Knot Categorification

and Mirror Symmetry

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In this talk, I will describe how mirror symmetry

solves the problem of categorifying

$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

link invariants.

The problem was introduced in '98,

by Khovanov showed how to associate to a link

a collection of bigraded vector spaces

$$\mathcal{H}_K^{*,*} = \bigoplus_{i,j\in\mathbb{Z}} \mathcal{H}_K^{i,j}$$

graded by a cohomological grading i and an "equivariant grading" j

such that their equivariant Euler characteristic is the Jones polynomial

$$J_K(\mathfrak{q}) = \sum_{i,j\in\mathbb{Z}} (-1)^i \mathfrak{q}^{j/2} \dim_{\mathbb{C}} \mathcal{H}_K^{i,j}$$

$\mathcal{H}_K^{i,j}$ are themselves link invariants.

The problem Khovanov initiated is to find

a physical, or at least geometric, meaning of

of Khovanov homology,

one that works uniformly for all gauge groups.

Edward Witten explained in ’88

that Jones polynomial comes from

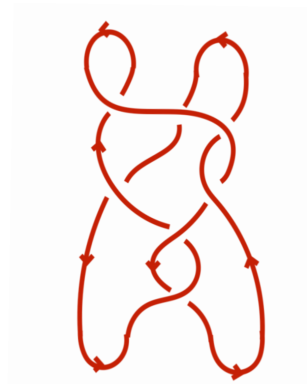
Chern-Simons theory with gauge group based on the Lie algebra

$$ {}^{L}\mathfrak{g} = \mathfrak{su}_2 $$

with (effective) Chern-Simons level κ related to $\mathfrak{q}$ by

$$ \mathfrak{q} = e^{\frac{2\pi i}{\kappa}} $$

This placed the Jones polynomial into a more general framework



which one gets by

considering Chern-Simons theory based on

different Lie algebras $^L\mathfrak{g}$ and the representations.

The relation of Witten's link invariants

to $U_{\mathfrak{q}}({}^L\mathfrak{g})$ quantum groups

was developed by Reshetikhin and Turaev in '89.

I will explain that Khovanov's homologies also

have origin in physics,

which places them into a more general framework

in parallel to what Witten did in '88.

The theory can be given two different descriptions,

related by a version of two dimensional homological mirror symmetry.

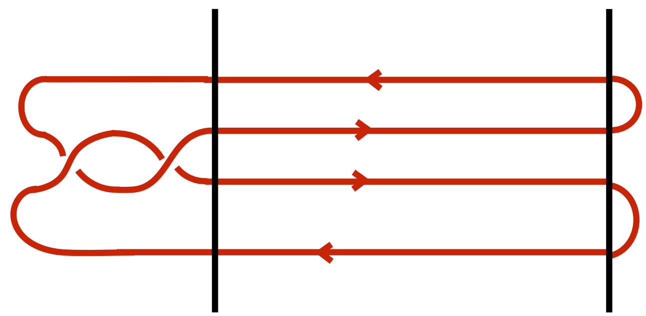
It is solvable explicitly,

by making homological mirror symmetry manifest.

Two-dimensional physics enters here because the descriptions we will get

come from two-dimensional theories associated to

$\text{link } K \times \text{time}$ in $\mathbb{R}^3 \times \text{time}$



More precisely, as it comes out of string theory,

the theory is naturally equipped to describe arbitrary links in

$\mathbb{R}^2 \times S^1 \times \text{time}$ and not just $\mathbb{R}^3 \times \text{time}$

The theories we will end up with

describe two dimensional defects in the six dimensional (0,2) theory

as anticipated from the works of Ooguri and Vafa in ’99,

and Gukov, Schwarz and Vafa in ’04.

The approach we will take is complementary to the approach

being developed by Witten,

in that it describes the same physics,

just from perspective of the defects, rather than the bulk 6d or 5d theory.

While string theory played a crucial role

in discovering the structure I will tell you about,

the final answer will not depend on understanding it.

One of the striking aspects of quantum knot invariants,

is the wealth of mathematics and physics connections

one gets to discover once one understands them.

The fact that this structure arises from a deeper theory,

will no doubt lead to many more connections.

We will see one of them here,

as a vast new family of examples of

homological mirror symmetry,

which connect it to representation theory.

In the same ’89 paper Witten also showed that

underlying Chern-Simons theory is a

two-dimensional conformal field theory associated to

${}^L\mathfrak{g}$ and κ

with affine

$$\widehat{{}^L\mathfrak{g}}_\kappa$$

Lie algebra symmetry.

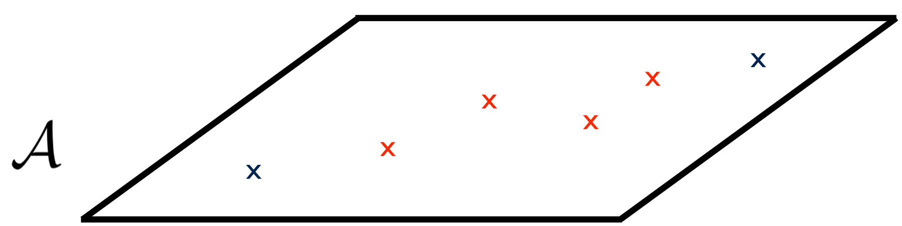
This, rather than Chern-Simons theory, is our starting point.

To eventually get invariants of knots in $\mathbb{R}^3$ or S^3

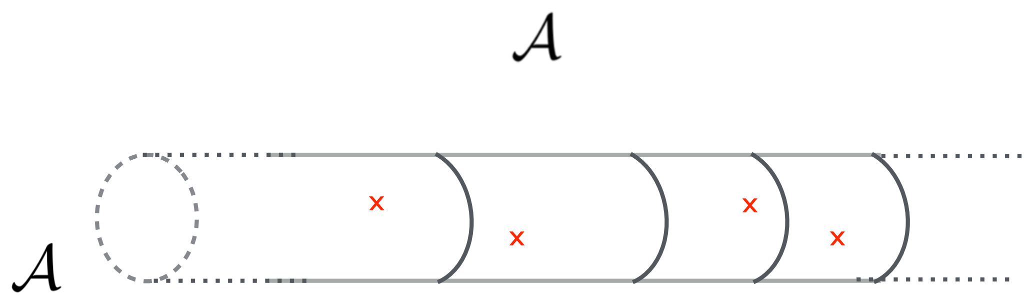
one typically starts with a Riemann surface

$$\mathcal{A}$$

which is a complex plane with punctures.



It is equivalent, but for our purpose better, to take



to be a punctured infinite cylinder.

This way, the theory will be able to describe links in

$\mathbb{R}^2 \times S^1$ and not just $\mathbb{R}^3$

To a puncture at a finite point

$$x = a_i$$

we will associate a finite dimensional representation

V_i of ${}^L\mathfrak{g}$.

which we will take to be minuscule.

To punctures at the two ends at infinity,

$\mathcal{A}$

we will associate a pair

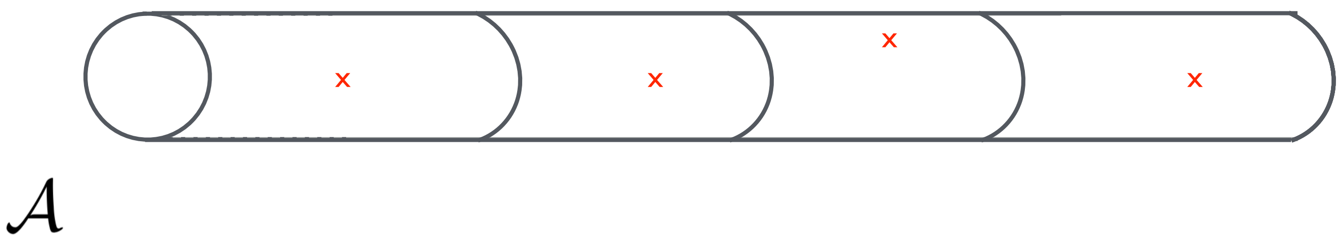
$$V_{\lambda} \quad , \quad V_{\lambda'}$$

of infinite dimensional, Verma module representations of

$${}^{L}\mathfrak{g}$$

whose highest weights are not integral.

To this data,



conformal field theory associates a

conformal block

$$\langle\lambda|\ \Phi_{V_1}(a_1)\cdots\Phi_{V_\ell}(a_\ell)\cdots\Phi_{V_n}(a_n)\ |\lambda'\rangle$$

obtained by sewing chiral vertex operators,

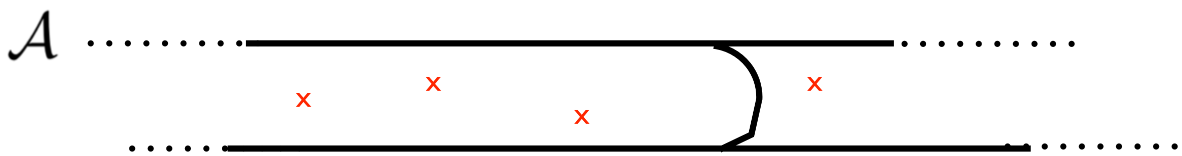
$$\Phi_{V_i}(a_i) : V_{\lambda_i} \ \rightarrow \ V_i(a_i)\otimes V_{\lambda_{i+1}}$$

which act as intertwiners between pairs of Verma module representations.

Rather than characterizing conformal blocks

$$\langle\lambda|\ \Phi_{V_1}(a_1)\cdots\Phi_{V_\ell}(a_\ell)\cdots\Phi_{V_n}(a_n)\ |\lambda'\rangle$$

in terms of vertex operators and sewing,



one can describe them as solutions to a differential equation.

The equation solved by conformal blocks of

$$\widehat{{}^L\mathfrak{g}}_\kappa$$

$\mathcal{A}$

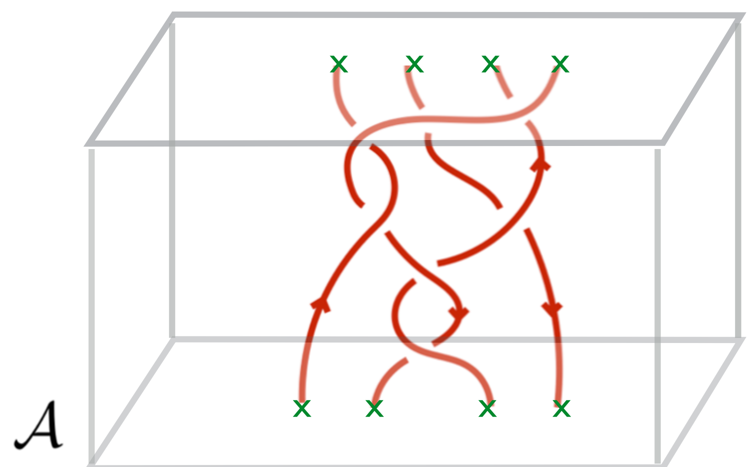
is the trigonometric version of the equation

discovered by Knizhnik and Zamolodchikov in ’84:

$$\kappa\, a_\ell \frac{\partial}{\partial a_\ell} \mathcal{V} = \sum_{j \neq \ell} r_{\ell i}(a_\ell / a_j)\, \mathcal{V}.$$

since the Riemann surface is an infinite cylinder.

By varying the positions of vertex operators on $\mathcal{A}$ as a function of "time"

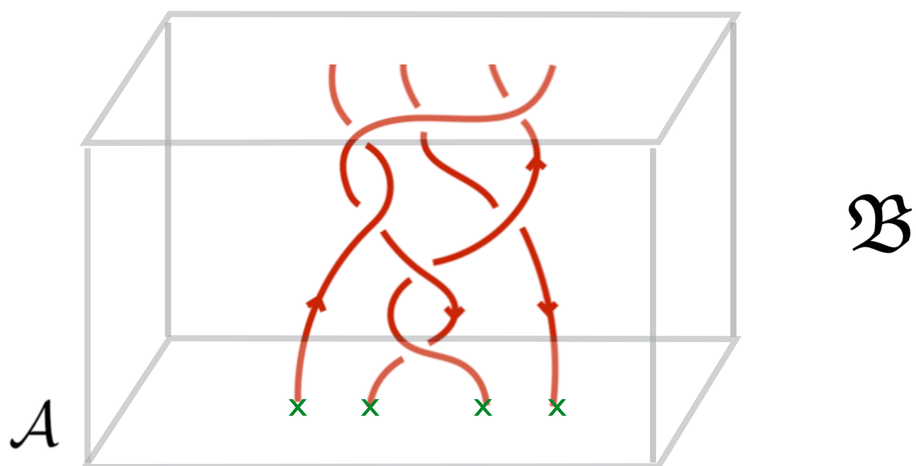


we get a colored braid B in $\mathcal{A} \times [0,1]$

From the perspective of the Knizhnik-Zamolodchikov equation

$$\kappa\, a_\ell \frac{\partial}{\partial a_\ell}\mathcal{V} = \sum_{j\neq\ell} r_{\ell i}(a_\ell/a_j)\,\mathcal{V}.$$

the braiding matrix



is the monodromy matrix along the path in the parameter space described by the braid.

Monodromy problem of the Knizhnik-Zamolodchikov equation

$$\kappa\, a_\ell \frac{\partial}{\partial a_\ell}\,\mathcal{V} = \sum_{j\neq\ell} r_{\ell i}(a_\ell/a_j)\,\mathcal{V}.$$

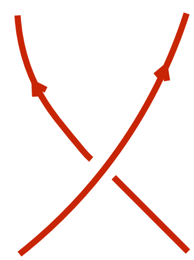
was solved by

Drinfeld in ’89 and by Kazhdan and Lustig

following works of Tsuchya, Kanye and Kohno in ’88.

They showed that monodromy matrix that reorders

a neighboring pair of vertex operators



of the affine current algebra

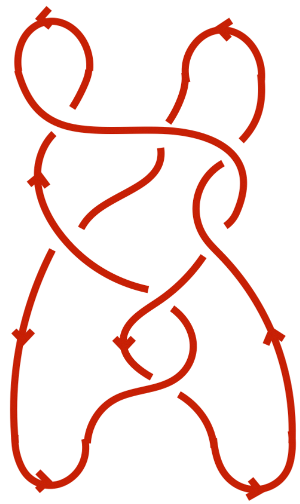
$$\widehat{{}^L\mathfrak{g}}_\kappa$$

is an R-matrix of the quantum group

$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

corresponding to ${}^L\mathfrak{g}$, whose construction is canonical.

Any link K can be represented as a



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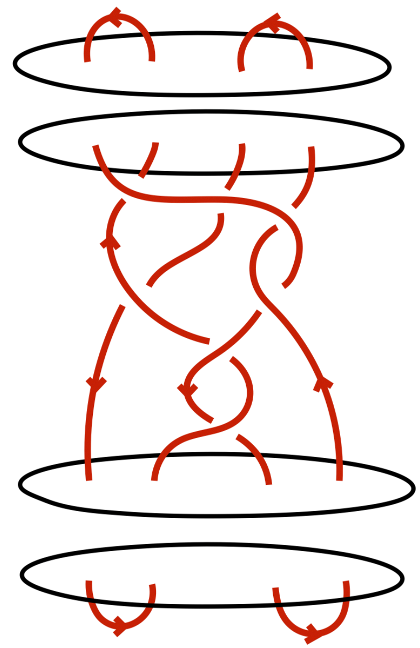
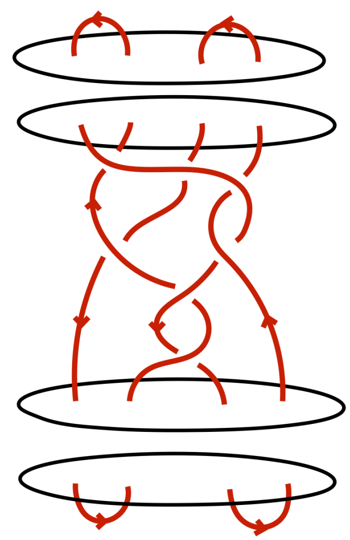


a closure of some braid.

The corresponding quantum link invariant is the matrix element

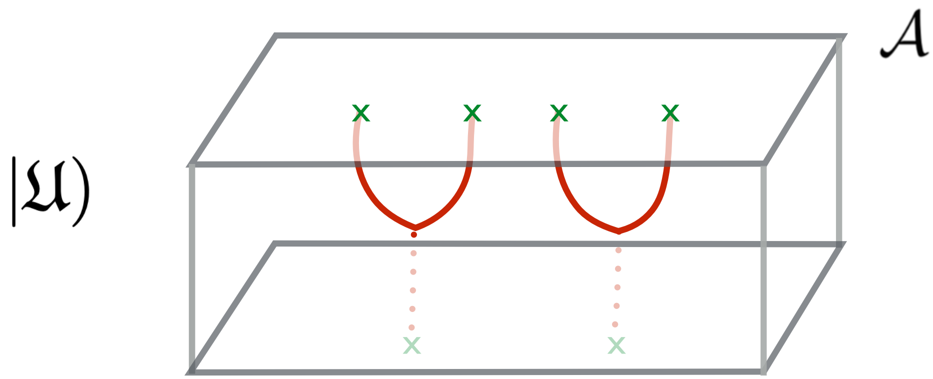


of the braiding matrix,

taken between a pair of conformal blocks

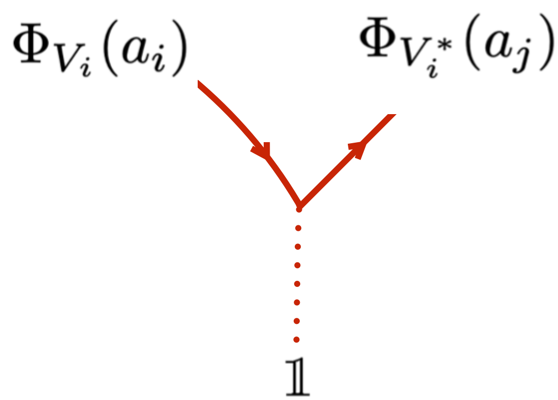
which correspond to the top and the bottom of the picture.

