Fukaya categories of Landau-Ginzburg models and Homological Mirror Symmetry

Denis Auroux (Harvard)

June 14, 2021 — String Math 2021 (IMPA)

Plan:
1. Fukaya categories of Landau-Ginzburg models
2. Homological mirror symmetry
3. Spherical functors

- part based on ideas of Seidel, Abouzaid, Ganatra, Sylvan, Hanlon, Jeffs, ...
- HMS: joint work in progress (?) with Abouzaid

essentially the same picture, in a different language: Nadler, Ganatra-Pardon-Shende, Gammage, ...
[further afield: Kontsevich, Soibelman, Kapranov, ... physics: Hori-Iqbal-Vafa, Gaiotto-Moore-Witten, ...]

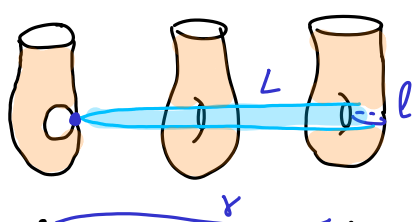
- **Symplectic Landau-Ginzburg model** := (Y,ω) symplectic, convex at infinity
  + $W: Y \to \mathbb{C}$ st. fibers $F_t = W^{-1}(t)$ are symplectic submanifolds (outside of critical locus)

  for example Y Kähler (complete) (quasiprojective), W holomorphic (regular function)

  $\mathcal{F}(Y,W)$ = Lagrangian submanifolds of Y rel. $W^{-1}(+\infty)$.

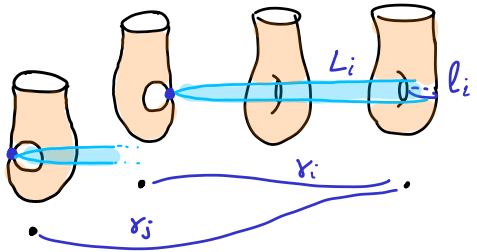
- Symplectic parallel transport along horizontal distribution $\mathcal{H} = (\ker dW)^{\perp_\omega} = \mathrm{span}(X_{\mathrm{Re}\,W}, X_{\mathrm{Im}\,W})$ over a path $\gamma: [0,1] \to \mathbb{C}$ avoiding $\mathrm{crit\,val}(W)$ gives symplectomorphisms $F_{\gamma(0)} \xrightarrow{\sim} F_{\gamma(1)}$
- transporting a Lagrangian $\ell \subset F_{\gamma(0)}$ gives a fibered Lagrangian $L \subset Y$ over γ.
- $\ell \subset F_{t_0}$ is a Lagrangian vanishing cycle for path $\gamma: t_0 \rightsquigarrow$ critval if parallel transport collapses ℓ entirely into $\mathrm{crit}(W)$.

  the fibered Lagrangian is then called a thimble.



Ex: **Lefschetz fibrations:** $\mathrm{Crit}(W)$ isolated nondegenerate (local model: $\mathbb{C}^n \xrightarrow{\sum z_i^2} \mathbb{C}$)
sing. fibers have ordinary double points; vanishing cycle = Lagrangian $S^{n-1} \subset F$.

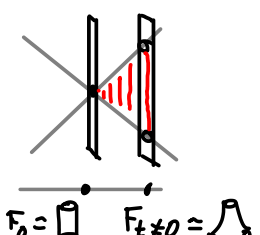
Seidel: the Fukaya cat. $\mathcal{F}(Y, W)$ of a Lefschetz fibration is generated by an exceptional collection of Lefschetz thimbles.



Our favorite examples, however, aren't Lefschetz fibrations:

- $\mathrm{Crit}(W)$ need not be isolated, or even proper. eg. $(\mathbb{C}^3, W = -z_1 z_2 z_3)$,
  $F_{t\neq 0} \simeq (\mathbb{C}^*)^2$, $F_0 = \bigcup_3 \mathbb{C}^2$ singular at $\mathrm{Crit}\, W = \bigcup_3 \mathbb{C}$.

- W may have "critical points at ∞" eg. $(\mathbb{C}^*)^2 \xrightarrow{W = z_1 + z_2} \mathbb{C}$



3]

The objects of the Fukaya category $\mathcal{F}(Y,W)$ are admissible Lagrangians

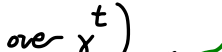
ie. properly embedded (or immersed) $L \hookrightarrow Y$ disjoint from the stop $Y_{-\infty} = \{\mathrm{Re}\, W \ll 0\}$

- with good control over holomorphic curves (maximum principle at ∂)
- unobstructed (no holom. discs with ∂ in L, or cancel by bounding cochain)
- equipped with spin structure, local system, grading, ...

