Painlevé equations, quantum operators and gauge theory

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SwissMAP
The Mathematics of Physics
National Centre of Competence in Research

FNSNF
Swiss National Science Foundation

UNIVERSITÉ DE GENÈVE

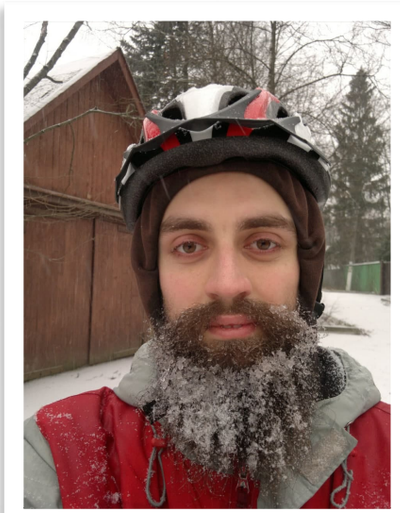
Mostly based on works in collaboration

with M.Bershtein & P.Gavrylenko: 2105.00985

Skoltech

&

HSE Moscow

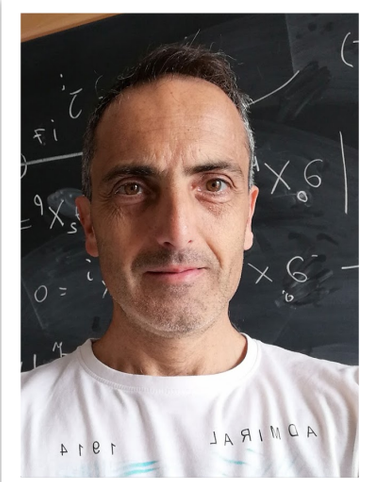


with G.Bonelli & A.Tanzini: 1710.11603 & 1603.01174

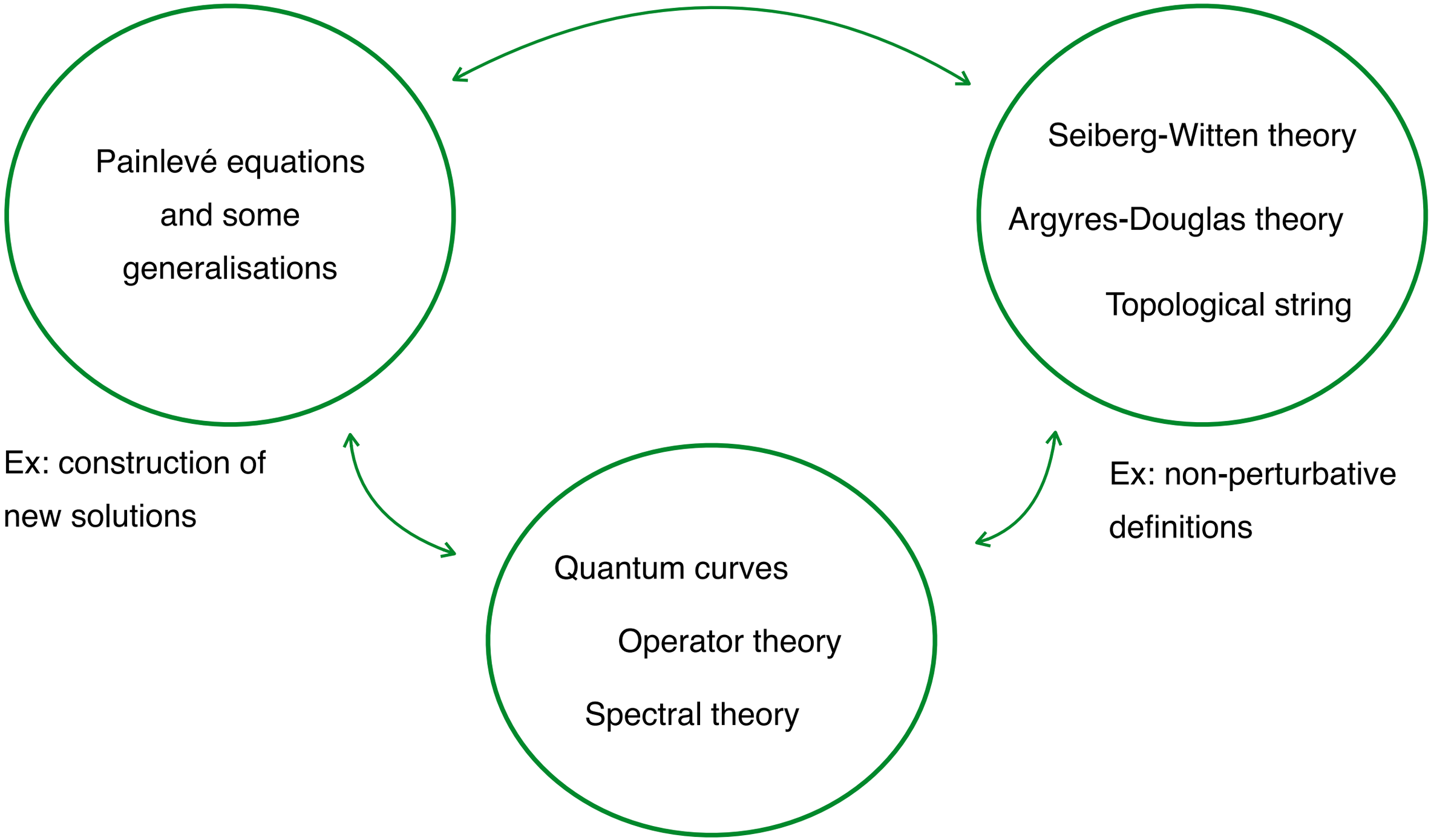
SISSA

&

INFN Trieste



main players in this talk



Ex: construction of new solutions

Ex: non-perturbative definitions

Ex: exact new results in spectral theory

Plan of the talk

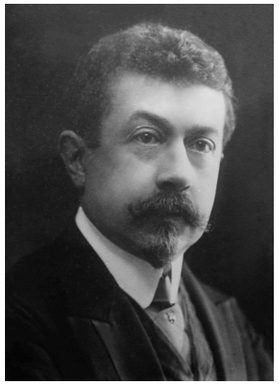
1. Painlevé equations, 4 dimensional rank 1 $\mathcal{N} = 2$ theories , Schrödinger operators

2. q-deformed Painlevé equations, topological string on toric CY3, functional-difference operators

First step:

Painlevé equations

S. Kovalevskaya



P. Painlevé

R.Fuchs

B. Gambier

Second order non-linear ODEs $\frac{d^2q}{dt^2} = R\left(q, \frac{dq}{dt}, t\right)$

whose solutions are free from movable critical points.

