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Generalized Hénon-Devaney Maps of the Plane

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"The saddest aspect of life right now is that science gathers knowledge faster than society gathers wisdom."

Isaac Asimov

"I must not fear. Fear is the mind-killer. Fear is the little-death that brings total obliteration. I will face my fear. I will permit it to pass over me and through me. And when it has gone past I will turn the inner eye to see its path. Where the fear has gone there will be nothing. Only I will remain."

Frank Herbert, *Dune*

Abstract

In this work we are going to consider the two-parameter family given by

$$\begin{aligned} f_{a,b} : \mathbb{R}^2 \setminus \{y=0\} &\to \mathbb{R}^2 \\ (x,y) &\mapsto \left(ax + \frac{1}{y}, by - \frac{b}{y} - abx \right) \end{aligned}$$

where $0 < a \leq b \leq 1$, if $a = b = 1$ this map is known as the "Hénon-Devaney map". Here we are going to give some dynamical and ergodic properties to these maps.

For all the parameters, we are going to exhibit two transversal invariant C^1-foliations.

For the case where $a < b \leq 1$ we are able to find a global attractor and, via the projection along the invariant manifolds, establish a one-dimensional map that gives a strong description in terms of dynamics and ergodic properties.

Even more, in the previous cases we can find some ergodic measures, finite or infinite depending on b, that are supported on the attractor for the map.

For the case $a = b = 1$, using the invariant foliation, we get a conjugation to a subshift providing a global understanding of the map's behavior.

Key-words: ergodic theory, dynamical systems, Hénon-Devaney map, infinite ergodic theory.

Contents

Introduction

The foundation of the Classic Ergodic Theory lies on the the kinetic theory of gases in the XIX century with L. Boltzmann, J. C. Maxwell and J. C. Gibbs. They were interested in understanding how typical orbits of an hamiltonian flow could cover a space.

Definition. *Let $(X, \mathcal{B}, \mu)$ be a measure space and $T : X \to X$. We say the measure μ is T invariant if for any $A \in \mathcal{B}$*

$$\mu(T^{-1}(A)) = \mu(A)$$

We also say that T is ergodic with respect to the measure μ if it does not have non-trivial invariant sets, i.e.

$$T^{-1}(A) = A \Rightarrow \mu(A) = 0 \text{ or } \mu(A^c) = 0 \qquad A \in \mathcal{B}$$

.

Ergodicity was the primal hypothesis that Boltzmann was looking for and the theory began to develop in order to decide if a given system is ergodic or not. One fundamental step was given in the 1930's by J. von Neumann and G. D. Birkhoff, they proved that time average exists for almost every orbit if the map is measure preserving, which is known as the *Ergodic Theorem*:

Theorem (Birkhoff's Ergodic Theorem)**.** *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic system and T a continuous transformation. If X is a probability space then the average time exists for almost every point $x \in X$ and for every $A \in \mathcal{B}$ of positive measure and it is proportional to the set A, that is*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_A(T^j(x)) = \mu(A)$$

However, the standard assumption that developed the study of this classic behavior is the fact that the measure associated to this kind of problem has to be *finite*. The *Poincarré Recurrence Theorem* is false if we remove this condition, i.e., we may not have any recurrence at all, and the previously mentioned Ergodic Theorem does not give any kind of useful information, the averages always vanish.

That is where Infinite Ergodic Theory started to deal with those problems. Recurrence is now asked as hypothesis and systems with infinite measure with this additional property are the new study objects. E. Hopf and J. Aaronson (see [A0]) were some of the pioneers to explore this subject. The questions that arose at first were regarding understanding the velocity of convergence: is it possible to take the averages in a different way in order to get something? Aaronson crushed that dream with his Ergodic Theorem. The averages always vanish.

The questions asked had changed. Mathematicians turned their attention into getting different "types" of ergodicity, understanding the distributional properties of the averages in the Birkhoff's theorem, trying to find new dynamical invariants to relate different systems and some other interesting questions we will discuss later.

Even though this subject is relatively unexplored, it has some intense research being done names like by F. Ledrappier, M. Lenci, O. Sarig, R. Zweimuller, A. Fisher and many other great mathematicians.

What about the toy models? Although there are some nice and well known examples like the Coin-Tossing Random Walk, some Hyperbolic Geodesic Flows and Boole's transformation (which we will discuss later), there still is a lack of diversity of examples with recurrence and that exhibits an interesting infinite measure.