Cups or caps



describe vertex operators coming together in pairs,

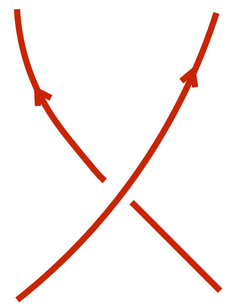
colored by complex conjugate representations

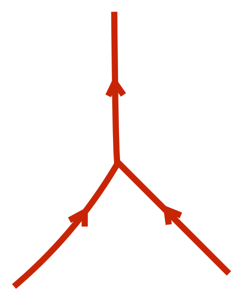


which can come together and "fuse" to disappear.

This way,

both braiding and fusion of conformal field theory





play an important role in the story.

Our starting point for categorification of link invariants

is a realization of conformal blocks

which comes from quantum field theory

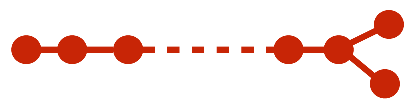
in two dimensions with N=2 supersymmetry.

The right theory is, ultimately, picked out by string theory.

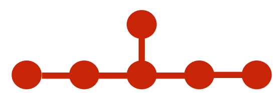
We will specialize ${}^{L}\mathfrak{g}$ to be a simply laced Lie algebra

so ${}^{L}\mathfrak{g} = \mathfrak{g}$ are one of the following types:

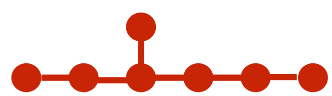
$\mathfrak{g} = A_n$



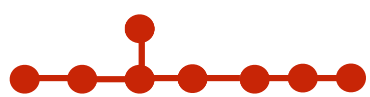
$\mathfrak{g} = D_n$



$\mathfrak{g} = E_6$



$\mathfrak{g} = E_7$



$\mathfrak{g} = E_8$

The generalization to non-simply laced Lie algebras

involves an extra step, which we will not have time to describe.

One description of the theory is the

the supersymmetric sigma model

with hyper-Kahler target

$$\mathcal{X}$$

which is a very special manifold.

The manifold

$$\mathcal{X}$$

we need can be described as the moduli space of

singular G monopoles, with prescribed Dirac singularities, on

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$$

G is the Lie group of adjoint type with Lie algebra $\mathfrak{g}$.

For every vertex operator on $\mathcal{A} = \mathbb{R} \times S^1$

$$\Phi_{V_i}(a_i)$$

place a singular G monopole

at the corresponding point on $\mathbb{R}$ in $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$

$$y_i = \log |a_i|$$

whose charge is the highest weight

$$\mu_i$$

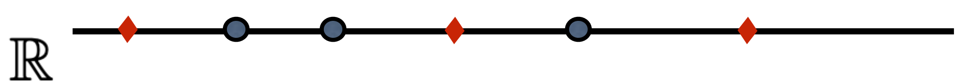
of the ${}^L\mathfrak{g}$ representation V_i

The charges of singular monopoles determine the representation

$$V = \bigotimes_{i=1}^{n} V_i$$

which the conformal blocks transform in.

The weight ν in that representation



determines the total monopole charge, including that of smooth monopoles.

The monopole moduli space

$$\mathcal{X}$$

is parameterized, in part, by positions of the smooth monopoles on

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$$

$\mathbb{R}$

keeping those of singular monopoles fixed,

$$\mathcal{X} \sim Sym^{\vec{d}}(S^1 \times \mathbb{R} \times \mathbb{C})$$

Our manifold $\mathcal{X}$ has several other useful descriptions.

The one familiar to mathematicians, $\mathcal{X}$ is as a resolution of

$$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu}$$

of a transversal slice in affine Grassmannian $\mathrm{Gr}_G = G((z))/G[[z]]$ of G

Here, the vector $\vec{\mu}$

encodes the singular monopole charges in order they appear

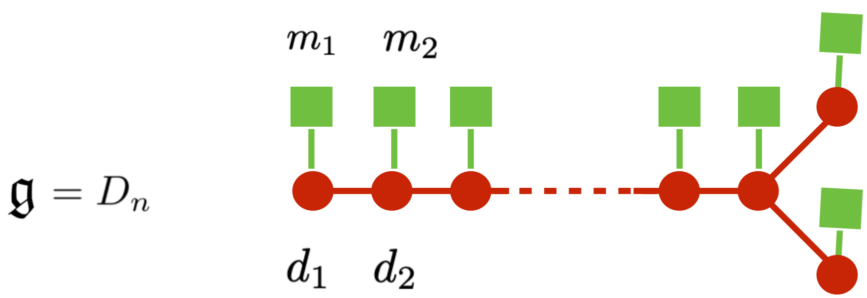
and ν is the total monopole charge.

To physicists,

$$\mathcal{X}$$

is the Coulomb branch of a certain

three dimensional quiver gauge theory



with N=4 supersymmetry, where singular and smooth monopole charges

determine the ranks of gauge and flavor symmetry groups.

The manifold

$$\mathcal{X} \sim Sym^{\vec{d}}(S^1 \times \mathbb{R} \times \mathbb{C})$$

is hyper-Kahler.

The positions of singular monopoles on

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$$

are the moduli of its metric.

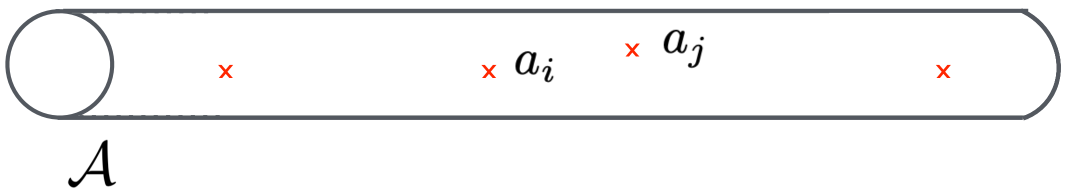
If all the representations V_i are minuscule,

and the positions of singular monopoles on

$\mathbb{R}$ 

are generic, $\mathcal{X}$ is smooth.

Complexified Kahler moduli of $\mathcal{X}$ are positions of singular monopoles on $\mathcal{A}$



We took the Riemann surface $\mathcal{A}$ to be a cylinder rather than a plane,

because the B-fields

that pair with the real Kahler moduli to get the complex ones,

are periodic.

By choosing all the singular monopoles to be at the origin of $\mathbb{C}$

$$\mathcal{X}$$

has a symmetry

$$\omega^{2,0} \to \mathfrak{q}\, \omega^{2,0}$$

We will work equivariantly with respect to a larger torus of symmetries

$$\mathrm{T} = \Lambda \times \mathbb{C}^{\times}_{\mathfrak{q}}$$

where Λ -action preserves the holomorphic symplectic form.

$$\langle \lambda | \, \Phi_{V_1}(a_1) \cdots \Phi_{V_\ell}(a_\ell) \cdots \Phi_{V_n}(a_n) \, | \lambda' \rangle$$

It encodes the Verma module weights.

Conformal blocks

$$\mathcal{V}(a_1, \ldots, a_\ell, \ldots, a_n) = \langle \lambda | \; \Phi_{V_1}(a_1) \cdots \Phi_{V_\ell}(a_\ell) \cdots \Phi_{V_n}(a_n) \; | \lambda' \rangle$$

arise as the partition functions of the supersymmetric sigma model,

D  $\dashrightarrow \mathcal{X}$

working equivariantly with respect to

$$\mathrm{T} = \Lambda \times \mathbb{C}^\times_{\mathfrak{q}}$$

The domain curve

D 

is best thought of an infinite cigar with an S^1 boundary at infinity.

In the interior of D , supersymmetry is preserved by an A-type twist

and at infinity, one places a B-type boundary condition.

The infinite length of the cigar makes the A-type supersymmetry preserved by the interior compatible with any supersymmetry on the boundary, even of B-type.

From perspective of $\mathcal{X}$, the Knizhnik-Zamolodchikov equation

is the "quantum differential equation:"

an equation for flat sections

$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta \, \mathcal{V}_\beta = 0.$$

of a connection on a vector bundle with fibers $H^*(\mathcal{X})$.

over the complexified Kahler moduli space,

introduced by Givental.

This is a recent theorem of I. Danilenko.

One gets different solutions to the quantum differential equation,

and to the Knizhnik-Zamolodchikov equation,

by choosing different B-type branes as

boundary conditions at infinity.

Boundary conditions form a category, and

the category of boundary conditions of the sigma model on $\mathcal{X}$,

preserving a B-type supersymmetry and

working equivariantly with respect to T is known as its

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

the derived category of $\mathrm{T}-$ equivariant coherent sheaves.

Picking a B-type brane

$$\mathcal{F} \in \mathscr{D}_{\mathcal{X}}$$

as the boundary condition at infinity,

the supersymmetric partition function computes Givental's J-function,

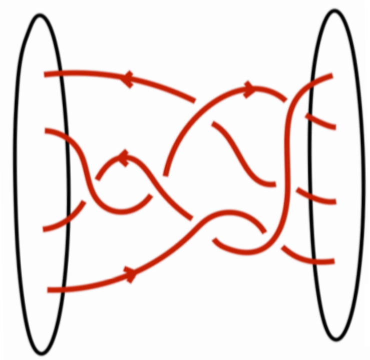
$$\mathcal{V}[\mathcal{F}]$$

or vertex function, defined via $\mathrm{T}-$ equivariant Gromov Witten theory of $\mathcal{X}$.

It depends on the brane only through its K-theory class

$$[\mathcal{F}] \in K_{\mathrm{T}}(\mathcal{X})$$

A braid B has a geometric interpretation as a path in complexified Kahler moduli that avoids singularities,



$\mathcal{A}$

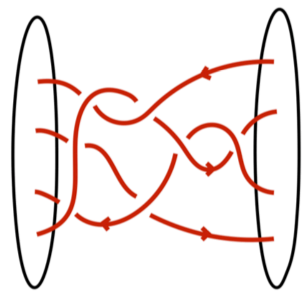
since the complexified Kahler moduli of

$$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu}$$

are the relative positions of vertex operators on $\mathcal{A}$

It follows the geometric realization of the action of

$$\mathfrak{B} \in U_{\mathfrak{q}}({}^L\mathfrak{g})$$



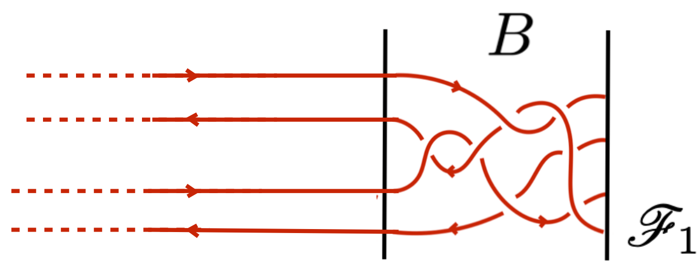
on the space of $\widehat{{}^L\mathfrak{g}}$ conformal blocks is

monodromy of the quantum differential equation of

$$\mathcal{X}$$

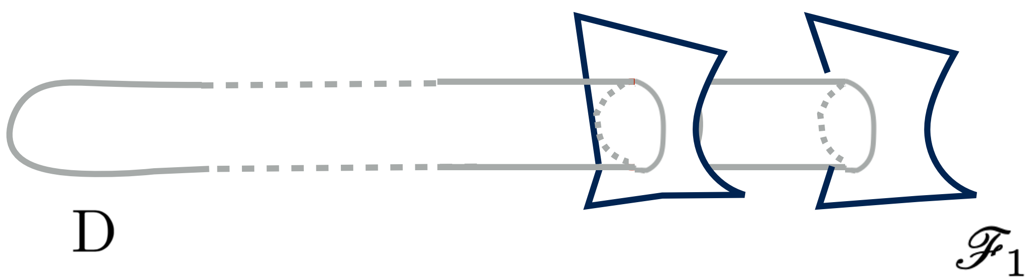
along the path in its Kahler moduli corresponding to the braid.

From the sigma model perspective, the monodromy problem arizes



by letting the moduli of the theory vary according to the braid,

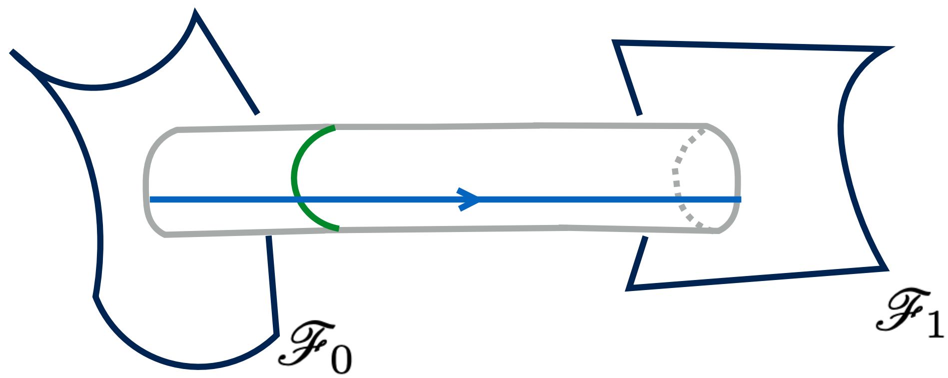
in the neighborhood of the boundary at infinity,



where the direction along the cigar coincides with the "time" along the braid.

It follows that the path integral of the sigma model

where time runs along the annulus



and moduli that vary according to the braid,

computes the matrix element of the monodromy

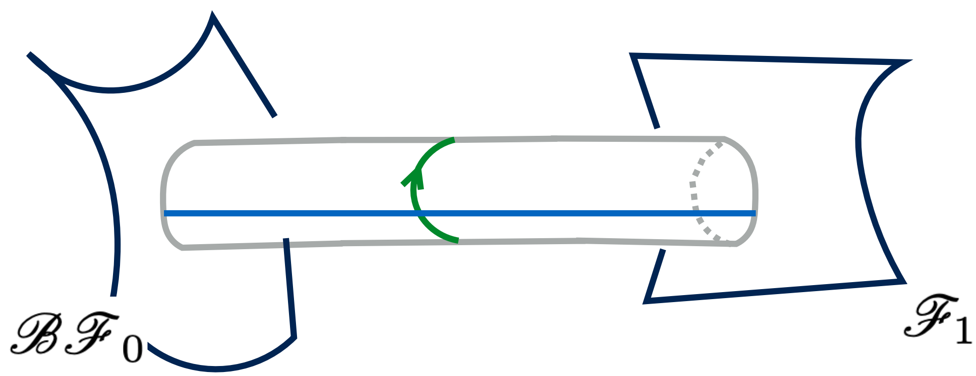
$\mathfrak{B}$

between pairs of conformal blocks picked out by the

B-branes at the two boundaries.

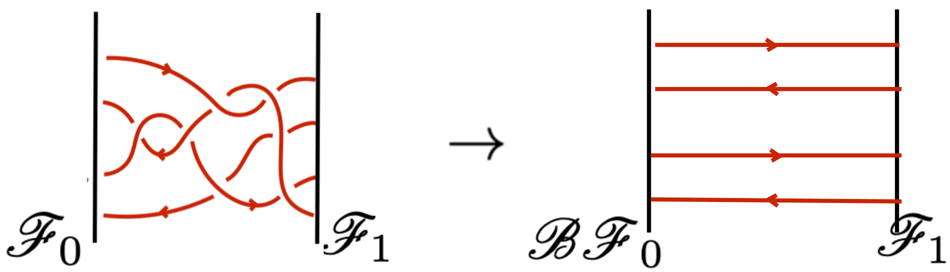
The same path integral

with time that runs around the S^1 ,



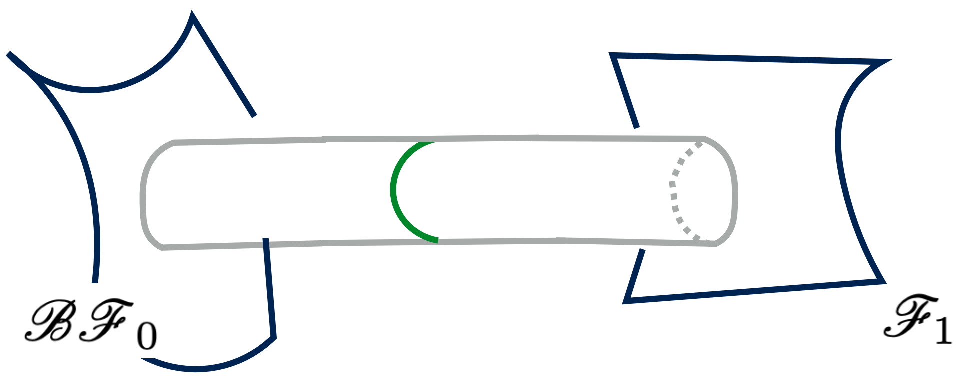
computes the index of the supercharge Q preserved by the two branes.

