Characterization of the Super-Hedging Pricing Rule

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Contents

Introduction

Literature Review

When dealing with frictionless, arbitrage-free and complete markets, it is well known that there exists a linear pricing rule and it is uniquely determined. Furthermore, this linear pricing rule induces a state-price density. The general model for the theory of linear asset pricing was formalized by Harrison and Kreps [1979], Duffie and fu Huang [1986], Harrison and Pliska [1981a] and Kreps [1981]. This general framework of linear asset pricing under the hypothesis of no arbitrage opportunities is based on Shubik [1961] and Cox and Ross [1976]. Shubik [1961] is the root of the theory of asset pricing under the condition that there are no arbitrage opportunitis, while Cox and Ross [1976] gives the linear price of any contingent claim as the expected value of its future returns with respect to the unique equivalent probability measure, the uniqueness being a consequence of the hypothesis of absence of arbitrage opportunities. We also have Black and Scholes [1973] in the roots of the theory of asset pricing, where the main idea is the linkage between the concept of arbitrage and trade in continuous time. It was introduced the "principle of no-arbitrage", allowing to derive, in certain mathematical frameworks of financial markets (for example, the Black-Scholes model in Samuelson [2015], also known as Samuelson model), unique prices for certain contingent claims, for example, options. The Black-Scholes formula is one particular example of a model that is embedded in the general theory constructed by Harrison and Kreps [1979], Duffie and fu Huang [1986], Harrison and Pliska [1981a] and Kreps [1981].

The main result in Harrison and Kreps [1979] and Duffie and fu Huang [1986] says that, under the absence of arbitrage opportunities, the there exists a probability measure under which the price processes of the assets avaliable to trade is a martingale. Any contingent claim can be perfectly replicated through the trade of an adequated portfolio and the cost to doing this is given by the expected value of its payoffs with respect to this probability measure. More than this, it was given a complete characterizations for certain types of market structures: a mathematical model of a financial market is arbitrage-free if and only if there exists a risk neutral probability that transforms it in a martingale. This important result in known in the literature of financial markets as The Fundamental Theorem of Asset Pricing, this nomenclature being given by Dybvig and Ross [1989]. This result is not just a theorem, but a principle that allows to construct a link between the no-arbitrage principle and the theory of martingales. With this link established, one can study precise informations about the cost of replicating and adquiring certain contingent claims.

The first result that turned the general principle of no-arbitrage in a precise theorem was established for the case of finite probability spaces by Harrison and Pliska [1981a]. A classical example of a model constructed taking as base a finite probability space is the binomial model, also known as the Cox-Ross-

Rubinstein model Cox, Ross, and Rubinstein [1979]. But the hypothesis of finite probability space was very restrictive, being inadequated for dealing even with the first models of the theory of asset pricing under no-arbitrage, such as the Brownian motion model established in Bachelier [1900] and the Black-Scholes model. So, the problem of achieve results that can be applyed to more general frameworks than the case of finite probability spaces remained open. To deal with more general frameworks, one approach, as used in the Black-Scholes and in the Bachelier model Bachelier [1900], is to use the theory of stochastic analysis. For example, one important result in this sense is the martingale representation theorem for Brownian motion, presented in Durrett [1993].

Taking the work Kreps [1981] as initial point, it was developed a long field of research that gives in results with strong mathematical rigor new versions of the Fundamental Theorem of Asset Pricing. In this line, we can citet Delbaen and Schachermayer [1994] andDelbaen, Schachermayer, Mathematik, Technische, and Zurich [2002]. In these papers it was presented a version concerned with general $\mathbb{R}^d$-valued semi martingales.

When frictions are present in the financial market in consideration, then the cost of replicating a given contingent claim is not more uniquely determined by the expected value of its payoffs with respect to an unique martingale measure. Moreover, not all securities can be exactly replicated anymore. In these situations, the problem is now formulated in terms of super-replication of contingent claims, instead of perfect replication. The precification of securities via super replication and its connection with the principle of no-arbitrage has been an active and important topic of research in the literature of financial markets. In the case of discrete time, we can citet the first established works, Bensaid, Lesne, Pagès, and Scheinkman [1992], Dalang, Morton, and Willinger [1990] and Jouini and Kallal [1995a]. In continuous time, we date back to Cvitanic and Karatzas [1993], Delbaen and Schachermayer [1994] and Delbaen and Schachermayer [1996].

In Bensaid, Lesne, Pagès, and Scheinkman [1992] the initial point was the standard binomial option pricing model. It was incorporated a friction in this model in the form of bid-ask spreads and then the optimization problem defining the super-replication price under the hypothesis of no-arbitrage was solved by dynamic programming. In Jouini and Kallal [1995a] it was considered an infinite state space framework and the main result provides a characterization of arbitrage-free financial markets with bid-ask spreads, in terms of martingale measures. Other examples of the incorporation of bid-ask spreads in the standard frictionless and arbitrage-free model of financial market are Yeoman [1992], Boyle and Vorst [1992], Dermody and Rockafellar [1991] and Edirisinghe, Naik, and Uppal [1993].

In Yeoman [1992] and Boyle and Vorst [1992] the binomial option pricing model of precification by replication is generalized to include bid-ask spread on the stock. In Dermody and Rockafellar [1991] and Edirisinghe, Naik, and Uppal [1993], it was remarked that the cost of perfect replication of contingent claims can be unnecesseraly high, because there may exist feasible strategies that dominate the payoff of the contingent claim at a lower initial cost when compared to perfect hedge.

An other type of possible friction is the presence of short sale constraints. In this scenario, the characterization under absence of arbitrage opportunities is given in Jouini and Kallal [1995b]. In this paper, it was showed that the financial market with this friction is arbitrage-free only and if only theres exists an equivalent probability measure and a numeraire such that the price processes of the assets avaliable to trade, normalized by the numeraire, are super martingales.

Still on the question of characterizing arbitrage-free financial markets through martingales, Naik

[1995] gives a characterization in a discrete time, event-tree framework. In both papers, Jouini and Kallal [1995b] and Naik [1995], it is assumed that there is no consumption in intermediate periods, that is, the assets does not pay dividends in the periods between the first one and the last one. In this sense, Ortu [2001] gives a contribution in a scenario that allows for intermediate dividend payment and positive bid-ask spreads on all assets. In this paper, it was showed that the absence of arbitrage opportunities is equivalent to the existence of mininum cost, super-replication strategies and underlying frictionless state-prices. The author uses a linearized super replication problem and its dual to supply alternative characterizations of absence of arbitrage opportunities. It is also presented a characterization of absence of arbitrage opportunities in terms of martingales. The framework of this paper is the one we are going to use in Chapter 1.

In the event-tree with dividends payment in the intermediate dates approach, we also have the results in Baccara, Battauz, and Ortu [2006]. In this paper, it was employed linear programming techniques in order to characterize absence of arbitrage opportunities in financial markets with bid-ask spreads. An other innovation of this paper concerned with linear programming approach is that it allows for bid-ask spreads at liquidation. An other contribution is that it was supplied a proof based on linear programming to the fact that absence of arbitrage opportunities imposes an upper bound on the bid and a lower bound on the ask price of a new security.

In a framework with uncertainty and two dates, today and one future period, Araujo, Chateauneuf, and Faro [2012] gives a characterization of the super-replication price of contingent claims supposing absence of arbitrage opportunities and incompleteness of a frictionless financial market with a riskless bond. The uncertainty is given assuming that there are a finite number of possible states of the world and the state occurred is revelead in the second period. Given a pricing rule

$$C : \mathbb{R}^S \to \mathbb{R},$$

that is, a function satisfying certain conditions that will be presented later in the present work, precifying contingent claims, it was defined the following sets, associated to this function:

$$F_C := \{x; C(x) + C(-x) = 0\}$$
$$L_C := \{x; y > x \Rightarrow C(y) > C(x)\}$$

It was showed that $C : \mathbb{R}^S \to \mathbb{R}$, where S is the number of possible states of the world, is the super-replication at minimum cost pricing rule of a frictionless and arbitrage-free financial market with a riskless bond if and only if $F_C = L_C$.

It is well known (see, for example, Theorem 4 in Araujo, Chateauneuf, and Faro [2018]) that a function $C : \mathbb{R}^S \to \mathbb{R}$ is a pricing rule if and only if there exists a unique convex and closed set $\mathcal{K} \subset \Delta^{S-1}$ of probability measures where at least one is strictly positive such that

$$C(x) = \max_{\mathbb{P} \in \mathcal{K}} \mathbb{E}_{\mathbb{P}}[x], \ \forall \ x \in \mathbb{R}^S$$

Taking this characterization, the authors in Araujo, Chateauneuf, Faro, and Holanda [2019] provided a geometric characterization for the set $\mathcal{K}$ of probability measures for the super-replication at mininum cost pricing rule of a frictionless and arbitrage-free financial market with a riskless asset and one uncertain future period.

The results in the present thesis are concerned with the same questions as in the last two papers citetd above. In Chapter 1, in a multi-period framework, it will be presented a characterization of the super-replication at minimum cost pricing rule in terms of an equality of sets, for the same market structure considered in Araujo, Chateauneuf, and Faro [2012], that is, without any kind of friction, arbitrage-free and with a risk free asset.

Objective

The objective of the present work is to characterize the super-replication pricing rule for different structures of financial markets under the hypothesis of absence of arbitrage opportunities. We analyse two types of configuration of financial market: in chapter 1 it is considered a multi-period frictionless and arbitrage-free financial market. We present a characterization for the minimum cost super-hedging pricing rule in terms of an equality of two specific sets, extending the result in Araujo, Chateauneuf, and Faro [2012] to the case of more than one future period. For this, we construct "submarkets" with only one future period and use the existent results to this case to derive our characterization. In chapter 2, we consider again a multi-period configuration but this time we incorporate stochastic interest rates for borrowing and lendind, varying between dates and between assets. Under the no-arbitrage principle, we show the existence of vector of discount factors to financial markets with this configuration and then use this result to write the minimum cost super-hedging pricing rule in this case.

Chapter 1

Markets Without Frictions

1.1 Framework and Definitions

The framework used in the present work is the one specified in Ortu [2001]. We consider a financial market with a finite time horizon indexed by $\mathbb{T} = \{0, 1, ..., T\}$. In each of the periods, agents can buy and sell $J + 1$ kinds of assets, $X_j, j \in \{0, 1, ..., J\}$ one of which is riskless and the other J are risky ones. The randomness of the risky assets' prices and dividend flows are based on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$, being defined as a pair $\{(q_j(t), d_j(t))\}_{t=0}^T$ of stochastic processes, for each asset $j \in \{1, ..., J\}$. The uncertainty in each of the dates will be inserted in the framework assuming that there are s_t states of the world in time $t = 1, ..., T$. We consider the σ-algebra of the parts of Ω, $\mathcal{F} = 2^\Omega$ and a strictly positive probability, $\mathbb{P}$, on $2^\Omega \setminus \varnothing$. The investors share an information flow described by the filtration $\{\mathcal{F}_t\}_{t=0}^T$ of $\mathcal{F}$, where $\mathcal{F}_0 = \Omega, \mathcal{F}_T = \mathcal{F}$.

The actions of the agents in the market are concretized through couples $\{\theta(t)\}_{t=0}^T = \{\theta^A(t), \theta^B(t)\}_{t=0}^T$ of stochastic processes, with $\theta^A(t) = \{\theta_j^A(t)\}_{j=0}^J, \theta^B(t) = \{\theta_j^B(t)\}_{j=0}^J$, where $\theta_j^A(t)$ is the quantity of units of asset j bought and $\theta_j^B(t)$ the quantity sold, in time t. So, as we are considering two stochastic processes to represent quantities bought and sold, instead of assuming negative values to that sold, we will say that a dynamic trading strategy is feasible if it is non-negative, and will denote the set of feasible dynamic trading strategy by Θ. An element $\theta \in \Theta$ gives to its owner a return of $x_\theta(t)$ in the date $t = 0, 1, ..., T$. The *cashflow process* generated in these terms is than given by:

$$x_\theta(t) = \begin{cases} -q(0) \cdot [\theta^A(0) - \theta^B(0)] & t = 0 \\ d(t) \cdot \sum_{\tau=0}^{t-1} [\theta^A(\tau) - \theta^B(\tau)] - q(t) \cdot [\theta^A(t) - \theta^B(t)] & t = 1, ..., T-1 \\ d(T) \cdot \sum_{\tau=0}^{T-1} [\theta^A(\tau) - \theta^B(\tau)] & t = T \end{cases}$$

In time 0, it is the negative of the initial cost imposed by the dynamic trading strategy θ. In the intermediate periods $t = 1, ..., T-1$, the investor starts with the dividends generated by the net positions that he owns on the $J + 1$ assets, referents to the previous dates, and has a cost to update these positions. So, the cashflow in these stages is the difference between these two quantities. As in the last we assume there are no more trades, the agents only liquidate their final net positions, gaining the revenue of the dividends created by these positions.

An important concept in the literature of financial markets is that of arbitrage, that means the possibility of risk-free gains, positive returns without costs. In our context, an arbitrage opportunity occurs when:

1. the investor has a positive cashflow in some of the intermediate periods or in the terminal one and a non-negative in all the other periods (less the initial one), but a null or negative cost, in the initial date, to trade the strategy $(\theta^A(0), \theta^B(0))$ or;

2. the investor has a negative cost in the initial date and a non-negative cashflow in all the intermediate periods and in the terminal one.

In formal terms we are saying that:

Definition 1.1. (Arbitrage)
A dynamic trading strategy $\theta \in \Theta$ such that $x_\theta(t) \geq 0 \ \forall \ t$ is an *arbitrage opportunity* if one of the following two conditions happens:

1. $x_\theta(t) > 0$ for some $t > 0$;

2. $x_\theta(0) > 0$.

The fundamental elements necessary to formally define a financial market are now presented and we can define:

Definition 1.2. (Frictionless Financial Market)
A *frictionless financial market* with T periods is a triple $\mathcal{M} = (x_j, \{q_j(t)\}_{t=0}^T, \{d_j(t)\}_{t=0}^T; 0 \leq j \leq J)$, where x_j are the assets and S_j, d_j are the assets' price processes and dividend processes, respectively.

Our interest in the present work is to study questions concerned with mathematical implications of the absence of arbitrage opportunities. Such issues will be discussed further on. By now, we impose the non-arbitrage condition in the framework:

Definition 1.3. We say the market $\mathcal{M}$ is *arbitrage-free* if $\nexists \ \theta \in \Theta$ such that θ is an arbitrage opportunity. That is, if

$$\theta \in \Theta, x_\theta(t) \geq 0 \ \forall \ t \Rightarrow x_\theta(t) = 0 \ \forall \ t$$

In Definition 1.3 the conditions 1. and 2. from Definition 1.1 are put together.

Hypothesis 1.1. The financial market $\mathcal{M}$ we consider here is arbitrage-free.

Now, we have all the components necessary to deal with the goals we are interested in inside the structure of the financial market being analysed. In the next section, we are going to construct through steps the problem on which the results are based on.

The main object necessary to specify how the trades made by the investors are precified is what is called *contingent claim* in the language of financial markets. So, first of all, let us construct the meaning of it.

Considering a financial market $\mathcal{M} = (X_j, \{q_j(t)\}_{t=0}^T, \{d_j(t)\}_{t=0}^T; 0 \leq j \leq J)$, the assets' payoffs is a random variable $x : \Omega \to \mathbb{R}^{\sum_{t=1}^T s_t}$ defined on the measurable space $(\Omega, \mathcal{F})$. Let us consider the σ-algebra of subsets of Ω generated by x, that is, denoting it by $\mathcal{G}$, we have $\mathcal{G} := x^{-1}(\mathcal{B}(\mathbb{R}^{\sum_{t=1}^T s_t}))$, where $\mathcal{B}(\mathbb{R}^{\sum_{t=1}^T s_t})$ is the Borel σ-algebra on $\mathbb{R}^{\sum_{t=1}^T s_t}$, adopting the Euclidean topology in $\mathbb{R}^{\sum_{t=1}^T s_t}$. Consider now the set $B(\mathcal{G})$ of all bounded (in the Euclidean norm in $\mathbb{R}^{\sum_{t=1}^T s_t}$), $\mathbb{R}^{\sum_{t=1}^T s_t}$-valued, $\mathcal{G}$-measurable functions on $(\Omega, \mathcal{F})$ equipped with the the sup norm ($\|\cdot\|_\infty$), that is, $B(\mathcal{G}) := \{f : \Omega \to \mathbb{R}^{\sum_{t=1}^T s_t}; \|f\|_2 \leq c$, for some $c \in \mathbb{R}; f\ \mathcal{G}$-measurable$\}$, where $\|f\|_\infty := \sup_{\omega \in \Omega} \|f(\omega)\|_2$. From Charalambos D. Aliprantis [2006], theorem 4.41, we have that $\forall f \in B(\mathcal{G})\ \exists\ g_f : \mathbb{R}^{\sum_{t=1}^T s_t} \to \mathbb{R}^{\sum_{t=1}^T s_t}$ Borel-measurable such that $f = g_f \circ x$. In this way, we identify the set of random payoffs of derivatives contingent on x with $B(\mathcal{G})$. So, we define:

Definition 1.4. (Contingent Claim)
A *contingent claim* is an element $f \in B(\mathcal{G})$.

These objects are those that will be priced in the financial market in consideration. We need to specify this pricing process. The following definition give us the way of doing this.

Definition 1.5. (Pricing Rule) A *pricing rule* is a $\mathbb{R}$-valued function $C : B(\mathcal{G}) \to \mathbb{R}$ that satisfies:

1. $C(\lambda x) = \lambda C(x)\ \forall\ \lambda \in \mathbb{R}$;
2. $C(x + y) \leq C(x) + C(y),\ \forall\ x, y \in B(\mathcal{G})$;
3. a) $x \geq 0 \Rightarrow C(x) \geq 0$
   b) $x > 0 \Rightarrow C(x) > 0$;
4. $C(x + k\mathbb{1}_{\{\Omega\}}) = C(x) + k,\ \forall\ x \in B(\mathcal{G}),\ \forall\ k \in \mathbb{R}$;
5. $C(\mathbb{1}_{\{\Omega\}}) = 1$;
6. $x, y \in B(\mathcal{G}), x(t) \geq y(t)\ \forall\ t \Rightarrow C(x) \geq C(y)$

Condition 1. says that $C(\cdot)$ positively homogeneous.

Condition 2. imposes to $C(\cdot)$ the property of subadditivity, which says that, given two contingent claims $x, y \in B(\mathcal{G})$, it is more expensive to buy them separately than the combination of both. The marketed contingent claims in the financial market are priced buy some exogenous pricing rule $C : B(\mathcal{G}) \to \mathbb{R}$. If an investor wants to buy a contingent claim $x \in B(\mathcal{G})$, $C(x)$ is the cost he has in doing this.

Condition 3. says that the pricing rule is arbitrage-free, that is, if an investor wants to buy a contingent claim with non-negative payoffs, then he has to spend a non-negative quantity of money when doing this. Moreover, if the claim has strictly positive returns than the cost of it is strictly positive.

Condition 4. is the property called constant additivity.

Condition 5. says that the pricing rule is normalized and condition 6. is the property of monotonicity, which says that contingent claims with better payoffs are more expensive.

1.2 Markets with Two Future Periods

When dealing with financial markets, the non-arbitrage hypothesis generates important mathematical consequences concerned with pricing rules. It is well know that in the case of frictionless complete markets, the ausence of arbitrage opportunities implies that the contigent claims can be perfectly hedged and their prices are given by the mathematical expectation of their returns with respect to the unique existent equivalent martingale measure. The claim is so priced by the called arbitrage pricing rule. This result was first proved in discrete time by Ross [1976] and in continuous time by Harrison and Pliska

[1981b]. The equivalent probability can be understood as prices in different states of nature, one period ahead, of one monetary unity. However, in cases of incomplete markets, not all contingent claims can be exactly replicated by dynamic trading strategies. Moreover, the equilavent martingale measure is not more unique. So, in these sceneries, it is necessary to reformulate the problem of finding prices to the contingent claims, substituting the perfect hedge and uniqueness of the expected payoffs by other proper notions.