Some researchers wanted something that could relate to the origins of Ergodic Theory, something that was related to physics. One remarkable example was studied by P. Cirilo, Y. Lima and E. Pujals in their paper [C] that describes a phenomenon that is linked to the Arnold diffusion. These physics driven problems got our attention, specially the *Hénon-Devaney map*.

Before stepping into that, let us recall the definition of the Boole's map

$$\begin{aligned} B : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto x - \frac{1}{x} \end{aligned}$$

which preserves the Lebesgue measure in the real line and the ergodicity of B was proved in 1973 by Adler and Weiss in [AW]. Some one-dimensional generalizations of this map were studied by S. Muñoz in his Ph.D. Thesis [M], by turning the "infinity" into something else depending on a parameter.

Hénon's Generating Families and his approach to the Three-Body Problem has been studied exhaustively by different areas. The asymptotic behavior to "truncated solutions" of the problem, presented in [H], is given by

$$\begin{aligned} f : \mathbb{R}^2 \setminus \{y = 0\} &\rightarrow \mathbb{R}^2 \\ (x, y) &\mapsto \left(x + \frac{1}{y}, y - \frac{1}{y} - x \right) \end{aligned}$$

This is known today as the *Hénon-Devaney map*, due the work done by Devaney in his paper [D], in which he constructed a topological conjugation of f to the Baker Transformation. It is clear the resemblance between B and f, and that is reason why f is considered to be the *two-dimensional version* of Boole's map. It is also easy to see that f preservers the Lebesgue measure in the plane and it is natural to ask about its ergodicity. This was asked by Devaney in his paper in 1981 and yet remains open.

Generalizing this map and proving some properties of the model is what S. Muñoz did in [M2]. He considered a two-parameter family that has the Hénon-Devaney map as the "limit" in the space of parameters. This is a good point to define the family of maps we are going to study, consider

$$\begin{aligned} f_{a,b} : \mathbb{R}^2 \setminus \{y=0\} &\rightarrow \mathbb{R}^2 \\ (x,y) &\mapsto \left(ax + \frac{1}{y}, by - \frac{b}{y} - abx\right) \end{aligned}$$

where $0 < a \leq b \leq 1$.

An initial remark regarding this family is given by "looking" towards the infinity, by taking a look at the differential of each $f_{a,b}$

$$Df_{a,b}(x,y) = \begin{pmatrix} a & -\dfrac{1}{y^2} \\ -ab & b + \dfrac{b}{y^2} \end{pmatrix}$$

and then making $|y| \to \infty$

$$Df_{a,b}(x,\infty) = \begin{pmatrix} a & 0 \\ -ab & b \end{pmatrix}$$

Here we have our first difference of behavior for different parameters, for different a, b we have three different possibilities for $Df_{a,b}(x,\infty)$

$a = b = 1:$ The matrix is parabolic with only one invariant direction;

$a < b \leq 1:$ The matrix has a splitting and a direction that is uniformly contractive;

But in the second case we also have two different behaviors at the infinity. For $a < b = 1$, the infinity matrix has a parabolic behavior along the center-unstable direction and for the other case it has some kind of repulsion.

Our initial goal is to establish a dominated splitting for each one of the parameters. The idea here is inspired by the classical proof given in [AW], that is, understanding the pre-images and "controlling" the differential. Our two-dimensional case requires an use of the graphic transformation and its differentials of higher order to get a better regularity of induced leaves

Theorem A. *For every $a \leq b \leq 1$, there exists two invariant cone fields for $f_{a,b}$. These cone fields induce the directions E^{cu} and E^{cs}. For $a = b = 1$, the splitting is not uniformly dominated. For $a < b \leq 1$, the splitting is dominated and E^{cs} is uniformly contractive. These that induces a C^1-foliation for the every map in the family, moreover if $a < b \leq 1$ then these leaves have a C^2-differentiability.*

The main ideas of the proof consist into looking at the graphic transformation induced by the differential of $f_{a,b}$ and its inverse. These graphic transformations have two invariant cone fields that are constant for every (x, y). The uniformity of the contraction rate is natural for the case $a < b \leq 1$, due the dominance of it by the parameter a, and that gets even more contractive if $b < 1$. The differentiability is achieved by looking at the differential of the graphic transformation and its dominance again by the parameters.

We actually think the case $a < b < 1$ can be taken a little bit further because we conjecture this last case might be an example of a robustly transitive family of maps with dominated splitting. The reason of that is related to the Theorem B and we are going to say a few more words about this after its statement.