The index stays the same if we take all the variation of the moduli to happen near one of two boundaries,



at the expense of changing the boundary condition,

$$\mathscr{F}_0 \to \mathscr{B}\mathcal{F}_0$$



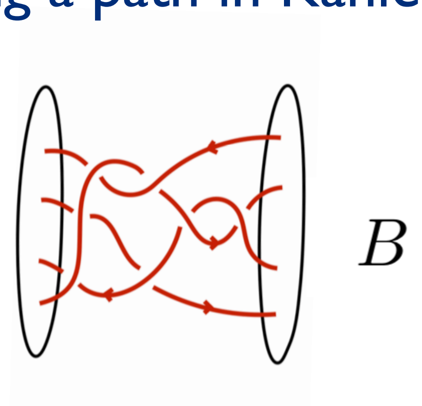
Braid group acts by

$$\mathcal{F} \to \mathscr{B}\mathcal{F}$$

an auto-equivalence $\mathscr{B}$ of the derived category,

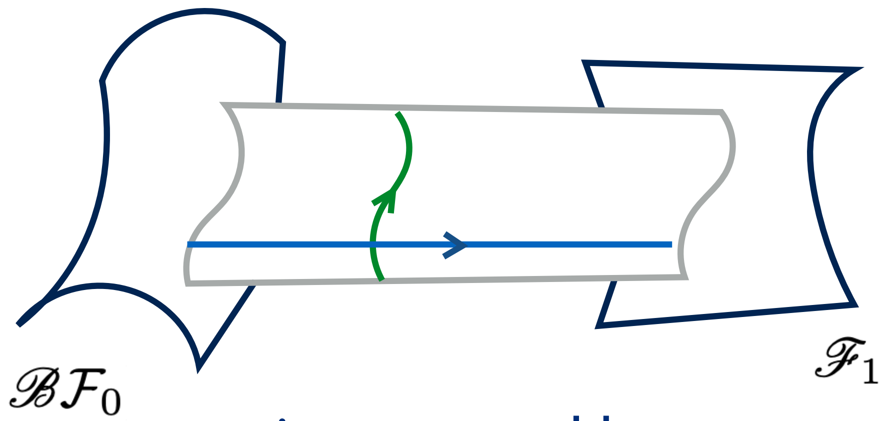
$$\mathscr{B} : D^b Coh_{\mathrm{T}}(\mathcal{X}) \to D^b Coh_{\mathrm{T}}(\mathcal{X})$$

since along a path in Kahler moduli



the category of B-type branes stays the same.

The cohomology of the supercharge Q



is computed by

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

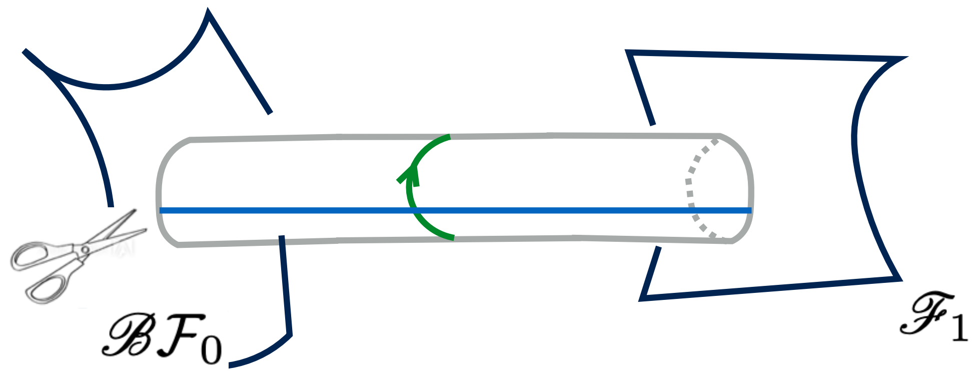
as its basic ingredient,

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{F}_0, \mathcal{F}_1)$$

the space of morphisms between a pair of branes.

The Euler characteristic of the homology theory

$$\chi(\mathscr{B}\mathcal{F}_0, \mathcal{F}_1) = \sum_{k\in\mathbb{Z}, J\in\mathbb{Z}^{\mathrm{rkT}}} (-1)^k \mathfrak{q}^{J/2} \dim_{\mathbb{C}} Hom(\mathscr{B}\mathcal{F}_0, \mathcal{F}_1[k]\{J\})$$



thus manifestly computes the monodromy matrix element

$$\chi(\mathscr{B}\mathcal{F}_0, \mathcal{F}_1) = (\mathfrak{B}\mathcal{V}_0 | \mathcal{V}_1)$$

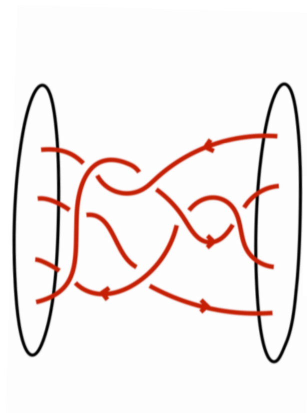
since we are free to think of either direction as time.

It follows that derived equivalence

$$\mathscr{B} : D^b Coh_{\mathrm{T}}(\mathcal{X}) \to D^b Coh_{\mathrm{T}}(\mathcal{X})$$

manifestly categorifies

the monodromy matrix $\mathfrak{B}$ of the Knizhnik-Zamolodchikov equation.



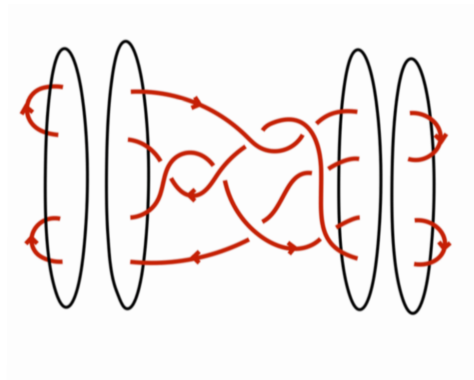
B

This explains a very difficult theorem of Bezrukavnikov and Okounkov,

which uses quantization of $\mathcal{X}$ in characteristic p.

The quantum invariants of links should also be categorified by

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$



since they too can be expressed as matrix elements of the braiding matrix

between pairs of conformal blocks.

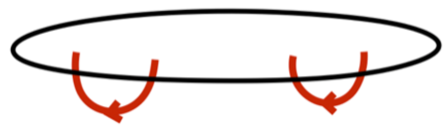
For this, we need to find objects of

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

which serve a cups and caps.

In looking for such objects of

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$



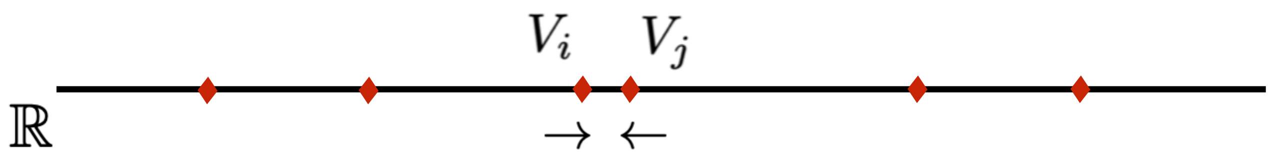
we will discover that not only braiding,

but also fusion has a geometric interpretation in terms of

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

$\mathcal{X}$ develops singularities

as a pair of singular monopoles approach each other



The singularities come from cycles in $\mathcal{X}$ that collapse at

the corresponding wall in Kahler moduli.

The collapsing cycles

$$F_{k_m}$$

turn out to be labeled by representations that occur in

$$V_i \otimes V_j = \bigotimes_{m=0}^{m_{max}} V_{k_m}$$

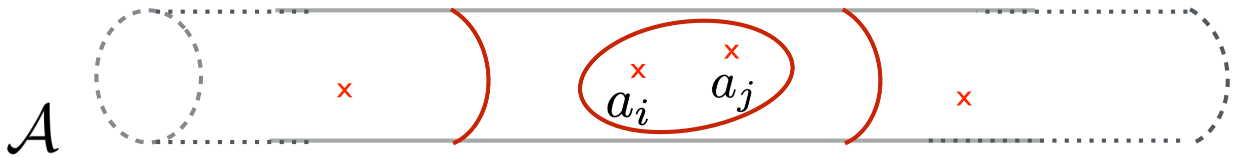
because singularity in the monopole moduli space is associated to

monopole bubbling phenomena:

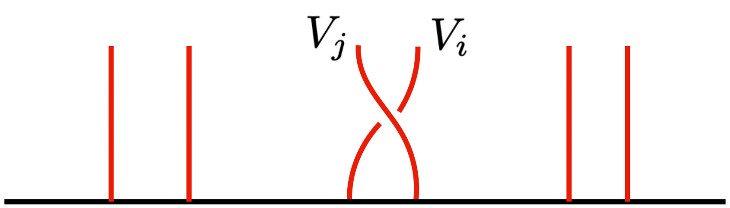
The highest weight of V_{k_m} is the charge of a single singular monopole

left behind after a number of smooth monopoles bubbles off.

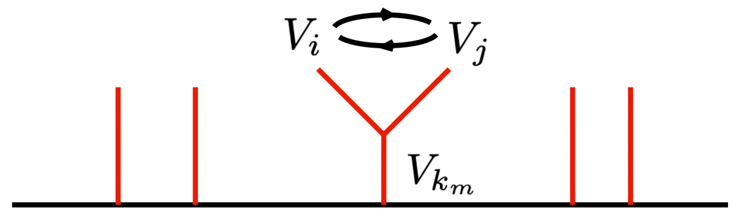
This reflects the fact that, as a pair of vertex operators approach



one gets a new natural basis of conformal blocks,



$\rightarrow$



which are eigenvectors of braiding, labeled by

$$V_i \otimes V_j = \bigotimes_{m=0}^{m_{max}} V_{k_m}$$

The branes

$$\mathcal{U}_i \in \mathscr{D}_{\mathcal{X}}$$

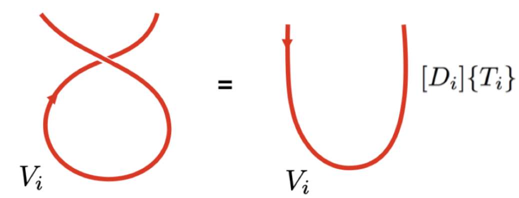
corresponding to cups turn out to be sheaves

supported on a vanishing cycle in $\mathcal{X}$ which is a minuscule Grassmannian,

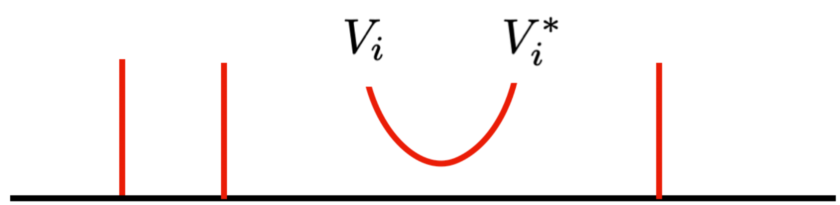
$$U_i = G/P_i$$

They are eigensheaves of braiding:

$$\mathscr{B}\mathcal{U}_i = \mathcal{U}_i[D_i]\{T_i\}$$



whose vertex functions are the conformal blocks:



By it origin in the sigma model on $\mathcal{X}$, the functor

$$\mathscr{B} : \mathscr{D}_{\mathcal{X}} \to \mathscr{D}_{\mathcal{X}}$$

comes from variations of stability condition on

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

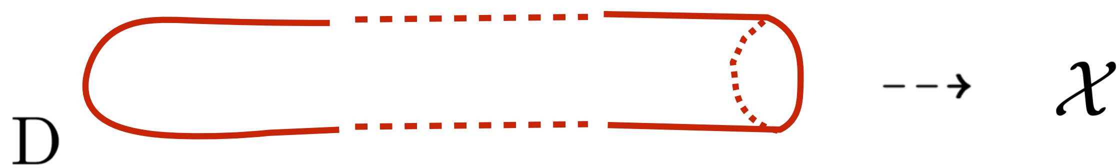
defined with respect to a central charge function

$$\mathcal{Z}^0[\mathcal{F}] \;\; : K(\mathcal{X}) \to \mathbb{C}$$

which is a close cousin of conformal blocks.

Like the conformal blocks $\mathcal{Z}^0[\mathcal{F}]$ can be computed by

Gromov-Witten theory



except with trivial insertion at the origin,

where one also turns off the equivariant parameters,

to get a map

$$\mathcal{Z}^0[\mathcal{F}] \ : K(\mathcal{X}) \to \mathbb{C}$$

depending only on Kahler moduli.

The stability condition defined with respect to

$$\mathcal{Z}^0[\mathcal{F}] \;: K(\mathcal{X}) \to \mathbb{C}$$

is known as the Pi stability condition,

discovered by Douglas.

Since in our case, $\mathcal{X}$ is hyper-Kahler,

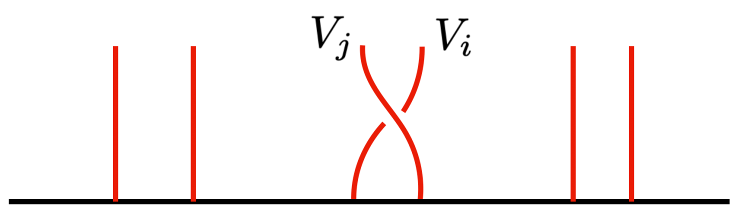
the exact central charge can be read off from classical geometry,

and the stability structure it gives rise to is extremely simple.

It is constant in a chamber in Kahler moduli which

corresponds to fixing the order of vertex operators,

and changes when a pair of them trade places.

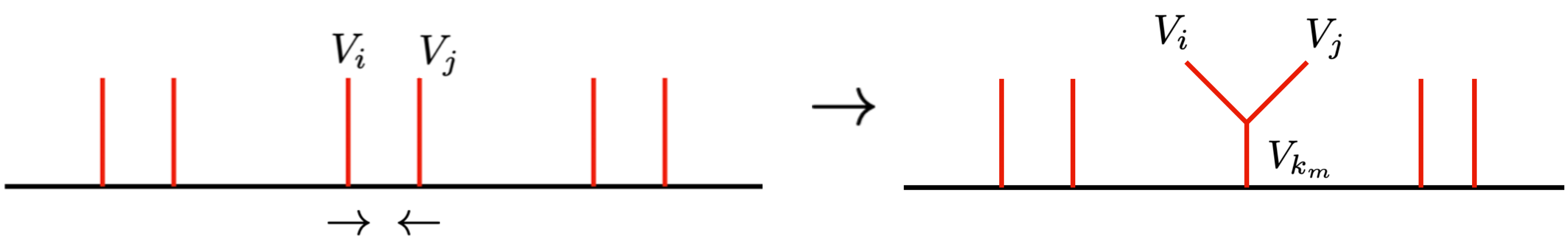


The theory at hand should give model examples

of Bridgeland stability.

Near the wall in Kahler moduli

we get vanishing cycles F_{k_m} corresponding to ways of fusing:



and objects $\mathcal{F}_k \in \mathscr{D}_{\mathcal{X}}$ whose central charge vanishes as

$$a_i \to a_j$$

as the dimension of the cycle

$$\mathcal{Z}^0_{k_m} = \mathcal{Z}^0(\mathcal{F}_{k_m}) \sim (a_i - a_j)^{\dim F_{k_m}} \times \text{finite}$$

Conformal blocks which diagonalize the action of braiding

do not in general come from of actual objects

of the derived category $\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$.

Eigensheaves of braiding $\mathcal{E} \subset \mathscr{D}_{\mathcal{X}}$ on which

the braiding functor acts as

$$\mathscr{B}\mathcal{E} = \mathcal{E}[-D_{\mathcal{E}}]\{C_{\mathcal{E}}\}$$

are rare.

What we get instead is a filtration

$$\mathscr{D}_{k_0} \subset \mathscr{D}_{k_1} \ldots \subset \mathscr{D}_{k_{max}} = \mathscr{D}_{\mathcal{X}}$$

on the derived category $\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$,

by the order of vanishing of $\mathcal{Z}^0$

with terms in the filtration labeled by distinct representations

$$V_i \otimes V_j = \bigotimes_{m=0}^{m_{max}} V_{k_m}$$

in the tensor product

In fact, one gets a pair of such filtrations,

$$\mathscr{D}_{k_0} \subset \mathscr{D}_{k_1} \ldots \subset \mathscr{D}_{k_{max}} = \mathscr{D}_{\mathcal{X}} \qquad \mathscr{D}'_{k_0} \subset \mathscr{D}'_{k_1} \ldots \subset \mathscr{D}'_{k_{max}} = \mathscr{D}'_{\mathcal{X}}$$

one on each side of the wall.

$\mathbb{R}$

Crossing the wall preserves the filtrations,

but it has the effect of mixing up objects of a given order of vanishing,

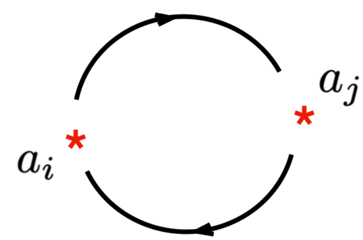
with those that vanish faster, and which belong to lower orders

in the filtration.

On the quotient subcategories, the derived equivalence is a degree shift

$$\mathscr{B} : \mathscr{D}_{k_m}/\mathscr{D}_{k_{m-1}} \to \mathscr{D}'_{k_m}/\mathscr{D}'_{k_{m-1}} \cong \mathscr{D}_{k_m}/\mathscr{D}_{k_{m-1}}[-D_m]\{C_m\}$$

which depends only on the order in the filtration,



and on the path around the singularity

and which comes from the equivariant central charge.

$$\mathcal{Z}_{k_m} \longrightarrow (-1)^{\pi i D_m}\, \mathfrak{q}^{-C_m/2}\, \mathcal{Z}_{k_m}$$

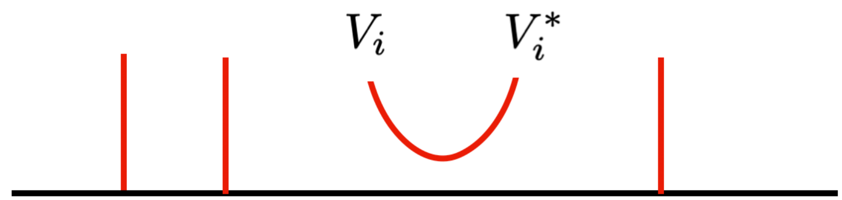
Derived equivalences of this type are the perverse equivalences

envisioned by

Rouquier and Chuang,

with few examples from geometry.

