Geometric Langlands program for 3-manifolds

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Based on works (in progress) joint with:
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What the talk is about

For mathematicians: Langlands duality patterns for 3-manifolds: homology of the space of connections on 3-manifolds, skein modules and complexified instanton Floer homology.

For physicists: state spaces of the GL twist of the 4d $\mathcal{N} = 4$ super Yang–Mills theory on 3-manifolds and the S-duality.

Basic example

$$\mathrm{LocSys}(S^1) \cong \mathrm{QCoh}(\mathbf{C}^\times) = \mathbf{C}[z, z^{-1}]\text{−mod}.$$

Indeed, the left-hand side is given by representations of $\pi_1(S^1) \cong \mathbf{Z}$; irreducible representations of $\mathbf{Z}$ are parametrized by $\mathbf{C}^\times$.

This can be understood as a mirror symmetry between the A-model into $\mathrm{T}^* S^1$ and the B-model into $\mathbf{C}^\times$.

Now suppose Σ is a closed Riemann surface, $H \cong (\mathbf{C}^\times)^n$ a torus and $H^\vee \cong (\mathbf{C}^\times)^n$ the dual torus, so that $\mathrm{Hom}(H, \mathbf{C}^\times) \cong \mathrm{Hom}(\mathbf{C}^\times, H^\vee)$. Then

$$\mathrm{LocSys}(\mathrm{Bun}_H(\Sigma)) \cong \mathrm{QCoh}(\mathrm{Loc}_{H^\vee}(\Sigma)).$$

- $\mathrm{Bun}_H(\Sigma)$ is the moduli space of H-bundles on Σ.
- $\mathrm{Loc}_{H^\vee}(\Sigma)$ is the moduli space of $H^\vee$-local systems (representations $\pi_1(\Sigma) \to H^\vee$).

Nonabelian version?

Let G be a reductive group. One can imagine a nonabelian version

$$\mathrm{LocSys}(\mathrm{Bun}_G(\Sigma)) \overset{???}{\cong} \mathrm{QCoh}(\mathrm{Loc}_{G^\vee}(\Sigma)),$$

where $G^\vee$ is the *Langlands dual group* to G:

$$G = \mathrm{SL}_n \leftrightarrow G^\vee = \mathrm{PGL}_n, \qquad G = \mathrm{SO}(2n+1) \leftrightarrow G^\vee = \mathrm{Sp}(2n).$$

Wrong. This equivalence fails in a subtle and interesting way: need to be careful about growth conditions for the A-model into the non-compact $\mathrm{T}^*\mathrm{Bun}_G(\Sigma)$ and need to be careful about the B-model into the singular $\mathrm{Loc}_{G^\vee}(\Sigma)$.

Example

Suppose X is a smooth variety with a function $f\colon X \to \mathbf{C}$ which has an isolated singularity at 0. Then we can enlarge $\mathrm{QCoh}(f^{-1}(0))$ to the category $\mathrm{IndCoh}(f^{-1}(0))$ of *ind-coherent sheaves* (Gaitsgory, Preygel, ...). The quotient

$$\mathrm{IndCoh}(f^{-1}(0))/\mathrm{QCoh}(f^{-1}(0)) = \mathrm{MF}(f)$$

is the category $\mathrm{MF}(f)$ of *matrix factorizations* of f (Orlov). This is closely related to the *LG/CY correspondence*.

Work of Kapustin and Witten

Kapustin and Witten (following Marcus) have analyzed a topological twist (the *GL twist*) of the 4d $\mathcal{N} = 4$ super Yang–Mills theory.

- This produces a 4-dimensional TQFT $Z_{G,\Psi}$ which depends on a parameter $\Psi \in \mathbf{CP}^1$.
- S-duality gives an equivalence $Z_{G,\Psi} \cong Z_{G^\vee,-1/\Psi}$.
- The compactification $Z_{G,0}(\Sigma \times -)$ on a Riemann surface Σ is the A-model into the Hitchin moduli space $\mathrm{Higgs}_G(\Sigma) \cong \mathrm{T}^*\mathrm{Bun}_G(\Sigma)$ with respect to ω_K.
- The compactification $Z_{G,\infty}(\Sigma \times -)$ is the B-model into the Hitchin moduli space $\mathrm{Higgs}_G(\Sigma) \cong \mathrm{Loc}_G(\Sigma)$ with respect to J.

1. The equivalence $Z_{G,0} \cong Z_{G^\vee,\infty}$ for the 2-categories of surface operators is the *local geometric Langlands* equivalence.
2. The equivalence of categories of boundary conditions $Z_{G,0}(\Sigma) \cong Z_{G^\vee,\infty}(\Sigma)$ is the *global geometric Langlands* equivalence.
3. The isomorphism of vector spaces of states $Z_{G,\Psi}(M^3) \cong Z_{G^\vee,-1/\Psi}(M^3)$ is the subject of this talk.