Next, we give some dynamical and ergodic properties, exhibiting some special sets for the maps. The first thing we do is find, for every $a < b \leq 1$, the set $\mathfrak{R}_{a,b}$: a non-compact region that is mapped inside itself. This is an attracting region that "grows" to the whole plane when a tends to 1.

The existence of this attracting region allows us to prove a bit more

Theorem B. *For every $a < b \leq 1$, there exists a global attractor*

$$\Lambda_{a,b} := \bigcap_{n \in \mathbb{N}} f^n_{a,b}(\mathfrak{R}_{a,b}) \subset \mathfrak{R}_{a,b}$$

which is not compact. This induces an one dimensional dynamics via the projection along the stable manifold to $\{x = 0\}$:

$a < b = 1:$ *the one dimensional map is "Boole-like";*

$a < b < 1:$ *the induced map is like the ones studied in [M], which we will refer as "Boole-like expanding" maps.*

To clarify what we mean by "Boole-like" maps it is necessary to make a definition. We say that a map $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is Boole-like if

(i) $g(x) < x$ for $x > 0$ and $g(x) > x$ for $x < 0$. Also, $\lim_{x \to \pm\infty} g(x) = \pm\infty$;

(ii) g is increasing for every $x \in \mathbb{R} \setminus \{0\}$;

(iii) $\lim_{|x| \to \infty} g'(x) = 1$ and $\lim_{|x| \to 0} g'(x) = \infty$;

(iv) $\lim_{|x| \to \infty} |g^{(k)}(x)|$ bounded for every $k \in \mathbb{N}$,

in roughly words, the graphic of the map is similar to the graphic of the Boole, a perturbation of it without creating fixed points.

We need to establish the notion of "Boole-like expanding" as well, a particular case of the maps known as *expanding alternating systems*. The formal definition consists in a map $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ that is almost a "Boole-like" but with some additional properties (i') Conditions (i), (ii) and (iv) as before;

(ii') $\lim_{|x|\to\infty} g'(x) \leq a < 1$ and $\lim_{|x|\to 0} g'(x) = \infty$;
And all of them imply that

(iii') there exists an interval I that contains the discontinuity such that the return map $g_I : I \to I$ is expanding.

The idea behind this definition is to think of something like the traditional Boole but instead of having the asymptote in the $\{x = y\}$ this map has an asymptote line that is below the identity. An example of it, taken form Corollary 22 in [M], is the map

$$\begin{aligned} B_a : \mathbb{R} \setminus \{0\} &\to \mathbb{R} \\ x &\mapsto ax - \frac{1}{x} \end{aligned}$$

for $a < 1$. This kind of map have some interesting dynamical properties, while Boole's map is transitive it is not robustly transitive but these maps are.

The main ideas behind theorem B consists into finding a non-compact region that is mapped inside itself and use it to determine where the attractor lies. The next step is to study the bahavior of the holonomies "near the infinity", that is, if y is big enough the holonomy at $Df_{a,b}$ can be approximated by the one induced by the straight lines that form the foliation of $Df_{a,b}(\infty)$ and are easier to work with. We use these simplified holonomies to project the map along the stable manifolds to the set $\{x = 0\}$. All that is left is to check the definitions of Boole-like and Boole-like expanding, that in this case, will only require to be checked the properties at the infinity, which will follow from the simple foliation induced by $D_{a,b}(\infty)$.

After this theorem the reason we think the case $a < b < 1$ is a robust transitive system get more evident. We already know that the Hénon-Devaney map is transitive but it is not robust transitive, by the same principles that do not let the Boole's map robust transitive. For the case where $a < b < 1$ the one-dimensional reduction is the map that are studied in [M] and it is shown there that these maps are robust transitive. So it is natural to think that small perturbations of the attractor when projected should be that perturbations of the Boole-like expanding maps, that are robust transitive.

