Instituto de Matemática Pura e Aplicada

Ramsey Theory for Sparse Graphs

Doctoral Thesis in Mathematics

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Abstract

In this thesis we address three problems in Graph Ramsey Theory: the size-Ramsey number of powers of trees, covering edge-colourings of random graphs by monochromatic trees, and monochromatic tiling in edge-coloured complete graphs.

Given a positive integer r, the *r-colour size-Ramsey number* of a graph H is the smallest integer m such that there exists a graph G with m edges for which any colouring of $E(G)$ with r colours has a monochromatic copy of H. In the first result in this thesis, we prove that for any positive integers k and r, the r-colour size-Ramsey number of the kth power of any n-vertex bounded degree tree is linear in n. As a corollary, we obtain that the r-colour size-Ramsey number of n-vertex graphs with bounded treewidth and bounded degree is linear in n.

In the second result in this thesis, we are interested in determining how many monochromatic trees are necessary to cover the vertices of an edge-coloured random graph. We show that if $p \gg n^{-1/6}(\ln n)^{1/6}$, then for every 3-edge-colouring of the random graph $G(n,p)$, there are three monochromatic trees such that their union covers all the vertices of $G(n,p)$. This improves, for three colours, a result of Bucić, Korándi and Sudakov.

In the third result of this thesis, we prove that for all integers $\Delta, r \geq 2$, there is a constant $C = C(\Delta, r) > 0$ such that the following holds for every sequence $\mathcal{F} = \{F_1, F_2, \ldots\}$ of graphs with $v(F_n) = n$ and $\Delta(F_n) \leq \Delta$: in every r-edge-coloured K_n, there is a collection of at most C monochromatic copies of graphs from $\mathcal{F}$ partitioning $V(K_n)$. This makes progress on a conjecture of Grinshpun and Sárközy.

Resumo

Nesta tese, abordamos três problemas na Teoria de Ramsey para Grafos: o número tamanho-Ramsey para potência de árvores, cobertura com árvores monocromáticas em colorações de arestas de grafos aleatórios, e azulejamento monocromático em grafos completos coloridos.

Dado um número inteiro positivo r, o *número tamanho-Ramsey com r cores* de um grafo H é o menor número inteiro m para o qual exista um grafo G com m arestas com a propriedade de que, em qualquer coloração de $E(G)$ com r cores, há uma cópia monocromática de H. No primeiro resultado desta tese, provamos que para quaisquer números inteiros positivos k e r, o número tamanho-Ramsey com r cores de uma k-potência de qualquer árvore com n vértices e grau máximo limitado é linear em n. Como corolário, obtemos que o número tamanho-Ramsey com r cores de grafos com n vértices e com largura de árvore limitada e grau máximo limitado é linear em n.

No segundo resultado desta tese, estamos interessados em determinar quantas árvores monocromáticas são necessários para cobrir os vértices de um grafo aleatório aresta-colorido. Mais precisamente, mostramos que se $p \gg n^{-1/6}(\ln n)^{1/6}$, então para cada 3-coloração das arestas do grafo aleatório $G(n,p)$ existem três árvores monocromáticas tais que a união delas cobre todos os vértices. Isso melhora, para três cores, um resultado de Bucić, Korándi and Sudakov.

No nosso terceiro resultado, provamos que para todos números inteiros $\Delta, r \geq 2$, existe uma constante $C = C(\Delta, r) > 0$, tal que o seguinte vale para toda sequência $\mathcal{F} = \{F_1, F_2, \ldots\}$ de grafos com $v(F_n) = n$ e $\Delta(F_n) \leq \Delta$: para toda r-aresta-coloração de K_n, existe uma coleção de no máximo C cópias monocromáticas de grafos em $\mathcal{F}$ particionando $V(K_n)$. Tal resultado é um progresso em uma conjectura de Grinshpun e Sárközy.

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Chapter 1

Introduction

The classical Ramsey problem for graphs asks whether there must exist monochromatic subgraphs in colourings of large graphs. Given graphs G and H and a positive integer r, we say that G is *r-Ramsey* for H, and we write $G \to (H)_r$, if in any r-colouring of the edges of G there is a monochromatic copy of H. For $r = 2$, we simply say that G is *Ramsey* for H and denote $G \to H$. The classical theorem of Ramsey [92] states that for every positive integers t and r, there exists an integer n such that $K_n \to (K_t)_r$. The *r-colour Ramsey number* $R_r(H)$ of a graph H is the minimum positive integer n such that $K_n \to (H)_r$. We denote by $R(H)$ the 2-colour Ramsey number of H.

Extensive research has been developed around Ramsey numbers, beginning with the work of Erdős and Szekeres [43], who in 1935 proved a recursion formula for the so called *off-diagonal Ramsey numbers* yielding the follow inequality:

$$R(K_t) \leq \binom{2t-2}{t-1}.$$

In particular, $R(K_t) \leq 2^{2t}$. In 1947, as one of the earliest application of the probabilistic method, Erdős [40] proved that $R(K_t) \geq 2^{t/2}$. Surprisingly, despite efforts of many researchers, the upper bound has only been improved by a sub-exponential factor (see [26]), and the lower bound has only been improved by Spencer [98] in 1975 by a factor of 2.

Ramsey numbers have been a vibrant research area in Combinatorics. The survey of Conlon, Fox and Sudakov [29] describes some of the results in the theory. Besides complete graphs, the most studied class of graphs has been the class of bounded-degree graphs. In 1983, Chvatál, Rödl, Szemerédi and Trotter [23], confirming a conjecture of Burr and Erdős [19], proved that for every positive integer Δ, there is a positive real number C such that if $\Delta(H) \leq \Delta$, then $R(H) \leq C|H|$. However, their proof, as an application of Szemerédi's regularity lemma, gave an upper bound for C that grows as a *tower* of height polynomial in Δ. This bound has been improved by Eaton [36], Graham, Rödl and Ruciński [52] and finally by Conlon, Fox and Sudakov [28], who proved in 2012 that there exists a constant c such that any graph H with maximum degree Δ satisfies $R(H) \leq 2^{c\Delta \log \Delta}|H|$. Conlon, Fox and Sudakov also conjectured (see [29]) that the logarithmic factor in the exponent is unnecessary.

Another important class of graphs that has been extensively explored in the literature is the class of graphs of bounded degeneracy. The *degeneracy* of a graph G is the smallest positive integer d such that every subgraph of G has minimum degree at most d. Burr and Erdős [19] conjectured in 1975 that for every positive integer d, there is a constant C_d such that for every graph H with degeneracy at most d we have $R(H) \leq C_d|H|$. This conjecture remained open for more than four decades. The first polynomial bound was established in 2004 by Kostochka and Rödl [75], who proved that $R(H) \leq C_d\Delta(H)|H|$, for every graph H with degeneracy at most d (in particular, this gives a quadratic upper bound). Kostochka and Sudakov [76] showed an almost linear upper bound using the dependent random choice technique and Fox and Sudakov [47] refined their method to prove that for every graph H with degeneracy at most d we have $R(H) \leq 2^{C_d\sqrt{\log|H|}}|H|$. The conjecture of Burr and Erdős was finally settled in 2017 by Lee [80] who proved that there exists a constant c for which every graph H with degeneracy at most d, chromatic number at most r and at least $2^{d^2 2^{cr}}$ vertices, satisfies $R(H) \leq 2^{d2^{cr}}|H|$. Since graphs with degeneracy at most d have chromatic number at most $d+1$, this gives the upper bound $R(H) \leq 2^{2^{Cd}}|H|$, for every graph H with degeneracy at most d (where C is an universal constant).

Substantial research has also been developed around the following asymmetric variant of the Ramsey numbers. Given graphs F and H, the *off-diagonal Ramsey number* of the pair (F, H), denoted by $R(F, H)$, is the smallest n such that every red-blue colouring of the edges of K_n contains a red copy of F or a blue copy of H. If F is connected, then $\chi(H)-1$ disjoint red cliques of order $|F|-1$ with all the edges between them coloured blue shows that $R(F, H) \geq (\chi(H)-1)(|F|-1)+1$. If we denote by $\sigma(H)$ the size of the smallest colour class in every optimal proper colouring of H, then we get the slightly better lower bound $R(F, H) \geq (\chi(H)-1)(|F|-1)+\sigma(H)$ by adding to the previous construction a red clique of order $\sigma(H)-1$ and colouring blue all the edges incident on this clique. This simple inequality due to Burr [18] has been shown to be tight for many pairs of graphs. We say that F is *H-good* if $R(F, H) = (\chi(H)-1)(|F|-1)+\sigma(H)$ and we say that F is *t-good* if it is K_t-good. Chvátal [24] showed that every tree is t-good, for every $t \in \mathbb{N}$. Burr and Erdős [20] showed that sufficiently large powers[1] of paths are t-good, while Allen, Brightwell and Skokan [3] generalized their result to H-goodness for every graph H (they in fact proved a more general result that covers many other classes of graphs besides powers of paths). Balla, Pokrovskiy and Sudakov [8] proved that sufficiently large bounded-degree trees are H-good, for every graph H. Fiz Pontiveros, Griffiths, Morris, Saxton and Skokan [44] showed that sufficiently large hypercubes are H-good, for every graph H. The reader can find more results about

[1] The kth power of a graph G is the graph G^k with vertex set $V(G)$ and edges consisting of pairs of vertices at distance at most k in G.

Ramsey goodness in the survey [29].

Historically, the theory of Ramsey numbers has been closely related to the theory of random sparse graphs. Indeed, the latter has been used to prove the existence of Ramsey graphs with peculiar structures. For instance, in 1986, Frankl and Rödl [48], motivated by a question of Erdős and Nešetřil [38], used the random graph $G(n, p)$ to construct a fairly small graph G such that $K_4 \nsubseteq G$ and $G \to K_3$. Those graphs were previously explicitly constructed by Nešetřil and Rödl [87] (in a more general context). However, the graphs they constructed were extremely large. Frankl and Rödl's result relied on proving that for every $\varepsilon > 0$, the random graph $G(n, p)$ on n vertices, where each edge is included independently with probability $p \geq n^{-1/2+\varepsilon}$, is Ramsey for K_3 with high probability. Łuczak, Ruciński and Voigt [84] improved this by showing that $n^{-1/2}$ is the threshold for the event $G(n, p) \to K_3$.

Since then, the study of Ramsey properties involving random graphs has become an active research area in combinatorics, with the most celebrated result being the theorem of Rödl and Ruciński [93] from 1995 that establishes the threshold for the symmetric Ramsey property $G(n, p) \to H$, for any graph H. In 1997, Kohayakawa and Kreuter [67] formulated a conjecture concerning the threshold for the asymmetric Ramsey property in $G(n, p)$. The conjectured upper bound for the threshold was proved under some assumptions by Kohayakawa, Schacht and Spöhel [71]. Recently, Mousset, Nenadov and Samotij [86] proved the upper bound in full generality, using the containers method of Balogh, Morris and Samotij [9] and Saxton and Thomason [97]. However, the conjectured lower bound for the threshold has only been proved for pairs of cycles [67] and pairs of cliques [85].

The *r-colour size-Ramsey number* $\hat{r}_r(H)$ of H is the minimum number of edges in a graph G such that $G \to (H)_r$. Erdős [39] asked in 1981 whether we have $\hat{r}_2(P_n) \gg n$. In 1983, Beck [10] answered Erdős' question negatively by proving that $\hat{r}_2(P_n) = O(n)$. His proof essentially consisted of showing that for some large constant C, with high probability, the random graph $G(Cn, n^{-1})$ is Ramsey for P_n. Alon and Chung [4] provided an explicit construction of graphs with $O(n)$ edges that are Ramsey for P_n. Beck also conjectured that for every positive integer Δ, there is a constant C such that for every tree T with $\Delta(T) \leq \Delta$, we have $\hat{r}_2(T) \leq C|T|$. This was proved by Friedman and Pippenger [49] in 1987, in a more general setting which also implies the corresponding result for arbitrarily many colours.

Recently, Clemens, Jenssen, Kohayakawa, Morrison, Mota and Reding [25] generalized Beck's result to powers of paths by proving that the 2-colour size-Ramsey number of the kth power of a path on n vertices is linear (as a function of n). This result was later extended to any fixed number r of colours by Han, Jenssen, Kohayakawa, Mota and Roberts [57]. In Chapter 2, in a work developed together with Berger, Kohayakawa, Maesaka, Martins, Mota, and Parczyk, we generalize the result from [57] to bounded powers of bounded degree trees. More precisely, we prove the following theorem.

Theorem I. *For every positive integers k, Δ and s, there exists $C > 0$ such that for any*

n-vertex tree T with $\Delta(T) \leq \Delta$, we have $\hat{r}_s(T^k) \leq Cn$.

Another important class of Ramsey-type problems concerns monochromatic covering and monochromatic partitioning of edge-coloured graphs. This line of research was initiated by Gerencsér and Gyárfás [51], who in 1967 proved, among other things, that for any 2-edge-colouring of K_n, there is a partition of $V(K_n)$ into 2 monochromatic paths. This result has been generalized in several ways. For instance, in 1979, Lehel (see [6]) conjectured that in every 2-edge-colouring of K_n, there are two monochromatic cycles[2] of different colours whose vertex sets partition $V(K_n)$. This conjecture was proved for sufficiently large n by Łuczak, Rödl and Szemerédi [83]; for smaller n, but still large, by Allen [2]; and finally, in 2010, Bessy and Thomassé [11] proved it for every n.

In a seminal paper from 1991, Erdős, Gyárfás and Pyber [42], in an attempt to generalize Gerencsér and Gyárfás' result, conjectured that for any r-edge-colouring of K_n there is a partition of $V(K_n)$ into r monochromatic paths. In 2014, Pokrovskiy [89] confirmed this conjecture for $r = 3$, however the conjecture is still open for larger values of r. Erdős, Gyárfás and Pyber conjectured further that one can partition $V(K_n)$ into r monochromatic cycles. For $r = 2$, this corresponds to Lehel's conjecture. Pokrovskiy [89] showed that this conjecture is false for $r \geq 3$ by providing an r-edge-colouring of the complete graph K_n such that any collection of r disjoint monochromatic cycles covers at most $n-1$ vertices. However, he conjectured that in every r-edge-colouring of K_n, there are r disjoint monochromatic cycles covering all but $O(1)$ vertices. Currently, the best result concerning partitions into monochromatic cycles is due to Gyárfás, Ruszinkó, Sárközy and Szemerédi [55] who proved in 2006 that in every r-edge-colouring of K_n, there are $O(r \log r)$ monochromatic cycles partitioning $V(K_n)$.

Erdős, Gyárfás and Pyber were also interested in generalizing the original result of Gerencsér and Gyárfás to partitioning into monochromatic trees instead of paths. Given a graph G and a positive integer r, let $\mathrm{tp}_r(G)$ denote the minimum number k for which in any r-edge-colouring of G, there are k monochromatic trees $T_1, \ldots, T_k$ such that their vertex sets partition $V(G)$, i.e.,

$$V(G) = V(T_1)\dot{\cup} \ldots \dot{\cup} V(T_k).$$

We define $\mathrm{tc}_r(G)$ analogously by not requiring the union above to be disjoint. In particular, $\mathrm{tc}_r(G) \leq \mathrm{tp}_r(G)$. An old remark commonly credited to Rado is that for every positive integer n we have $\mathrm{tp}_2(K_n) = 1$. Erdős, Gyárfás and Pyber proved that $\mathrm{tp}_3(K_n) = 2$ and they conjectured that for every $r \geq 2$, we have $\mathrm{tp}_r(K_n) \leq r-1$. Haxell and Kohayakawa [58] proved that for every $r \geq 3$, there exists n_0 such that $\mathrm{tp}_r(K_n) \leq r$, for $n \geq n_0$. Bal and DeBiasio [7] generalized Haxell and Kohayakawa's result by showing that for every

[2] In this thesis, we adopt the convention that a vertex corresponds to a cycle of size one, while an edge corresponds to a cycle of size two.

positive integer r there exists n_0 such that for every graph G with $n \geq n_0$ vertices and $\delta(G) \geq (1 - 1/er!)n$, we have $\mathrm{tp}_r(G) \leq r$. On the other hand, it is easy to see that $\mathrm{tc}_r(K_n) \leq r$, for every n. However, even a weaker version of Erdős, Gyárfás and Pyber's conjecture stating that $\mathrm{tc}_r(K_n) \leq r-1$ remains open for $r \geq 4$.

Gyárfás [54] noticed that a well-known conjecture of Ryser is equivalent to the statement that for every graph G and positive integer r we have $\mathrm{tc}_r(G) \leq (r-1)\alpha(G)$, where $\alpha(G)$ denotes the independence number of G. Ryser's conjecture for $r = 2$ is equivalent to König-Egerváry's theorem and for $r = 3$ has been proved by Aharoni [1] in 2001; however, it remains open for larger values of r. Haxell and Scott [61] proved in 2012 a weaker version of Ryser's conjecture for $r = 4$ and $r = 5$. They proved that there is some $\varepsilon > 0$ such that we have $\mathrm{tc}_r(G) \leq (r - \varepsilon)\alpha(G)$, for every graph G and $r \in \{4, 5\}$.

In 2017, Bal and DeBiasio [7], motivated by the work of Rödl and Ruciński [93] on the Ramsey property of random graphs, initiated the study of covering random graphs by monochromatic trees. They conjectured that for any $r \geq 2$, the threshold for the event $\mathrm{tc}_r(G(n,p)) \leq r$ has order $(\log n/n)^{1/r}$. This conjecture was verified for $r = 2$ by Kohayakawa, Mota and Schacht [68] (they actually showed that $tp_2(G(n,p)) \leq 2$ for the conjectured range of p). However, Ebsen, Mota and Schnitzer[3] showed that it does not hold for larger values of r.

Korándi, Mousset, Nenadov, Škorić and Sudakov [74] investigated the problem of covering random graphs by monochromatic cycles. They proved that for $p \geq n^{-1/r+\varepsilon}$, with high probability, in any r-edge-colouring of $G = G(n,p)$, there is a collection of at most $O(r^8 \log r)$ monochromatic cycles covering $V(G)$. Lang and Lo [79] proved that for $p \geq n^{-1/2r}$, with high probability in every r-edge-colouring of $G = G(n,p)$, there is a collection of at most $O(r^4 \log r)$ monochromatic cycles partitioning $V(G)$.

In a recent work, Bucić, Korándi and Sudakov [17] analysed the behaviour of $\mathrm{tc}_r(G(n,p))$ for every $r \geq 2$. In Chapter 3, in a work developed together with Kohayakawa, Mota and Schülke, we improve their results for $r = 3$. More precisely, we show the following:

Theorem II. *If $p = p(n)$ satisfies $p \gg \big(\frac{\log n}{n}\big)^{1/6}$, then with high probability we have*

$$\mathrm{tc}_3\big(G(n,p)\big) \leq 3.$$

It is not hard to see that Theorem II cannot be improved by reducing the number of trees unless p is very close to 1. Indeed, let $\{v_1, v_2, v_3\}$ be an independent set in $G(n,p)$, then colour all the edges incident on v_i with the colour i, for $i \in \{1, 2, 3\}$, and colour all the remaining edges of $G(n,p)$ in any way. This colouring shows that with high probability we have $\mathrm{tc}_3(G(n,p)) \geq 3$, for $p \leq 1 - O(n^{-1})$. However, we believe that the lower bound for p in Theorem II can be improved to $\big(\frac{\log n}{n}\big)^{1/4}$.

[3] Their proof is described in [68].

As we mentioned earlier, Erdős, Gyárfás and Pyber [42] proved that for every r-edge-colouring of K_n, it is possible to partition $V(K_n)$ into a bounded number (depending on r) of monochromatic paths, trees or even cycles. Grinshpun and Sárközy [53] extended this result to more general sequences of graphs. Let $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ be an infinite sequence of graphs with $|F_i| = i$, for each $i \in \mathbb{N}$. Given an r-colouring of the edges of the complete graph K_n, a *monochromatic $\mathcal{F}$-tiling* of size t is a collection of monochromatic vertex-disjoint graphs $G_1, \ldots, G_t$, each of which is isomorphic to some member of $\mathcal{F}$, and such that

$$V(K_n) = V(G_1)\dot{\cup} \cdots \dot{\cup} V(G_t).$$

Let us write $\tau_r(n, \mathcal{F})$ for the minimum $t \in \mathbb{N}$ such that for every r-edge-colouring of the edges of K_n, there is a monochromatic $\mathcal{F}$-tiling of size at most t. The *r-colour tiling number* of $\mathcal{F}$ is defined as

$$\tau_r(\mathcal{F}) := \sup_{n\in\mathbb{N}} \tau_r(n, \mathcal{F}).$$

Grinshpun and Sárközy [53] proved that for every positive integer Δ, there is a positive number C such that if $\mathcal{F}$ is a sequence of graphs with maximum degree at most Δ, then $\tau_2(\mathcal{F}) \leq 2^{C\Delta \log \Delta}$. In particular, the 2-colour tiling number of a sequence of bounded-degree graphs is finite. They conjectured that the r-colour tiling number of a sequence of bounded-degree graphs should also be finite and have at most an exponential growth with Δ. In Chapter 4, in a joint work with Corsten, we prove that the r-colour tiling number of a sequence of bounded-degree graphs is indeed finite by establishing a triple-exponential bound. More precisely, we prove the following.

Theorem III. *There is an absolute constant $K > 0$ such that for all integers $r, \Delta \geq 2$, we have*

$$\tau_r(\mathcal{F}) \leq \exp^2\left(r^{Kr\Delta^3}\right),$$

for every sequence $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ of graphs with $|F_i| = i$ and $\Delta(F_i) \leq \Delta$, for each $i \in \mathbb{N}$.

The proof of Theorem III combines ideas from the absorption method as in the original paper of Erdős, Gyárfás and Pyber [42] with some modern approaches involving the blow-up lemma and the weak regularity lemma of Duke, Lefmann and Rödl [35].

Chapter 2

Size-Ramsey Number of Powers of Bounded Degree Trees

2.1 Introduction

Given graphs G and H and a positive integer s, we denote by $G \to (H)_s$ the property that any s-colouring of the edges of G contains a monochromatic copy of H. We are interested in the problem proposed by Erdős, Faudree, Rousseau and Schelp [41] of determining the minimum integer m for which there is a graph G with m edges such that property $G \to (H)_s$ holds. Formally, the *s-colour size-Ramsey number* $\hat{r}_s(H)$ of a graph H is defined as follows:

$$\hat{r}_s(H) = \min\{e(G)\colon G \to (H)_s\}.$$

Answering a question posed by Erdős [39], Beck [10] showed that $\hat{r}_2(P_n) = O(n)$ by means of a probabilistic proof. Alon and Chung [4] proved the same fact by explicitly constructing a graph G with $O(n)$ edges such that $G \to (P_n)_2$. In the last decades many successive improvements were obtained in order to determine the size-Ramsey number of paths (see, e.g., [10, 12, 34] for lower bounds, and [10, 33, 81, 34] for upper bounds). The best known bounds for paths are $5n/2 - 15/2 \leq \hat{r}_2(P_n) \leq 74n$ from [34]. For any $s \geq 2$ colours, Dudek and Prałat [34] and Krivelevich [78] proved that there are positive constants c and C such that $cs^2n \leq \hat{r}_s(P_n) \leq Cs^2(\log s)n$.

Moving away from paths, Beck [10] asked whether $\hat{r}_2(H)$ is linear for any bounded degree graph. This question was later answered negatively by Rödl and Szemerédi [94], who constructed a family $\{H_n\}_{n\in\mathbb{N}}$ of n-vertex graphs of maximum degree $\Delta(H_n) \leq 3$ such that $\hat{r}_2(H_n) = \Omega(n \log^{1/60} n)$. The current best upper bound for the size-Ramsey number of graphs with bounded degree was obtained in [70] by Kohayakawa, Rödl, Schacht and

The work described in this chapter was developed in a joint project with Sören Berger, Yoshiharu Kohayakawa, Giulia Satiko Maesaka, Taísa Martins, Guilherme Oliveira Mota and Olaf Parczyk.

Szemerédi, who proved that for any positive integer Δ there is a constant c such that, for any graph H with n vertices and maximum degree Δ, we have

$$\hat{r}_2(H) \leq cn^{2-1/\Delta}\log^{1/\Delta} n.$$

For more results on the size-Ramsey number of bounded degree graphs see [30, 49, 59, 60, 66, 69].

Let us turn our attention to powers of bounded degree graphs. Let H be a graph with n vertices and let k be a positive integer. The *kth power* H^k of H is the graph with vertex set $V(H)$ in which there is an edge between distinct vertices u and v if and only if u and v are at distance at most k in H. Recently it was proved that the 2-colour size-Ramsey number of powers of paths and cycles is linear [25]. This result was extended to any fixed number s of colours in [57], i.e.,

$$\hat{r}_s(P_n^k) = O_{k,s}(n) \quad \text{and} \quad \hat{r}_s(C_n^k) = O_{k,s}(n). \tag{2.1}$$

The main result in this chapter (Theorem I) extends (2.1) to bounded powers of bounded degree trees. We prove that for any positive integers k and s, the s-colour size-Ramsey number of the kth power of any n-vertex bounded degree tree is linear in n.

Theorem I. *For every positive integers k, Δ and s, there exists $C > 0$ such that for any n-vertex tree T with $\Delta(T) \leq \Delta$, we have $\hat{r}_s(T^k) \leq Cn$.*

We remark that Theorem I is equivalent to the following result for the 'general' or 'off-diagonal' size-Ramsey number $\hat{r}(H_1, \ldots, H_s) = \min\{e(G) \colon G \to (H_1, \ldots, H_s)\}$: if $H_i = T_i^k$ for $i = 1, \ldots, s$ where $T_1, \ldots, T_s$ are bounded degree trees, then $\hat{r}(H_1, \ldots, H_s)$ is linear in $\max_{1 \leq i \leq s} v(H_i)$. To see this, it is sufficient to apply Theorem I to a tree containing the disjoint union of $T_1, \ldots, T_s$.

The graph that we present to prove Theorem I does not depend on T, but only on Δ, k and n. Moreover, our proof not only gives a monochromatic copy of T^k for a given T, but a monochromatic subgraph that contains a copy of the kth power of every n-vertex tree with maximum degree at most Δ. That is, we prove the existence of so called 'partition universal graphs' with $O_{k,\Delta,s}(n)$ edges for the family of powers T^k of n-vertex trees with $\Delta(T) \leq \Delta$.

Recently, Kamčev, Liebenau, Wood, and Yepremyan [64] proved, among other things, that the 2-colour size-Ramsey number of an n-vertex graph with bounded degree and bounded treewidth is $O(n)$[1]. This is equivalent to our result for $s = 2$. Indeed, any graph with bounded treewidth and bounded maximum degree is contained in a suitable blow-up of some bounded degree tree [32, 99] and a blow-up of a bounded degree tree is contained in the power of another bounded degree tree. Conversely, bounded powers of bounded degree trees

[1] They in fact formulate this for the general 2-colour size-Ramsey number $\hat{r}(H_1, H_2)$.

have bounded treewidth and bounded degree. Therefore, we obtain the following equivalent version of Theorem I, which generalises the result from [64] and answers one of their main open questions (Question 5.2 in [64]).

Corollary 2.1.1. *For any positive integers k, Δ and s and any n-vertex graph H with treewidth k and $\Delta(H) \leq \Delta$, we have*

$$\hat{r}_s(H) = O_{k,\Delta,s}(n).$$

The proof of Theorem I follows the strategy developed in [57], proving the result by induction on the number of colours s. Very roughly speaking, we start with a graph G with suitable properties and, given any s-colouring of the edges of G ($s \geq 2$), either we obtain a monochromatic copy of the power of the desired tree in G, or we obtain a large subgraph H of G that is coloured with at most $s-1$ colours; moreover, the graph H that we obtain is such that we can apply the induction hypothesis on it. Naturally, we design the requirements on our graphs in such a way that this induction goes through. As it turns out, the graph G will be a certain blow-up of a random-like graph. While this approach seems uncomplicated upon first glance, the proof requires a variety of additional ideas and technical details.

To implement the above strategy, we need, among other results, two new and key ingredients which are interesting on their own: (i) a result that states that for any sufficiently large graph G, either G contains a large expanding subgraph or there is a given number of reasonably large disjoint subsets of $V(G)$ without any edge between any two of them (see Lemma 2.3.4); (ii) an embedding result that states that in order to embed a power T^k of a tree T in a certain blow-up of a graph G it is enough to find an embedding of an auxiliary tree T' in G (see Lemma 2.3.6).

2.2 Auxiliary results

In this section we state a few results which will be needed in the proof of Theorem I. The first lemma guarantees that, in a graph G that has edges between large subsets of vertices, there exists a long "transversal" path along a constant number of large subsets of vertices of G. Denote by $e_G(X,Y)$ the number of edges between two disjoint sets X and Y in a graph G.

Lemma 2.2.1 ([25, Lemma 3.5])**.** *For every integer $\ell \geq 1$ and every $\gamma > 0$ there exists $d_0 = 2 + 4/(\gamma(\ell+1))$ such that the following holds for any $d \geq d_0$. Let G be a graph on dn vertices such that for every pair of disjoint sets $X, Y \subseteq V(G)$ with $|X|, |Y| \geq \gamma n$ we have $e_G(X,Y) > 0$. Then for every family $V_1, \ldots, V_\ell \subseteq V(G)$ of pairwise disjoint sets each of size* *at least γdn, there is a path $P_n = (x_1, \ldots, x_n)$ in G with $x_i \in V_j$ for all $1 \leq i \leq n$, where $j \equiv i \pmod{\ell}$.*

We will also use the classical Chernoff's inequality and Kővári–Sós–Turán theorem.