The objects $\mathcal{U} \in \mathscr{D}_{\mathcal{X}}$ corresponding to cups



belong to the lowest term of the filtration

$$\mathscr{D}_{k_0} \subset \mathscr{D}_{k_1} \ldots \subset \mathscr{D}_{k_{max}} = \mathscr{D}_{\mathcal{X}}$$

so they are necessarily eigensheaves the braiding functor

$$\mathscr{B}\mathcal{U} = \mathcal{U}[-D_0]\{C_0\}$$

Even then, they are extremely special ones,

for the same reason the identity representation is special.

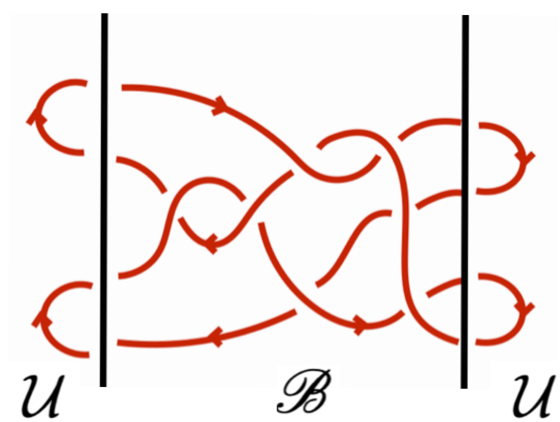
Using very special properties of perverse filtrations

and these vanishing cycle branes

it is not hard to show that not only do the homology groups

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{U},\mathcal{U})$$

manifestly categorify the corresponding $U_{\mathfrak{q}}(^L\mathfrak{g})$ link invariants,



they are themselves link invariants.

Recently, Ben Webster proved that link invariants

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{U}, \mathcal{U})$$

that come from

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

are equivalent to invariants he defined in '13

KLRW algebras studied by

Khovanov and Lauda, by Rouquier and by himself.

As stated, neither the approach by

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

nor by KRLW algebras is very computation friendly.

In the rest of the talk I want to describe how physics

let's one reformulate the problem,

and solve the theory.

The resulting description is completely new.

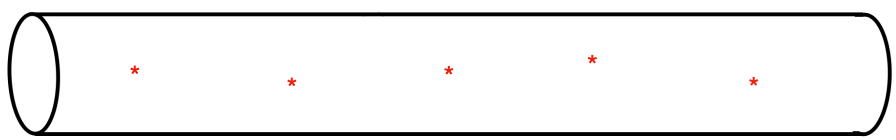
The second description is based on a Landau-Ginsburg model

whose target is, to a first approximation,

$$Y \sim Sym^{\vec{d}}(\mathcal{A})\backslash F_0$$

an open subset of symmetric product of copies of

of the Riemann surface where conformal blocks live



$$\mathcal{A} \cong \mathbb{C}^{\times} \cong \text{infinite cylinder}$$

with potential W .

The relation between

$\mathcal{X}$ and Y

is a cousin of ordinary 2d mirror symmetry.

Mirror symmetry relating them cannot be ordinary mirror symmetry

since Y is half the dimension of $\mathcal{X}$

Instead, the monopole moduli space $\mathcal{X}$ has a "core" locus

$$X$$

which is half dimensional and contains all the information about the geometry,

and whose mirror is

$$Y$$

Viewing

$$\mathcal{X} \sim Sym^{\vec{d}}(S^1 \times \mathbb{R} \times \mathbb{C})$$

as the moduli space of monopoles on $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$

its core

$$X \sim Sym^{\vec{d}}(S^1 \times \mathbb{R})$$

is the locus where all monopoles,

singular or not, are at the origin of $\mathbb{C}$ and at points in $\mathbb{R}$

X is preserved by the $\mathbb{C}^\times_{\mathfrak{q}} \subset \mathrm{T}$ action which scales $\omega^{2,0} \to \mathfrak{q}\, \omega^{2,0}$

Since the bottom row has as much information about the geometry as the top,

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, leftrightarrow, "\text{mirror}"] \arrow[dr, leftrightarrow, "\text{equivariant mirror}"] & \mathcal{Y} \arrow[d] \\
X \arrow[u] \arrow[r, leftrightarrow, "\text{mirror}"] & Y
\end{tikzcd}
$$

we will call Y the equivariant mirror of $\mathcal{X}$.

While X embeds into $\mathcal{X}$ as a

holomorphic Lagrangian submanifold of dimension $D = \dim_{\mathbb{C}} \mathcal{X}/2$

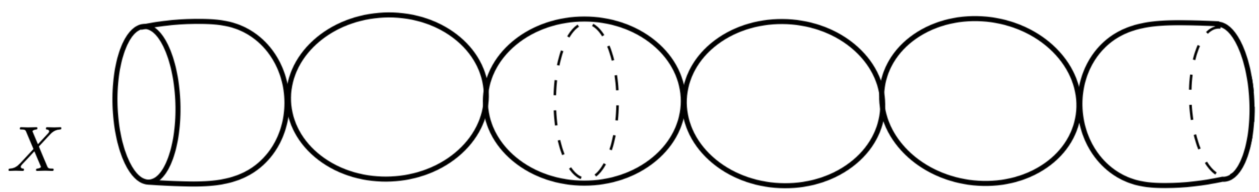
$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{X} \arrow[r, leftrightarrow, "\text{mirror}"] \arrow[dr, leftrightarrow, "\text{equivariant mirror}"] & \mathcal{Y} \arrow[d] \\
X \arrow[u] \arrow[r, leftrightarrow, "\text{mirror}"] & Y
\end{tikzcd}
$$

$\mathcal{Y}$ fibers over Y with holomorphic Lagrangian $(\mathbb{C}^\times)^D$ fibers

A model example to keep in mind is

$\mathcal{X}$ which is an A_{m-1} surface.

Its core X looks like



it is a collection of $m-1$ $\mathbb{P}^1$'s with a pair of infinite discs attached.

$\mathcal{X}$ is the moduli space of a single smooth $G = SU(2)/\mathbb{Z}_2$ monopole,

in presence of m singular ones.

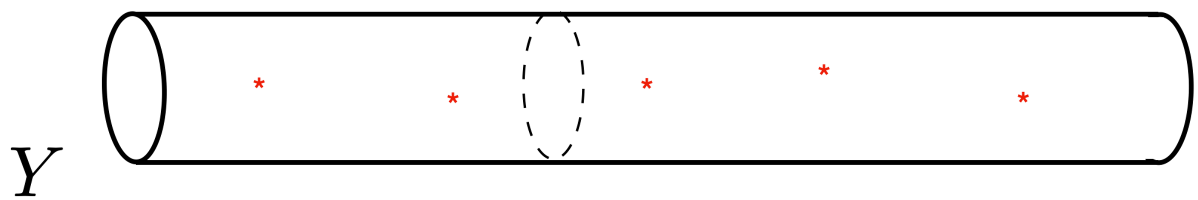
The ordinary mirror of $\mathcal{X}$ which is an A_{m-1} surface,

is $\mathcal{Y}$ which is a "multiplicative" A_{m-1} surface,

with a potential which we will not need.

The "multiplicative" A_{m-1} surface $\mathcal{Y}$,

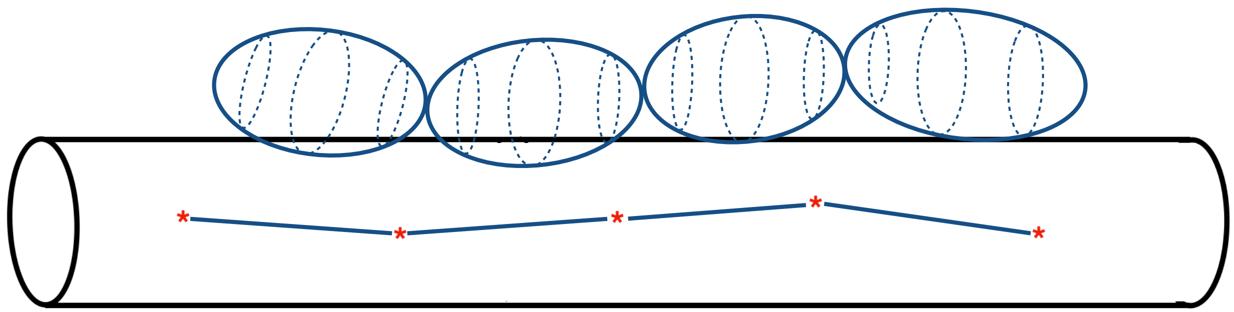
is a $\mathbb{C}^{\times}$-fibration over



Y is an infinite cylinder with m marked points in the interior.

At the marked points, the $\mathbb{C}^{\times}$ fibers of $\mathcal{Y}$ degenerate .

There are $m-1$ Lagrangian spheres in $\mathcal{Y}$

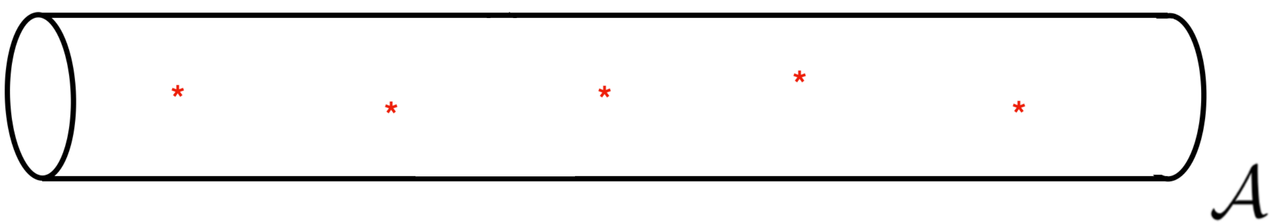


which are mirror to $m-1$ vanishing $\mathbb{P}^1$'s in $\mathcal{X}$.

They project to Lagrangians in Y that begin and end at the punctures.

Y is a single copy of the Riemann surface

where the conformal blocks live:

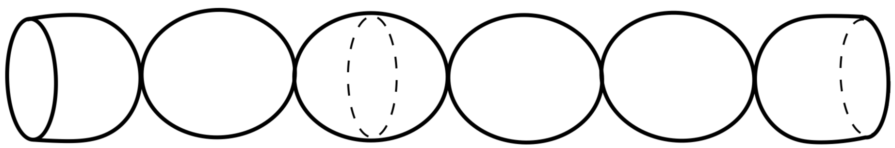


The positions of vertex operator correspond to the marked points

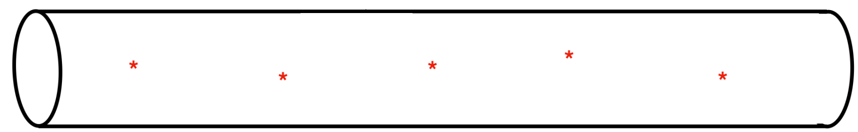
where the $\mathbb{C}^{\times}$ fibration $\mathcal{Y} \to Y$ degenerates.

By SYZ mirror symmetry,

the mirror pair



X



Y

share a common base,

$\mathbb{R}$

which is the moduli space of one smooth monopole

on $\mathbb{R}$ in presence of m singular ones.

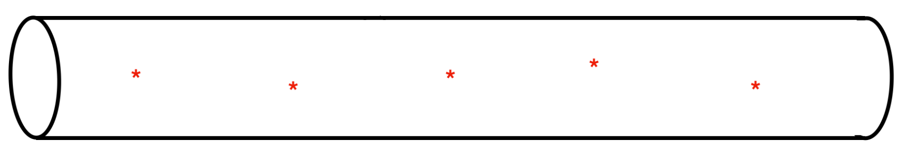
More generally, the equivariant mirror of

$$\mathcal{X} = Gr^{\vec{\mu}}{}_{\nu}$$

and the ordinary mirror of its core X, is

$$Y = \pi^*(Sym^{\vec{d}}(\mathcal{A})\backslash F_0)$$

where $\mathcal{A}$ is our Riemann surface with punctures,



and where $\vec{d} = (d_1, \ldots, d_{\mathrm{rk}})$ encodes the numbers of smooth monopoles.

Projecting to the common SYZ base of

X and of Y

is the same as projecting $\mathcal{X}$,

the moduli space of singular monopoles on

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$$

to

$\mathbb{R}$

Including an equivariant $\mathrm{T}-$ action

on $\mathcal{X}$ and on X

corresponds to adding to the sigma model on

$$Y = \pi^*(Sym^{\vec{d}}(\mathcal{A})\backslash F_0)$$

a specific potential,

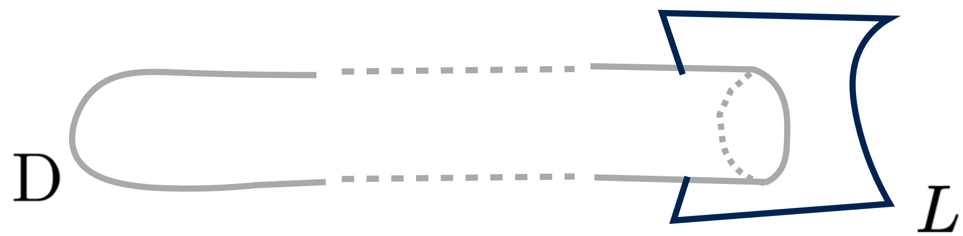
$$W$$

which is a multi-valued complex function on Y .

From the mirror perspective, the conformal block of

$$\widehat{{}^L\mathfrak{g}}$$

is the partition function of the B-twisted theory on D ,



with A-type boundary condition at infinity, corresponding to the Lagrangian L in Y .

Such amplitudes have the following form

$$\mathcal{V}_\alpha[L] = \int_L \Phi_\alpha \, \Omega \, e^{-W}$$

where Ω is the top holomorphic form on Y ,

$$\Omega = \bigwedge_{a=1}^{\mathrm{rk}} \bigwedge_{\alpha=1}^{d_a} \frac{dy_{\alpha,a}}{y_{\alpha,a}}$$

is the Landau-Ginsburg potential,

and Φ 's are the chiral ring operators.

We are have re-discovered here, from mirror symmetry,

the integral formulation of conformal blocks of

$$\widehat{L_{\mathfrak{g}}}$$

which goes back to work of Feigin and E.Frenkel in the ’80’s

and Schechtman and Varchenko.

There is a reconstruction theory,

due to Givental and Teleman, which says that

starting with the solution of quantum differential equation,

one gets to reconstruct all genus topological string amplitudes

of a semi-simple 2d field theory.

Thus, the B-twisted the Landau-Ginsburg model (Y, W)

and A-twisted sigma model on $\mathcal{X}$

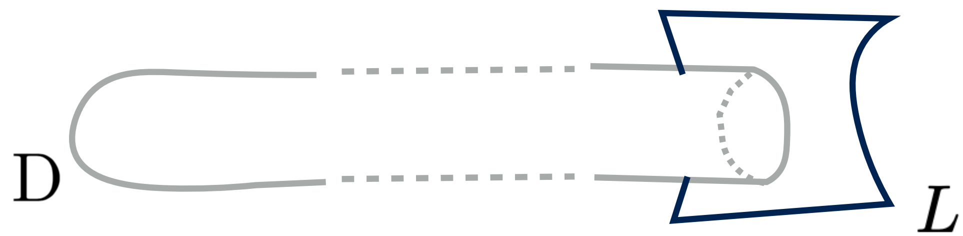
working equivariantly with respect to T

are equivalent to all genus.

Corresponding to a solution of the

Knizhnik-Zamolodchikov equation

is an A-brane at the boundary of D at infinity,



The brane is an object of the category of A-branes

$$\mathscr{D}_Y = D(\mathcal{FS}(Y, W))$$

the derived Fukaya-Seidel category of Y with potential W

The set one needs to delete

$$Y = \pi^*(Sym^{\vec{d}}(\mathcal{A})\backslash F_0)$$

gives rise to a set of one forms on Y

$$c^0 = dW^0/2\pi i, \qquad c^a = dW^a/2\pi i \ \in \ H_1(Y, \mathbb{Z})$$

with integer periods

which are responsible for equivariant gradings

of both branes and Homs between them in

$$\mathscr{D}_Y = D(\mathcal{FS}(Y, W))$$

The potential W is a multi-valued holomorphic function on Y :

$$W = W^0/\kappa + \sum_{a=1}^{\mathrm{rk}} \lambda_a W^a.$$

It mirrors the

$$\mathrm{T} = \Lambda \times \mathbb{C}^\times_{\mathfrak{q}}$$

equivariant action on X .

Mirror symmetry

helps us understand exactly which questions we need to ask

$$
\begin{tikzcd}
\mathcal{X} \arrow[rr, leftrightarrow, "\text{mirror}"] \arrow[rrd, leftrightarrow, "\text{equivariant mirror}"] & & \mathcal{Y} \arrow[d] \\
X \arrow[u] \arrow[rr, leftrightarrow, "\text{mirror}"] & & Y
\end{tikzcd}
$$

to recover homological knot invariants from Y .

