Quantum Field Theory and Beyond

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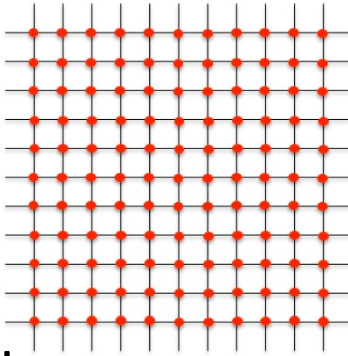
Lattice vs. continuum QFT

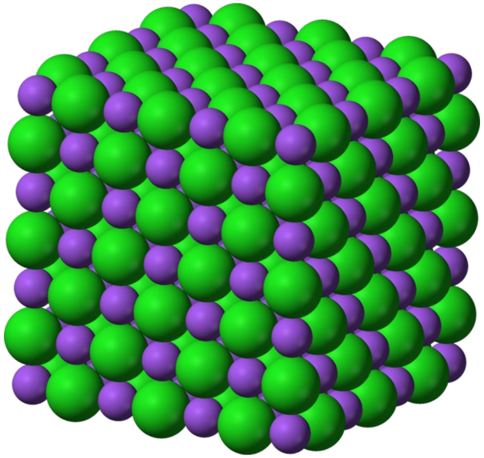
QFT is enormously successful. Yet, it is not mathematically rigorous.

One approach is to regularize it by placing it on a lattice.



- Then, the functional integral is well defined.
- Continuum limit: introduce a lattice spacing a, take $a \to 0$ and the number of sites to infinity holding all the lengths fixed.
  - Compute correlation functions at positions $a \ll x$.

In condensed matter physics, the problem is defined on a lattice and the goal is to find its low-energy/long-distance limit.



- It is expected to be described by an effective continuum field theory.

Challenges in using a lattice to define a given continuum QFT

- Does the limit exist and is it independent of the details of the lattice theory?
- Some continuum theories depend on the topology of field space, which relies on continuity of the fields. How is this captured by the lattice theory?
  - This issue affects various topological terms in the action, certain global symmetries, anomalies, etc.. More below.
- Some QFTs (e.g., theories with self-dual forms or fermions) do not admit a suitable Euclidean lattice action.
- Some QFTs do not even have a continuum Lagrangian, let alone a lattice version of it.

Challenges in finding a continuum low-energy QFT of a given lattice model

Exotic models, e.g., XY-plaquette model [Paramekanti, Balents, Fisher; …] (see below), fracton models [Chamon; Haah; Vijay, Haah, and Fu; …], do not have a standard continuum limit.

- They are characterized by an exact or emergent subsystem global symmetries […; Lawler, Fradkin; …] – separate symmetry group element for different subspaces. (More below.)
  - Observables vary at the lattice scale a, and hence they are discontinuous (and even singular) in the continuum limit.
  - Infinite ground state degeneracy in the continuum limit (sometimes no well defined limit).

UV/IR mixing – long-distance phenomena depend on short-distance details – reminiscent of some string theory constructions.

Exotic continuum QFT [NS, Shao]

Elements in our continuum theories

- Spacetime symmetries (in addition to translations)
  - No Lorentz invariance
  - No rotation symmetry (only discrete rotations)
- Impose exotic global symmetries and then gauge them
- Discontinuous fields and gauge parameters
  - Not as discontinuous as on the lattice – the allowed discontinuities are restricted
  - Universal – independent of most of the details at the lattice scale

Does this make sense?

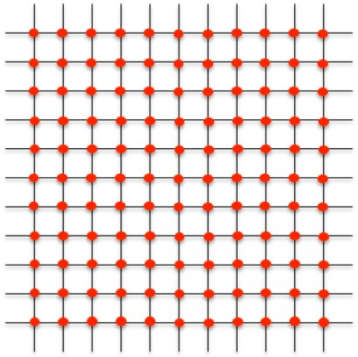
In order to explore it, let us review a more standard case…

Canonical example of lattice vs. continuum: XY-model in 1+1d

[…; Jose, Kadanoff, Kirkpatrick, Nelson; …]

Use a Euclidean-time, Lagrangian formulation.