Theorem 2.2.2 (Chernoff's inequality). *Let $0 < \varepsilon \leq 3/2$. If X is a sum of independent Bernoulli random variables then*

$$\mathbb{P}(|X - \mathbb{E}[X]| > \varepsilon\mathbb{E}[X]) \leq 2 \cdot e^{-(\varepsilon^2/3)\mathbb{E}[X]}.$$

Theorem 2.2.3 (Kővári-Sós–Turán [77]). *Let $k \geq 1$ and let G be a bipartite graph with x vertices in each vertex class. If G contains no copy of $K_{2k,2k}$, then G has at most $4x^{2-1/(2k)}$ edges.*

2.3 Bijumbledness, expansion and embedding of trees

In this section we provide the necessary tools to obtain the desired monochromatic embedding of a power of a tree in the proof of Theorem I. We start by defining the expanding property of a graph.

Property 2.3.1 (Expanding). *A graph G is (n, a, b)-*expanding *if for all $X \subseteq V(G)$ with $|X| \leq a(n-1)$, we have $|N_G(X)| \geq b|X|$.*

Here $N_G(X)$ is the set of neighbours of X, i.e. all vertices in $V(G)$ that share an edge with some vertex from X. The following embedding result due to Friedman and Pippenger [49] guarantees the existence of copies of bounded degree trees in expanding graphs.

Lemma 2.3.2. *Let n and Δ be positive integers and G a non-empty graph. If G is $(n, 2, \Delta+1)$-expanding, then G contains any n-vertex tree with maximum degree Δ as a subgraph.*

Owing to Lemma 2.3.2, we are interested in graph properties that guarantee expansion. One such property is bijumbledness, defined below.

Property 2.3.3 (Bijumbledness). *A graph G on N vertices is (p, θ)-*bijumbled *if, for all disjoint sets X and $Y \subseteq V(G)$ with $\theta/p < |X| \leq |Y| \leq pN|X|$, we have $\big|e_G(X,Y) - p|X||Y|\big| \leq \theta\sqrt{|X||Y|}$.*

Note that bijumbledness immediately implies that

$$\text{for all disjoint sets } X, Y \subseteq V(G) \text{ with } |X|, |Y| > \theta/p \text{ we have } e_G(X,Y) > 0. \tag{2.2}$$

Moreover, a simple averaging argument guarantees that in a (p,θ)-bijumbled graph G on N vertices we have

$$\left| e(G) - p\binom{N}{2} \right| \leq \theta N. \tag{2.3}$$

We now state the first main novel ingredient in the proof of our main result (Theorem I). The following lemma ensures that in a sufficiently large graph we get an expanding subgraph with appropriate parameters or we get reasonably large disjoint subsets of vertices that span no edges between them. This result was inspired by [88, Theorem 1.5]. Furthermore, we remark that similar results have been proved in [91, 90].

Lemma 2.3.4. *Let $f \geq 0$, $D \geq 0$, $\ell \geq 2$ and $\eta > 0$ be given and let $A = (\ell-1)(D+1)(\eta + f) + \eta$.*

If G is a graph on at least An vertices, then

(*i*) *there is a non-empty set $Z \subseteq V(G)$ such that $G[Z]$ is (n,f,D)-expanding, or*

(*ii*) *there exist disjoint $V_1,\ldots,V_\ell \subseteq V(G)$ such that $|V_i| \geq \eta n$ for $1 \leq i \leq \ell$ and $e_G(V_i,V_j) = 0$ for $1 \leq i < j \leq \ell$.*

Proof. Let us assume that (*i*) does not hold. Since G is not (n,f,D)-expanding, we can take $V_1 \subseteq V(G)$ of maximum size satisfying that $|V_1| \leq (\eta+f)n$ and $|N_G(V_1)| < D|V_1|$. We claim that $|V_1| \geq \eta n$. Assume, for the sake of contradiction that $|V_1| < \eta n$. Let

$$W_1 = V(G) \setminus (V_1 \cup N_G(V_1)).$$

Then $|W_1| > An - (D+1)\eta n > 0$. Applying that (*i*) does not hold, we get $X \subseteq W_1$ such that $|X| \leq f(n-1)$ and $|N_{G[W_1]}(X)| < D|X|$. Note that $N_G(X) \subseteq N_{G[W_1]}(X) \cup N_G(V_1)$. Thus

$$|N_G(X \dot\cup V_1)| = |N_{G[W_1]}(X) \cup N_G(V_1)| < D(|X| + |V_1|).$$

Also $|X \dot\cup V_1| \leq (\eta+f)n$, deriving a contradiction to the maximality of V_1.

Let $1 \leq k \leq \ell - 2$ and suppose we have disjoint sets $(V_1,\ldots,V_k)$ such that

(I) $|V_i| \geq \eta n$, for $1 \leq i \leq k$;

(II) $e(V_i,V_j) = 0$, for $1 \leq i < j \leq k$;

(III) $|\bigcup_{i=1}^k (V_i \cup N_G(V_i))| < k(D+1)(\eta+f)n$.

We can increase this sequence in the following way. Let $W_k = V(G) \setminus \bigcup_{i=1}^k (V_i \cup N_G(V_i))$ and note that

$$|W_k| \overset{\text{(III)}}{\geq} An - (\ell-2)(D+1)(\eta+f)n \geq (D+1)(\eta+f)n + \eta n > 0.$$

Since (i) does not hold, there exists $V_{k+1} \subseteq W_k$ of maximum size with $|V_{k+1}| \le (\eta + f)n$ such that $|N_{G[W_k]}(V_{k+1})| < D|V_{k+1}|$. Note that $e_G(V_i, V_{k+1}) \le e_G(V_i, W_k) = 0$, for every $1 \le i \le k$. Therefore we have that (II) holds for the sequence $(V_1, \dots, V_{k+1})$. Furthermore, note that

$$N_G(V_{k+1}) \subseteq \bigcup_{i=1}^{k} N_G(V_i) \cup N_{G[W_k]}(V_{k+1})\,. \tag{2.4}$$

This gives us (III) for the sequence $(V_1, \dots, V_{k+1})$, since

$$\begin{aligned}\left|\bigcup_{i=1}^{k+1}(V_i \cup N_G(V_i))\right| &\overset{(2.4)}{=} \left|\bigcup_{i=1}^{k}(V_i \cup N_G(V_i)) \cup V_{k+1} \cup N_{G[W_k]}(V_{k+1})\right| \\ &< (k+1)(D+1)(\eta + f)n.\end{aligned}$$

To see that $(V_1, \dots, V_{k+1})$ satisfies (I), define

$$W_{k+1} = V(G) \setminus \bigcup_{i=1}^{k+1}(V_i \cup N_G(V_i)) \overset{(2.4)}{=} W_k \setminus (V_{k+1} \cup N_{G[W_k]}(V_{k+1})).$$

Assume that $|V_{k+1}| < \eta n$ and derive a contradiction as before.

Therefore, we generate a sequence $(V_1, \dots, V_{\ell-1})$ with the properties required by (ii). To complete the sequence, note that (III) gives that $|W_{\ell-1}| \ge \eta n$ and set $V_\ell = W_{\ell-1}$.

□

As a corollary of the previous lemma, we get the following lemma that says that sufficiently large bijumbled graphs contain a non-empty expanding subgraph.

Lemma 2.3.5 (Bijumbledness implies expansion). *Let f, θ, D and $c \ge 1$ be positive numbers with $c \ge 4(D+2)\theta$ and $a \ge 2(D+1)f$. If G is a $(c/(an), \theta)$-bijumbled graph with an vertices, then there exists a non-empty subgraph H of G that is (n, f, D)-expanding.*

Proof. Let $p = c/(an)$ and let G be a (p, θ)-bijumbled graph. Suppose for a contradiction that no subgraph of G is (n, f, D)-expanding. We apply Lemma 2.3.4 with $\ell = 2$ and $\eta = \frac{2\theta a}{c}$. Note that since $a \ge 2(D+1)f$ and $c \ge 4(D+2)\theta$ and from the choice of η we have

$$a \ge (D+1)f + \frac{a}{2} \ge (D+1)f + \frac{2(D+2)\theta a}{c} \ge (D+1)f + (D+2)\eta = (D+1)(f+\eta) + \eta.$$

Then, we get two disjoint sets $V_1, V_2 \subseteq V(G)$ with $|V_1| = |V_2| = \eta n > \theta/p$ such that $e_G(V_1, V_2) = 0$. On the other hand, by (2.2), we have $e_G(V_1, V_2) > 0$, a contradiction. Therefore, there is some subgraph of G that is (n, f, D)-expanding. □

The next lemma is crucial for embedding the desired power of a tree. Let G be a graph and $\ell \ge r$ be positive integers. An (ℓ, r)-*blow-up* of G is a graph obtained from G by replacing each vertex of G by a clique of size ℓ and for every edge of G arbitrarily adding a complete bipartite graph $K_{r,r}$ between the cliques corresponding to the vertices of this edge.

Lemma 2.3.6 (Embedding lemma for powers of trees)**.** *Given positive integers k and Δ, there exists r_0 such that the following holds for every n-vertex tree T with maximum degree Δ. There is a tree $T' = T'(T,k)$ on at most $n+1$ vertices and with maximum degree at most Δ^{2k} such that for every graph J with $T' \subseteq J$ and any (ℓ, r)-blow-up J' of J with $\ell \geq r \geq r_0$ we have $T^k \subseteq J'$.*

Proof. Given positive integers k, Δ, take $r_0 = \Delta^{4k}$. Let T be an n-vertex tree with maximum degree Δ. Let x_0 be any vertex in $V(T)$ and consider T as rooted at x_0. For each vertex $v \in V(T)$, let $D(v)$ denote the set of *descendants* of v in T (including v itself). Let $D^i(v)$ be the set of vertices $u \in D(v)$ at distance at most i from v in T.

Let T' be a tree with vertex set consisting of a special vertex x^* and the vertices $x \in V(T)$ such that the distance between x and x_0 is a multiple of $2k$. The edge set of T' consists of the edge x^*x_0 and the pairs of vertices $x, y \in V(T') \setminus \{x^*\}$ for which $x \in D^{2k}(y)$ or $y \in D^{2k}(x)$. That is,

$$\begin{aligned} V(T') &= \{x \in V(T) \colon \mathrm{dist}_T(x_0, x) \equiv 0 \pmod{2k}\} \cup \{x^*\} \\ E(T') &= \left\{ xy \in \binom{V(T') \setminus \{x^*\}}{2} : x \in D^{2k}(y) \text{ or } y \in D^{2k}(x) \right\} \cup \{x^*x_0\}. \end{aligned}$$

In particular, note that $\Delta(T') \leq \Delta^{2k}$ and $|V(T')| \leq n+1$. Let us consider T' as a tree rooted at x^*.

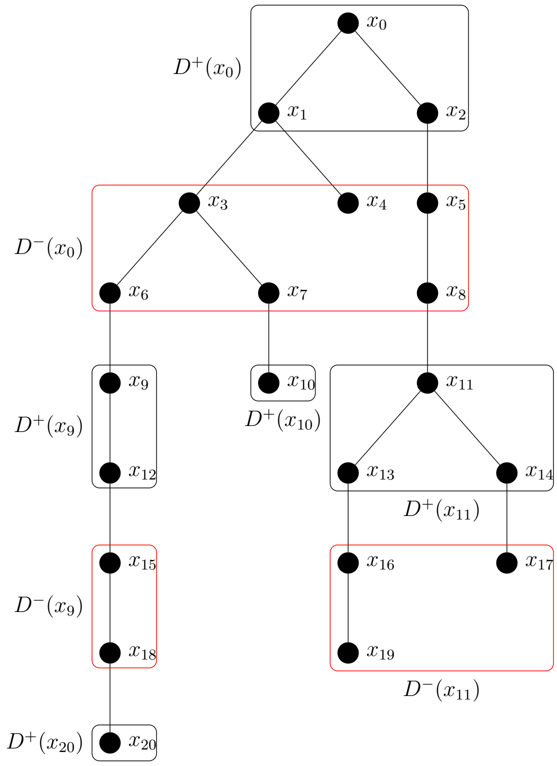
Now suppose that J is a graph such that $T' \subseteq J$ and J' is an (ℓ, r)-blow-up of J with $\ell \geq r \geq r_0$. Our goal is to show that $T^k \subseteq J'$. First, since J' is an (ℓ, r)-blow-up of J, there is a collection $\{K(x) : x \in V(J)\}$ of disjoint ℓ-cliques in J' such that for each edge $xy \in E(J)$, there is a copy of $K_{r,r}$ between the vertices of $K(x)$ and $K(y)$. Let us denote by $K(x,y)$ such copy of $K_{r,r}$.

For each $x \in V(T') \setminus \{x^*\}$, let $D^+(x) = D^{k-1}(x)$ and $D^-(x) = D^{2k-1}(x) \setminus D^{k-1}(x)$. In order to fix the notation, it helps to think of $D^+(x)$ and $D^-(x)$ as the *upper* and *lower half of close descendants* of x, respectively. We denote by x^+ the parent of x in T'. Suppose that there exists an injective map $\varphi : V(T) \to V(J')$ such that for every $x \in V(T') \setminus \{x^*\}$, we have

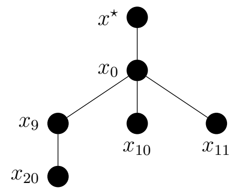
(1) $\varphi(D^+(x)) \subseteq K(x, x^+) \cap K(x^+)$;

(2) $\varphi(D^-(x)) \subseteq K(x, x^+) \cap K(x)$.

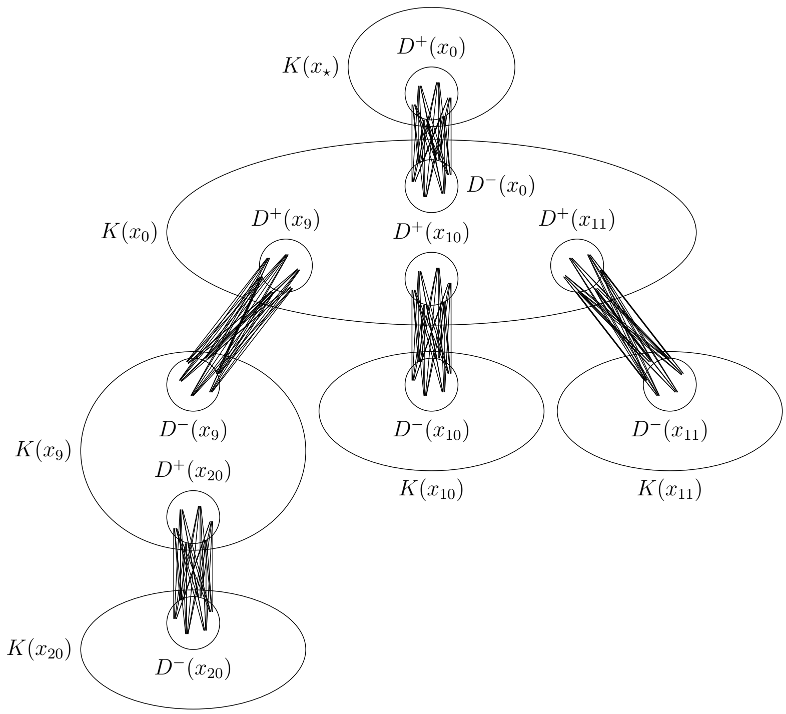
Then we claim that such map is in fact an embedding of T^k into J'. Figure 2.1 should help to visualize the concepts developed so far.



(a) Tree T.



(b) Corresponding T'.



(c) Embedding T^k into an (ℓ, r)-blow-up of T'.

Figure 2.1: Illustration of the concepts and notation used throughout the proof of Lemma 2.3.6 when $\Delta = 3$ and $k = 2$.

Claim 2.3.7. *If $\varphi : V(T) \to V(J')$ is an injective map such that for all $x \in V(T') \setminus \{x^*\}$, the properties (1) and (2) hold, then φ is an embedding of T^k into J'.*

Proof. We want to show that if u and v are distinct vertices in T at distance at most k, then $\varphi(u)\varphi(v)$ is an edge in J'. Let $\tilde{u}$ and $\tilde{v}$ be vertices in $V(T') \setminus \{x^*\}$ with $u \in D^{2k-1}(\tilde{u})$ and $v \in D^{2k-1}(\tilde{v})$. If $\tilde{u} = \tilde{v}$, then by properties (1) and (2), we have $\varphi(u)$ and $\varphi(v)$ adjacent in J', once all the vertices in $\varphi(D^{2k-1}(\tilde{u}))$ are adjacent in J' either by edges from $K(\tilde{u})$, $K(\tilde{u}^+)$ or $K(\tilde{u}, \tilde{u}^+)$. If $\tilde{u} = \tilde{v}^+$, then we must have $u \in D^-(\tilde{u})$ and $v \in D^+(\tilde{v})$ and properties (1) and (2) give us $\varphi(u), \varphi(v) \in K(\tilde{u})$. Analogously, if $\tilde{v} = \tilde{u}^+$, then $v \in D^-(\tilde{v})$ and $u \in D^+(\tilde{u})$ and properties (1) and (2) imply that $\varphi(u), \varphi(v) \in K(\tilde{v})$. If $\tilde{u}^+ = \tilde{v}^+$ (with $\tilde{u} \neq \tilde{v}$), then we have $u \in D^+(\tilde{u})$ and $v \in D^+(\tilde{v})$ and property (1) give us $\varphi(u), \varphi(v) \in K(\tilde{u}^+)$.

Therefore we may assume that $\tilde{u}$ and $\tilde{v}$ are at distance at least 2 in T' and do not share a parent. But this implies that

$$\min\{\operatorname{dist}_T(x, y) : x \in D^{2k-1}(\tilde{u}), y \in D^{2k-1}(\tilde{v})\} \geq 2k + 1,$$

contradicting the fact that u and v are at distance at most k in T. □

We conclude the proof by showing that such a map exists.

Claim 2.3.8. *There is an injective map $\varphi : V(T) \to V(J')$ for which (1) and (2) hold for every $x \in V(T') \setminus \{x^*\}$.*

Proof. We just need to show that for every $x \in V(T')$, there is enough room in $K(x)$ and in $K(x, x^+)$ to guarantee that (1) and (2) hold. In order to do so, $K(x)$ should be large enough to accommodate the set

$$D^-(x) \cup \bigcup_{\substack{y \in V(T') \\ y^+ = x}} D^+(y). \tag{2.5}$$

Since T' has maximum degree at most Δ^{2k} and T has maximum degree Δ, we have that the set in (2.5) has at most Δ^{4k} vertices. And since $|K(x)| = \ell \geq r_0 = \Delta^{4k}$, $K(x)$ is large enough to accommodate the set in (2.5). Finally, since $|K(x, x^+) \cap K(x)| = |K(x, x^+) \cap K(x^+)| = r \geq r_0 = \Delta^{4k}$ the set $K(x, x^+)$ is also large enough to accommodate $D^-(x)$ or $D^+(x)$ as in properties (1) and (2). □

□

We end this section discussing a graph property that needs to be inherited by some subgraphs when running the induction in the proof of Theorem I.

Definition 2.3.9. For positive numbers n, a, b, c, ℓ and θ, let $\mathcal{P}_n(a, b, c, \ell, \theta)$ denote the class of all graphs G with the following properties, where $p = c/(an)$.

(*i*) $|V(G)| = an$,

(*ii*) $\Delta(G) \leq b$,

(*iii*) G has no cycles of length at most 2ℓ,

(*iv*) G is (p, θ)-bijumbled.

Only mild conditions on a, b, c, ℓ and θ are necessary to guarantee the existence of a graph in $\mathcal{P}_n(a, b, c, \ell, \theta)$ for sufficiently large n. These conditions can be seen in (*i*)–(*iii*) in Definition 2.3.10 below. In order to keep the induction going in our main proof we also need a condition relating k and Δ, which represents, respectively, the power of the tree T we want to embed and the maximum degree of T (see (*iv*) in the next definition).

Definition 2.3.10. A 7-tuple $(a, b, c, \ell, \theta, \Delta, k)$ is *good* if

(*i*) $a \geq 3$,

(*ii*) $c \geq \theta\ell$,

(*iii*) $b \geq 9c$,

(*iv*) $\ell \geq 21\Delta^{2k}$.

Next we prove that conditions (*i*)–(*iii*) in Definition 2.3.10 together with $\theta \geq 32\sqrt{c}$ are enough to guarantee that there are graphs in $\mathcal{P}_n(a, b, c, \ell, \theta)$ as long as n is large enough. We remark that next lemma is stated for a good 7-tuple, but condition (*iv*) of Definition 2.3.10 is not necessary and, therefore, also Δ and k are irrelevant.

Lemma 2.3.11. *If $(a, b, c, \ell, \theta, \Delta, k)$ is a good 7-tuple with $\theta \geq 32\sqrt{c}$, then for sufficiently large n the family $\mathcal{P}_n(a, b, c, \ell, \theta)$ is non-empty.*

Proof. Let $(a, b, c, \ell, \theta, \Delta, k)$ be a good 7-tuple with $\theta \geq 32\sqrt{c}$ and let n be sufficiently large. Put $N = an$ and let $G^* = G(3N, p)$ be the binomial random graph with $3N$ vertices and edge probability $p = c/N$. From Chernoff's inequality (Theorem 2.2.2) we know that almost surely

$$e(G^*) \leq 2p\binom{3N}{2} \leq 9cN. \tag{2.6}$$

From [60, Lemma 8], we know that almost surely G^* is $(p, e^2\sqrt{6p(3N)})$-bijumbled, i.e. the following holds almost surely: for all disjoint sets X and $Y \subseteq V(G^*)$ with $e^2\sqrt{18N}/\sqrt{p} < |X| \leq |Y| \leq p(3N)|X|$, we have

$$\left|e_{G^*}(X, Y) - p|X||Y|\right| \leq (e^2\sqrt{6})\sqrt{p(3N)|X||Y|}. \tag{2.7}$$

The expected number of cycles of length at most 2ℓ in G^* is given by $\mathbb{E}(C_{\leq 2\ell}) = \sum_{i=3}^{2\ell} \mathbb{E}(C_i)$, where C_i is the number of cycles of length i. Then,

$$\mathbb{E}(C_{\leq 2\ell}) = \sum_{i=3}^{2\ell} \binom{3an}{i} \frac{(i-1)!}{2} p^i \leq \sum_{i=3}^{2\ell} (3c)^i \leq 2\ell(3c)^{2\ell}.$$

Then, from Markov's inequality, we have

$$\mathbb{P}\big(C_{\leq 2\ell} \geq 4\ell(3c)^{2\ell}\big) \leq \frac{1}{2}. \tag{2.8}$$

Since (2.6) and (2.7) hold almost surely and the probability in (2.8) is at most $1/2$, for sufficiently large n there exists a $(p, e^2\sqrt{18c})$-bijumbled graph G' with $3N$ vertices that contains less than $4\ell(3c)^{2\ell}$ cycles of length at most 2ℓ and $e(G') \leq 2p\binom{3N}{2} \leq 9cN$. Then, by removing $4\ell(3c)^{2\ell}$ vertices we obtain a graph G'' with no such cycles such that

$$|V(G'')| = 3an - 4\ell(3c)^{2\ell} \geq 2an \quad \text{and} \quad e(G'') \leq 9cN.$$

To obtain the desired graph G in $\mathcal{P}_n(a,b,c,\ell,\theta)$, we repeatedly remove vertices of highest degree in G'' until N vertices are left, obtaining a subgraph $G \subseteq G''$ such that $\Delta(G) \leq 9c \leq b$, as otherwise we would have deleted more than $e(G'')$ edges. Note that deleting vertices preserves the bijumbledness. Therefore, for all disjoint sets X and $Y \subseteq V(G)$ with $e^2\sqrt{18N}/\sqrt{p} < |X| \leq |Y| \leq p(3N)|X|$ we have

$$\big|e_G(X,Y) - p|X||Y|\big| \leq (e^2\sqrt{6})\sqrt{p(3N)|X||Y|} \leq (32\sqrt{pN})\sqrt{|X||Y|} \leq \theta\sqrt{|X||Y|}. \tag{2.9}$$

We obtained a graph G on N vertices and maximum degree $\Delta(G) \leq b$ such that G contains no cycles of length at most 2ℓ and is (p,θ)-bijumbled, for $p = c/N$. Therefore, the proof of the lemma is complete. □

2.4 Proof of Theorem I

We derive Theorem I from Proposition 2.4.1 below. Before continuing, given an integer $\ell \geq 1$, let us define what we mean by a *sheared complete blow-up* $H\{\ell\}$ of a graph H: this is any graph obtained by replacing each vertex v in $V(H)$ by a complete graph $C(v)$ with ℓ vertices, and by adding all edges *but a perfect matching* between $C(u)$ and $C(v)$, for each $uv \in E(H)$. We also define the *complete blow-up* $H(\ell)$ of a graph H analogously, but by adding all the edges between $C(u)$ and $C(v)$, for each $uv \in E(H)$.

Proposition 2.4.1. *For all integers $k \geq 1$, $\Delta \geq 2$, and $s \geq 1$ there exists r_s and a good 7-tuple $(a_s, b_s, c_s, \ell_s, \theta_s, \Delta, k)$ with $\theta_s \geq 32\sqrt{c_s}$ for which the following holds. If n is sufficiently* *large and* $G \in \mathcal{P}_n(a_s, b_s, c_s, \ell_s, \theta_s)$ *then, for any tree* T *on* n *vertices with* $\Delta(T) \le \Delta$*, we have*

$$G^{r_s}\{\ell_s\} \to (T^k)_s.$$

Theorem I follows from Proposition 2.4.1 applied to a certain subgraph of a random graph.

Proof of Theorem I. Fix positive integers k, Δ and s and let T be an n-vertex tree with maximum degree Δ. Proposition 2.4.1 applied with parameters k, Δ and s gives r_s and a good 7-tuple $(a_s, b_s, c_s, \ell_s, \theta_s, \Delta, k)$ with $\theta_s \ge 32\sqrt{c_s}$.

Let n be sufficiently large. By Lemma 2.3.11, since $\theta_s \ge 32\sqrt{c_s}$, there exists a graph $G \in \mathcal{P}_n(a_s, b_s, c_s, \ell_s, \theta_s)$. Let χ be an arbitrary s-colouring of $E(G^{r_s}\{\ell_s\})$. Then, Proposition 2.4.1 gives that $G^{r_s}\{\ell_s\} \to (T^k)_s$. Since $|V(G)| = a_s n$, the maximum degree of G is bounded by the constant b_s, and since r_s and ℓ_s are constants, we have $e(G^{r_s}\{\ell_s\}) = O_{k,\Delta,s}(n)$, which concludes the proof of Theorem I. $\square$

The proof of Proposition 2.4.1 follows by induction in the number of colours. Before we give this proof, let us state the results for the base case and the induction step.

Lemma 2.4.2 (Base Case)**.** *For all integers* $h \ge 1$*,* $k \ge 1$ *and* $\Delta \ge 2$ *there is an integer* r *and a good 7-tuple* $(a, b, c, \ell, \theta, \Delta, k)$ *with* $\theta \ge 2^{h-1}32\sqrt{c}$ *such that if* n *is sufficiently large, then the following holds for any* $G \in \mathcal{P}_n(a, b, c, \ell, \theta)$*. For any* n*-vertex tree* T *with* $\Delta(T) \le \Delta$*, the graph* $G^r\{\ell\}$ *contains a copy of* T^k*.*

Lemma 2.4.3 (Induction Step)**.** *For any positive integers* $\Delta \ge 2$*,* $s \ge 2$*,* k*,* r*,* $h \ge 1$ *and any good 7-tuple* $(a, b, c, \ell, \theta, \Delta, k)$ *with* $\theta \ge 2^h 32\sqrt{c}$*, there is a positive integer* r' *and a good 7-tuple* $(a', b', c', \ell', \theta', \Delta, k)$ *with* $\theta' \ge 2^{h-1}32\sqrt{c'}$ *such that the following holds. If* n *is sufficiently large then for any graph* $G \in \mathcal{P}_n(a', b', c', \ell', \theta')$ *and any* s*-colouring* χ *of* $E(G^{r'}\{\ell'\})$

- *(i) there is a monochromatic copy of* T^k *in* $G^{r'}\{\ell'\}$ *for any* n*-vertex tree* T *with* $\Delta(T) \le \Delta$*, or*
- *(ii) there is* $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$ *such that* $H^r\{\ell\} \subseteq G^{r'}\{\ell'\}$ *and* $H^r\{\ell\}$ *is coloured with at most* $s-1$ *colours under* χ*.*

Now we are ready to prove Proposition 2.4.1.

Proof of Proposition 2.4.1. Fix integers $k \ge 1$, $\Delta \ge 2$ and $s \ge 1$ and define $h_i = s - i$ for $1 \le i \le s$. Let r_1 and a good 7-tuple $(a_1, b_1, c_1, \ell_1, \theta_1, \Delta, k)$ with $\theta_1 \ge 2^{h_1}32\sqrt{c_1}$ be given by Lemma 2.4.2 applied with s, k and Δ.

We will prove the proposition by induction on the number of colours $i \in \{1, \dots, s\}$ with the additional property that if the colouring has i colours then $\theta_i \ge 2^{h_i}32\sqrt{c_i}$.

Notice that Lemma 2.4.2 implies that for sufficiently large n, if $G \in \mathcal{P}_n(a_1, b_1, c_1, \ell_1, \theta_1)$, then $G^{r_1}\{\ell_1\} \to (T^k)_1$. Therefore, since $\theta_1 \geq 2^{h_1}32\sqrt{c_1}$, if $i = 1$, we are done.