Since Y is an ordinary mirror of X ,

we should start by understanding how to recover

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, leftrightarrow, "\text{mirror}"] \arrow[drr, leftrightarrow, "\text{equivariant mirror}"] & & \mathcal{Y} \arrow[d] \\
X \arrow[u] \arrow[rr, leftrightarrow, "\text{mirror}"] & & Y
\end{tikzcd}$$

homological knot invariants from X , instead of $\mathcal{X}$

Every B-brane on $\mathcal{X}$ which is relevant to us

"comes from"

a B-brane on the core X

via a functor,

$$f_* : \mathscr{D}_X \rightarrow \mathscr{D}_{\mathcal{X}}$$

that interprets a brane F on X, an object of $\mathscr{D}_X$

as a brane $\mathcal{F} = f_* F$ on $\mathcal{X}$, an object of $\mathscr{D}_{\mathcal{X}}$

The functor f_* has an adjoint f^* that goes the other way,

$$\begin{tikzcd} \mathscr{D}_{\mathcal{X}} \arrow[d, shift right, "f^*"'] \\ \mathscr{D}_X \arrow[u, shift right, "f_*"'] \end{tikzcd}$$

Adjointness means that given any pair of branes on $\mathcal{X}$ that come from X

the Hom between them, computed upstairs, in $\mathscr{D}_{\mathcal{X}}$

$$\mathcal{F} = f_* F, \quad \mathcal{G} = f_* G$$

agrees with the Hom downstairs, in $\mathscr{D}_X$,

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) = Hom^{*,*}_{\mathscr{D}_X}(f^* f_* F, G)$$

after replacing F with $f^* f_* F$

By mirror symmetry, for every pair of B-type branes

$$\mathcal{F} = f_* F, \ \ \mathcal{G} = f_* G$$

on $\mathcal{X}$ which come from X, there is a pair of A-branes

$$k^* k_* L_F \quad , \qquad L_G$$

on Y which are mirror to

$$f^* f_* F \quad , \quad G$$

such that Hom's on Y agree with those on $\mathcal{X}$.

$$Hom_{\mathscr{D}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) = Hom_{\mathscr{D}_{(Y,W)}}(k^* k_* L_F, L_G)$$

The functors k_* and k^* that enter

$$Hom_{\mathscr{D}_\mathcal{X}}(\mathcal{F},\mathcal{G}) = Hom_{\mathscr{D}_{(Y,W)}}(k^* k_* L_F, L_G)$$

relate objects on Y and on $\mathcal{Y}$,

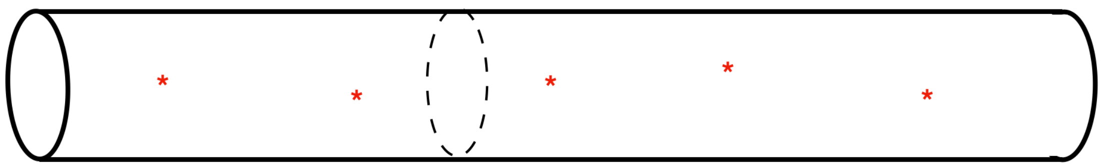
in a way that mirrors f^* and f_* ,

$$\begin{tikzcd} \mathscr{D}_\mathcal{Y} \arrow[d, shift right, "k^*"'] \\ \mathscr{D}_Y \arrow[u, shift right, "k_*"'] \end{tikzcd}$$

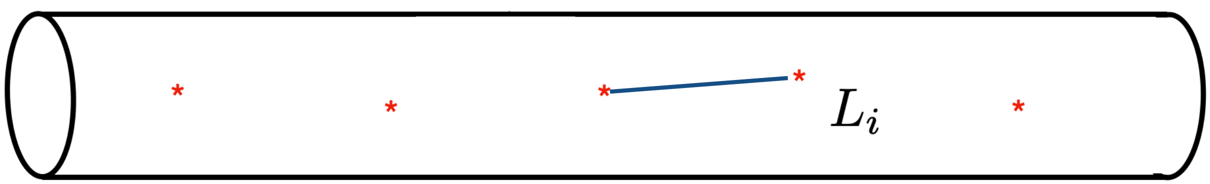
Their construction is joint work with Michael McBreen and Vivek Shende.

Recall our example, Y the equivariant mirror to

$\mathcal{X}$ which is the A_n surface.



Mirror to i-th vanishing $\mathbb{P}^1$ in X is the Lagrangian



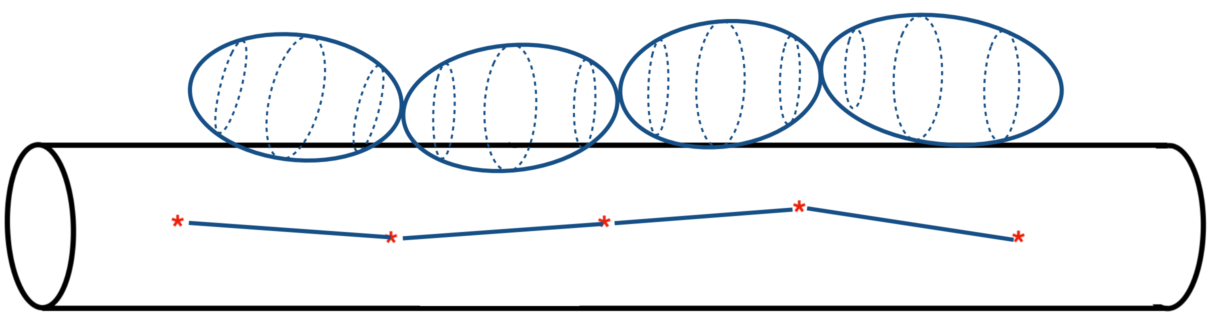
The functor going up

$$\begin{array}{c} \mathscr{D}_{\mathcal{Y}} \\ \uparrow k_* \\ \mathscr{D}_Y \end{array}$$

amounts to pairing the brane downstairs with the

$$(S^1)^D \subset (\mathbb{C}^\times)^D$$

fiber over it, which is how one gets this picture:



The functor going the other way

$$k^* : \mathscr{D}_{\mathcal{Y}} \to \mathscr{D}_Y$$

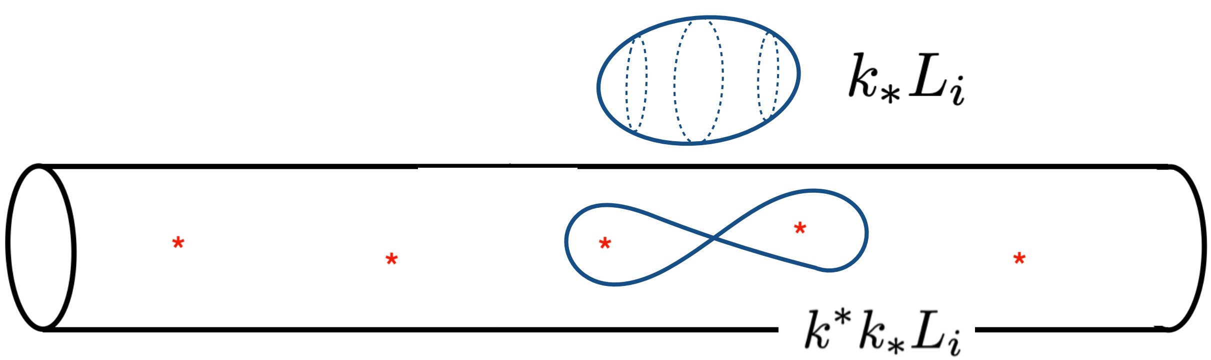
does not send the vanishing sphere $k_* L_i$ back to L_i :

$$k^* k_* L_i \neq L_i$$

Instead, either computing it either from mirror symmetry,

or its via its definition (coming from a Lagrangian correspondence),

one finds a figure eight Lagrangian

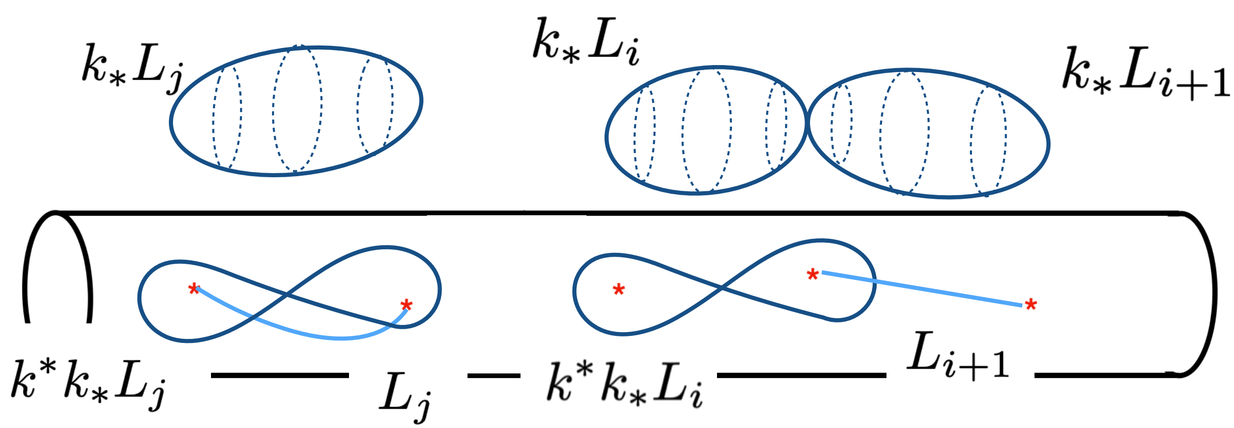


The basic virtue of the pair of adjoint functors,

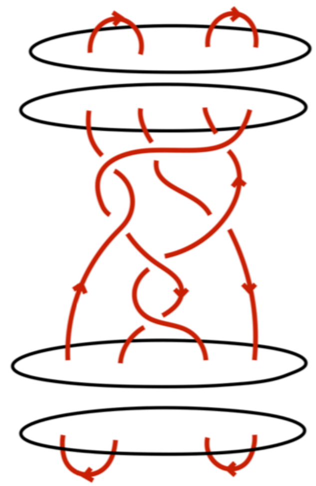
is that one ends up preserving Hom's.

$$Hom_{\mathscr{D}_{\mathcal{Y}}}(k_*L_i, k_*L_j) = Hom_{\mathscr{D}_Y}(k^*k_*L_i, L_j)$$

It is not difficult to see that this indeed is the case



The example we just gave is relevant for Khovanov homology.



$d = 2$

For a link obtained by closing of a braid with $m = 2d$ strands,

$\mathcal{X}$ is the moduli space of d smooth $G = SU(2)/\mathbb{Z}_2$ monopoles,

in presence of m singular ones.

The same $\mathcal{X}$ can be described as

$$\mathcal{X} \sim Sym^d(A_{m-1})$$

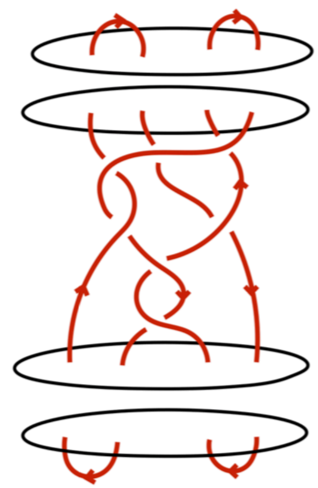
an open subspace of the d -fold symmetric product

(more precisely of the Hilbert scheme of d points)

of the A_{m-1} surface with $m = 2d$,

by a theorem of Manolescu.

In $\mathcal{X} \sim Sym^d(A_{m-1})$ the branes that close off the strands



are supported on a vanishing cycle $U \subset \mathcal{X}$ which is

a product of d non-intersecting $\mathbb{P}^1$'s,

$$U = \mathbb{P}^1 \times \ldots \times \mathbb{P}^1$$

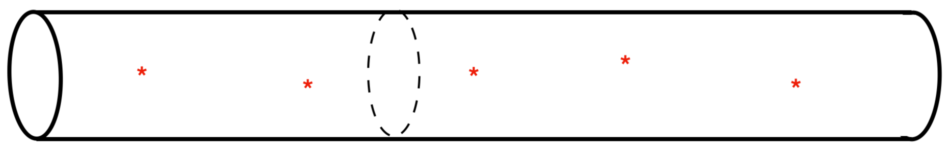
Its equivariant mirror

is an open subset of symmetric product,

$$Y \sim Sym^d(\mathcal{A})\backslash F^0$$

corresponding to configuration space of d unordered points

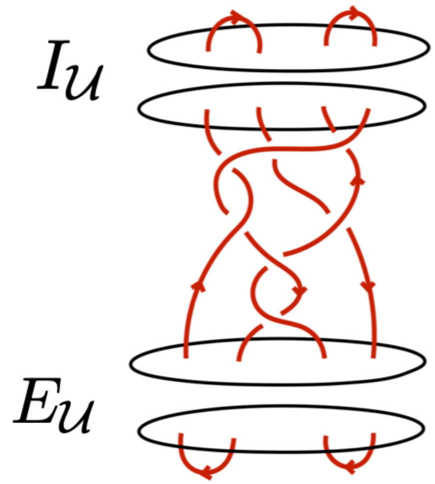
on the surface where the conformal blocks live

$\mathcal{A}$ 

with potential

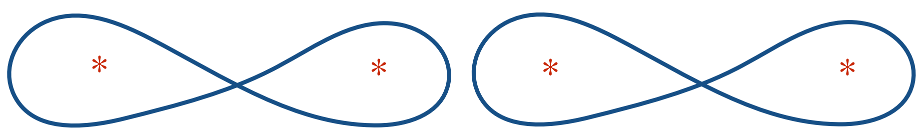
$$W = \sum_{\alpha} \lambda \ln(y_\alpha) + \sum_{i,\alpha} \ln(1 - a_i/y_\alpha)/\kappa - \sum_{\beta\neq\alpha} \ln(1 - y_\beta/y_\alpha)/\kappa.$$

By equivariant mirror symmetry,



the d cups are products of d non-intersecting figure eight Lagrangians:

$E_{\mathcal{U}}$:



and the d caps, the d interval Lagrangians:

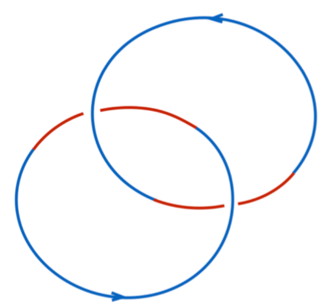
$I_{\mathcal{U}}$:

In the Landau-Ginsburg description,

both the Lagrangians and the action of braiding on them are geometric.

so we can start with a projection of a link to a the surface $\mathcal{A}$.

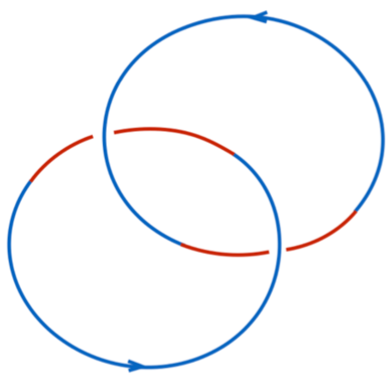
To translate it to a pair of Lagrangians, choose a bicoloring,

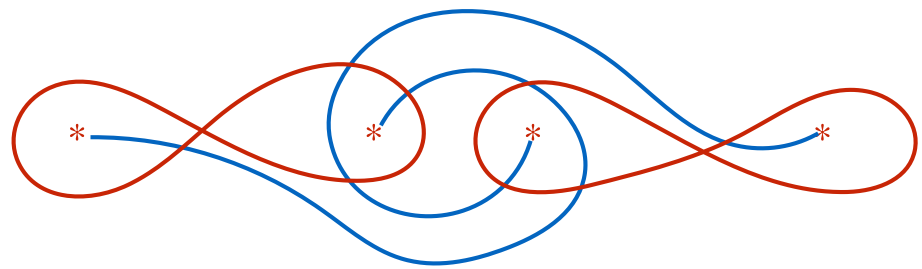


by equal number d of segments of each color,

such that red always underpasses the blue.

The mirror Lagrangians $I_{\mathcal{U}}$ and $\mathscr{B}E_{\mathcal{U}}$ are obtained by replacing all the blue segments by simple intervals, and the red segments by figure eight branes:



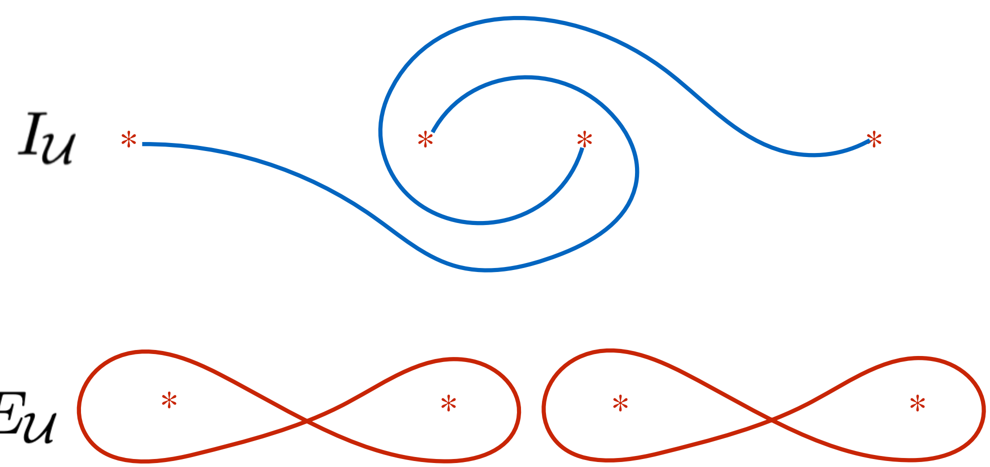


By mirror symmetry, the homological link invariant is

the space of morphisms

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \bigoplus_{M\in\mathbb{Z},\vec{J}\in\mathbb{Z}^2} Hom_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}[M]\{\vec{J}\})$$

between the pair of branes.



The spaces of morphisms

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \bigoplus_{M\in\mathbb{Z},\vec{J}\in\mathbb{Z}^2} Hom_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}[M]\{\vec{J}\})$$

are defined by Floer theory

which is modeled after Morse theory approach to supersymmetric quantum mechanics.

The starting point is the Floer complex

$$CF^{*,*}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \bigoplus_{\mathcal{P} \in \mathscr{B}E_{\mathcal{U}} \cap I_{\mathcal{U}}} \mathbb{C}\,\mathcal{P}$$

spanned by the intersection points of the two Lagrangians,

and graded by cohomological (or Maslov) and the equivariant degrees.

