ds\right)+C_2,$$

for two positive constants C_1 and C_2. Using Gronwall's inequality and estimate (3.11) we see that there is a constant $C>0$ such that

$$\int_0^t M_\theta(s)ds\le C(1+T).$$

□

As a direct corollary from this lemma we have the upper bound for density

$$m_\rho(t) \geq \frac{C_2}{1+C(1+T)}, \tag{3.27}$$

thus proving the main result of this section:

Proposition 3. *There are positive constants m_1 and M_1 such that*

$$m_1 \leq \rho(x,t) \leq M_1 \tag{3.28}$$

for all $x \in \Omega$ and all $t \in [0,T]$.

3.3 Estimates on the temperature

Note that in the calculations of the previous section we only used the finiteness of T and the positiveness of θ. In this section we are going to show that θ also does not approach zero and therefore the estimates above are true for arbitrary $T > 0$.

Lemma 6. *There is a constant $C > 0$ such that*

$$m_\theta(t) \geq \frac{C}{1+t}, \tag{3.29}$$

for all $t \in [0,T]$

Proof. Consider equation (1.4) for temperature. By adding and subtracting $\frac{R^2}{4C_\vartheta\mu}\theta^2$ and rearranging the resulting terms we get

$$\theta_t = \frac{\kappa}{C_\vartheta}(\rho\theta_x)_x + \rho\left(\left(\frac{\mu}{C_\vartheta}\right)^{1/2} u_x - \frac{R}{2C_\vartheta^{1/2}\mu^{1/2}}\theta\right)^2 - \frac{R^2}{4C_\vartheta\mu}\theta^2\rho.$$

Dividing this equation by θ^2 we get

$$\begin{aligned}
\frac{\theta_t}{\theta^2} &= \frac{\kappa}{C_\vartheta}\frac{(\rho\theta_x)_x}{\theta^2} + \frac{\rho}{\theta^2}\left(\left(\frac{\mu}{C_\vartheta}\right)^{1/2} u_x - \frac{R}{2C_\vartheta^{1/2}\mu^{1/2}}\theta\right)^2 - \frac{R^2}{4C_\vartheta\mu}\rho \\
&= -\frac{\kappa}{C_\vartheta}\left(\rho\left(\frac{1}{\theta}\right)_x\right)_x + \frac{2\kappa}{C_\vartheta}\rho\theta\left(\frac{1}{\theta}\right)_x^2 + \rho\frac{1}{\theta^2}\left(\left(\frac{\mu}{C_\vartheta}\right)^{1/2} u_x - \frac{R}{2C_\vartheta^{1/2}\mu^{1/2}}\theta\right)^2 \\
&\quad -\frac{R^2}{4C_\vartheta\mu}\rho,
\end{aligned}$$

which is equivalent to

$$\sigma_t = \frac{\kappa}{C_\vartheta}(\rho\sigma_x)_x - \left[\frac{2\kappa}{C_\vartheta}\rho\theta\sigma_x^2 + \rho\sigma^2\left(\left(\frac{\mu}{C_\vartheta}\right)^{1/2} u_x - \frac{R}{2C_\vartheta^{1/2}\mu^{1/2}}\theta\right)^2\right] + \frac{R^2}{4C_\vartheta\mu}\rho, \quad (3.30)$$

for the function $\sigma = \frac{1}{\theta}$. Multiply (3.30) by σ^{2p-1}, where $p > 1$ is arbitrary, integrate over Ω and, since the expression in the square brackets is non-negative, we arrive to

$$\frac{1}{2p}\frac{d}{dt}||\sigma||_{L^{2p}(\Omega)}^{2p} \leq \frac{R^2}{4C_\vartheta\mu}\int_\Omega \rho\sigma^{2p-1}dx.$$