Morphisms: $\hom(L,L') = \varinjlim CF(L^t, L')$, L^t = push L by a positive Hamiltonian isotopy: $\begin{cases} \to \text{ increase } \arg(W) \text{ on ends of } L, \text{ without crossing the stop.} \\ \to \text{ if } L \text{ isn't fiberwise proper, fiberwise wrapping.} \end{cases}$

There are many competing definitions – 3 main flavors:

(1) (Seidel, Abouzaid, ...): assume L is fibered & fiberwise proper outside a compact subset

(eg. thimbles in Lefschetz fibrations: L^t fibers over γ^t)  (Seidel)

(This suffices if the fibration is loc. trivial at infinity
then fiberwise proper objects generate and fiberwise wrapping isn't needed).

(2) When Y is exact Liouville (eg. affine variety), can use Sylvan's partially wrapped Fukaya category or view $Y \setminus Y_{-\infty}$ as a sector (Ganatra-Pardon-Shende)

ie: assume L is conical at infinity & modify the Hamiltonian perturbations of wrapped Floer theory to avoid crossing the stop.

* This is the most versatile, but the non-exact case hasn't been developed yet
* we don't consider arbitrary sectors - the stops of LG models are swappable
* Wrapping is usually NOT consistent with the fibration W: L^t not necess. fibered at ∞.

(3) Monomial admissibility (Hanlon, Abouzaid-A.) (so far: Y toric)

ie: consider a finite collection of distinguished functions (eg. toric monomials) z^ν s.t. $\{z^\nu\}: Y \to \mathbb{C}^N$ is proper, and cover $Y \setminus \text{compact} = \bigcup U_\nu$ (eg. $U_\nu = \{|z^\nu| >> 1\}$)

+ require L to be admissible, ie. $\arg(z^\nu_{|L \cap U_\nu})$ = locally constant at ∞.

Hamiltonian perturbations: L^t = increase value of $\arg(z^\nu)$ at ∞.

★ Monomial admissibility makes it possible to consider decompositions of W into a sum of two terms – a "main" term w_0, and an "auxiliary" term w_F defining stops on the fibers of w_0. Ie., we decompose the stop of (Y, W) into two components, and explore the geometry of Y relative to the part of the stop which corresponds to w_0. We change notation and consider F = fiber of w_0, rather than the whole fiber/stop.

★ Equivalently: consider Landau-Ginzburg models / sectors w/ swappable stops whose fibers are themselves L-G. models / sectors (F, w_F).

Requirement: parallel transport between fibers of w_0 preserves the fiberwise stop & admissibility of Lagrangians in F (eg. preserve $\arg(w_F)$ at $|w_F| \to \infty$).

Conj: can make a similar definition outside of toric setting & the resulting "two-stage category" (whose objects are fibered Lagrangians wrt w_0, rather than $w_0 + w_F$), agrees with other versions.

6)

The fiberwise wrapped Fukaya category of $(Y, w_0 + w_F)$

(Y toric, w_0 monomial)

(Abouzaid-A., see also Hanlon)

Objects: properly embedded Lagrangians $L \subset Y$ (+ extra data: spin str., grading, ...)

which are **unobstructed + monomially admissible**: (cf. A. Hanlon's thesis)

1) for $|w_0| \gg 1$, $\arg(w_0|_L)$ is loc. constant $\in (-\frac{\pi}{2}, \frac{\pi}{2})$ (ie. $w_0|_L \in$ union of radial arcs)

2) inside the fibers F_t of w_0 (again toric!), impose admissibility w.r.t. a collection of toric monomials z^ν (incl. all terms in w_F), ie. $\arg(z^\nu)$ = loc. constant over subsets U_ν.

Note:
- monomial admissibility gives control over discs in Floer products via maximum principle
- use a specific toric Kähler form for which $\{\log w_0, \log z^\nu\}$ Poisson-commute in U_ν.