Painlevé equations

$$\text{VI}: \quad \frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\frac{dq}{dt} + \frac{2q(q-1)(q-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{q^2} + \frac{\gamma(t-1)}{(q-1)^2} + \frac{\delta t(t-1)}{(q-t)^2}\right)$$

$$\text{V}: \quad \frac{d^2q}{dt^2} = \left(\frac{1}{2q} + \frac{1}{q-1}\right)\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{(q-1)^2}{t^2}\left(\alpha q + \frac{\beta}{q}\right) + \frac{\gamma q}{t} - \frac{1}{2}\frac{q(q+1)}{q-1}$$

$$\text{IV}: \quad \frac{\mathrm{d}^2\mathrm{q}}{\mathrm{dt}^2} = \frac{1}{2\mathrm{q}}\left(\frac{\mathrm{dq}}{\mathrm{dt}}\right)^2 + \frac{3}{2}\mathrm{q}^3 + 4\mathrm{tq}^2 + 2(\mathrm{t}^2 - \alpha)\mathrm{q} + \frac{\beta}{\mathrm{q}}$$

$$\text{III}_1: \quad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{q^2(\alpha + 4q)}{4t^2} + \frac{\beta}{4t} - \frac{1}{q}$$

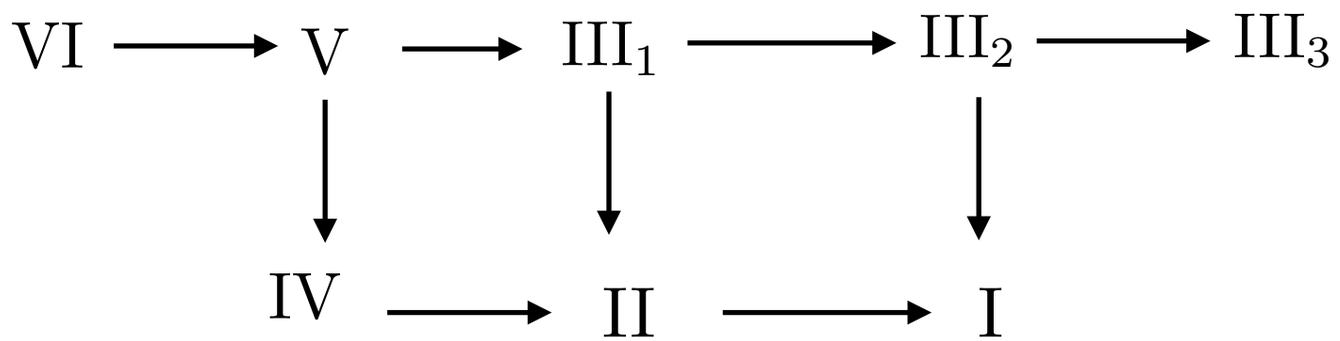
$$\text{III}_2: \quad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{2q^2}{t^2} + \frac{\alpha}{4t} - \frac{1}{q}$$

$$\text{III}_3: \quad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{2q^2}{t^2} - \frac{2}{t}$$

$$\text{II}: \quad \frac{d^2q}{dt^2} = 2q^3 + tq + \alpha$$

$$\text{I}: \quad \frac{d^2q}{dt^2} = 6q^2 + t$$

Painlevé equations can be organised into a confluence diagram



Example:

$$\mathrm{III}_2: \quad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{2q^2}{t^2} + \frac{\alpha}{4t} - \frac{1}{q}$$

$$\mathrm{III}_3: \quad \frac{d^2q}{ds^2} = \frac{1}{q}\left(\frac{dq}{ds}\right)^2 - \frac{1}{s}\frac{dq}{ds} + \frac{2q^2}{s^2} - \frac{2}{s}$$

$$t = s\epsilon$$

$$\alpha = -4/\epsilon$$

$$\epsilon \to 0$$

Painlevé equations appear in physics in several contexts (2d gravity, 2d Ising model,..).

This interplay has known a new beautiful development in the last 10 years thanks to the Painlevé/gauge correspondence (or Isomonodromy/CFT/gauge theory correspondence or Kiev construction).

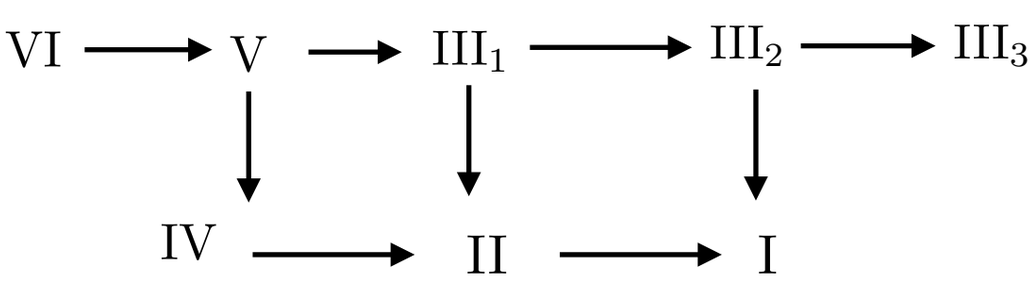
Q1: why should we expect a relation between these two subjects?

Q2: why is this connection interesting?

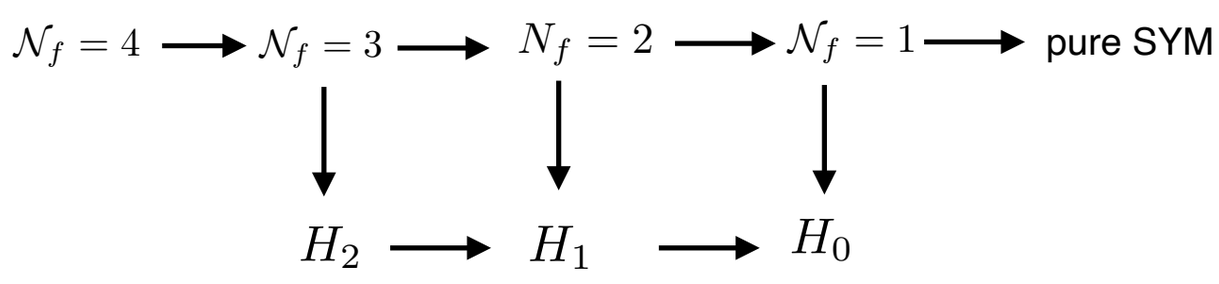
[Yamada et al 2002 & 2004]

Q1: why should we expect a relation between these two subjects?

Painlevé equations



four dimensional SU(2) Seiberg-Witten (SW) and Argyres-Douglas (AD) theories



Painlevé free parameters $\sim$ masses of hypermultiplets/mass deformations

time $\sim$ dynamical scale Λ

Q2: why is this connection interesting?