On the lattice, phases $e^{i\phi}$ at the sites with the action



$$S = -\beta \sum_{links} \cos(\Delta_\mu \phi)$$

Global $U(1)$ symmetry (momentum)

$$\phi(x,\tau) \to \phi(x,\tau) + \alpha$$

The continuum theory (same β when it is large)

$$S = \frac{\beta}{2} \int d\tau dx (\partial_\mu \phi)^2$$

XY-model in 1+1d – the continuum theory

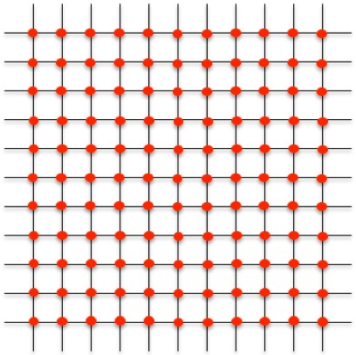
$$S = \frac{\beta}{2}\int d\tau dx \left(\partial_\mu \phi\right)^2$$

- This is the famous $c = 1$ compact boson. Free (quadratic action).
- Global symmetries $\partial_\mu j_\mu = 0$
  - $U(1)^m$ momentum $j_\mu^m = -i\beta\partial_\mu\phi$
  - $U(1)^w$ winding (vorticity), emergent $j_\mu^w = \frac{\epsilon_{\mu\nu}}{2\pi}\partial_\nu\phi$
  - Mixed ‘t Hooft anomaly between them
- Exact self-duality (T-duality): exchanging $\beta \leftrightarrow \frac{1}{(2\pi)^2\beta}$ and $U(1)^m \leftrightarrow U(1)^w$. Not present on the lattice.
- How much of this continuum discussion can be present on the lattice?

XY-model in 1+1d – modify the lattice theory

Following [...; Gross, Klebanov; ...; Sachdev, Park; ...], "suppress the vortices" on the lattice

Use the Villain formulation – replace $\phi \in S^1$ with $\phi \in \mathbb{R}$ coupled to a $\mathbb{Z}$ gauge field n_μ on the links



$$S_{Villain} = \frac{\beta}{2} \sum_{links} \left(\Delta_\mu \phi - 2\pi n_\mu\right)^2$$

$$\phi \sim \phi + 2\pi\, m$$

$$n_\mu \sim n_\mu + \Delta_\mu m$$

Suppress the vortices by adding the curvature square

$$\kappa \sum_{plaq} (\Delta_\tau n_x - \Delta_x n_\tau)^2$$

XY-model in 1+1d – getting closer to the continuum [Gorantla, Lam, NS, Shao]

$$\frac{\beta}{2}\sum_{links}\left(\Delta_\mu\phi - 2\pi n_\mu\right)^2 + \kappa\sum_{plaq}(\Delta_\tau n_x - \Delta_x n_\tau)^2$$

For $\kappa \to \infty$, the field strength of the $\mathbb{Z}$ gauge field, $\Delta_\tau n_x - \Delta_x n_\tau$ vanishes – the gauge field is flat.

We can replace the action by the modified Villain action

$$S_{mod.Villain} = \frac{\beta}{2}\sum_{links}\left(\Delta_\mu\phi - 2\pi n_\mu\right)^2 + i\sum_{plaq}\tilde{\phi}(\Delta_\tau n_x - \Delta_x n_\tau)$$

with a Lagrange multiplier field $\tilde{\phi} \sim \tilde{\phi} + 2\pi$.

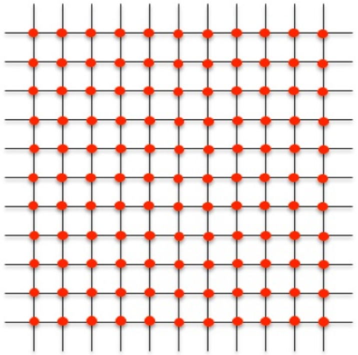
This lattice theory is similar to the continuum theory…

XY-model in 1+1d – getting closer to the continuum [Gorantla, Lam, NS, Shao]

$$S_{mod.Villain} = \frac{\beta}{2} \sum_{links} \left(\Delta_\mu \phi - 2\pi n_\mu\right)^2 + i \sum_{plaq} \tilde{\phi}(\Delta_\tau n_x - \Delta_x n_\tau)$$

- Free
- Exact global symmetries
  - $U(1)^m$ momentum, $\phi \to \phi + \alpha$, $\quad j_\mu^m = -i\,\beta(\Delta_\mu \phi - 2\pi n_\mu)$
  - $U(1)^w$ winding, $\quad \tilde{\phi} \to \tilde{\phi} + \tilde{\alpha}$, $\quad j_\mu^w = \frac{\epsilon_{\mu\nu}}{2\pi}(\Delta_\nu \phi - 2\pi n_\nu)$
  - ‘t Hooft anomaly. The symmetries act locally. But the Lagrangian density is not invariant; only e^{-S} is invariant.
- Using Poisson resummation, self-duality: $\phi \leftrightarrow \tilde{\phi}$, $\beta \leftrightarrow \frac{1}{(2\pi)^2 \beta}$

An exotic theory: XY-plaquette model in 2+1d [Paramekanti, Balents, Fisher; …]



We will use a Euclidean-time, Lagrangian formulation.