The left side of this inequality can be written as

$$\frac{1}{2p}\frac{d}{dt}||\sigma||_{L^{2p}(\Omega)}^{2p} = ||\sigma||_{L^{2p}(\Omega)}^{2p-1}\frac{d}{dt}||\sigma||_{L^{2p}(\Omega)},$$

while the right side of the inequality can be estimated using Hölder's inequality by

$$\frac{R^2}{4C_\vartheta\mu}\int_\Omega \rho\sigma^{2p-1}dx \leq \frac{R^2}{4C_\vartheta\mu}||\rho||_{L^{2p}(\Omega)}||\sigma||_{L^{2p}(\Omega)}^{2p-1}.$$

In this way we get

$$\frac{d}{dt}||\sigma||_{L^{2p}(\Omega)} \leq \frac{R^2}{4C_\vartheta\mu}||\rho||_{L^{2p}(\Omega)}.$$

And so,

$$||\sigma||_{L^{2p}(\Omega)} \leq ||\sigma(.,0)||_{L^{2p}(\Omega)} + \frac{R^2}{4C_\vartheta\mu}\int_0^t ||\rho||_{L^{2p}(\Omega)}ds.$$

Letting $p \to \infty$ (remembering that $0 < m \leq \theta_0$) we see that

$$\begin{aligned}\limsup_{p\to\infty}||\sigma||_{L^{2p}(\Omega)} &\leq \frac{1}{m} + \frac{R^2}{4C_\vartheta\mu}\int_0^t M_\rho(s)ds \\ &\leq \frac{1}{m} + \frac{R^2}{4C_\vartheta\mu}M_1 t.\end{aligned}$$

This implies that

$$\frac{1}{m_\theta(t)} = ||\sigma||_\infty \leq \frac{1}{m} + \frac{R^2}{4C_\vartheta\mu}M_1 t,$$

thus proving the lemma. □

Let us prove one final lemma on the temperature before passing on to the a priori estimates on the norms of the solution functions.

Lemma 7. *For all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$M_\theta^2(t) \leq \varepsilon \int_\Omega \rho \theta_x^2 dx + C_\varepsilon. \tag{3.31}$$

Proof. For $x \in [N, N+1]$ define

$$\psi = \psi_N(x,t) = \theta(x,t) - \int_N^{N+1} \theta(x,t)dx.$$

As

$$\int_N^{N+1} \psi(x,t)dx = 0,$$

there is a point $b = b_N(t) \in [N, N+1]$ such that $\theta(b,t) = 0$. So we can write

$$\begin{aligned}
|\psi(\cdot,t)|^{3/2} &= \int_b^{\cdot} \frac{\partial}{\partial x} |\psi(x,t)|^{3/2} dx \\
&= \frac{3}{2} \int_b^{\cdot} |\psi(x,t)|^{1/2} sign(\psi(x,t)) \psi_x(x,t)dx \\
&= \frac{3}{2} \int_b^{\cdot} \left(\frac{1}{\rho(x,t)}\right)^{1/2} |\psi(x,t)|^{1/2} sign(\psi(x,t)) \rho^{1/2}(x,t) \psi_x(x,t)dx.
\end{aligned}$$

Since, $\psi_x = \theta_x$, by Cauchy-Schwarz inequality we have

$$|\psi(\cdot,t)|^{3/2} \leq \frac{3}{2} \left(\frac{1}{m_1} \int_N^{N+1} |\psi(x,t)| dx\right)^{1/2} \left(\int_\Omega \rho \theta_x^2(x,t)dx\right)^{1/2}.$$

Note that

$$\int_N^{N+1} |\psi(x,t)| dx \leq 2 \int_N^{N+1} \theta(x,t)dx.$$

Also, as was shown before, this last quantity is bounded uniformly by the constant $M(E_0)$ given by lemma 3. So, for a constant $C_1 > 0$ large enough, we have

$$|\psi(x,t)|^{3/2} \leq C_1 \left(\int_\Omega \rho \theta_x^2(x,t)dx\right)^{1/2}.$$

Thus,

$$\begin{aligned}
\theta^2(x,t) &= \left(\psi(x,t) + \int_N^{N+1} \theta(x,t)dx\right)^2 \\
&\leq 2M(E_0)^2 + 2C_1 \left(\int_\Omega \rho \theta_x^2(x,t)dx\right)^{2/3}.
\end{aligned}$$