Painlevé equations

$$\text{VI}: \quad \frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\frac{dq}{dt} + \frac{2q(q-1)(q-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{q^2} + \frac{\gamma(t-1)}{(q-1)^2} + \frac{\delta t(t-1)}{(q-t)^2}\right)$$

$$\text{V}: \quad \frac{d^2q}{dt^2} = \left(\frac{1}{2q} + \frac{1}{q-1}\right)\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{(q-1)^2}{t^2}\left(\alpha q + \frac{\beta}{q}\right) + \frac{\gamma q}{t} - \frac{1}{2}\frac{q(q+1)}{q-1}$$

$$\text{IV}: \quad \frac{\mathrm{d}^2\mathrm{q}}{\mathrm{dt}^2} = \frac{1}{2\mathrm{q}}\left(\frac{\mathrm{dq}}{\mathrm{dt}}\right)^2 + \frac{3}{2}\mathrm{q}^3 + 4\mathrm{tq}^2 + 2(\mathrm{t}^2 - \alpha)\mathrm{q} + \frac{\beta}{\mathrm{q}}$$

$$\text{III}_1: \quad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{q^2(\alpha + 4q)}{4t^2} + \frac{\beta}{4t} - \frac{1}{q}$$

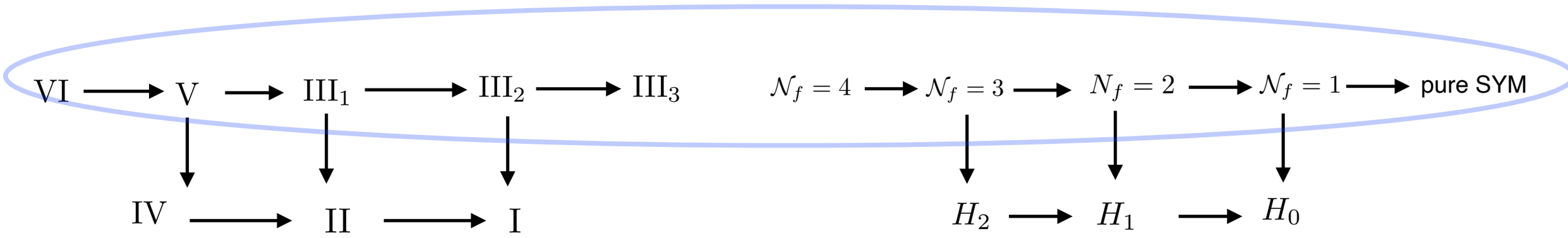
$$\text{III}_2: \quad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{2q^2}{t^2} + \frac{\alpha}{4t} - \frac{1}{q}$$

$$\text{III}_3: \quad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{2q^2}{t^2} - \frac{2}{t}$$

$$\text{II}: \quad \frac{d^2q}{dt^2} = 2q^3 + tq + \alpha$$

$$\text{I}: \quad \frac{d^2q}{dt^2} = 6q^2 + t$$

for the moment we focus on the upper part



We can construct general solutions in an explicit form to Painlevé equations by using the Nekrasov partition function in the self-dual phase of Ω background ($\epsilon_1 + \epsilon_2 = 0$) for a corresponding Seiberg-Witten theory.

[Gamayun, Iorgov, Lisovyy: 2012]

“ Kiev construction”

+ further developments in: Its, Lisovyy, Tykhyy - Bonelli, Lisovyy, Maruyoshi, Sciarappa, Tanzini - Its, Lisovyy, Prokhorov - Nagoya- Bonelli, AG, Tanzini - Lencses, Novaes - Bershtein, Gavrylenko, Marshakov - Coman, Longhi, Teschner - Cunha, Novaes - Mizoguchi, Otsuka, Tashiro - Mironov, Morozov - Bonelli, Del Monte, Gavrylenko, Tanzini- Lisovyy, Naidiuk - . . .

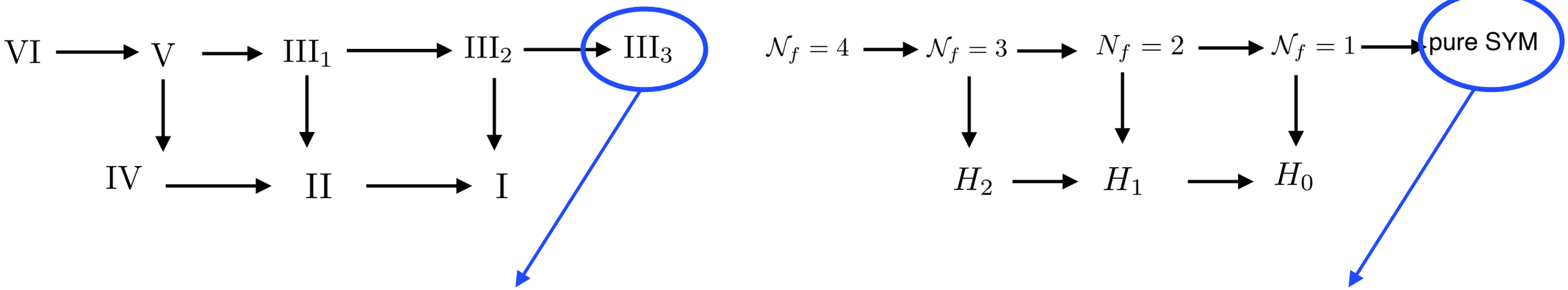
Kiev construction is now a **theorem**. First proofs in

[Iorgov, Lisovyy, Teschner 2014]

[Bershtein, Shchechkin 2014]

+ other derivations/proofs in: Gavrylenko, Lisovyy - Bonelli, Del Monte, Gavrylenko, Tanzini - Del Monte, Desiraju, Gavrylenko - Cafasso, Gavrylenko, Lisovyy - Nekrasov, Jeong - Nekrasov - Bershtein, Gavrylenko, AG . . .

Example:



$$\text{III}_3: \quad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{2q^2}{t^2} - \frac{2}{t}$$

Two integration constants (σ, η)

four dimensional $\mathcal{N} = 2$ SU(2) super Yang-Mills theory

Theorem: [Gamayun,Iorgov,Lisovyy - Its, Lisovyy, Tykhyy- Bershtein,Shchechkin - Gavrylenko,Lisovyy]

$$q(t,\sigma,\eta) = \sqrt{t}\mathrm{e}^{-2\pi\mathrm{i}\eta}\left(\frac{\tau^{\mathrm{GIL}}(t,\sigma,\eta)}{\tau^{\mathrm{GIL}}(t,\sigma+\frac{1}{2},\eta)}\right)^2$$

solves Painlevé III_3

$$\mathrm{III}_3: \quad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{2q^2}{t^2} - \frac{2}{t}$$

with initial conditions specified by σ and η.

$$\tau^{\mathrm{GIL}}(t,\sigma,\eta) = \sum_{n\in\mathbb{N}} \mathrm{e}^{2\pi \mathrm{i} n\eta} Z(\sigma+n,t)$$

where $Z(\sigma,t) = t^{\sigma^2} \dfrac{\mathscr{B}(\sigma,t)}{G(1+2\sigma)G(1-2\sigma)}$

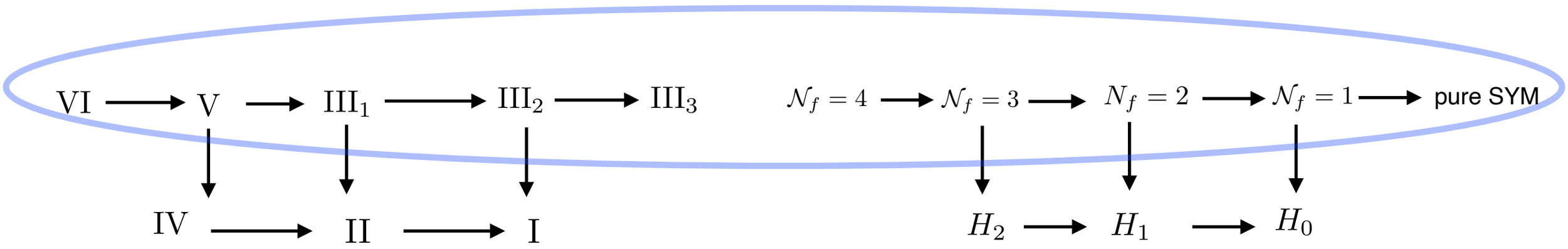
with $\mathscr{B}(\sigma,t) =$ Nekrasov instanton function for the pure 4 dim $SU(2)$ $\mathcal{N}=2$ SYM theory (in the self-dual phase $\epsilon_1 = -\epsilon_2 = \epsilon$)