In the literature of financial markets, the most common alternativa of dealing with the difficult of pricing contingent claims under market frictions is to consider the super-hedging problem. Instead of only a perfect replication, it is now considered that the duplication will be equal or greater than the claim. In the following, we define the problem in mathematical terms.

$$\begin{aligned} \pi(x) = \inf_{\theta \in \Theta} & -x_\theta(0) \\ & \text{s.t. } x_\theta(t) \geq x(t) \ \forall \ t \in \{1, ..., T\} \end{aligned} \tag{1.1}$$

In words, $\pi(x)$ is the minimum cost to construct a dynamic trading strategy that has a return of at least the return of the contingent claim x, in each period. First of all, let us verify that $\pi(\cdot)$, as defined above, is a pricing rule according to Definition 1.5.

Lemma 1.1. The function value $\pi(\cdot)$ defined in (1.1) is a pricing rule.

Given a contingent claim $x = (x(1), ..., x(T)) \in B(\mathcal{G})$, the solution $\pi(x)$ to the problem above is called the *super hedging price* of x. Our interest is to study certain consequences of the ausence of arbitrage hypothesis to the prices defined in this way. By Lemma 1.1 we have that $\pi(\cdot)$ is a pricing rule.

In markets with only one future period, we have by Peter J. Huber [2009], Proposition 10.1, the following characterization of pricing rules:

Theorem 1.1. *The function $C : \mathbb{R}^S \to \mathbb{R}$ is a pricing rule if and only if there exists a convex and closed set $\mathcal{K}$ of probability measures where at least one of them is strictly positive such that:*

$$C(x) = \max_{\mathbb{P} \in \mathcal{K}} \mathbb{E}_{\mathbb{P}}(x) \ \forall \ x \in \mathbb{R}^S \tag{1.2}$$

In our context of multi-period financial markets, through the indetification $\times_{t=1}^{T} \mathbb{R}^{S_t} \cong \mathbb{R}^{\sum_{t=1}^{T} S_t}$ and after some symplex normalizations we obtain the result below:

Proposition 1.1. The function $C : B(\mathcal{G}) \to \mathbb{R}$ is a pricing rule as defined in Definition 1.5 if and only if there exist convex and closed sets $\mathcal{K}_t \subseteq \Delta^{S_t - 1}, t = 1, ..., T, \mathcal{K} = \times_{t=1}^{T} \mathcal{K}_t$ of probability measures $\mathbb{P}_t$ defined on $(\Omega_t, \mathcal{A}_t)$ and $\mathbb{P} := (\mathbb{P}_1, ..., \mathbb{P}_T)$ such that

$$C(x) = \max_{\mathbb{P}_t \in \mathcal{K}_t} \sum_{t=1}^{T} c_t \mathbb{E}_{\mathbb{P}_t}[x(t)] \ \forall \ x = (x(1), ..., x(T)) \in B(\mathcal{G}),$$

where $c_t > 0$ are constants.

So, any contingent claim $x \in B(\mathcal{G})$ can be precified as the sum of its expected returns in each period, multiplied by a constant.

We are interested in understanding the set $\mathcal{K} = \times_{t=1}^{T} \mathcal{K}_t$ associated to the superhedging pricing rule of a frictionless and arbitrage-free financial markets with two or more future periods. In the case of only one future period, Araujo, Chateauneuf, and Faro [2012] showed that $C : \mathbb{R}^S \to \mathbb{R}$ written in the form of equation (1.2) is the superhedging pricing rule of a frictionless and arbitrage-free financial market with one bond if and only if $F_C = L_C$, where

$$F_C = \{x; C(x) + C(-x) = 0\} \tag{1.3}$$

$$L_C = \{x; y > x \Rightarrow C(y) > C(x)\} \tag{1.4}$$

The set (1.3) is called the set of *frictionless* securities, that is, the securities whose bid and ask prices are the same.

The set (1.4) is called the set of *undominated* securities. These are the securities such that if we take an other security which payoff is greater than that one, than it is more expensive then the original one.

A first question that appears is: In the context of multi period financial markets, as defined previously, do we have the same characterization of the super hedging pricing rule in the absence of arbitrage opportunities? That is, can we say that a pricing rule $C : B(\mathcal{G}) \to \mathbb{R}$ satisfies $F_C = L_C$ if and only if it is the super hedging pricing rule of the market in consideration? We will see that the answer is no. We will see that, in aa multi-period framework, the set F_C of frictionless securities is replaced by a new set G_C, that is, the characterization will be given by $G_C = L_C$.

We are going to see that we still have an equality of sets that characterizes a pricing rule being a super hedging pricing rule of a frictionless and arbitrage-free multi period financial market, but instead of the set F_C we will have another one, G_C, to be defined soon.

The idea to obtain this result is, first, given a financial market with two future periods, we are going to define markets with only one future period and use them to get conclusions about the multi period financial market. Then, by an induction argument we use this result to prove the case with three or more future periods. It will be clear in the next pages after some definitions and specifications. So, let us doing this.

Consider a financial market with 3 periods $\mathcal{M} = (x_j, \{q_j(t)\}_{t=0}^{2}, \{d_j(t)\}_{t=0}^{2}, 0 \le j \le J)$.

Define the following two markets with 2 periods:

$\mathcal{M}^1 := (x_j, q_j^1, d_j^1, 0 \le j \le J)$, where $q_j^1 := q_j(0), d_j^1 := d_j(1)$;

$\mathcal{M}^2 := (x_j, q_j^2, d_j^2, 0 \le j \le J)$, where $q_j^2 := q_j(0), d_j^2 := d_j(2)$.

With these markets defined, we have the following equivalence:

Lemma 1.2. The financial market $\mathcal{M}$ is frictionless and arbitrage-free if and only if $\mathcal{M}^1, \mathcal{M}^2$ are frictionless and arbitrage-free.

Through Lemma 1.2, we can verify if a financial market with three periods is frictionless and arbitrage-free checking if the “submarkets” with two periods is frictionless and arbitrage free.

In the following example we illustrate Lemma 1.2.

Example 1.1. Consider the financial market $\mathcal{M} = (x_j, \{q_j(t)\}_{t=0}^2, \{d_j(t)\}_{t=0}^2, 0 \le j \le J)$ such that

$$d_j(1), d_j(2) \backsim \mathcal{U}(0,1) \ \forall \ j \ne 0$$
$$q_j(1) \backsim \mathcal{U}(1,2)$$

The cashflow process for the market $\mathcal{M}$ is

$$x_\theta(t) = \begin{cases} -q(0) \cdot [\theta^A(0) - \theta^B(0)] & t = 0 \\ d(1) \cdot [\theta^A(0) - \theta^B(0)] - q(1) \cdot [\theta^A(1) - \theta^B(1)] & t = 1 \\ d(T) \cdot \sum_{\tau=0}^{1} [\theta^A(\tau) - \theta^B(\tau)] & t = 2 \end{cases}$$

The financial market $\mathcal{M}^1 = (x_j, q_j^1, d_j^1)$ is specified by

$$q_j^1 := q_j(0) \ \forall j \in \{0, ..., J\}$$
$$d_j^1 := d_j(1) \ \forall j \in \{0, ..., J\},$$

with cashflow process

$$x_\theta^1(t) = \begin{cases} -q(0) \cdot [\theta^A(0) - \theta^B(0)] & t = 0 \\ d(1) \cdot [\theta^A(0) - \theta^B(0)] & t = 1 \end{cases}$$

Suppose the market $\mathcal{M}^1$ is not arbitrage-free. Then exists $\theta \in \Theta$ such that

$$x_\theta^1(0) = -q(0) \cdot [\theta^A(0) - \theta^B(0)] \ge 0;$$
$$d(1) \cdot [\theta^A(0) - \theta^B(0)] > 0$$

Define

$$\bar{\theta}^A(1) := \theta^B(0) \ \forall j \in \{0, ..., J\} \text{ and}$$
$$\bar{\theta}^B(1) := \theta^A(0) \ \forall j \in \{0, ..., J\}$$

Then we have

$$\bar{\theta}^A(1) - \bar{\theta}^B(1) = -[\theta^A(0) - \theta^B(0)] \ \forall j$$
$$\Rightarrow q_j(1)[\theta_j^A(1) - \theta_j^B(1)] = -q_j(1)[\theta^A(0) - \theta^B(0)] \ \forall j$$
$$\Rightarrow d_j(1)[\theta^A(0) - \theta^B(0)] - q_j(1)[\bar{\theta}^A(1) - \bar{\theta}^B(1)] = d_j(1)[\theta^A(0) - \theta^B(0)] + q_j(1)[\theta^A(0) - \theta^B(0)]$$

As $q_j(1) > d_j(1)$ a.s. $\forall j$, the last inequality implies that

$$d_j(1)[\theta^A(0) - \theta^B(0)] + q_j(1)[\theta^A(0) - \theta^B(0)] > 2d_j[\theta_j^A(0) - \theta_j^B(0)],$$

which implies

$$x_{\bar{\theta}}(1) = d(1) \cdot [\bar{\theta}^A(0) - \bar{\theta}^B(0)]$$
$$= 2 \sum_{j=0}^{J} d_j(1)[\theta_j^A(0) - \theta_j^B(0)] > 0$$

So, in summary, we got:

$$x_{\bar{\theta}}(0) \ge 0;$$
$$x_{\bar{\theta}}(1) > 0,$$

that is, $\bar{\theta} = (\bar{\theta}^A, \bar{\theta}^B)$ is an arbitrage opportunity in the financial market $\mathcal{M}$.

Now, let us link the super hedging pricing rule of the market with two future periods with the super hedging pricing rules of the markets with only one future period, as defined previously.

Given a pricing rule $C : B(\mathcal{G}) \to \mathbb{R}$ in the financial market $\mathcal{M}$, writing

$$C(x) = \max_{\mathbb{P}_t \in \mathcal{K}_t} \left\{ \sum_{t=1}^{2} c_t \mathbb{E}_{\mathbb{P}_t}[x(t)] \right\},$$

define the following functions:

$C_t : \mathbb{R}^{S_t} \to \mathbb{R}$ $t = 1, 2$ such that

$C_1(x) := \frac{1}{c_1} C((x; 0)) \forall x \in \mathbb{R}^{S_1}$;

$C_2(x) := \frac{1}{c_2} C((0; x)) \forall x \in \mathbb{R}^{S_2}$.

These functions are important because they are the super-replication at mininum cost pricing rules in the markets with two periods. As we will see, we can use them to know if a given pricing rule $C : B(\mathcal{G}) \to \mathbb{R}$ is the super-replication at minimum cost price in the multi-period financial market $\mathcal{M}$ in which we are interested in.

Also, define the following sets:

$F_{C_t} := \{x \in \mathbb{R}^{S^t}; C_t(x) + C_t(-x) = 0\}, t = 1, 2$

$G_C := \{x = (x(1), x(2)) \in B(\mathcal{G}); C((x(1), -x(2))) + C((-x(1), x(2))) = 0\}$

$L_C^1 := \{x = (x(1), x(2)) \in B(\mathcal{G}); y(1) > x(1) \Rightarrow C(y) > C(x)\}$

$L_C^2 := \{x = (x(1), x(2)) \in B(\mathcal{G}); y(2) > x(2) \Rightarrow C(y) > C(x)\}$

$\mathcal{L}_C := L_C^1 \cap L_C^2$

$L_C := \{x; y > x \Rightarrow C(y) > C(x)\}$

In the sets above, $y(t) > x(t)$ means $y_s(t) \geq x_s(t)$ $\forall$ $s \in \mathcal{S}_t, y_{s^*}(t) > x_{s^*}(t)$ for some $s \in \mathcal{S}_t$. That is, if we replace, in some state, the payoff of the contingent claim x by a better payoff from a claim y then y is strictly more expensive than x.

Observe that the sets F_{C_t} are the sets of frictionless securities (as defined previously by Araujo, Chateauneuf, and Faro [2012]) under the pricing rule C_t in the market $\mathcal{M}^t, t = 1, 2$, respectively.

Also, observe that L_C^t is the projection of L_C in $\mathbb{R}^{S_t}, t = 1, 2$. Indeed, $\mathcal{L}_C = L_C$, as we prove in the following Lemma:

Lemma 1.3. $\mathcal{L}_C = L_C$.

An important and interesting characteristic of the set G_C previously defined is that it has an structure of linear subspace.

Lemma 1.4. G_C is a linear subspace.

An important fact is that the securities in the set G_C are such that if $x = (x(1), x(2)) \in G_C$ then, if we look to each coordinate, it is a frictionless security in each of the markets with one future period, that is, $x(1)$ is a frictionless security in the market $\mathcal{M}^1$ and $x(2)$ is a frictionless one in the market $\mathcal{M}^2$.

Lemma 1.5. $x = (x(1), x(2)) \in G_C \Leftrightarrow x(t) \in F_{C_t}, t = 1, 2$

We also have important relations between the super-replication pricing rule of the market with two future periods, $\mathcal{M}$ and the super-replication prices of the markets with one future period, $\mathcal{M}^1$ and $\mathcal{M}^2$.

It is interesting that we can recuperate the super-replication at minimum cost pricing rules of the markets $\mathcal{M}^1$ and $\mathcal{M}^2$ through the super-replication price of the financial market $\mathcal{M}$. This is the statement of Lemma 1.6.

Lemma 1.6. Suppose the frictionless market $\mathcal{M} = (x_j, \{S_j\}_{t=0}^2, \{d_j(t)\}_{t=0}^2, 0 \le j \le J)$ is arbitrage-free. If the function $C : B(\mathcal{G}) \to \mathbb{R}$ is the super-replication at minimum cost pricing rule in the financial market $\mathcal{M}$ then $C_t : \mathbb{R}^{s_t} \to \mathbb{R}$ is the super-replication at minimum cost pricing rule in the financial market $\mathcal{M}^t$, $t = 1, 2$, respectively.

We also have that, if $C_1(\cdot), C_2(\cdot)$ as previously defined are the super-replication at minum cost pricing rule of the markets $\mathcal{M}^1, \mathcal{M}^2$, respectively, then $C(\cdot)$ is the super-replication at minimum cost pricing rule of the financial market $\mathcal{M}$. This result will be constructed by steps, through the next Lemmas.

Lemma 1.7. If $C_t : \mathbb{R}^{S_t} \to \mathbb{R}$ defined by $C_1(x) := \left(\frac{1}{c_1}\right) C(x; 0),\ \forall\ x \in \mathbb{R}^{S_1}, C_2(y) := \left(\frac{1}{c_2}\right) C(0; y)\ \forall\ y \in \mathbb{R}^{S_2}$ is the super-replication at minnimum cost pricing rule of the frictionless and arbitrage-free financial market $\mathcal{M}^1, \mathcal{M}^2$, respectively, then $C : B(\mathcal{G}) \to \mathbb{R}$ is linear and strictly posive in G_C and has the form

$$C(x) = \max_{\mathbb{P} \in \mathcal{Q}_C} \{c_1 \mathbb{E}_{\mathbb{P}_1}[x(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[x(2)]\}\ \forall\ x = (x(1), x(2)) \in \mathbb{R}^{S_1} \times \mathbb{R}^{S_2},$$

where

$$\mathcal{Q}_C := \{\mathbb{P} = (\mathbb{P}_1; \mathbb{P}_2) \in \Delta_{++}^{S_1 - 1} \times \Delta_{++}^{S_2 - 1}; c_1 \mathbb{E}_{\mathbb{P}_1}[x(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[x(2)] = C(x)\ \forall\ x = (x(1), x(2)) \in G_C)\},$$

$c_1, c_2 > 0$ constants

Lemma 1.8. If the function $C : \mathbb{R}^{S_1} \times \mathbb{R}^{S_2} \to \mathbb{R}$ is linear and strictly positive in G_C and has the form

$$C(x) = \max_{\mathbb{P} \in \mathcal{Q}_C} \{c_1 \mathbb{E}_{\mathbb{P}_1}[x(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[x(2)]\}\ \forall\ x = (x(1), x(2)) \in \mathbb{R}^{S_1} \times \mathbb{R}^{S_2},$$

where

$$\mathcal{Q}_C := \{\mathbb{P} = (\mathbb{P}_1; \mathbb{P}_2) \in \Delta_{++}^{S_1 - 1} \times \Delta_{++}^{S_2 - 1}; c_1 \mathbb{E}_{\mathbb{P}_1}[x(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[x(2)] = C(x)\ \forall\ x = (x(1), x(2)) \in G_C)\},$$

$c_1, c_2 > 0$ constants, then $C : \mathbb{R}^{S_1} \times \mathbb{R}^{S_2} \to \mathbb{R}$ is the super-replication at mininum cost pricing rule of the financial market $\mathcal{M}$.

Lemma 1.7 and Lemma 1.8 together implies the following result, which allows to know, given a pricing rule $C(\cdot)$ in the market $\mathcal{M}$, if it is the super-replication at minimum cost price through the super-replication at minimum cost price of the markets $\mathcal{M}^t, t = 1, 2$:

Lemma 1.9. Consider the two frictionless and arbitrage-free financial markets $\mathcal{M}^1, \mathcal{M}^2$. If $C_t : \mathbb{R}^{S_t} \to \mathbb{R}$ is the super-replication at minimum cost pricing rule of the financial market $\mathcal{M}^t, t = 1, 2$, respectively, then $C : B(\mathcal{G}) \to \mathbb{R}$ is the super-replication at minimum cost pricing rule of the financial market $\mathcal{M}$.

Finally, joining Lemma 1.6 and Lemma 1.9 we have the following Proposition:

Proposition 1.2. Consider the frictionless and arbitrage-free financial market $\mathcal{M}$. Then $C : B(\mathcal{G}) \to \mathbb{R}$,

$$C(x) = \max_{\mathbb{P}_t \in \mathcal{K}_t} \sum_{t=1}^{2} c_t \mathbb{E}_{\mathbb{P}_t}[x(t)]$$

is the super-replication at minimum cost pricing rule of $\mathcal{M}$ if and only if $C_t : \mathbb{R}^{S_t} \to \mathbb{R}, t = 1, 2$ defined by $C_1(x) := \frac{1}{c_1} C(x; 0) \ \forall \ x \in \mathbb{R}^{S_1}, C_2(y) := \frac{1}{c_2} C(0; y) \ \forall \ y \in \mathbb{R}^{S_2}$ is the super-replication at minimum cost pricing rule of $\mathcal{M}^1, \mathcal{M}^2$, respectively.

Corollary 1.1. $x = (x(1), x(2)) \in G_C \Leftrightarrow x \in L_C$

Finally, as a consequence of the previous proposition and corollary, we have:

Proposition 1.3. The function $C : B(\mathcal{G}) \to \mathbb{R}$ is the super-replication at minimum cost pricing rule of a frictionless and arbitrage-free financial market with 3 periods $\mathcal{M} = (x_j, \{q_j(t)\}_{t=0}^2, \{d_j(t)\}_{t=0}^2, 0 \le j \le J)$ if and only if $G_C = L_C$.

The following example is an adaptation of an example in Castagnoli, Maccheroni, and Marinacci [2002].

