

SUPERSTRING MEASURE AND SUPERPERIODS

Giovanni Felder, ETH Zurich

joint work with David Kazhdan and Alexander Polishchuk

1905.12805

2006.13271

StringMath 2021, 17 June 2021

PLAN

1. Perturbative superstring amplitudes
2. Supercurves and their moduli
3. Mumford isomorphism, super period matrix and superstring measure
4. Regularity of the superstring measure

Superstring perturbation theory (RNS)

Scattering amplitudes in superstring theory are roughly given by

$$A_h = \sum_{g=0}^{\infty} \lambda^{2g-2} A_{g,h}$$

$$A_{g,h} = \int_{M_{g,n}} \psi_{g,n}$$

↗ superstring measure
 ↘ Section of the Berezinian
linebundle

Moduli space of supercurves of genus g
(super-Riemann surfaces) with n
marked points

- Well understood in genus ≤ 2
- Subtleties : $M_{g,n}$ is not split in higher genus [DONAGI-WITTEN],
- Integrals are not absolutely convergent [WITTEN 2012-2013]

Today:

Construction of $\psi_{g,n}$ for $n=0$, (Type I superstring in 10D)
Regularity of $\psi_g = \psi_{g,0}$ for $g \leq 11$

Complex supergeometry

Language of ringed spaces

- A complex supermanifold is a locally ringed space over \mathbb{C}

$$X = (|X|, \mathcal{O}_X)$$

top. space $|X|$ \uparrow sheaf of \mathbb{C} -algebras

locally isomorphic to $\mathbb{C}^{n|m} = (\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n} \otimes_{\mathbb{C}} \Lambda(\theta_1, \dots, \theta_m))$

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- Morphisms are morphisms of ringed spaces over \mathbb{C}
- $I_X \subset \mathcal{O}_X$ nilpotent ideal (locally generated by $\theta_1, \dots, \theta_n$)

I_X is the vanishing ideal of the reduced space

$$X_{\text{red}} = (|X|, \mathcal{O}_X/I_X) \hookrightarrow X,$$

a complex manifold

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- Superschemes locally $\mathcal{O}_X \cong \mathbb{Z}_2\text{-graded comm. ring}$, X_{red} scheme

Canonical bundle

- The role of the canonical bundle $\Lambda^n \Omega_X^1$ of an n -dimensional complex manifold is taken by the Berezinian line bundle $\omega_X = \text{Ber}(\Omega_X^1)$ on an $n|m$ -dimensional complex supermanifold
- Local coordinates $z_1, \dots, z_n, \theta_1, \dots, \theta_m$ define a local basis $dz_1, \dots, d\theta_m$ of Ω_X^1 and a local basis

$$[dz_1, \dots, dz_n \mid d\theta_1, \dots, d\theta_m]$$

of $\text{Ber}(\Omega_X^1)$. Under change of coordinates

$$[dw/d\eta] = \text{Ber}\left(\frac{\partial(w, \eta)}{\partial(z, \theta)}\right) [dz/d\theta]$$

$$\text{Ber} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C) \det(D)^{-1}$$

Supercurves

- A smooth supercurve C is a complex supervariety of dimension $1|1$ with a distribution $\mathcal{D} \subset T_C$ of dimension $0|1$ such that $\frac{1}{2} [,] : \mathcal{D} \otimes \mathcal{D} \rightarrow T_C / \mathcal{D}$ is an isomorphism (of \mathbb{Q} -modules)

Supercurves

- A smooth supercurve G is a complex supervariety of dimension $1|1$ with a distribution $\mathcal{D} \subset T_G$ of dimension $0|1$ such that $\frac{1}{2} [\cdot, \cdot] : \mathcal{D} \otimes \mathcal{D} \rightarrow T_G/\mathcal{D}$ is an isomorphism (of \mathbb{Q} -modules)
- In local coordinates z, θ , $\mathcal{D} = \mathcal{O}_G \mathcal{D}$ where

$$\mathcal{D} = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z} \in T_G = \mathcal{O}_G \frac{\partial}{\partial \theta} + \mathcal{O}_G \frac{\partial}{\partial z}$$

$$\frac{1}{2} [\mathcal{D}, \mathcal{D}] = \frac{\partial^2}{\partial z^2}$$
 generates T_G/\mathcal{D}

More generally we need to consider families $G \rightarrow S$ over a superbase S , distributions $\mathcal{D} \subset T_G/S$.