Conjecture (Ben-Zvi–Nadler)

There is an equivalence of categories

$$\mathrm{Shv}_{\mathcal{N}_G}(\mathrm{Bun}_G(\Sigma)) \cong \mathrm{IndCoh}_{\mathcal{N}_{G^\vee}}(\mathrm{Loc}_{G^\vee}(\Sigma)).$$

- $\mathcal{N}_G \subset \mathrm{T}^*\mathrm{Bun}_G(\Sigma)$ is the *global nilpotent cone*, a conical Lagrangian. The left-hand side is a partially wrapped Fukaya category of $\mathrm{T}^*\mathrm{Bun}_G(\Sigma)$.
- $\mathcal{N}_{G^\vee} \subset \mathrm{T}^*[-1]\mathrm{Loc}_{G^\vee}(\Sigma)$ is another conical subset. The right-hand side is an enlargement of $\mathrm{QCoh}(\mathrm{Loc}_{G^\vee}(\Sigma))$.

Let $G = \mathrm{SL}_2$ and

$$\mathrm{Shv}_{\mathcal{N}_G}(\mathrm{Bun}_G(\Sigma))^{temp} = \mathrm{Ker}\left(\mathrm{Shv}_{\mathcal{N}_G}(\mathrm{Bun}_G(\Sigma)) \longrightarrow \mathrm{LocSys}(\mathrm{Bun}_G(\Sigma))\right)$$

be the *tempered* part.

Conjecture

There are equivalences

$$\begin{array}{ccccc} \mathrm{Shv}_{\mathcal{N}_G}(\mathrm{Bun}_G(\Sigma))^{temp} & \longrightarrow & \mathrm{Shv}_{\mathcal{N}_G}(\mathrm{Bun}_G(\Sigma)) & \longrightarrow & \mathrm{LocSys}(\mathrm{Bun}_G(\Sigma)) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathrm{QCoh}(\mathrm{Loc}_{G^\vee}(\Sigma)) & \longrightarrow & \mathrm{IndCoh}_{\mathcal{N}_{G^\vee}}(\mathrm{Loc}_{G^\vee}(\Sigma)) & \longrightarrow & \mathrm{IndCoh}_{\mathcal{N}_{G^\vee}}(\mathrm{Loc}_{G^\vee}(\Sigma))/\mathrm{QCoh}(\mathrm{Loc}_{G^\vee}(\Sigma)) \end{array}$$

Boundary conditions and Betti Langlands

The topological theory has the following boundary conditions. $\Psi = 0$:

1. Zero section. In the 2d σ-model this corresponds to the zero section of $\mathrm{T}^*\mathrm{Bun}_G(\Sigma)$ (a (B, A, A) brane).
2. Automorphic Whittaker sheaf (*D5 brane* = *Nahm pole* boundary condition). In the 2d σ-model this corresponds to the (A, B, A) brane of opers.

$\Psi = \infty$:

1. Spectral Whittaker sheaf. In the 2d σ-model this corresponds to a (B, B, B) brane supported at the trivial local system in $\mathrm{Loc}_{G^\vee}(\Sigma)$.
2. Structure sheaf (*NS5 brane* boundary condition). In the 2d σ-model this corresponds to an (A, B, A) brane supported everywhere.

S-duality swaps these boundary conditions. Comparing the braided monoidal categories of line operators on these boundary conditions we get true facts

$$\mathrm{LocSys}(\mathrm{B}G) \cong \mathrm{QCoh}(J_{G^\vee,0}[-1]), \qquad \mathrm{Whit}(\mathrm{Gr}_G) \cong \mathrm{Rep}(G^\vee),$$

where $J_{G^\vee,0}$ is the centralizer of a principal nilpotent and $\mathrm{Whit}(\mathrm{Gr}_G)$ is the category of *Whittaker sheaves* on the affine Grassmannian.

Theorem

$$\int_\Sigma \mathrm{LocSys}(\mathrm{B}G) \cong \mathrm{LocSys}(\mathrm{Bun}_G(\Sigma)), \qquad \int_\Sigma \mathrm{Rep}(G^\vee) \cong \mathrm{QCoh}(\mathrm{Loc}_{G^\vee}(\Sigma)).$$

Betti geometric Langlands for 3-manifolds

Let M be a closed 3-manifold, $H \cong (\mathbf{C}^*)^n$ and $\mathrm{Conn}_H(M)$ the moduli space of H-bundles with a connection.

Theorem

$$\mathrm{H}_\bullet(\mathrm{Conn}_H(M)) \cong \mathbf{R}\Gamma(\mathrm{Loc}_{H^\vee}(M), \mathcal{O}).$$

Suppose $G = \mathrm{SL}_2$.