Example 1.2. Consider we have a financial market with three periods: today ($t = 0$) and two future dates ($t = 1, 2, T = 3$). We assume that in the second and in the third period there are two possible states of the world, that is, $S_1 = S_2 = 2$. Define the following probabilities $\mathbb{P} = (\mathbb{P}_1; \mathbb{P}_2) \in \Delta_{++}^{S_1 - 1}, \Delta_{++}^{S_2 - 1}$:

$$\mathbb{P}_t^1 := \left(\frac{1}{2}, \frac{1}{2}\right), t = 1, 2$$
$$\mathbb{P}_t^2 := \left(\frac{1}{3}, \frac{2}{3}\right), t = 1, 2$$

And the pricing rule $C : B(\mathcal{G}) \to \mathbb{R}$ by

$$C(x) : \max_{\mathbb{P} = (\mathbb{P}_1; \mathbb{P}_2) \in \mathcal{K}} \{\mathbb{E}_{\mathbb{P}_1}[x(1)] + \mathbb{E}_{\mathbb{P}_2}[x(2)]\}$$

where

$$\mathcal{K} = \{\alpha \mathbb{P}^1 + (1 - \alpha)\mathbb{P}^2; \alpha \in [0, 1]\}$$

Then

$$C(x) = \max_{\alpha \in [0,1]} \left\{\frac{\alpha}{2}[x_1(1) + x_2(1)] + \frac{1 - \alpha}{3}[x_1(2) + x_2(2)]\right\}, \ \forall \ x = (x(1), x(2)) \in B(\mathcal{G})$$

For any $x = (x(1), x(2)) \in B(\mathcal{G})$ we have

$$C((x(1), -x(2))) = \max_{\alpha \in [0,1]} \left\{\frac{\alpha}{2}[x_1(1) + x_2(1)] - \frac{1 - \alpha}{3}[x_1(1) + x_2(2)]\right\}$$

$$C((-x(1),x(2))) = \max_{\alpha\in[0,1]}\left\{-\frac{\alpha}{2}[x_1(1)+x_2(1)] + \frac{1-\alpha}{3}[x_1(1)+x_2(2)]\right\}$$

Let $\alpha_1, \alpha_2 \in [0,1]$ such that

$C((x(1),-x(2))) = \frac{\alpha_1}{2}[x_1(1)+x_2(1)] - \frac{1-\alpha_1}{3}[x_1(1)+x_2(2)]$;

$C((-x(1),x(2))) = -\frac{\alpha_2}{2}[x_1(1)+x_2(1)] + \frac{1-\alpha_2}{3}[x_1(1)+x_2(2)]$

Then we have $C((x(1),-x(2))) + C((-x(1),x(2))) = 0$ if and only if

$$\begin{aligned}
&\frac{\alpha_1}{2}[x_1(1)+x_2(1)] - \frac{1-\alpha_1}{3}[x_1(1)+x_2(2)] - \frac{\alpha_2}{2}[x_1(1)+x_2(1)] + \frac{1-\alpha_2}{3}[x_1(1)+x_2(2)] = 0\\
&\Leftrightarrow \frac{\alpha_1-\alpha_2}{2}[x_1(1)+x_2(1)] + \frac{\alpha_1 - 2}{3}[x_1(2)+x_2(2)] = 0\\
&\Leftrightarrow \frac{3(\alpha_1-\alpha_2)[x_1(1)+x_2(1)] + 2(\alpha_1-\alpha_2[x_1(2)+x_2(2)])}{6} = 0\\
&\Leftrightarrow 3(\alpha_1-\alpha_2)[x_1(1)+x_2(1)] = -2(\alpha_1-\alpha_2[x_1(2)+x_2(2)]\\
&\Leftrightarrow [x_1(1)+x_2(1)] = -\frac{2}{3}[x_1(2)+x_2(2)]
\end{aligned}$$

Then

$$G_C = \left\{x = (x(1),x(2)) \in B(\mathcal{G}); [x_1(1)+x_2(1)] = -\frac{2}{3}[x_1(2)+x_2(2)]\right\}$$

By analogous calculations we obtain

$$F_C = \left\{x = (x(1),x(2)) \in B(\mathcal{G}); [x_1(1)+x_2(1)] = \frac{2}{3}[x_1(2)+x_2(2)]\right\}$$

So, we have

$$G_C \neq L_C$$

Now, take $x = ((1,1),(2,2))$. It is easy to verify that $x \in L_C$.

Take $y \in B(\mathcal{G})$ such that $y_1(1) > x_1(1), y_2(1) \geq x_2(1), y(2) \geq x(2)$. Then

$$y_1(1) + y_2(1) > x_1(1) + x_2(1),$$

Implying that

$$\frac{\alpha}{2}[y_1(1)+y_2(1)] > \frac{\alpha}{2}[x_1(1)+x_2(1)] \tag{1.5}$$

Moreover

$$y_1(2) + y_2(2) \geq x_1(2) + x_2(2)$$

Implying that

$$\frac{1-\alpha}{3}[y_1(2)+y_2(2)] \geq \frac{1-\alpha}{3}[x_1(2)+x_2(2)] \tag{1.6}$$

(1.5) and (1.6) together implies

$$C(y) > C(x),$$

By analogous arguments we show that if $y_2(1) > x_2(1), y_1(1) \geq x_1(1), y(2) \geq x(2)$ or $y(1) \geq x(1), y(2) > x(2)$ then

$$C(y) > C(x),$$

that is, $x \in L_C$.

So, in summary, $x \notin G_C, x \in L_C$, so $G_C \neq L_C$.

Therefore, by Proposition 1.3, $C(\cdot,\cdot)$ is not the super-replication at minimum cost pricing rule of the financial market in question.

1.3 Markets with More than Two Future Periods

Now, consider a financial market with 4 periods $\mathcal{M} = \left(x_j, \{q_j(t)\}_{t=0}^3, \{d_j(t)\}_{t=0}^3, 0 \leq j \leq J\right)$.

Giving a pricing rule $C : B(\mathcal{G}) \to \mathbb{R}$ in the market $\mathcal{M}$, define the following sets:

$G_C^1 := \{C(xD_1) + C(-xD_1) = 0\}$

$G_C^2 := \{C(xD_2) + C(-xD_2) = 0\}$

$G_C^3 := \{C(xD_3) + C(-xD_3) = 0\}$,

where $D_t, t = 1, 2, 3$ are the diagonal matrices defined by

$$D_t[ii] := \begin{cases} 1 & \text{if } S_{t-1} + 1 \leq i \leq S_t \\ -1 \text{ otherwise} & \end{cases}$$

Similarly to the case with three periods, we are going to define, for the case with four periods, two financial markets with three periods, in the following way:

$\mathcal{M}^1 := (x_j, \{q_j^1(t)\}_{t=0}^2, \{d_j^1(t)\}_{t=0}^2, 0 \leq j \leq J)$;

$q_j^1(t) := q_j(t), d_j^1(t) := d_j(t), \forall t, \forall j$

$\mathcal{M}^2 := (x_j, \{q_j^2(t)\}_{t=0}^2, \{d_j^2(t)\}_{t=0}^2, 0 \leq j \leq J)$;

$q_j^2(t) := q_j(t), d_j^2(t) := d_j(t), t = 0, 1$;

$S_j^2(2) := q_j(3), d_j^2(2) := d_j(3), \forall j$

And then define the following functions:

$C_1 : \mathbb{R}^{S_1} \times \mathbb{R}^{S_2} \to \mathbb{R}$; $C_2 : \mathbb{R}^{S_1} \times \mathbb{R}^{S_3} \to \mathbb{R}$ such that

$C_1((x(1);(x(2)))) := C((x(1);x(2);0))$;

$C_2((x(1);(x(3)))) := C((x(1);0;x(3)))$

And the sets

$G_{C_1} := \{x \in \times_{t=1}^{2} \mathbb{R}^{S_t}; C_1((x(1),-x(2))) + C_1(-x(1);x(2)) = 0\}$

$G_{C_2} := \{x \in \mathbb{R}^{S_1} \times \mathbb{R}^{S_3}; C_2((x(1);-x(3))) + C_2((-x(1);x(3))) = 0\}$

We have then the following results, analogous to the previous case with 3 periods:

Lemma 1.10. The financial market $\mathcal{M}$ is frictionless and arbitrage-free if and only if $\mathcal{M}^1, \mathcal{M}^2$ are frictionless and arbitrage-free.

Proposition 1.4. Suppose the market $\mathcal{M}$ is arbitrage-free. Then the function $C : B(\mathcal{G}) \to \mathbb{R}$ is the super-replication at minimum cost pricing rule in the financial market $\mathcal{M}$ if and only if $C_1 : \mathbb{R}^{s_1} \times \mathbb{R}^{s_2} \to \mathbb{R}, C_2 : \mathbb{R}^{s_1} \times \mathbb{R}^{s_3} \to \mathbb{R}$ is the super-replication at minimum cost pricing rule in the financial market $\mathcal{M}^1, \mathcal{M}^2$, respectively.

Lemma 1.11. $x = (x(1);x(2);x(3)) \in G_C \Leftrightarrow (x(1),x(2)) \in G_{C_1}$ and $(x(1);x(3)) \in G_{C_2}$

Corollary 1.2. $x = (x(1);x(2);x(3)) \in G_C = G_C^1 \cap G_C^2 \cap G_C^3 \Leftrightarrow x \in L_C = L_C^1 \cap L_C^2 \cap L_C 3$.

As consequence of Lemma 1.11 and Corollary 1.2, we have the result for the case with 4 periods (3 future periods).

Proposition 1.5. The function $C : B(\mathcal{G}) \to \mathbb{R}$ is the super-replication at minimum cost pricing rule of a frictionless and arbitrage-free financial market with 4 periods $\mathcal{M} = \left(x_j, \{q_j(t)\}_{t=0}^3, \{d_j(t)\}_{t=0}^3, 0 \le j \le J\right)$ if and only if $G_C = L_C$.

This bring us to the general case with T periods, defining

$$G_C^t := \{x; C(xD_t) + C(-xD_t)\} = 0, t = 1, ..., T,$$

where $D_t, t = 1, ..., T$ are the diagonal matrices defined by

$$D_t[ii] := \begin{cases} 1 & \text{if } S_{t-1} + 1 \le i \le S_t \\ -1 \text{ otherwise} & \end{cases}$$

$G_C := \bigcap_{t=1}^T G_C^t, L_C := \bigcap_{t=1}^T L_C^t$.

Notice that each set G_C^t as defined above suggests the intuition that, if we consider contingent claims in the format $y = xD_t$, then G_C^t would be the set of frictionless securities of a market with only one future period and claims in the form $y = xD_t$.

So, by an induction argument, we have the following result for the general case with T periods:

Theorem 1.2. *The function $C : B(\mathcal{G}) \to \mathbb{R}$ is the super-replication at minimum cost pricing rule of a frictionless and arbitrage-free financial market with T periods $\mathcal{M} = \left(x_j, \{q_j(t)\}_{t=0}^{T}, \{d_j(t)\}_{t=0}^{T}, 0 \leq j \leq J\right)$ if and only if $G_C = L_C$.*

To see that Theorem 1.2 is a generalization on Theorem 5 in Araujo, Chateauneuf, and Faro [2012], let us show that, in Examples 1.2 and 1.3, that $G_C \neq F_C$.

We saw that

$$G_C = \left\{x = (x(1), x(2)) \in B(\mathcal{G}); [x_1(1) + x_2(1)] = -\frac{2}{3}[x_1(2) + x_2(2)]\right\}$$

By analogous calculation to that used to find G_C we obtain

$$F_C = \left\{x = (x(1), x(2)) \in B(\mathcal{G}); [x_1(1) + x_2(1)] = \frac{2}{3}[x_1(2) + x_2(2)]\right\}$$

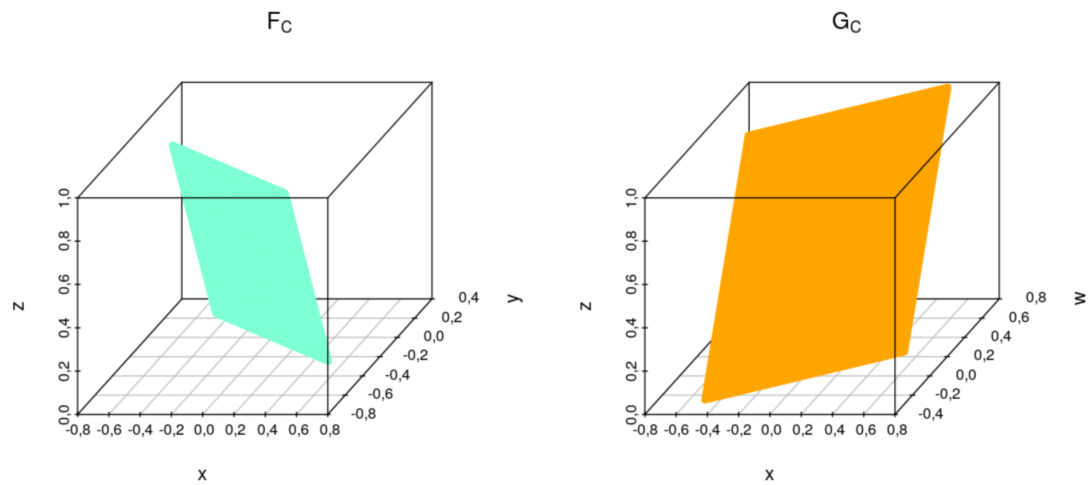
So, we have

$$G_C \neq L_C$$

Example 1.3. Consider a financial market with two future periods but uncertainty only in the last period, in which we have two states of nature, that is, $S_1 = 1, S_2 = 2$. Also, consider a pricing rule given by

$$C(x) = \max_{\alpha \in [0, \frac{1}{2}]} \{\alpha x(1) + (1 - \alpha)[x_1(2) + x_2(2)]\}$$

Graphically, we can see that the sets F_C and G_C are different:



Araujo, Chateauneuf, and Faro [2012] gave a geometric characterization for the super-hedging at minimum cost pricing rule of a frictionless and arbitrage-free market with one future period, in terms of a property called *non-expansibility*.

In the next, we present this property and we give an other characterization for the super-hedging at minimum cost pricing rule for our market framework, in terms of this property.

Definition 1.6. (Non-expansible Set) We say that $\mathcal{K} \subset \Delta$ is *non-expansible* if:

$$\mathbb{P},\ \mathbb{Q} \in \mathcal{K} \Rightarrow \{\alpha\mathbb{P} + (1-\alpha)\mathbb{Q}; \alpha \in \mathbb{R}\} \cap \Delta \subset \mathcal{K}$$

Now, in order to give a characterization in terms of the non-expansibility property we will rewrite the pricing rule, through the following Proposition:

Proposition 1.6. The function $C : \times_{t=1}^{T} \mathbb{R}^{S_t} \to \mathbb{R}$ is a pricing rule if and only if there exist convex and closed sets $\mathcal{K}_t \subseteq \Delta^{S_t-1}, t = 1, ..., T, \mathcal{K} = \times_{t=1}^{T} \mathcal{K}_t$ of probability measures $\mathbb{P}_t$ defined on $(\Omega_t, \mathcal{A}_t)$ and $\mathbb{P} := (\mathbb{P}_1, ..., \mathbb{P}_T)$ such that

$$C(x) = \max_{\mathbb{P}_t \in \mathcal{K}_t} \sum_{t=1}^{T} c_t \mathbb{E}_{\mathbb{P}_t}[x(t)]\ \forall\ x = (x(1), ..., x(T)) \in \bigtimes_{t=1}^{T} \mathbb{R}^{S_t},$$

where $c_t > 0$ are constants.

In Araujo et al. [2019], it was proved that, in a one future period framework, that:

Proposition 1.7. Let K be a non-expansible polytope with at least one interior point, then

$$C(X) := \max_{\mathbb{P} \in \mathcal{K}} \mathbb{E}_{\mathbb{P}}[X] \tag{1.7}$$

satisfy $L_C = F_C$. Also, if C is a pricing rule satisfying $F_C = L_C$, then $\mathcal{K}$ is a non-expansible polytope with at least one interior point.

As a consequence of the Theorem 1.2 , Proposition 1.6, Proposition 1.7, Lemma 1.3 and 1.5 together we have (because we than have several markets with only one future period satisfying conditions of Theorem 5 in Araujo et al. [2012] and so $F_C = L_C$ for each of them):

Corollary 1.3. A pricing rule

$$C(x) = \max_{\mathbb{P}_t \in \mathcal{K}_t} \sum_{t=1}^{T} c_t \mathbb{E}_{\mathbb{P}_t}[x(t)], \forall x = (x(1); \cdots; x(T)) \in \bigtimes_{t=1}^{T} \mathbb{R}^{S_t}$$

is the super-hedging at minimum cost pricing rule of a frictionless and arbitrage-free financial market with T periods if and only if the sets $\mathcal{K}_t \subset \Delta^{S_t-1}$ are non-expansible.

Chapter 2

Markets with Interest Rates

2.1 Framework and Definitions

In the previous section, we dealt with frictionless financial markets. Now, we impose a type of friction: transaction costs. We study the implications of ausence of arbitrage opportunities to the assets' prices. First, we define the structure of this new market, considering the friction mentioned.

As previously, we consider $J+1$ assets, where one is the frictionless bond and the other J are risky assets. But now the return of the assets are specified through different interest rates for borrowing and lending. Based on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and filtration $\{\mathcal{F}_t\}_{t=0}^T$ defined in the previous section, we mantain the adapted stochastic processes $\{q_j(t)\}_{t=0}^T$ representing the prices of the assets, $j \in \{0, ..., J\}$ and introduce new processes $\{\alpha_j(t)\}_{t=0}^T, \{\beta_j(t)\}_{t=0}^T$ also adapted to the filtration $\{\mathcal{F}_t\}_{t=0}^T$ denoting the interest rates of each asset j, with $\alpha_j(t) \geq 0$ and $1 > \beta_j(t) \geq 0$. The investor gets $(1-\beta_j(t))q_j(t)$ if selling one unit of asset j in time t and pays $(1+\alpha_j(t))q_j(t)$ if buying one unit of asset j.

We assume again *dynamic trading strategies* as couples $\{\theta_j(t)\}_{t=0}^T = \{\theta_j^A(t),$ $\theta_j^B(t)\}_{t=1}^T \in \Theta$ of stochastic processes adapted to $\{\mathcal{F}_t\}_{t=0}^T$, where $\theta_j^A(t), \theta_j^B(t)$ are the quantities of units of asset j bought and sold, respectively, in time $t \in \{0, ..., T\}$.

For each asset j and each period t we create the functions $\phi_j^t : \mathbb{R}^{J+1} \times \mathbb{R}^{J+1} \to \mathbb{R}$ defined in the following way:

$\phi_j^t : \mathbb{R}^{J+1} \times \mathbb{R}^{J+1} \to \mathbb{R}$ defined by

$$\phi_j^t(e, f) := (1+\beta_j(t))e - (1+\alpha_j(t))f \;\forall\; e, f \in \mathbb{R} \times \mathbb{R}$$

And the map $\phi : \mathbb{R}^{J+1} \times \mathbb{R}^{J+1} \to \mathbb{R}^{J+1}$ specified by

$$\phi^t(x, y) := (\phi_0^t(x, y), ..., \phi_J^t(x, y)) \;\forall\; x, y \in \mathbb{R}^{J+1} \times \mathbb{R}^{J+1}$$

We suppose the investor has an endowment $\nu \in \mathbb{R}$ in time $t = 0$ and then with this motivation let us define the cone:

$$V_0 := \{(a, \theta) \in \mathbb{R} \times \Theta; a \geq q(0) \cdot \phi^0(\theta^A(0), \theta^B(0))\} \tag{2.1}$$

In time 0, one pays $x_\theta(0)$ to construct the portfolio $\theta(0)$. So, supposing he has a in cash in time 0, what condition (2.1) says is that a trader needs to have a quantity in cash that is equal or higher than the cost of imposing the portfolio $\theta(0)$.