Finally, using Young's inequality with ε we have

$$\theta^2(x,t) \le C_\varepsilon + \varepsilon \int_\Omega \rho \theta_x^2(x,t)dx$$

and this holds for all $x \in \Omega$. □

3.4 First estimates on derivatives

Because of the coupling, the estimates on the derivatives become a little more complicated than those contained in [1]. This section will be devoted to prove the following inequality

$$\eta'(t) \le C(1+M_\theta(t))(1+\eta(t)), \quad t \in [0,T], \tag{3.32}$$

where

$$\eta(t) = ||z||^2_{L^2(\Omega)} + ||u||^4_{L^4(\Omega)} + ||u||^2_{L^2(\Omega)} + \alpha||g(1/\rho)h(|w|^2)||_{L^1(\Omega)} + ||\,|w|^4||_{L^1(\Omega)} \\ + ||w_x||^2_{L^2(\Omega)} + ||\beta||^2_{L^2(\Omega)} + \int_0^t\int_\Omega \rho\theta_x^2 dxds + \int_0^t\int_\Omega \rho u_x^2 dxds$$

and $z = \frac{1}{2}u^2 + C_\vartheta(\theta - \theta_*)$, $\beta = u + \mu\rho_x/\rho$ and $C > 0$ is a constant. Before proving this inequality (which will take some time and effort) let us show some consequences of it.

By lemma 5 we have that $M_\theta \in L^1([0,T])$ so, by Gronwall's inequality there is a constant $L > 0$, which depends only on the initial data, such that $\eta(t) \le L$ for all $t \in [0,T]$.

From (3.8) we already had the estimates

$$\max_{t\in[0,T]} ||u||^2_{L^2(\Omega)} + \max_{t\in[0,T]} ||w_x||^2_{L^2(\Omega)} \le E_0 \tag{3.33}$$

which are reaffirmed by (3.32). Now, from (3.32) we also have

$$\max_{t\in[0,T]} ||u||^4_{L^4(\Omega)} + \int_0^t ||\theta_x||^2_{L^2(\Omega)}ds + \int_0^t ||u_x||^2_{L^2(\Omega)}ds \le L + \frac{L}{m_1}. \tag{3.34}$$

Moreover, writing $\rho_x^2 = \frac{\rho^2}{\mu^2}(\beta - u)^2 \le \frac{2M_1^2}{\mu^2}(\beta^2 + u^2)$ we have

$$\max_{t\in[0,T]} ||\rho_x||^2_{L^2(\Omega)} \le \frac{4M_1^2}{\mu^2}L. \tag{3.35}$$

Similarly, $(\theta-\theta_*)^2 = \frac{1}{C_\vartheta}(z-\frac{1}{2}u^2)^2 \le \frac{2}{C_\vartheta}(z^2+u^4)$ and so,

$$\max_{t\in[0,t]} ||\theta-\theta_*||^2_{L^2(\Omega)} \le \frac{4}{C_\vartheta} L \tag{3.36}$$

Note that in these last estimates we made use of (3.28). Finally, multiplying (1.2) by $\rho-\rho_*$ and taking (3.34) into account we obtain the estimate

$$\max_{t\in[0,T]} ||\rho-\rho_*||^2_{L^2(\Omega)} \le C, \tag{3.37}$$

for a constant $C>0$ depending only on the initial data.

The proof of (3.32) will rely on a series of lemmas which we prove below. Estimate (3.28) is of particular importance in the proof. Let us begin.

First, multiplying (1.3) by u, (1.4) by C_ϑ and adding the resulting equations we see that the function $z=\frac{1}{2}u^2+C_\vartheta(\theta-\theta_*)$ satisfies the equation

$$z_t = \mu(\rho z_x)_x + (\kappa-\mu C_\vartheta)(\rho\theta_x)_x - R(\rho\theta u)_x - \alpha(g'(1/\rho)h(|w|^2))_x u. \tag{3.38}$$

Lemma 8. *Given $\varepsilon_1, \varepsilon_2 > 0$ arbitrary, there exist positive constants $C_{\varepsilon_1}, C_{\varepsilon_2}$ such that*