Conjecture

There are isomorphisms

$$\begin{array}{ccccc} \mathrm{H}^{\mathcal{N}_G}_\bullet(\mathrm{Conn}_G(M))^{temp} & \longrightarrow & \mathrm{H}^{\mathcal{N}_G}_\bullet(\mathrm{Conn}_G(M)) & \longrightarrow & \mathrm{H}_\bullet(\mathrm{Conn}_G(M)) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathbf{R}\Gamma(\mathrm{Loc}_{G^\vee}(M), \mathcal{O}) & \longrightarrow & \mathbf{R}\Gamma(\mathrm{Loc}_{G^\vee}(M), \omega_{\mathcal{N}_{G^\vee}}) & \longrightarrow & \dots \end{array}$$

- The definition of $\mathrm{H}^{\mathcal{N}_G}_\bullet(\mathrm{Conn}_G(M)), \omega_{\mathcal{N}_{G^\vee}}$ is a work in progress by Ben-Zvi–Gunningham–S.
- The more general theory of $\omega_{\dots}$ is a work in progress by Beraldo.

The conjecture is not entirely trivial, but true for $M = S^3$.

The case of generic (irrational) Ψ is simpler.
Expectations:

- For generic Ψ the tempered part captures the whole TQFT.
- Deforming away from $\Psi = \infty$ corresponds to a Batalin–Vilkovisky quantization of $\mathbf{R}\Gamma(\mathrm{Loc}_G(M), \mathcal{O})$ (S–Williams). Here $\mathbf{R}\Gamma(\mathrm{Loc}_G(M), \mathcal{O})$ has a Batalin–Vilkovisky antibracket since $\mathrm{Loc}_G(M)$ is the critical locus of the Chern–Simons functional.

- Bussi, Joyce et al. have introduced a perverse sheaf $P_{\mathrm{Loc}_G(M)}$ on $\mathrm{Loc}_G(M)$ (more generally, on oriented (-1)-shifted symplectic stacks).
- Abouzaid–Manolescu: the ***complexified instanton Floer homology*** of M to be

$$\mathrm{HP}^\bullet_G(M) = \mathbf{R}\Gamma(\mathrm{Loc}_G(M), P_{\mathrm{Loc}_G(M)}).$$

  The idea is that it is an $\mathrm{SL}_2(\mathbf{C})$ (rather than an $\mathrm{SU}(2)$) version of the instanton Floer homology of M, i.e. the Morse homology of the complexified Chern–Simons functional. In the complexified setting the instanton effects are expected to vanish, so the Morse homology localizes into a perverse sheaf $P_{\mathrm{Loc}_G(M)}$ on the critical locus.

Expectation: $Z_{G,\Psi}(M^3) \cong \mathrm{HP}^\bullet_G(M^3)$ *for generic* Ψ.

S-duality and the previous observations suggest the following conjecture.

Conjecture

There is an isomorphism

$$\mathrm{HP}^\bullet_G(M^3) \cong \mathrm{HP}^\bullet_{G^\vee}(M^3).$$

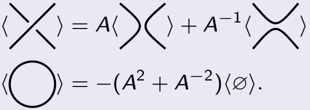
True in the case $\pi_1(M)$ is finite.

Question: can we make this Langlands duality statement more concrete?

Idea: the ***skein module*** $\mathrm{Sk}_G(M^3)$ is the vector space of operators generated by the Wilson lines in the analytically continued Chern–Simons theory.

Definition

Let M be an oriented 3-manifold and $A \in \mathbf{C}^\times$ (think: $A = \sqrt{q} = \exp(\pi i \Psi)$). The ***skein module*** $\mathrm{Sk}_{\mathrm{SL}_2}(M)$ is the $\mathbf{C}$-vector space spanned by isotopy classes of framed unoriented links in M modulo the following local relations:



Theorem (Gunningham–Jordan–S)

For q (Ψ) generic $\mathrm{Sk}_G(M)$ is a finite-dimensional space.

The computation of dimensions of $\mathrm{Sk}_G(M)$ is an interesting and largely open problem. In the last few years there has been a significant progress for $G = \mathrm{SL}_2$ (Carrega, Gilmer–Masbaum, Detcherry–Wolfe, ...)

Theorem (Gunningham–S, in progress)

There is an isomorphism $\mathrm{HP}^0_G(M) \cong \mathrm{Sk}_G(M)$.

Langlands duality for skein modules

Conjecture

For q generic there is an isomorphism

$$\mathrm{Sk}_G(M) \cong \mathrm{Sk}_{G^\vee}(M).$$

There is a further refinement of the conjecture if one introduces electric and magnetic charges on both sides.
The conjecture is true for $G = H \cong (\mathbf{C}^\times)^n$ (Przytycki).

Theorem (Gunningham–Jordan–S, in progress)

The conjecture is true for $G = \mathrm{SL}_2$ and $M = \Sigma \times S^1$.

The claim is quite nontrivial. For instance, it asserts the existence of an embedding

$$\mathrm{H}^{\mathsf{middle}}(\mathrm{Loc}^{tw}_{\mathrm{SL}_2}(\Sigma)) \subset \mathrm{Sk}_{\mathrm{SL}_2}(\Sigma \times S^1)$$

with the image picked out by fixing the electric charge, where $\mathrm{Loc}^{tw}_{\mathrm{SL}_2}(\Sigma)$ is the twisted character variety. The cohomology groups of twisted character varieties have been studied extensively in the last 10 years.