Berezinian of supercurves

The role of the canonical bundle (holomorphic 1-forms) of curves is taken by the Berezinian $\omega_{C/S} = \text{Ber}(\Omega^1_{C/S})$ of supercurves.

The exact sequence

$$0 \rightarrow \mathcal{D} \rightarrow T_C \rightarrow \mathcal{D}^2 \rightarrow 0$$

dualizes to

$$0 \rightarrow \mathcal{D}^{-2} \rightarrow \Omega^1_C \xrightarrow{\delta} \mathcal{D}^{-1} \rightarrow 0$$

Taking Berezinians we get

$$\omega_{C/S} = \text{Ber}(\Omega^1_{C/S}) \cong \mathcal{D}^{-1}$$

δ can be viewed as a derivation

$$\delta : \mathcal{O}_C \rightarrow \omega_{C/S}$$

In local coordinates :

$$\delta f = \left(\frac{\partial}{\partial \theta} + \theta \frac{\partial^2}{\partial z^2} \right) f [dz | d\theta]$$

Superconformal vector fields and deformations of supercurves

Superconformal vector fields are vector fields preserving the distribution

$$T_{C/S}^{sc} = \{ v \in T_{C/S} \mid [v, \omega] \subset \omega \}$$

Lemma The map $T_C \rightarrow \mathcal{D}^2$ restricts to an isomorphism $T_{C/S}^{sc} \xrightarrow{\sim} \mathcal{D}^2 \cong \omega_{C/S}^{-2}$

In particular $T_{C/S}^{sc}$ is a line bundle

Kodaira-Spencer theory: infinitesimal deformations of a supercurve C are classified by

$$H^1(C, T_{C/S}^{sc}) = H^1(C, \omega_{C/S}^{-2}) \xrightarrow{\sim} H^0(C, \omega_{C/S}^3)^\vee$$

Serre duality

Supercurves over an even base

Over S even (e.g. $S = \{pt\}$) supercurves have a classical description in terms of **Spin structures** (theta characteristics) on ordinary curves C_0 , i.e. line bundles \mathcal{L} with an isomorphism $\mathcal{L}^2 \xrightarrow{\cong} \omega_{C_0}$

$$\mathcal{O}_C = \mathcal{O}_{C_0} \oplus \mathcal{L}$$

$$\omega_C \cong \mathcal{L} \oplus (\mathcal{L}^2 = \omega_{C_0})$$

$$f(z, \theta) = f_0(z) + f_1(z)\theta$$

$$f [dz | d\theta] = f_0 [dz | d\theta] + f_1 \theta [dz, d\theta]$$

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$$f [dz | d\theta] = f_0 [dz | d\theta] + f_1 \theta [dz, d\theta]$$

$\delta: \mathcal{O}_C \rightarrow \Omega^1_C$ is the identity on \mathcal{L} and the de Rham differential $f \mapsto df$ on \mathcal{O}_{C_0}

Thus for any family $C \rightarrow S$ of supercurves, the restriction to $S \cap C(S)$ is a family of curves with spin structure

Moduli of supercurves

M_g = moduli space of supercurves of genus $g \geq 2$
 $= M_g^+ \cup M_g^-$

$(M_g^\pm)_{\text{red}}$ = moduli space of curves of genus g with
 even/odd spin structure

smooth Deligne-Mumford stack of dimension $3g-3/2g-2$

$$T_C^* M_g = H^0(C, \omega_C^3) = \prod H^0(C_{\text{red}}, \mathcal{L}^3) \oplus H^0(C_{\text{red}}, \mathcal{L}^4)$$

$\xrightarrow{(2g-2)\text{-dimensional}}$

\nwarrow quadratic differentials
 \nearrow $(3g-3)$ -dim.

Mumford isomorphism

$p: C \rightarrow S = M_g$ universal curve

For any line bundle E on C we have a cohomology sheaf $R^{\bullet} p_* E$. Its Berezinian is a line bundle $Ber(R^{\bullet} p_* E)$ on M_g .

Mumford isomorphisms relate the line bundles $Ber(R^{\bullet} p_* \omega_{C/S}^k)$, $\bullet \in \mathbb{Z}$.