Example 2.1. Suppose we are inside a financial market with three periods: today, tomorrow and after tomorrow and some trader has \$ x in cash in her bank account today. She wants to protect herself against losses due to fluctuations over some financial instrument, an investment, for example. To this, she constructs a strategy $\theta(0)$ today that will give some return tomorrow and after tomorrow and costs \$ 200. The condition in the cone V_0 specifies that it is allowed only if $x > 200$.

We also define the sets:

$$Z_t := \{\theta \in \Theta; q(t) \cdot \phi^t(\theta^A(t) - \theta^A(t-1), \theta^B(t) - \theta^B(t-1)) \leq 0\}, t \in \mathbb{T} \backslash \{0, T\} \tag{2.2}$$

The relation in (2.2) is a *self-financing* condition: when liquidating, in period t, the portfolio $\theta(t-1)$ obtained in period $t-1$, its value needs to be at least the cost $q(t)\phi^t(\theta)$ of constructing the portfolio $\theta(t)$ in time t. That is, a portfolio $\theta(t-1)$ can be *rebalanced*, in time t, to a portfolio $\theta(t)$ if $\theta \in Z_t$.

Definition 2.1. (Self-financing Strategies)
We will say that a dynamic trading strategy $\theta \in \Theta$ is a *self-financing strategy* if $\theta \in Z_t\ \forall\ t \in \mathbb{T} \backslash \{0\}$.

Now, let us define cones with the motivation of imposing conditions to superhede contingent claims.

$$V_t := \{(\theta, b) \in \Theta \times \mathbb{R}^{S_t}; q(t) \cdot \phi^t(\theta^A(t-1), \theta^B(t-1)) \geq b\}, \forall\ t \in \mathbb{T} \backslash \{0\} \tag{2.3}$$

If an investor wants to hedge a certain contigent claim b in period t, then he needs that the return obtained liquidating the portfolio constructed in the previous period be at least the return of the contingent claim. This is the condition in (2.3).

Definition 2.2. (Hedging Strategy)
$(\nu, \theta, x) \in \mathbb{R} \times \Theta \times \times_{t=1}^{T} \mathbb{R}^{S_t}$ such that θ is a self-financing strategy is an *hedging strategy* if $(\nu, \theta) \in V_0$ a.s. and $(\theta, x(t)) \in V_t$ a.s., $\forall t \in \mathbb{T} \backslash \{0\}$.

Let us denote by $\mathcal{H}$ the set of all hedging strategies:

$$\mathcal{H} := \left\{(\nu, \theta, x) \in \mathbb{R} \times \Theta \times \bigtimes_{t=1}^{T} \mathbb{R}^{S_t}; (\nu, \theta, x) \text{ is an hedging strategy}\right\}$$

Definition 2.3. (Arbitrage Opportunity)
$(\nu, \theta, x) \in \mathcal{H}$ such that $\nu \leq 0$ and $x \geq 0$ is an *arbitrage opportunity* if one of the following two conditions is true:

1. $\nu < 0$;

2. $x(t) > 0$ for some $t \in \{1, ..., T\}$.

Definition 2.4. (arbitrage-free)
We say the market $\mathcal{M} = \left(\{x_j\}_{j=0}^{J}, \mathcal{H}\right)$ is *arbitrage-free* if there is no arbitrage opportunities, that is, if:

$$\nu \leq 0, x(t) \geq 0\ \forall\ t \in \mathbb{T} \backslash \{0\} \Rightarrow \nu = 0, x(t) = 0\ \forall\ t \in \mathbb{T} \backslash \{0\}, \forall (\nu, \theta, x) \in \mathcal{H}$$

Let $m_0 \in \mathbb{R}_{++}$, $m_t : \Omega_t \to \mathbb{R}^{S_t}_{++}$ $\mathcal{F}_t$-measurable such that $\mathbb{E}_{\mathbb{P}_t}[m_t x(t)]$ is well defined and finite $\forall\ t \in \{1, ..., T\}$, where $m_t x(t) = (m_1 x_1(t), ..., q_{t,S_t} x_{S_t}(t)) \in \mathbb{R}^{S_t}$. Then we define

Definition 2.5. (Vector of Discount Factors)
We say that $(m_0, ..., m_T) \in \times_{t=0}^{T} \mathbb{R}^{S_t}_{++}$ is a *vector of discount factors* if

$$\sum_{t=1}^{T} \mathbb{E}_{\mathbb{P}_t}[m_t x(t)] - m_0 \nu \leq 0, \forall\ (\nu, \theta, x) \in \mathcal{H}$$

We will denote the set of vectors of discount factors by $\mathcal{Q}$:

$$\mathcal{Q} := \left\{ m \in \bigtimes_{t=0}^{T} \mathbb{R}^{S_t}_{++}; m \text{ is a vector of discount factors} \right\} \tag{2.4}$$

Consider a financial market with three periods: today, tomorrow and after tomorrow. Assume there are two states of the world in both future periods: high prices for a certain good and low prices, and the price of this good in some way has influence over the returns of some investiment. In all periods, participants of the financial market can trade two assets: one riskless aset and a risky one. Suppose the price process of the stock has uniform distribution over the interval $(1, 2)$ at future periods. About the interest rates, the lending rate has a uniform distribution over $(\frac{1}{2}, 1)$ and the borrowing rate is uniform in $(1, 2)$. Let us write these ideas in mathematical terms, according to our previously defined framework:

$\mathcal{M} = (\{x_j\}_{j=0}^{1}, \mathcal{H})$

$S_1 = S_2 = 2$

$\alpha_0(t) = 0, t = 0, 1, 2$

$\beta_0(t) = 0, t = 0, 1, 2$

$\alpha_1(t) \sim \mathcal{U}(1, 2), t = 0, 1, 2$

$\beta_1(t) \sim \mathcal{U}(\frac{1}{2}, 1), t = 0, 1, 2$

$S_1(t) \sim \mathcal{U}(1, 2), t = 0, 1, 2$

Let us find the set of state-prices vectors. For this, according to inequality (**??**), we need first to know the cone of hedging strategies $\mathcal{H}$.

By definition **??**, (ν, θ, x) is an hedging strategy, that is, $(\nu, \theta, x) \in \mathcal{H}$ if:

1. $(\nu, \theta) \in V_0$;
2. $(\theta, x(t)) \in V_t, t = 1, 2$;
3. $\theta \in Z_t, t = 1, 2$.

So, let us write these three conditions.

$(\nu,\theta) \in V_0 : \nu \geq [\theta_0^A(0) - \theta_0^B(0)] + [2\theta_1^A(0) - \frac{1}{2}\theta_1^B(0)]$

Multiplying by $q_0 > 0$ we get:

$$q_0\nu \geq q_0[\theta_0^A(0) - \theta_0^B(0)] + q_0[2\theta_1^A(0) - \frac{1}{2}\theta_1^B(0)] \tag{2.5}$$

Equation (2.5) together with (**??**) give us that the following is necessary to (q_0, q_1, q_2) be a state-prices vector, that is, $(q_0, q_1, q_2) \in \mathcal{Q}$:

$$q_0[\theta_0^A(0) - \theta_0^B(0)] + q_0[2\theta_1^A(0) - \frac{1}{2}\theta_1^B(0)] \geq \sum_{t=1}^{2}\langle q_t, x(t)\rangle, \forall\ (\nu,\theta,x) \in \mathcal{H} \tag{2.6}$$

Let us derive the second condition in order to obtain the cone of hedging strategies $\mathcal{H}$:

$(\theta, x(t)) \in V_t, t = 1,2 :$ First, for $t = 1$, we need $(\theta, x(1)) \in V_1$, that is:

$$-\sum_{j=0}^{1} S_j(1)\phi_j^1(\theta_j^A(0), \theta_j^B(0)) \geq x(1)$$

$$\Rightarrow -[(\theta_0^A(0) - \theta_0^B(0)) + S_1(1)((1+\alpha_1(1))\theta_1^A(0) - (1-\beta_1(1))\theta_1^B(0))] \geq x(1) \tag{2.7}$$

As $S_1(1) > 1, 1+\alpha_1(1) > 2$ and $1 - \beta_1(1) > \frac{1}{2}$, we have:

$$-[(\theta_0^A(0)-\theta_0^B(0))+S_1(1)((1+\alpha_1(1))\theta_1^A(0)-(1-\beta_1(1))\theta_1^B(0))] > -[\theta_0^A(0)-\theta_0^B(0)]-2\theta_1^A(0)+\frac{1}{2}\theta_1^B(0) \tag{2.8}$$

So, inequalities (2.7) and (2.8) together give us that $(\theta, x(1)) \in V_1$ if and only if:

$$-[\theta_0^A(0) - \theta_0^B(0)] - 2\theta_1^A(0) + \frac{1}{2}\theta_1^B(0) \geq x_s(1), s = 1,2 \tag{2.9}$$

Similarly, for $t = 2$, we have that $(\theta, x(2)) \in V_2$ if and only if:

$$-[\theta_0^A(1) - \theta_0^B(1)] - 2\theta_1^A(1) + \frac{1}{2}\theta_1^B(1) \geq x_s(2), s = 1,2 \tag{2.10}$$

Now, let us study the condition that guarantees $\theta \in \Theta$ is a self-financing strategy, that is, $\theta \in Z_t, t = 1,2$:

$\theta \in Z_t, t = 1,2 :$ For $t = 1$, by the definition of the cone $Z - 1$ we need:

$$\sum_{j=0}^{1} S_j(1)\phi_j^1(\theta_j^A(1) - \theta_j^A(0), \theta_j^B(1) - \theta_j^B(0)) \leq 0$$

$$\Rightarrow \sum_{j=0}^{1} S_j(1)[(1+\alpha_j(1))(\theta_j^A(1) - \theta_j^A(0)) - (1-\beta_j(1))(\theta_j^B(1) - \theta_j^B(0))] \leq 0 \tag{2.11}$$

As $S_1(1) < 2, \alpha_1(1) < 2, \beta_1(1) > \frac{1}{2}$, we get:

$$\sum_{j=0}^{1} S_j(1)[(1+\alpha_j(1))(\theta_j^A(1)-\theta_j^A(0))-(1-\beta_j(1))(\theta_j^B(1)-\theta_j^B(0))] < 2[3(\theta_j^A(1)-\theta_j^A(0))-\frac{1}{2}(\theta_j^B(1)-\theta_j^B(0))] \tag{2.12}$$

Inequalities (2.11) and (2.12) together give us that $\theta \in Z_1$ if and only if:

$$[(\theta_0^A(1) - \theta_0^A(0)) - (\theta_0^B(1) - \theta_0^B(0))] + 2[3(\theta_j^A(1) - \theta_j^A(0)) - \frac{1}{2}(\theta_j^B(1) - \theta_j^B(0))] \leq 0 \tag{2.13}$$

Similarly, for $t = 2$, we have that $\theta \in Z_2$ if and only if:

$$[(\theta_0^A(2) - \theta_0^A(1)) - (\theta_0^B(2) - \theta_0^B(1))] + 2[3(\theta_j^A(2) - \theta_j^A(1)) - \frac{1}{2}(\theta_j^B(2) - \theta_j^B(1))] \leq 0 \tag{2.14}$$

2.2 Results

In this section, we first give an equivalence between the hypothesis of absence of arbitrage opportunities and existence of at least one vector of discount factors. Then we define what is the superhedging at minimum cost pricing rule in this framework with interest rates and writes it in function of vectors of discount factors. Finally, we give a characterization to it.

Proposition 2.1. The financial market $\mathcal{M} = \left(\{x_j\}_{j=0}^J, \mathcal{H}\right)$ is arbitrage-free if and only if there exists a vector of regular discount factors $(m_0, ..., m_T)$.

Definition 2.6. (Pricing Rule)
A *Pricing Rule* is a $\mathbb{R}$-valued function $C : \times_{t=1}^T \mathbb{R}^{S_t} \to \mathbb{R}$ that satisfies:

1. $C(\lambda x) = \lambda C(x), \forall \lambda \in \mathbb{R}_+$;
2. $C(x+y) \leq C(x) + C(y), \forall x, y \in \times_{t=1}^T \mathbb{R}^{S_t}$;
3. a) $x \geq 0 \Rightarrow C(x) \geq 0$;
   b) $x > 0 \Rightarrow C(x) > 0$
4. $C(x + k\mathbb{1}_{\{\Omega\}}) = C(x) + k, \forall\ x \in \times_{t=1}^T \mathbb{R}^{S_t}, \forall\ k \in \mathbb{R}$;
5. $C(\mathbb{1}_{\{\Omega\}}) = 1$;
6. $x, y \in \times_{t=1}^T \mathbb{R}^{S_t}; x(t) \geq y(t)$ (componentwise) $\forall\ t \Rightarrow C(x) \geq C(y)$

The pricing rule we are interested in is the super-hedging pricing rule, defined as

$$\pi(x) := \inf_{\nu \in \mathcal{V}_x} \nu = \inf_{(\nu,\theta,x) \in \mathcal{H}} \nu,$$

where

$$\mathcal{V}_x := \{\nu \in \mathbb{R}; \exists\ \theta \in \Theta \text{ such that } (\nu, \theta, x) \in \mathcal{H}\}$$

By duality, we have:

$$\pi(x) = \max_{\lambda \geq 0} \quad \lambda \sum_{t=1}^{T} m_t \cdot x(t)$$
$$\text{s.t.} \qquad m_0 \lambda = 1$$

As $m \in \mathcal{Q}$, then $m > 0$. So, we can rewrite the restriction as $\lambda = \frac{1}{m_0}$ and obtain

$$\pi(x) = \max_{m_0 > 0} \frac{1}{m_0} \sum_{t=1}^{T} m_t \cdot x(t)$$

$\Rightarrow$

$$\pi(x) = \max_{m \in \mathcal{Q}} \frac{1}{m_0} \sum_{t=1}^{T} m_t \cdot x(t) \tag{2.15}$$

Using expression (2.15), we can manipulate the set of vectors of discount factors and obtain the following characterization for the superhedging pricing rule:

Theorem 2.1. $C : \times_{t=1}^{T} \mathbb{R}^{S_t} \to \mathbb{R}$ *is the superhedging at minimum cost pricing rule of an arbitrage-free market* $\mathcal{M} = (\{x_j\}_{j=0}^{J}, \mathcal{H})$ *with interest rates* $\{\alpha_j(t)\}_{t=0}^{T}, \{\beta_j(t)\}_{t=0}^{T}$ *if and only if* $\exists\ \gamma_j(t) \in [\alpha_j(t), \beta_j(t)]$ *such that*

$$C(x) = \max_{m \in \mathcal{L}} \frac{1}{m_0} \sum_{t=1}^{T} m_t \cdot x(t),$$

where

$$\mathcal{L} = \left\{ m \in \bigtimes_{t=0}^{S_t} \mathbb{R}^{S_t} : \left\{ \left(\sum_{s=1}^{S_t} m_s, t \right) [1 + \gamma(t)] q(t) \right\}_{t=0}^{T} \text{ is a martingale} \right\}$$

Appendix A

Appendix

Proof. of Lemma 1.2
$(\Rightarrow)$Let $\theta^{(1)} \in \Theta^{(1)} := \Pi^{(1)}(\Theta)$

$$x_{\theta^{(1)}}(t) = \begin{cases} x_\theta(0) & t = 0; \\ \\ d(1) \cdot [\theta^A(0) - \theta^B(0)] & t = 1. \end{cases}$$

Let $\theta^{(1)} \in \mathbb{R}^{2(J+1)}$ such that $\sum_{j=0}^J d_j(1) \cdot (\theta_j^A(0) - \theta_j^B(0)) > 0$.

If $\sum_{j=0}^J \theta_j^{(1)} \cdot q_j^{(1)} = 0$, then $x_\theta(0) = 0$. So, defining

$\theta_j^A(1) := c \in \mathbb{R}, \theta_j^B(1) := -\theta_j^A(1);$

$\theta_j^A(2) := c \in \mathbb{R}$, $\theta_j^B(2) := -\theta_j^A(2)$, $\forall j \in \{0, \dots, J\}$, we have:

$$\begin{cases} x_\theta(1) > 0; \\ x_\theta(2) = 0; \\ x_\theta(0) = 0, \end{cases}$$

that is, θ is an arbitrage oportunity in the 3-period market. So, we must have

$\sum_{j=0}^J \theta_j^{(1)} \cdot q_j^{(1)} = x_\theta(0) > 0.$

Now, let $\theta^{(1)} \in \mathbb{R}^{2(J+1)}$ such that $\sum_{j=0}^J d_j(1) \cdot (\theta_j^A(0) - \theta_j^B(0)) = 0$.

If $\sum_{j=0}^J \theta_j^{(1)} \cdot q_j^{(1)} > 0$, then $x_\theta(0) > 0$. So, defining $\theta_j^A(1) = \theta_j^B(1) := c \in \mathbb{R}, \forall j$, we have:

$x_\theta(1) = x_\theta(2) = 0; x_\theta(0) > 0,$

that is, $x_\theta \geq 0 \nRightarrow x_\theta = M\theta = 0$

So, θ is an arbitrage oportunity in $\mathcal{M}$. Then, we must have $x_\theta(0) = \sum_{j=0}^{J} \theta_j \cdot q_j = 0$. So, we proved:

$\mathcal{M}$ arbitrage-free $\Rightarrow \mathcal{M}^1, \mathcal{M}^2$ arbitrage-free.

($\Leftarrow$) Now, suppose $\mathcal{M}^1, \mathcal{M}^2$ arbitrage-free.

If $\exists\, \theta \in \Theta$ such that $x_\theta = M\theta > 0$, then:

$\sum_{j=0}^{J} d_j(2) \cdot [\theta_j^A(0) - \theta_j^B(0)] + \sum_{j=0}^{J} d_j(2) \cdot [\theta_j^A(1) - \theta_j^B(1)] > 0$

Then, we have ne of the following four cases:

1. $\sum_{j=0}^{J} d_j(2)[\theta_j^A(0) - \theta_j^B(0)] < 0 < \sum_{j=0}^{J} d_j(2)[\theta_j^A(1) - \theta_j^B(1)]$;
2. $\sum_{j=0}^{J} d_j(2)[\theta_j^A(1) - \theta_j^B(1)] < 0 < \sum_{j=0}^{J} d_j(2)[\theta_j^A(0) - \theta_j^B(0)] \Rightarrow x_\theta^{(2)} > 0$;
3. $\sum_{j=0}^{J} d_j(2)[\theta_j^A(0) - \theta_j^B(0)] > \sum_{j=0}^{J} d_j(2)[\theta_j^A(1) - \theta_j^B(1)] > 0 \Rightarrow x_\theta^{(2)} > 0$;
4. $\sum_{j=0}^{J} d_j(2)[\theta_j^A(1) - \theta_j^B(1)] > \sum_{j=0}^{J} d_j(2)[\theta_j^A(0) - \theta_j^B(0)] > 0 \Rightarrow x_\theta^{(2)} > 0$

The first condition implies that $\sum_{j=0}^{J} d_j(2)[\theta_j^A(1) - \theta_j^B(1)] - \sum_{j=0}^{J} d_j(2)[\theta_j^A(0) - \theta_j^B(0)] > 0$

Define ${}^*\theta_j^A(0) := \theta_j^B(0), {}^*\theta_j^B(0) := \theta_j^A(0)$.