$$\mathscr{B}(\sigma,t) = 1 + \sum_{n\geq 1} c_n(\sigma) t^n = 1 + \frac{t}{2\sigma^2} + \frac{8\sigma^2+1}{4\sigma^2(4\sigma^2-1)} t^2 + \cdots$$

gauge theory language: $\sigma = a/\epsilon$: vev of scalars in vector multiplet

$t = \Lambda^4/\epsilon^4$: instanton counting parameter

This construction can be easily generalised to all the upper part of the diagram



Known results in physics → New results in mathematics

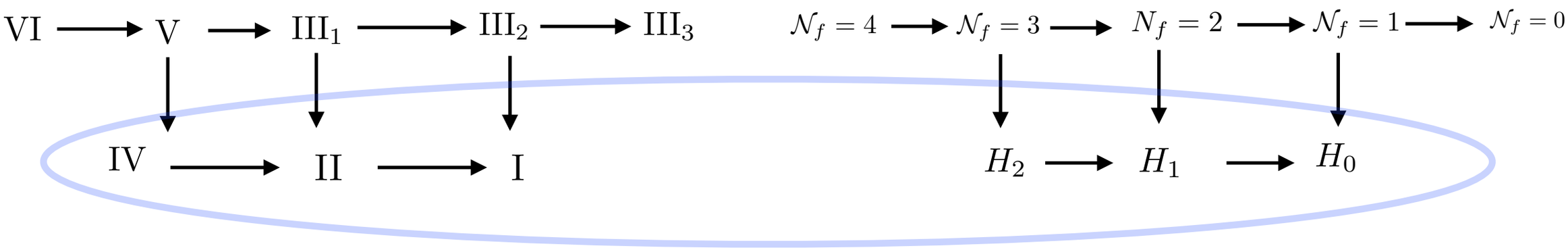
What about the other way around?

Example: use Painlevé knowledge to resum instanton expansion in Nekrasov function. This is useful to study the strong coupling (or magnetic) frame.

Bonelli, AG, Tanzini 2016

Comment:

It is possible to extend this correspondence to the lower line in the confluence diagram

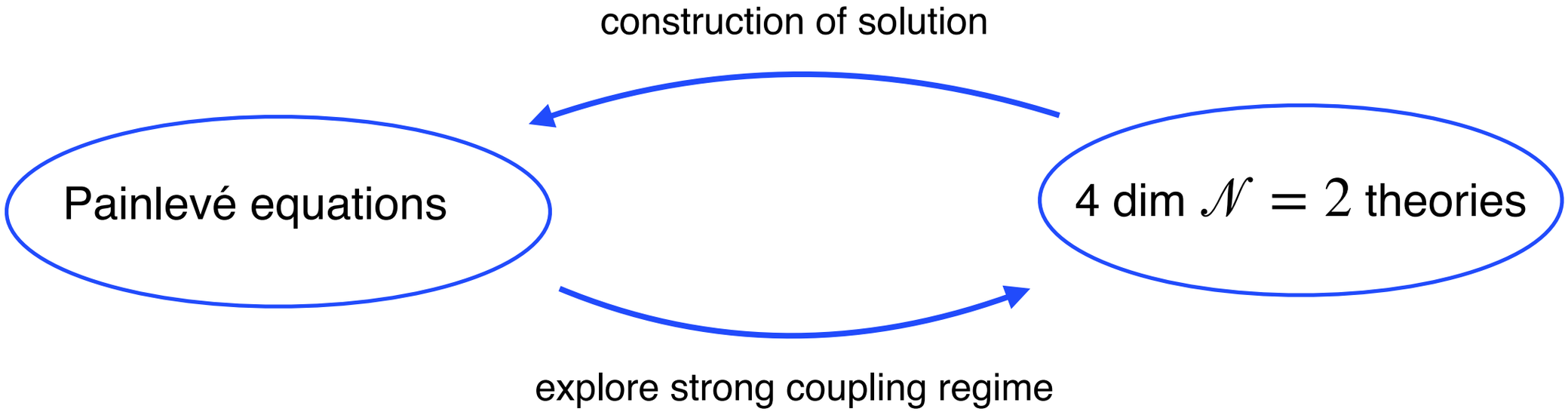


Its, Lisovyy, Prokhorov 2016 - Bonelli, Lisovyy, Maruyoshi, Sciarappa, Tanzini 2016

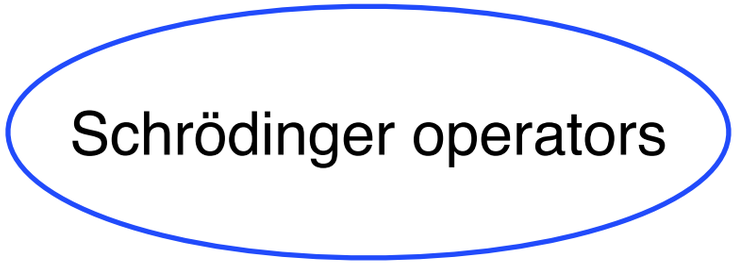
Lisovyy, Roussillon 2016 - Nagoya 2015, 2019 - AG, Gu 2018 - Itoyama, Oota, Yano 2018

. . .

So far we discussed:



Next: application to the study



Another interesting aspect of Painlevé equations is their connection to Schrödinger operators.

Novokshenov 1986 + Lukyanov 2011 + . . . :