In the Floer theory approach to the A-model,

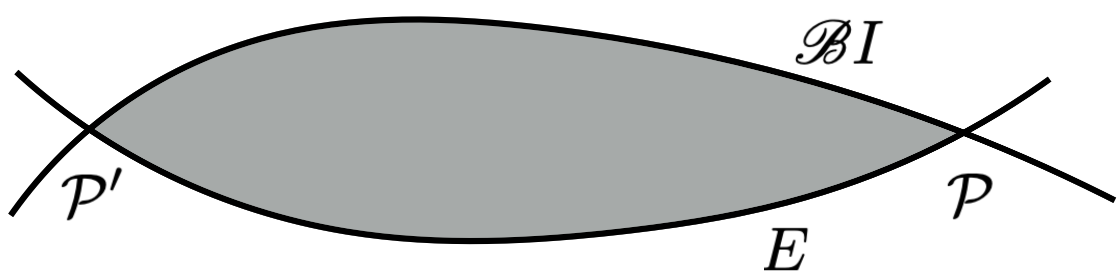
the action of the differential

$$Q : CF^{*,*} \to CF^{*,*}$$

is obtained by counting holomorphic disk instantons in Y

interpolating from $\mathcal{P}$ to $\mathcal{P}'$, of Fermion number one

and equivariant degree zero.



The differential Q

$$Q : CF^{*,*} \to CF^{*,*}$$

squares to zero $Q^2 = 0$ in absence of anomalies.

The cohomology of the resulting complex is the space of exact ground states,

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \mathrm{Ker}\, Q / \mathrm{Im}\, Q$$

and the space of morphisms between the branes in $\mathscr{D}_Y$.

In our case, the theory can be described explicitly

thanks to the fact it is a close cousin

of Heegard-Floer theory.

We would get Heegard-Floer theory by replacing

$$^{L}\mathfrak{g} = \mathfrak{su}_2$$

with

$$^{L}\mathfrak{g} = \mathfrak{gl}_{1|1}$$

and leads to the Alexander polynomial, rather than Jones'.

The Heegard-Floer theory has a target which is a

a (different) open subset of the symmetric product of d copies of $\mathcal{A}$,

with a similar but different potential.

The differences can be accounted by thinking of

Heegard-Floer theory as a theory of fermions on $\mathcal{A}$

while ours is of anyons.

One gets to rephrase the A-model

$$y : \mathrm{D} \to \mathrm{Y}$$

in terms of counting to holomorphic curves

$$S \subset \mathrm{D} \times \mathcal{A}$$

with a pair of projections - to D as a d-fold cover and to $\mathcal{A}$

as a domain with boundaries on one dimensional Lagrangians

is known as the "cylindrical approach" to Floer theory.

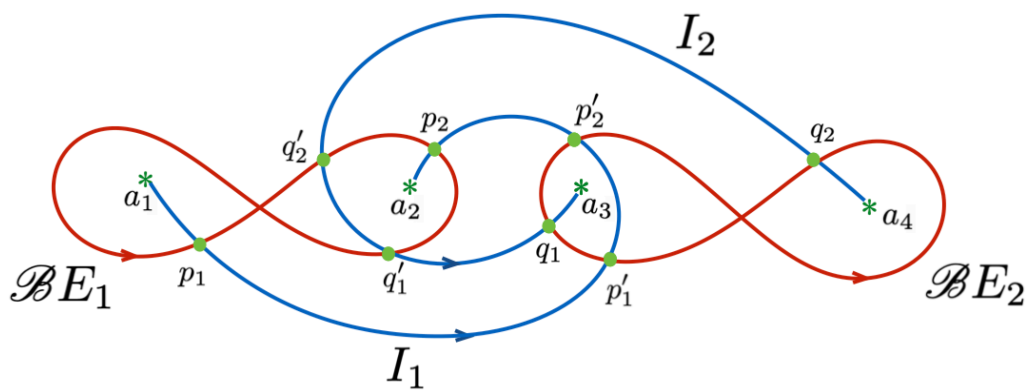
Lagrangians in Y are products of d one dimensional Lagrangians on $\mathcal{A}$,

and the intersection points of a pair of Lagrangians

$$\mathcal{P} = (p_1, \ldots, p_d), \qquad p_\alpha \in \mathscr{B}E_\alpha \cap I_{\sigma(\alpha)}$$

are d-touples of intersections of one dimensional Lagrangians

taken up to permutations $\sigma \in S_d$

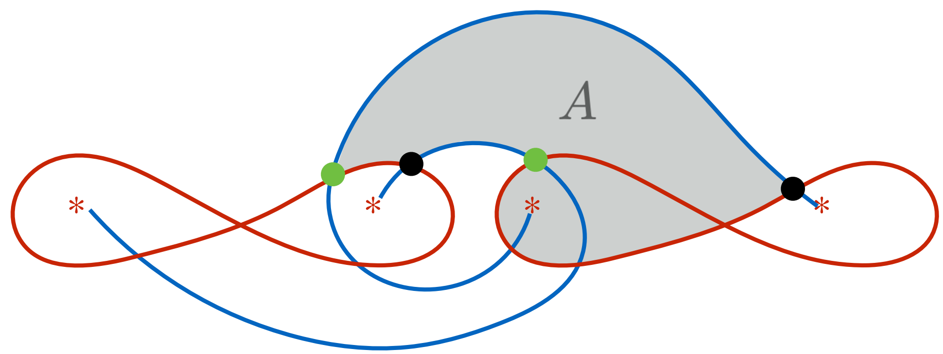


In this example, these are $\mathcal{P}_{ij} = p_i q_j$ and $\mathcal{P}'_{ij} = p'_i q'_j$.

A holomorphic map from a disk to Y

$$y : \mathrm{D} \to Y$$

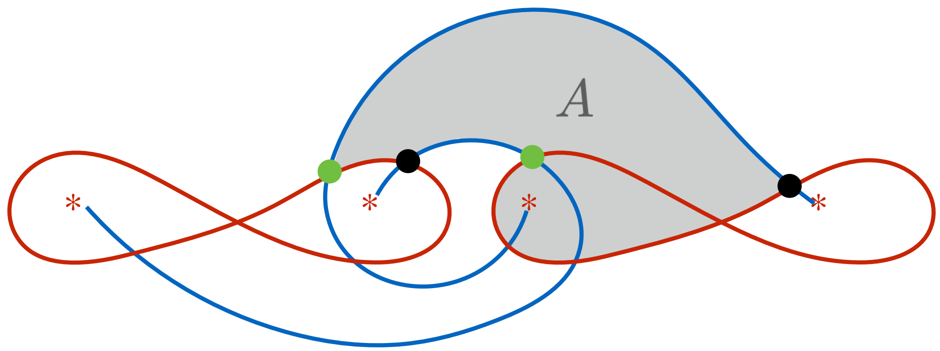
projects with non-negative multiplicities, to domains A on



on $\mathcal{A}$, with boundaries on the one dimensional Lagrangians

and vertices at their intersection points.

As in Heegard-Floer theory, one can read off



from the domain A the fermion number,

and the equivariant degree of the map to

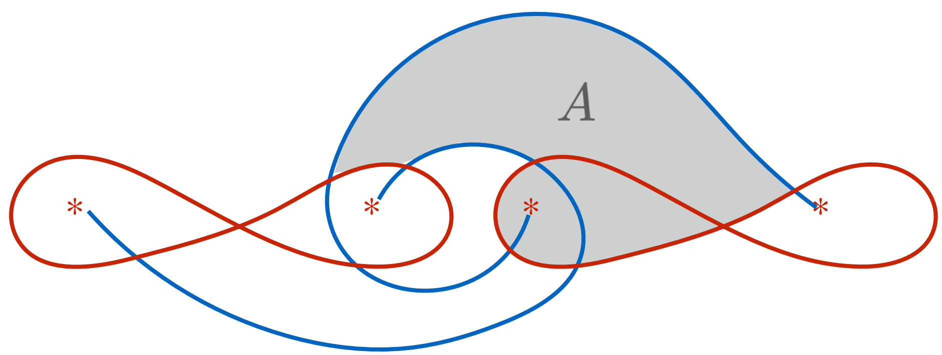
$$y : \mathrm{D} \to Y$$

For example, fermion number, or Maslov index of a disk is

$$\mathrm{ind}[y] = 2e(A)$$

where

$$e(A) = \chi(A) - \#\mathrm{acute}/4 + \#\mathrm{obtuse}/4,$$



The above disk has Maslov index one.

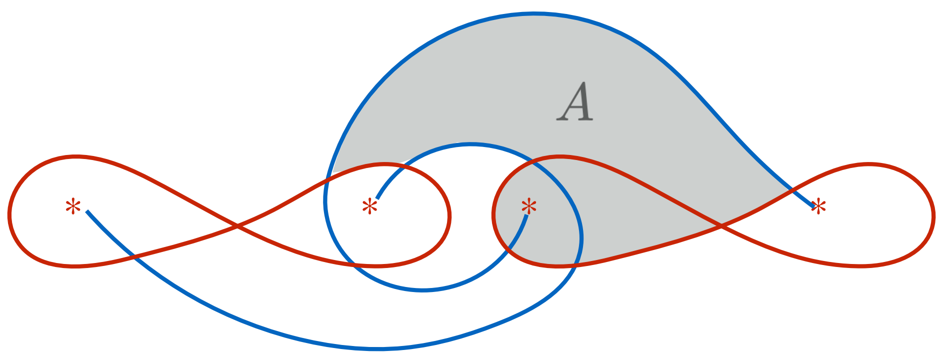
The cylindrical approach to Floer theory reduces the problem of

counting holomorphic maps

$$y : \mathrm{D} \to \mathrm{Y}$$

to a well defined

problem in complex analysis, one for each domain A



solution of which amounts to exercises in Riemannian mapping theorem.

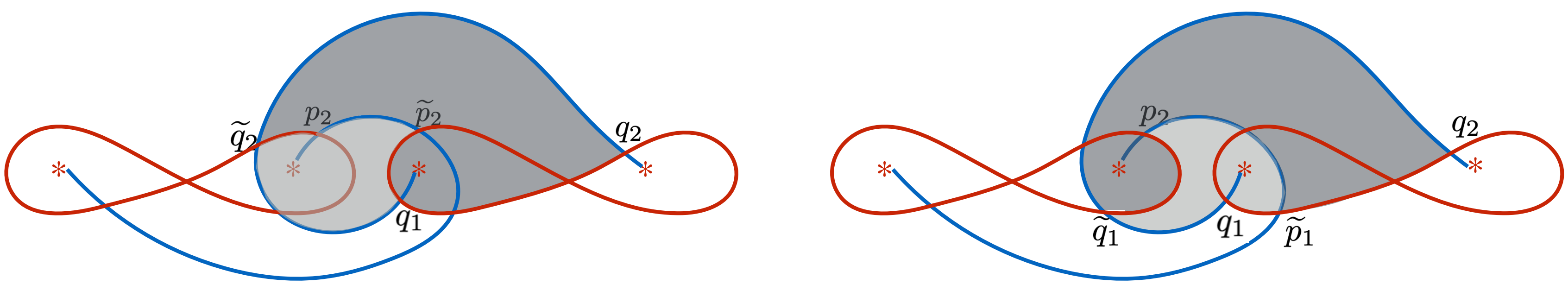
The fact the differential squares to zero

comes, as usual from contributions to Q^2, cancelling in pairs,

coming from broken maps

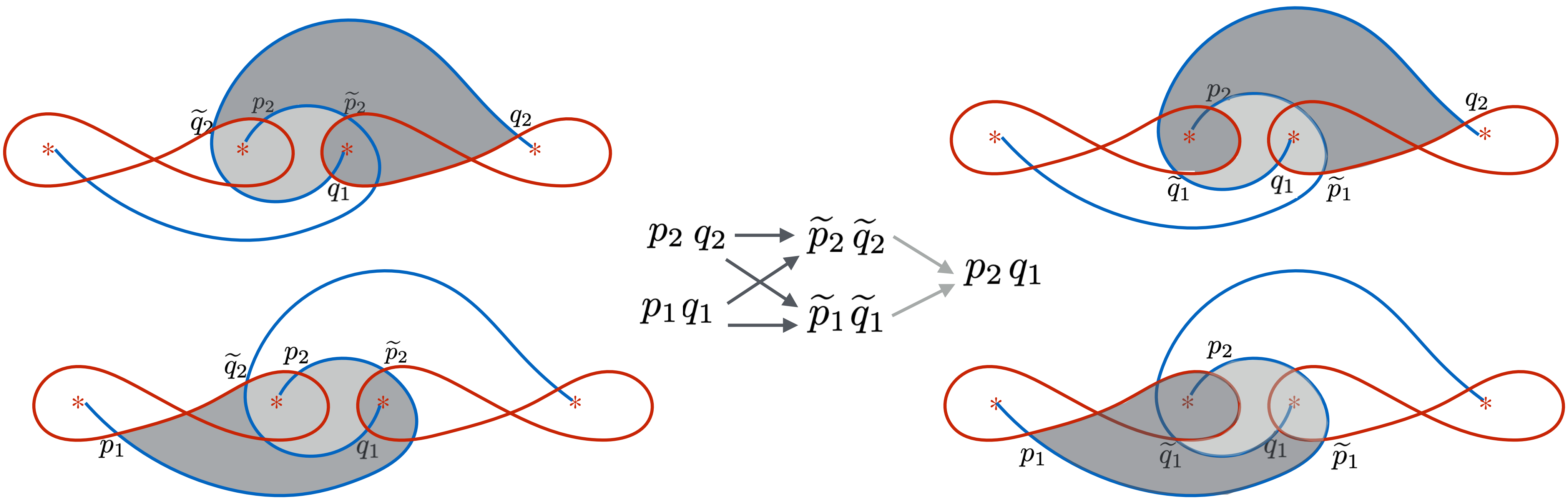
which are boundaries of moduli of

Maslov index 2 disks, with equivariant degree 0.



$$\begin{tikzcd} p_2\, q_2 \arrow[r] \arrow[dr] & \widetilde{p}_2\, \widetilde{q}_2 \arrow[dr] & \\ & \widetilde{p}_1\, \widetilde{q}_1 \arrow[r] & p_2\, q_1 \end{tikzcd}$$

In this example, corresponding to the right-handed Hopf link, there are 8 intersection points, and 6 domains that can contribute non-trivially to the differential.



The cohomology of the complex above, together with the 3 remaining points are the Khovanov homology of the right-handed Hopf link.

The dimensions of vector spaces in the complex

$$CF^{*,*}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \bigoplus_{\mathcal{P} \in \mathscr{B}E_{\mathcal{U}} \cap I_{\mathcal{U}}} \mathbb{C}\,\mathcal{P}$$

that counts holomorphic maps

grows polynomially with the number of crossings in our case,

which should be compared to exponential growth

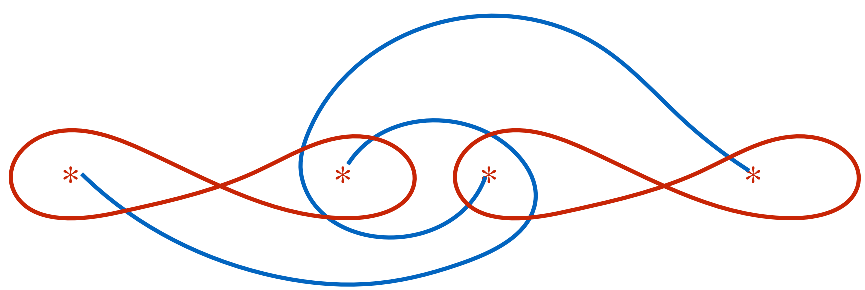
in Khovanov's case.

To evaluate the Euler characteristic

$$\chi(E, \mathscr{B}I) = \sum_{M,J\in\mathbb{Z}} (-1)^M \mathfrak{q}^J Hom_{\mathscr{D}_Y}(E, \mathscr{B}I[M]\{J\})$$

one simply counts the intersection points of Lagrangians, keeping track of gradings

$$\chi(E, \mathscr{B}I) = \sum_{\mathcal{P}\in E\cap\mathscr{B}I} (-1)^{M(\mathcal{P})} \mathfrak{q}^{J(\mathcal{P})}$$



The fact this computes the Jones polynomial is a theorem of Bigelow from the '90s.

The disk counting problem can be solved,

in the process making homological mirror symmetry that relates

$$\mathscr{D}_X \cong \mathscr{D}_Y ,$$

derived categories of the mirror pair

$$X \xleftrightarrow{\text{mirror}} Y ,$$

manifest.

This holds uniformly for any simply laced ${}^L\mathfrak{g}$

Working with the derived Fukaya-Seidel category,

$$\mathscr{D}_Y = D(\mathcal{F}\mathcal{S}(Y, W))$$

as opposed to Fukaya-Seidel category $\mathcal{F}\mathcal{S}(Y, W)$ itself

is actually simpler.

For one $\mathscr{D}_Y$ has far fewer objects,

as any deformation of branes that does not change the amplitudes is

an equivalence.

By contrast, in $\mathcal{F}\mathcal{S}(Y, W)$, one only gets

Hamiltonian isotopies of individual branes.

In particular,

one can generate $\mathscr{D}_Y = D(\mathcal{FS}(Y, W))$ by a finite set of branes,

which are thimbles of the potential

$$W = W^0/\kappa + \sum_{a=1}^{\mathrm{rk}} \lambda_a W^a.$$

For every a critical point $\mathcal{C} \in Y$ of the potential

$$\partial_y W(\mathcal{C}) = 0$$

we get a pair of “left” and “right” thimbles

$$T_{\mathcal{C}},\ I_{\mathcal{C}} \ \in \ \mathscr{D}_Y$$

which are, respectively, the set of all initial conditions for upward and downward gradient flows of $\mathrm{Re}W$ on which $\mathrm{Im}W$ is constant.