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}||z||^2_{L^2(\Omega)} + \left(\frac{\kappa C_\vartheta}{2}-\varepsilon_1\right)\int_\Omega \rho\theta_x^2 dx \le \frac{C_\vartheta}{\kappa}\int_\Omega \rho\theta^2u^2dx + C_{\varepsilon_2}||u||^2_{L^2(\Omega)} \\ + C_{\varepsilon_1}\left(||z||^2_{L^2(\Omega)} + ||u||^4_{L^4(\Omega)}\right) + C_0\int_\Omega \rho u^2u_x^2dx + \varepsilon_2\int_\Omega \rho u_x^2 dx.\end{aligned} \tag{3.39}$$

Here, $C_0>0$ is a constant.

Proof. Multiply equation (3.39) by z and integrate over Ω

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}||z||^2_{L^2(\Omega)} + \mu\int_\Omega \rho z_x^2 dx + (\kappa-\mu C_\vartheta)\int_\Omega \rho\theta_x z_x dx \\ = R\int_\Omega \rho\theta u z_x dx + \alpha\int_\Omega g'(1/\rho)h(|w|^2)(uz)_x dx.\end{aligned} \tag{3.40}$$

By Young's inequality, for any $\delta>0$ we have

$$R\int_\Omega \rho\theta u z_x dx \le \delta\int_\Omega \rho z_x^2 dx + \frac{R^2}{4\delta}\int_\Omega \rho\theta^2u^2dx.$$

Concerning the last term in (3.40) we have

$$\alpha \int_\Omega g'(1/\rho)h(|w|^2)(uz)_x dx = \alpha \int_\Omega g'(1/\rho)h(|w|^2)(3u^2u_x + C_\vartheta(\theta_x u + (\theta-\theta_*)u_x)dx.$$

Multiplying and dividing by ρ, using (3.28) and Young's inequality with ε we have

$$\begin{aligned}\alpha \int_\Omega g'(1/\rho)h(|w|^2)(uz)_x dx \leq \frac{A}{2m_1}\left(3+\frac{AC_\vartheta}{2\varepsilon_2}\right)||u||^2_{L^2(\Omega)} + \frac{3A}{2m_1}\int_\Omega \rho u^2 u_x^2 dx \\ + \frac{A^2}{4m_1\varepsilon_1}||\theta-\theta_*||^2_{L^2(\Omega)} + \varepsilon_1 \int_\Omega \rho\theta_x dx + \varepsilon_2 \int_\Omega \rho u_x^2 dx,\end{aligned}$$

where $A = \alpha \max |g'| \max |h|$. Note that

$$C_\vartheta^2 (z-u^2)^2 \leq 2(z^2+u^4)$$

and so,

$$||\theta-\theta_*||^2_{L^2(\Omega)} \leq C\left(||z||^2_{L^2(\Omega)} + ||u||^4_{L^4(\Omega)}\right).$$

Putting all of this together with (3.40) we get

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}||z||^2_{L^2(\Omega)} + \int_\Omega \rho\left[(\mu-\delta)z_x^2 + (\kappa-\mu C_\vartheta)\theta_x z_x\right]dx \leq \frac{R^2}{4\delta}\int_\Omega \rho\theta^2 u^2 dx + C_{\varepsilon_2}||u||^2_{L^2(\Omega)} \\ + C_{\varepsilon_1}\left(||z||^2_{L^2(\Omega)} + ||u||^4_{L^4(\Omega)}\right) + \varepsilon_1 \int_\Omega \rho\theta_x dx + \varepsilon_2 \int_\Omega \rho u_x^2 dx.\end{aligned} \tag{3.41}$$

Observe that

$$(\mu-\delta)z_x^2 + (\kappa-\mu C_\vartheta)\theta_x z_x = (4\mu-4\delta)u^2u_x^2 + 2(\mu C_\vartheta + \kappa - 2\delta C_\vartheta)u_x\theta_x + (\kappa - 2\delta C_\vartheta)C_\vartheta\theta_x^2.$$