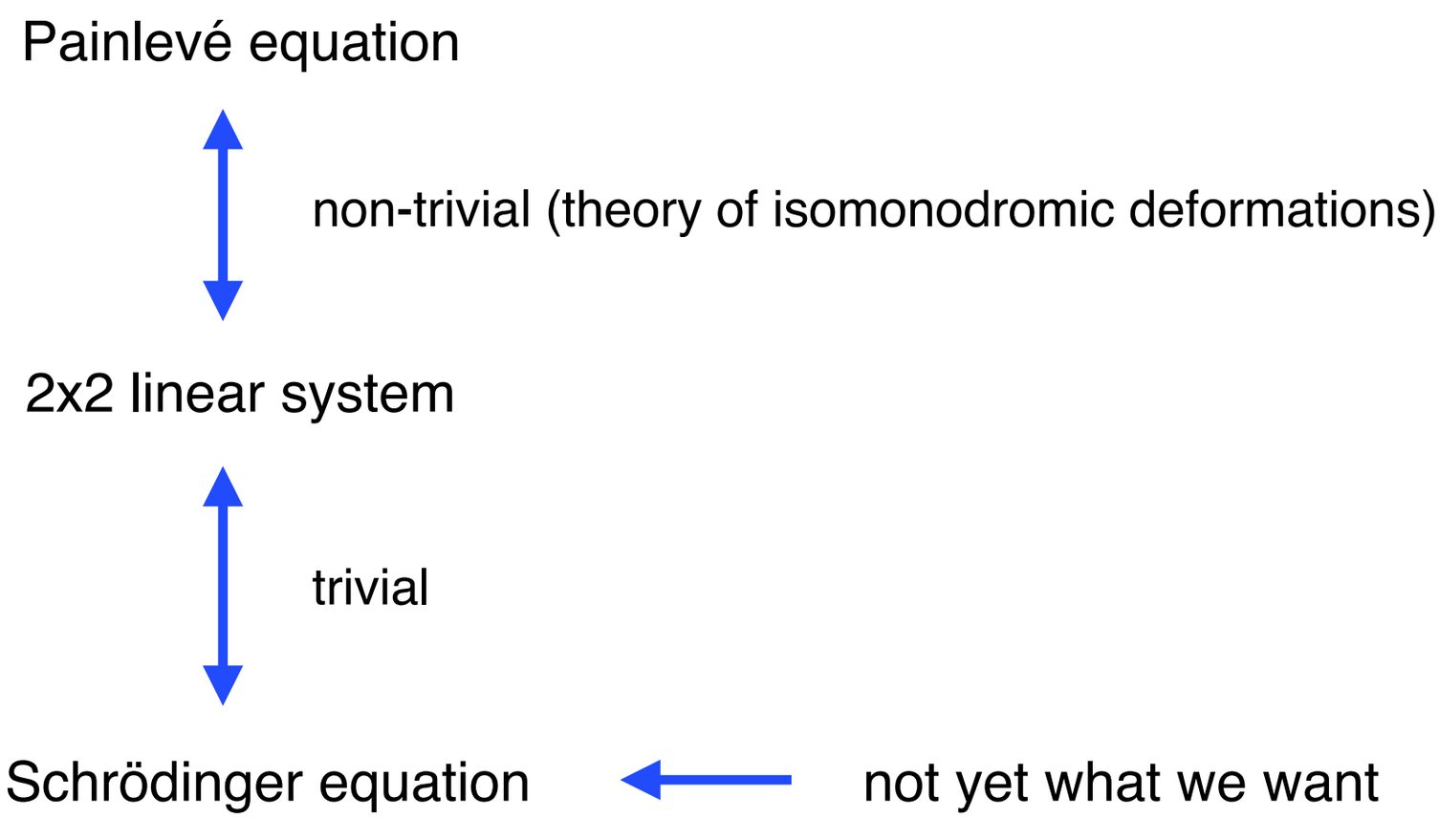
Distribution of movable poles in Painlevé III_3 $\sim$ distribution of eigenvalues in (modified) Mathieu

Masoero 2009 + Masoero, Roffelsen 2018 + . . . :

Distribution of movable poles in Painlevé I, II $\sim$ distribution of eigenvalues in cubic, quartic potential

Q: can we understand this relation systematically by using the underlying $\mathcal{N} = 2$ theories?

how do we associate a Schrödinger equation to each Painlevé equation?



Start from a 2 x 2 linear system

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} Y_1(z) \\ Y_2(z) \end{pmatrix} = \begin{pmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{pmatrix}\begin{pmatrix} Y_1(z) \\ Y_2(z) \end{pmatrix} \quad \rightarrow \quad \left.\begin{array}{l} Y_2 = f_1\left(Y_1, \frac{\mathrm{d}}{\mathrm{dz}} Y_1\right) \\ \frac{\mathrm{d}}{\mathrm{dz}} Y_2 = f_2\left(Y_1, Y_2\right) \end{array}\right\}$$

2nd order equation for Y_1

Define $\tilde{Y}_1(z) = Y_1(z)/\sqrt{A_{12}(z)}$ $\rightarrow$ $\left(-\partial_z^2 + W(z)\right)\widetilde{Y}_1(z) = 0$

where $W(z) = \left(-\det A + A'_{11} - \frac{A_{11}A'_{12}}{A_{12}} - \frac{2A_{12}A''_{12} - 3(A'_{12})^2}{4A_{12}^2}\right)$

Suppose $A_{12}(z)$ has zero at $z = z_0$ $\rightarrow$ Additional singularity at $z = z_0$

We want to avoid this (singularity matching condition).

In Painlevé context this leads to:

$\rightarrow$ $W(z)$ reduces to $U(z)$

$\rightarrow$ vanishing condition for (combination of) τ functions.

Example:

$$\mathrm{PIII}_3: \quad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{2q^2}{t^2} - \frac{2}{t}$$

The associated 2 x 2 linear system is (see Fokas, Its, Kapaev, Novokshenov)

$$A(z) = z^{-2}\begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} + z^{-1}\begin{pmatrix} -p/q & t/q \\ -1 & p/q \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where $q = q(t, \sigma, \eta)$ solves PIII_3 and $p = p(t, \sigma, \eta) = \frac{t}{2}\frac{\mathrm{d}q}{\mathrm{d}t} - \frac{q}{2}$

The potential is ($z = \mathrm{e}^x$) :

$$W(x) = \frac{p^2 + pq - q\left(t + q^2\right)}{q^2} + \frac{t(p+q)}{q(q\mathrm{e}^x - t)} + \frac{3t^2}{4(t - q\mathrm{e}^x)^2} + \frac{t}{\mathrm{e}^x} + \mathrm{e}^x + \frac{1}{4}$$

we have $A_{12}(z) = \dfrac{t}{qz} - 1$ → zeros at $z = \mathrm{e}^x = t/q$

→ avoid additional singularities if $q = 0$ or $q = \infty$

recall $q(t, \sigma, \eta) \sim \left(\dfrac{\tau^{\mathrm{GIL}}(t, \sigma, \eta)}{\tau^{\mathrm{GIL}}(t, \sigma + \frac{1}{2}, \eta)} \right)^2$

$\tau^{\mathrm{GIL}}(t, \sigma + \dfrac{1}{2}, \eta) = 0$

$$\left(-\partial_x^2 + W(x)\right)\tilde{Y}_1(x) = 0 \quad \longrightarrow \quad (1) \quad \left(-\partial_x^2 + \sqrt{t}\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right) + E(t)\right)\tilde{Y}_1(x) = 0$$

where $\quad E(t) = t\frac{\mathrm{d}}{\mathrm{dt}}\log\tau^{\mathrm{GIL}}(t,\sigma,\eta)$

$$\longrightarrow \quad (2) \quad \tau^{\mathrm{GIL}}(t,\sigma+\frac{1}{2},\eta) = 0$$

Suppose $x \in \mathbb{R}$, $\quad t \in \mathbb{R}_+ :\ \tilde{Y}_1(x) \in L^2(\mathbb{R}) \quad \longleftrightarrow \quad \eta = 0$

Hence, by using the Painlevé framework we can easily derive the exact spectrum of modified Mathieu as:

$$E_n(t) = t\frac{\mathrm{d}}{\mathrm{dt}}\log\tau^{\mathrm{GIL}}(t,\sigma_n,0) \qquad \text{where} \quad \tau^{\mathrm{GIL}}(t,\sigma_n+\frac{1}{2},0) = 0$$

Recall: $\tau^{\mathrm{GIL}}(t,\sigma,\eta)$ is computed exactly by using Kiev formula and in particular the self-dual phase of the Ω background ($\epsilon_1 = -\epsilon_2$).