On the lattice, phases $e^{i\phi}$ at the sites with the action

$$S = -\beta_0 \sum_{\tau-links} \cos(\Delta_\tau \phi) - \beta \sum_{xy-plaq} \cos(\Delta_x \Delta_y \phi)$$

Global $U(1)$ subsystem (momentum) symmetry

$$\phi(x,y,\tau) \to \phi(x,y,\tau) + \alpha_x(x) + \alpha_y(y)$$

Related continuum theory

$$S = \int d\tau dx dy \left(\frac{\mu_0}{2} (\partial_\tau \phi)^2 + \frac{1}{2\mu} \left(\partial_x \partial_y \phi\right)^2 \right)$$

An exotic theory: XY-plaquette model in 2+1d [NS, Shao]

$$S = \int d\tau dx dy \left(\frac{\mu_0}{2} (\partial_\tau \phi)^2 + \frac{1}{2\mu} \left(\partial_x \partial_y \phi\right)^2 \right)$$

- Free
- Because of the derivative structure, some discontinuous field configurations are not suppressed. Some observables are discontinuous and even singular. UV/IR mixing.
- Subsystem global symmetries $\qquad \partial_\tau j_\tau = \partial_x \partial_y j_{xy}$
  - $U(1)^m$ momentum $\qquad j_\tau^m = i\mu_0 \partial_\tau \phi\,,\ \ j_{xy}^m = \frac{i}{\mu} \partial_x \partial_y \phi$
  - $U(1)^w$ winding (vorticity), emergent

    $$j_\tau^w = \frac{1}{2\pi} \partial_x \partial_y \phi\,,\ \ j_{xy}^w = \frac{1}{2\pi} \partial_\tau \phi$$
  - Mixed 't Hooft anomaly between them

An exotic theory: XY-plaquette model in 2+1d [NS, Shao]

- Exact self-duality – T-duality (not present on the lattice)
  - $\mu_0 \leftrightarrow \frac{\mu}{(2\pi)^2}$
  - $U(1)^m \leftrightarrow U(1)^w$

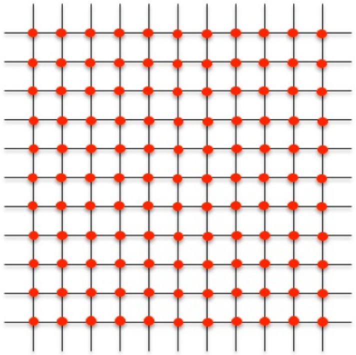
Many questions:

- How should we treat more precisely such a continuum field theory with discontinuous fields, discontinuous observables, and other peculiarities? Make the treatment more rigorous.
- How much of that depends on the continuum limit? Can we find these phenomena (winding subsystem symmetry, 't Hooft anomaly, self-duality, etc.) on the lattice?

XY-plaquette model in 2+1d – getting closer to the continuum [Gorantla, Lam, NS, Shao]

Repeat the discussion of the 1+1d XY-model for this model.

$$S = -\beta_0 \sum_{\tau-links} \cos(\Delta_\tau \phi) - \beta \sum_{xy-plaq} \cos\left(\Delta_x \Delta_y \phi\right)$$



Use the Villain form

$$S_{Villain} = \frac{\beta_0}{2} \sum_{\tau-links} (\Delta_\tau \phi - 2\pi n_\tau)^2 + \frac{\beta}{2} \sum_{xy-plaq} \left(\Delta_x \Delta_y \phi - 2\pi n_{xy}\right)^2$$

Here $\phi \in \mathbb{R}$, $n_\tau, n_{xy} \in \mathbb{Z}$ with the $\mathbb{Z}$ tensor gauge symmetry

$$\phi \sim \phi + 2\pi\, m$$
$$n_\tau \sim n_\tau + \Delta_\tau m$$
$$n_{xy} \sim n_{xy} + \Delta_x \Delta_y m$$

XY-plaquette model in 2+1d – getting closer to the continuum [Gorantla, Lam, NS, Shao]

- Add to the action the gauge invariant term

$$\kappa \sum_{cubes} \left(\Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau\right)^2$$