By completing squares we see that for $\delta < \frac{\mu C_\vartheta + \kappa}{2C_\vartheta}$ we have

$$(\mu-\delta)z_x^2 + (\kappa-\mu C_\vartheta)\theta_x z_x \geq -\left(4\delta + \frac{1}{\delta}(\mu C_\vartheta+\kappa)^2\right)u^2u_x^2 + (\kappa-2\delta C_\vartheta)C_\vartheta\theta_x^2.$$

Choosing $\delta = \frac{\kappa}{4C_\vartheta}$ and replacing in inequality (3.41) we get the result. □

Corollary 3. *By (possibly) redefining the constant C_{ε_2} we have that*

$$\frac{d}{dt}\left(\frac{1}{2}||z||^2_{L^2(\Omega)}+\frac{C_0}{4\mu}||u||^4_{L^4(\Omega)}\right)+\left(\frac{\kappa C_\vartheta}{2}-\varepsilon_1\right)\int_\Omega \rho\theta_x^2dx-\varepsilon_2\int_\Omega \rho u_x^2dx$$
$$\leq\left(\frac{C_\vartheta}{\kappa}+\frac{3RC_0}{4\mu^2}\right)\int_\Omega \rho\theta^2u^2dx+C_{\varepsilon_2}||u||^2_{L^2(\Omega)}+C_{\varepsilon_1}\left(||z||^2_{L^2(\Omega)}+||u||^4_{L^4(\Omega)}\right). \tag{3.42}$$

Proof. Multiply (1.3) by u^3 and integrate. Applying Young's inequality with ε we get

$$\begin{aligned}\frac{d}{dt}\frac{1}{4}||u||^4_{L^4(\Omega)}+3\mu\int_\Omega \rho u^2u_x^2dx &= 3R\int_\Omega \rho\theta u^2u_xdx-3\alpha\int_\Omega g'(1/\rho)h(|w|^2)u^2u_xdx\\ &\leq \frac{3R}{2\varepsilon_3}\int_\Omega \rho\theta^2u^2dx+\frac{\varepsilon_3}{2}\int_\Omega \rho u^2u_x^2dx\\ &\quad+\frac{\varepsilon_3}{2}\int_\Omega \rho u^2u_x^2dx+\frac{3A}{2\varepsilon_3 m_1}||u||^2_{L^2(\Omega)},\end{aligned}$$

where $A=\alpha\max|g'|\max|h|$, as in the proof of the previous lemma, and $\varepsilon_3>0$ si arbitrary. Choosing $\varepsilon_3=2\mu$ we have

$$\frac{d}{dt}\frac{1}{4}||u||^4_{L^4(\Omega)}+\mu\int_\Omega \rho u^2u_x^2dx\leq\frac{3R}{8\mu}\int_\Omega \rho\theta^2u^2+\frac{3A}{4\mu m_1}||u||^2_{L^2(\Omega)}.$$

Multiplying this last inequality by $\frac{C_0}{\mu}$ and adding the result to (3.39) we get (3.42). □

Lemma 9. *Given $\varepsilon_4>0$ arbitrary we have*

$$\frac{d}{dt}\left(\frac{1}{2}||u||^2_{L^2(\Omega)}+\alpha||g(1/\rho)h(|w|^2)||_{L^1(\Omega)}+||\,|w|^4||_{L^1(\Omega)}+||w_x||^2_{L^2(\Omega)}\right)+\mu\int_\Omega \rho u_x^2dx$$
$$\leq\frac{R}{2}||\rho_x||^2_{L^2(\Omega)}+\frac{R}{2m_1}\int_\Omega \rho\theta^2u^2dx+\varepsilon_4\int_\Omega \rho\theta_x^2dx+\frac{R^2M_1}{4\varepsilon_4}||u||^2_{L^2(\Omega)}. \tag{3.43}$$

Proof. Multiply (1.3) by u and integrate

$$\frac{d}{dt}\frac{1}{2}||u||^2_{L^2(\Omega)}+\mu\int_\Omega \rho u_x^2dx+\alpha\int_\Omega g'(1/\rho)h(|w|^2)u_xdx=-R\int_\Omega(\rho_x\theta u+\rho\theta_xu)\,dx.$$