Then

$\sum_{j=0}^{J} d_j(2)[{}^*\theta_j^A(0) - {}^*\theta_j^B(0)] = \sum_{j=0}^{J} d_j(2)[\theta_j^B(0) - \theta_j^A(0)] = -\sum_{j=0}^{J} d_j(2)[\theta_j^A(0) - \theta_j^B(0)] > 0$

$x_{^*\theta}(0) = -q(0) \cdot [\theta^B(0) - \theta^A(0)] = q(0) \cdot [\theta^A(0) - \theta^B(0)] = -x_\theta(0) < 0$

Let ${}^0\theta_j^A(0), {}^0\theta_j^B(0) \in [0, +\infty)$ such that:

- $-\sum_{j=0}^{J} q_j(0)[{}^*\theta_j^A(0) - {}^*\theta_j^B(0)] - \sum_{j=0}^{J} q_j(0)[{}^0\theta_j^A(0) - {}^0\theta_j^B(0)] = -\sum_{j=0}^{J} q_j(0)[(\theta_j^B(0) + {}^0\theta_j^A(0)) - (\theta_j^A(0) + {}^0\theta_j^B(0))] > 0$ and
- $\sum_{j=0}^{J} d_j(2)[{}^0\theta_j^A(0) - {}^0\theta_j^B(0)] > 0 \Rightarrow \sum_{j=0}^{J} d_j(2)[(\theta_j^B(0) + {}^0\theta_j^A(0)) - (\theta_j^A(0) + {}^0\theta_j^B(0))] = \sum_{j=0}^{J} d_j(2)[\theta_j^B(0) - \theta_j^A(0)] + \sum_{j=0}^{J} d_j(2)[{}^0\theta_j^A(0) - {}^0\theta_j^B(0)] = \sum_{j=0}^{J} d_j(2)[{}^*\theta_j^A(0) - {}^*\theta_j^B(0)] + \sum_{j=0}^{J} d_j(2)[{}^0\theta_j^A(0) - {}^0\theta_j^B(0)] > 0$

So, defining $(\tilde{\theta}^A; \tilde{\theta}^B) := ({}^*\theta_j^A + {}^0\theta_j^A; {}^*\theta_j^B + {}^0\theta_j^B) = (\theta^B + {}^0\theta^A; \theta^A + {}^0\theta^B)$, we have:

$x^{(2)}_{{}^0\theta} > 0 \Rightarrow {}^0\theta$ is an arbitrage-oportunity in $\mathcal{M}^2$,

contradicting the hypothesis that $\mathcal{M}^2$ is arbitrage-free. Then, we must have $x_\theta = M\theta = 0 \ \forall \ \theta \in \Theta$ such that $x_\theta = M\theta \geq 0$, that is, $\mathcal{M}$ is arbitrage-free. So, we proved:

$\mathcal{M}^1, \mathcal{M}^2$ arbitrage-free $\Rightarrow \mathcal{M}$ arbitrage-free. □

Proof. of Lemma 1.7
Let $x = (x(1), x(2)), y = (y(1); y(2)) \in G_C$. Then, by Lemma 1.5, we have $x(1), y(1) \in F_{C_1}, x(2), y(2) \in F_{C_2}$.

As G_C is a linear subspace then $x + y = (x(1) + y(1); x(2) + y(2)) \in G_C$. So, we can avaliate $C(\cdot;\cdot)$ in $x + y$.

Also, by Lemma 3 in Araujo, Chateauneuf, and Faro [2012], we know that $x(1) + y(1) \in F_{C_1}, x(2) + y(2) \in F_{C_2}$. So, we can avaliate $C_1(\cdot)$ in $x(1) + y(1)$ and $C_2(\cdot)$ in $x(2) + y(2)$.

By Lemma 21 in Araujo, Chateauneuf, and Faro [2012] we know that $C_1|_{F_{C_1}}$ and $C_2|_{F_{C_2}}$ are both linear. Moreover they have the following representation:

$$C_1(x) = \max_{\mathbb{P}_1 \in \mathcal{Q}_{C_1}} \mathbb{E}_{\mathbb{P}_1} \ \forall \ x \in \mathbb{R}^{S_1},$$

where $\mathcal{Q}_{C_1} = \{\mathbb{P}_1 \in \Delta^{S_1-1}_{++}; \mathbb{E}_{\mathbb{P}_1}[x] = C_1(x) \ \forall \ x \in F_{C_1}\}$

And

$$C_2(y) = \max_{\mathbb{P}_2 \in \mathcal{Q}_{C_2}} \mathbb{E}_{\mathbb{P}_2} \ \forall \ y \in \mathbb{R}^{S_2},$$

where $\mathcal{Q}_{C_2} = \{\mathbb{P}_2 \in \Delta^{S_2-1}_{++}; \mathbb{E}_{\mathbb{P}_2}[y] = C_2(y) \ \forall \ y \in F_{C_2}\}$

First, take $x = (x(1), x(2)) \in G_C$. Then $x(1) \in F_{C_1}, x(2) \in F_{C_2}$.

$$\begin{aligned}
C(x) =& C(x(1), x(2)) \\
&= C((x(1); 0) + (0; x(2))) \\
&\leq C((x(1), 0)) + C((0, x(2))) \\
&= c_1 C_1(x(1)) + c_2 C_2(x(2)) \\
&= c_1 \mathbb{E}_{\mathbb{P}_1}[x(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[x(2)] \ \forall \ \mathbb{P}_1 \in \mathcal{Q}_{C_1}, \mathbb{P}_2 \in \mathcal{Q}_{C_2} \\
&\leq \max_{\mathbb{P}_1 \in \mathcal{Q}_{C_1}, \mathbb{P}_2 \in \mathcal{Q}_{C_2}} \{c_1 \mathbb{E}_{\mathbb{P}_1}[x(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[x(2)]\}
\end{aligned}$$

Suppose that

$$C(x) < \max_{\mathbb{P}_1 \in \mathcal{Q}_{C_1}, \mathbb{P}_2 \in \mathcal{Q}_{C_2}} \{c_1 \mathbb{E}_{\mathbb{P}_1}[x(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[x(2)]\}$$

Then

$$
\begin{aligned}
C_1(x(1)) &= \left(\frac{1}{c_1}\right) C((x(1),0)) \\
&= C((x(1),x(2)) + (0;-x(2))) \\
&\leq C(x(1),x(2)) + C(0;-x(2)) \\
&= C(x(1),x(2)) + C_2(-x(2))
\end{aligned}
$$

$$< \max_{\mathbb{P}_1 \in \mathcal{Q}_{C_1}, \mathbb{P}_2 \in \mathcal{Q}_{C_2}} \{c_1 \mathbb{E}_{\mathbb{P}_1}[x(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[x(2)]\} + C_2(-x(2))$$

As $-x(2) \in F_{C_2}$ because $x(2) \in F_{C_2}$ and F_{C_2} is as linear subspace, then the above inequality implies

$$
\begin{aligned}
C_1(x(1)) &< \max_{\mathbb{P}_1 \in \mathcal{Q}_{C_1}, \mathbb{P}_2 \in \mathcal{Q}_{C_2}} \{c_1 \mathbb{E}_{\mathbb{P}_1}[x(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[x(2)]\} + \mathbb{E}_{\mathbb{P}_2}[-x(2)] \\
&= \max_{\mathbb{P}_1 \in \mathcal{Q}_{C_1}, \mathbb{P}_2 \in \mathcal{Q}_{C_2}} \{c_1 \mathbb{E}_{\mathbb{P}_1}[x(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[x(2)]\} - \mathbb{E}_{\mathbb{P}_2}[x(2)] \\
&= \max_{\mathbb{P}_1 \in \mathcal{Q}_{C_1}, \mathbb{P}_2 \in \mathcal{Q}_{C_2}} \{c_1 \mathbb{E}_{\mathbb{P}_1}[x(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[x(2)]\} - C_2(x(2)) \\
&= \max_{\mathbb{P}_1 \in \mathcal{Q}_{C_1}, \mathbb{P}_2 \in \mathcal{Q}_{C_2}} \{C_1(x(1)) + C_2(x(2))\} - C_2(x(2)) \\
&= C_1(x(1)) + C_2(x(2)) - C_2(x(2)) \\
&= C_1(x(1)),
\end{aligned}
$$

because $x(2) \in F_{C_2} \Rightarrow C_2(x(2)) - C_2(x(2)) = 0$.

So, we arrive at $C_1(x(1)) < C_1(x(1))$, an absurd. Then we must have

$$
\begin{aligned}
C(x) &= \max_{\mathbb{P}_1 \in \mathcal{Q}_{C_1}, \mathbb{P}_2 \in \mathcal{Q}_{C_2}} \{c_1 \mathbb{E}_{\mathbb{P}_1}[x(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[x(2)]\} \\
&= \max_{\mathbb{P}=(\mathbb{P}_1,\mathbb{P}_2) \in \mathcal{Q}_C} \max_{\mathbb{P}_1 \in \mathcal{Q}_{C_1}, \mathbb{P}_2 \in \mathcal{Q}_{C_2}} \{c_1 \mathbb{E}_{\mathbb{P}_1}[x(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[x(2)]\} \ \forall \ x = (x(1),x(2)) \in G_C,
\end{aligned}
$$

where

$$\mathcal{Q}_C \coloneqq \{\mathbb{P} = (\mathbb{P}_1; \mathbb{P}_2) \in \Delta_{++}^{S_1-1} \times \Delta_{++}^{S_2-1}; c_1 \mathbb{E}_{\mathbb{P}_1}[x(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[x(2)] = C(x) \ \forall \ x = (x(1),x(2)) \in G_C)\},$$

$c_1, c_2 > 0$ constants

As $C_t|_{F_{C_t}}, t = 1,2$ are both strictly positive, then for all $x \in G_C$ we have

$$
\begin{aligned}
C(x) &= c_1 \mathbb{E}_{\mathbb{P}_1}[x(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[x(2)] \\
&= c_1 C_1(x(1)) + c_2 C_2(x(2)) \\
&> 0 + 0 \\
&= 0 \ \forall \ x \in G_C,
\end{aligned}
$$

that is, $C|_{G_C}$ is strictly positive.

Let us check that $C|_{G_C}$ is linear.

Let $\bar{x}, \bar{y} \in G_C$. Then $\bar{x}+\bar{y} \in G_C, \bar{x}(1), \bar{y}(1) \in F_{C_1}, \bar{x}(2), \bar{y}(2) \in F_{C_2}, \bar{x}(1)+\bar{y}(1) \in F_{C_1}, \bar{x}(2)+\bar{y}(2) \in F_{C_2}$ and

$$\begin{aligned} C(\bar{x}+\bar{y}) &= \max_{\mathbb{P}_1 \in \mathcal{Q}_{C_1}, \mathbb{P}_2 \in \mathcal{Q}_{C_2}} \{c_1 \mathbb{E}_{\mathbb{P}_1}[\bar{x}(1)+\bar{y}(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[\bar{x}(2)+\bar{y}(2)]\} \\ &= \max_{\mathbb{P}_1 \in \mathcal{Q}_{C_1}, \mathbb{P}_2 \in \mathcal{Q}_{C_2}} \{c_1 \mathbb{E}_{\mathbb{P}_1}[\bar{x}(1)] + c_1 \mathbb{E}_{\mathbb{P}_1}[\bar{y}(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[\bar{x}(2)] + c_2 \mathbb{E}_{\mathbb{P}_2}[\bar{y}(2)]\} \end{aligned}$$

As $\mathbb{E}_{\mathbb{P}_t}[x] = C_t(x)\ \forall\ x \in F_{C_t}, t = 1, 2$ and $\bar{x}(1), \bar{y}(1) \in F_{C_1}, \bar{x}(2), \bar{y}(2) \in F_{C_2}$ then

$$\begin{aligned} C(\bar{x}+\bar{y}) &= \max_{\mathbb{P}_1 \in \mathcal{Q}_{C_1}, \mathbb{P}_2 \in \mathcal{Q}_{C_2}} \{c_1 C_1(\bar{x}(1)) + c_1 C_1(\bar{y}(1)) + c_2 C_2(\bar{x}(2)) + c_2 C_2(\bar{y}(2))\} \\ &= c_1 C_1(\bar{x}(1)) + c_1 C_1(\bar{y}(1)) + c_2 C_2(\bar{x}(2)) + c_2 C_2(\bar{y}(2)) \\ &= (c_1 \mathbb{E}_{\mathbb{P}_1}[\bar{x}(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[\bar{x}(2)]) + (c_1 \mathbb{E}_{\mathbb{P}_1}[\bar{y}(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[\bar{y}(2)]) \end{aligned}$$

But, as $\bar{x} = (\bar{x}(1), \bar{x}(2)), \bar{y} = (\bar{y}(1), \bar{y}(2)) \in G_C$ and $\mathbb{P} := (\mathbb{P}_1, \mathbb{P}_2) \in \mathcal{Q}_{C_1} \times \mathcal{Q}_{C_2} \Rightarrow \mathbb{P} \in \mathcal{Q}_C$, the above inequality is equal to

$$\begin{aligned} C(\bar{x}+\bar{y}) &= C(\bar{x}(1); \bar{x}(2)) + C(\bar{y}(1); \bar{y}(2)) \\ &= C(\bar{x}) + C(\bar{y}) \end{aligned}$$

Now, since $C|_{G_C}$ is linear and strictly positive, then, through the indentification $\mathbb{R}^{S_1} \times \mathbb{R}^{S_2} \simeq \mathbb{R}^{S_1+S_2}$, by theorem 6 in Clark [1993] we have that there exists a linear and strictly positive extension h of $C|_{G_C}$ to $\mathbb{R}^{S_1+S_2}$. Then, h is a linear and strictly positive extension of $C|_{G_C}$ to $\mathbb{R}^{S_1} \times \mathbb{R}^{S_2}$.

Finally, consider a base $\{b_0, ..., b_J\}$ of G_C with $b_0 = \mathbb{1}_{\{\Omega\}}$ (because $\mathbb{1}_{\{\Omega\}} \in G_C$) and take $y = (y(1); y(2)) = \sum_{j=0}^J \lambda_j b_j \in B(\mathcal{G} \backslash G_C)$.

As $h(\mathbb{1}_{\{\Omega\}}) = 1$, $\exists \mathbb{P} = (\mathbb{P}_1^*; \mathbb{P}_2^*)$ such that $\mathbb{E}_{\mathbb{P}_1^*}[x(1)] + \mathbb{E}_{\mathbb{P}_2^*}[x(2)] = h(x) = C(x)\ \forall\ x = (x(1), x(2)) \in \mathbb{R}^{S_1} \times \mathbb{R}^{S_2}$. In particular, $\mathbb{E}_{\mathbb{P}_1^*}[b_j(1)] + \mathbb{E}_{\mathbb{P}_2^*}[b_j(2)] = h(b_j) = C(b_j)\ \forall\ j$. Then

$$\begin{aligned} C(y) = h(y) &= h(\sum_{j=0}^J \lambda_j b_j) \\ &= \sum_{j=0}^J \lambda_j h(b_j) \\ &= \sum_{j=0}^J \{\lambda_j \mathbb{E}_{\mathbb{P}_1^*}[b_j(1)] + \lambda_j \mathbb{E}_{\mathbb{P}_2^*}[b_j(2)]\} \\ &= \max_{\mathbb{P}=(\mathbb{P}_1;\mathbb{P}_2) \in \mathcal{Q}_C} \{\sum_{j=0}^J [\lambda_j \mathbb{E}_{\mathbb{P}_1}[b_j(1)] + \lambda_j \mathbb{E}_{\mathbb{P}_2}[b_j(2)]]\} \\ &= \max_{\mathbb{P}=(\mathbb{P}_1;\mathbb{P}_2) \in \mathcal{Q}_C} \{\sum_{j=0}^J [\mathbb{E}_{\mathbb{P}_1}[\lambda_j b_j(1)] + \mathbb{E}_{\mathbb{P}_2}[\lambda_j b_j(2)]]\} \\ &= \max_{\mathbb{P}=(\mathbb{P}_1;\mathbb{P}_2) \in \mathcal{Q}_C} \{\mathbb{E}_{\mathbb{P}_1}[y(1)] + \mathbb{E}_{\mathbb{P}_2}[y(2)]\}, \end{aligned}$$

where

$$\mathcal{Q}_C := \max_{\mathbb{P}_1 \in \mathcal{Q}_{C_1}, \mathbb{P}_2 \in \mathcal{Q}_{C_2}} \{c_1 \mathbb{E}_{\mathbb{P}_1}[x(1)] + c_2 \mathbb{E}_{\mathbb{P}_2}[x(2)] = C(x)\ \forall\ x = (x(1), x(2)) \in G_C\},$$

$c_1, c_2 > 0$ constants. □

Proof. *of Lemma 1.3*
Let $x = (x(1), ..., x(T)) \in \mathcal{L}$. Suppose $x \notin L_C$. Then there exists $y \in \mathbb{R}^L, L = \sum_{t=1}^T S_t$ such that $y_{s^*} > x_{s^*}$, $y_s \geq x_s \forall s \in \mathcal{S} = \bigcup_{t=1}^T \mathcal{S}$ and $C(y) \leq C(x)$.

Through the identification $\mathbb{R}^L \cong \times_{t=1}^T \mathbb{R}^{S_t}$ we can see $y = (y_1, ..., y_L) \in \mathbb{R}^L$ as $y = (y(1); ...; y(T)) \in \times_{t=1}^T \mathbb{R}^{S_t}$.

Moreover, define

$$t^* := t \in \{1, ..., T\} \text{ such that } s^* \in \mathcal{S}_{t^*}$$

So, we have

$$y(t^*) > x(t^*) \text{ with } C(y) \leq C(x),$$

contradicting the hypothesis that $x \in L_C^{t^*}$. So, we must have $x \in L_C$. This concludes the proof of the inclusion $\mathcal{L}_C \subset L_C$.

Now, take $x \in L_C$ and suppose $x \notin \mathcal{L}_C$. Then there exists $\bar{t} \in \{1, ..., T\}$ such that $x \notin L_C^{\bar{t}}$.

Then there exists $y \in \times_{t=1}^T \mathbb{R}^{S_t}$ such that

$$y(\bar{t}) > x(\bar{t}), C(y) \leq C(x)$$

Again through the identification $\mathbb{R}^L \cong \times_{t=1}^T \mathbb{R}^{S_t}$ we can see $y = (y(1); ...; y(T)) \in \times_{t=1}^T \mathbb{R}^{S_t}$ as $y = (y_1, ..., y_L) \in \mathbb{R}^L$.

Then, $y > x$ and $C(y) \leq C(x)$, contradicting the hypothesis that $x \in L_C$. So we must have $x \in \mathcal{L}_C$. This concludes the proof of the inclusion $L_C \subset \mathcal{L}_C$. □

Proof. *of Lemma 1.4*
Let $x = (x(1), x(2)) \in G_C, \lambda > 0$ scalar.

By positive homogeneity of $C(\cdot, \cdot)$ we have

$$\begin{aligned} C(\lambda(x(1); -x(2))) + C(\lambda(-x(1); x(2))) &= \lambda C((x(1), -x(2))) + \lambda C((-x(1), x(2))) \\ &= \lambda[C((x(1), -x(2))) + C((-x(1), x(2)))] \\ &= \lambda \times 0 \\ &= 0 \end{aligned} \tag{A.1}$$

Then we have $\lambda x \in G_C$ for $\lambda > 0$.

Now, if $\lambda < 0$ then $-x \in G_C$ and $-\lambda > 0$, implying that, by the positive homogeneity, that $\lambda x = (-\lambda)(-x) \in G_C$.

So, we conclude that $\lambda x \in G_C \ \forall \ \lambda \in \mathbb{R}$.