The critical point equations

$$\partial_y W(\mathcal{C}) = 0$$

are a variant of Bethe ansatz equations with an irregular singularity at $y = 0$

$$\sum_i \frac{\langle \mu_i, {}^L e_a \rangle}{y_{a,\alpha} - a_i}\, a_i - \sum_{(b,\beta)\neq(a,\alpha)} \frac{\langle {}^L e_b, {}^L e_a \rangle}{y_{a,\alpha} - y_{b,\beta}} (y_{a,\alpha} + y_{b,\beta})/2 = \lambda_a/\lambda_0,$$

They are isolated and non-degenerate

and labeled by the weights

$$\mathcal{C} \in (V_1 \otimes \ldots \otimes V_m)_\nu$$

in the weight space the conformal blocks transform in.

The set of thimbles depend on the chamber in equivariant parameter space

$$(\lambda, \kappa)$$

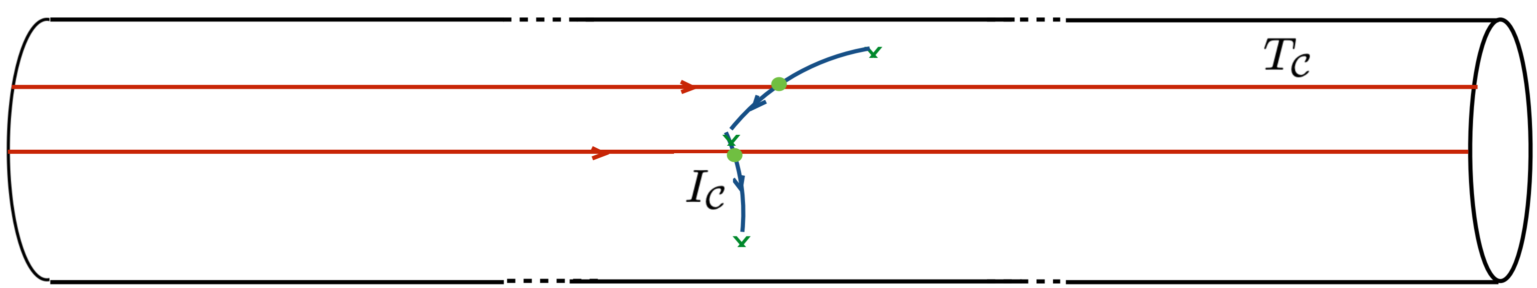
There is a choice of chamber which is suggested by mirror symmetry,

in which the left thimbles

$$T_{\mathcal{C}}$$

are products of real line Lagrangians,

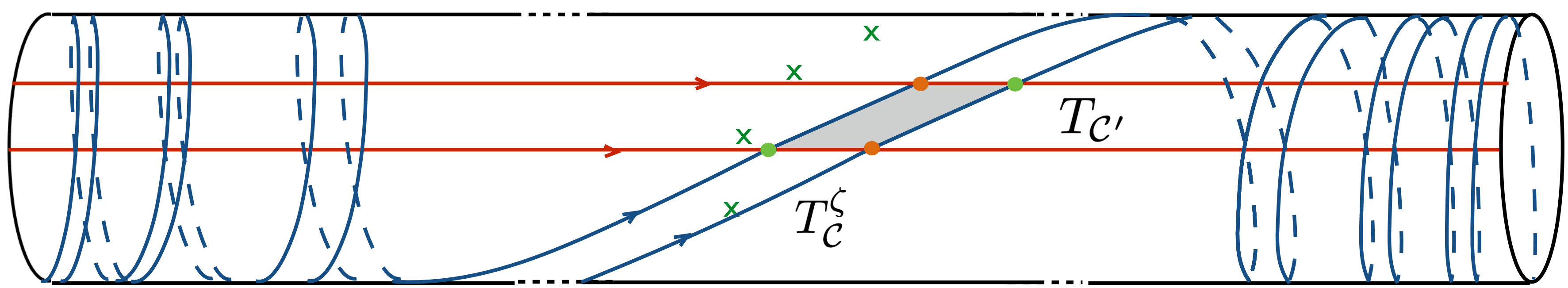
$$T_{\mathcal{C}} = T_{i_1} \times T_{i_2} \times \ldots \times T_{i_d}$$



which are mirror to vector bundles on X

To formulate Hom's between such branes

one has to deform one of them:



after which the intersection points become isolated.

There are still infinitely many of them,

since the branes mirror vector bundles on a non-compact space.

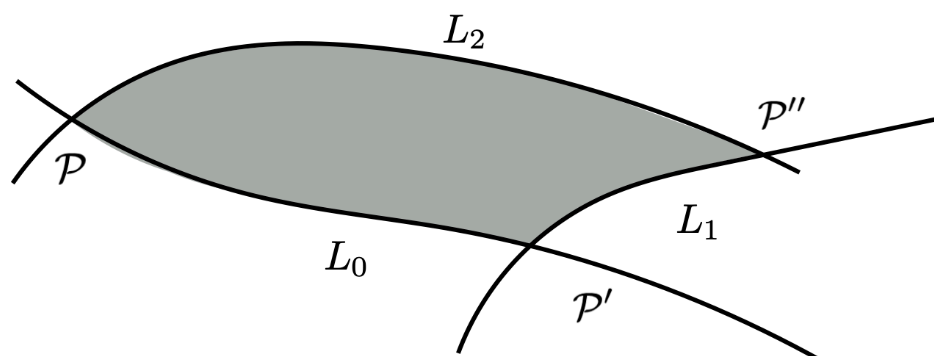
The thimbles $T = \oplus_{\mathcal{C}} T_{\mathcal{C}}$ generate an algebra

$$A = Hom^{*}_{\mathscr{D}_Y}(T,T) = \bigoplus_{\mathcal{C},\mathcal{C}'} \bigoplus_{\vec{J}\in\mathbb{Z}^2} Hom_{\mathscr{D}_Y}(T_{\mathcal{C}}, T_{\mathcal{C}'}\{\vec{J}\})$$

since the vector space A ,

$$A \times A \;\rightarrow\; A$$

inherits a product from Floer theory



All the algebra elements turn out to have cohomological degree zero

$$Hom_{\mathscr{D}_Y}(T_{\mathcal{C}}, T_{\mathcal{C}'}[n]\{\vec{d}\}) = 0, \text{ for all } n \neq 0, \text{ and all } \vec{d}.$$

in particular, the action of the differential is trivial,

This is not an accident, but a reflection of mirror symmetry which maps $T = \oplus_{\mathcal{C}} T_{\mathcal{C}}$ to the tilting generator of $\mathscr{D}_X$

The algebra A is an ordinary associative algebra

$$A = \bigoplus_{\mathcal{C},\mathcal{C}'} \bigoplus_{\vec{J}\in\mathbb{Z}^2} Hom_{\mathscr{D}_Y}(T_{\mathcal{C}}, T_{\mathcal{C}'}\{\vec{J}\})$$

graded by equivariant degrees.

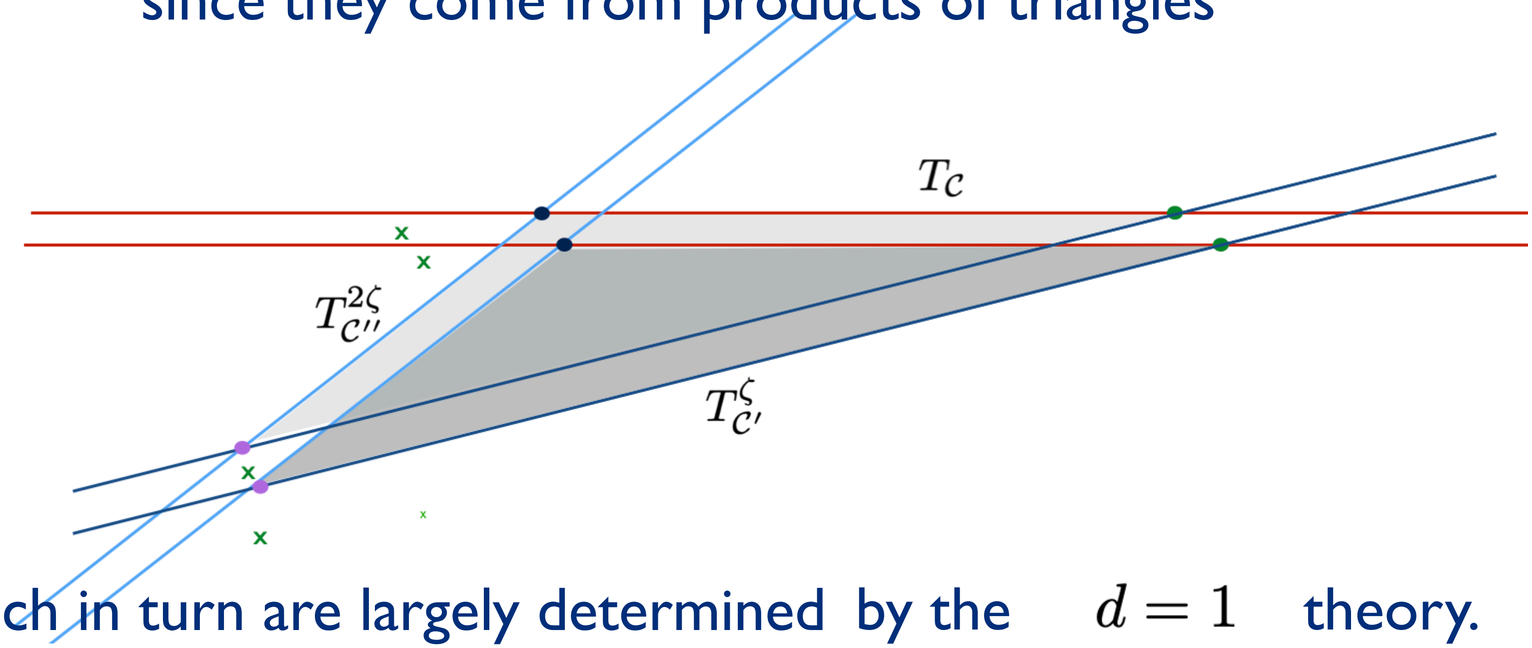
While there are not many holomorphic disk counts

that one can evaluate explicitly,

it turns out that all the ones that can contribute to

algebra products are computable,

since they come from products of triangles



which in turn are largely determined by the $d = 1$ theory.

Since the thimbles generate $\mathscr{D}_Y$

and everything there is to know about the thimbles

is contained in the algebra A

we get an equivalence of derived categories

$$\mathscr{D}_A \cong \mathscr{D}_Y$$

where $\mathscr{D}_A$ is the derived category of A -modules

which comes from the functor

$$Hom^*_{\mathscr{D}_Y}(T, -) : \mathscr{D}_Y \to \mathscr{D}_A$$

where $T = \bigoplus_{\mathcal{C}} T_{\mathcal{C}}$

The algebra A can always be thought of a path algebra of a quiver

whose nodes correspond to the critical points $\mathcal{C}$

and where paths from node $\mathcal{C}$ to node $\mathcal{C}'$ encode

$$\bigoplus_{\{\vec{d}\}} Hom_{\mathscr{D}_Y}(T_{\mathcal{C}'}, T_{\mathcal{C}}\{\vec{d}\})$$

For us these quivers always have closed loops,

in contrast to simpler theories with coming from

single valued potentials.

This is results in richer representation theory, and richer derived category.

For Y which is the equivariant mirror of the A_{m-1} surface

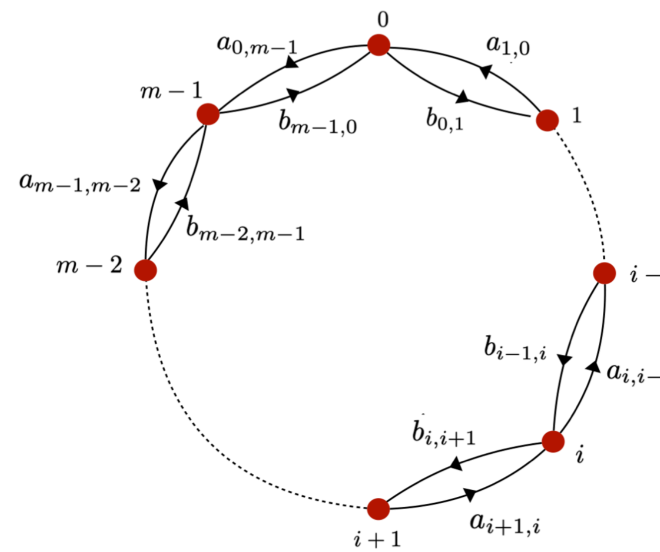
the algebra is the path algebra of the familiar quiver

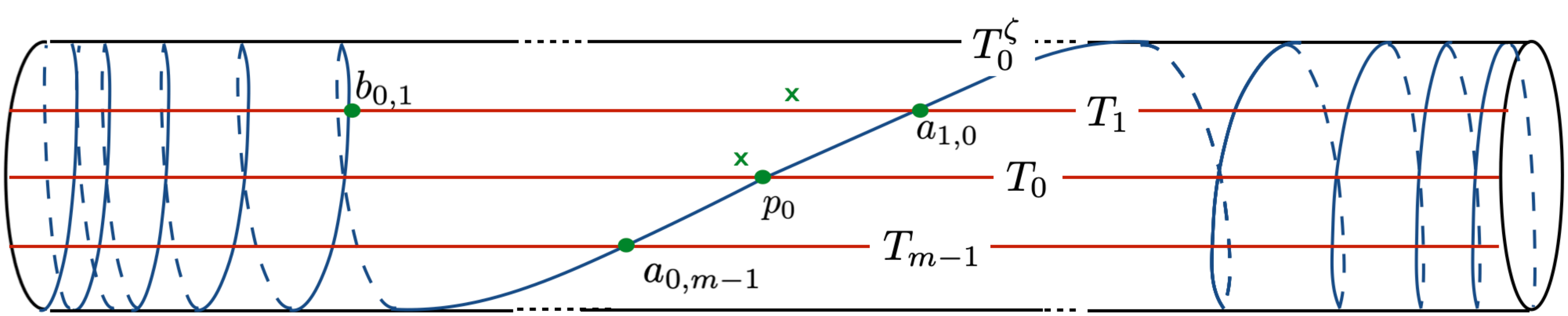
with unfamiliar relations

$$a_{i+1,i}\, b_{i,i+1} = 0, \quad \forall i.$$

which capture thimble intersection points and

relations between them:





The familiar quiver arizes in studying B-type branes on

the A_{m-1} surface, which is $\mathcal{X}$, the equivariant mirror of Y,

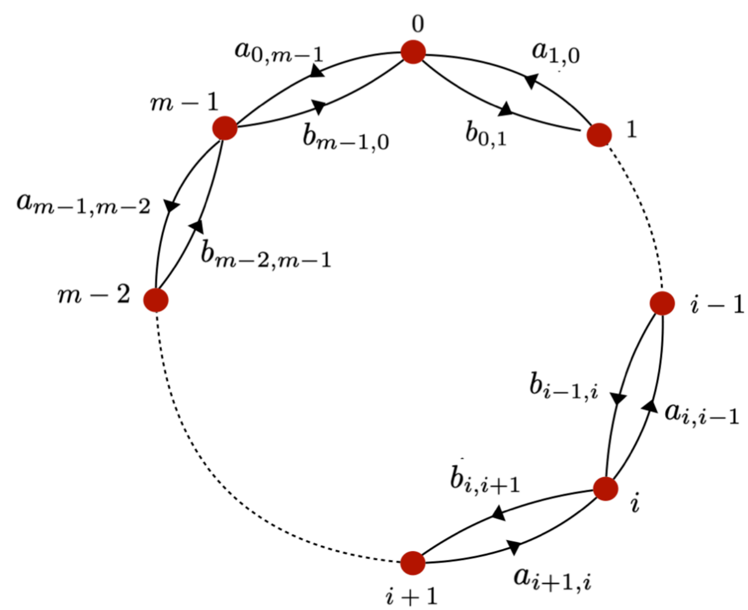
where one has the equivalence

$$\mathscr{D}_{\mathscr{A}} \cong \mathscr{D}_{\mathcal{X}}$$

with algebra $\mathscr{A}$ which is a path algebra

of the same quiver, with different relations:

$$b_{i,i+1}\, a_{i+1,i} = a_{i,i-1}\, b_{i-1,i},$$



Imposing instead the relation

$$a_{i+1,i}\, b_{i,i+1} = 0, \quad \forall i.$$

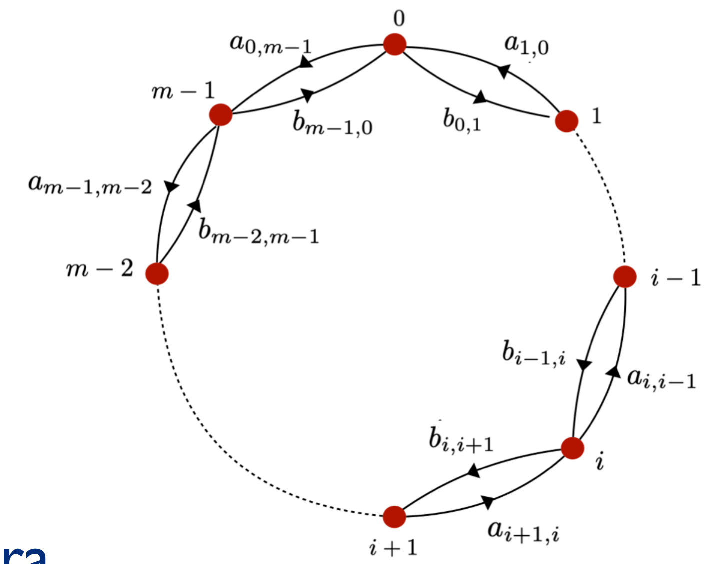
corresponds to restricting $\mathcal{X}$ to its core X

whose category of branes

$$\mathscr{D}_X \cong \mathscr{D}_A$$

is the category of modules of the smaller algebra

$$A = \mathscr{A}/\mathcal{I}$$



In this way, homological mirror symmetry becomes manifest:

$$\mathscr{D}_X \cong \mathscr{D}_A \cong \mathscr{D}_Y$$

This models how mirror symmetry should be understood more generally.