The proof of corollary 2 can be easily adapted to show that

$$\alpha\int_\Omega g'(1/\rho)h(|w|^2)u_xdx=\frac{d}{dt}\left(||\,|w|^4||_{L^1(\Omega)}+||w_x||^2_{L^2(\Omega)}\right).$$

Now,

$$R\int_\Omega (\rho_x\theta u+\rho\theta_x u)\,dx \le \frac{R}{2}||\rho_x||^2_{L^2(\Omega)}+\frac{R}{2m_1}\int_\Omega \rho\theta^2u^2dx+\varepsilon_4\int_\Omega \rho\theta_x^2dx+\frac{R^2}{4\varepsilon_4}M_1||u||^2_{L^2(\Omega)}.$$

Thus, proving the lemma. □

Lemma 10. *(i) The function w satisfies*

$$\frac{d}{dt}||w||^2_{L^2(\Omega)}=0 \tag{3.44}$$

and

$$||w(t)||_\infty \le 2^{1/2}||w||^{1/2}_{L^2(\Omega)}||w_x||^{1/2}_{L^2(\Omega)} \tag{3.45}$$

In particular (3.33), (3.44) and (3.45) imply

$$\max_{t\in[0,T]}||w(t)||_\infty \le C \tag{3.46}$$

for a constant $C>0$ which depends only on the initial data.

(ii) Let $\beta = u+\mu(ln\rho)_x = u+\mu\rho_x/\rho$. Then given $\varepsilon_6>0$ arbitrary, there is a constant $C_{\varepsilon_6}>0$ such that

$$\frac{d}{dt}\frac{1}{2}||\beta||^2_{L^2(\Omega)} \le C_{\varepsilon_6}(1+M_\theta(t))(1+||\beta||^2_{L^2(\Omega)})+\varepsilon_6\int_\Omega \rho\theta_x^2dx, \tag{3.47}$$

where $C>0$ is a constant.

Proof. Equality (3.44) is easily obtained by multiplying (1.5) by $\overline{w}$, taking imaginary part and integrating over Ω. The proof of (3.45) is identical as of lemma 2. Observe that theorem 2 guarantees that w is zero on the boundary of Ω. This proves (i).

Let us prove (ii). Note that equation (1.2) can be written as

$$(ln\rho)_t = \rho u_x.$$

Replacing this equality in (1.3) we find that β satisfies the equation

$$\begin{aligned}\beta_t &= -R(\rho\theta)_x+\alpha(g'(1/\rho)h(|w|^2))_x\\ &= -R\rho\left(\frac{\beta-u}{\mu}\theta+\theta_x\right)-\alpha g''(1/\rho)h(|w|^2)\frac{\beta-u}{\mu}\rho+\alpha g'(1/\rho)h'(|w|^2)(|w|^2)_x.\end{aligned}$$

Multiplying by β and applying Young's inequality with ε we see that there is a constant

$C_{\varepsilon_6} > 0$ such that

$$\begin{aligned}\frac{d}{dt}\beta^2 &\leq \frac{M_1}{\mu}\left(RM_\theta(t)+\tilde{A}\right)(\beta^2+|u\beta|)+\varepsilon_6\rho\theta^2+C_{\varepsilon_6}\beta^2+\tilde{\tilde{A}}(2|Re(ww_x)\beta|)\\ &\leq \frac{M_1}{\mu}\left(RM_\theta(t)+\tilde{A}\right)\left(\beta^2+\frac{u^2}{2}+\frac{\beta^2}{2}\right)+\varepsilon_6\rho\theta^2+C_{\varepsilon_6}\beta^2\\ &\qquad +2\tilde{\tilde{A}}||w||_\infty\left(\frac{1}{2}|w_x|^2+\frac{1}{2}\beta^2\right),\end{aligned}$$

where $\tilde{A} = \alpha\max|g''|\max|h|$ and $\tilde{\tilde{A}} = \alpha\max|g'|\max|h'|$. Integrating this inequality over Ω, using (3.33) and (3.46) and redefining C_{ε_6} we obtain (3.47). □