AG, Gu, Mariño 2019- Gavrylenko, Marshakov, Stoyan 2020 - Bershtein, Gavrylenko, AG 2021

This can be generalised to the full Painlevé diagram.

For example:

elliptic form of PVI $\longleftrightarrow$ quantum Calogero-Moser operator

($\mathcal{N} = 2^*$ theory)

Bershtein, Gavrylenko, AG 2021

Painlevé V $\longleftrightarrow$ confluent Heun equation

($N_f = 3$ theory)

Cunha,Cavalcante 2021

. . . .

→ we can exploit the recent progress in Painlevé equations

to obtain exact results for Schrödinger operators.

Comments:

The Schrödinger operators arising in this construction can also be obtained from the quantization of **four** dimensional Seiberg-Witten curves.

There is an alternative way to study the spectrum of these operators by using the Nekrasov-Shatashvili phase $\epsilon_2 = 0,\ \epsilon_1 = \hbar$ of the Ω background.

We can relate these two phases of the Ω background by using Nakajima-Yoshioka blowup relations.

AG, Gu 2016 - AG, Mariño, Gu 2019 - Gavrylenko, Marshakov, Stoyan 2020 - Nekrasov 2020 - Nekrasov, Jeong 2020 - Bershtein, Gavrylenko, AG 2021

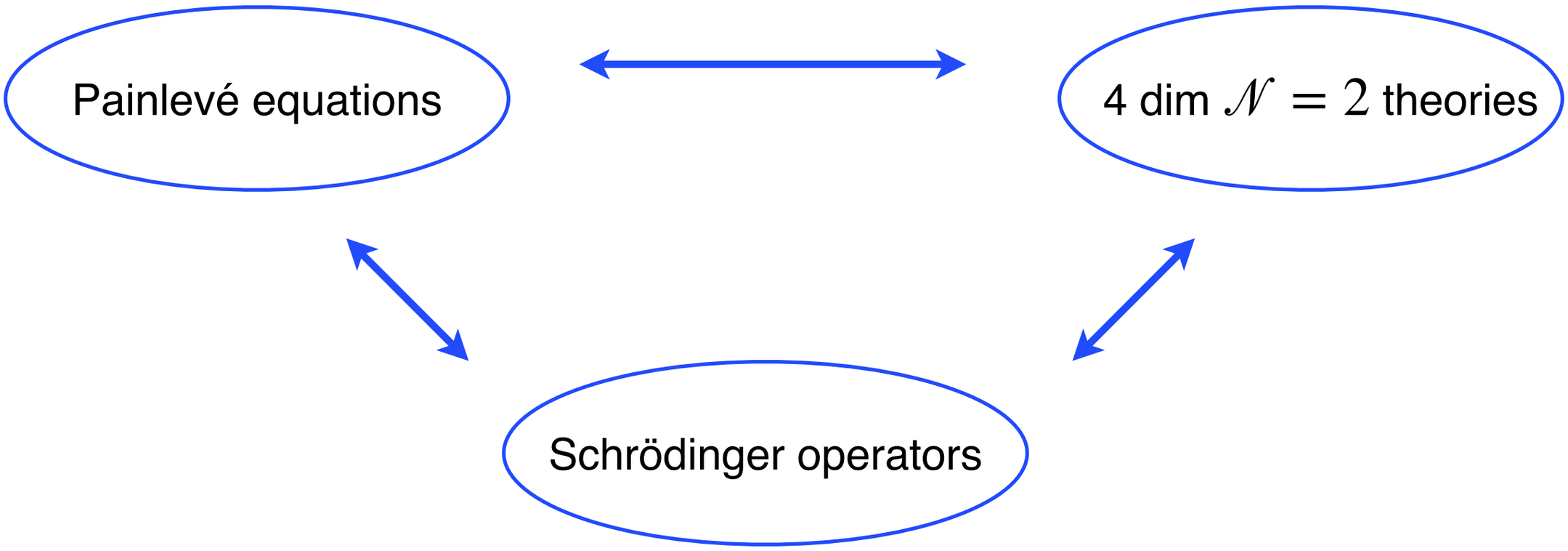
Comments:

An interesting aspect of the Painlevé approach is the generalisation to the five dimensional picture and the framework of difference equations (or relativistic integrable systems).

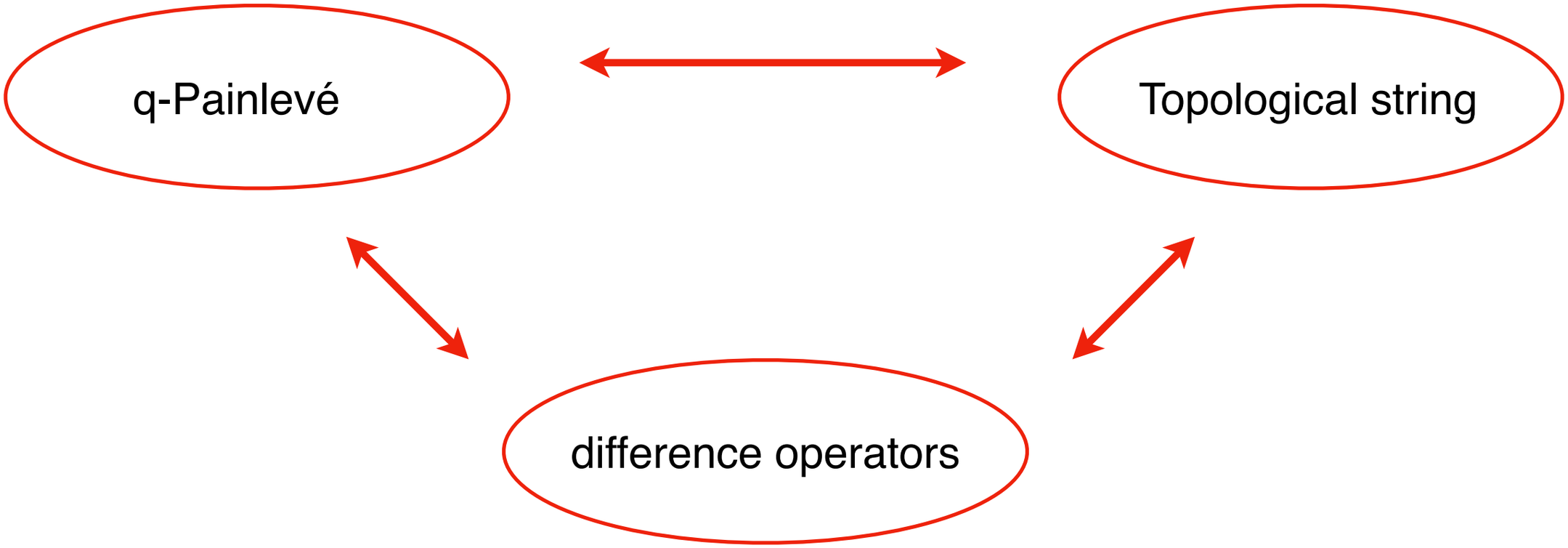
Indeed in this context the Nekrasov-Shatashvili uplift to five dimensions needs additional non-perturbative corrections. [AG, Hatsuda, Mariño 2014]

Such corrections arise naturally in the Painlevé approach.