The algebra

$$A = Hom^*_{\mathscr{D}_Y}(T,T)$$

is computable explicitly for any ${}^L\mathfrak{g}$.

It has flavors of the algebras

that appeared in works Khovanov, Lauda and Rouquier

which Webster generalized,

in his algebraic approach to categorification.

The algebra, and description of how link invariants arise from it

which we will end up with is far simpler.

The virtue of the equivalence

$$\mathscr{D}_A \cong \mathscr{D}_Y$$

is that any brane

$$L \in \mathscr{D}_Y$$

has a "projective resolution"

$$L \cong L(T)$$

as a complex ,

$$L(T) = \dots \xrightarrow{t_2} L_2(T) \xrightarrow{t_1} L_1(T) \xrightarrow{t_0} L_0(T)$$

every term of which is a direct sum of thimble $T_{\mathcal{C}}$ branes.

The maps

$$L(T) = \ldots \xrightarrow{t_2} L_2(T) \xrightarrow{t_1} L_1(T) \xrightarrow{t_0} L_0(T)$$

encode a prescription for how to obtain the brane

$$L \in \mathscr{D}_Y$$

by starting with the direct sum of thimbles

$$\bigoplus_k L_k(T)[k]$$

and gluing them by deforming the differential away from the trivial to:

$$Q = \sum_k \; t_k$$

giving expectation values

$$t_i \in A = Hom^*_{\mathscr{D}_Y}(T,T)$$

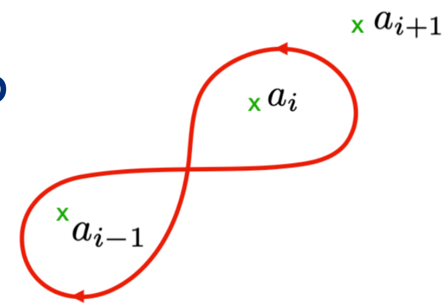
to tachyons at their intersections.

For example, in the case of

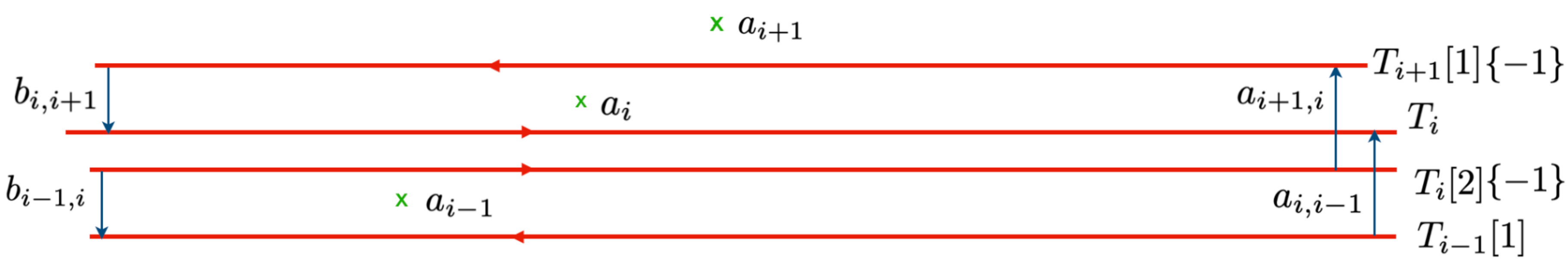
Y which is the equivariant mirror of the A_{m-1} surface

the figure eight brane which serves as a cap

has the resolution



$$E_i \cong T_i\{-1\} \xrightarrow{\begin{pmatrix} a_{i+1,i} \\ -b_{i-1,i} \end{pmatrix}} \begin{matrix} T_{i+1}\{-1\} \\ \oplus \\ T_{i-1} \end{matrix} \xrightarrow{\begin{pmatrix} b_{i,i+1} & a_{i,i-1} \end{pmatrix}} T_i$$



where one glues the branes at corresponding intersections.

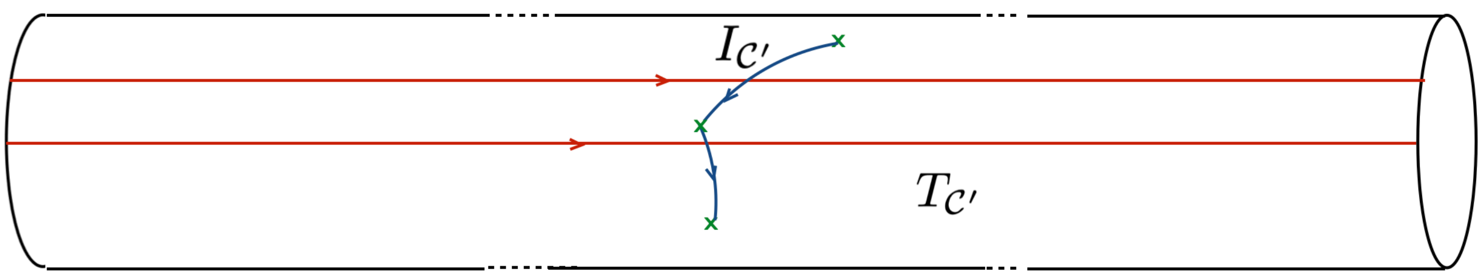
The derived category has a second description

in terms of the right thimbles $I = \bigoplus_{\mathcal{C}} I_{\mathcal{C}}$ instead

$$A^{\vee} = Hom^{*,*}_{\mathscr{D}_Y}(I, I)$$

so we get a pair of equivalences

$$\mathscr{D}_A \cong \mathscr{D}_Y \cong \mathscr{D}_{A^{\vee}}$$



The right thimbles are all compact, and dual to the left

$$Hom_{\mathscr{D}_Y}(T_{\mathcal{C}}, I_{\mathcal{C}'}) = \delta_{\mathcal{C},\mathcal{C}'} = Hom_{\mathscr{D}_Y}(I_{\mathcal{C}}, T_{\mathcal{C}'}[d])$$

which implies that

the algebra $A^{\vee}$ is related to A by Koszul duality,

which gives an algebraic way to understand the equivalence

$$\mathscr{D}_A \cong \mathscr{D}_Y \cong \mathscr{D}_{A^{\vee}}$$

The important part of this for us today is that

among the right thimble branes

$I_{\mathcal{U}}$

are the branes that serve as caps.

This has a striking consequence.

It means that we get a second, purely classical, description

of knot homology groups

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \bigoplus_{J\in\mathbb{Z}} Hom^{*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}\{J\})$$

which we can read off from the description of the brane

$$\mathscr{B}E_{\mathcal{U}} \cong \ldots \xrightarrow{e_1} \mathscr{B}E_1(T) \xrightarrow{e_0} \mathscr{B}E_0(T)$$

as a bound state of thimbles,

without any further work.

From the complex describing the brane

$$\mathscr{B}E_{\mathcal{U}} \cong \ldots \xrightarrow{e_1} \mathscr{B}E_1(T) \xrightarrow{e_0} \mathscr{B}E_0(T)$$

we get for free a complex of vector spaces

$$0 \to hom_A(\mathscr{B}E_0, I_{\mathcal{U}}\{J\}) \xrightarrow{e_0} hom_A(\mathscr{B}E_1, I_{\mathcal{U}}\{J\}) \xrightarrow{e_1} \ldots .$$

with the action of the differential

$$Q = \sum_k e_k$$

that squares to zero.

The space of ground states, and the link homology,

$$Hom^*_{\mathscr{D}_Y}(E_{\mathcal{U}}, \mathscr{B}I_{\mathcal{U}}\{\vec{J}\}) = H^*\big(hom_A(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}\{J\})\big)$$

is the cohomology of this complex.

Per construction, the vector space one gets

$$0 \to hom_A(\mathscr{B}E_0, I_{\mathcal{U}}\{J\}) \xrightarrow{e_0} hom_A(\mathscr{B}E_1, I_{\mathcal{U}}\{J\}) \xrightarrow{e_1} \ldots.$$

as the k-term in the complex

is spanned by the intersection points

$$\mathcal{P} \in \mathscr{B}E_{\mathcal{U}} \cap I_{\mathcal{U}}$$

of equivariant degree J and fermion number $M = k$

as is the Floer complex

$$CF^{*,*}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \bigoplus_{\mathcal{P} \in \mathscr{B}E_{\mathcal{U}} \cap I_{\mathcal{U}}} \mathbb{C}\,\mathcal{P}$$

The differential $Q = \sum_k e_k$ constructed classically,

$$\mathscr{B}E_{\mathcal{U}} \cong \ldots \xrightarrow{e_1} \mathscr{B}E_1(T) \xrightarrow{e_0} \mathscr{B}E_0(T)$$

from the geometry of the brane

sums up the action of instantons on it.

This is how the equivalence

$$\mathscr{D}_A \cong \mathscr{D}_Y$$

solves the knot categorification problem.

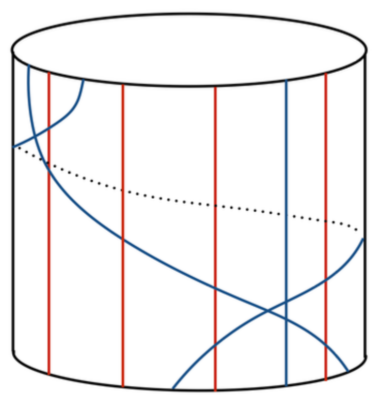
The algebra

$$A = Hom^*_{\mathscr{D}_Y}(T,T)$$

is best represented graphically.

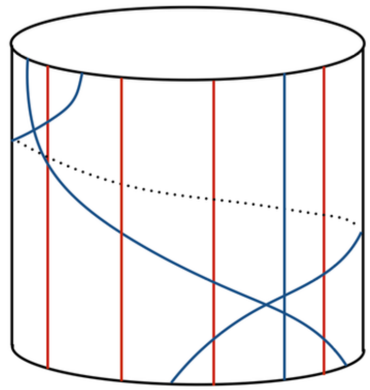
For ${}^L\mathfrak{g} = \mathfrak{su}_2$, the arbitrary element of the algebra is a configuration of

d blue strings

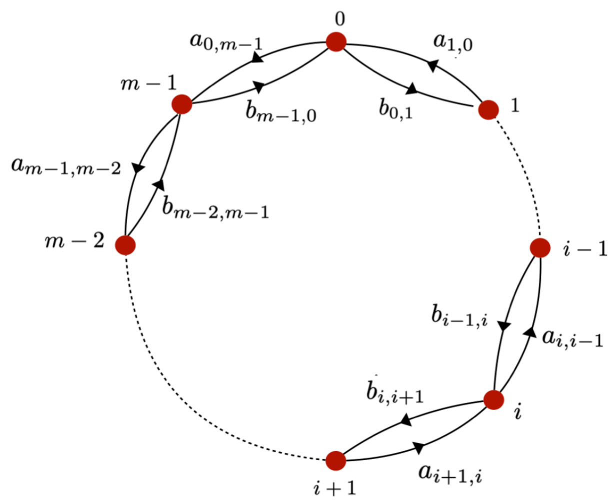


and m red ones, on a cylinder $\mathrm{C} = S^1 \times [0,1]$

A single string in the diagram

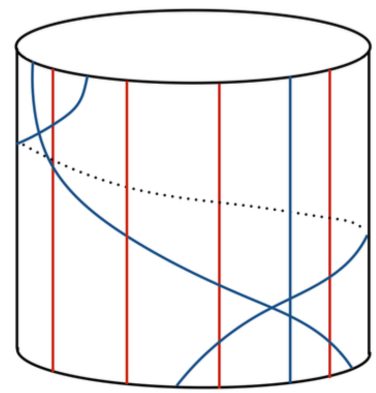


describes paths on the $d = 1$ quiver,



configurations of strings paths on the quiver one gets for general d .

Every string diagram is made of string bits,



which are one of four types:

p_i $\quad a_{i+1,i}$ $\quad b_{i,i+1}$ $\quad x_i$

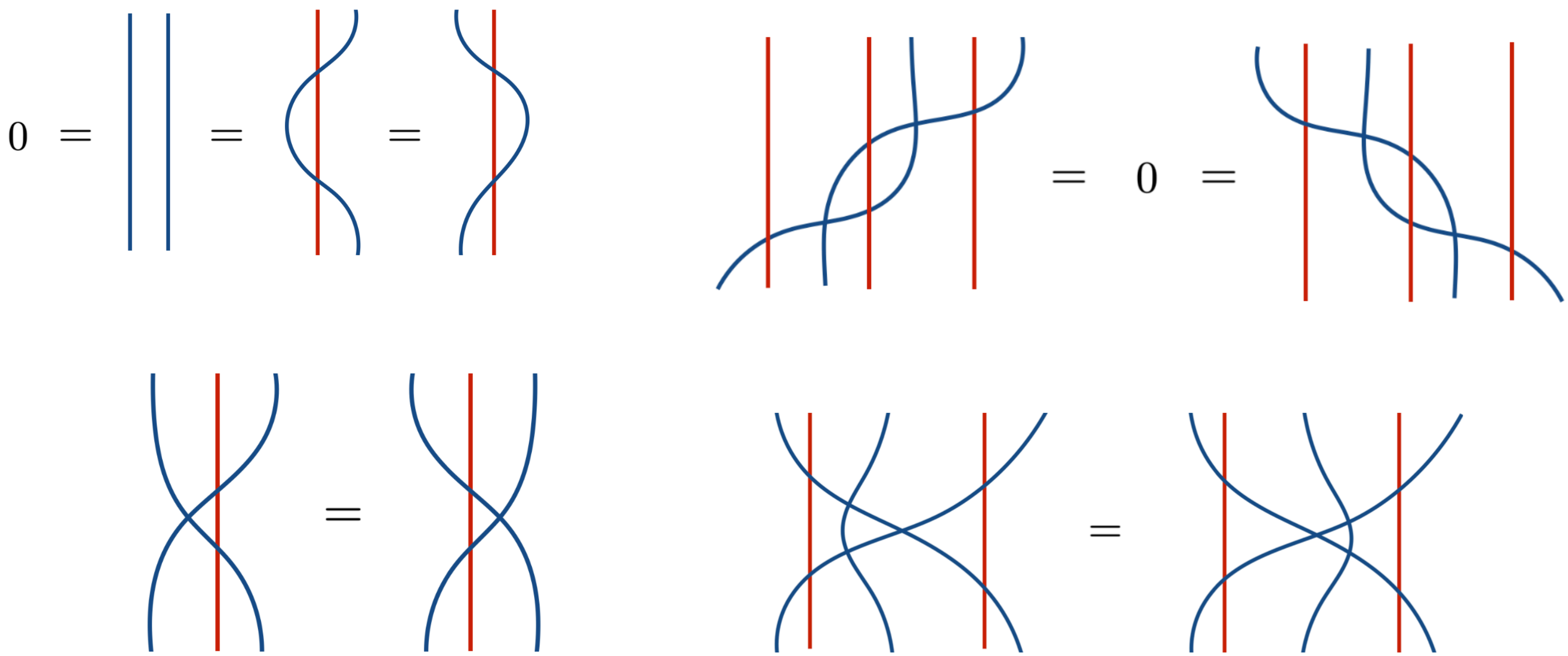
which carry specific equivariant degrees, and where the first three come from the one dimensional problem.

Algebra multiplication

$$A \times A \;\rightarrow\; A$$

is stacking cylinders.

The relations are



They say the blue strings need to be taut or the algebra element vanishes.

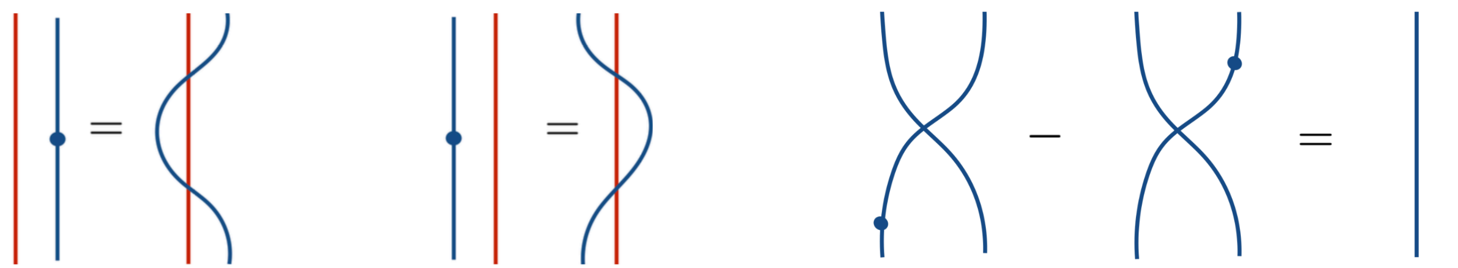
The KLRW algebra

$$\mathscr{A}$$

of Khovanov, Lauda, Roquier and Webster

are given in similar terms

but with more generators and different relations.



By a result of Webster, it describes the category of branes on $\mathcal{X}$,

so it is naturally more complicated.

At the moment at least,

the prescriptions for obtaining link invariants from the

KLRW

algebras are more opaque than

what I described,

perhaps because one lacked the geometric understanding

of the branes that serve as cups and caps.