Now, let $y = (y(1); y(2)), z = (z(1); z(2)) \in G_C$. Then

$$\begin{cases} C((y(1); -y(2))) + C((-y(1); y(2))) = 0 \\ C((z(1); -z(2))) + C((-z(1); z(2))) = 0 \end{cases}$$

By sublinearity we get

$$\begin{aligned}C((y(1)+z(1);-y(2)-z(2))) &= C((y(1);-y(2))+(z(1);-z(2)))\\ &\le C((y(1);-y(2)))+C(z(1);-z(2))\end{aligned}$$

And

$$\begin{aligned}C((-y(1)-z(1);y(2)+z(2))) &= C((-y(1);y(2))+(-z(1);z(2)))\\ &\le C((-y(1);y(2)))+C(-z(1);z(2))\end{aligned}$$

Summing both inequalities we have

$C((y(1)+z(1);-y(2)-z(2)))+C((-y(1)-z(1);y(2)+z(2))) \le C((y(1);-y(2)))+C(z(1);-z(2))+$
$C((-y(1);y(2)))+C(-z(1);z(2)) = [C((y(1);-y(2)))+C((-y(1);y(2)))]+$
$[C(z(1);-z(2))+C(-z(1);z(2))] = 0+0 = 0$

By other side we have

$C((-y(1)-z(1);y(2)+z(2)))+C((y(1)+z(1);-y(2)-z(2))) \ge C((-y(1)-z(1));y(2)+z(2))+$
$(y(1)z(1);-y(2)-z(2)) = C(0;0) = 0$

So we arrived at

$$0 \ge C((y(1)+z(1);-y(2)-z(2)))+C((y(1)+z(1);-y(2)-z(2))) \le 0$$

$\therefore C((y(1)+z(1);-y(2)-z(2)))+C((y(1)+z(1);-y(2)-z(2))) = 0$, that is, $y+z \in G_C$. □

Proof. of Lemma 1.5
($\Rightarrow$) Let $x = (x(1), x(2)) \in G_C$. Then

$$C((x(1),-x(2)))+C((-x(1),x(2))) = 0$$

As $C_1(x(1)) = C(x(1);0)$, we can rewrite it as

$$C_1(x(1)) = C((x(1);-x(2))+(0;x(2)))$$

As $C(\cdot,\cdot)$ is a pricing rule, it is sublinear, so we get

$$C_1(x(1)) \le C((x(1),-x(2)))+C((0,x(2))) \tag{A.2}$$

Analogously, we can rewrite $C_1(-x(1)) = C((-x(1),0))$ as

$$C_1(-x(1)) = C((-x(1);x(2))+(0;-x(2))),$$

implying that

$$C_1(-x(1)) \le C((-x(1),x(2)))+C((0,-x(2))) \tag{A.3}$$

Summing equations (A.2) and (A.3) we obtained

$$\begin{aligned}&C_1(x(1))+C_1(-x(1)) \le [C((x(1),-x(2)))+C((0,x(2)))]+[C((-x(1),x(2)))+C((0,-x(2)))]\\ \Rightarrow\ &C_1(x(1))+C_1(-x(1)) \le [C((x(1),-x(2)))+C((-x(1),x(2)))]+[C((0,x(2)))+C((0,-x(2)))]\\ &\quad = [C((x(1),-x(2)))+C((-x(1),x(2)))]+[C_2(x(2))+C_2(-x(2))]\end{aligned}$$

As $x = (x(1), x(2)) \in G_C$, $C((x(1), -x(2))) + C((-x(1), x(2)))$, implying that

$$C_1(x(1)) + C_1(-x(1)) \leq C_2(x(2)) + C_2(-x(2)) \tag{A.4}$$

By other side, we can rewrite $C_2(x(2)) = C((0, x(2)))$ as

$$\begin{aligned} C_2(x(2)) &= C((-x(1); x(2)) + (x(1); 0)) \\ &\leq C((-x(1), x(2))) + C(x(1); 0) \\ &= C((-x(1), x(2))) + C_1(x(1)) \end{aligned} \tag{A.5}$$

And

$$\begin{aligned} C_2(-x(2)) &= C((x(1); -x(2)) + (-x(1); 0)) \\ &\leq C((x(1), -x(2))) + C(-x(1); 0) \\ &= C((x(1), -x(2))) + C_1(-x(1)) \end{aligned} \tag{A.6}$$

Summing (A.5) and (A.6) we get

$$C_2(x(2)) + C_2(-x(2)) \leq [C((-x(1), x(2))) + C((x(1), -x(2)))] + [C_1(x(1)) + C_1(-x(1))]$$

As $x = (x(1), x(2)) \in G_C$, $C((-x(1), x(2))) + C((x(1), -x(2))) = 0$. Then the inequality above implies that

$$C_1(x(1)) + C_1(-x(1)) \geq C_2(x(2)) + C_2(-x(2)) \tag{A.7}$$

Equations (A.4) and (A.7) together implies that

$$C_1(x(1)) + C_1(-x(1)) = C_2(x(2)) + C_2(-x(2))$$

By the sublinearity of the pricing rules, we know that

$$C_t(x(t)) + C_t(-x(t)) \geq 0, t = 1, 2$$

As we are supposing $x(1) \notin F_{C_1}, x(2) \notin F_{C_2}$, we are supposing

$$C_1(x(1)) + C_1(-x(1)) = C_2(x(2)) + C_2(-x(2)) > 0,$$

that is,

$$C((x(1), 0)) + C((-x(1), 0)) = C((0, x(2))) + C((0, -x(2))) > 0,$$

But, by the sublinearity of $C(\cdot, \cdot)$, we have

$$\begin{aligned} 0 &< C((0, -x(2))) + C((0, x(2))) \\ &= [C((x(1); -x(2)) + (-x(1); 0))] + [C((-x(1); x(2)) + (x(1); 0))] \\ &\leq [C((x(1), -x(2))) + C((-x(1), 0))] + [C((-x(1), x(2))) + C((x(1), 0))] \\ &= [C((x(1), -x(2))) + C((-x(1), x(2)))] + [C((-x(1), 0)) + C((x(1), 0))] \end{aligned}$$

Then

$$\begin{aligned} C((x(1), -x(2))) + C((-x(1), x(2))) &> -[C((-x(1), 0)) + C((x(1), 0))] \\ &> -0 \\ &= 0 \end{aligned}$$

So, we obtained

$$C((x(1), -x(2))) + C((-x(1), x(2))) > 0,$$

contradicting the hypothesis $x = (x(1), x(2)) \in G_C$. Then we must have $x(1) \in F_{C_1}$ and $x(2) \in F_{C_2}$.

($\Leftarrow$) Suppose $x(1) \in F_{C_1}$ and $x(2) \in F_{C_2}$. Then

$$C_1(x(1)) + C_1(-x(1)) = 0;$$
$$C_2(x(2)) + C_2(-x(2)) = 0$$

Summing both equations we have

$$C_1(x(1)) + C_1(-x(1)) + C_2(x(2)) + C_2(-x(2)) = 0 \tag{A.8}$$

We want to show that

$$C((x(1), -x(2))) + C((-x(1), x(2))) = 0$$

We know by the sublinearity of $C(\cdot, \cdot)$ that

$$C((x(1), -x(2))) + C((-x(1), x(2))) \geq 0$$

We can rewrite $C((x(1), -x(2))) + C((-x(1), x(2)))$ as

$$\begin{aligned} C((x(1), -x(2))) + C((-x(1), x(2))) &= [C((x(1); 0) + (0; -x(2)))] + [C((-x(1); 0) + (0; x(2)))] \\ &\geq [C((x(1), 0)) + C((0, -x(2)))] + [C((-x(1), 0)) + C((0, x(2)))] \\ &= [C((x(1), 0)) + C((-x(1), 0))] + [C((0, -x(2))) + C((0, x(2)))] \end{aligned}$$

If $C((x(1), -x(2))) + C((-x(1), x(2))) > 0$ we have

$$[C((x(1), 0)) + C((-x(1), 0))] + [C((0, -x(2))) + C((0, x(2)))] > 0,$$

contradicting (A.8). So, we must have

$$C((x(1), -x(2))) + C((-x(1), x(2))) = 0,$$

that is, $x = (x(1), x(2)) \in G_C$. □

Proof. of Corollary 1.1
We know from Lemma 1.5 that $x = (x(1), x(2)) \in G_C$ if and only if $x(1) \in F_{C_1}, x(2) \in F_{C_2}$.

As $C_1 : \mathbb{R}^{S_1} \to \mathbb{R}, C_2 : \mathbb{R}^{S_2} \to \mathbb{R}$ are the super-replication at minimum cost pricing rule of the markets $\mathcal{M}^1, \mathcal{M}^2$, respectively, We know by Araujo, Chateauneuf, and Faro [2012] that $x(1) \in F_{C_1}$ if and only if $x(1) \in L_{C_1}$ and $x(2) \in F_{C_2}$ if and only if $x(2) \in L_{C_2}$.

So, we have that $x = (x(1), x(2)) \in G_C$ if and only if $x \in L_C$. □

Proof. *of Lemma 1.6*
Let $C : B(\mathcal{G}) \to \mathbb{R}$ the super-replication at minimum cost pricing rule in the market

$$\mathcal{M} = \left(x_j, \{q_j(t)\}_{t=0}^2, \{d_j(t)\}_{t=0}^2, 0 \leq j \leq J\right),$$

that is,

$$\begin{aligned} C((x(1);(x(2)))) = \pi((x(1),x(2))) = \inf_{\theta \in \Theta} & -x_\theta(0) \\ \text{s.t. } & x_\theta(1) \geq x(1); \\ & x_\theta(2) \geq x(2) \end{aligned}$$

The super-replication at minimum cost pricing rule of the market $\mathcal{M}^2$ is given by

$$\begin{aligned} \pi^2(x) = \inf_{\theta \in \Theta} & -x_\theta(0) \\ \text{s.t. } & x_\theta(2) \geq x \end{aligned}$$

As

$$\begin{aligned} C_2(x(2)) = C((0,x(2))) = \inf_{\theta \in \Theta} & -x_\theta(0) \\ \text{s.t. } & x_\theta(1) \geq x(1) = 0; \\ & x_\theta(2) \geq x(2), \end{aligned}$$

the constraint $x_\theta(1) \geq 0$ is unecessary, because $x = (x(1), x(2)) \geq 0 \ \forall \ x \in B(\mathcal{G})$. This implies that $C_2(x(2)) = \pi^2(x(2)) \forall x(2) \in \mathbb{R}^{S_2}$, that is, $C_2(\cdot)$ is the super-replication at minimum cost pricing rule of the market $\mathcal{M}^2$.

Now, let us verify for $C_1(\cdot)$.

The super-replication at minimum cost pricing rule of the market $\mathcal{M}^1$ is given by

$$\begin{aligned} \pi^1(x) = \inf_{\theta \in \Theta} & -x_\theta(0) \\ \text{s.t. } & d(1) \cdot [\theta^A(0) - \theta^B(0)] \geq x, \end{aligned}$$

As $C_1(x(1)) = C((x(1), 0))$, we have

$$\begin{aligned} C_1(x(1)) = \inf_{\theta \in \Theta} & -x_\theta(0) \\ \text{s.t. } & x_\theta(1) \geq x(1) \\ & x_\theta(2) \geq 0 \end{aligned}$$

As $x \geq 0 \ \forall \ x = (x(1), x(2)) \in B(\mathcal{G})$, the restriction $x_\theta(2) \geq 0$ is irrelevant. So, we get

$$\begin{aligned} C_1(x(1)) = \inf_{\theta \in \Theta} & -x_\theta(0) \\ \text{s.t. } & x_\theta(1) \geq x(1) \end{aligned}$$

Substituting $x_\theta(1) = d(1) \cdot [\theta^A(0) - \theta^B(0)] - q(1) \cdot [\theta^A(1) - \theta^B(1)]$ we get:

$$\begin{aligned} C_1(x(1)) = \inf_{\theta \in \Theta} & -x_\theta(0) \\ \text{s.t. } & d(1) \cdot [\theta^A(0) - \theta^B(0)] \geq x(1) + q(1) \cdot [\theta^A(1) - \theta^B(1)] \\ & x_\theta(2) = d(2) \cdot \sum_{\tau=0}^{1} [\theta^A(\tau) - \theta^B(\tau)] \geq 0 \end{aligned}$$

In order to have $C_1(x(1)) = \pi^1(x(1))$, we need that $q(1) \cdot [\theta^A(1) - \theta^B(1)] \geq 0$, implying that the restriction $d(1) \cdot [\theta^A(0) - \theta^B(0)]$ in the problem defining $\pi^1(\cdot)$ is satisfied.

Suppose $q(1) \cdot [\theta^A(1) - \theta^B(1)] < 0$. Then exists $j^* \in \{0, ..., J\}$ such that $S_{j^*}[\theta^A_{j^*}(1) - \theta^B_{j^*}(1)] < 0$ a.s.. As $S_{j^*}(1)$ a.s., we have

$$S_{j^*}[\theta^A_{j^*}(1) - \theta^B_{j^*}(1)] < 0 \Leftrightarrow \theta^A_{j^*}(1) - \theta^B_{j^*}(1) < 0,$$

implying that

$$d_{j^*2}(2)[\theta^A_{j^*}(1) - \theta^B_{j^*}(1)] < 0 \text{ a.s},$$

because $d_{j^*}(2) > 0$ a.s..

Define

$$J^* := \{j \in \{0, ..., J\}; q_j(1)[\theta^A_j(1) - \theta^B_j(1)] < 0 \text{ a.s.}\}$$

And

$$\bar{J} := J \backslash J^*$$

Then $q(1) \cdot [\theta^A(1) - \theta^B(1)] < 0 \Leftrightarrow \mid \sum_{j \in J^*} q_j(1)[\theta^A_j(1) - \theta^B_j(1)] \mid > \sum_{j \in \bar{J}} q_j(1)[\theta^A_j(1) - \theta^B_j(1)]$.

But if this happens, as $\mathcal{M}$ is arbitrage-free, we have

$$\sum_{j \in J^*} d_j(2)[\theta^A(1) - \theta^B(1)] + \sum_{j \in \bar{J}} d_j(2)[\theta^A(1) - \theta^B(1)] < 0,$$

which implies, together with $\mathcal{M}$ being arbitrage-free, that $\sum_{j \in J} d_j(2) \cdot [\theta^A(0) - \theta^B(0)] < 0$, contradicting the restriction $\sum_{j \in J} d_j(2) \cdot [\theta^A(0) - \theta^B(0)] < 0$ in the problem defining $C_2(x(2)) = C((0, x(2)))$. So, we must have $q(1) \cdot [\theta^A(1) - \theta^B(1)] \geq 0$ a.s., implying that

$$d(1) \cdot [\theta^A(0) - \theta^B(0)] \geq x(1) + q(1) \cdot [\theta^A(1) - \theta^B(1)] \geq x(1),$$

that is, the restriction of the problem defining $C_1(\cdot)$ is the restriction of the problem defining $\pi^1(\cdot)$. So

$$C_1(x) = \pi^1(x) \ \forall \ x \in \mathbb{R}^{S_1},$$

that is, $C_1(\cdot)$ is the super-replication at minimum cost pricing rule of the financial market $\mathcal{M}^1$.

Proof. of Lemma 1.8
Let $x = (x(1), x(2)) \in B(\mathcal{G})$.

For $\mathbb{P} = (\mathbb{P}_1, \mathbb{P}_2) \in \mathcal{Q}_C$ the equality in the set $\mathcal{Q}_C$ is

$\mathbb{E}_{\mathbb{P}_1}[x(1)] + \mathbb{E}_{\mathbb{P}_2}[x(2)] = c_1 \sum_{m=1}^{S_1} \mathbb{P}_{1,m} x_m(1) + c_2 \sum_{n=1}^{S_2} \mathbb{P}_{2,n} x_n(2)$

Defining

$$\begin{cases} \phi_m := c_1 \mathbb{P}_{1,m} & 1 \leq m \leq S_1 \\ \rho_n := c_2 \mathbb{P}_{2,n} & 1 \leq n \leq S_2 \end{cases}$$

And then

$$\psi_i := \begin{cases} \phi_i & 1 \leq i \leq S_1 \\ \rho_{i-n} & S_1 + 1 \leq i \leq S_1 + S_2 \end{cases},$$

As $c_1 > 0, c_2 > 0$, we have $\psi \in \mathbb{R}_{++}^{S_1+S_2}$ and through the identification $\mathbb{R}^{S_1} \times \mathbb{R}^{S_2} \simeq \mathbb{R}^{S_1+S_2}$, writing $x = (x(1), x(2)) \in B(\mathcal{G})$ as $x = (x_1, ..., x_{S_1+S_2})$ we can rewrite the above equality as

$$\begin{aligned} \mathbb{E}_{\mathbb{P}_1}[x(1)] + \mathbb{E}_{\mathbb{P}_2}[x(2)] &= \sum_{m=1}^{S_1} \phi_m x_m(1) + \sum_{n=1}^{S_2} \rho_n x_n(2) \\ &= \sum_{i=1}^{S_1} \psi_i x_i + \sum_{i=S_1+1}^{S_1+S_2} \psi_i x_i \\ &= \sum_{i=1}^{S_1+S_2} \psi_i x_i \\ &= \psi \cdot x \end{aligned}$$

Then

$$C(x) = \max_{\psi \in \Psi} \psi \cdot x = \max_{(1,\tilde{\psi})^T \in \mathcal{D}},$$

where

$$\Psi = \{\psi \in \mathbb{R}_{++}^{S_1+S_2}; \psi \cdot x = C(x) \ \forall \ x \in G_C\}$$

$$\mathcal{D} = \left\{ \begin{pmatrix} 1 \\ \psi \end{pmatrix}; \psi \in \Psi \right\}$$

$$\mathcal{D} = \left\{ \begin{pmatrix} 1 \\ \psi \end{pmatrix}; \psi \in \mathbb{R}_{++}^{S_1+S_2}, \psi \cdot x = C(x) \ \forall \ x \in G_C \right\}$$

As $C|_{G_C}$ is linear and strictly positive, then, by Theorem 6 in Clark [1993] there exists an extension h linear and strictly positive of $C|_{G_C}$ to all $\mathbb{R}^{S_1+S_2}$.

As $\mathbb{1}_{\{\Omega\}} \in G_C$ we can consider $\{b_0, ..., b_J\}$ a basis of G_C with $b_0 = \mathbb{1}_{\{\Omega\}}$.

As $h(\mathbb{1}_{\{\Omega\}}) = 1$, there exist $\psi^0 \in \mathbb{R}_{++}^{S_1+S_2}$ such that

$$\psi^0 \cdot x = h(x) = C(x) \ \forall \ x = (x(1), x(2)) = (x_1, ..., x_{S_1+S_2}) \in \mathbb{R}^{S_1+S_2}$$

Then

$$\max_{(1,\psi)^T \in \mathcal{D}} \psi \cdot x = C(x) = \psi^0 \cdot x \ \forall \ x \in \mathbb{R}^{S_1+S_2}$$

Then $\forall x$ is $\mathbb{R}^{S_1+S_2}$ it is true that

$$C(x) = \psi^0 \cdot x \geq \psi \cdot x \ \forall \ (1,\psi)^T \in \mathcal{D} \tag{A.9}$$

But this statement is equivalent to

$$\psi^0 \cdot x = \max_{(1,\psi)^T \in \tilde{\mathcal{D}}} \psi \cdot x = C(x) \ \forall \ x \in \mathbb{R}^{S_1+S_2},$$

where

$$\tilde{\mathcal{D}} = \left\{ \begin{pmatrix} 1 \\ \psi \end{pmatrix}; \psi \in \mathbb{R}^{S_1+S_2}_{++}; \psi \cdot x = C(x) \ \forall \ x \in \mathbb{R}^{S_1+S_2} \right\}$$

Denote by $\pi(\cdot)$ the super-replication at minimum cost pricing rule of the financial market $\mathcal{M}$. Let us see that $C(x) \leq \pi(x) \ \forall \ x \in \mathbb{R}^{S_1+S_2}$.

Suppose $C(y) > \pi(y)$ for some $y \in \mathbb{R}^{S_1+S_2}$.