Relevant for integration: $Ber(\Omega_S^1) = Ber(p_* \omega_{C/S}^3)$

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Theorem (Voronov, Rosly-Schwarz-Voronov)

There is a canonical isomorphism

$$i: (\text{Ber}(R^1 p_* w_{C/S}))^5 \longrightarrow \text{Ber}(R^1 p_* w_{C/S}^3)$$

(super-)Abelian differentials

$$= w_S = \text{Ber}(\Omega_S^1)$$

$$(R^1 p_* w_{C/S}^3 = 0)$$

Stable supercurves

- M_g admits a compactification \overline{M}_g , the moduli space of stable supercurves [Deligne]
- \overline{M}_g is a smooth Deligne-Mumford superstack [Deligne, Moosaviah-Zhou, FKP]
- M_g is the complement of a normal crossings divisor in \overline{M}_g

$$\Delta = \Delta_{NS} \cup \Delta_R$$
- The Mumford isomorphism extends to an isomorphism

$$\omega_{\overline{M}_g} \cong \text{Ber}^5(R_{P_*}^{WC/\overline{M}_g}) (-2\Delta_{NS} - \Delta_R) \quad [\text{FKP}]$$

Superstring measure

(for the vacuum amplitude in 10D Type II superstring)

$$\psi_g \in \Gamma(U, \text{Ber}(\Omega^1_{M_g^t \times M_g^{t,c}})) \xrightarrow{\text{complex conj.}} = \Gamma(U, \text{Ber} \Omega^1_{M_g^t} \boxtimes \text{Ber} \Omega^1_{M_g^{t,c}})$$

defined on some neighbourhood U of the (quasi-)diagonal $\Delta \subset M_g^t \times M_g^{t,c}$
as follows:

* pairs of complex conjugate curves with possibly different spin structures

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defined on some neighbourhood U of the (quasi-)diagonal $\Delta \subset M_g^t \times M_g^{+c}$ as follows:

- On $H^0(C, \omega_C)$ we have a canonical sesquilinear pairing

$$\langle \alpha, \bar{\beta} \rangle = \int_C \alpha \wedge \bar{\beta}, \text{ defining a section}$$

$$s \in \Gamma(U, \text{Ber } p_* \omega_{CS} \boxtimes \text{Ber } p_* \omega_{\overline{CS}}) \text{ defined near } \Delta_g$$

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$s \in \Gamma(U, \text{Ber } p_* \omega_{CS} \boxtimes \text{Ber } p_* \omega_{CS})$ defined near Δ_g

- The superstring measure is the image of s^5 by the Mumford isomorphism

$$\psi_g = (i \otimes \bar{i}) / s^5 \in \Gamma(\Delta_g \setminus D_g^{(2)}, \text{Ber}(\Omega^1_{M_g^+}) \boxtimes \text{Ber}(\Omega^1_{M_g^{+c}}))$$

↑ Mumford isomorphism ↑ Theta-null divisor

Theta null divisor

- A spin structure \mathcal{L} on a curve C_0 is called even if $h^0(\mathcal{L}) := \dim H^0(C_0, \mathcal{L})$ is even ($h^0(\mathcal{L}) \bmod 2$ is constant in families)

$$M_g^{\text{spin}} = M_g^+ \cup M_g^- \quad (\text{disjoint union})$$

At generic points of the moduli space M_g^* of even spin curves

$$H^0(C_0, \mathcal{L}) \simeq H^1(C_0, \mathcal{L}) = 0$$

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- On the Theta-null divisor $D_g \subset M_g^+$, $h^0(\mathcal{L}) > 0$

The corresponding supercurve has

$$\dim H^0(C_0, \omega_{C_0}) = (1/2k), \quad \dim H^0(C_0, \omega_{C_0}^2) = (g/2k)$$

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The corresponding supercurve has

$$\dim H^0(C, \Omega_C) = (1/2k), \quad \dim H^0(C, \omega_C) = (g/2k)$$

- The superstring measure has potential singularities as we approach the Theta null divisor since $P_* \omega_{C/S}$ is not locally free there.

Behaviour of the superstring measure around D_g

• $\psi_g = i \otimes \bar{i} (s^5)$, $s \in \text{Ber}(R^\circ p_* \omega_{C/S}) \otimes \text{Ber}(R^\circ p'_* \bar{\omega}_{\bar{C}/\bar{S}})$

Away from D_g , $H^0(C, \omega_C)$ is of dimension $g|0$ Basis' $\omega_1, \dots, \omega_g$
 $H^1(C, \omega_C) \cong \mathbb{C}$

$$S_{q_i} \omega_j = \delta_{ij}$$

$$s = \frac{1}{\det \langle \omega_1, \bar{\omega}_j \rangle} \omega_1 \wedge \dots \wedge \omega_g \otimes \bar{\omega}_1 \wedge \dots \wedge \bar{\omega}_g = \frac{1}{2\pi \Omega} \theta \otimes \bar{\theta}$$

$$\Omega_{ij} = \int_{\theta_i} \omega_j$$

Behaviour of the superstring measure around D_g

- $\psi_g = i \otimes \bar{i} (s^5) , s \in \text{Ber}(R^* p_* \omega_{C/S}) \otimes \text{Ber}(R^* p'_* \bar{\omega}_{\bar{C}/\bar{S}})$