Assume $2 \leq i \leq s$ and suppose the statement holds for $i-1$ colours with the additional property that $\theta_{i-1} \geq 2^{h_{i-1}}32\sqrt{c_{i-1}}$, where r_{i-1} and a good 7-tuple $(a_{i-1}, b_{i-1}, c_{i-1}, \ell_{i-1}, \theta_{i-1}, \Delta, k)$ are given by the induction hypothesis. Therefore, for any tree T on n vertices with $\Delta(T) \leq \Delta$, we know that for a sufficiently large n

$$H^{r_{i-1}}\{\ell_{i-1}\} \to (T^k)_{i-1} \quad \text{for any} \quad H \in \mathcal{P}_n(a_{i-1}, b_{s-1}, c_{i-1}, \ell_{i-1}, \theta_{i-1}). \tag{2.10}$$

Note that since $i \leq s$, we have $h_{i-1} = s-(i-1) \geq 1$. Then, since $\theta_{i-1} \geq 2^{h_{i-1}}32\sqrt{c_{i-1}}$, we can apply Lemma 2.4.3 with parameters $\Delta, s, k, r_{i-1}, h_{i-1}$ and $(a_{i-1}, b_{i-1}, c_{i-1}, \ell_{i-1}, \theta_{i-1}, \Delta, k)$, obtaining r_i and $(a_i, b_i, c_i, \ell_i, \theta_i, \Delta, k)$ with $\theta_i \geq 2^{h_i}32\sqrt{c_i}$.

Let $G \in \mathcal{P}_n(a_i, b_i, c_i, \ell_i, \theta_i)$ and let n be sufficiently large. Now let χ be an arbitrary i-colouring of $E(G^{r_i}\{\ell_i\})$. From Lemma 2.4.3, we conclude that either (i) there is a monochromatic copy of T^k in $G^{r_i}\{\ell_i\}$ for any tree T on n vertices with $\Delta(T) \leq \Delta$, in which case the proof is finished, or (ii) there exists a graph $H \in \mathcal{P}_n(a_{i-1}, b_{i-1}, c_{i-1}, \ell_{i-1}, \theta_{i-1})$ such that $H^{r_{i-1}}\{\ell_{i-1}\} \subseteq G^{r_i}\{\ell_i\}$ and $H^{r_{i-1}}\{\ell_{i-1}\}$ is coloured with at most $s-1$ colours under χ. In case (ii), the induction hypothesis (2.10) implies that we find the desired monochromatic copy of T^k in $H^{r_{i-1}}\{\ell_{i-1}\} \subseteq G^{r_i}\{\ell_i\}$. $\square$

The proof of Lemma 2.4.2 follows by proving that for a good 7-tuple $(a, b, c, \ell, \theta, \Delta, k)$ with $\theta \geq 2^{h-1}32\sqrt{c}$, large graphs G in $\mathcal{P}_n(a, b, c, \ell, \theta)$ are expanding (using Lemma 2.3.5). Then, we use Lemma 2.3.2 to conclude that G contains the desired tree T. After this step we greedily find an embedding of T^k in $G^k\{\ell\}$.

Proof of the base case (Lemma 2.4.2). Let $h \geq 1$, $k \geq 1$ and $\Delta \geq 2$ be integers. Let

$$r = k, \quad \ell = 21\Delta^{2k}, \quad \theta = 4^h 256\ell, \quad c = \theta\ell, \quad b = 9c$$

and put $D = \Delta + 1$. Note that $\theta \geq 2^{h-1}32\sqrt{c}$ and let

$$a \geq 4(D+1).$$

Since $\ell \geq 4(\Delta + 3)$, we have $c \geq 4(D+2)\theta$. From the lower bounds on c and a we know that we can use the conclusion of Lemma 2.3.5 applying it with $f = 2$, θ, $D = \Delta + 1$ and c.

Note that from our choice of constants, $(a, b, c, \ell, \theta, \Delta, k)$ is a good tuple. Let n be sufficiently large and let T be a tree on n vertices with $\Delta(T) \leq \Delta$. Let $G \in \mathcal{P}_n(a, b, c, \ell, \theta)$. From Lemma 2.3.5 we know that G has an $(n, 2, \Delta+1)$-expanding subgraph and, therefore, from Lemma 2.3.2 we conclude that G contains a copy of T. Clearly, the graph G^k contains a copy of T^k. It remains to prove that the graph $G^k\{\ell\}$ also contains a copy of T^k.

Let $\{v_1, \ldots, v_n\}$ be the vertices of T_n and denote by T_j the subgraph of T induced by $\{v_1, \ldots, v_j\}$. Given a vertex $v \in V(G)$, let $C(v)$ denote the ℓ-clique in $G^k\{\ell\}$ that corresponds to v. Suppose that for some $1 \leq j < k$ we have embedded T_j^k in $G^k\{\ell\}$ where, for each $1 \leq i \leq j$, the vertex v_i was mapped to some $w_i \in C(v_i)$.

By the definition of $G^k\{\ell\}$, every neighbour v of v_{j+1} in G^k is adjacent to all but one vertex of $C(v_{j+1})$. Therefore, since $\Delta(T^k) \leq \Delta^k$ and $|C(v_{j+1})| = \ell \geq \Delta^k + 1$, we may thus find a vertex $w_{j+1} \in C(v_{j+1})$ such that w_{j+1} is adjacent in $G^k\{\ell\}$ to every w_i with $1 \leq i \leq j$ such that $v_i v_{j+1} \in E(T_{j+1}^k)$. From that we obtain a copy of T_{j+1}^k in $G^k\{\ell\}$ where $w_i \in C(v_i)$ for $1 \leq i \leq j+1$. Therefore, starting with any vertex w_1 in $C(v_1)$, we may obtain a copy of T^k in $G^k\{\ell\}$ inductively, which proves the lemma. $\square$

The core of the proof of Theorem I is the induction step (Lemma 2.4.3). We start by presenting a sketch of its proof.

Sketch of the induction step (Lemma 2.4.3). We start by fixing suitable constants r', a', b', c', ℓ' and θ'. Let n be sufficiently large and let $G \in \mathcal{P}_n(a', b', c', \ell', \theta')$ be given. Consider an arbitrary colouring χ of the edges of a sheared complete blow-up $G^{r'}\{\ell'\}$ of $G^{r'}$ with s colours. We shall prove that either there is a monochromatic copy of T^k in $G^{r'}\{\ell'\}$, or there is a graph $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$ such that a sheared complete blow-up $H^r\{\ell\}$ of H^r is a subgraph of $G^{r'}\{\ell'\}$ and this copy of $H^r\{\ell\}$ is coloured with at most $s-1$ colours under χ.

First, note that, by Ramsey's theorem, if ℓ' is large then each ℓ'-clique $C(v)$ of $G^{r'}\{\ell'\}$ contains a large monochromatic clique. Let us say that blue is the most common colour of these monochromatic cliques. Let these blue cliques be $C'(v) \subseteq C(v)$. Then we consider a graph $J \subseteq G^{r'}$ induced by the vertices v corresponding to the blue cliques $C'(v)$ and having only the edges $\{u, v\}$ such that there is a blue copy of a large complete bipartite graph under χ in the bipartite graph induced between the blue cliques $C'(u)$ and $C'(v)$ in $G^{r'}\{\ell'\}$.

Then, by Lemma 2.3.4 applied to J, either there is a set $\varnothing \neq Z \subseteq V(J)$ such that $J[Z]$ is expanding, or there are large disjoint sets $V_1, \ldots, V_\ell$ with no edges between them in J. In the first case, Lemma 2.3.6 guarantees that there is a tree T' such that, if $T' \subseteq J[Z]$, then there is a blue copy of T^k in $G^{r'}\{\ell'\}$. To prove that $T' \subseteq J[Z]$, we recall that $J[Z]$ is expanding and use Lemma 2.3.2. This finishes the proof of the first case.

Now let us consider the second case, in which there are large disjoint sets $V_1, \ldots, V_\ell$ with no edges between them in J. The idea is to obtain a graph $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$ such that $H^r\{\ell\} \subseteq G^{r'}\{\ell'\}$ and, moreover, $H^r\{\ell\}$ does not have any blue edge. For that we first obtain a path Q in G with vertices $(x_1, \ldots, x_{2a\ell n})$ such that $x_i \in V_j$ for all i where $i \equiv j \bmod \ell$. Then we partition Q into $2an$ paths $Q_1, \ldots, Q_{2an}$ with ℓ vertices each, and consider an auxiliary graph H' on $V(H') = \{Q_1, \ldots, Q_{2an}\}$ with $Q_iQ_j \in E(H')$ if and only $E_G(V(Q_i), V(Q_j)) \neq \varnothing$. To ensure that H' inherits properties from G we use that there can bet at most one edge between Q_i and Q_j in G, because there are no cycles of length less than 2ℓ in G.

We obtain a subgraph $H'' \subseteq H'$ by choosing edges of H' uniformly at random with a suitable probability p. Then, successively removing vertices of high degree, we obtain a graph $H \subseteq H''$ with $H \in \mathcal{P}_n(a,b,c,\ell,\theta)$. It now remains to find a copy of $H^r\{\ell\}$ in $G^{r'}\{\ell'\}$ with no blue edges. To do so, we first observe that the paths $Q_i \in V(H')$ give rise to ℓ-cliques in $G^{r'}$ ($r' \geq \ell$). One can then prove that there is a copy of $H^r\{\ell\}$ in $G^{r'}$ that avoids the edges of J. By applying the Lovász local lemma we can further deduce that there is a copy of $H^r\{\ell\}$ in $G^{r'}\{\ell'\}$ with no blue edges. $\square$

Proof of the induction step (Lemma 2.4.3). We start by fixing positive integers $\Delta \geq 2$, $s \geq 2$, k, r, h and a good 7-tuple $(a,b,c,\ell,\theta,\Delta,k)$ with

$$\theta \geq 2^h 32\sqrt{c}.$$

Recall that from the definition of good 7-tuple, we have

$$b \geq 9c.$$

Let d_0 be obtained from Lemma 2.2.1 applied with ℓ and $\gamma = 1/(2\ell)$ (note that $d_0 \leq 10$). Further let

$$a'' = \ell(\Delta^{2k}+2)(2a \cdot d_0 + 2).$$

Notice that a'' is an upper bound on the value A given by Lemma 2.3.4 applied with $f = 2$, $D = \Delta^{2k}+1$, ℓ and $\eta = 2a \cdot d_0$.

Let r_0 be given by Lemma 2.3.6 on input Δ and k. We may assume r_0 is even. Furthermore, let

$$t = \max\{r_0, \big(40(\ell b^{r+1}+\ell)\big)^{r_0}\} \quad \text{and} \quad \ell' = \max\{4s\ell^2, r_s(t)\},$$

where $r_s(t) = R_s(K_t)$ denotes the s-colour Ramsey number for cliques of order t. Let $a' = \ell' a$ and note that $a'/s \geq 2a''$ because $\ell \geq 21\Delta^{2k}$. Define constants c^*, c' and r' as follows.

$$c^* = 2\ell' c, \quad c' = \frac{\ell'}{2\ell^2}c^* = \frac{\ell'^2}{\ell^2}c, \quad r' = \ell r. \tag{2.11}$$

Put

$$b' = 9c' \quad \text{and} \quad \theta' = \frac{c^*}{4c\ell}\theta = \frac{\ell'}{2\ell}\theta$$

Claim 2.4.4. *$(a',b',c',\ell',\theta',\Delta,k)$ is a good 7-tuple and $\theta' \geq 2^{h-1}32\sqrt{c'}$.*

Proof. We have to check all conditions in Definition 2.3.10. Clearly $a' \geq 3$, $b' \geq 9c'$ and $\ell' \geq \ell \geq 21\Delta^{2k}$. Below we prove that the other conditions hold

- $c' \geq \theta'\ell'$:
$$c' = \frac{\ell'^2}{\ell^2} c \geq \frac{\ell'^2}{\ell}\theta = 2\theta'\ell' > \theta'\ell'.$$

- $\theta' \geq 2^{h-1} 32\sqrt{c'}$:
$$\theta' = \frac{\ell'}{2\ell}\theta \geq \frac{\ell'}{2\ell} 2^h 32\sqrt{c} = 2^{h-1} 32\sqrt{c'}.$$

□

Let G be a graph in $\mathcal{P}_n(a', b', c', \ell', \theta')$. Assume

$$N_G = a'n \quad \text{and} \quad p_G = c'/N_G$$

and let T be an arbitrary tree with n vertices and maximum degree Δ and consider an arbitrary s-colouring $\chi\colon E(G^{r'}\{\ell'\}) \to [s]$ of the edges of $G^{r'}\{\ell'\}$. We shall prove that either there is a monochromatic copy of T^k in $G^{r'}\{\ell'\}$, or there is a graph $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$ such that a sheared complete blow-up $H^r\{\ell\}$ of H^r is a subgraph of $G^{r'}\{\ell'\}$ and this copy of $H^r\{\ell\}$ is coloured with at most $s-1$ colours under χ.

By Ramsey's theorem (see, for example, [29]), since $\ell' \geq r_s(t)$, each ℓ'-clique $C(w)$ in $G^{r'}\{\ell'\}$ (for $w \in V(G)$) contains a monochromatic clique of size at least t. Without loss of generality, let us assume that most of those monochromatic cliques are blue. Let $W \subseteq V(G)$ be the set of vertices w such that there is a blue t-clique $C'(w) \subseteq C(w)$. We have

$$|W| \geq \frac{|V(G)|}{s} = \frac{a'n}{s} \geq 2a''n. \tag{2.12}$$

Define J as the subgraph of $G^{r'}$ with vertex set W and edge set

$$E(J) = \left\{ uv \in E(G^{r'}[W]) : \text{ there is a blue copy of } K_{r_0,r_0} \text{ in } G^{r'}\{\ell'\}[C'(u), C'(v)] \right\}.$$

That is, J is the subgraph of $G^{r'}$ induced by W and the edges uv such that there is a blue copy of K_{r_0,r_0} under χ in the bipartite graph induced by $G^{r'}\{\ell'\}$ between the vertex sets of the blue cliques $C'(u)$ and $C'(v)$.

We now apply Lemma 2.3.4 with $f = 2$, $D = \Delta^{2k} + 1$, ℓ, and $\eta = 2a \cdot d_0$ to the graph J (notice that $|V(J)| \geq 2a''n$ is large enough so we can apply Lemma 2.3.4), splitting the proof into two cases:

(i) there is $\varnothing \neq Z \subseteq V(J)$ such that $J[Z]$ is $(n+1, 2, \Delta^{2k}+1)$-expanding,

(ii) there exist $V_1, \ldots, V_\ell \subseteq V(J)$ such that $|V_i| \geq 2ad_0 n$ for $1 \leq i \leq \ell$ and $J[V_i, V_j]$ is empty for any $1 \leq i < j \leq \ell$.

In case $J[Z]$ is $(n+1, 2, \Delta^{2k}+1)$-expanding, we first notice that Lemma 2.3.6 applied to the graph $J[Z]$ implies the existence of a tree $T' = T'(T, \Delta, k)$ of maximum degree at most Δ^{2k} with at most $n+1$ vertices such that if $J[Z]$ contains T', then $T^k \subseteq J'$ for any (r_0, r_0)-blow-up J' of J. But since $J[Z]$ is $(n+1, 2, \Delta^{2k}+1)$-expanding, Lemma 2.3.2 implies that $J[Z]$ contains a copy of T'. Therefore, the graph $G^{r'}\{\ell'\}$ contains a blue copy of T^k, as we can consider J' as the subgraph of $G^{r'}\{\ell'\}$ containing only edges inside the blue cliques $C'(u)$ (which have size $t \geq r_0$) and the edges of the complete blue bipartite graphs K_{r_0,r_0} between the blue cliques $C'(u)$. This finishes the proof of the first case.

We may now assume that there are subsets $V_1, \ldots, V_\ell \subseteq V(J)$ with $|V_i| \geq 2ad_0n$ for $1 \leq i \leq \ell$ and $J[V_i, V_j]$ is empty for any $1 \leq i < j \leq \ell$. We want to obtain a graph $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$ such that $H^r\{\ell\} \subseteq G^{r'}\{\ell'\}$ and contains no blue edges.

Let $J' = J[V_1 \cup \cdots \cup V_\ell]$, $G' = G[V_1 \cup \cdots \cup V_\ell]$ and note that $|V(G')| = |V(J')| \geq d_0 \cdot 2a\ell n$, where we recall that d_0 is the constant obtained by applying Lemma 2.2.1 with ℓ and $\gamma = 1/(2\ell)$. We want to use the assertion of Lemma 2.2.1 to obtain a transversal path of length $2a\ell n$ in G' and so we have to check the conditions adjusted to this parameter.

First note, that we have $|V_i| \geq 2ad_0n \geq \gamma d_0 \cdot 2a\ell n$ for $1 \leq i \leq \ell$. Moreover, since G' is an induced subgraph of G and $G \in \mathcal{P}_n(a', b', c', \ell, \theta')$, we know by (2.2) that for all $X, Y \subseteq V(G')$ with $|X|, |Y| > \theta' a' n / c'$ we have $e_{G'}(X, Y) > 0$. Observe that $\theta' a' n / c' < an = \gamma \cdot 2a\ell n$ once $a' = \ell' a$ and $c' > \theta' \ell'$. Therefore, we may use Lemma 2.2.1 to conclude that G' contains a path $P_{2a\ell n} = (x_1, \ldots, x_{2a\ell n})$ with $x_i \in V_j$ for all i, where $j \equiv i \pmod{\ell}$.

We split the obtained path $P_{2a\ell n}$ of G' into consecutive paths $Q_1, \ldots, Q_{2an}$ each on ℓ vertices. More precisely, we let $Q_i = (x_{(i-1)\ell+1}, \ldots, x_{i\ell})$ for $i = 1, \ldots, 2an$. The following auxiliary graph is the base of our desired graph $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$.

> H' is the graph on $V(H') = \{Q_1, \ldots, Q_{2an}\}$ such that $Q_iQ_j \in E(H')$ if and only if there is an edge in G between the vertex sets of Q_i and Q_j.

Claim 2.4.5. $H' \in \mathcal{P}_n(2a, \ell b', c^*, \ell, \ell\theta')$.

Proof. We verify the conditions of Definition 2.3.9. Since H' has $2an$ vertices, condition (i) clearly holds. Since $\Delta(G) \leq b'$ and for any $Q_i \in V(H')$ we have $|Q_i| = \ell$ (as a subset of $V(G)$), there are at most $\ell b'$ edges in G with an endpoint in Q_i. Then, $\Delta(H') \leq \ell b'$.

For condition (iii), recall that any vertex of H' corresponds to a path on ℓ vertices in G. Thus, a cycle of length at most 2ℓ in H' implies the existence of a cycle of length at most $2\ell^2$ in G. Since $2\ell' \geq 2\ell^2$ and G has no cycles of length at most $2\ell'$, we conclude that H' contains no cycle of length at most 2ℓ, which verifies condition (iii).

Let $N_{H'} = 2an$ and

$$p_{H'} = \frac{c^*}{N_{H'}} = \frac{c^*}{2an}. \tag{2.13}$$

Let us verify condition (iv), i.e., we shall prove that H' is $(p_{H'}, \ell\theta')$-bijumbled.

Consider arbitrary sets X and Y of $V(H')$ with $\ell\theta'/p_{H'} < |X| \leq |Y| \leq p_{H'}N_{H'}|X|$. For simplicity, we may assume that $X = \{Q_1, \ldots, Q_x\}$ and $Y = \{Q_{x+1}, \ldots, Q_{x+y}\}$. Let $X_G = \bigcup_{j=1}^{x} Q_j \subseteq V(G)$ and $Y_G = \bigcup_{j=x+1}^{x+y} Q_j \subseteq V(G)$. Note that $|X_G| = \ell|X|$ and $|Y_G| = \ell|Y|$. As there are no cycles of length smaller than 2ℓ in G, we only have at most one edge between the vertex sets of Q_i and Q_j. Therefore we have

$$e_{H'}(X, Y) = e_G(X_G, Y_G). \tag{2.14}$$

We shall prove that $|e_{H'}(X, Y) - p_{H'}|X||Y|| \leq \ell\theta'\sqrt{|X||Y|}$. From the choice of c', we have

$$p_{H'}|X||Y| = \frac{c^*}{2an}|X||Y| = \frac{c'}{a'n}\ell|X|\ell|Y| = \frac{c'}{a'n}|X_G||Y_G| = p_G|X_G||Y_G|. \tag{2.15}$$

From the choice of θ', c', and $p_{H'}$, since $\ell\theta'/p_{H'} < |X| \leq |Y| \leq p_{H'}N_{H'}|X|$, we obtain

$$\frac{\theta'}{p_G} < |X_G| \leq |Y_G| \leq p_G N_G |X_G|.$$

Combining (2.15) with (2.14) and the fact that G is (p_G, θ')-bijumbled, we get that

$$|e_{H'}(X, Y) - p_{H'}|X||Y|| = |e_G(X_G, Y_G) - p_G|X_G||Y_G|| \leq \theta'\sqrt{|X_G||Y_G|} = \ell\theta'\sqrt{|X||Y|}. \tag{2.16}$$

Therefore, H' is $(p_{H'}, \ell\theta')$-bijumbled, which verifies condition (iv). □

The parameters for $\mathcal{P}_n(2a, \ell b', c^*, \ell, \ell\theta')$ are tightly fitted such that we can find the following subgraph of H'.

Claim 2.4.6. *There exists $H \subseteq H'$ such that $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$.*

Proof. We first obtain $H'' \subseteq H'$ by picking each edge of H' with probability

$$p = \frac{2c}{c^*} = \frac{1}{\ell'}$$

independently at random. Note that $p \leq 1/2$.

From (2.3), we get

$$e(H') \leq p_{H'}\binom{2an}{2} + \ell\theta' 2an \leq (c^* + 2\ell\theta')an \leq (c^* + 2\ell\frac{c'}{\ell'})an \leq 2c^*an$$

From Chernoff's inequality, we then know that almost surely we have

$$e(H'') \leq 2p \cdot e(H') \leq 2 \cdot \left(\frac{2c}{c^*}\right) \cdot 2c^*an \leq 8acn \leq abn. \tag{2.17}$$

Let $N_{H''} = 2an$ and
$$p_{H''} = p \cdot p_{H'} = \frac{c}{an}.$$

We shall prove that H'' is $(p_{H''}, \theta)$-bijumbled almost surely. For that, we will first prove by using Chernoff's inequality (Theorem 2.2.2) that, for any arbitrary sets X and Y of $V(H')$ with $\theta/p_{H''} < |X| \leq |Y| \leq p_{H'}N_{H'}|X|$ we have

$$|e_{H''}(X,Y) - p \cdot e_{H'}(X,Y)| \leq \frac{\theta}{2}\sqrt{|X||Y|}. \tag{2.18}$$

Note that for such sets X and Y, since $|X| > \theta/p_{H''} \geq \ell\theta'/p_{H'}$, we can use (2.16).

Since $|X|, |Y| > \theta/p_{H''}$, we have $\sqrt{|X||Y|} > \theta an/c$. From $\sqrt{|X||Y|} > \theta an/c$, we obtain that $\ell'\theta < \frac{2\ell' c\sqrt{|X||Y|}}{2an}$ from which we can conclude that $2\ell\theta' < p_{H'}\sqrt{|X||Y|}$. Thus, we get $\ell\theta'\sqrt{|X||Y|} < p_{H'}|X||Y|/2$. Therefore, combining this with (2.16) we have

$$\frac{p_{H'}|X||Y|}{2} < e_{H'}(X,Y) < 2p_{H'}|X||Y|. \tag{2.19}$$

Let $\varepsilon = \theta\sqrt{|X||Y|}/(2p \cdot e_{H'}(X,Y))$ and note that from (2.19) we have $\varepsilon < 1$. Since $\theta \geq 10\sqrt{c}$, also from (2.19) we obtain

$$\frac{\varepsilon^2 p \cdot e_{H'}(X,Y)}{3} = \frac{|X||Y|\ell'\theta^2}{12 \cdot e_{H'}(X,Y)} > 4an.$$

Therefore, by using Chernoff's inequality, since there are at most 2^{4an} choices of pairs of sets $\{X, Y\}$, almost surely we have that for any disjoint subsets X and Y of vertices of H'' with $\theta/p_{H''} < |X| \leq |Y| \leq p_{H'}N_{H'}|X|$, inequality (2.18) holds.

Observe that $p_{H''}N_{H''}|X| = 2c|X| \leq c^*|X| = p_{H'}N_{H'}|X|$. Therefore, H'' is almost surely $(p_{H''}, \theta)$-bijumbled, as by (2.16) and (2.18) we get

$$\begin{aligned}
|e_{H''}(X,Y) - p_{H''}|X||Y|| &\leq |e_{H''}(X,Y) - p \cdot e_{H'}(X,Y)| + |p \cdot e_{H'}(X,Y) - p_{H''}|X||Y|| \\
&\overset{(2.18)}{\leq} \frac{\theta}{2}\sqrt{|X||Y|} + p(|e_{H'}(X,Y) - p_{H'}|X||Y||) \\
&\overset{(2.16)}{\leq} \frac{\theta}{2}\sqrt{|X||Y|} + \frac{\ell\theta'}{\ell'}\sqrt{|X||Y|} \\
&= \theta\sqrt{|X||Y|}.
\end{aligned}$$

Therefore, there exists a $(p_{H''}, \theta)$-bijumbled graph H'' as above. We fix such a graph and construct the desired graph H from this H'' by sequentially removing the an vertices of highest degree. Notice that H has maximum degree at most b, otherwise this would imply that H'' has more than abn edges, contradicting (2.17). Since H is a subgraph of H', and H' does not contain cycles of length at most 2ℓ, the same holds for H. Finally, since deleting vertices preserves the bijumbledness property, we conclude that $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$. $\square$

Recall that J is the subgraph of $G^{r'}$ induced by W, with $|W| \geq a'n/s$ and edges uv such that there is a blue copy of K_{r_0,r_0} under χ in the bipartite graph induced by the vertex sets of blue cliques $C'(u)$ and $C'(v)$ in $G^{r'}\{\ell'\}$. Furthermore, recall that there are subsets $V_1, \ldots, V_\ell \subseteq V(J)$ with $|V_i| \geq 2ad_0n$ for $1 \leq i \leq \ell$ and $J[V_i, V_j]$ is empty for any $1 \leq i < j \leq \ell$, and we defined $J' = J[V_1 \cup \cdots \cup V_\ell]$ and $G' = G[V_1 \cup \cdots \cup V_\ell]$. Lastly, recall that $Q_i = (x_{(i-1)\ell+1}, \ldots, x_{i\ell})$ for $i = 1, \ldots, 2an$, where the vertices x_i belong to G'. Assume, without loss of generality, $V(H) = \{Q_1, \ldots, Q_{an}\}$. In what follows, when considering the graph $H^r(\ell)$, the ℓ-clique corresponding to Q_i is composed of the vertices $x_{(i-1)\ell+1}, \ldots, x_{i\ell}$, and hence one can view $V\big(H^r(\ell)\big)$ as a subset of $V(G')$.

Claim 2.4.7. $H^r(\ell) \subseteq G^{r'}$. *Moreover, $G^{r'}$ contains a copy of $H^r\{\ell\}$ that avoids the edges of J.*

Proof. We will prove that $H^r(\ell) \subseteq G^{r'}$ where $Q_1, \ldots, Q_{an} \subseteq V(J)$ are the ℓ-cliques of $H^r(\ell)$. Suppose that Q_i and Q_j are at distance at most r in the graph H. Without loss of generality, let $Q_i = Q_1$ and $Q_j = Q_m$ for some $m \leq r$. Moreover, let $(Q_1, Q_2, \ldots, Q_m)$ be a path in H. Note that there exist vertices $u_1, \ldots, u_{m-1}$ and $u'_2, \ldots, u'_m$ in $V(G')$ such that $u_1 \in Q_1$, $u'_m \in Q_m$, $u_j, u'_j \in Q_j$ for all $j = 2, \ldots, m-1$ and $\{u_i, u'_{i+1}\}$ is an edge of G' for $i = 1, \ldots, m-1$.

Let $u'_1 \in Q_1$ and $u_m \in Q_m$ be arbitrary vertices. Since for any j, the set Q_j is spanned by a path on ℓ vertices in G', it follows that u_j and u'_j are at distance at most $\ell - 1$ in G' for all $1 \leq j \leq m$. Therefore, u'_1 and u_m are at distance at most $(\ell-1)m + (m-1) < \ell r \leq r'$ in G' and hence $u'_1 u_m$ is an edge in $G^{r'}[V_1 \cup \ldots \cup V_\ell] \subseteq G^{r'}$. Since the vertices u'_1 and u_m were arbitrary, we have shown that if Q_i and Q_j are adjacent in H^r (i.e., Q_i and Q_j are at distance at most r in H) then (Q_i, Q_j) gives a complete bipartite graph $C(Q_i, Q_j)$ in $G^{r'}$. Moreover, taking $i = j$ we see that each Q_i in $G^{r'}$ must be complete. This implies that $H^r(\ell)$ is a subgraph of $G^{r'}$.