We know that, for any $(1,\psi)^T \in \tilde{\mathcal{D}}$ it is true that $\psi \cdot x = C(x) \ \forall \ x \in \mathbb{R}^{S_1+S_2}$. Then, in particular for $x = \tilde{M}\theta$ we have

$$\psi^T(\tilde{M}\theta) = C(\tilde{M}\theta) \ \forall \ \theta \in \Theta \tag{A.10}$$

Then if exists $y \in \mathbb{R}^{S_1+S_2}$ such that $C(y) > \pi(y)$, there exists, from $y = \tilde{M}\theta^0$, $\theta^0 \in \Theta$ such that

$$\psi^T(\tilde{M}\theta^0) = C(\tilde{M}\theta^0) > \psi^T(\tilde{M}\theta) \ \forall \ \theta \in \Theta$$

But this contradicts the restriction set from Theorem 2 in Ortu [2001], because this set,

$$\Psi = \left\{ \begin{pmatrix} 1 \\ \tilde{\psi} \end{pmatrix}; \tilde{\psi} \in \mathbb{R}^{S_1+S_2}_{++}, \tilde{\psi} \cdot x \leq \pi(x) \ \forall \ x \in \mathbb{R}^{S_1+S_2} \right\},$$

which is equivalent to

$$\Psi = \left\{ \begin{pmatrix} 1 \\ \tilde{\psi} \end{pmatrix}; \tilde{\psi} \in \mathbb{R}^{S_1+S_2}_{++}, \tilde{\psi}^T \cdot \tilde{M}\theta \leq \pi(\tilde{M}\theta) \ \forall \ \theta \in \Theta \right\}$$

Therefore, we conclude that $\tilde{\mathcal{D}} = \Psi$, implying that $C(x) = \pi(x) \ \forall \ x \in \mathbb{R}^{S_1+S_2}$. In particular, $C(x) = \pi(x) \ \forall \ x \in B(\mathcal{G})$, that is, $C : B(\mathcal{G}) \to \mathbb{R}$ is the super-replication at minimum cost pricing rule of the financial market $\mathcal{M}$. □

Proof. of Lemma 1.10
($\Rightarrow$) Suppose $\mathcal{M}^1$ is not arbitrage-free.

Define the financial markets with two periods $\mathcal{M}^{1,1} = (x_j, q_j^{1,1}, d_j^{1,1}, 0 \leq j \leq J), \mathcal{M}^{1,2} = (x_j, q_j^{1,2}, d_j^{1,2}, 0 \leq j \leq J), \mathcal{M}^{1,3} = (x_j, q_j^{1,3}, d_j^{1,3}, 0 \leq j \leq J)$ in the following way:

$$q_j^{1,1} := q_j(0), d_j^{1,1} := d_j(1), \forall j$$
$$q_j^{1,2} := q_j(0), d_j^{1,2} := d_j(2), \forall j$$
$$q_j^{1,3} := q_j(0), d_j^{1,3} := d_j(3), \forall j$$

The cashflow process in the market $\mathcal{M}^{1,1}, \mathcal{M}^{1,2}, \mathcal{M}^{1,3}$ is given, respectively, by:

$$x_\theta^{1,1}(t) = \begin{cases} -q(0) \cdot [\theta^A(0) - \theta^B(0)] & t = 0 \\ d(1) \cdot [\theta^A(0) - \theta^B(0)] & t = 1 \end{cases}$$

$$x_\theta^{1,2}(t) = \begin{cases} -q(0) \cdot [\theta^A(0) - \theta^B(0)] & t = 0 \\ d(2) \cdot [\theta^A(0) - \theta^B(0)] & t = 1 \end{cases}$$

$$x_\theta^{1,3}(t) = \begin{cases} -q(0) \cdot [\theta^A(0) - \theta^B(0)] & t = 0 \\ d(3) \cdot [\theta^A(0) - \theta^B(0)] & t = 1 \end{cases}$$

By Lemma 1.2, if $\mathcal{M}^1$ is not arbitrage-free, then $\mathcal{M}^{1,1}, \mathcal{M}^{1,2}$ and $\mathcal{M}^{1,3}$ are not arbitrage-free.

If $\mathcal{M}^{1,1}$ is not arbitrage-free, then exists ${}^1\theta$ such that $x_{{}^1\theta}^{1,1}(t) \geq 0 \ \forall \ t \in \{0, 1\}$ and

1. $x_{{}^1\theta}^{1,1}(0)$ or;

2. $x_{{}^1\theta}^{1,1}(1) > 0$

Let us analyse the two cases.

1. $x_{{}^1\theta}^{1,1}(0) > 0$

Remember that the cashflow process for a dynamic trading strategy $\theta \in \Theta$ in the financial market $\mathcal{M} = (x_j, \{q_j(t)\}_{t=0}^3, \{d_j(t)\}_{t=0}^3)$ is given by

$$x_\theta(t) = \begin{cases} -q(0) \cdot [\theta^A(0) - \theta^B(0)] & t = 0 \\ d(t) \cdot \sum_{\tau=0}^{t-1} [\theta^A(\tau) - \theta^B(\tau)] - q(t) \cdot [\theta^A(t) - \theta^B(t)] & t = 1, 2 \\ d(3) \cdot \sum_{\tau=0}^{2} [\theta^A(\tau) - \theta^B(\tau)] & t = 3 \end{cases}$$

Define $\tilde{\theta}$ in the following way:

- $\tilde{\theta}(0) := \theta(0)$

  $\Rightarrow x_{\tilde{\theta}}(0) = x_{{}^1\theta}^1(0) > 0$

- $\tilde{\theta}_1^A(1) := \frac{d_j(1)^1\theta_j^A(0)}{2q_j(1)}, \tilde{\theta}_1^B(1) := \frac{d_j(1)^1\theta_j^B(0)}{2q_j(1)}$

$$\begin{aligned} \Rightarrow x_{\tilde{\theta}}(1) &= d(1) \cdot [{}^1\theta^A(0) -{}^1 \theta^B(0)] - q(1) \cdot [\tilde{\theta}_1^A(1) - \tilde{\theta}_1^B(1)] \\ &= \frac{1}{2} d(1) \cdot [{}^1\theta^A(0) -{}^1 \theta^B(0)] \\ &= \frac{1}{2} x_{{}^1\theta}^{1,1}(1) \\ &\geq 0 \end{aligned}$$

- $\tilde{\theta}_j^A(2) := \frac{d_j(2)[\tilde{\theta}_j^A(0)+\tilde{\theta}_j^A(1)]}{q_j(2)}, \tilde{\theta}_j^A(2) := \frac{d_j(2)[\tilde{\theta}_j^B(0)+\tilde{\theta}_j^B(1)]}{q_j(2)}$

$$\begin{aligned}\Rightarrow x_{\tilde{\theta}}(2) &= d(2)\cdot[(\tilde{\theta}_j^A(0)-\tilde{\theta}_j^B(0))+(\tilde{\theta}_j^A(1)-\tilde{\theta}_j^B(1))]- \\ &\sum_{j=0}^{J}\left[\frac{q_j(2)d_j(2)[(\tilde{\theta}_j^A(0)-\tilde{\theta}_j^B(0))+(\tilde{\theta}_j^A(1)-\tilde{\theta}_j^B(1))]}{q_j(2)}\right] \\ &= 0\end{aligned}$$

- $\tilde{\theta}_j^A(3) := d_j(3)^3\theta_j^A(0) + \frac{d_j(2)^2\theta_j^A(0)+d_j(1)^1\theta_j^A(0)}{d_j(3)}$
  $\tilde{\theta}_j^B(3) := d_j(3)^3\theta_j^B(0) + \frac{d_j(2)^2\theta_j^B(0)+d_j(1)^1\theta_j^B(0)}{d_j(3)}$

$$\begin{aligned}x_{\tilde{\theta}}(3) &= d(3)\cdot\sum_{\tau=0}^{2}[\tilde{\theta}^A(\tau)-\tilde{\theta}^B(\tau)] \\ &= x_{1\theta}^{1,1}(1)+x_{2\theta}^{1,2}(1)+x_{3\theta}^{1,3}(1) \\ &\geq 0,\end{aligned}$$

that is, $\tilde{\theta}$ is an arbitrage opportunity in the market $\mathcal{M}$, contradicting the hypothesis that $\mathcal{M}$ is arbitrage-free.

Now, let us analyse the second case.

2. $x_{1\theta}^{1,1}(1) > 0$
Using the same $\tilde{\theta}$ constructed in the case 1 and by the same arguments we have:

$$\begin{cases} x_{\tilde{\theta}(0)} = x_{1\theta}^{1,1}(1) \geq 0; \\ x_{\tilde{\theta}(1)} = \frac{1}{2}x_{1\theta}^{1,1}(1) > 0; \\ x_{\tilde{\theta}(2)} = 0; \\ x_{\tilde{\theta}}(3) = x_{1\theta}^{1,1}(1)+x_{2\theta}^{1,2}(1)+x_{3\theta}^{1,3}(1) > 0, \end{cases}$$

that is, $\tilde{\theta}$ is again an arbitrage opportuniy in the financial market $\mathcal{M}$.

So, we must have $\mathcal{M}^1$ arbitrage-free.

By analogous calculations, we can show that if $\mathcal{M}^2$ or $\mathcal{M}^3$ is not arbitrage-free, then we can construct an arbitrage opportunity in the financial market $\mathcal{M}$. So, we also must have $\mathcal{M}^1, \mathcal{M}^2$ arbitrage-free.

$(\Leftarrow)$ Suppose that $\mathcal{M}$ is not arbitrage-free.
Then exists $\theta \in \Theta$ such that $x_\theta(t) \geq 0\ \forall\ t \in \{0,1,2,3\}$ and

1. $x_\theta(0) > 0$ or;
2. $x_\theta(t) > 0$ for some $t \in \{1,2,3\}$

Let us analyse the two cases.

1. $x_\theta(0) > 0$
Consider the market $\mathcal{M}^{1.1}$ defined in the same way as in the implication $\Rightarrow$ of the proof.

Define a dynamic trading strategy $\tilde{\theta}$ in the financial market $\mathcal{M}^{1.1}$ in the following way:

- $\tilde{\theta}(0) := \theta(0)$
  $\Rightarrow x^{1,1}_{\tilde{\theta}}(0) = x_\theta(0) > 0$
- $x^{1,1}_{\tilde{\theta}}(1) = x_\theta(1) + q(1) \cdot [\theta^A(1) - \theta^B(1)]$
  If $q(1) \cdot [\theta^A(1) - \theta^B(1)] < 0$, then we can construct an arbitrage opportunity in the market $\mathcal{M} = (x_j, \{q^1_j(t)\}^2_{t=0}, \{d^1_j(t)\}^2_{t=0})$ defined by

  $q^1_j(t) : q_j(t) \ \forall\ t \in \{0,1,2\},\ \forall\ j$
  $d^1_j(t) := d_j(t) \ \forall\ t \in \{0,1,2\},\ \forall\ j,$

  contradicting the hypothesis that $\mathcal{M}^1$ is arbitrage-free. Then we must have $q(1) \cdot [\theta^A(1) - \theta^B(1)] \geq 0$, implying that

  $$x^{1,1}_{\tilde{\theta}}(1) = x_\theta(1) + q(1) \cdot [\theta^A(1) - \theta^B(1)],$$

  that is, defining $\tilde{\theta}(1) := \theta(1)$ we have that $\tilde{\theta}$ is an arbitrage opportunity in $\mathcal{M}^{1,1}$. But this implies, by Lemma 1.2, that $\mathcal{M}^1$ is not arbitrage-free, contradicting the hypothesis. So, we must have $\mathcal{M}^1$ arbitrage-free.

Now let us verify the second case.

2. $x_\theta(1) > 0$
Defining $\tilde{\theta} := \theta$ in the market $\mathcal{M}^1$ we have:

$$\begin{cases} x^1_{\tilde{\theta}}(0) = x_\theta(0) \geq 0; \\ x^1_{\tilde{\theta}}(1) = x_\theta(1) > 0; \\ x^1_{\tilde{\theta}}(2) = x_\theta(2) + q(2) \cdot [\theta^A(2) - \theta^B(2)]; \end{cases}$$

If $q(2) \cdot [\theta^A(2) - \theta^B(2)] < 0$ we can construct an arbitrage opportunity in the market

$$\mathcal{M}^2 = (x_j, \{q^2_j(t)\}^2_{t=0}, \{d^2_j(t)\}^2_{t=0})$$

defined by

$q^2_j(t) := q_j(t)$ for $t = 0, 1$ and for all j

$S^2_j(3) := q_j(3) \ \forall\ j$

$d^2_j(t) := d_j(t)$ for $t = 0, 1$ and for all j

$d_j^2(3) := d_j(3)\ \forall\ j,$

contradicting the hypothesis that $\mathcal{M}^2$ is arbitrage-free. So, we must have $q(2) \cdot [\theta^A(2) - \theta^B(2)] \geq 0$, implying that

$$x^1_{\tilde{\theta}}(2) = x_\theta(2) + q(2) \cdot [\theta^A(2) - \theta^B(2)] \geq 0$$

In summary, we have

$$\begin{cases} x^1_{\tilde{\theta}}(0) = x_\theta(0) \geq 0; \\ x^1_{\tilde{\theta}}(1) = x_\theta(1) > 0; \\ x^1_{\tilde{\theta}}(2) = x_\theta(2) + q(2) \cdot [\theta^A(2) - \theta^B(2)] \geq 0; \end{cases}$$

, that is, $\tilde{\theta}$ is an arbitrage opportunity in the financial market $\mathcal{M}^1$, contradicting the hypothesis.

By analogous arguments, we show that if $x_\theta(2) > 0$ or $x_\theta(3) > 0$, we can construct an arbitrage opportunity in one of the financial markets $\mathcal{M}^1, \mathcal{M}^2$ or $\mathcal{M}^3$, contradicting the htpothesis.

So, we can conclude that we must have $\mathcal{M}$ arbitrage-free. This finishes the proof of the Lemma. □

Proof. of Proposition 1.4
First, write $C : B(\mathcal{G}) \to \mathbb{R}$ in the form

$$C(x) = \max_{\mathbb{P}_t \in \mathcal{K}_t} \sum_{t=1}^{T} c_t \mathbb{E}_{\mathbb{P}_t}[x(t)]$$

By Lemma 1.10, the markets $\mathcal{M}^t, t = 1, 2$ are frictionless and arbitrage-free.

Define the following financial markets with one future period:

$\mathcal{N}^1 := \left(x_j, \{l_j^1(t)\}_{t=0}^1, v_j^1(t)_{t=0}^1, 0 \leq j \leq J\right)\ l_j(t) := q_j(t), l_j^1(t) := d_j^1(t), t = 0, 1$

$\mathcal{N}^2 := \left(x_j, \{l_j^2(t)\}_{t=0}^1, v_j^2(t)_{t=0}^1, 0 \leq j \leq J\right)\ l_j^2(0) := q_j(0), l_j^2(0) := q_j(0), l_j^2(1) := d_j(2), l_j^2(1) := q_j(2)$

$\mathcal{N}^3 := \left(x_j, \{l_j^3(t)\}_{t=0}^1, v_j^3(t)_{t=0}^1, 0 \leq j \leq J\right)\ l_j^3(0) := q_j(0), l_j^3(0) := q_j(0), l_j^3(1) := d_j(3), l_j^2(1) := q_j(3)$

By Lemma 1.2, $\mathcal{N}^t, t = 1, 2, 3$ is frictionless and arbitrage-free.

Also, by Proposition 1.2, $f_t : \mathbb{R}^{S_t} \to \mathbb{R}, t = 1, 2, 3$ defined by

$f_1(x) := C(x; 0; 0)$

$f_2(y) := C(0; y; 0)$

$f_3(z) := C(0; 0; z)$

are the super-replication at minimum cost pricing pricing of the markets $\mathcal{M}^t, t = 1, 2, 3$, respectively, if and only if $C : B(\mathcal{G} \to \mathbb{R})$ is the super-replication at minimum cost pricing rule of $\mathcal{M}$.

But $f_1(x) = \frac{1}{c_1}C_1(x;0), f_2(y) = \frac{1}{c_2}C_2(0;y), f_3(z) = \frac{1}{c_3}C_3(0;z)$.

Therefore, again by Proposition 1.2, C is the super-replication at minimum cost pricing rule of $\mathcal{M}$ if and only if C_1, C_2 are the is the super-replication at minimum cost pricing rule of $\mathcal{M}^t, t = 1,2$, respectively. $\square$

Proof. of Lemma 1.11
Take $x = (x(1);x(2);x(3)) \in G_C = \bigcap_{t=1}^3 G_C^t$.

First of all, let us note that

$[C_1(-x(1);x(2))+C_1(x(1);-x(2))]+[C_2(x(1);-x(3))+C_2(-x(1);x(3))]+[C_3(x(2);-x(3))+C(-x(2);x(3))] \geq C_1(0;0)+C_2(0;0)+C_3(0;0) = 0$.

If the above inequality is strictly positive than at least one of the three terms is strictly positive (observe that each one is bigger than or equal to zero, by property of pricing rules). Suppose, without loss of generality, that it is the first one.

We have the following inequalities, by sublinearity of pricing rules:

$C(-x(1);x(2);x(3)) + C((x(1);-x(2);0)) \geq C((0;0;x(3)))$;
$C(x(1);-x(2);-x(3)) + C((-x(1);x(2);0)) \geq C((0;0;-x(3))$

By other side,

$C((-x(1),x(2),x(3))) \leq C((-x(1);x(2);0)) + C((0;0;x(3)))$; $C((x(1),-x(2),-x(3))) \leq C((x(1);-x(2);0)) + C((0;0;-x(3)))$

These four inequalities together implies

$C((-x(1),x(2),x(3)))+C((x(1),-x(2),-x(3))) = [C((0;0;x(3)))+C((0;0;-x(3)))]+[C((-x(1);x(2);0))+C((x(1);-x(2);0))] > 0$,

contradicting the hypothesis that $x \in G_C^1$.

By the same arguments we can show that the second or the third terms in the inequality is strictly positive, than $x \notin G_C^2$ or $x \notin G_C^3$, contradicting the hypothesis that $x \in G_C$. So, we must have

$$C_1((-x(1),x(2))) + C_1((x(1),-x(2))) = 0$$

$$C_2((-x(1);x(3))) + C_1((x(1);-x(3))) = 0$$

$$C_3((-x(2);x(3))) + C_1((x(2);-x(3))) = 0$$

Therefore, we conclude that $x \in G_{C_t}, t = 1,2,3$. This proofs that $x \in G_C \Rightarrow (x(1),x(2)) \in G_{C_1}, (x(1);x(3)) \in G_{C_2}, (x(2);x(3)) \in G_{C_3}$.

Now, let us prove the other direction.

We know that

$$\begin{aligned}C((x(1),-x(2),-x(3))) &= C((x(1);-x(2);0)+(0;0;-x(3)))\\ &\le C((x(1);-x(2);0))+C((0;0;-x(3)))\\ &= C_1((x(1),-x(2)))+C_3((0;-x(3)))\end{aligned}$$

$$\begin{aligned}C((-x(1),x(2),x(3))) &= C((-x(1);x(2);0)+(0;0;x(3)))\\ &\le C((-x(1);x(2);0))+C((0;0;x(3)))\\ &= C_1((-x(1),x(2)))+C_3((0;x(3)))\end{aligned}$$

Joining both inequalities we get $C((x(1),-x(2),-x(3)))+C((-x(1),x(2),x(3))) \le [C_1((x(1),-x(2)))+C_1((-x(1),x(2)))]+[C_3((0;-x(3)))+C_3((0;x(3)))]$

As $(x(1),x(2)) \in G_{C_1}$ we have $C_1((x(1),-x(2)))+C_1((-x(1),x(2)))=0$.