What we discussed so far:

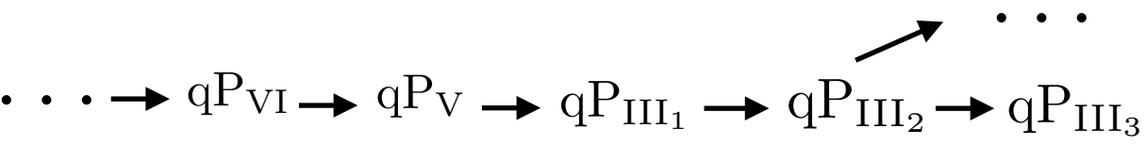


There is an interesting generalisation of this framework to topological string theory, q-deformed Painlevé equations and difference operators.

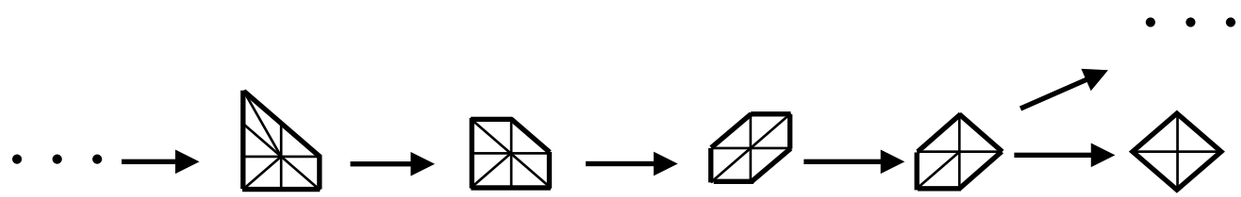


Bershtein, Shchechkin 2016 - Bonelli, AG, Tanzini 2017 - Bershtein, Gavrylenko, Marshakov 2017- Jimbo, Nagoya, Sakai 2017- Matsuhira, Nagoya 2018- Bonelli, Del Monte, Tanzini 2020 - Nosaka 2021- Moriyama, Yamada 2021, ….

q- deformed Painlevé equations



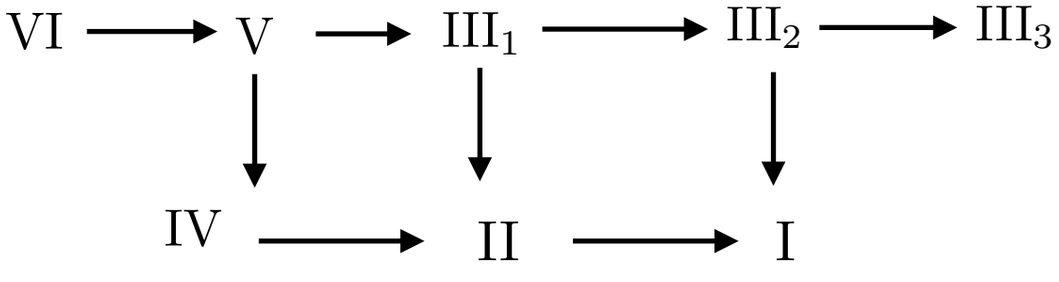
Topological string on toric CY3



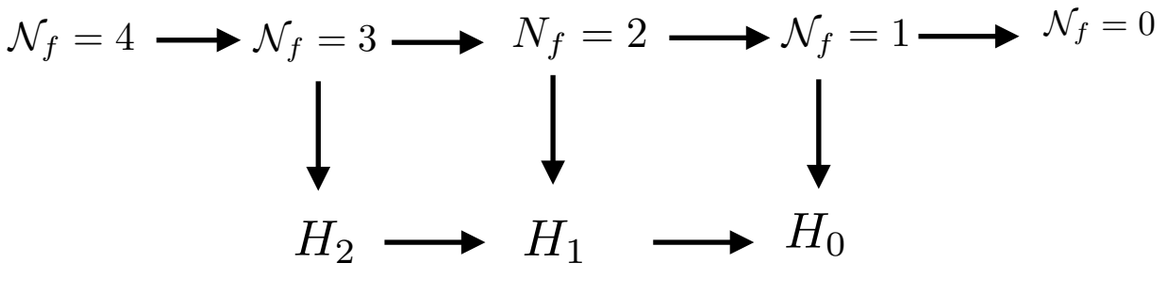
Ramani, Grammaticos - Jimbo, Sakai - Sakai - Yamada et al …

Klemm, Lerche, Mayr, Vafa- Katz, Klemm, Vafa - …

Painlevé equations



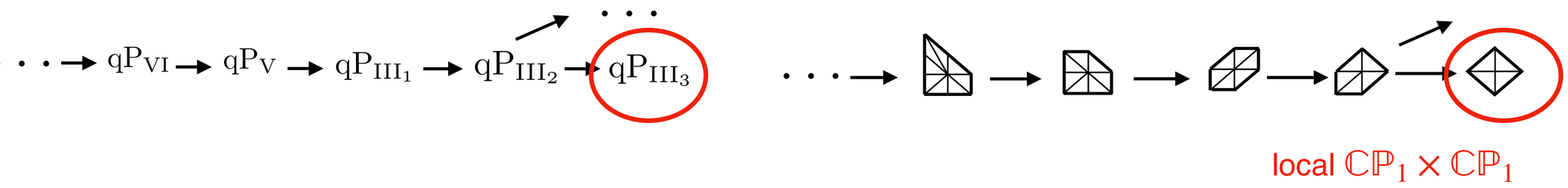
4d SU(2) Seiberg-Witten theory



Claim: Fredholm determinant of quantum mirror curves to such geometries solves a corresponding q-Painlevé

[Bonelli, AG, Tanzini 2017]

Example:



The quantum mirror curve in this example is

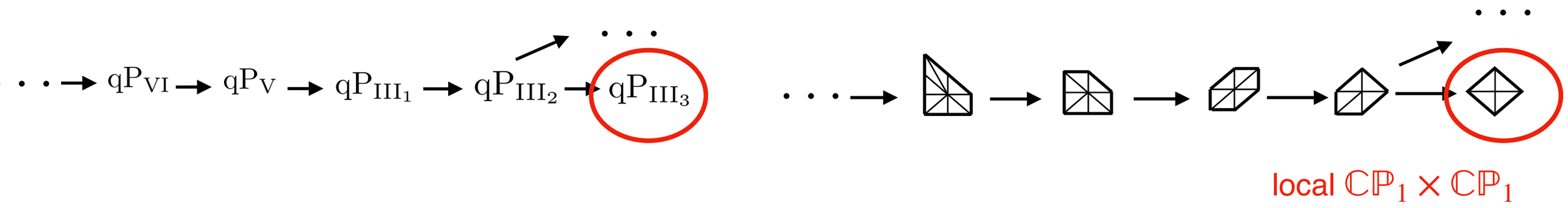
$$\mathrm{O} = \mathrm{e}^{\hat{p}} + \mathrm{e}^{-\hat{p}} + m\mathrm{e}^{\hat{x}} + \mathrm{e}^{-\hat{x}}, \qquad [\hat{x}, \hat{p}] = \mathrm{i}\hbar$$

[Hori, Vafa - Batyrev -Aganagic, Dijkgraaf, Klemm, Mariño, Vafa - Mironov, Morozov - Nekrasov, Shatashvili -…]

such that

$$\mathrm{O}\psi(x) = \psi(x + \mathrm{i}\hbar) + \psi(x - \mathrm{i}\hbar) + (m\mathrm{e}^{x} + \mathrm{e}^{-x})\psi(x)$$