For the second part of the claim we consider which of the edges of this copy of $H^r(\ell)$ can also be edges of J. Recall from the definition of J' that we found subsets $V_1, \ldots, V_\ell \subseteq J$ such that no edge of J lies between different parts. Moreover each set $Q_i \subseteq J$ takes precisely one vertex from each set $V_1, \ldots, V_\ell$. It follows that each Q_i is independent in J. Now let us say we have $x \in Q_i$ and $y \in Q_j$ $(i \neq j)$ that are adjacent in J. We can not have x and y in different parts of the partition $\{V_1, \ldots, V_\ell\}$. Thus x and y lie in the same part. Therefore edges from J between Q_i and Q_j must form a matching. Then we can find a copy of $H^r\{\ell\}$ that avoids J by removing a matching between the l-cliques from $H^r(\ell)$.

$\square$

To complete the proof of Lemma 2.4.3, we will embed a copy of the graph $H^r\{\ell\} \subseteq G^{r'}$ found in Claim 2.4.7 in $G^{r'}\{\ell'\}$ in such a way that $H^r\{\ell\}$ uses at most $s-1$ colours.

Claim 2.4.8. *$G^{r'}\{\ell'\}$ contains a copy of $H^r\{\ell\}$ with no blue edges.*

Proof. Recall that each vertex u in J corresponds to a clique $C'(u) \subseteq G^{r'}\{\ell'\}$ of size t and that this clique is monochromatic in blue in the original colouring χ of $E(G^{r'}\{\ell'\})$. Recall also that if an edge $\{u, v\}$ of $G^{r'}[W]$ is not in J, then there is no blue copy of K_{r_0,r_0} in the bipartite graph between $C'(u)$ and $C'(v)$ in $G^{r'}\{\ell'\}$. By the Kővári–Sós–Turán theorem (Theorem 2.2.3), there are at most $4t^{2-1/r_0}$ blue edges between $C'(u)$ and $C'(v)$. Recall further that $C'(u)$ and $C'(v)$ are, respectively, subcliques of the ℓ'-cliques $C(u)$ and $C(v)$ in $G^{r'}\{\ell'\}$. Since $\{u, v\}$ is an edge of $G^{r'}$, there is a complete bipartite graph with a matching removed between $C(u)$ and $C(v)$ in $G^{r'}\{\ell'\}$ and so there is a complete bipartite graph with at most a matching removed for $C'(u)$ and $C'(v)$. It follows that there are at least

$$t^2 - t - 4t^{2-1/r_0}$$

non-blue edges between $C'(u)$ and $C'(v)$.

Using the copy of $H^r\{\ell\} \subseteq G^{r'}$ avoiding edges of J obtained in Claim 2.4.7 as a 'template', we will embed a copy of $H^r\{\ell\}$ in $G^{r'}\{\ell'\}$ with no blue edges. For each vertex $u \in V(H^r\{\ell\}) \subseteq V(J)$ we will pick precisely one vertex from $C'(u) \subseteq G^{r'}\{\ell'\}$ in our embedding. The argument proceeds by the Lovász Local Lemma.

For each $u \in V(H^r\{\ell\}) \subseteq V(J)$ let us choose $x_u \in C'(u)$ uniformly and independently at random. Let $e = \{u, v\}$ be an edge of our copy of $H^r\{\ell\}$ in $G^{r'}$ that is not in J. As pointed out above, we know that there are at least $t^2 - t - 4t^{2-1/r_0}$ non-blue edges between $C'(u)$ and $C'(v)$. Letting A_e be the event that $\{x_u, x_v\}$ is a blue edge or a non-edge in $G^{r'}\{\ell'\}$, we have that

$$\mathbb{P}[A_e] \leq \frac{t + 4t^{2-1/r_0}}{t^2} \leq 5t^{-1/r_0}.$$

The events A_e are not independent, but we can define a dependency graph D for the collection of events A_e by adding an edge between A_e and A_f if and only if $e \cap f \neq \varnothing$. Then, $\Delta = \Delta(D) \leq 2\Delta(H^r\{\ell\}) \leq 2(b^{r+1}\ell + \ell)$. From our choice of t we get that

$$4\Delta\mathbb{P}[A_e] \leq 40(b^{r+1}\ell + \ell^2)t^{-1/r_0} \leq 1$$

for all e. Then the Local Lemma [5, Lemma 5.1.1] tells us that $\mathbb{P}\big[\bigcap_e \bar{A}_e\big] > 0$, and hence a simultaneous choice of the x_u's ($u \in V(H^r\{\ell\})$) is possible, as required. This concludes the proof of Claim 2.4.8. □

The proof of Lemma 2.4.3 is now complete. □

2.5 Concluding Remarks

In Chapter 2, in order to prove Theorem I we needed to show that the family $\mathcal{P}_n(a, b, c, \ell, \theta)$ is non-empty given a good 7-tuple $(a, b, c, \ell, \theta, \Delta, k)$ with $\theta \geq 32\sqrt{c}$. We prove this in Lemma 2.3.11 using the binomial random graph. Alternatively, it is possible to replace this by using explicit constructions of high girth expanders. For example, the Ramanujan graphs constructed by Lubotzky, Phillips, and Sarnak [82] can be used to prove Lemma 2.3.11.

As pointed out in Section 2.1, every graph with maximum degree and bounded treewidth is contained in some bounded power of a bounded degree tree and vice versa. This implies that Corollary 2.1.1 is equivalent to Theorem I. For bounded degree graphs, bounded treewidth is equivalent to bounded cliquewidth and also to bounded rankwidth [65]. Therefore, Corollary 2.1.1 also holds with treewidth replaced by any of these parameters.

An obvious direction for further research concerning the size-Ramsey number is to investigate the size-Ramsey number of powers T^k of trees T when k and $\Delta(T)$ are no longer bounded. Haxell and Kohaykawa [59] showed that for every positive integer s, there exists a constant C_s such that for any tree T with maximum degree at most Δ we have $\hat{r}_s(T) \leq C_s\Delta|T|$. Our proof of Theorem I actually shows that $\hat{r}_s(T^k) \leq r_s(2^{\Delta^{5k}})|T|$, where $r_s(t) = R_s(K_t)$ denotes the s-colour Ramsey number of K_t. It is known (see [29]) that $r_s(t)$ grows as a tower of t of height s. It would be nice to improve Theorem I to a much smaller constant. In particular, we conjecture that for every positive integer s, there exists a constant C_s such that for every tree T with maximum degree at most Δ we have $\hat{r}_s(T^k) \leq C_s 2^{\Delta^{5k}}|T|$.

Chapter 3

Covering the Random Graph by Monochromatic Trees

3.1 Introduction

Given a graph G and a positive integer r, let $\mathrm{tc}_r(G)$ denote the minimum number k such that in any r-edge-colouring of G, there are k monochromatic trees $T_1, \ldots, T_k$ such that the union of their vertex sets covers $V(G)$, i.e.,

$$V(G) = V(T_1) \cup \cdots \cup V(T_k).$$

We define $\mathrm{tp}_r(G)$ analogously by requiring the union above to be disjoint.

It is easy to see that $\mathrm{tp}_2(K_n) = 1$ for all $n \geq 1$, and Erdős, Gyárfás and Pyber [42] proved that $\mathrm{tp}_3(K_n) = 2$ for all $n \geq 1$, and conjectured that $\mathrm{tp}_r(K_n) = r - 1$ for every n and r. Haxell and Kohayakawa [58] showed that $\mathrm{tp}_r(K_n) \leq r$ for all sufficiently large $n \geq n_0(r)$. We remark that it is easy to see that $\mathrm{tc}_r(K_n) \leq r$ (just pick any vertex $v \in V(K_n)$ and let T_i, for $i \in [r]$, be a maximal monochromatic tree of colour i containing v), but it is not even known whether or not $\mathrm{tc}_r(K_n) \leq r - 1$ for every n and r (as would be implied by the conjecture of Erdős, Gyárfás and Pyber).

Concerning general graphs instead of complete graphs, Gyárfás [54] noted that a well-known conjecture of Ryser on matchings and transversal sets in hypergraphs is equivalent to the following bound on $\mathrm{tc}_r(G)$.

Conjecture 3.1.1 (Gyárfás's reformulation of Ryser's conjecture)**.** *For every graph G and integer $r \geq 2$, we have*

$$\mathrm{tc}_r(G) \leq (r-1)\alpha(G). \tag{3.1}$$

The work described in this chapter was developed in a joint project with Yoshiharu Kohayakawa, Guilherme Oliveira Mota and Bjarne Schülke.

In particular, Ryser's conjecture, if true, would imply that $\mathrm{tc}_r(K_n) \le r-1$, for every $n \ge 1$ and $r \ge 2$. Ryser's conjecture was proved in the case $r = 3$ by Aharoni [1], but for $r \ge 4$ very little is known. For example, Haxell and Scott [61] proved (in the context of Ryser's original conjecture) that there exists $\epsilon > 0$ such that for $r \in \{4, 5\}$, we have $\mathrm{tc}_r(G) \le (r-\epsilon)\alpha(G)$, for any graph G.

Bal and DeBiasio [7] initiated the study of covering and partitioning random graphs by monochromatic trees. They proved that if $p \ll \big(\frac{\log n}{n}\big)^{1/r}$, then with high probability[1] we have $\mathrm{tc}_r(G(n,p)) \to \infty$. They conjectured that for any $r \ge 2$, this was the correct threshold for the event $\mathrm{tp}_r(G(n,p)) \le r$. Kohayakawa, Mota and Schacht [68] proved that this conjecture holds for $r = 2$, while Ebsen, Mota and Schnitzer[2] showed that it does not hold for more than two colours.

Bucić, Korándi and Sudakov [17] proved that if $p \ll \big(\frac{\log n}{n}\big)^{\sqrt{r}/2^{r-2}}$, then w.h.p. we have $\mathrm{tc}_r(G(n,p)) \ge r+1$, which implies that the threshold for the event $\mathrm{tc}_r(G) \le r$ is in fact significantly larger than the one conjectured by Bal and DeBiasio when r is large. Bucić, Korándi and Sudakov also proved that w.h.p. we have $\mathrm{tc}_r(G(n,p)) \le r$ for $p \gg \big(\frac{\log n}{n}\big)^{1/2^r}$. They were also able to roughly determine the typical behaviour of $\mathrm{tc}_r(G(n,p))$ in terms of the range where p lies in (see [17, Theorems 1.3 and 1.4]).

Considering colourings with three colours, the general results from [17], as stated, imply that if $p \gg \big(\frac{\log n}{n}\big)^{1/8}$, then w.h.p. we have $\mathrm{tc}_3(G(n,p)) \le 3$, and if $p \gg \big(\frac{\log n}{n}\big)^{1/6}$, then w.h.p. $\mathrm{tc}_3(G(n,p)) \le 88$ (the methods from [17] may actually give a somewhat better upper bound than 88, if one optimizes their calculations). Our main theorem in this chapter improves these bounds.

Theorem II. *If $p = p(n)$ satisfies $p \gg \big(\frac{\log n}{n}\big)^{1/6}$, then with high probability we have*

$$\mathrm{tc}_3\big(G(n,p)\big) \le 3.$$

It can be easily seen that if $1-p \ll n^{-1}$, then w.h.p. there is a 3-edge-colouring of $G(n,p)$ for which 3 monochromatic trees are needed to cover all vertices — it suffices to consider three non-adjacent vertices x_1, x_2 and x_3, and colour the edges incident to x_i with colour i and colour all the remaining edges with any colour. Therefore, the bound for $tc_3(G(n,p))$ in Theorem II is the best possible as long as p is not too close to 1.

We remark that, from the example described in [68], we know that for $p \ll \big(\frac{\log n}{n}\big)^{1/4}$, we have w.h.p. $\mathrm{tc}_3(G(n,p)) \ge 4$. It would be very interesting to describe the behaviour of $\mathrm{tc}_3(G(n,p))$ when $\big(\frac{\log n}{n}\big)^{1/4} \ll p \ll \big(\frac{\log n}{n}\big)^{1/6}$.

This chapter is organized as follows. In Section 3.2 we present some definitions and auxiliary results that we will use in the proof of Theorem II, which is outlined in Section 3.3. The details of the proof of Theorem II are given in Section 3.4.

[1] We will write shortly *w.h.p.* for *with high probability*.

[2] A description of this construction can be found in [68].

3.2 Preliminaries

Most of our notation is standard (see [13, 15, 31] and [14, 63]). However, we will mention in the following few definitions regarding hypergraphs that will play a major role in our proofs just for completeness.

We say that a set A of vertices in a hypergraph $\mathcal{H}$ is a *vertex cover* if every hyperedge of $\mathcal{H}$ contains at least one element of A. The *covering number* of $\mathcal{H}$, denoted by $\tau(\mathcal{H})$, is the smallest size of a vertex cover in $\mathcal{H}$. A *matching* in $\mathcal{H}$ is a collection of disjoint hyperedges in $\mathcal{H}$. The *matching number* of $\mathcal{H}$, denoted by $\nu(\mathcal{H})$, is the largest size of a matching in $\mathcal{H}$. An immediate relationship between $\tau(\mathcal{H})$ and $\nu(\mathcal{H})$ is the inequality $\nu(\mathcal{H}) \leq \tau(\mathcal{H})$. If additionally $\mathcal{H}$ is r-uniform, then we have $\tau(\mathcal{H}) \leq r\nu(\mathcal{H})$. A conjecture due to Ryser (which first appeared in the thesis of his Ph.D. student, Henderson [62]) states that for every r-uniform r-partite hypergraph $\mathcal{H}$, we have $\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$. Note that the König–Egerváry theorem corresponds to Ryser's conjecture for $r = 2$. Aharoni [1] proved that Ryser's conjecture holds for $r = 3$, but the conjecture remains open for $r \geq 4$.

Given a vertex v in a 3-uniform hypergraph $\mathcal{H}$, the *link graph* of $\mathcal{H}$ with respect to v is the graph $L_v = (V, E)$ with vertex set $V = V(\mathcal{H})$ and edge set $E = \{xy : \{x, y, v\} \subseteq \mathcal{H}\}$.

We will use the following theorem due to Erdős, Gyárfás and Pyber [42] in the proof of our main result.

Theorem 3.2.1 (Erdős, Gyárfás and Pyber)**.** *For any 3-edge-colouring of a complete graph K_n, there exists a partition of $V(K_n)$ into 2 monochromatic trees.*

We will also use the following lemma, which is a simple application of Chernoff's inequality. For a proof of the first item see [74, Lemma 3.8]. The second item is an immediate corollary of [74, Lemma 3.10].

Lemma 3.2.2. *Let $\varepsilon > 0$. If $p = p(n) \gg \left(\frac{\log n}{n}\right)^{1/6}$, then w.h.p. $G \in G(n,p)$ has the following properties.*

(i) For any disjoint sets $X, Y \subseteq V(G)$ with $|X|, |Y| \gg \frac{\log n}{p}$, we have

$$|E_G(X, Y)| = (1 \pm \varepsilon)p|X||Y|.$$

(ii) Every vertex $v \in V(G)$ has degree $d_G(v) = (1 \pm \varepsilon)pn$ and every set of $i \leq 6$ vertices has $(1 \pm \varepsilon)p^i n$ common neighbours.

3.3 A sketch of the proof

In this section we will give an overview of the proof of Theorem II. Let $G = G(n, p)$, with $p \gg \left(\frac{\log n}{n}\right)^{1/6}$, and let $\varphi : E(G) \to \{\text{red, green, blue}\}$ be any 3-edge-colouring of G. We consider an auxiliary graph F, with $V(F) = V(G)$ and $ij \in E(F)$ if and only if there is, in the colouring φ, a monochromatic path in G connecting i and j. Then we define a 3-edge-colouring φ' of F with $\varphi'(ij)$ being the colour of any monochromatic path in G connecting i and j. Note that any covering of F with monochromatic trees with respect to the colouring φ' corresponds to a covering of G with monochromatic trees with respect to the colouring φ with the same number of trees.

Next, we consider different cases depending on the value of $\alpha(F)$. If $\alpha(F) = 1$, then F is a complete 3-edge-coloured graph and by a theorem of Erdős, Gyárfás and Pyber (see Theorem 3.2.1), there exists a partition of $V(F)$ into 2 monochromatic trees. The remaining proof now is divided into the cases $\alpha(F) \geq 3$ and $\alpha(F) = 2$.

Case $\alpha(F) \geq 3$. From the condition on the independence number of G, there exist three vertices $r, b, g \in V(G)$ that pairwise do not have any monochromatic path connecting them. With high probability, they have a common neighbourhood in G of size at least $np^3/2$. Let X_{rbg} be the largest subset of this common neighbourhood such that for each $i \in \{r, b, g\}$, the edges from i to X_{rbg} in G are all coloured with one colour. Then, since there are no monochromatic paths between any two of r, b, g, we have $|X_{rbg}| \geq np^3/12$ and moreover we may assume that all edges between r and X_{rbg} are red, all between b and X_{rbg} are blue and those between g and X_{rbg} are green. Now we notice that all vertices that have a neighbour in X_{rbg} are covered by the union of the spanning trees of the red component of r, the blue component of b and the green component of g.

We are done in the case where every vertex has a neighbour in X_{rbg}, as the vertices in $X_{rbg} \cup N_G(X_{rbg})$ are covered by the red, blue and green component containing r, b and g, respectively. Otherwise, w.h.p. any vertex $y \in V \setminus (X_{rbg} \cup N_G(X_{rbg}))$ has many common neighbours with r, b and g in G that are also neighbours of some vertex in X_{rbg}. An analysis of the possible colourings of the edges between X_{rbg} and the common neighbourhood of the vertices r, b, g and y yields the following: for some $i \in \{r, b, g\}$, let us say $i = r$, every vertex $y \in X_{rbg}$ can be connected to r by a monochromatic path in colour red or either to g or b by a monochromatic path in the colour blue or green, respectively.

This already gives us that all vertices in G can be covered by 5 monochromatic trees, since all the vertices in $N_G(X_{rbg})$ lie in the red component of r, or the green component of g, or in the blue component of b and every vertex in $V \setminus N_G(X_{rbg})$ lies in the red component of r, in the blue component of g or in the green component of b. By analysing the colours of edges to the common neighbourhood of carefully chosen vertices, we are able to show that actually three of those five trees already cover all the vertices of G.

Case $\alpha(F) = 2$. Let us consider a 3-uniform hypergraph $\mathcal{H}$ defined as follows (this definition is inspired by a construction of Gyárfás [54] and also appears in [17]). The vertices of $\mathcal{H}$ are the monochromatic components of F and three vertices form a hyperedge if the corresponding three components have a vertex in common (in particular, those three monochromatic components must be of different colours). Hence, $\mathcal{H}$ is a 3-uniform 3-partite hypergraph.

We observe that if A is a vertex cover of $\mathcal{H}$, then the monochromatic components associated with the vertices in A cover all the vertices of G. This implies that $\mathrm{tc}_3(G) \leq \tau(\mathcal{H})$. Also, it is easy to see that $\nu(\mathcal{H}) \leq \alpha(F) = 2$. Now, recall that Aharoni's result [1] (which corresponds to Ryser's conjecture for $r = 3$) states that for every 3-uniform 3-partite hypergraph $\mathcal{H}$ we have $\tau(\mathcal{H}) \leq 2\nu(\mathcal{H})$. Together with the previous observation, this implies $\mathrm{tc}_3(G) \leq 4$. But our goal is to prove that $\mathrm{tc}_3(G) \leq 3$. To this aim, we analyse the hypergraph $\mathcal{H}$ more carefully, reducing the situation to a few possible settings of components covering all vertices. In each of those cases, we can again analyse the possible colouring of edges of common neighbours of specific vertices, inferring that indeed there are 3 monochromatic components which cover all vertices.

3.4 Proof of Theorem II

Instead of analysing the colouring of the graph $G = G(n,p)$, it will be helpful to analyse the following auxiliary graph.

Definition 3.4.1 (Shortcut graph)**.** Let G be a graph and φ be a 3-edge-colouring of G. The *shortcut graph* of G (with respect to φ) is the graph $F = F(G,\varphi)$ that has $V(G)$ as the vertex set and the following edge set:

$$\{uv : u, v \in V(G) \text{ and } u \text{ and } v \text{ are connected in } G \text{ by a path monochromatic under } \varphi\}.$$

Let us consider an edge-multicolouring φ' of $F = F(G,\varphi)$ which assigns to an edge $uv \in E(F(G,\varphi))$ the list of all the colours of monochromatic paths connecting u and v in G under the colouring φ. We will say that φ' is the *inherited colouring*[3] of $F(G,\varphi)$. We say that an edge $e \in F(G,\varphi)$ *has colour* ρ (or *is coloured with* ρ) if ρ belongs to the list of colours assigned to e by φ'. We say that a subgraph H of $F(G,\varphi)$ is *monochromatic under* φ' if all the edges of H are coloured with a common colour. Let $\mathrm{tc}(F,\varphi')$ be the minimum number k such that there are k trees $T_1,\dots,T_k$ which are monochromatic under φ' such that $V(F) = V(T_1) \cup \dots \cup V(T_k)$. Note that any covering of $F(G,\varphi)$ with monochromatic trees under φ' corresponds to a covering of G with monochromatic trees under the colouring φ. In particular, if we show that for every 3-edge-colouring φ of G, we have $\mathrm{tc}(F,\varphi') \leq 3$, where $F = F(G,\varphi)$ is the shortcut graph of G with respect to φ, and φ' is the inherited colouring of F, then we have shown that $\mathrm{tc}_3(G) \leq 3$. Therefore, Theorem II follows from the following lemma.

[3]Although φ' is a multicolouring, in the sense that we assigned several colours to each edge, we will refer to it as colouring, for simplicity.

Lemma 3.4.2. *Let* $p \gg \left(\frac{\log n}{n}\right)^{1/6}$ *and let* $G = G(n,p)$. *The following holds with high probability. For any* 3*-edge-colouring* φ *of* G*, we have* $\mathrm{tc}(F, \varphi') \leq 3$*, where* F *is the shortcut graph* $F = F(G, \varphi)$ *and* φ' *is the inherited colouring of* F*.*

The proof of Lemma 3.4.2 is divided into two different cases, depending on the independence number of F. Subsections 3.4.1 and 3.4.2 are devoted, respectively, to the proof of Lemma 3.4.2 when $\alpha(F) \geq 3$ and $\alpha(F) \leq 2$.

From now on, we fix $\varepsilon > 0$ and assume that $p \gg \left(\frac{\log n}{n}\right)^{1/6}$ and n is sufficiently large. Then, by Lemma 3.2.2, we may assume that the following holds w.h.p.:

1. There is an edge between any two sets of size $\omega\left((\log n)/p\right)$.

2. Every vertex $v \in V(G)$ has degree $d_G(v) = (1 \pm \varepsilon)pn$.

3. Every set of $i \leq 6$ vertices has $(1 \pm \varepsilon)p^i n$ common neighbours.

3.4.1 Shortcut graphs with independence number at least three

Proof of Lemma 3.4.2 for $\alpha(F) \geq 3$. Since $\alpha(F) \geq 3$, there exist three vertices $r, b, g \in V(G)$ that pairwise do not have any monochromatic path connecting them in G. In particular, if v is a common neighbour of r, b and g in G, then the edges vr, vb and vg have all different colours. The common neighbourhood of r, b and g in G has size at least $np^3/2$. Let X_{rbg} be the largest subset of this common neighbourhood such that for each $i \in \{r, b, g\}$, the edges between i and the vertices of X_{rbg} are all coloured with the same colour in G. Then $|X_{rbg}| \geq np^3/12$. Without loss of generality, assume that all edges between r and the vertices of X_{rbg} are red, between b and the vertices of X_{rbg} are blue and those between g and the vertices of X_{rbg} are green. Let $C_{\text{red}}(r)$, $C_{\text{blue}}(b)$ and $C_{\text{green}}(g)$ be respectively the red, blue and green components in G containing r, g and b.

Notice that all vertices of F that have a neighbour in X_{rbg} are covered by $C_{\text{red}}(r)$, $C_{\text{blue}}(b)$ or $C_{\text{green}}(g)$. Therefore, the proof would be finished if every vertex had a neighbour in X_{rbg}. If this is not the case, we fix an arbitrary vertex $y \in V \setminus (X_{rbg} \cup N_G(X_{rbg}))$. By our choice of p, there are at least $np^4/2$ common neighbours of y, r, b and g. Let X_{yrbg} be the largest subset of the common neighbourhood of y, r, b and g such that for each $i \in \{r, b, g\}$, the edges between i and X_{yrbg} are all coloured the same. Then $|X_{yrbg}| \geq np^4/12$. Note that since $y \notin N_G(X_{rbg})$, the sets X_{yrbg} and X_{rbg} are disjoint. Furthermore, since $|X_{yrbg}|, |X_{rbg}| \gg \frac{\log n}{p}$, we have

$$|E_G(X_{yrbg}, X_{rbg})| \geq 1.$$

We now analyse the colours between r, b, g and the set X_{yrbg}. Again, since there is no monochromatic path connecting any two of r, b and g, all $i \in \{r, b, g\}$ have to connect

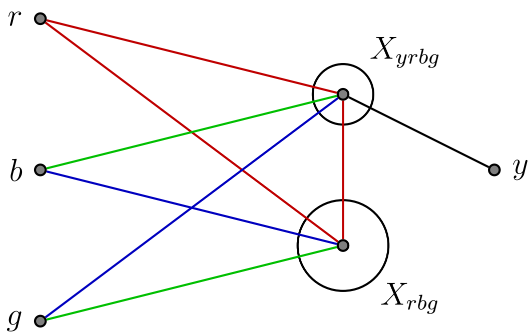


Figure 3.1: Analysis of the colouring of the edges incident on X_{rbg} and on X_{yrbg}.

to X_{yrbg} in different colours. Since X_{yrbg} is disjoint from X_{rbg}, by the maximality of X_{rbg} we cannot have r, b and g being simultaneously connected to X_{yrbg} by red, blue and green edges, respectively. Assume first that for each $i \in \{r, b, g\}$, the edges between i and X_{yrbg} have different colours from the edges between i and X_{rbg}. Then let uv be an edge between X_{yrbg} and X_{rbg} and notice that whatever the colour of uv is, we will have a monochromatic path connecting two of the vertices in $\{r, g, b\}$. Therefore, we can assume that for some $i \in \{r, g, b\}$, we have that all the edges between i and X_{rbg} and all the edges between i and X_{yrbg} coloured the same. Without loss of generality, we may say that such i is r. In this case, the edges between b and X_{yrbg} are green and the edges between g and X_{yrbg} are blue. Finally, all the edges between X_{yrbg} and X_{rbg} are red, otherwise we would be able to connect b and g by some monochromatic path. Figure 3.1 shows the colouring of the edges that we have analysed so far.

Let us now consider any further vertex $x \in V \setminus (X_{rbg} \cup N_G(X_{rbg}))$ with $x \neq y$, if such a vertex exists. We define X_{xrbg} analogously to X_{yrbg} and observe that the colour pattern from r, b, g to X_{xrbg} must be the same as the one to X_{yrbg}. Indeed, if this is not the case, then a similar analysis of the colours of the edges between $\{r, b, g\}$ and X_{xrbg} yields that for some $i \in \{b, g\}$, we know that the edges connecting i to X_{xrbg} are of the same colour as the edges connecting i to X_{rbg}. Without loss of generality, let us say that i is g. Then the edges between b and X_{xrbg} are red and the edges between r and X_{xrbg} are green, otherwise X_{xrbg} and X_{rbg} would not be disjoints sets. Figure 3.2 shows the colouring of the edges incident to X_{yrbg} and X_{xrbg}. Since $|X_{yrbg}|, |X_{xrbg}| \gg \frac{\log n}{p}$, we have that there is some edge uv between X_{yrbg} and X_{xrbg}. But then however we colour uv, we will get a monochromatic path connecting two vertices in $\{r, b, g\}$, which is a contradiction. Thus, the colour pattern of edges between $\{r, b, g\}$ and X_{xrbg} is the same as the colour pattern of the edges between $\{r, b, g\}$ and X_{yrbg}.

Therefore, we have that each vertex in $X_{rbg} \cup N_G(X_{rbg})$ belongs to one of the monochromatic components $C_{\text{red}}(r)$, $C_{\text{blue}}(b)$ or $C_{\text{green}}(g)$, while a vertex in $V(G) \setminus (X_{rbg} \cup N_G(X_{rbg}))$ belongs to one of the monochromatic components $C_{\text{red}}(r)$, $C_{\text{green}}(b)$ or $C_{\text{blue}}(g)$ where the

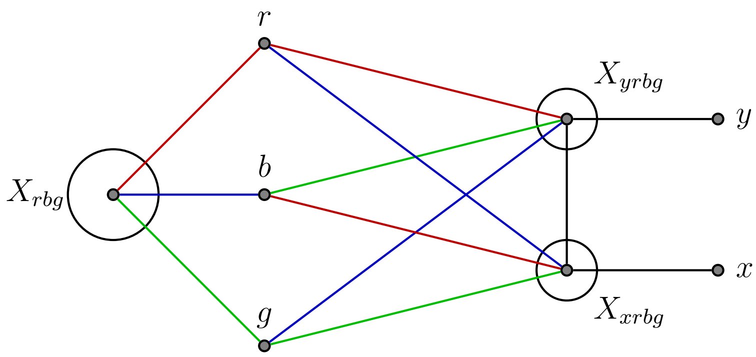


Figure 3.2: Analysis of the colour of the edges incident on X_{yrbg} and on X_{xrbg}.

latter two are the green component containing b and the blue component containing g, respectively. This gives a covering of G with five monochromatic trees. Next we will show that actually three of those trees already cover all the vertices.