L admissible $\rightsquigarrow$ flow L^t (Ham. isotopic to L; admissible)

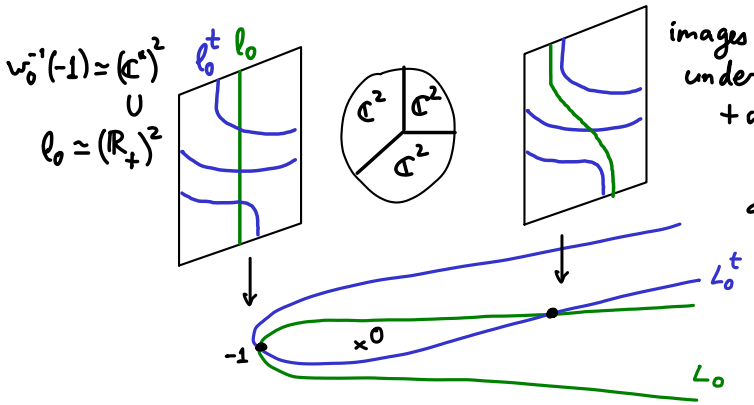
The flow increases the values of $\arg(w_0)$ and $\arg(z^\nu)_{|U_\nu}$ at ∞.

(within $\arg \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for w_0 and terms of w_F: STOP at $-\infty$; else $\arg \uparrow \infty$: WRAP)

Define $\hom(L_0, L_1) := \varinjlim_{t \to \infty} CF^*(L_0^t, L_1)$ under natural continuation maps.

Example: $(\mathbb{C}^3, w_0 = -z_1 z_2 z_3)$ (mirror of $= \{(x_1,x_2) \in (\mathbb{C}^*)^2 \mid 1 + x_1 + x_2 = 0\}$).

L_0 = parallel transport $\ell_0 = (\mathbb{R}_+)^2 \subset (\mathbb{C}^*)^2 \simeq w_0^{-1}(-1)$ along U-shaped arc.



$$\hom(L_0, L_0) \simeq CW^*(\ell_0, \ell_0) \oplus CW^*(\ell_0, \ell_0)[-1]$$

$\Bbbk[x_1^{\pm 1}, x_2^{\pm 1}]$ ∂ = multiplication by $1 + x_1 + x_2$

(Abouzaid-A.)

$$H^* \hom(L_0, L_0) \underset{\text{(ring iso.)}}{\simeq} \Bbbk[x_1^{\pm 1}, x_2^{\pm 1}]/(1 + x_1 + x_2) \simeq \hom(\mathcal{O}, \mathcal{O}) \text{ in } D^b\mathrm{Coh}\left(\text{ }\right) \quad (\rightsquigarrow \text{HMS})$$

(similarly for general hypersurfaces $f^{-1}(0) \subset (\mathbb{C}^*)^n$ and their toric LG mirrors, get $\Bbbk[x_i^{\pm 1}]/(f)$)

Construction: (Abouzaid - A. - Katzarkov, see earlier work of Hori-Vafa, P. Clarke, ..., also Chan-Lau-Leung, ...)

$H = f^{-1}(0) \subset (\mathbb{K}^*)^n$, $f(x) = \sum_{\alpha \in A \subset \mathbb{Z}^n} c_\alpha t^{\rho(\alpha)} x^\alpha$ Laurent polynomial $(t \to 0)$, $\rho(\alpha) \in \mathbb{R}$

↑ this means $x_1^{\alpha_1} \dots x_n^{\alpha_n}$

or $H = f^{-1}(0) \subset V$ toric var., f section of $\mathcal{L} \to V$ with associated polytope $\simeq$ Conv Hull (A)

$\rightsquigarrow$ to find mirror (Y, W), let $\varphi = \mathrm{Trop}(f): \mathbb{R}^n \to \mathbb{R}$, $\varphi(\xi) = \max_{\alpha \in A} \left(\langle \alpha, \xi \rangle - \rho(\alpha) \right)$.

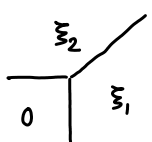
Let Y = toric Kähler (CY) var. with moment polytope $\Delta_Y = \{(\xi, \eta) \in \mathbb{R}^{n+1} \mid \eta \geq \varphi(\xi)\}$.

and $W = w_0 = -z^{(0,\dots,0,1)}$ toric monomial vanishing to order 1 on each toric divisor $\subset Y$.

resp $W = w_0 + w_F$: w_F = one monomial for each toric divisor of V)

Ex:

$H: \{1 + x_1 + x_2 = 0\} \rightsquigarrow$



$\Delta_Y: \eta \geq \max(0, \xi_1, \xi_2)$

$Y \simeq \mathbb{C}^3$, $w_0 = -z_1 z_2 z_3$

$\{x_0 + x_1 + x_2 = 0\} \subset \mathbb{P}^2 \rightsquigarrow$ $(\mathbb{C}^3, \underbrace{-z_1 z_2 z_3}_{w_0} + \underbrace{T(z_1 + z_2 + z_3)}_{w_F})$