Let us make two observations. First note that $\rho_x^2 = \frac{\rho^2}{\mu^2}(\beta-u)^2 \leq \frac{2M_1^2}{\mu^2}(\beta^2+u^2)$. So,

$$||\rho_x||^2_{L^2(\Omega)} \leq C(1+||\beta||^2_{L^2(\Omega)}).$$

Second, observe that

$$\int_\Omega \rho\theta^2u^2dx \leq M_1M_\theta^2(t)||u||^2_{L^2(\Omega)} \leq CM_\theta^2(t).$$

From lemma 7 given $\varepsilon_5 > 0$ there exists $C_{\varepsilon_5} > 0$ such that

$$M_\theta^2(t) \leq C_{\varepsilon_5} + \varepsilon_5\int_\Omega \rho\theta_x^2. \tag{3.48}$$

With these observations at hand, the proof of (3.32) follows directly form corollary 3 and lemmas 9 and 10 by adding inequalities (3.42), (3.43) and (3.47) and choosing $\varepsilon_1,...,\varepsilon_6$ small enough.

3.5 Last estimates and global existence

The above estimates have been carefully proven so that they do not depend on the domain $\Omega = \Omega_k$ as well as on T. In this section we discuss the final estimates which, through a compactness argument, will allow us to prove local existence of solutions to the Cauchy problem. The a priori estimates on the bounded domain problem will continue to hold on the limit so that global (in time) existence is guaranteed.

With estimates (3.33) and (3.35) at hand, the proof of lemma 3.4 in [4] can be adapted to our needs, with no major difficulties, in order to prove the estimate

$$||u_x||^2_{L^2(\Omega)} + \int_0^t ||u_{xx}||^2_{L^2(\Omega)} ds \le c(t) \tag{3.49}$$

where $c \in ([0,\infty))$ is a positive function depending only on the initial data. After this, directly from equations (1.2) and (1.3) the following estimates also hold

$$\max_{t\in[0,T]} ||\rho_t||^2_{L^2(\Omega)} \le C, \quad \int_0^T ||u_t||^2 ds \le C, \tag{3.50}$$

where $C > 0$ depends only on the data. This follows by multiplying (1.2) and (1.3) by ρ_t and u_t respectively and bounding the resulting terms as we have been doing throughout this work. Observe that estimates (3.35), (3.37) and (3.50) imply that $\rho \in C([0,T], H^1(\Omega))$. Finally, the analogues for the temperature of the estimates (3.49) and (3.50) also hold, by a similar argument. That is,

$$||\theta_x||^2_{L^2(\Omega)} + \int_0^t ||\theta_{xx}||^2_{L^2(\Omega)} + ||\theta_t||^2 ds \le C(t). \tag{3.51}$$

Since all estimates found in this chapter are uniform in k, by applying an argument similar to the one explained in section 2.3, there exist $t_0 > 0$ and a subsequence $(\rho_{k_m}, u_{k_m}, \theta_{k_m}, w_{k_m})$ of $(\rho_k, u_k, \theta_k, w_k)$ such that $\rho_{k_m} \to \rho$, $u_k \to u_x$, $u_{k_m x} \to u_x$, $\theta_{k_m} \to \theta$, $\theta_{k_m x} \to \theta_x$ and $w_{k_m} \to w$ in $L^2_{loc}(\mathbb{R} \times [0,t_0])$ and a.e., where (ρ, u, θ, w) is a solution to the Cauchy problem (1.2)-(1.5), (1.6). Note that all estimates from this chapter continue to hold for the limit functions. Since they do not depend on t, we can extend our solution to any time interval $[0,T]$ in the following way: suppose that $T_0 < \infty$ is the maximal time of existence of the solution. Then we can apply the local existence result just proven, with initial data $(\rho(\cdot,T_0), u(\cdot,T_0), \theta(\cdot,T_0), w(\cdot,T_0))$, thus extending our solution to an interval $[0, T_0 + t_0]$. This contradicts the maximality of T_0 and the proof of theorem 1 is complete.