Suppose that at least four among the components $C_{\text{red}}(r)$, $C_{\text{blue}}(b)$, $C_{\text{green}}(b)$, $C_{\text{green}}(g)$, and $C_{\text{blue}}(g)$ are needed to cover all vertices. Since there does not exist any monochromatic path between any two of r, b, g, we know that for each $i \in \{r, b, g\}$, any monochromatic component containing i does not intersect $\{r, g, b\} \setminus \{i\}$. Hence, for each $i \in \{r, b, g\}$, one of these components contains i. Also, one element in $\{r, b, g\}$ belongs to two of these components. Without loss of generality, let us say that b belongs to two of these components. Therefore, $C_{\text{red}}(r)$, $C_{\text{blue}}(b)$ and $C_{\text{green}}(b)$ are three of these at least four components needed to cover all the vertices. Now there are two cases regarding the fourth component: we need $C_{\text{green}}(g)$ as the fourth component or we need $C_{\text{blue}}(g)$ (those two cases might intersect).

We begin with the first case, where we need the components $C_{\text{red}}(r)$, $C_{\text{blue}}(b)$, $C_{\text{green}}(b)$ and $C_{\text{green}}(g)$ to cover all the vertices of G. Let

$$\tilde{b} \in C_{\text{blue}}(b) \setminus (C_{\text{red}}(r) \cup C_{\text{green}}(b) \cup C_{\text{green}}(g))$$

and let

$$\tilde{g} \in C_{\text{green}}(b) \setminus (C_{\text{red}}(r) \cup C_{\text{blue}}(b) \cup C_{green}(g))\,.$$

Then let $X_{\tilde{b}\tilde{g}rbg}$ be the maximum set of common neighbours of $\tilde{b}, \tilde{g}, r, g, b$ such that for each $i \in \{\tilde{b}, \tilde{g}, r, b, g\}$, the edges from i to $X_{\tilde{b}\tilde{g}rbg}$ are all coloured the same. Since $|X_{\tilde{b}\tilde{g}rbg}| \geq np^5/240 \gg \frac{\log n}{p}$, we have

$$|E_G(X_{\tilde{b}\tilde{g}rbg}, X_{yrbg})| \geq 1 \quad \text{and} \quad |E_G(X_{\tilde{b}\tilde{g}rbg}, X_{rbg})| \geq 1.$$

We will analyse the possible colours of the edges between the specified vertices and $X_{\tilde{b}\tilde{g}rbg}$. If for each of r, b, g, the colour it sends to $X_{\tilde{b}\tilde{g}rbg}$ is different from the colour it sends to X_{rbg}, then any edge between $X_{\tilde{b}\tilde{g}rbg}$ and X_{rbg} ensures a monochromatic path between two of r, b, g (in the colour of that edge). Similarly, it cannot happen that for each of r, b, g, the colour it sends to $X_{\tilde{b}\tilde{g}rbg}$ is different from the colour it sends to X_{yrbg}. Thus, since r sends red to both X_{rbg} and X_{yrbg} while the colours from b (and g) to X_{rbg} and X_{yrbg} are switched, the colour of the edges between r and $X_{\tilde{b}\tilde{g}rbg}$ is red.

Now note that, by the choice of $\tilde{b}$ and $\tilde{g}$, the edges between each of them and $X_{\tilde{b}\tilde{g}rbg}$ can not be red. Further, the choice implies that an edge between $\tilde{b}$ and $X_{\tilde{b}\tilde{g}rbg}$ can not be of the same colour (green or blue) as an edge between $\tilde{g}$ and $X_{\tilde{b}\tilde{g}rbg}$. If g would send blue (and hence b would send green) edges to $X_{\tilde{b}\tilde{g}rbg}$, there would either be a blue path between b and g (if the edges between $\tilde{b}$ and $X_{\tilde{b}\tilde{g}rbg}$ are blue) or $\tilde{b}$ would lie in $C_{\text{green}}(b)$ (if the edges between $\tilde{b}$ and $X_{\tilde{b}\tilde{g}rbg}$ are green). Since both those situations would mean a contradiction, we may assume that each of r, b, g sends edges with that colour to $X_{\tilde{b}\tilde{g}rbg}$ as it does to X_{rbg}. But then $X_{\tilde{b}\tilde{g}rbg}$ is actually a subset of X_{rbg} and since $\tilde{g}$ has an edge to X_{rbg}, it lies in one of $C_{\text{red}}(r)$, $C_{\text{blue}}(b)$, or $C_{\text{green}}(g)$; a contradiction.

In the case where the forth component that we need is $C_{\text{blue}}(g)$, we repeat the construction of $X_{\tilde{b}\tilde{g}rbg}$ similarly as before by letting

$$\tilde{b} \in C_{\text{blue}}(b) \setminus (C_{\text{red}}(r) \cup C_{\text{green}}(b) \cup C_{\text{blue}}(g))$$

and

$$\tilde{g} \in C_{\text{green}}(b) \setminus (C_{\text{red}}(r) \cup C_{\text{blue}}(b) \cup C_{\text{blue}}(g)).$$

Also as before, we end up with $X_{\tilde{b}\tilde{g}rbg}$ being part of X_{rbg}. From the choice of $\tilde{g}$, the edges it sends to $X_{\tilde{b}\tilde{g}rbg}$ have to be green, since otherwise it would be in $C_{\text{red}}(r)$ or $C_{\text{blue}}(b)$. But that gives a green path between b and g, a contradiction.

Summarising, we infer that three components among $C_{\text{red}}(r)$, $C_{\text{blue}}(b)$, $C_{\text{green}}(b)$, $C_{\text{green}}(g)$ and $C_{\text{blue}}(g)$ cover the vertex set of G. $\square$

3.4.2 Shortcut graphs with independence number at most two

Proof of Lemma 3.4.2 for $\alpha(F) \leq 2$. We start by noticing that if $\alpha(F) = 1$, then the graph F together with the colouring φ' is a complete 3-coloured graph and therefore, by Theorem 3.2.1, there exists a partition of $V(F)$ into 2 monochromatic trees. Thus, we may assume that $\alpha(F) = 2$.

Let $\mathcal{H}$ be the 3-uniform hypergraph with $V(\mathcal{H})$ being the collection of all the monochromatic components of F under the colouring φ' and three monochromatic components form a hyperedge in $\mathcal{H}$ if they share a vertex. Notice that $\mathcal{H}$ is 3-partite, since distinct monochromatic components of the same colour do not have a common vertex and therefore they can not belong to the same hyperedge. In other words, the colour of each component give us a 3-partition of the vertex set of $\mathcal{H}$. We denote by V_{red},V_{blue} and V_{green} the set of vertices of $V(\mathcal{H})$ that correspond to, respectively, red, blue and green components. Such construction was inspired by a construction due to Gyárfás [54] and it was also used in [17].

Note that every vertex v of F is contained in a monochromatic component for each one of the colours (a monochromatic component could consist only of v). Therefore, any vertex cover of $\mathcal{H}$ corresponds to a covering of the vertices of F with monochromatic trees. Indeed, if A is a vertex cover of $\mathcal{H}$, then consider the monochromatic components corresponding to each vertex in A. If any vertex v of F is not covered by those components, then the vertices in $\mathcal{H}$ corresponding to the red, green and blue components in F containing v do not belong to A and they form an hyperedge. But this contradicts the fact that A is a vertex cover of $\mathcal{H}$. Therefore,

$$\mathrm{tc}(F, \varphi') \leq \tau(\mathcal{H}). \tag{3.2}$$

The inequality (3.2) corresponds to Proposition 4.1 in [17] in our setting.

Let $L = \bigcup_{s \in V_{\text{red}}} L_s$ be the union of the link graphs L_s of all vertices $s \in V_{\text{red}}$. Any vertex cover of this bipartite graph L corresponds to a vertex cover of $\mathcal{H}$ of the same size. Therefore,

$$\tau(\mathcal{H}) \leq \tau(L). \tag{3.3}$$

Furthermore, by the König–Egerváry theorem we know that $\tau(L) = \nu(L)$. Thus, if $\nu(L) \leq 3$, then by (3.2) and (3.3), we have

$$\mathrm{tc}(F, \varphi') \leq \tau(\mathcal{H}) \leq \tau(L) = \nu(L) \leq 3.$$

Therefore, we may assume that $\nu(L) \geq 4$, and fix a matching M_L of size at least four in L. Let us say that M_L consists of the edges G_1B_1, G_2B_2, G_3B_3, and G_4B_4, where $\{G_1, G_2, G_3, G_4\} \subseteq V_{\text{green}}$ and $\{B_1, B_2, B_3, B_4\} \subseteq V_{\text{blue}}$.

Now we give an upper bound for $\nu(\mathcal{H})$. Note that any matching $M_{\mathcal{H}}$ in $\mathcal{H}$ gives us an independent set I in F. Indeed, for each hyperedge $e \in M_{\mathcal{H}}$, let $v_e \in V(F)$ be any vertex in the intersection of those monochromatic components associated to the vertices in e and let $I = \{v_e : e \in M_{\mathcal{H}}\}$. We claim that I is an independent set in F. Indeed, if v_e and v_f were adjacent vertices in I, then e and f intersect, as the edge connecting v_e to v_f in F will connect the monochromatic components containing v_e and v_f of that colour that is given to the edge $v_e v_f$. Therefore, since $\alpha(F) = 2$, we have

$$\nu(\mathcal{H}) \leq \alpha(F) = 2. \tag{3.4}$$

Now, if there are three different edges in M_L that are edges in the link graphs of three different vertices of V_{red}, then there would be a matching of size 3 in $\mathcal{H}$, contradicting (3.4). Therefore, we may assume that M_L is contained in the union of at most two link graphs, say L_{R_1} and L_{R_2}, of vertices $R_1, R_2 \in V_{\text{red}}$. Now we are left with three cases: (Case 1) two edges of M_L belong to L_{R_1} and two belong to L_{R_2}; (Case 2) three edges of M_L belong to L_{R_1} and one to L_{R_2}; (Case 3) the four edges of M_L belong to L_{R_1}. Without loss of generality, we can describe each of those three cases as follows (see Figures 3.3, 3.4 and 3.5):

Case 1: The edges G_1B_1 and G_2B_2 belong to L_{R_1} and the edges G_3B_3 and G_4B_4 belong to L_{R_2}. That means that all the following four sets are non-empty:

$$\begin{aligned} J_1 &:= R_1 \cap G_1 \cap B_1, \\ J_2 &:= R_1 \cap G_2 \cap B_2, \\ J_3 &:= R_2 \cap G_3 \cap B_3, \\ J_4 &:= R_2 \cap G_4 \cap B_4. \end{aligned}$$

Case 2: The edges G_1B_1, G_2B_2 and G_3B_3 belong to L_{R_1} and the edge G_4B_4 belongs to L_{R_2}. That means that all the following four sets are non-empty:

$$\begin{aligned} J_1 &:= R_1 \cap G_1 \cap B_1, \\ J_2 &:= R_1 \cap G_2 \cap B_2, \\ J_3 &:= R_1 \cap G_3 \cap B_3, \\ J_4 &:= R_2 \cap G_4 \cap B_4. \end{aligned}$$

Case 3: The edges G_1B_1, G_2B_2, G_3B_3 and G_4B_4 belong to L_{R_1}. That means that all the following four sets are non-empty:

$$\begin{aligned} J_1 &:= R_1 \cap G_1 \cap B_1, \\ J_2 &:= R_1 \cap G_2 \cap B_2, \\ J_3 &:= R_1 \cap G_3 \cap B_3, \\ J_4 &:= R_1 \cap G_4 \cap B_4. \end{aligned}$$

In this case, let R_2 be any other red component different from R_1 and let B and G be, respectively, a blue and a green component with $R_2 \cap B \cap G \neq \varnothing$. Suppose that $G \notin \{G_1, G_2, G_3, G_4\}$. Then the three of the edges G_1B_1, G_2B_2, G_3B_3 and G_4B_4 are not incident to GB (because B must be different from at least three of the sets B_1, B_2, B_3 and B_4) and these three edges together with GB may be analysed just as in Case 2. Therefore, we may suppose that $G \in \{G_1, G_2, G_3, G_4\}$. Let us say, without loss of generality, that $G = G_4$. If $B \notin \{B_1, B_2, B_3\}$, then the edges G_1B_1, G_2B_2 and G_3B_3 belong to L_{R_1}, the edge GB belongs to L_{R_2} and this case may be analysed, again, just as in Case 2. Therefore, we may assume that $B \in \{B_1, B_2, B_3\}$. Let us say, without loss of generality that $B = B_3$.

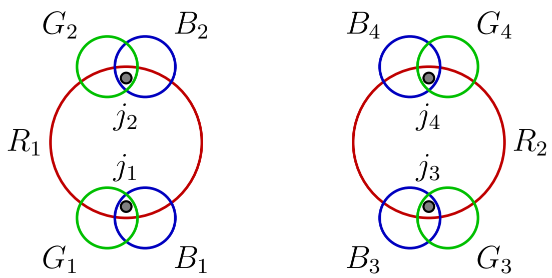


Figure 3.3: Case 1

Then let J_5 be the following non-empty set:

$$J_5 := R_2 \cap G_4 \cap B_3. \tag{3.5}$$

Let us further remark that, since $\nu(\mathcal{H}) \leq 2$, in each of the three cases above, we have

$$V(F) = R_1 \cup R_2 \cup G_1 \cup G_2 \cup G_3 \cup G_4 \cup B_1 \cup B_2 \cup B_3 \cup B_4.$$

Otherwise, for any uncovered vertex $v \in V(F)$, the hyperedge given by the red, blue and green components containing v together with the hyperedges $R_1B_1G_1$ and $R_2B_3G_3$ (in Cases 1 and 2) or $R_2B_3G_4$ (in Case 3) is a matching of size 3 in $\mathcal{H}$.

Let us start with Case 1.

Proof in Case 1: We will prove that R_1 and R_2 together with possibly one further monochromatic component cover $V(F)$. For each $i \in \{1,2,3,4\}$, let $\tilde{B}_i = B_i \setminus (R_1 \cup R_2)$ and $\tilde{G}_i = G_i \setminus (R_1 \cup R_2)$.

Pick vertices $j_i \in J_i$, with $i \in \{1,2,3,4\}$, arbitrarily. Consider a vertex $o \in \tilde{B}_1$ (if such a vertex exists). Since $\alpha(F) = 2$, there is an edge connecting two of o, j_2, j_3. Because j_2 and j_3 belong to different components of each colour, such an edge must be incident to o. So let us say that such edge is oj_i, for some $i \in \{2,3\}$. Since $o \notin R_1 \cup R_2$, the edge oj_i cannot be red. And since $o \in B_1$, oj_i cannot be blue either, otherwise we would connect the blue components B_1 and B_i. Now assume that o and j_2 are not adjacent. Then oj_3 is a green edge in F. By analogously analysing the edge between o, j_2 and j_4 together with the supposition that oj_2 is not an edge in F, we get that oj_4 must be a green edge in F. But then we have a green path j_3oj_4 connecting j_3 to j_4, a contradiction. Therefore oj_2 is an edge in F and it is green. That implies that $o \in G_2$. Therefore $\tilde{B}_1 \subseteq G_2$. Analogously, we can conclude the following:

$$\begin{aligned} \tilde{B}_1 &\subseteq G_2, & \tilde{G}_1 &\subseteq B_2, \\ \tilde{B}_2 &\subseteq G_1, & \tilde{G}_2 &\subseteq B_1, \\ \tilde{B}_3 &\subseteq G_4, & \tilde{G}_3 &\subseteq B_4, \\ \tilde{B}_4 &\subseteq G_3, & \tilde{G}_4 &\subseteq B_3. \end{aligned} \tag{3.6}$$

Claim 3.4.3. *We have* $\tilde{B}_1 \cup \tilde{G}_1 \cup \tilde{B}_2 \cup \tilde{G}_2 = \varnothing$ *or* $\tilde{B}_3 \cup \tilde{G}_3 \cup \tilde{B}_4 \cup \tilde{G}_4 = \varnothing$.

Proof. Suppose for a contradiction that there exist $o_1 \in \tilde{B}_1 \cup \tilde{G}_1 \cup \tilde{B}_2 \cup \tilde{G}_2$ and $o_2 \in \tilde{B}_3 \cup \tilde{G}_3 \cup \tilde{B}_4 \cup \tilde{G}_4$. Recall that from our choice of p, there is some $z \in N(j_1, j_2, j_3, j_4, o_1, o_2)$. Two of the edges zj_i,for $i \in \{1, 2, 3, 4\}$, have the same colour. Since each j_i belongs to different green and blue components, those two edges are red. Since $\{j_1, j_2\} \in R_1$ and $\{j_3, j_4\} \in R_2$, those two red edges are either zj_1 and zj_2 or zj_3 and zj_4. Let us say that zj_1 and zj_2 are red (the other case is similar). Then one of the edges zj_3 and zj_4 has to be green and the other blue. Now, since $o_1 \notin R_1$, the edge zo_1 is either green or blue. Then one of the paths o_1zj_3 or o_1zj_4 is green or blue. This implies that $o_1 \in B_3 \cup G_3 \cup B_4 \cup G_4$. On the other hand, (3.6) implies that $o_1 \in (B_1 \cup B_2) \cap (G_1 \cup G_2)$. But then we reached a contradiction, since that would mean that o_1 belongs to two different components of the same colour. □

We may assume without loss of generality that $\tilde{B}_3 \cup \tilde{G}_3 \cup \tilde{B}_4 \cup \tilde{G}_4$ is empty. Then, recalling that $\nu(\mathcal{H}) \le 2$ and in view of (3.6), the union of the components R_1, B_1, G_1 and R_2 covers every vertex of F. If we show that $B_1 \subseteq G_1 \cup R_1 \cup R_2$ or that $G_1 \subseteq B_1 \cup R_1 \cup R_2$, then we get three monochromatic components covering the vertices of F. Our next claim states precisely that.

Claim 3.4.4. *We have* $\tilde{B}_1 \setminus G_1 = \varnothing$ *or* $\tilde{G}_1 \setminus B_1 = \varnothing$.

Proof. Suppose that there exist two distinct vertices $b \in \tilde{B}_1 \setminus G_1$ and $g \in \tilde{G}_1 \setminus B_1$. Let $z \in N(j_1, j_2, j_3, j_4, b, g)$. As before, either zj_1 and zj_2 or zj_3 and zj_4 are red edges. First assume that zj_1 and zj_2 are red. Then one of the edges zj_3 and zj_4 has to be green and the other blue. Now, since $b \notin R_1$, the edge zb is either green or blue. Then one of the paths bzj_3 or bzj_4 is green or blue. This implies that $b \in B_3 \cup G_3 \cup B_4 \cup G_4$. On the other hand, (3.6) implies that $b \in B_1 \cap G_2$. Then we reached a contradiction, since that would mean that b belongs to two different components of the same colour.

Therefore, the edges zj_3 and zj_4 are red and one of the edges zj_1 and zj_2 is green and the other is blue. First let us say that zj_1 is green and zj_2 is blue. Since $b \notin (R_1 \cup R_2)$, the edge zb cannot be red. Also the edge zb cannot be blue otherwise the path bzj_2 would connect the components B_1 and B_2. Finally, zb cannot be green, otherwise the path bzj_1 would gives us that $b \in G_1$. Therefore, zj_1 is blue and zj_2 is green. But this case analogously leads to a contradiction (with g and G_i instead of b and B_i and green and blue switched). □

We proceed to the proof of Case 2.

Proof in Case 2: As in Case 1, pick vertices $j_i \in J_i$, with $i \in \{1, 2, 3, 4\}$ arbitrarily. We claim that $V(F) \subseteq R_1 \cup R_2 \cup B_4 \cup G_4$. Indeed, let $o \in V(F) \setminus (R_1 \cup R_2)$. Notice that since $\alpha(F) = 2$, there is an edge in each of the following sets of three vertices: $\{o, j_4, j_1\}$, $\{o, j_4, j_2\}$, and $\{o, j_4, j_3\}$. We claim that oj_4 is an edge of F. Indeed,

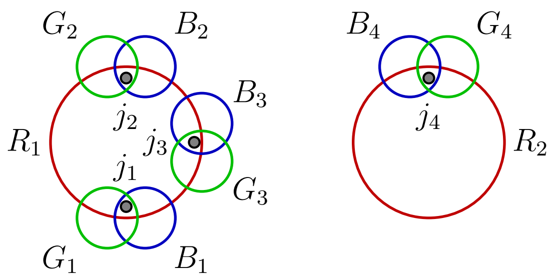


Figure 3.4: Case 2

if this was not the case, then since there cannot be an edge between j_4 and j_i for $i = 1, 2, 3$, we would have the edges oj_1, oj_2 and oj_3 and all of them would be coloured green or blue. Thus, two of them would be coloured the same, connecting two distinct components of one colour in this colour, a contradiction. So $oj_4 \in E(F)$ and since oj_4 cannot be red, we conclude that $o \in (B_4 \cup G_4)$. Therefore, R_1, R_2, B_4 and G_4 cover all vertices of F.

If $B_4 \setminus (R_1 \cup R_2 \cup G_4) = \varnothing$ or $G_4 \setminus (R_1 \cup R_2 \cup B_4) = \varnothing$, then we get three monochromatic components covering $V(F)$. So let us assume that there exist $b \in B_4 \setminus (R_1 \cup R_2 \cup G_4)$ and $g \in G_4 \setminus (R_1 \cup R_2 \cup B_4)$. If b and g are not adjacent, then since each of the sets $\{b, g, j_i\}$, for $i = 1, 2, 3$, has to induce at least one edge, there are two edges between b and $\{j_1, j_2, j_3\}$ or two edges between g and $\{j_1, j_2, j_3\}$. However, from the choice of b, we know that all the edges between b and $\{j_1, j_2, j_3\}$ are green, and therefore, two of such edges would give us a green connection between two different green components, a contradiction. Similarly, from the choice of g, we know that all the edges between b and $\{j_1, j_2, j_3\}$ are blue, and two of such edges would give us a blue connection between two different blue components, again a contradiction.

Hence, we conclude that $bg \in F$ for any $b \in B_4 \setminus (R_1 \cup R_2 \cup G_4)$ and any $g \in G_4 \setminus (R_1 \cup R_2 \cup B_4)$ and any such edge bg is red. Therefore, there is a red component R_3 covering $(B_4 \triangle G_4) \setminus (R_1 \cup R_2)$, where $B_4 \triangle G_4$ denotes the symmetric difference. If $(B_4 \cap G_4) \setminus (R_1 \cup R_2) = \varnothing$, then R_1, R_2 and R_3 cover $V(F)$ and we are done. Therefore, suppose there is a vertex $x \in (B_4 \cap G_4) \setminus (R_1 \cup R_2)$. If $R_2 \setminus (B_4 \cup G_4) = \varnothing$, then R_1, B_4, G_4 cover $V(F)$ and we are done. Therefore, suppose there is a vertex $y \in R_2 \setminus (B_4 \cup G_4)$. Note that $xy \notin E(F)$, since x and y belong to different components in each of the colours. Also, $xj_i \notin E(F)$, for $i \in \{1, 2, 3\}$, since otherwise two different components of the same colour would be connected in that colour by the edge xj_i. Now $\alpha(F) = 2$ implies that $yj_i \in E(F)$, for $i \in \{1, 2, 3\}$ (otherwise, $\{x, y, j_i\}$ would be an independent set). But these edges must all be green or blue, hence two of them are of the same colour, connecting two different components of one colour in that colour, a contradiction.

We arrived at the last case, Case 3.

Proof in Case 3: Similarly to the previous cases, let us pick vertices $j_i \in J_i$, with $i \in \{1, 2, 3, 4, 5\}$ arbitrarily. We will show first that we can cover all vertices of F with four

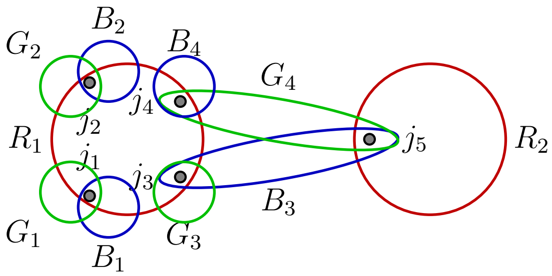


Figure 3.5: Case 3

monochromatic components. Let $o_1, o_2 \in V(F) \setminus (R_1 \cup B_3 \cup G_4)$ and let z be a vertex in $N(j_1, j_2, j_3, o_1, o_2, j_5)$. At least one of the edges zj_1, zj_2 and zj_3 is red, as otherwise we would connect two distinct components of one colour in that colour. Therefore, $z \in R_1$. Since $o_1, o_2, j_5 \notin R_1$, the edges zo_1, zo_2 and zj_5 cannot be red. Furthermore, o_1z and o_2z are coloured with a colour different from the colour of the edge j_5z, as otherwise they would belong to B_3 or G_4. Thus, o_1 and o_2 are connected by a monochromatic path in green or blue. Hence, we showed that any two vertices of $V(F) \setminus (R_1 \cup B_3 \cup G_4)$ are connected by a monochromatic path in green or blue. We infer that there is a green or blue component covering $V(F) \setminus (R_1 \cup B_3 \cup G_4)$. Therefore, R_1, B_3, G_4 and one further blue or green component C cover all vertices of G. Let us assume that C is a green component; the case where C is a blue component is analogous.

We claim that $R_1 \cup B_3 \cup C$, or $R_1 \cup G_4 \cup C$, or $R_1 \cup B_3 \cup G_4$ covers $V(F)$. Indeed, suppose for the sake of contradiction that there exist vertices $g \in G_4 \setminus (R_1 \cup B_3 \cup C)$, $b \in B_3 \setminus (R_1 \cup G_4 \cup C)$ and $c \in C \setminus (R_1 \cup B_3 \cup G_4)$. Let $z \in N(j_1, j_2, j_3, g, b, c)$ and note that one of zj_1, zj_2 and zj_3 is red. Consequently gz, cz and bz are not red. Notice, however, that gz and bz can not be both green and neither both blue. Now let us say cz is green. Since $c \notin G_4$ and $g \in G_4$, we would have gz blue in this case. But then bz must be green and since $c \in C$ and C is a green component, we have $b \in C$, which is a contradiction. Therefore, cz must be blue. Then, since $c \notin B_3$ and $b \in B_3$, the edge bz should be green. Thus the edge gz is blue. Since this argument holds for any $g \in G_4 \setminus (R_1 \cup B_3 \cup C)$ and $c \in C \setminus (R_1 \cup B_3 \cup G_4)$, we conclude that $V(F) \setminus (R_1 \cup B_3)$ can be covered by one blue tree. Hence, G can be covered by the three monochromatic trees. This finishes the last case and thereby the proof of Lemma 3.4.2. □

3.5 Concluding Remarks

The prove Theorem II relied mainly on the fact the random graph $G(n, p)$ has the properties stated in Lemma 3.2.2. It is easy to see that if G is graph on n vertices with $\delta(G) \geq (1 - \varepsilon)n$, for some $\varepsilon > 0$, then every set of at most 6 vertices in G has a common neighbour. Furthermore, for every sufficiently large sets $X, Y \subseteq V(G)$, we will have $e(X, Y) > 0$. This allows us to prove, by following the same ideas from the proof of Theorem II, that for sufficiently small $\varepsilon > 0$, every graph G with $\delta(G) \geq (1-\varepsilon)n$ is such that $\mathrm{tc}_3(G) \leq 3$. It would be interesting to determine the maximum ε for which this is still true. Bal and DeBiasio [7] proved that $\varepsilon \geq 1/(6e)$ and they notice that ε cannot be larger than $1/4$ (they in fact generalized this to r colours obtaining the bounds $1/(er!) \leq \varepsilon \leq 1/(r+1)$). Our proof, however, does not yield a better value of ε.

The proof of Theorem II was divided into two cases: $\alpha(F) \geq 3$ and $\alpha(G) \leq 2$. In order to generalize Theorem II for $r > 3$, one could consider the cases $\alpha(F) \geq r$ and $\alpha(F) \leq r-1$. Each of those cases has its own difficulty and it is not clear how to systematically generalize our arguments in those cases for larger values of r. Notice that our approach to the second case relied on analysing a construction of Gyárfás, proving a better upper bound than the one given by Ryser's conjecture, which for $r = 3$ corresponds to a theorem of Aharoni [1]. However we did not need to use Aharoni's result per se and perhaps in order to generalize our arguments for larger value of r one can also avoid Ryser's conjecture.

Chapter 4

Tiling Edge-coloured Complete Graphs

4.1 Introduction

A conjecture of Lehel states that the vertices of any 2-edge-coloured complete graph can be partitioned into two monochromatic cycles of different colours. Here, single vertices and edges are considered cycles. This conjecture first appeared in [6], where it was also proved for some special types of colourings of K_n. Łuczak, Rödl and Szemerédi [83] proved Lehel's conjecture for sufficiently large n using the regularity method. Allen [2] gave an alternative proof, with a better bound on n. Finally, Bessy and Thomassé [11] proved Lehel's conjecture for all integers $n \geq 1$.

For colourings with more colours, Erdős, Gyárfás and Pyber [42] proved that the vertices of every r-edge-coloured complete graph on n vertices can be partitioned into $O(r^2 \log r)$ monochromatic cycles. They further conjectured that r cycles should be enough. The currently best-known upper bound is due to Gyárfás, Ruszinkó, Sárközy and Szemerédi [55], who showed that $O(r \log r)$ cycles suffice. However, the conjecture was refuted by Pokrovskiy [89], who showed that, for every $r \geq 3$, there exist infinitely many r-edge-coloured complete graphs which cannot be vertex-partitioned into r monochromatic cycles. Nevertheless, Pokrovskiy conjectured that in every r-edge-coloured complete graph one can find r vertex-disjoint monochromatic cycles which cover all but at most c_r vertices for some $c_r \geq 1$ only depending on r (in his counterexample $c_r = 1$ is possible).