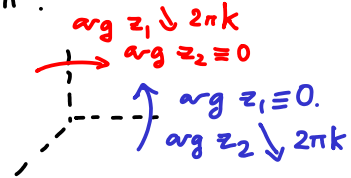
9)

Example (cont.): $(\mathbb{C}^3, w_0 + w_F = -z_1z_2z_3 + T(z_1+z_2+z_3))$ (mirror of $H: x_0+x_1+x_2=0 \subset \mathbb{P}^2$)

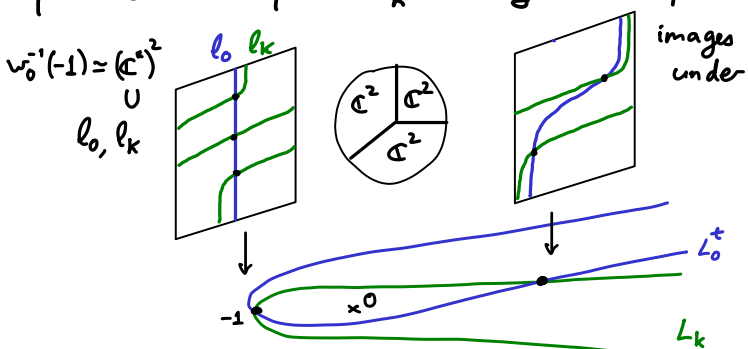
Fibers $\{w_0 = -c\}$ are $\simeq \left((\mathbb{C}^*)^2, w_F = T\left(z_1 + z_2 + \frac{c}{z_1 z_2}\right)\right) \simeq$ mirror to $\mathbb{P}^2$.

Start with $\ell_k \subset w_0^{-1}(-1) \simeq (\mathbb{C}^*)^2$, $\arg(z_i) = f_i(\log|z|)$ twist by $2\pi k$ across

admissible wrt w_F, mirror to $\mathcal{O}_{\mathbb{P}^2}(k)$. $HF^*(\ell_0, \ell_k) \simeq H^*(\mathcal{O}_{\mathbb{P}^2}(k))$



L_k = parallel transport ℓ_k along U-shaped arc.



images of ℓ_0, ℓ_k under monodromy $\simeq \ell_0, \ell_{k-1}$ ← The monodromy of w_0 around origin is mirror to $-\otimes \mathcal{O}(1)$ [in general, $-\otimes \mathcal{O}(H)$]

$\Rightarrow hom(L_0, L_k) \simeq$

$CF^*(\ell_0, \ell_k) \oplus CF^*(\ell_0, \ell_{k-1})[-1]$

$\cdot(x_0+x_1+x_2)$

(in general, $\times f =$ def. section of H)

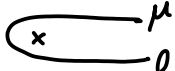
$\Rightarrow H^0 hom(L_0, L_k) = H^0(\mathbb{P}^2, \mathcal{O}(k))/(x_0+x_1+x_2) \cong H^0\left(\overset{H}{\underset{\{x_0+x_1+x_2=0\}}{\bigcirc}}, \mathcal{O}(k)\right)$.

(Abuzaid-A.) (see also Cannizzo!)

In general: match U-shaped Lagrangians $\longleftrightarrow \mathcal{L}_{|H}$ for $\mathcal{L} \to V$ line bundle. ($\Rightarrow$ HMS mod generation)

The Fukaya categories of (Y, w_0+w_F) and $(F=w_0^{-1}(t), w_F)$ are related by functors

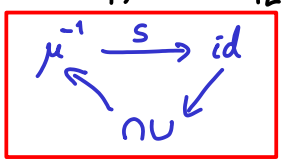
monodromy of w_0 around origin $\mu \circlearrowright \mathcal{F}(F, w_F) \underset{\cap}{\overset{\cup}{\rightleftarrows}} \mathcal{F}(Y, w_0+w_F) \circlearrowright \sigma$ wrap once past the stop $w_0 \to -\infty$. (spherical functor)

$\cup \ell$ = parallel-transport $\ell \subset F$ (admissible) along U-shape 

$=: \cup \ell$

$\cap L$ = ends of $L \subset Y$ at $w_0 \to \infty$ (actually $\in Tw\, \mathcal{F}(F, w_F)$ if $w_0|_L$ has more than one end).