Appendix A

Eulerian coordinates vs Lagrangian coordinates

The one dimensional Navier Stokes equations from gas dynamics are most commonly stated in Eulerian coordinates as

$$\rho_t + (\rho u)_x = 0, \tag{A.1}$$

$$\rho(u_t + uu_x) + p_x = \mu u_{xx} + \rho F, \tag{A.2}$$

$$C_\vartheta \rho(\theta_t + u\theta_x) + \theta p_\theta u_x = \kappa\theta_{xx} + \mu u_x^2, \tag{A.3}$$

where u, ρ and θ are the fluid's velocity, density and temperature respectively, p is the pressure, F is an external force and μ, κ and C_ϑ are positive constants. In this work we considered the case where pressure p is given by $p = p(\rho, \theta) = R\rho\theta$. Here our variables (t,x) take values in a square $[0,T] \times \Omega$ where $T > 0$ and $\Omega = (a,b)$ for certain values $-\infty \leq a < b \leq \infty$.

If we consider the vector field $H(x,t,z) = (\rho(x,t), -\rho(x,t)u(x,t), 0)$, being z a newly introduced variable, then, as long as equation (A.1) is satisfied, we have that $curl H = 0$. As a consequence there exists a function $y = y(x,t,z)$ whose gradient is equal to H (this is a particular case of the well known fact that every zero curl vector field defined on a simply connected domain is conservative, i.e., is the gradient of some function). Let us forget about the variable z (for the function y does not depend on it) and consider the function $y(x,t)$ defined as above, satisfying $y_x = \rho$ and $y_t = -\rho u$. The Lagrangian transformation Y can be defined by $Y(x,t) = (y(x,t),t)$. Applying the chain rule we arrive at the following system:

$$\begin{aligned}
&\rho_t + \rho^2 u_y = 0,\\
&u_t + p_y = \mu(\rho u_y)_y + F \circ Y,\\
&\theta_t + \frac{1}{C_\vartheta}\theta p_\theta u_y = \frac{1}{C_\vartheta}\kappa(\rho\theta_y)_y + \frac{\mu}{C_\vartheta}\rho u_y^2,
\end{aligned}$$

which is precisely the system of consideration in chapter 1 where the variable x therein is the Lagrangian variable.

In the literature, Lagrangian variables are the most common approach when studying these equations and one difficulty when dealing with the three dimensional case is that the Lagrangian transformation is not as simple as in the one dimensional case we are concerned with.

After solving the problem in the Lagrangian variables it is necessary to recover the original Eulerian system. For this it is sufficient to show that the Lagrangian transformation is not singular. Now, the Jacobian matrix J of the Lagrangian transformation is given by

$$J = \begin{pmatrix} \rho & \rho u \\ 0 & 1 \end{pmatrix}. \tag{A.4}$$

Thus, in order to show non-singularity of Y it suffices to show that the density ρ does not approach to zero in finite time. Boundedness form above of ρ is also of essential in the calculations. This justifies some of the statements in the beginning of chapter 1.

References

[1] S.N. Antontsev, A.V. Kazhikhov and V.M. Monakhov. “Boundary Value Problems in Mechanics of Nonhomogeneous Fluids”. Studies in Mathematics and Its Applications, Vol. 22, North-Holland, Amsterdam, 1990.

[2] D.J. Benney. *A general theory for interactions between short and long waves*. Studies in Applied Mathematics **56** (1977), 81–94.

[3] G.Q. Chen and D. Wang *Global Solutions of Nonlinear Magnetohydrodynamics with Large Initial Data*, Journal of Differential Equations 182, 344–376 (2002).

[4] J.P Dias and H. Frid *Short wave-long wave interactions for compressible Navier–Stokes equations*, SIAM J. Math. Anal., 43 (2011), pp. 764–787.

[5] H. Frid, R. Pan and W. Zhang. *Global smooth solutions in R3 to short wave-long wave interactions systems for viscous compressible fluids*, SIAM J. Math. Anal., Vol. 46, No. 3 (2014), pp. 1946–1968.