In this chapter, we study similar problems in which we are given a family of graphs $\mathcal{F}$ and an edge-coloured complete graph K_n and our goal is to partition $V(K_n)$ into monochromatic copies of graphs from $\mathcal{F}$. All families of graphs $\mathcal{F}$ we consider here are of the form $\mathcal{F} = \{F_1, F_2, \ldots\}$, where F_i is a graph on i vertices for every $i \in \mathbb{N}$. We call such a family a *sequence of graphs*. A collection $\mathcal{H}$ of vertex-disjoint subgraphs of a graph G is an $\mathcal{F}$-*tiling* of G if $\mathcal{H}$ consists of copies of graphs from $\mathcal{F}$ with $V(G) = \bigcup_{H \in \mathcal{H}} V(H)$. If G is edge-coloured, we say that $\mathcal{H}$ is *monochromatic* if every $H \in \mathcal{H}$ is monochromatic. Let $\tau_r(\mathcal{F}, n)$

The work described in this chapter was developed in a joint project with Jan Corsten.

be the minimum $t \in \mathbb{N}$ such that for every r-edge-coloured K_n, there is a monochromatic $\mathcal{F}$-tiling of size at most t. We define the *tiling number* of $\mathcal{F}$ as

$$\tau_r(\mathcal{F}) = \sup_{n\in\mathbb{N}} \tau_r(\mathcal{F}, n).$$

Using this notation, the results of Pokrovskiy [89] and of Gyárfás, Ruszinkó, Sárközy and Szemerédi [55] mentioned above imply that $r+1 \leq \tau_r(\mathcal{F}_{\text{cycles}}) = O(r\log r)$, where $\mathcal{F}_{\text{cycles}}$ is the family of cycles. Note that, in general, it is not clear at all that $\tau_r(\mathcal{F})$ is finite and it is a natural question to ask for which families this is the case.

The study of such tiling problems for general families of graphs was initiated by Grinshpun and Sárközy [53]. The *maximum degree* $\Delta(\mathcal{F})$ of a sequence of graphs $\mathcal{F}$ is given by $\sup_{F\in\mathcal{F}} \Delta(F)$, where $\Delta(F)$ is the maximum degree of F. We denote by $\mathcal{F}_\Delta$ the collection of all sequences of graphs $\mathcal{F}$ with $\Delta(\mathcal{F}) \leq \Delta$. Grinshpun and Sárközy proved that $\tau_2(\mathcal{F}) \leq 2^{O(\Delta\log\Delta)}$ for all $\mathcal{F} \in \mathcal{F}_\Delta$. In particular, $\tau_2(\mathcal{F})$ is finite whenever $\Delta(\mathcal{F})$ is finite. They also proved that $\tau_2(\mathcal{F}) \leq 2^{O(\Delta)}$ for every sequence of bipartite graphs $\mathcal{F}$ of maximum degree at most Δ, and showed that this is best possible up to a constant factor in the exponent (see also Section 4.8 for a more detailed discussion on the lower bound).

Sárközy [95] further proved that $\tau_2(\mathcal{F}_{k\text{-cycles}}) = O(k^2\log k)$, where $\mathcal{F}_{k\text{-cycles}}$ denotes the family of kth power of cycles[1]. For more than two colours less is known. Answering a question of Elekes, Soukup, Soukup and Szentmiklóssy [37], Bustamante, Corsten, Frankl, Pokrovskiy, and Skokan [21] proved that $\tau_r(\mathcal{F}_{k\text{-cycles}})$ is finite for all $r, k \in \mathbb{N}$. Grinshpun and Sárközy [53] conjectured that the same should be true for all families of graphs of bounded degree with an exponential bound.

Conjecture 4.1.1 (Grinshpun-Sárközy [53], 2016)**.** *For every $r, \Delta \in \mathbb{N}$ and $\mathcal{F} \in \mathcal{F}_\Delta$, $\tau_r(\mathcal{F})$ is finite. Moreover, there is some $C_r > 0$ such that $\tau_r(\mathcal{F}) \leq \exp(\Delta^{C_r})$.*

The main theorem in this chapter shows that $\tau_r(\mathcal{F})$ is indeed finite. For a given positive integer k, we denote by $\exp^k$ the kth-composition of the exponential function.

Theorem III. *There is an absolute constant $K > 0$ such that for all integers $r, \Delta \geq 2$, we have*

$$\tau_r(\mathcal{F}) \leq \exp^2\left(r^{Kr\Delta^3}\right),$$

for every sequence $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ of graphs with $|F_i| = i$ and $\Delta(F_i) \leq \Delta$, for each $i \in \mathbb{N}$.

In order to prove Theorem III, we shall prove an absorption lemma (see Lemma 4.5.1) whose proof relies on a *density increment argument*. This is responsible for the double exponential bound in Theorem III.

[1] The k-th power of a graph H is the graph obtained from H by adding an edge between any two vertices at distance at most k

The chapter is organized as follows. In Section 4.2, we present an overview of the proof of Theorem III and the proof of our absorption lemma. In Section 4.3 we collect a few lemmas regarding regular pairs and regular cylinders that we shall use repeatedly in later sections. The proof of our absorption lemma and Theorem III can be found in Section 4.5 and Section 4.6, respectively. Finally, we finish the chapter with some concluding remarks in Section 4.8.

4.2 Proof overview

The proof of Theorem III, similarly to the proof of the two colour result of Grinshpun and Sárközy [53], combines ideas from the absorption method as in the original paper of Erdős, Gyárfás and Pyber [42] with some modern approaches involving the blow-up lemma and the weak regularity lemma of Duke, Lefmann and Rödl [35]. However, in order to extend these ideas to more colours, we need to prove a significantly more complicated *absorption lemma*, requiring new ideas involving a density increment argument.

Our absorption lemma (Lemma 4.5.1) states that if we have $k := \Delta + 2$ disjoint sets of vertices $V_1, \dots, V_k$ with $|V_i| \geq 2|V_1|$ for all $i = 2, \dots, k$ such that every vertex in V_1 belongs to at least $\delta|V_2| \cdots |V_k|$ monochromatic k-cliques *transversal*[2] in $(V_1, \dots, V_k)$, then it is possible to cover the vertices in V_1 with a constant number (depending on δ, r and Δ) of monochromatic vertex disjoint copies of graphs from $\mathcal{F}$. Furthermore, we can choose such a covering using no more than $|V_1|$ vertices in each $V_2, \dots, V_k$.

To deduce Theorem III from the absorption lemma, we need to partition $V(K_n)$ in a similar fashion as in [21]: first we find $k-1$ monochromatic *super-regular cylinders* $Z_1, \dots, Z_{k-1}$ covering a positive proportion of the vertices of K_n (see Section 4.3 for the definition of super-regular cylinders). Then we apply a result of Fox and Sudakov [46] to *greedily* cover with few disjoint monochromatic copies of graphs from $\mathcal{F}$ almost all of the vertices in $V(K_n) \setminus (Z_1 \cup \dots \cup Z_{k-1})$, leaving uncovered a set R of size much smaller than $|Z_{k-1}|$ (see Proposition 4.4.2).

Now we split R into two sets: the set R_1 of vertices belonging to at least $\delta|Z_1| \cdots |Z_{k-1}|$ monochromatic k-cliques transversal in $(R, Z_1, \dots, Z_{k-1})$, and the set $R_2 = R \setminus R_1$. Using our absorption lemma we can cover the vertices in R_1 using no more than $|R_1|$ vertices of each of the cylinders $Z_1, \dots, Z_{k-1}$. For each $i = 1, \dots, k-1$, let Z_i' be the set of vertices in Z_i that has not been used to cover R_1. Since we $|R_1|$ is significantly smaller than $|Z_i|$, it follows that each Z_i' is still a super-regular cylinder. Now, if the set R_2 was empty, then we would be done. Indeed, a consequence of the blow-up lemma (Lemma 4.3.3) guarantees that we can cover each of the cylinders $Z_1', \dots, Z_{k-1}'$ with $k+1$ copies of vertex disjoint monochromatic graphs from $\mathcal{F}$.

[2] A k-clique is transversal in $(V_1, \dots, V_k)$ if it contains one vertex in each one of the sets $V_1, \dots, V_k$.

So let us consider the case where R_2 is non-empty. In this case, we repeat the process above. This time we first find a reasonably large regular cylinder Z_k in R_2, then we greedily cover most of the vertices in $R_2 \setminus Z_k$ and apply the absorption lemma to those vertices that have not yet been covered and belong to many monochromatic k-cliques transversal in R_2 and $k-1$ of the cylinders $Z'_1, \ldots, Z'_{k-1}, Z_k$. The set of leftover vertices, which we denote by R_3, is either empty (and in this case we are done, as above) or is non-empty, in which case we repeat the process to cover R_3. Finally, using a lemma from [21] (see Lemma 4.6.1) and Ramsey's theorem, we can show that this process must stop after $R_r(K_k)$ many iterations, where $R_r(K_k)$ denotes the r-colour Ramsey number of the graph K_k.

In order to prove the absorption lemma, we employ a density increment argument. This is the most difficult part of the proof and the key new idea in this result. First, we partition V_1 into r sets $V_1^{(1)}, \ldots, V_1^{(r)}$ so that for every $j \in [r]$, every $v \in V_i^{(j)}$ is incident to at least $d/r \cdot |V_2| \cdots |V_k|$ monochromatic cliques of colour j which are transversal in $(V_1, \ldots, V_k)$. We will cover each of these sets separately, making sure not to repeat vertices. Let us illustrate how to cover $V_1^{(1)}$.

We start by finding a large k-cylinder $Z = (U_1, \ldots, U_k)$ with $U_1 \subset V_1^{(1)}, U_2 \subset V_2, \ldots U_k \subset V_k$ which is super-regular in colour 1. We shall use Z as an *absorber* at the end of the proof to cover any small set of leftovers. Next, we greedily cover most of $V_1^{(1)} \setminus U_1$ by monochromatic copies of $\mathcal{F}$ until the set of uncovered vertices R has size much smaller then $|U_1|$. To cover the set R, we will find a partition $R = S \cup T_2 \cup \ldots \cup T_k$, where each vertex in S belongs to many monochromatic k-cliques of colour 1 which are transversal in $(S, U_2, \ldots, U_k)$ (allowing S to be absorbed into the cylinder Z at the end of the proof) and each vertex in T_i, for $i \in \{2, \ldots, k\}$, belongs to at least $(\delta + \eta)|V_2| \cdots |V_{i-1}||U_i| \cdots |U_k|$ monochromatic k-cliques transversal in $(T_i, V_2, \ldots, V_i, U_{i+1}, \ldots, U_k)$, for some $\eta \ll \delta$.

To cover the vertices in each T_i, with $i \in \{2, \ldots, k\}$, we repeat the argument with $(V_1, \ldots, V_k)$ replaced by $(T_i, V_2, \ldots, V_i, U_{i+1}, \ldots, U_k)$ and δ replaced by $\delta + \eta$. This is our density increment argument. Since every time we repeat the argument we significantly increase the density of k-cliques, we can bound the number of required repetitions in terms of the initial density of k-cliques.

While covering each of the sets $T_2, \ldots, T_k$, we shall guarantee that the set of vertices $X_i \subseteq U_i$ that we use to cover them has size much smaller than $|U_i|$ for all $i = 2, \ldots, k$. This way, the cylinder $Z' = (U_1 \cup S, U_2 \setminus X_2, \ldots, U_k \setminus X_k)$ will be super-regular in colour 1 and thus we can cover Z' using the blow-up lemma. Repeating this for every colour $j \in [r]$, we get a covering of V_1 with $O_{\delta,r,\Delta}(1)$ many monochromatic disjoint copies of graphs from $\mathcal{F}$.

4.3 Regularity

In this section, we will gather all the notations and results related to the classical regularity technique which we require for the proof. We start by introducing some relevant notations. Let $G = (V_1, V_2, E)$ be a bipartite graph with parts V_1 and V_2. For any $U_i \subseteq V_i$, $i = 1, 2$, the density of the pair (U_1, U_2) in G is given by

$$d(U_1, U_2) = \frac{e(U_1, U_2)}{|U_1||U_2|}.$$

We say that G (or the pair (V_1, V_2)) is *ε-regular* if for all $U_i \subseteq V_i$ with $|U_i| \geq \varepsilon |V_i|$, $i = 1, 2$, we have

$$|d(U_1, U_2) - d(V_1, V_2)| \leq \varepsilon.$$

If additionally we have $d(V_1, V_2) \geq d$ and $\deg(v, V_i) \geq \delta |V_i|$ for all $v \in V_{3-i}$, $i = 1, 2$, then we say that G (or (V_1, V_2)) is *(ε, d, δ)-super-regular*. We often say that G is *(ϵ, d)-super-regular* instead of (ϵ, d, d)-super-regular.

We begin with some simple facts about super-regular pairs. The first one is known as the slicing lemma and roughly says that if we take a large induced subgraph in a dense regular pair we still get a dense regular pair. Its proof is straightforward from the definition of a regular pair.

Lemma 4.3.1 (Slicing lemma)**.** *Let $\beta > \varepsilon > 0$, $d \in [0, 1]$ and let (V_1, V_2) be an $(\epsilon, d, 0)$-super-regular pair. Then any pair (U_1, U_2) with $|U_i| \geq \beta |V_i|$ and $U_i \subseteq V_i$, $i = 1, 2$, is $(\epsilon', d', 0)$-super-regular with $\epsilon' = \max\{\epsilon/\beta, 2\epsilon\}$ and $d' = d - \epsilon$.*

The following lemma essentially says that after removing few vertices from a super-regular pair and adding few new vertices with large degree, we still have a super-regular pair. The reader can find a proof of it in Section 4.7.

Lemma 4.3.2. *Let $0 < \varepsilon < 1/2$ and let $d, \delta \in [0, 1]$ so that $\delta \geq 4\varepsilon$. Let (V_1, V_2) be an (ϵ, d, δ)-super-regular pair in a graph G. Let $X_i \subseteq V_i$ for $i \in \{1, 2\}$, and let Y_1, Y_2 be disjoint subsets of $V(G) \setminus (V_1 \cup V_2)$. Suppose that for each $i \in \{1, 2\}$ we have $|X_i|, |Y_i| \leq \varepsilon^2 |V_i|$ and $\deg(v, V_i) \geq \delta |V_i|$ for every $v \in Y_{3-i}$. Then the pair $((V_1 \setminus X_1) \cup Y_1, (V_2 \setminus X_2) \cup Y_2)$ is $(8\epsilon, d - 8\varepsilon, \delta/2)$-super-regular.*

Let $k \geq 2$ be an integer and let G be a graph. Given disjoint sets of vertices $V_1, \ldots, V_k \subseteq V(G)$, we call $Z = (V_1, \ldots, V_k)$ a *k-cylinder* and often identify it with the induced k-partite subgraph $G[V_1, \ldots, V_k]$. We write $V_i(Z) = V_i$ for every $i \in [k]$. We say that Z is *ε-balanced* if

$$\max_{i \in [k]} |V_i(Z)| \leq (1 + \varepsilon) \min_{i \in [k]} |V_i(Z)|$$

and *balanced* if it is 0-balanced. Furthermore, we say that Z is ϵ-regular if all the $\binom{k}{2}$ pairs (V_i, V_j) are ϵ-regular. If G is an r-edge-coloured graph and $i \in [r]$, we say that Z is ε-regular in colour i if it is ε-regular in G_i, the graph consisting of all edges of G with colour i. Similarly, we define (ε, d)-regular and (ϵ, d, δ)-super-regular cylinders.

As sketched in Section 4.2, we will use super-regular cylinders as absorbers. The following lemma, which Grinshpun and Sárközy [53] deduced from the blow-up lemma [73, 72, 96] and the Hajnal-Szemerédi theorem [56],[3] allows us to do this.

Lemma 4.3.3. *There is a constant K, such that for all $0 \leq \delta \leq d \leq 1/2$, $\Delta \in \mathbb{N}$, $k = \Delta + 2$, $0 < \varepsilon \leq (\delta d^{\Delta})^K$, and $\mathcal{F} \in \mathcal{F}_\Delta$, the following is true for every (ε, d, δ)-super-regular k-cylinder $Z = (V_1, \dots, V_k)$.*

(i) If Z is ε-balanced, then its vertices can be partitioned into at most $\Delta + 3$ copies of graphs from $\mathcal{F}$.

(ii) If $|V_i| \geq |V_1|$ for all $i = 2, \dots, k$, then there is a copy of a graph from $\mathcal{F}$ covering V_1 and at most $|V_1|$ vertices of each of $V_2, \dots, V_k$.

It is important in the proof of Theorem III that we can find super-regular k-cylinders which cover linearly many vertices. The existence of such a pair follows readily from the regularity lemma. Conlon and Fox [27, Lemma 5.3] used the weak regularity lemma of Duke, Lefmann, and Rödl [35] to obtain better constants. We shall use the following coloured version of their result, the proof of which is very similar and can be found in Section 4.7. See also [53, Lemma 2] for a 2-coloured version which follows readily from the non-coloured version.

Lemma 4.3.4. *Let $k, r \geq 2$, $0 < \varepsilon < 1/(rk)$ and $\gamma = \varepsilon^{r^{8rk}\varepsilon^{-5}}$. Then every r-edge-coloured complete graph on $n \geq 1/\gamma$ vertices contains, in one of the colours, a balanced $(\varepsilon, 1/2r)$-super-regular k-cylinder $Z = (U_1, \dots, U_k)$ with parts of size at least γn.*

The following lemma further guarantees that this remains possible as long as the host-graph has many k-cliques. It is also a straightforward consequence of the weak regularity lemma of Duke, Lefmann, and Rödl and we provide a proof in Section 4.7.

Lemma 4.3.5. *Let $k \geq 2$, and let $0 < \varepsilon < 1/2$ and $2k\varepsilon \leq d \leq 1$. Let $\gamma = \varepsilon^{k^2\varepsilon^{-12}}$. Suppose that G is a k-partite graph with parts $V_1, \dots, V_k$ with at least $d|V_1| \cdots |V_k|$ cliques of size k. Then there is some $\gamma' \in [\gamma, \varepsilon]$ and an $(\varepsilon, d/2)$-super-regular k-cylinder $Z = (U_1, \dots, U_k)$ in G with $U_i \subset V_i$ and $|U_i| = \lfloor \gamma' |V_i| \rfloor$ for every $i \in [k]$.*

[3]The second part of the theorem is not explicitly stated in [53] but follows readily from the blow-up lemma and the Hajnal-Szemerédi theorem.

4.4 Greedily covering most vertices

In the proof, we will use the following theorem of Fox and Sudakov [46] about r-colour Ramsey numbers of bounded-degree graphs.

Theorem 4.4.1 ([46, Theorem 4.3]). *Let $k, \Delta, r, n \in \mathbb{N}$ with $r \geq 2$ and let $H_1, \ldots, H_r$ be k-partite graphs with n vertices and maximum degree at most Δ. Then*

$$R(H_1, \ldots, H_r) \leq r^{2rk\Delta} n.$$

Recall that $\mathcal{F}_\Delta$ denotes the collection of all sequences of graphs $\mathcal{F}$ with $\Delta(F) \leq \Delta$, for every $F \in \mathcal{F}$, and let $\mathcal{F}_{\Delta,k}$ be the collection of sequences $\mathcal{F} \in \mathcal{F}_\Delta$ such that F is k-partite, for every $F \in \mathcal{F}$. Note that $\mathcal{F}_\Delta = \mathcal{F}_{\Delta,\Delta+1}$. The following consequence of the previous theorem states that, for each $\mathcal{F} \in \mathcal{F}_{k,\Delta}$, we can cover almost all vertices of K_n with monochromatic copies of graphs from $\mathcal{F}$. The proof basically follows by *greedily* taking a large monochromatic copy of a graph in $\mathcal{F}$ covering vertices that has not been covered yet.

Proposition 4.4.2. *Let $\Delta, k, r \in \mathbb{N}$, let $\gamma > 0$ and let $C = 4r^{2rk\Delta} \log(1/\gamma)$. Then, for every $\mathcal{F} \in \mathcal{F}_{\Delta,k}$ and every r-edge-coloured K_n with $n \geq r^{-2rk\Delta}$, it is possible to cover all but γn vertices of K_n with at most C vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$.*

Proof. Let $\mathcal{F} = \{F_1, F_2, \ldots\} \in \mathcal{F}_{\Delta,k}$, $t = r^{-2rk\Delta}$, $C = (4/t)\log(1/\gamma)$ and $n \geq r^{-2rk\Delta}$. Consider $n_1 = \lfloor tn \rfloor \geq tn/2$. By Theorem 4.4.1, since $R_r(F_{n_1}) \leq t^{-1} n_1 \leq n$, there is a monochromatic copy of F_{n_1} in K_n. Let H_1 be such copy and let $V_1 = V \setminus V(H_1)$. Note that $|V_1| = n - n_1 \leq (1 - t/2)n$.

Suppose that we have inductively found vertex-disjoint monochromatic graphs $H_1, \ldots, H_i \subseteq K_n$ that are copies of graphs in $\mathcal{F}$ and such that $V_i := V(K_n) \setminus (V(H_1) \cup \cdots \cup V(H_i))$ has at most $(1 - t/2)^i n$ vertices. If $|V_i| \leq 2/t$, then we cover the vertices in V_i with single vertices and stop the process. Therefore, suppose that $|V_i| \geq 2/t$. Then let $n_{i+1} = \lfloor t|V_i| \rfloor \geq t|V_i|/2$. Again by Theorem 4.4.1, since $R_r(F_{n_{i+1}}) \leq t^{-1} n_{i+1} \leq |V_i|$, there is a monochromatic copy of $F_{n_{i+1}}$ contained in V_i. Let H_{i+1} be such a copy. Thus the set $V_{i+1} := V(K_n) \setminus (V(H_1) \cup \cdots \cup V(H_{i+1}))$ has size

$$|V_{i+1}| = |V_i| - n_{i+1} \leq (1 - t/2)|V_i| \leq (1 - t/2)^{i+1} n.$$

Now, after $C/2$ steps, we have covered all but at most

$$(1 - t/2)^{C/2} n \leq e^{-(t/4)C} n \leq \gamma n$$

vertices of K_n using at most $C/2 + 2/t \leq C$ vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$. □

In particular, by choosing $\gamma = 1/n$, we get the following corollary.

Corollary 4.4.3. *Let $\Delta, k, r \in \mathbb{N}$ and let $C = 4r^{2rk\Delta}\log n$. Then, for every $\mathcal{F} \in \mathcal{F}_{\Delta,k}$ and every r-edge-coloured K_n, there is a collection of at most C monochromatic vertex-disjoint copies of graphs from $\mathcal{F}$ whose vertex-sets partition $V(G)$.*

4.5 The Absorption Lemma

Given a graph G and $U \subseteq V$, recall that we denote by $G[U]$ the subgraph of G induced by U. Given disjoint sets $V_1, \dots, V_k \subseteq V(G)$, with $k \geq 2$, we denote by $G[V_1, \dots, V_k]$ the subgraph of G with vertex set $V_1 \cup \dots \cup V_k$ containing only edges that are between two of the sets $V_1, \dots, V_k$. Furthermore, for each $v \in V_1$, let

$$\deg_G(v, V_2, \dots, V_k) = |\{(v_2, \dots, v_k) \in V_2 \times \dots \times V_k : \{v, v_2, \dots, v_k\} \text{ is a } k\text{-clique in } G\}|$$

and

$$\mathrm{d}_G(v, V_2, \dots, V_k) := \frac{\deg_G(v, V_2, \dots, V_k)}{|V_2| \cdots |V_k|}.$$

If additionally, we have an edge colouring $\chi \colon E(G) \to [r]$ of $E(G)$, then we denote by $\deg_{G,i}(v, V_2, \dots, V_k) = \deg_{G_i}(v, V_2, \dots, V_k)$, where G_i is the graph with vertex set $V(G)$ consisting of the edges of G with colour i. We define $\mathrm{d}_{G,i}(v, V_2, \dots, V_k)$ similarly and denote $\mathrm{d}_{G,I}(v, V_2, \dots, V_k) := \sum_{i \in I} \mathrm{d}_{G,i}(v, V_2, \dots, V_k)$, for each $I \subseteq [r]$. If the graph G is clear from context, we may drop the G in the subscript.

Given a set V, we denote by $K(V)$ the complete graph with vertex set V. Given disjoint sets $V_1, \dots, V_k$, we denote by $K(V_1, \dots, V_k)$ the complete k-partite graph with parts $V_1, \dots, V_k$. Let $G = K(V_1) \cup K(V_1, \dots, V_k)$ and let $\mathcal{H}$ be a collection of subgraphs of G. We denote by $\cup\mathcal{H}$ the graph with edge set $\bigcup_{H \in \mathcal{H}} E(H)$ and vertex set $V(\mathcal{H}) := \bigcup_{H \in \mathcal{H}} V(H)$. We say that $\mathcal{H}$ *canonically covers* V_1 if $V_1 \subseteq V(\mathcal{H})$ and

$$|V(\mathcal{H}) \cap V_i| \leq |V(\mathcal{H}) \cap V_1|$$

for all $i \in [2, k]$.[4] The following lemma is the key ingredient of the proof of Theorem III.

Lemma 4.5.1 (Absorption Lemma)**.** *There is some absolute constant $K > 0$, such that the following is true for all $d > 0$, all integers $\Delta, r \geq 2$ and for every $\mathcal{F} \in \mathcal{F}_\Delta$. Let $k = \Delta + 2$ and let*

$$C = \exp^2\left(\left(\frac{r}{d}\right)^{K\Delta}\right).$$

Consider k disjoint sets $V_1, \dots, V_k$ with $|V_i| \geq 4|V_1|$, for all $i \in [2, k]$, and let $G = K(V_1) \cup K(V_1, \dots, V_k)$. Suppose that $\chi : E(G) \to [r]$ is a colouring in which for every $v \in V_1$

[4] Here, we denote by $[i, j]$ the set of integers z with $i \leq z \leq j$.

we have $d_{[r]}(v, V_2, \ldots, V_k) \geq d$*. Then, there is a collection of at most* C *vertex-disjoint monochromatic copies of graphs from* $\mathcal{F}$ *in* G *which canonically covers* V_1*.*

The edges of G inside V_1 will only be used to find copies from $\mathcal{F}$ which lie entirely in V_1 in order to greedily cover most vertices of V_1. The difficult part is finding monochromatic copies in $K(V_1, \ldots, V_k)$ covering the remaining vertices. To do so, we will reduce the problem to only one colour within $K(V_1, \ldots, V_k)$ and then deduce Lemma 4.5.1 from the following lemma.

Lemma 4.5.2. *There is some absolute constant* $K > 0$*, such that the following is true for all* $d > 0$*, all integers* $\Delta, r \geq 2$ *and for every* $\mathcal{F} \in \mathcal{F}_\Delta$*. Let* $k = \Delta + 2$ *and let*

$$C = \exp^2\left(\left(\frac{r}{d}\right)^{K\Delta}\right).$$

Consider k *disjoint sets* $V_1, \ldots, V_k$ *with* $|V_i| \geq 2|V_1|$*, for all* $i \in [2, k]$ *and let* $G = K(V_1) \cup K(V_1, \ldots, V_k)$*. Suppose that* $\chi : E(G) \to [r]$ *is a colouring in which for every* $v \in V_1$ *we have* $d_1(v, V_2, \ldots, V_k) \geq d$*. Then, there is a collection of at most* C *vertex-disjoint monochromatic copies of graphs from* $\mathcal{F}$ *in* G *which canonically covers* V_1*.*

Lemma 4.5.1 follows routinely from Lemma 4.5.2.

Proof of Lemma 4.5.1. Let K' be the absolute constant from Lemma 4.5.2 and let $d' = d/(2r)$, $\gamma = d'/(kr)$, and $C' = \exp^2\left((r/d')^{K'\Delta}\right)$. Partition $V_1 = U_1 \cup \ldots \cup U_r$ such that for each $j \in [r]$ we have $d_j(v, V_2, \ldots, V_k) \geq 2d'$, for all $v \in U_j$. We will inductively cover U_j, for each $j \in [k]$.

Let us first consider the base case, i.e., $j = 1$. From Proposition 4.4.2, there is a collection $\mathcal{H}'$ of at most[5] C' disjoint monochromatic copies of graphs from $\mathcal{F}$ covering all but $\gamma|U_1| \leq \gamma|V_1|$ vertices of $G[U_1]$. Let $V_1' = U_1 \setminus V(\mathcal{H}')$. By applying Lemma 4.5.2 to $G' := G[V_1' \cup V_2 \cup \cdots \cup V_k]$ (with d'), there is a collection $\mathcal{H}''$ of at most C' disjoint monochromatic copies of graphs from $\mathcal{F}$ in G' which canonically covers V_1'. Let $\mathcal{H}_1 = \mathcal{H}' \cup \mathcal{H}''$. Note that $\mathcal{H}_1$ canonically covers U_1 and covers at most $\gamma|V_1|$ vertices of V_i, for each $i \in [2, k]$.