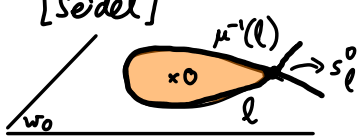
+ exact triangle of functors on $\mathcal{F}(F, w_F)$:



(Abouzaid-Ganatra, Sylvan).

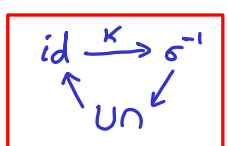
where S = section-counting natural transformation from μ^{-1} to id [Seidel]

$\forall \ell \subset F$, $s^0_\ell \in CF^0(\mu^{-1}(\ell), \ell)$ counts holom. sections of w_0 over

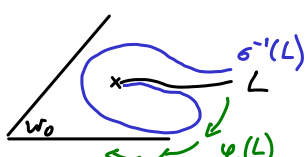


+ exact triangle of functors on $\mathcal{F}(Y, w_0+w_F)$:

κ = continuation element for Hamiltonian isotopy $id \longrightarrow \sigma^{-1}$



ie. κ^0_L counts holom. discs



monodromy of w_0 around origin $\mu \circlearrowright \mathcal{F}(F, w_F) \underset{\cap}{\overset{\cup}{\rightleftarrows}} \mathcal{F}(Y, w_0+w_F) \circlearrowleft \sigma$ wrap once past the stop $w_0 \to -\infty$. (spherical functor)

This yields proofs of homological mirror symmetry & its functoriality in at least 2 settings:

① HMS for log Fano (X, D_0), $D_0 + D' = -K_X$

$\rightsquigarrow (Y, W = w_0 + w_F)$ mirror to X [toric case: Hanlon-Hicks]

$(F = w_0^{-1}(t), w_F)$ mirror to D_0

$$\begin{array}{ccc} \mu \circlearrowright \mathcal{F}(F, w_F) & \underset{\cap}{\overset{\cup}{\rightleftarrows}} & \mathcal{F}(Y, w_0+w_F) \circlearrowleft \sigma \\ \text{HMS for } D_0 \Big\| & & \Big\| \text{ HMS for } X \\ \mathrm{Perf}(D_0) & \underset{i^*}{\overset{i_*}{\rightleftarrows}} & \mathrm{Perf}(X) \end{array}$$

Ex: $\mathcal{F}\left((\mathbb{C}^*)^2, z_1+z_2+\frac{q}{z_1 z_2}\right) \rightleftarrows \mathcal{F}\left(\triangle\right)$ mirror to $x_0x_1x_2 = 0$

mirror to $\mathbb{P}^2$: $O(1) \to 0 \to O(-1) \cdots$

or $\mathcal{F}\left(\triangle, w_F\right) \leftrightarrow$ (stops)

② HMS for hypersurfaces $H \subset V$ [Abouzaid-A.]

$\rightsquigarrow (Y, W = w_0 + w_F)$ mirror to H

$(F = w_0^{-1}(t), w_F)$ mirror to V

$$\begin{array}{ccc} \mu \circlearrowright \mathcal{F}(F, w_F) & \underset{\cap}{\overset{\cup}{\rightleftarrows}} & \mathcal{F}(Y, w_0+w_F) \circlearrowleft \sigma \\ \text{HMS for } V \Big\| & & \Big\| \text{ HMS for } H \\ \mathrm{Perf}(V) & \underset{i_*}{\overset{i^*}{\rightleftarrows}} & \mathrm{Perf}(H) \end{array}$$

Ex: $\mathcal{F}(\mathbb{C}^3, -z_1z_2z_3) \rightleftarrows W\left((\mathbb{C}^*)^2\right)$

mirror to $\{1+x_1+x_2=0\}$ mirror to $(\mathbb{C}^*)^2$

or compactification to $\subset \mathbb{P}^2$:

$\mathcal{F}(\mathbb{C}^3, -z_1z_2z_3 + T(z_1+z_2+z_3)) \rightleftarrows \mathcal{F}\left((\mathbb{C}^*)^2, z_1+z_2+\frac{q}{z_1z_2}\right)$

- In this language, the above calculation for mirror (Y, w_0+w_F) of hypersurface $H \subset V$ is:

$$\hom_Y(Ul, Ul') \simeq \hom_F(l, \cap Ul') \simeq \mathrm{Cone}\left(\hom_F(l, \mu^{-1}(l')) \xrightarrow{s} \hom_F(l, l')\right)$$