We know that $C_3((0;-x(3)))+C_3((0;x(3))) \ge 0$. If it is strictly positive we have

$$\begin{aligned}C_3((0;x(3)))+C_3((0;-x(3))) &= [C_3((-x(2);x(3))+(x(2);0))]+[C_3((x(2);-x(3))+(-x(2);0))]\\ &\le C_3((-x(2);x(3)))+C_3((x(2);-x(3)))+C_3((x(2);0))+C_3((-x(2);0))\end{aligned}$$

Then

$$\begin{aligned}C_3((-x(2);x(3)))+C_3((x(2);-x(3))) &\ge [C_3((0;x(3)))+C_3((0;-x(3)))]-[C_3((x(2);0))+C_3((-x(2);0))]\\ &> 0-0=0,\end{aligned}$$

contradicting $(x(2);x(3)) \in G_{C_3}$. So, we must have $C_3((0;-x(3)))+C_3((0;x(3)))=0$, implying that

$$C((x(1),-x(2),-x(3)))+C((-x(1),x(2),x(3))) \le 0 \tag{A.11}$$

By other side, we have

$$C((x(1),-x(2),-x(3)))+C((-x(1),x(2),x(3))) \ge C((0;0;0))=0 \tag{A.12}$$

(A.11) and (A.12) together implies

$$C((x(1),-x(2),-x(3)))+C((-x(1),x(2),x(3)))=0,$$

that is, $x \in G_C^1$.

By analogous arguments we prove that $x \in G_C^2$ and $x \in G_C^3$. □

Proof. of Corollary 1.2
By Lemma 1.11, we know that $x \in G_C$ if and only if $(x(1),x(2)) \in G_{C_1}$, $(x(1);x(3)) \in G_{C_2}$, $(x(2),x(3)) \in G_{C_3}$.

By Lemma 1.5, $(x(1),x(2)) \in G_{C_1}$ if and only if $x(t) \in F_{f_t}, t = 1,2$, $(x(1);x(3)) \in G_{C_2}$ if and only if $x(t) \in F_{f_t}, t = 1,3$ and $(x(2);x(3)) \in G_{C_3}$ if and only if $x(t) \in F_{f_t}, t = 2,3$.

This implies, by Corollary 1.1, that $x \in L^1_{C_1} \cap L^2_{C_1}, x \in L^2_{C_2}$.

As $L^1_{C_1} = L^1_C, L^2_{C_1} = L^2_C, L^2_{C_2} = L^3_C$, we conclude that $x \in \bigcap_{t=1}^3 L^t_C$. □

Proof. of Theorem 1.2
We know from Proposition 1.5 that it is true for $T = 4$.

Suppose it is true for $t = T > 4$. Denoting the superhedging pricing rule of this market by $\bar{C}$, we then have $G_{\bar{C}} = \bigcap_{t=1}^T G^t_{\bar{C}} = \bigcap_{t=1}^T L^t_{\bar{C}} = L_{\bar{C}}$.

Let $x \in G_C = \bigcap_{t=1}^{T+1} G^t_C$.

We can rewrite G_C as $G_C = \left(\bigcap_{t=1}^T G^T_C\right) \cap G^{T+1}_C$.

Identifying $\times_{t=1}^{T+1} \mathbb{R}^{S_t}$ with $\mathbb{R}^L, L = \sum_{t=1}^{T+1} S_t$, we can consider $x = (x(1);...;x(T+1)) \in \times_{t=1}^{T+1} \mathbb{R}^{S_t}$ as $(x(1);...;x(T);x(T+1)) \in \times_{t=1}^{T} \mathbb{R}^{S_t} \times \mathbb{R}^{T+1}$. So, defining appropriately a financial market with T periods, we have that $(x(1),...,x(T)) \in G_{\bar{C}}$, $\bar{C}$ is the super-replication at minimum cost pricing rule of this market with T periods. Then, by the induction hypothesis, $(x(1),...,x(T)) \in L_{\bar{C}}$. But $(x(1),...,x(T)) \in L_{\bar{C}} = \bigcap_{t=1}^T L^t_{\bar{C}}$ implies that $x = (x(1),...,x(T)) \in \bigcap_{t=1}^T L^T_C$ (because $\bar{C}(x(1),...,x(T)) = \frac{1}{c_T} C(x)$).
But we also know that $x \in G^{T+1}_C$, where

$$G^{T+1}_C = \{x : C(xD_{T+1}) + C(-xD_{T+1}) = 0\}$$

Through the identification $\times_{t=1}^{T+1} \mathbb{R}^{S_t} \cong \mathbb{R}^L, L = \sum_{t=1}^{T+1} S_t$, we can see G^{T+1}_C as the set of frictionless securities of a frictionless and arbitrage-free market with one future period, where the contingent claims are of the form $y = xD_{T+1} \in \mathbb{R}^L$. That is, denoting the super-replication at minimum cost pricing rule of this market by P we have

$$G^{T+1}_C = F_P = \{y = xD_{T+1} \in \mathbb{R}^L : P(y) + P(-y) = 0\}$$

Therefore, by Theorem 5 in Araujo, Chateauneuf, and Faro [2012], $G^{T+1}_C = F_P = L_P$.

But $L_P = L^{T+1}_C$. As we also have $x \in \bigcap_{t=1}^T$, we conclude that $x \in \bigcap_{t=1}^{T+1} L^t_C = \mathcal{L}_C$. So, by Lemma 1.3 we conclude that $x \in L_C$. □

Appendix B

Appendix

Proof. of Proposition 2.1

($\Rightarrow$) If there exists $(m_0, ..., m_T)$ a vector discount factors than $\mathcal{H}$ is arbitrage-free.

Define the following cone:

$$\mathcal{C} := \{(\nu, x) \in \mathbb{R} \times \bigtimes_{t=1}^{T} \mathbb{R}^{S_t}; \nu \leq 0, x \geq 0\}$$

Then the hypothesis of absence of arbitrage opportunities is equivalent to

$$\mathcal{H} \cap \mathcal{C} = \{0\}$$

Let $(m_0, ..., m_T)$ a vector of discount factors and $(\nu, x) \in \mathcal{H} \cap \mathcal{C}$.

We need to show that $(\nu, x) = 0$.

We have $\nu \leq 0, x \geq 0 \Rightarrow -\nu \geq 0, x \geq 0$

By Definition 2.5 we know that

$$\sum_{t=1}^{T} \mathbb{E}_{\mathbb{P}_t}[m_t x(t)] - \mathbb{E}_{\mathbb{P}_0}[m_0 \nu] \leq 0$$

As $(m_0, ..., m_T)$ is strictly positive and $-\nu \geq 0, x \geq 0$, the only way of the above inequality be true is if $-\nu = 0, x = 0$, that is, $\mathcal{H} \cap \mathcal{C} = \{0\}$. So, $\mathcal{H}$ is arbitrage-free.

($\Leftarrow$) If $\mathcal{H}$ is arbitrage-free then there exists $(m_0, ..., m_T)$ a vector of regular discount factors. Consider tha same cone $\mathcal{C}$ as in ($\Rightarrow$). Since $\mathcal{C}$ is closed, proper, we are supposing $\mathcal{H}$ closed and the hypothesis of absence of arbitrage opportunities is equivalent to

$$\mathcal{H} \cap \mathcal{C} = \{0\},$$

applying Corollary 11.4.2 of Rockafellar [1970] we have:

$$\exists\ L \in \mathcal{C}^+ \text{ such that } L \cdot h \leq 0\ \forall\ h \in \mathcal{H},$$

where $\mathcal{C}^+ := \{z \in \mathcal{C}^*; z \cdot y \geq 0\ \forall\ y \in \mathcal{C}\}$, $\mathcal{C}^*$ being the dual cone of $\mathcal{C}$, that is, $\mathcal{C}^* = \{z; z \cdot y \geq 0\ \forall\ \ \in \mathcal{C}\}$.

Any linear functional F in $\mathbb{R} \times \bigtimes_{t=1}^{T} \mathbb{R}^{S_t}$ can be written as

$$F(x) = \sum_{t=1}^{T} \mathbb{E}_{\mu_t}[m_t x(t)] - \mathbb{E}_{\mu_0}[m_0 x(0)] \ \forall \ x = (x(0); ...; x(T)) \in \mathbb{R} \times \bigtimes_{t=1}^{T} \mathbb{R}^{S_t},$$

for some $m_0 \in \mathbb{R}, (m_1, ..., m_T) \in \bigtimes_{t=1}^{T} \mathbb{R}^{S_t}$.

Then, writting the functional L as

$$L((\nu, x)) = \sum_{t=1}^{T} \mathbb{E}_{\mu_t}[m_t x(t)] - \mathbb{E}_{\mu_0}[m_0 \nu],$$

we have that $L \in \mathcal{C}^+$ if and only if m_0 and $(m_1, ..., m_T)$ are strictly positive. But $(m_0, ..., m_T)$ strictly positive implies

$$L((\nu, x)) = \sum_{t=1}^{T} \mathbb{E}_{\mu_t}[m_t x(t)] - \mathbb{E}_{\mu_0}[m_0 \nu] \le 0 \ \forall \ (\nu, \theta, x) \in \mathcal{H},$$

which is exactly the definition of vector of regular discount factors (Definition 2.5). □

Proof. of Theorem 2.1
First, we can rewrite the cones Z_t as

$$Z_t = \{\theta \in \Theta : \theta(t-1) - \theta(t) \in M_t, \theta(t-1) \in \mathbb{R}_+^{2(J+1)}\}, \tag{B.1}$$

for some closed cone $M_t \subset \mathbb{R}_+^{2(J+1)}$.

Define the cone Z_t^* by

$$Z_t^* := \{(c, d) \in \mathbb{R}+^{J+1} \times \mathbb{R}+^{J+1} : d \cdot \theta(t-1) \le c \cdot \theta(t) \ \forall \theta \in Z_t\}$$

Take $m = (m_0, ..., m_T)$ a vector of discount factors. Then

$$\sum_{t=1}^{T} \mathbb{E}_{\mathbb{P}_t}[m_t(x)] - q_0 \nu \le 0 \ \forall \ (\nu, \theta, x) \in \mathcal{H} \tag{B.2}$$

Define the optimization problem

$$\max_{(\nu,\theta,x) \in \mathcal{H}} f((\nu, x)),$$

where

$$f((\nu, x)) := \sum_{t=1}^{T} \mathbb{E}_{\mathbb{P}_t}[m_t(x)] - q_0 \nu \ \forall \ (\nu, \theta, x) \in \mathcal{H}$$

We can rewrite the problem as

$$\begin{aligned} &\max_{(\nu,x)} f((\nu, x)) \\ &(\nu, \theta) \in V_0 \\ \text{s.t. } &(\theta, x(t)) \in V_t, t = 1, ..., T \\ &\theta \in Z_t, t = 1, ..., T \end{aligned}$$

Then

$$\begin{aligned}
&\max_{(\nu,x)} f((\nu,x))\\
&\nu \geq q(0)\cdot\phi^0(\theta^A(0),\theta^B(0))\\
\text{s.t. } &q(t)\cdot\phi^t(\theta^A(t)-\theta^A(t-1),\theta^B(t)-\theta^B(t-1)) \geq x(t), t=1,...,T\\
&q(t)\cdot\phi^t(\theta^A(t)-\theta^A(t-1),\theta^B(t)-\theta^B(t-1)) \leq 0, t=1,...,T
\end{aligned}$$

As $m=(m_0,...,m_T)$ is a vector of discount factors, 0 is a maximum value for the function $f((\nu,x))$. By Kuhn-Tucker theorem, there exist $p_0,p_1,...,p_T$ Lagrange multipliers such that

$$f((\nu,x))+\mathbb{E}[p_0[\nu-q(0)\cdot\phi^0(\theta^A(0),\theta^B(0))]]+\sum_{t=1}^{T}\mathbb{E}[p_t q(t)\cdot\phi^t(\theta^A(t-1)-\theta^A(t),\theta^B(t-1)-\theta^B(t))-x(t)] \leq 0 \tag{B.3}$$

Rearranging terms in (B.3) we have

$$\begin{aligned}
&f((\nu,x))+\mathbb{E}[p_0[\nu-q(0)\cdot\phi^0(\theta^A(0),\theta^B(0))]]+\mathbb{E}[p_1 q(1)\cdot\phi^1(\theta^A(0)-\theta^A(1),\theta^B(0)-\theta^B(1)-x(1))]+\\
&+\sum_{t=1}^{T}\mathbb{E}[p_{t+1}q(t+1)\cdot\phi^{t+1}(\theta^A(t)-\theta^A(t+1),\theta^B(t)-\theta^B(t+1))-x(t+1)] \leq 0
\end{aligned} \tag{B.4}$$

As $\theta\in Z_t$ we have that $(\theta,0)\in V_t$. Then (B.7) is valid "without" the terms $x(t)$ (because $x=0$). Then we have

$$\begin{aligned}
&f((\nu,x))+\mathbb{E}[p_0[\nu-q(0)\cdot\phi^0(\theta^A(0),\theta^B(0))]]+\mathbb{E}[p_1 q(1)\cdot\phi^1(\theta^A(0)-\theta^A(1),\theta^B(0)-\theta^B(1))]+\\
&\sum_{t=1}^{T}\mathbb{E}[p_{t+1}q(t+1)\cdot\phi^{t+1}(\theta^A(t)-\theta^A(t+1),\theta^B(t)-\theta^B(t+1))] \leq 0
\end{aligned} \tag{B.5}$$

The inequality (B.5) is true if and only if each term is non-positive. In particular:

$$\mathbb{E}[p_t q(t)\cdot\phi^{t+1}(\theta^A(t)-\theta^A(t+1),\theta^B(t)-\theta^B(t+1))] \leq 0,$$

implying that

$$p_t q(t)\cdot\phi^{t+1}(\theta^A(t),\theta^B(t))-\mathbb{E}[p_{t+1}q(t+1)\cdot\phi^{t+1}(\theta^A(t+1),\theta^B(t+1)|\mathcal{F}_t)] \leq 0$$ [1]

Then

$$[q(t+1)\cdot\phi^{t+1}(\theta^A(t),\theta^B(t))]-q(t+1)\cdot\phi^{t+1}(\theta^A(t+1),\theta^B(t+1))\mathbb{E}[p_{t+1}|\mathcal{F}_t] \leq 0$$

So,

$$[\theta^A(t)-\theta^B(t)]p_t-[\theta^A(t+1)-\theta^B(t+1)]\mathbb{E}[p_{t+1}|\mathcal{F}_t] \leq 0, \tag{B.6}$$

which is equivalent to

$$[\theta^A(t)-\theta^B(t)]\cdot p_t \leq [\theta^A(t+1)-\theta^B(t+1)]\mathbb{E}\cdot[p_{t+1}|\mathcal{F}_t]$$

That is,

$$(p_t,\mathbb{E}[p_{t+1}|\mathcal{F}_t])\in Z_t^*$$

[1] $p_t q(t)=(p_t^0 q_0(t),...,p_t^J q_J(t))$

But as Z_t has the form (B.1), it implies that we can rewrite Z_t^* as

$$Z_t^* = \{(x,y) \in \mathbb{R}_+^{J+1} \times \mathbb{R}_+^{J+1} : -c \in M_t^*, c-d \in (\mathbb{R}_+^{J+1})^*\}$$

As $(\mathbb{R}_+^{J+1})^* = \{0\}$ and $(p_t, \mathbb{E}[p_{t+1}|\mathcal{F}_t]) \in Z_t^*$, we conclude that $p_t - \mathbb{E}[p_{t+1}|\mathcal{F}_t] \in \{0\}$, that is

$$p_t - \mathbb{E}[p_{t+1}|\mathcal{F}_t] = 0$$

Therefore, $\{p_t\}_{t=0}^T$ is a martingale.

Now, let us define the following maximization problem

$$\begin{aligned}&\max_{\theta}\{p_t \cdot [\theta^B(t-1) - \theta^A(t-1) - (\theta^B(t) - \theta^A(t))]\}\\ &\text{s.t. } -q(t) \cdot \phi^t(\theta^A(t) - \theta^A(t-1), \theta^B(t) - \theta^B(t-1)) \geq 0\end{aligned}$$

As $p_t \cdot [\theta^B(t-1) - \theta^A(t-1) - (\theta^B(t) - \theta^A(t))] \leq 0$ (because $(\theta \in Z_t)$ implies $(\theta, 0) \in V_t$) we have that 0 is the maximum value of the function

$$g(\theta) := p_t \cdot [\theta^B(t-1) - \theta^A(t-1) - (\theta^B(t) - \theta^A(t))]$$

By Kuhn-Tucker theorem, there exists λ_t such that

$$p_t \cdot [\theta^B(t-1) - \theta^A(t-1) - (\theta^B(t) - \theta^A(t))] - \lambda_t q(t) \cdot \phi^t(\theta^A(t) - \theta^A(t-1), \theta^B(t) - \theta^B(t-1)) \leq 0, \forall\ \theta \in \Theta$$

It is true if and only if

$$p_t^j \cdot [\theta_j^B(t-1) - \theta_j^A(t-1) - (\theta_j^B(t) - \theta_j^A(t))] - \lambda_t q(t) \cdot \phi^t(\theta_j^A(t) - \theta_j^A(t-1), \theta_j^B(t) - \theta_j^B(t-1)) \leq 0\ \forall\ j$$

Then

$$p_t^j(b-a) - \lambda_t q_j(t) \phi_j^t(a,b) \leq 0\ \forall\ a, b \in \mathbb{R} \tag{B.7}$$

For $r = b - a < 0$, (B.7) is equivalent to

$$\lambda_t q_j(t)(1 + \beta_j(t)) \leq p_t^j,\ \forall\ j$$

For $r = b - a > 0$, (B.7) is equivalent to

$$\lambda_t q_j(t)(1 + \alpha_j(t)) \geq p_t^j,\ \forall\ j$$

By the above inequality and the restrictions defining the cone $\mathcal{H}$ of hedging strategies, we have

$$\left(\sum_{s=1}^{S_t} m_t, s\right) q_j(t)(1 + \alpha_j(t)) \leq p_t^j \leq \left(\sum_{s=1}^{S_t} m_t, s\right) q_j(t)(1 + \beta_j(t)), t = 1, ..., T \tag{B.8}$$

So, there exists $\gamma_j(t) \in [\alpha_j(t), \beta_j(t)]$ such that $p_t^j = \left(\sum_{s=1}^{S_t} m_t, s\right) q_j(t)(1 + \gamma_j(t)), \forall j$. As $\{p_t\}$ is a martingale, we have that $\{\left(\sum_{s=1}^{S_t} m_t, s\right) q_j(t)(1 + \gamma_j(t))\}_{t=0}^T$ is a martingale.

So, we proved that if $m = (m_0, ..., m_T)$ is a vector of discount factors if and only if $\{\left(\sum_{s=1}^{S_t} m_t, s\right) q_j(t)(1 + \gamma_j(t))\}_{t=0}^T$ is a martingale. Then, from (2.15), we have the implication $\Rightarrow$ of the theorem.

To prove the converse, we know that $\left(\sum_{s=1}^{S_t} m_t, s\right) q_j(t)(1 + \gamma_j(t))\}_{t=0}^T$ is a martingale if and only if $m = (m_0, ..., m_T)$ is a vector of discount factors. Taking $\{\alpha_j(t)\}_{t=0}^T, \{\beta_j(t)\}_{t=0}^T$ stochastic processes such that $\alpha_j(t) \leq \gamma_j(t) \leq \beta_j(t)\ \forall j, \forall t$ and defining $_x := C(x), \forall x \in \times_{t=1}^T \mathbb{R}^{S_t}$, we have that $C(\cdot)$ is the superhedging at minimum cost pricing rule of the financial market $\mathcal{M} = (\{x_j\}_{j=0}^J, \mathcal{H})$ with price processes $\{q_j(t)\}_{t=0}^T$ and interest rates $\{\alpha_j(t)\}_{t=0}^T, \{\beta_j(t)\}_{t=0}^T$. $\square$

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