Now consider $j \geq 2$ and suppose that we have found a collection $\mathcal{H}_{j-1}$ of at most $2(j-1)C'$ disjoint monochromatic copies of graphs from $\mathcal{F}$ in G that canonically covers $U_1 \cup \cdots \cup U_{j-1}$ and covers at most $(j-1)\gamma|V_1|$ vertices of V_i, for each $i \in [2, k]$. From Proposition 4.4.2, there is a collection $\mathcal{H}'$ of at most C' disjoint monochromatic copies of graphs from $\mathcal{F}$ covering all but $\gamma|U_j| \leq \gamma|V_1|$ vertices of $G[U_j]$. Let $V_1' = U_j \setminus V(\mathcal{H}')$ and let $V_i' := V_i \setminus V(\mathcal{H}_{j-1})$, for each $i \in [2, k]$. Note that

$$|V_i'| \geq |V_i| - (j-1)\gamma|V_1| \geq 4|V_1| - r\gamma|V_i| \geq 2|V_1| \geq 2|V_1'|.$$

[5]Note that the constant from Proposition 4.4.2 is smaller than C'.

Also, for each $v \in V_1'$, we have

$$\deg_j(v, V_2', \dots, V_k') \geq \deg_j(v, V_2, \dots, V_k) - k(j-1)\gamma |V_2| \cdots |V_k|.$$

Consequently,

$$\mathrm{d}_j(v, V_2', \dots, V_k') \geq \mathrm{d}_j(v, V_2, \dots, V_k) - kr\gamma \geq 2d' - d' \geq d'.$$

Therefore, we can apply Lemma 4.5.2 to $G' := G[V_1' \cup \cdots \cup V_k']$ and get a collection $\mathcal{H}''$ of at most C' disjoint monochromatic copies of graphs from $\mathcal{F}$ in G that canonically covers V_1'. In particular, $\mathcal{H}''$ covers at most $|V_1'| \leq \gamma |V_1|$ vertices of V_i, for each $i \in [2, k]$. Let $\mathcal{H}_j = \mathcal{H}_{j-1} \cup \mathcal{H}' \cup \mathcal{H}''$. Then $\mathcal{H}_j$ is a collection of at most $2jC'$ disjoint monochromatic copies of graphs from $\mathcal{F}$ in G that canonically covers $U_1 \cup \cdots \cup U_j$ and covers at most $j\gamma|V_1|$ vertices of V_i, for each $i \in [2, k]$.

In the end, we have found a collection $\mathcal{H}_r$ of disjoint monochromatic copies of graphs from $\mathcal{F}$ that canonically covers V_1. Furthermore, $\mathcal{H}_r$ has at most $2rC' \leq \exp^2\left((r/d)^{4K'\Delta}\right)$ graphs, finishing the proof. □

The proof of Lemma 4.5.2 is quite long and technical (see Section 4.2 for a sketch), and we will therefore break it up into smaller claims. We use ▫ to denote the end of the proof of a claim and □ to denote the end of the main proof.

Proof of Lemma 4.5.2. Let Δ and r be given positive integers, $k = \Delta + 2$ and $\mathcal{F} \in \mathcal{F}_\Delta$. For each $d > 0$, let $C(d)$ be the smallest positive integer C such that the following holds:

> ($\star$) Let $V_1, \dots, V_k$ be disjoint sets with $|V_i| \geq 2|V_1|$ for all $i \in [2, k]$, let $H \subset K(V_1, \dots, V_k)$ be a graph with $\mathrm{d}_H(v, V_2, \dots, V_k) \geq d$ for every $v \in V_1$ and $G = K(V_1) \cup H$. Let $\chi : E(G) \to [r]$ be a colouring such that every edge in $E(H)$ receives colour 1. Then, there is a collection $\mathcal{H}$ of at most C vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$ contained in G that canonically covers V_1.

Note that $C(d)$ is a decreasing function in d, and that $C(d) = 0$ for every $d > 1$. Our goal is to show that $C(d)$ is finite for every $d > 0$. We will do this by establishing a recursive upper bound (see Equation (4.1)).

Let us first define all relevant constants used in the proof. Let K' be the universal constant given by Lemma 4.3.3 and fix some $0 < d \leq 1$. Define

$$\varepsilon = \left(\frac{d}{100}\right)^{2K'\Delta}, \quad \gamma = \tfrac{1}{r} \cdot \varepsilon^{k^2 \varepsilon^{-12}} \quad \text{and} \quad \eta = \frac{d\gamma^k}{2}.$$

It might be of benefit for the reader to have in mind that those constants obey the following hierarchy:

$$1 \geq d \gg \varepsilon \gg \gamma \gg \eta > 0.$$

Furthermore, define

$$P(d) := 4r^{4rk^2} \log(2/\eta^2) + 1.$$

We will prove that for every $d' \geq d$ we have

$$C(d') \leq P(d) + kC\left(d' + \eta\right). \tag{4.1}$$

Since $C(d') = 0$ if $d' > 1$, it follows by iterating that $C(d) \leq (2k)^{2/\eta} P(d)$. Furthermore, we have

$$2/\eta \leq \gamma^{-2k} \leq \varepsilon^{-2rk^3\varepsilon^{-12}} \leq \exp\left(r\varepsilon^{-20}\right) \leq \exp\left((r/d)^{400K'\Delta}\right).$$

It follows that

$$C(d) \leq \exp^2\left((r/d)^{500K'\Delta}\right) P(d) \leq \exp^2\left((r/d)^{1000K'\Delta}\right)$$

concluding the proof of Lemma 4.5.2.

It remains to prove Equation (4.1). Let $d' \geq d$ be fixed now and let $V_1, \ldots, V_k$, G and $\chi : E(G) \to [r]$ be as in ($\star$) (with d' playing the role of d). By assumption, there are at least $d|V_1||V_2| \cdots |V_k|$ cliques of size k in $G[V_1, V_2, \ldots, V_k]$ each of which is monochromatic in colour 1. Since $\gamma = \varepsilon^{k^2\varepsilon^{-12}}$ and $d \geq 2k\varepsilon$, we can apply Lemma 4.3.5 to get some $\gamma' \geq \gamma$ and a k-cylinder $Z = (U_1, \ldots, U_k)$ which is $(\varepsilon, d/2)$-super-regular with $U_i \subset V_i$ and $|U_i| = \lfloor \gamma'|V_i| \rfloor$ for every $i \in [k]$. Without loss of generality we may assume that $\gamma|V_i|$ is an integer for every $i \in [k]$ and that we have $\gamma' = \gamma$. By Proposition 4.4.2, there is a collection $\mathcal{H}_R$ of at most $4r^{4rk^2} \log(2/\eta^2)$ vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$ contained in $K(V_1 \setminus U_1)$ covering all vertices in $V_1 \setminus U_1$ except for a set R with $|R| \leq \eta^2|V_1|$. We remark here that

$$|R| \leq \eta/(4k) \cdot |U_1| \leq \varepsilon^2 |U_1|. \tag{4.2}$$

It remains now to cover the vertices in R. For each $i \in [k]$, let

$$d_i = \frac{1 - \gamma^i}{1 - \gamma^k} \cdot d' \tag{4.3}$$

and note that $(1-\gamma)d' \leq d_1 \leq \cdots \leq d_k = d'$. For $i \in [2,k]$, let $\tilde{V}_i = V_i \setminus U_i$ and define

$$S_i = \{v \in R : \mathrm{d}(v, V_2, \ldots, V_{i-1}, V_i, U_{i+1}, \ldots, U_k) \geq d_i\},$$
$$T_i = \{v \in R : \mathrm{d}(v, V_2, \ldots, V_{i-1}, \tilde{V}_i, U_{i+1}, \ldots, U_k) > d' + 2\eta\}.$$

We will prove Equation (4.1) using a series of claims, which we shall prove at the end.

Claim 4.5.3. *We have $R = S_1 \cup T_2 \cup \ldots \cup T_k$.*

Without loss of generality, we may assume that $S_1, T_2, \ldots, T_k$ are pairwise disjoint (more formally, we can define $T_i' := T_i \setminus (S_1 \cup T_2 \cup \ldots \cup T_{i-1})$ for all $i \in [2,k]$ and continue the proof with these sets). Our goal now is to cover each of the sets $S_1, T_2, \ldots, T_k$ one by one using the following claims.

Claim 4.5.4. *For every $i \in [2,k]$ and every set $A \subseteq V(G) \setminus T_i$ with $|A \cap V_s| \leq |R|$ for all $s \in [2,k]$, there is a collection $\mathcal{H}_i$ of at most $C(d'+\eta)$ monochromatic disjoint copies of graphs from $\mathcal{F}$ in G, such that*

(i) $V(\mathcal{H}_i) \cap V_1 = T_i$,

(ii) $V(\mathcal{H}_i) \cap A = \varnothing$, and

(iii) $|V(\mathcal{H}_i) \cap V_j| \leq |T_i|$ for all $j \in [2,k]$.

Claim 4.5.5. *For every set $A \subseteq V(G) \setminus (S_1 \cup U_1)$ with $|A \cap V_s| \leq |R|$ for all $s \in [2,k]$, there is a monochromatic copy H_1 of a graph from $\mathcal{F}$ in G, such that*

(i) $V(H_1) \cap V_1 = S_1 \cup U_1$,

(ii) $V(H_1) \cap A = \varnothing$ and

(iii) $|V(H_1) \cap V_j| \leq |S_1 \cup U_1|$ for all $j \in [2,k]$.

With these claims at hand, we can now prove Equation (4.1). First, we apply Claim 4.5.4 repeatedly to get collections $\mathcal{H}_2, \ldots, \mathcal{H}_k$ of at most $C(d'+\eta)$ disjoint monochromatic copies of graphs from $\mathcal{F}$ that canonically covers $T_2, \ldots, T_k$, respectively, as follows. Let $i \in \{2, \ldots, k\}$ and suppose we have constructed $\mathcal{H}_2, \ldots, \mathcal{H}_{i-1}$. Let $A_i := V(\mathcal{H}_2) \cup \ldots \cup V(\mathcal{H}_{i-1})$ and note that $|A_i \cap V_s| \leq |T_2| + \cdots + |T_{i-1}| \leq |R|$ for all $s \in [2,k]$. Apply now Claim 4.5.4 for i and $A = A_i$ to get the desired collection $\mathcal{H}_i$.

Next, we apply Claim 4.5.5 with $A = V(\mathcal{H}_2) \cup \ldots \cup V(\mathcal{H}_k)$ to get a copy H_1 of a graph from $\mathcal{F}$ with the desired properties. Note that, similarly as above, we have $|A \cap V_s| \leq |R|$ for all $s \in [2,k]$. By construction $V(H_1), V(\mathcal{H}_2), \ldots, V(\mathcal{H}_k)$ and $V(\mathcal{H}_R)$ are disjoint and cover V_1. Moreover, for every $s \in [2,k]$, we have

$$|(V(H_1) \cup \ldots \cup V(\mathcal{H}_k) \cup V(\mathcal{H}_R)) \cap V_s| \leq |S_1 \cup U_1| + |T_1| + |T_2| + \cdots + |T_k|$$

$$\leq |U_1 \cup R| \leq |V_1|.$$

Hence, $\{H_1\} \cup \ldots \cup \mathcal{H}_k \cup \mathcal{H}_R$ canonically covers V_1. Finally, we have $|\{H_1\} \cup \ldots \cup \mathcal{H}_k \cup \mathcal{H}_R| \leq P(d) + kC\big(d' + \eta\big)$, proving Equation (4.1). It remains now to prove Claims 4.5.3 to 4.5.5.

Proof of Claim 4.5.3. Since $S_k = R$, it suffices to show $S_i \subseteq S_{i-1} \cup T_i$ for each $i \in [2, k]$. Let $i \in [2, k]$ and let $v \in S_i \setminus S_{i-1}$. We have

$$\begin{aligned}\deg(v, V_2, \ldots, V_{i-1}, \tilde{V}_i, U_{i+1}, \ldots, U_k) = {} & \deg(v, V_2, \ldots, V_{i-1}, V_i, U_{i+1}, \ldots, U_k) \\ & - \deg(v, V_2, \ldots, V_{i-1}, U_i, U_{i+1}, \ldots, U_k).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{d}(v, V_2, \ldots, V_{i-1}, \tilde{V}_i, U_{i+1}, \ldots, U_k) &= \mathrm{d}(v, V_2, \ldots, V_{i-1}, V_i, U_{i+1}, \ldots, U_k)\frac{|V_i|}{|\tilde{V}_i|} \\ &\quad - \mathrm{d}(v, V_2, \ldots, V_{i-1}, U_i, U_{i+1}, \ldots, U_k)\frac{|U_i|}{|\tilde{V}_i|} \\ &> d_i\frac{|V_i|}{|\tilde{V}_i|} - d_{i-1}\frac{|U_i|}{|\tilde{V}_i|} \\ &= \frac{d_i - \gamma d_{i-1}}{1-\gamma} \\ &= \frac{(1-\gamma^i)d' - \gamma(1-\gamma^{i-1})d'}{(1-\gamma)(1-\gamma^k)} \\ &= \frac{d'}{1-\gamma^k} \geq d' + 2\eta,\end{aligned}$$

where we use Equation (4.3) and the definition of η to obtain the last identities. Thus $v \in T_i$ and hence $S_i \subseteq S_{i-1} \cup T_i$. □

Proof of Claim 4.5.4. Let $V_s' := V_s \setminus A$ for all $s \in [2, i-1]$, $\tilde{V}_i' := \tilde{V}_i \setminus A$ and $U_s' := U_s \setminus A$ for all $s \in [i+1, k]$. Then, by Equation (4.2), we have

$$\begin{aligned}|V_s'| &\geq |V_s| - |R| \geq \big(1 - \tfrac{\eta}{4k}\big)|V_s| \geq \frac{|V_s|}{2}, \text{ for } s = 2, \ldots, i-1, \\ |\tilde{V}_i'| &\geq |\tilde{V}_i| - |R| \geq \big(1 - \tfrac{\eta}{4k}\big)|\tilde{V}_i| \geq \frac{|\tilde{V}_i|}{2}, \text{ and} \\ |U_s'| &\geq |U_s| - |R| \geq \big(1 - \tfrac{\eta}{4k}\big)|U_s| \geq \frac{|U_j|}{2}, \text{ for } s = i+1, \ldots, k.\end{aligned}$$

In particular, it follows that

$$\begin{aligned}|V_s \setminus V_s'| &\leq |R| \leq \frac{\eta}{4k}|V_s| \leq \frac{\eta}{2k}|V_s'|, \text{ for } s = 2, \ldots, i-1, \\ |V_i \setminus V_i'| &\leq |R| \leq \frac{\eta}{4k}|V_i| \leq \frac{\eta}{2k}|V_i'|, \text{ and}\end{aligned}$$

$$|U_s \setminus U_s'| \le |R| \le \frac{\eta}{4k}|U_s| \le \frac{\eta}{2k}|U_s'|, \text{ for } s = i+1, \dots, k.$$

Therefore, for every $v \in T_i$, we have

$$\begin{aligned}
&\mathrm{d}(v, V_2', \dots, V_{i-1}', \tilde{V}_i', U_{i+1}', \dots, U_k') \\
&\ge d' + 2\eta - \sum_{s=2}^{i-1} \frac{|V_s \setminus V_s'|}{|V_s'|} - \frac{|\tilde{V}_i \setminus \tilde{V}_i'|}{|\tilde{V}_i'|} - \sum_{s=i+1}^{k} \frac{|U_s \setminus U_s'|}{|U_s'|} \\
&\ge d' + 2\eta - (k-1)\frac{\eta}{2k} \ge d' + \eta.
\end{aligned}$$

Hence, by definition of $C(d'+\eta)$ (see ($\star$)), there exists a collection $\mathcal{H}_i$ of at most $C(d'+\eta)$ monochromatic copies of graphs from $\mathcal{F}$ that canonically covers T_i in the graph

$$K(T_i) \cup K(T_i, V_2', \dots, V_{i-1}', \tilde{V}_i', U_{i+1}', \dots, U_k').$$

By construction, $\mathcal{H}_i$ satisfies the requirements of the claim (note that *(iii)* holds since $\mathcal{H}_i$ is a canonical covering). □

Proof of Claim 4.5.5. Let $Y_1 = S_1$ and, for each $i \in [2,k]$, let $X_i = U_i \cap A$. Observe that $|Y_1| \le |R| \le \varepsilon^2|U_1|$ and $|X_i| \le |R| \le \varepsilon^2|U_i|$ for all $i \in [2,k]$. Let $U_1' = U_1 \cup Y_1$ and, for each $i \in [2,k]$, let $U_i' := U_i \setminus X_i$. We now consider the cylinder $Z' := (U_1', \dots, U_k')$. By definition of S_1, we have $\mathrm{d}(v, U_2, \dots, U_k) \ge d_1 \ge d/2$ and in particular $\deg(v, U_i) \ge d/2 \cdot |U_i|$ for all $v \in Y_1$ and $i \in [2,k]$.

Hence, by Lemma 4.3.2, Z' is $(8\varepsilon, d/4)$-super-regular. Furthermore, we have $|U_1'| \le |U_i'|$ for all $i \in [k]$. Thus, by Lemma 4.3.3, there is a monochromatic copy H_1 of a graph from $\mathcal{F}$ in Z that covers $U_1' = U_1 \cup S_1$ and at most $|U_1'|$ vertices from each of $U_2', \dots, U_k'$. By construction, this copy satisfies the requirements of the claim. □

This finishes the proof of Lemma 4.5.2. □

4.6 Proof of Theorem III

In this section, we will finish the proof of Theorem III. We will make use of the following lemma from [21] and follow the same proof technique. Since our proof of this lemma is short, we include it here for completeness. Given a k-uniform hypergraph $\mathcal{H}$, a vertex $v \in V(\mathcal{H})$ and sets $B_2, \dots, B_k \subseteq V(\mathcal{H})$, we define

$$\deg_{\mathcal{H}}(v, B_2, \dots, B_k) := |\{(v_2, \dots, v_k) \in B_2 \times \dots \times B_k : \{v, v_2, \dots, v_k\} \in E(\mathcal{H})\}|.$$

Lemma 4.6.1. *Let k and N be positive integers and let $\mathcal{H}$ be a k-uniform hypergraph. Suppose that $B_1, \ldots, B_N \subseteq V(\mathcal{H})$ are non-empty disjoint sets such that for every $1 \leq i_1 < \cdots < i_k \leq N$ we have*

$$\deg_{\mathcal{H}}(v, B_{i_2}, \ldots, B_{i_k}) < \binom{N}{k}^{-1} |B_{i_2}| \cdots |B_{i_k}|$$

for all $v \in B_{i_1}$. Then, there exists an independent set $\{v_1, \ldots, v_N\}$ with $v_i \in B_i$, for each $i \in [N]$.

Proof. For each $i \in [N]$, let v_i be chosen uniformly at random from B_i. Let $I = \{v_1, \ldots, v_N\}$. Then we have

$$\begin{aligned}
\mathbb{P}\left[I \text{ is not an independent set}\right] &\leq \sum_{1 \leq i_1 < \cdots < i_k \leq N} \mathbb{P}\left[\{v_{i_1}, \ldots, v_{i_k}\} \in E(\mathcal{H})\right] \\
&= \sum_{1 \leq i_1 < \cdots < i_k \leq N} \frac{1}{|B_{i_1}|} \sum_{v \in B_1} \mathbb{P}\left[\{v_{i_1}, \ldots, v_{i_k}\} \in E(\mathcal{H}) \mid v_{i_1} = v\right] \\
&= \sum_{1 \leq i_1 < \cdots < i_k \leq N} \frac{1}{|B_{i_1}|} \sum_{v \in B_1} \frac{\deg_{\mathcal{H}}(v, B_{i_2}, \ldots, B_{i_k})}{|B_{i_2}| \cdots |B_{i_k}|} \\
&< \sum_{1 \leq i_1 < \cdots < i_k \leq N} \binom{N}{k}^{-1} = 1.
\end{aligned}$$

Therefore, there exists an independent set $\{v_1, \ldots, v_N\}$ with $v_i \in B_i$, for each $i \in [N]$. $\square$

We are now able to prove Theorem III. The main idea is to find reasonably large cylinders that are super-regular for one of the colours, greedily cover most of the remaining vertices using Proposition 4.4.2 and then apply the Absorption Lemma (Lemma 4.5.1) to the set of remaining vertices that share many monochromatic cliques with the cylinders. We then iterate this process until no vertices remain. Using Lemma 4.6.1, we will show that a bounded number of iterations suffices.

Proof of Theorem III. Fix $r, \Delta \geq 2$, $\mathcal{F} \in \mathcal{F}_\Delta$. Let G be an r-edge-coloured complete graph on n vertices. Let

$$k = \Delta + 2, \quad N = r^{rk}, \quad \delta = \binom{N}{k}^{-1} \quad \text{and} \quad d = \frac{1}{2r}.$$

In order to use Lemma 4.3.3 and Lemma 4.3.4, respectively, consider the constants

$$\varepsilon = \left(\delta d^{\Delta}\right)^{2K'} \quad \text{and} \quad \gamma = \varepsilon^{r^{8rk} \varepsilon^{-5}},$$

where K' is the universal constant given by Lemma 4.3.3. Consider also the constants

$$\alpha = \varepsilon^2 \quad \text{and} \quad C_1 = 4r^{2rk\Delta} \log\left(\frac{4}{\alpha\gamma}\right)$$

in order to use Proposition 4.4.2. Finally, let

$$C_2 = \exp^2((2r/\delta)^{\tilde{K}\Delta}) \le \exp^2\left(r^{16\tilde{K}r\Delta^3}\right),$$

where $\tilde{K}$ is the universal constant from Lemma 4.5.1, and let $K = 20\tilde{K}$.

We will build a framework consisting of many k-cylinders working as absorbers and small sets that can be absorbed by them. More precisely, our goal is to define sets with the following properties (Figure 4.1 should help the reader to understand the structure of those sets as we define them):

> **Framework.** There are sets $Z_1, \dots, Z_N$, $S_{k-1}, \dots, S_N$, $R_k, \dots, R_{N+1}$, $R'_k, \dots, R'_{N+1}$ with the following properties.
>
> (F.1) $V(G) = \bigcup_{i=1}^{N} Z_i \cup \bigcup_{i=k-1}^{N} S_i \cup \bigcup_{i=k}^{N+1} R'_i$ is a partition.
>
> (F.2) $Z_1, \dots, Z_N$[6] are k-cylinders which are (ε, d)-super-regular in one of the colours (or empty).
>
> (F.3) $S_{k-1}, \dots, S_N$ are sets of vertices which we will cover greedily by monochromatic copies of graphs from $\mathcal{F}$.
>
> (F.4) For each $i \in [k, N+1]$, R'_i can be partitioned into sets $R'_{i,I}$ for all $I \in \binom{[i-1]}{k-1}$, such that, for each $I = \{i_1, \dots, i_{k-1}\} \subseteq [i]$, we have $\mathrm{d}_{[r]}(u, Z_{i_1}, \dots, Z_{i_{k-1}}) \ge \delta$ for all $u \in R'_{i+1,I}$.
>
> (F.5) For each $k \le i < j \le N+1$, we have $S_j \cup Z_j \cup R'_j \subseteq R_i$ and $|R_i| \le \alpha |Z_{i-1}|$.

So let us construct those sets from the framework. First, if $n < 1/4\gamma$, then Corollary 4.4.3 gives a covering with at most C_2 monochromatic vertex-disjoint copies of graphs from $\mathcal{F}$. Therefore we may assume that $n \ge 1/4\gamma$. Hence, by applying Lemma 4.3.4 multiple times, we find $k-1$ vertex-disjoint k-cylinders $Z_1, \dots, Z_{k-1}$ such that each of them is (ε, d)-super-regular in some colour (not necessarily the same) and $|Z_1| \ge \dots \ge |Z_{k-1}| \ge \gamma n/2$. Let $V_{k-1} = V(G) \setminus (Z_1 \cup \dots \cup Z_{k-1})$. By Proposition 4.4.2, there is a collection of at most C_1 monochromatic vertex-disjoint copies from $\mathcal{F}$ in V_{k-1} covering a set S_{k-1} such that the leftover vertices $R_k = V_{k-1} \setminus S_{k-1}$ satisfies $|R_k| \le \alpha\gamma n/2 \le \alpha |Z_{k-1}|$. Let $R'_k \subseteq R_k$ be the set of vertices $u \in R_k$ with $\mathrm{d}_{[r]}(u, Z_1, \dots, Z_{k-1}) \ge \delta$. Let $R'_{k,[k-1]} = R'_k$ and $V_k = R_k \setminus R'_k$.

Inductively, for each $i = k, \dots, N$, we do the following. If $|V_i| < 1/4\gamma$, we use Corollary 4.4.3 to cover V_i using at most C_2 monochromatic vertex-disjoint copies from $\mathcal{F}$ and

[6]We shall identify the cylinders with their vertex-set.

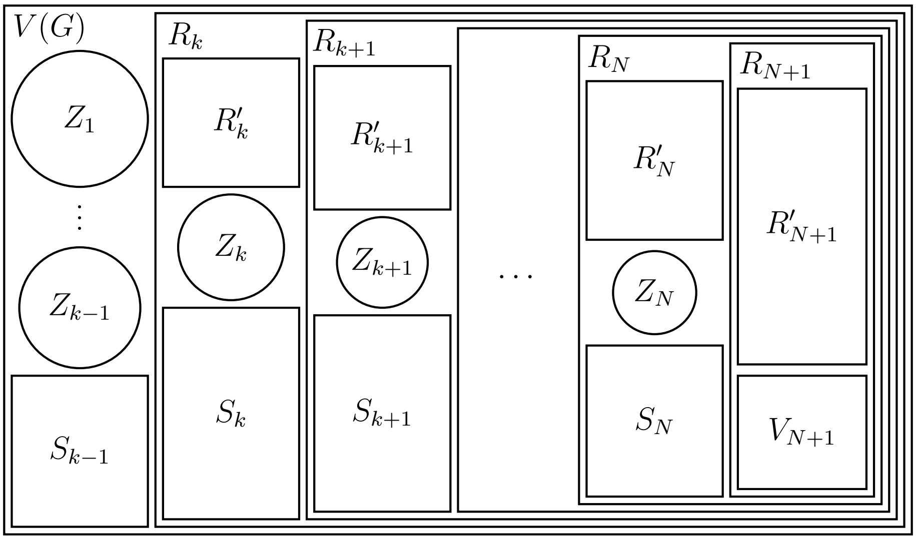


Figure 4.1: A partition of $V(G)$. Each set in the picture is much smaller than the closest cylinder Z_i to the left.

let $Z_i = S_i = R_{i+1} = R'_{i+1} = V_{i+1} = \varnothing$. Otherwise, we apply Lemma 4.3.4 to find a monochromatic (ε, d)-super-regular k-cylinder Z_i contained in V_i with $|Z_i| \geq \gamma |V_i|$. By Proposition 4.4.2, there is a collection of at most C_1 monochromatic, vertex-disjoint copies from $\mathcal{F}$ in $V_i \setminus Z_i$ covering a set $S_i \subseteq V_i$, so that the set of leftover vertices $R_{i+1} = V_i \setminus S_i$ has size at most $\alpha\gamma|V_i| \leq \alpha |Z_i|$.

Let R'_{i+1} be the set of vertices u in R_{i+1} for which there is a set $I = \{i_1, \dots, i_{k-1}\} \subseteq [i]$ such that $\mathrm{d}_{[r]}(u, Z_{i_1}, \dots, Z_{i_{k-1}}) \geq \delta$. Let

$$R'_{i+1} = \bigcup_{I \in \binom{[i]}{k-1}} R'_{i+1,I}$$

be a partition of R'_{i+1} so that, for each $I = \{i_1, \dots, i_{k-1}\} \subseteq [i]$, we have $\mathrm{d}_{[r]}(u, Z_{i_1}, \dots, Z_{i_{k-1}}) \geq \delta$ for all $u \in R'_{i+1,I}$. Finally, let $V_{i+1} = R_{i+1} \setminus R'_{i+1}$.

The following claim implies that these sets partition $V(G)$ as in Item (F.1).

Claim 4.6.2. *The set V_{N+1} is empty.*

Proof. Define a k-uniform hypergraph $\mathcal{H}$ with vertex set $U = Z_1 \cup \ldots \cup Z_N \cup V_{N+1}$ and hyperedges corresponding to monochromatic k-cliques in $G[U]$. If V_{N+1} is non-empty, then so are $Z_1, \dots, Z_N$. Since for each $i = k, \dots, N$ we have $Z_i \subseteq R_i \setminus R'_i$ and $V_{N+1} = R_{N+1} \setminus R'_{N+1}$, it follows that $\mathcal{H}$ satisfies the hypothesis of Lemma 4.6.1. Therefore, there is an independent set $\{v_1, \dots, v_{N+1}\}$ in $\mathcal{H}$ of size $N+1$. On the other hand, since $N \geq R_r(K_k)$, it follows that the set $\{v_1, \dots, v_{N+1}\}$ has a monochromatic k-clique in $G[U]$, which is a contradiction. □

The vertices in $S_{k-1} \cup \dots \cup S_N$ are already covered by monochromatic copies of graphs from $\mathcal{F}$. Our goal now is to cover the sets $R'_k, \dots, R'_{N+1}$ using Lemma 4.5.1 without using too many vertices from the cylinders $Z_1, \ldots, Z_N$. This way, we can cover the remaining vertices in $Z_1 \cup \cdots \cup Z_N$ using Lemma 4.3.3.

Claim 4.6.3. *Let $i \in \{k, \ldots, N+1\}$ and $I = \{i_2, \ldots, i_k\} \subseteq [i-1]$. Let $A \subseteq V(G) \setminus R_{i,I}$ be a set with $|A \cap Z_j| \leq \alpha |Z_j|$ for each $j \in I$. Then there is a collection of at most C_2 monochromatic vertex-disjoint copies of graphs from $\mathcal{F}$ in*

$$G' = K(R'_{i,I}) \cup K(R'_{i,I}, Z_{i_2}, \ldots, Z_{i_k})$$

which are disjoint from A and canonically cover $R'_{i,I}$.