Away from D_g , $H^0(C, \omega_C)$ is of dimension $g/0$ Basis' $\omega_1, \dots, \omega_g$
 $H^1(C, \omega_C) \cong \mathbb{C}$

$$S_{q_i} \omega_j = \delta_{ij}$$

$$S = \frac{1}{\det(\omega_1, \bar{\omega}_j)} \omega_1 \wedge \dots \wedge \omega_g \otimes \bar{\omega}_1 \wedge \dots \wedge \bar{\omega}_g = \frac{1}{\text{Im } \Omega} \theta \otimes \bar{\theta}$$

$$\Omega_{ij} = S_{\theta_i} \omega_j$$

- We need to understand the behaviour of $\theta = \omega_1 \wedge \dots \wedge \omega_g$ and Ω as we approach D_g .

Regularity of the superstring measure in low genus

Theorem 1 (FKP) Locally around a point of the divisor $D_g \subset (\mathcal{M}_g^+)_\text{red} = \mathcal{M}_g^{\text{spin}+}$ there is an even function $f \in \Omega_{\mathcal{M}_g^+}$ such that

- $f|_{(\mathcal{M}_g^+)_\text{red}}$ vanishes on D_g to first order
- $\theta = \omega_1 \wedge \dots \wedge \omega_g = f^2 \alpha$ with $\alpha \neq 0$
- $\int \Omega$ is regular on D_g .

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- $f|_{(\mathcal{M}_g^+)_\text{red}}$ vanishes on D_g to first order
- $\theta = \omega_1 \wedge \dots \wedge \omega_g = f^2 \alpha$ with $\alpha \neq 0$
- $S\Omega$ is regular on D_g .

Theorem 2 (FKP) The superstring measure ψ_g is regular on a neighbourhood $U \subset \mathcal{M}_g^+ \times \mathcal{M}_g^+$ of the diagonal Δ if $g \leq 11$

Proof of the regularity Theorem 2, given Theorem 1

- ψ_g is the image of $s^5 = \frac{\theta^5 \otimes \bar{\theta}^5}{(\det \operatorname{Im} \Omega)^5}$ by the Mumford isomorphism
- $\theta^5 = f^{10} \alpha^5, \quad \alpha \neq 0.$
- $\Omega = \Omega_{\text{reg}} + \frac{A}{f}, \quad A_{ij} \in I^2$ at least quadratic in odd variables
- $I^m = 0 \quad m > 2g-2 \quad (\dim M_g = 3g-3/2g-2)$
- $\frac{1}{(\det \operatorname{Im} \Omega)^5} = \frac{1}{(\det \operatorname{Im} \Omega_{\text{reg}})^5} \sum_{ij=1}^{g-1} \frac{a_{ij}}{f^i \bar{f}^j}$
- $s \sim f^{11-g} \bar{f}^{11-g}$ regular
if $g \leq 11$

Equation of the hull-divisor and Lagrangian intersection

Let \mathcal{L} be an even spin structure on C ($\mathcal{L}^{\otimes 2} \xrightarrow{\sim} \omega_C$)
 $P \in C$, $n \in \mathbb{N}$ large.

$$V = H^0(C, \mathcal{L}(nP) / \mathcal{L}(-nP)) = \text{Span}(z^j dz^{n-j}, j=-m, \dots, n-1)$$

has a non-degenerate symmetric bilinear form

$$\langle s_1, s_2 \rangle = \text{Res}_P(s_1 s_2)$$

(i.e. a symplectic form on TV).

$$L_1 = H^0(C, \mathcal{L}(nP)), \quad L_2 = H^0(C, \mathcal{L}/\mathcal{L}(-nP))$$

are maximal isotropic (Lagrangian),

$$H^0(C, \mathcal{L}) = L_1 \cap L_2 \quad H^*(C, \mathcal{L}) = H^*(L_1 \rightarrow V/L_2)$$

In families $p: C \rightarrow S$, $Rp_* \mathcal{L}$ is quasi-isomorphic to
 a complex $[M \xrightarrow{\phi} M^\vee]$ with ϕ skew-symmetric $s = \text{Pf}(\phi)$

Thank you