- For $\kappa \to \infty$ the field strength of the $\mathbb{Z}$ tensor gauge field (n_τ, n_{xy}), $\Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau$ vanishes. We can replace the action by the modified Villain action

$$S_{mod.Villain}$$
$$= \frac{\beta_0}{2} \sum_{\tau-links} (\Delta_\tau \phi - 2\pi n_\tau)^2 + \frac{\beta}{2} \sum_{xy-plaq} \left(\Delta_x \Delta_y \phi - 2\pi n_{xy}\right)^2$$
$$+ i \sum_{cubes} \tilde{\phi}(\Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau)$$

with a Lagrange multiplier $\tilde{\phi} \sim \tilde{\phi} + 2\pi$.

XY-plaquette model in 2+1d – getting closer to the continuum [Gorantla, Lam, NS, Shao]

$$S_{mod.\,Villain} = \frac{\beta_0}{2} \sum_{\tau-links} (\Delta_\tau \phi - 2\pi n_\tau)^2 + \frac{\beta}{2} \sum_{xy-plaq} \left(\Delta_x \Delta_y \phi - 2\pi n_{xy}\right)^2 + i \sum_{cubes} \tilde{\phi} (\Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau)$$

Similar to the continuum version of the XY-plaquette model:

- Free
- Exact subsystem global symmetries with mixed 't Hooft anomaly
  - $U(1)^m$ momentum $\phi(x,y,\tau) \to \phi(x,y,\tau) + \alpha_x(x) + \alpha_y(y)$
  - $U(1)^w$ winding $\tilde{\phi}(x,y,\tau) \to \tilde{\phi}(x,y,\tau) + \tilde{\alpha}_x(x) + \tilde{\alpha}_y(y)$
- Using Poisson resummation, self-duality: $\phi \leftrightarrow \tilde{\phi},\ \beta_0 \leftrightarrow \frac{1}{(2\pi)^2 \beta}$

Modified Villain versions of various standard lattice models [Gorantla, Lam, NS, Shao]

We analyzed many standard models with ordinary global or gauge (of various forms) $U(1)$ and $\mathbb{Z}_N$ symmetries in diverse dimensions.

Their modified Villain versions make manifest

- the emergent symmetries of the continuum theories and their ‘t Hooft anomalies
- the duality relations of the continuum theories

Modified Villain versions of various exotic lattice models [Gorantla, Lam, NS, Shao]

We have also analyzed many exotic models with subsystem symmetries

- Gapless, $U(1)$ subsystem symmetry
  - XY-plaquette in 3+1d (ϕ-theory) is dual to an exotic gauge theory ($\hat{A}$-theory)
  - tensor gauge theory (A-theory) in 3+1d is dual to a non-gauge theory ($\hat{\phi}$-theory)
  - $U(1)$ anisotropic model
- Gapped, $\mathbb{Z}_N$ subsystem symmetry
  - $\mathbb{Z}_N$ versions of these theories including the celebrated X-cube model of [Vijay, Haah, Fu].

Modified Villain versions of various exotic lattice models [Gorantla, Lam, NS, Shao]

In all these cases, the emergent symmetries and the dualities of the continuum theories are manifest on the lattice.

These lattice models provide a rigorous justification for the treatment of discontinuous fields in the continuum field theory descriptions of these models.

They confirm all our previous continuum results.

Summary

- It is common to use a lattice theory to define a continuum quantum field theory.
- The low-energy limit of a lattice theory is expected to be a continuum quantum field theory.
- Exotic lattice models are challenging counter examples, primarily because of their UV/IR mixing:
  - Subsystem global symmetry
  - Large ground state degeneracy
  - Discontinuous and even singular observables
- These seem incompatible with the standard framework of continuum QFT.
- The continuum field theory descriptions of these exotic models necessarily involve discontinuous fields.

Summary

- We considered previously studied, standard and exotic lattice models.
- We deformed them slightly by writing them in the Villain form. This makes them free theories (quadratic).
- A more significant deformation constrains the integer Villain gauge fields to be flat.
- The resulting lattice models, the modified Villain form of the original models, exhibit
  - new exact global symmetries and ‘t Hooft anomalies
  - new exact dualities
- These properties make these modified Villain models very close to the continuum models.

Summary of the summary

The continuum field theory descriptions of exotic models with subsystem global symmetries necessarily involve discontinuous fields.

The modified Villain versions of these models provide a rigorous justification for the analysis of these continuum field theories.

Thank you

Stay healthy