Proof. Let $\tilde{V}_1 = R'_{i,I}$ and for $j \in [k] \setminus \{1\}$, let $\tilde{V}_j = Z_{i_j} \setminus A$. Note that $|\tilde{V}_j| \geq 4|\tilde{V}_1|$ for every $j \in [k] \setminus \{1\}$ and

$$\begin{aligned} \deg_{[r]}(v, \tilde{V}_2, \ldots, \tilde{V}_k) \geq& \deg_{[r]}(v, Z_{i_2}, \ldots, Z_{i_k}) - k\alpha |Z_{i_2}| \cdots |Z_{i_k}| \\ \geq& (\delta - k\alpha)|Z_{i_2}| \cdots |Z_{i_k}| \\ \geq& \delta/2 \cdot |Z_{i_2}| \cdots |Z_{i_k}| \end{aligned}$$

for every $v \in \tilde{V}_1$. Hence, by Lemma 4.5.1, there is a collection of at most C_2 vertex-disjoint copies from $\mathcal{F}$ in $\tilde{V}_1 \cup \ldots \cup \tilde{V}_k$ that canonically covers $\tilde{V}_1$, finishing the proof. □

We will use Claim 4.6.3 now to cover $\bigcup_{i=k}^{N+1} R'_i$. Let $\prec$ be a linear order on $\mathcal{I} := \left\{(i, I) : i \in [k, N+1], I \in \binom{[i-1]}{k-1}\right\}$. Let $(i, I) \in \mathcal{I}$ and suppose that, for all $(i', I') \in \mathcal{I}$ with $(i', I') \prec (i, I)$, we have already constructed a family $\mathcal{H}_{i'.I'}$ of monochromatic copies of graphs from $\mathcal{F}$ which canonically covers $R'_{i',I'}$ in $K(R'_{i',I'}) \cup K(R'_{i',I'}, Z_{i'_2}, \ldots, Z_{i'_k})$, where $I' = \{i'_2, \ldots, i'_k\}$, and such that the sets $V(\mathcal{H}_{i',I'})$, for $(i', I') \prec (i, I)$, are disjoint.

Let $A = \bigcup_{(i',I') \prec (i,I)} V(\mathcal{H}_{i',I'})$ be the set of already covered vertices. We claim that

$$|A \cap Z_j| \leq \alpha |Z_j| \tag{4.4}$$

for each $j \in [N]$. Indeed, given some $j \in [N]$, for all $(i', I') \in \mathcal{I}$ with $i' \leq j$, we have $V(\mathcal{H}_{i',I'}) \cap Z_j = \varnothing$, since $\mathcal{H}_{i',I'}$ canonically covers $R'_{i',I'}$ in $K(R'_{i',I'}) \cup K(R'_{i',I'}, Z_{i'_2}, \ldots, Z_{i'_k})$. Now for all $(i', I') \in \mathcal{I}$ with $i' > j$, we have $|V(\mathcal{H}_{i',I'}) \cap Z_j| \leq |R'_{i',I'}|$, again because $\mathcal{H}_{i,I}$ canonically covers $R'_{i',I'}$. Therefore,

$$|A \cap Z_j| \leq \sum_{(i',I') \prec (i,I)} |V(\mathcal{H}_{i',I'}) \cap Z_j| \leq \sum_{(i',I') \in \mathcal{I} \,:\, i' > j} \left|R'_{i',I'}\right| \leq |R_{j+1}|,$$

since the sets $\{R'_{i',I'} : (i', I') \in \mathcal{I}, i > j\}$ are disjoint subsets of R_{j+1}. Finally, since $|R_{j+1}| \leq \alpha |Z_j|$, this implies Equation (4.4). In particular, by Claim 4.6.3, there is a collection $\mathcal{H}_{i,I}$ of monochromatic copies of graphs from $\mathcal{F}$ that canonically covers $R'_{i,I}$ in

$K(R'_{i,I}) \cup K(R'_{i,I}, Z_{i_2}, \ldots, Z_{i_k})$, where $I = \{i_2, \ldots, i_k\}$, and such that $V(\mathcal{H}_{i,I})$ is disjoint from A.

It remains to cover $\bigcup_{i=1}^{N} Z_i$. Let $A := \bigcup_{(i,I)\in\mathcal{I}} V(\mathcal{H}_{i,I})$ be the set of vertices covered in the previous step. Note that, similarly as in Equation (4.4), we have $|A \cap Z_j| \leq \alpha |Z_j|$ for all $j \in [N]$. Therefore, by Lemma 4.3.2, the cylinder $\tilde{Z}_j$ obtained from Z_j by removing all vertices in A is $(8\varepsilon, d/2)$-super-regular and ε-balanced for every $j \in [N]$. It follows from Lemma 4.3.3 that, for every $j \in [N]$, there is a collection $\mathcal{H}_j$ of at most $\Delta+3$ monochromatic vertex-disjoint copies of graphs from $\mathcal{F}$ contained in Z_j covering $V(Z_j)$.

In total, the number of monochromatic copies we used to cover $V(G)$ is at most

$$\begin{aligned} N \cdot C_1 + N^k \cdot C_2 + N \cdot (\Delta + 3) &\leq 2N^k C_2 \\ &\leq 2r^{rk^2} \cdot \exp^2\left(r^{16\tilde{K}r\Delta^3}\right) \\ &\leq \exp^2\left(r^{Kr\Delta^3}\right). \end{aligned}$$

This concludes the proof of Theorem III. □

4.7 Proofs of the auxiliary lemmas

In this section, we shall prove the lemmas stated in Section 4.3 for which we could not find a proof in the literature. Their proofs however are standard and not difficult.

Proof of Lemma 4.3.2. Let $U_i = (V_i \setminus X_i) \cup Y_i$ for $i \in \{1,2\}$. We will show that (U_1, U_2) is $(8\varepsilon, d - 8\varepsilon, \delta/2)$-super-regular. Let now $Z_i \subseteq U_i$ with $|Z_i| \geq 8\varepsilon |U_i|$, and let $Z'_i = Z_i \setminus Y_i$ and $Z''_i = Z_i \cap Y_i$ for $i \in \{1,2\}$. Note that we have

$$|Z_i| \geq 8\varepsilon |U_i| \geq \varepsilon |V_i|, \tag{4.5}$$

$$|Z''_i| \leq |Y_i| \leq \varepsilon^2 |V_i| \overset{(4.5)}{\leq} \varepsilon |Z_i| \text{ and} \tag{4.6}$$

$$|Z'_i| = |Z_i| - |Z''_i| \overset{(4.6)}{\geq} (1-\varepsilon)|Z_i| \tag{4.7}$$

for both $i \in \{1,2\}$. We therefore have

$$e(Z_1, Z_2) \leq e(Z'_1, Z'_2) + e(Z''_1, Z_2) + e(Z_1, Z''_2) \overset{(4.6)}{\leq} e(Z'_1, Z'_2) + 2\varepsilon |Z_1||Z_2|$$

and thus

$$d(Z_1, Z_2) \leq d(Z'_1, Z'_2) + 2\varepsilon.$$

On the other hand, we have

$$d(Z_1,Z_2)=\frac{e(Z_1,Z_2)}{|Z_1||Z_2|}\geq\frac{e(Z_1',Z_2')}{|Z_1'||Z_2'|}\cdot\frac{|Z_1'||Z_2'|}{|Z_1||Z_2|}$$
$$\stackrel{(4.7)}{\geq} d(Z_1',Z_2')(1-\varepsilon)^2\geq d(Z_1',Z_2')-2\varepsilon$$

and hence $d(Z_1,Z_2)=d(Z_1',Z_2')\pm 2\varepsilon$. Furthermore, by ε-regularity of (V_1,V_2), we have $d(Z_1',Z_2')=d(V_1,V_2)\pm\varepsilon$ and we conclude

$$d(Z_1,Z_2)=d(V_1,V_2)\pm 3\varepsilon.$$

This holds in particular for $Z_1=U_1$ and $Z_2=U_2$ and therefore the pair (U_1,U_2) is $(8\varepsilon,d-8\varepsilon,0)$-super-regular. Let $u_1\in U_1$ now. By assumption, we have $\deg(u_1,V_2)\geq\delta|V_2|$ and therefore

$$\deg(u_1,U_2)\geq\deg(u_1,V_2\setminus X_2)\geq(\delta-\varepsilon^2)|V_2|$$
$$\geq(\delta-\varepsilon^2)|U_2|\geq\delta/2\cdot|U_2|.$$

A similar statement is true for every $u_2\in U_2$ finishing the proof. □

The following consequence of the slicing lemma will be useful when we prove Lemmas 4.3.4 and 4.3.5.

Lemma 4.7.1. *Let k be a positive integer and $d,\varepsilon>0$ with $\varepsilon\leq 1/(2k)$. If $Z=(V_1,\dots,V_k)$ is an ε-regular k-cylinder and $d(V_i,V_j)\geq d$ for all $1\leq i<j\leq k$, then there is some $\gamma\leq k\varepsilon$ and sets $\tilde V_1\subseteq V_1,\dots,\tilde V_k\subseteq V_k$ with $|\tilde V_i|=\lceil(1-\gamma)|V_i|\rceil$ for all $i\in[k]$ so that the k-cylinder $\tilde Z=(\tilde V_1,\dots,\tilde V_k)$ is $(2\varepsilon,d-k\varepsilon)$-super-regular.*

Proof. For $i\neq j\in[k]$, let $A_{i,j}:=\{v\in V_i:\deg(v,V_j)<(d-\varepsilon)|V_j|\}$. By definition of ε-regularity, we have $|A_{i,j}|<\varepsilon|V_i|$ for every $i\neq j\in[k]$. For each $i\in[k]$, let $A_i=\bigcup_{j\in[k]\setminus\{i\}}A_{i,j}$. Clearly $|A_i|<(k-1)\varepsilon|V_i|$ for every $i\in[k]$, so we can add arbitrary vertices from $V_i\setminus A_i$ to A_i until $|A_i|=\lfloor(k-1)\varepsilon|V_i|\rfloor$ for every $i\in[k]$. Let now $\tilde V_i=V_i\setminus\tilde A_i$ for every $i\in[k]$ and let $\tilde Z=(\tilde V_1,\dots,\tilde V_k)$. Observe that $|\tilde V_i|=\lceil(1-\gamma)|V_i|\rceil$ for all $i\in[k]$, where $\gamma=(k-1)\varepsilon$. It follows from Lemma 4.3.1 and definition of A_i that $\tilde Z$ is $(2\varepsilon,d-\varepsilon,d-k\varepsilon)$-super-regular. □

Given k disjoint sets $V_1,\dots,V_k$, we call a cylinder $(U_1,\dots,U_k)$ *relatively balanced* (w.r.t. $(V_1,\dots,V_k)$) if there exists some $\gamma>0$ so that $U_i\subset V_i$ with $|U_i|=\lfloor\gamma|V_i|\rfloor$ for every $i\in[k]$. We say that a partition $\mathcal{K}$ of $V_1\times\cdots\times V_k$ is *cylindrical* if each partition class is of the form $W_1\times\cdots\times W_k$ (which we associate with the k-cylinder $Z=(W_1,\dots,W_k)$) with $W_j\subseteq V_j$ for every $j\in[k]$. Finally, we say that $\mathcal{K}=\{Z_1,\dots,Z_N\}$ is ε-regular if

(i) $\mathcal{K}$ is a cylindrical partition of $V_1\times\cdots\times V_k$,

(ii) each Z_i, $i \in [k]$, is a relatively balanced w.r.t. $(V_1, \dots, V_k)$, and

(iii) all but $\varepsilon|V_1| \cdots |V_k|$ of the k-tuples $(v_1, \dots, v_k) \in V_1 \times \cdots \times V_k$ are in ε-regular cylinders.

For technical reasons, we will allow some of the sets $V_1, \dots, V_k$ to be empty. In this case $(A, \varnothing)$ is considered ε-regular for every set A and $\varepsilon > 0$. If G is an r-edge-coloured graph and $i \in [r]$, we say that a cylinder $\mathcal{K}$ is ε-regular in colour i if is ε-regular in G_i (the graph on $V(G)$ with all edges of colour i).

In [27], Conlon and Fox used the weak regularity lemma of Duke, Lefmann and Rödl [35] to find a reasonably large balanced k-cylinder in a k-partite graph. In order to prove a coloured version of Conlon and Fox's result, we will need the following coloured version of the weak regularity lemma of Duke, Lefmann and Rödl. Note that, like the weak regularity lemma of Frieze and Kannan [50], we get an exponential bound on the number of cylinders, in contrast to the much worse tower-type bound required by Szemerédi's regularity lemma (see [45]).

Theorem 4.7.2 (Duke–Lefmann–Rödl [35])**.** *Let $0 < \varepsilon < 1/2$, $k, r \in \mathbb{N}$ and let $\beta = \varepsilon^{rk^2\varepsilon^{-5}}$. Let G be an r-edge-coloured k-partite graph with parts $V_1, \dots, V_k$. Then there exist some $N \le \beta^{-k}$, sets $R_1 \subseteq V_1, \dots, R_k \subseteq V_k$ with $|R_i| \le \beta^{-1}$ and a partition $\mathcal{K} = \{Z_1, \dots, Z_N\}$ of $(V_1 \setminus R_1) \times \cdots \times (V_k \setminus R_k)$ so that $\mathcal{K}$ is ε-regular in every colour and $V_i(Z_j) \ge \lfloor \beta |V_i| \rfloor$ for every $i \in [k]$ and $j \in [N]$.*

Although the original statement of Duke, Lefmann and Rödl [35, Proposition 2.1] does not involve the colouring and assume that sets $V_1, \dots, V_k$ have the same size, their proof can be easily adapted to prove Theorem 4.7.2.

We are now ready to prove Lemmas 4.3.4 and 4.3.5.

Proof of Lemma 4.3.4. Let $k, r \ge 2$, $0 < \varepsilon < 1/(rk)$ and $\gamma = \varepsilon^{r^{8rk}\varepsilon^{-5}}$. Let $n \ge 1/\gamma$ and suppose we are given an r-edge coloured K_n. Let $\tilde{k} = r^{rk}$ and let $V_1, \dots, V_{\tilde{k}} \subseteq [n]$ be disjoint sets of size $\lfloor n/\tilde{k} \rfloor$ and let G be the $\tilde{k}$-partite subgraph of K_n induced by $V_1, \dots, V_{\tilde{k}}$ (inheriting the colouring). Let $\tilde{\varepsilon} = \varepsilon/2$ and $\beta = \tilde{\varepsilon}^{r^{2rk+1}\tilde{\varepsilon}^{-5}}$. We apply Theorem 4.7.2 to get some $N \le \beta^{-\tilde{k}}$, sets $R_1 \subseteq V_1, \dots, R_{\tilde{k}} \subseteq V_{\tilde{k}}$ each of which of size at most β^{-1} and a partition $\mathcal{K} = \{Z_1, \dots, Z_N\}$ of $(V_1 \setminus R_i) \times \cdots \times (V_{\tilde{k}} \setminus R_{\tilde{k}})$ which is $\tilde{\varepsilon}$-regular in every colour, and with $V_i(Z_j) \ge \lfloor \beta |V_i| \rfloor \ge 2\gamma n$ for every $i \in [\tilde{k}]$ and $j \in [N]$. Note that one of the cylinders (say Z_1) must be $\tilde{\varepsilon}$-regular in every colour and, since $(V_1, \dots, V_k)$ is balanced, so is Z_1. We consider now the complete graph with vertex-set $\{V_1(Z_1), \dots, V_{\tilde{k}}(Z_1)\}$ and colour every edge $V_i(Z_1)V_j(Z_1)$, $1 \le i < j \le \tilde{k}$, with a colour $c \in [r]$ so that the density of the pair $(V_i(Z_1), V_j(Z_1))$ in colour c is at least $1/r$. By Ramsey's theorem [92, 43], there is a colour, say 1, and k parts (say $V_1(Z_1), \dots, V_k(Z_1)$) so that the cylinder $(V_1(Z_1), \dots, V_k(Z_1))$ is $(\tilde{\varepsilon}, 1/r, 0)$-super-regular in colour 1. By Lemma 4.7.1, there is an $(\varepsilon, 1/(2r))$-super-regular balanced subcylinder $\tilde{Z}_1$ with parts of size at least γn. □

Proof of Lemma 4.3.5. Let $k \geq 2$, and let $d, \varepsilon > 0$ with $2k\varepsilon \leq d \leq 1$. Let $\gamma = \varepsilon^{k^2\varepsilon^{-12}}$ and let G be a k-partite graph with parts $V_1, \ldots, V_k$. Let $\tilde{\varepsilon} = \varepsilon/4$ and $\beta = \tilde{\varepsilon}^{k^2\tilde{\varepsilon}^{-5}}$. We may assume that $|V_i| \geq 1/\gamma$ for every $i \in [k]$ (otherwise we set $U_i := \varnothing$ for all $i \in [k]$ with $|V_i| < 1/\gamma$). In particular, we have $|V_i| \geq k/(\tilde{\varepsilon}\beta)$ for all $i \in [k]$.

We apply Theorem 4.7.2 (with $r = 1$) to get some $N \leq \beta^{-k}$, sets $R_1 \subseteq V_1, \ldots, R_k \subseteq V_k$, each of which of size at most β^{-1}, and an $\tilde{\varepsilon}$-regular partition $\mathcal{K} = \{Z_1, \ldots, Z_N\}$ of $(V_1 \setminus R_1) \times \cdots \times (V_k \setminus R_k)$ with $V_i(Z_j) \geq \lfloor \beta |V_i| \rfloor$ for every $i \in [k]$ and $j \in [N]$.

Note that the number of cliques of size k incident to $R = R_1 \cup \ldots \cup R_k$ is at most

$$\sum_{i=1}^{k} \beta^{-1} \prod_{j \in [k] \setminus \{i\}} |V_j| \leq \tilde{\varepsilon} |V_1| \cdots |V_k|.$$

Furthermore, since $\mathcal{K}$ is $\tilde{\varepsilon}$-regular, there are at most $\tilde{\varepsilon}|V_1| \cdots |V_k|$ cliques of size k in G that belong to a cylinder of $\mathcal{K}$ that is not ε-regular. Suppose that each cylinder $Z \in \mathcal{K}$ has at most $(d - 2\tilde{\varepsilon})|V_1(Z)| \cdots |V_k(Z)|$ cliques of size k. Then the number of k-cliques in G is at most

$$\tilde{\varepsilon}|V_1| \cdots |V_k| + \sum_{Z \in \mathcal{K}} (d - 2\tilde{\varepsilon})|V_1(Z)| \cdots |V_k(Z)| \leq (d - \tilde{\varepsilon})|V_1| \cdots |V_k|,$$

which contradicts our hypothesis over G. Therefore, there is a cylinder $\tilde{Z}$ in $\mathcal{K}$ that contains at least $(d - 2\tilde{\varepsilon})|V_1(\tilde{Z})| \cdots |V_k(\tilde{Z})|$ cliques of size k. In particular, $\tilde{Z}$ is $(\tilde{\varepsilon}, d - 2\tilde{\varepsilon}, 0)$-super-regular and relatively balanced with parts of size at least $\lfloor \beta |V_i| \rfloor$. Finally, we apply Lemma 4.7.1 (and possibly delete a single vertex from some parts) to get a relatively balanced $(\varepsilon, d - (k+2)\tilde{\varepsilon})$-super-regular k-cylinder Z with parts of size at least $\frac{\beta}{2}|V_i| \geq \gamma |V_i|$. This completes the proof since $(k+2)\tilde{\varepsilon} \leq k\varepsilon \leq d/2$. $\square$

4.8 Concluding Remarks

We were able to prove that sequences of graphs with maximum degree Δ have finite r-colour tiling number for every $r \geq 3$, but our bound is super-exponential in Δ. Grinshpun and Sárközy [53] conjectured that it is possible to prove an upper bound which is essentially exponential in Δ (see Conjecture 4.1.1). The problem becomes somewhat easier when restricted to bipartite graphs. In fact, our proof gives a double exponential upper bound in Δ for r-colour tiling numbers of sequences of bipartite graph with maximum degree Δ. Indeed, the factor k in the recursive bound Equation (4.1) can be dropped for bipartite graphs. It would be very interesting to confirm Conjecture 4.1.1 for sequences of bipartite graphs.

Another interesting problem is to prove a version of Theorem III for other sequences of graphs. Given a sequence of graphs $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ with $|F_i| = i$, for every $i \in \mathbb{N}$, let $\rho_r(\mathcal{F}) = \sup_{i\in\mathbb{N}} R_r(F_i)/i$. If $\rho_r(\mathcal{F})$ is finite, then we say that $\mathcal{F}$ has linear r-colour Ramsey number. If $\mathcal{F}$ is *increasing*[7], then it follows from the pigeon-hole principle that $\tau_r(\mathcal{F}) \geq \rho_r(\mathcal{F})$. Indeed, for each $n \in \mathbb{N}$, every r-edge-coloured K_n contains a monochromatic copy from $\mathcal{F}$ of size at least $i = \lceil n/\tau_r(\mathcal{F})\rceil$. In particular, since $\mathcal{F}$ is increasing, there is a monochromatic copy of F_i in every r-edge colouring of K_n. This implies that $R_r(F_i) \leq \tau_r(\mathcal{F}) \cdot i$, and therefore $\rho_r(\mathcal{F}) \leq \tau_r(\mathcal{F})$.

Graham, Rödl and Ruciński [52] proved that there exists a sequence of bipartite graphs $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ with $\rho_2(\mathcal{F}) \geq 2^{\Omega(\Delta)}$. Grinshpun and Sárközy observed that one can make this sequence increasing, thereby showing that $\tau_2(\mathcal{F}) \geq 2^{\Omega(\Delta)}$ as well. Conlon, Fox and Sudakov [28] proved that for every sequence of graphs with degree at most Δ, we have $\rho_2(\mathcal{F}) \leq 2^{O(\Delta \log \Delta)}$ while Grinshpun and Sárközy [53] proved that $\tau_2(\mathcal{F}) \leq 2^{O(\Delta \log \Delta)}$. For more colours, Fox and Sudakov [46] proved that for every sequence of graphs with degree at most Δ, we have $\rho_r(\mathcal{F}) \leq 2^{O_r(\Delta^2)}$, while Theorem III shows that $\tau_r(\mathcal{F}) \leq \exp^3(O_r(\Delta^3))$.

With these results in mind, one can naturally ask if there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that for every sequence of graphs $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ we have $\tau_r(\mathcal{F}) \leq f(\rho_r(\mathcal{F}))$. That is, if it is possible to bound $\tau_r(\mathcal{F})$ in terms of $\rho_r(\mathcal{F})$. In particular, this would imply that sequences of graphs with linear Ramsey number have finite tiling number. However, the following example due to Alexey Pokrovskiy (personal communication) shows that $\tau_r(\mathcal{F})$ cannot be bounded by $\rho_r(\mathcal{F})$ in general. Let S_i be a star with i vertices and let $\mathcal{S} = \{S_i : i \in \mathbb{N}\}$ be the family of stars. It follows readily from the pigeonhole principle that $R_r(S_i) \leq r(i-2)+2$, for every $i \in \mathbb{N}$, and thus $\rho_r(\mathcal{S}) \leq r$. However, the following shows that $\tau_r(\mathcal{S}) = \infty$, for every $r \geq 2$.

Example 4.8.1. For all $r \geq 2$ and all sufficiently large n, we have $\tau_r(\mathcal{S}, n) \geq r \cdot \log(n/8)$.

Proof. Let $\tau = r\log(n/8)$ and colour $E(K_n)$ uniformly at random with r colours. Given a vertex $v \in [n]$ and a colour c, let $S_c(v)$ be the star centred at v formed by all the edges of colour c incident on v. Note that there is a monochromatic $\mathcal{S}$-tiling of size at most τ if and only if there are distinct vertices $v_1, \ldots, v_\tau$ and colours $c_1, \ldots, c_\tau \in [r]$ such that $\bigcup_{i\in[\tau]} V(S_{c_i}(v_i)) = [n]$.

Fix distinct vertices $v_1, \ldots, v_\tau \in [n]$ and colours $c_1, \ldots, c_\tau \in [r]$. Let U be the random set $U = \bigcup_{i\in[\tau]} V(S_{c_i}(v_i))$. Note that the events $\{v \in U\}$, for $v \in [n] \setminus \{v_1, \ldots, v_\tau\}$, are independent and each has probability $1-(1-1/r)^\tau$. Therefore, using $e^{-x/(1-x)} \leq 1-x \leq e^x$ for all $x \leq 1$, we get

$$\begin{aligned}\mathbb{P}[U = [n]] &= (1-(1-1/r)^\tau)^{n-\tau} \\ &\leq \exp\left(-(n-\tau)(1-1/r)^\tau\right) \\ &\leq \exp\left(-n(1-1/r)^{\tau+1}\right)\end{aligned}$$

[7]That is, $F_i \subseteq F_{i+1}$, for every $i \in \mathbb{N}$.

$$\begin{aligned} &\le \exp\left(-n\exp\left(-4\tau/r\right)\right) \\ &\le \exp\left(-\sqrt{n}\right). \end{aligned}$$

Taking a union bound over all choices of $v_1, \ldots, v_\tau$ and $c_1, \ldots, c_\tau$, we conclude that the probability that there is a monochromatic $\mathcal{S}$-tiling of size τ is at most

$$(rn)^{-\tau} \cdot e^{-\sqrt{n}} < 1$$

for all sufficiently large n. Hence, there exists an r-colouring of $E(K_n)$ without a monochromatic $\mathcal{S}$-tiling of size at most τ, finishing the proof. □

Lee [80] proved that graphs with bounded degeneracy[8] have linear Ramsey number. Example 4.8.1 shows however that it is not possible to extend this result to a tiling result. Nevertheless, it may be possible to allow unbounded degrees in this case.

Question 1. *Is there a function $\omega : \mathbb{N} \to \infty$ with $\lim_{n\to\infty} \omega(n) = \infty$, such that the following is true for all integers $r, d \ge 2$? If $\mathcal{F} = \{F_1, F_2, \ldots\}$ is a sequence of d-degenerate graphs with $v(F_n) = n$ and $\Delta(F_n) \le \omega(n)$ for all $n \in \mathbb{N}$, then $\tau_r(\mathcal{F}) < \infty$.*

Böttcher, Kohayakawa and Taraz [16] proved an extension of the blow-up lemma to graphs H of bounded arrangeability[9] with $\Delta(H) \le \sqrt{n}/\log(n)$. Using their result, it is possible to prove the following strengthening of Theorem III.

Theorem 4.8.2. *For all integers $r, a \ge 2$ and all sequences of a-arrangeable graphs $\mathcal{F} = \{F_1, F_2, \ldots\}$ with $|F_n| = n$ and $\Delta(F_n) \le \sqrt{n}/\log(n)$ for all $n \in \mathbb{N}$, we have $\tau_r(\mathcal{F}) < \infty$.*

The proof is almost identical, with the following two differences. First, instead of Lemma 4.3.3, we need to use the blow-up lemma mentioned above together with the following alternative to Hajnal's and Szemerédi's theorem which guarantees balanced partitions of graphs with small degree. Given a sequence $\mathcal{F} = \{F_1, F_2, \ldots\}$ of a-arrangeable graphs with $\Delta(F_n) \le \sqrt{n}/\log(n)$ for every $n \in \mathbb{N}$, we define another sequence of graphs $\tilde{\mathcal{F}} = \{\tilde{F}_1, \tilde{F}_2, \ldots\}$ as follows. Since every a-arrangeable graph is $(a+2)$-colourable, we can fix a partition of $V(F_n) = V_1(F_n) \cup \ldots \cup V_k(F_n)$ into independent sets, where $k = a+2$. Then, for every $j \in \mathbb{N}$, we define $\tilde{F_{jk}}$ to be the disjoint union of k copies of F_j. Note that each $\tilde{F}_{jk}$ has a k-partition into parts of equal sizes (by rotating each copy around). Finally, for each $j \in \mathbb{N} \cup \{0\}$ and every $i \in [k-1]$, we define $\tilde{F}_{jk+i}$ to be the disjoint union of $\tilde{F}_{jk}$ and i isolated vertices (here $\tilde{F}_0$ is the empty graph). Observe that all $\tilde{F}_n$ have k-partitions into

[8] A graph G is d-degenerate if there is an ordering of its vertices so that every $v \in V(G)$ is adjacent to at most d vertices which come before v.

[9] A graph G is called *a-arrangeable* for some $a \in \mathbb{N}$ if its vertices can be ordered in such a way that for every $v \in V(G)$, there are at most a vertices to the left of v that have some common neighbour with v to the right of v.

parts of almost equal sizes. Furthermore, every $\tilde{\mathcal{F}}$-tiling $\mathcal{T}$ corresponds to an $\mathcal{F}$-tiling $\tilde{\mathcal{T}}$ of size at most $(2k-1)|\mathcal{T}|$. Therefore, it suffices to prove Theorem 4.8.2 for graphs with balanced $(a+2)$-partitions.

Second, we need to replace Theorem 4.4.1 with a similar theorem for a-arrangeable graphs G with $\Delta(G) \leq \sqrt{n}/\log(n)$, where $n = v(G)$. For two colours, such a theorem was proved by Chen and Schelp [22]. For more than two colours, this was (to the best of the author's knowledge) never explicitly stated, but is easy to obtain using modern tools (for example, by applying the above mentioned blow-up lemma for a-arrangeable graphs).

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