& homological mirror symmetry is proved by matching this with $D^b\mathrm{Coh}(H) \underset{i^*}{\overset{i_*}{\rightleftarrows}} D^b\mathrm{Coh}(V)$

$$\hom_H(i^*\mathcal{L}, i^*\mathcal{L}') \simeq \hom_V(\mathcal{L}, i_*i^*\mathcal{L}') \simeq \mathrm{Cone}\left(\hom_V(\mathcal{L}, \mathcal{L}' \otimes \mathcal{O}(-H)) \xrightarrow{f} \hom_V(\mathcal{L}, \mathcal{L}')\right)$$

[Abouzaid-A., see also Cannizzo].

- The two functors $\cup$, $\cap$ are defined differently, but play almost symmetric roles spherical functor package. Conj: Fiber and total space can be swapped around by stabilization: $(Y, W = w_0 + w_F)$ $F = w_0^{-1}(t)$ $(t \gg 0)$ $\rightsquigarrow$ $\left(\tilde{Y} = Y \times \underset{w}{\underset{z}{\mathbb{C}}}, \tilde{W} = z(1 - t^{-1}w_0) + w_F\right)$
  - "A-model Knörrer periodicity" (M. Jeffs) $\Rightarrow \mathcal{F}(Y \times \mathbb{C}, \tilde{W}) \simeq \mathcal{F}(F, w_F)$
  - the levels of $\tilde{w}_0 := z$ are $\simeq (Y, w_0 + w_F)$ by considering thimbles for $z(1 - t^{-1}w_0)$ (Morse-Bott along $F \times \{0\}$)

Localizations / quotients:

monodromy around 0 $\mu \circlearrowright \mathcal{F}(F, w_F) \underset{\cap}{\overset{\cup}{\rightleftarrows}} \mathcal{F}(Y, w_0 + w_F) \circlearrowleft \sigma$ wrap once past stop

$\cup \ell$ = parallel-transport $\ell \subset F$ (admissible) along $\cup$-shape, $\cap L$ = ends of $L \subset Y$ at $w_0 \to \infty$

On $\mathcal{F}(F, w_F)$, $\cap \cup \simeq \mathrm{Cone}(\mu^{-1} \xrightarrow{S} \mathrm{id})$ On $\mathcal{F}(Y, w_0 + w_F)$, $\cup \cap \simeq \mathrm{Cone}(\mathrm{id} \xrightarrow{K} \sigma^{-1})$

---

① Localizing $\mathcal{F}(Y, w_0 + w_F)$ wrt $\mathrm{id} \xrightarrow{K} \sigma^{-1}$ $\Leftrightarrow$ quotient by $\mathrm{Im}(\cup)$ $\Leftrightarrow$ stop removal at w_0

(Abouzaid-Seidel, Sylvan, GPS)

yields $\mathcal{F}(Y, w_F)$ (if no fiberwise stop, $\mathcal{W}(Y)$ – the fully wrapped Fukaya category)

In HMS for log CY / Fano, with fiber mirror to $D_0 \subset X$, $D_0 + D' = -K_X$, σ^{-1} corresponds to $-\otimes \mathcal{O}(D_0)$ and $\cup$ to i_*; localization gives mirror to $X - D_0$

② Localizing $\mathcal{F}(F, w_F)$ wrt $\mu^{-1} \xrightarrow{S} \mathrm{id}$ $\Leftrightarrow$ quotient by $\mathrm{Im}(\cap)$ (vanishing cycles)

when $\mathrm{critval}(w_0) = \{0\}$, can be interpreted as the Fukaya cat. of the singular fiber F_0.

Eg. $\mathcal{W}(\{xy=0\} \subset \mathbb{C}^2) = \mathcal{W}(T^*S^1/\langle S^1 \rangle) \simeq \mathrm{Perf}\left(\bullet \circlearrowleft\right)$, $\mathcal{W}(\{xyz=0\}) \simeq \mathrm{Perf}\left((\mathbb{C}^*)^2 - \{1 + x_1 + x_2 = 0\}\right)$. 2d pants

Thm (Jeffs): (Knörrer periodicity) $\mathcal{W}(F_0) \simeq \mathcal{F}(Y \times \mathbb{C}, zw_0)$ (expect $\mathcal{F}(F_0, w_F) \simeq \mathcal{F}(Y \times \mathbb{C}, zw_0 + w_F)$)