Example:



The kernel of the inverse operator $\rho = \mathrm{O}^{-1}$ on $L^2(\mathbb{R})$ is

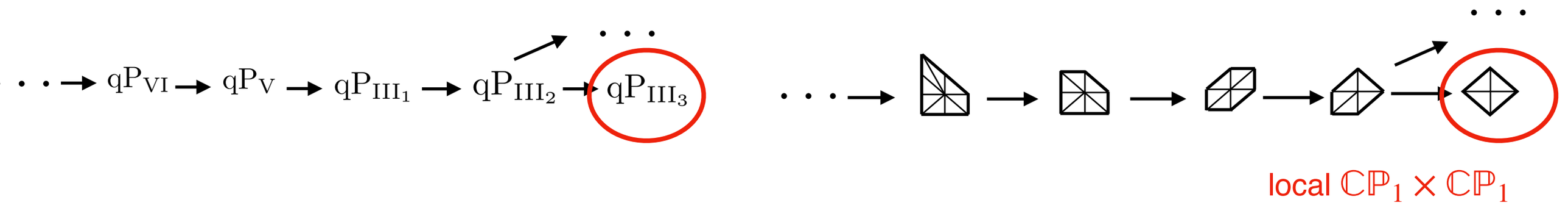
$$\rho(x,y) = \frac{\mathrm{e}^{-u(x,m,\hbar)-u(y,m,\hbar)}}{4\pi \cosh\left(\frac{x-y}{2}\right)}$$

where $$u(x,m,\hbar) \sim -\log\left|\frac{\phi_{\mathrm{b}}\left(\frac{bx}{2\pi} - \frac{1}{2\pi b}\log m + ib/4\right)}{\phi_{\mathrm{b}}\left(\frac{bx}{2\pi} + \frac{1}{2\pi b}\log m - ib/4\right)}\right|, \quad b^2 = \hbar$$

ϕ_{b} : Faddeev quantum dilogarithm

[Kashaev, Mariño, Zakany 2015]

Example:



Theorem: ρ has a discrete spectrum $\{E_n\}_{n\geq 0}$ on $L^2(\mathbb{R})$ and it is of trace class.

AG, Hatsuda, Mariño 2014

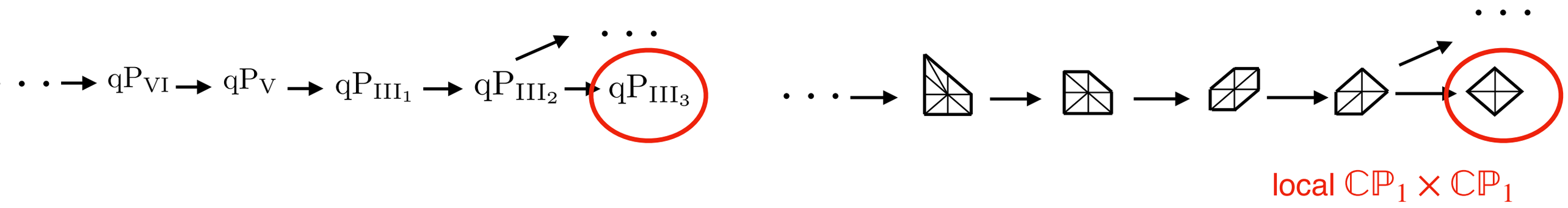
Kashaev, Mariño 2015

Laptev, Schwimmer, Takhtajan 2015

Hence we can define the Fredholm determinant as:

$$\det\left(1+\kappa\,\rho\right)=\prod_{n\geq 0}\left(1+\kappa E_n\right)$$

Example:

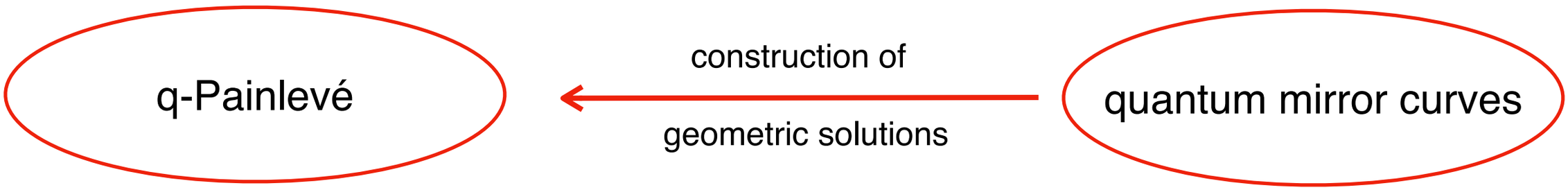


The Fredholm determinant $\tau_q(\kappa, \xi) \sim \det(1 + \kappa\ \rho)$

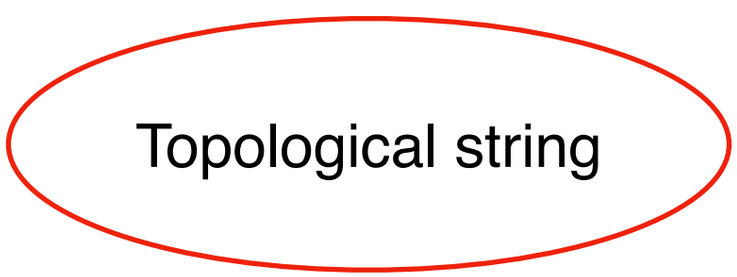
where $q = \mathrm{e}^{\frac{4\pi^2 \mathrm{i}}{\hbar}}$, $\xi = \log m$, solves q-Painlevé III_3

$$\tau_q\left(-\kappa, \xi - \frac{4\pi^2\mathrm{i}}{\hbar}\right)\tau_q\left(-\kappa, \xi + \frac{4\pi^2\mathrm{i}}{\hbar}\right)\left(1 + \mathrm{e}^{-\xi/2}\right) = \tau_q(\kappa, \xi)^2 + \mathrm{e}^{-\xi/2}\tau_q(-\kappa, \xi)^2$$

[Bonelli, AG, Tanzini 2017]


q-Painlevé
construction of
geometric solutions
quantum mirror curves

Next:


Topological string

Fredholm determinants of quantum mirror curves play a central role in a non-perturbative definition of topological string on toric CY3 [AG, Hatsuda, Mariño 2014]

$$Z_X(N,\hbar) = \oint \mathrm{d}\kappa \, \det(1+\kappa\rho_X)\kappa^{1-N}$$

$$\log Z_X(N,\hbar) \xrightarrow[t=\frac{N}{\hbar} \quad \text{fixed}]{\hbar\ ,\ N\to\infty} \sum_{g\geq 0} F_g^X(t)\ \hbar^{2-2g}$$

F_g^X : genus g free energy of topological string on X

Conclusions

There is a deep connection between Painlevé equations, Seiberg-Witten theories, quantum operators and their respective generalisations. Many results have already been obtained but the full picture is still under construction.

→ topological string (e.g. proofs, extensions, connection with BPS quivers, long distance expansions, …)

→ PI , PII, PIV (e.g. poles vs spectrum, applications to AD theories . .)

. . .

Thank you!