The ergodic properties are given over the attractor we found, the measure that is object of our interest comes naturally using the differentiability the foliation, a projection along the leaves and some standard methods of finding ergodic measures for generalized Boole's map, which gives up the following theorem

Theorem C. *For $a < b \leq 1$, there exists a SRB measure $\mu_{a,b}$ supported in an attractor $\Lambda_{a,b}$ such that it can be disintegrated over the unstable manifold $\mathcal{W}^u(\Lambda_{a,b})$ of the attractor inside $\mathfrak{R}_{a,b}$ and we also have that $\mu^u_{a,b}$ is absolute continuous with respect to the one-dimensional Lebesgue measure. Also*

$a < b < 1:$ *The measure $\mu_{a,b}$ is finite and the attractor $\Lambda_{a,b}$ lies inside a non-compact region;*

$a < b = 1:$ *The measure* $\mu_{a,b}$ *is infinite and the attractor* $\Lambda_{a,b}$ *lies in a non-compact region.*

We have two cases to be dealt with and the proofs differ in essence but for both, the proof is based in the projection we got in the previous theorem. There exists a compactification to the projected map and it has a natural finite measure. Here is where the difference lies, if $b = 1$ the induced measure is infinite and we cannot say if the measure is ergodic or not for this case. However, if we look at the case $b < 1$ the map that induces the compactification integrable and the measure before compactificating is finite and ergodic. Then, we pull the measure back using the holonomy and can conclude the proof for $a < b < 1$

To the other case, the idea is the same but the methods are different. The projection will induce an Boole-like map in the real line and this induced map has a natural construction to find a infinite measure: the Boole-like maps have a set that are delimited by first two pre-images of the discontinuity and this set have the property that every point is mapped inside it at finite time, just like the original Boole. We use this compact set to find an ergodic measure there and then spread it to all the real line using the dynamics. The measure that was spread is *infinite and ergodic*. Pulling the measure again via the holonomy, we get what we wanted.

It is important to notice that this one-dimensional reduction allow us to completely understand the dynamics for the parameters $a < b \leq 1$. However for the limit case, we do not know if it is possible to make this reduction for an one-dimensional dynamic. But to fully understand the dynamics, the stable and unstable foliations will give us a tool to decode the map and see exactly how the map behaves.

To get that, we focus on trying to understand how the images and pre-images of each discontinuity spread throughout the plane. Understanding how they spread give us some important information about how the invariant manifolds are, that they lie within these images of the discontinuities and help us to code the Hénon-Devaney map in a way that it describes exactly where each orbit is in the plane.

Theorem D. *There exists* $\Sigma_i, \Sigma_j \subsetneq \Sigma := \{-2, -1, 0, 1, 2\}^{\mathbb{Z}}$ *and* $h : \mathbb{R}^2 \to \Sigma_i \times \Sigma_j$ *an homeomorphism such that the following diagram commutes*

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "f"] \arrow[d, "h"'] & \mathbb{R}^2 \arrow[d, "h"] \\
\Sigma_i \times \Sigma_j \arrow[r, "\sigma"'] & \Sigma_i \times \Sigma_j
\end{tikzcd}$$

where

$$\sigma : \Sigma_i \times \Sigma_j \to \Sigma_i \times \Sigma_j$$

is the product of the usual shift maps restricted to each one of its respective spaces, that induces a subshift for $\Sigma_i \times \Sigma_j$.

The proof starts by looking at the invariant manifolds and their behavior with respect to the discontinuities turning them into a new system of coordinates of the plane.

We study of where each region comes from and is mapped to with respect these new coordinates, identifying a region where the dynamics changes drastically. The coding comes naturally with these coordinates, 1 representing the region where the dynamics change, 2 the region outside of it, -1 and -2 the reflected regions and 0 the points which will not have any other forward or backward iterate.

Even tough the ergodicity of the Hénon-Devaney map remains an open problem, if one can manage to prove that the map f is recurrent or, equivalently, that Lebesgue almost every point enters the "interesting" region $\mathcal{R}_{1,1}$ (the special region for the coding), the ergodicity should come naturally using some classical arguments.... but once again.... that is just a big "if"...

Chapter 1

The Two-Parameter Family

One of the most famous examples in Infinite Ergodic Theory is the Boole's Map defined by

$$\begin{array}{rcl} B:\mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x-\dfrac{1}{x} \end{array}$$

and it is easy to see that it preserves Lebesgue measure in the real line. In 1973, Adler and Weiss studied the ergodicity of this map in their paper [AW]. The main idea here, in simple words, is to understand how the pre-images of the discontinuity spread out through the real line and how it is possible to control the differential map.

Some one-dimensional generalizations of this map were studied by S. Muñoz in his PhD. Thesis [M], in which he finds some topological properties for the maps with a different behavior of the "infinity", that is, turning the indifferent fixed point that infinity is into something else depending on the parameter.

Hénon introduced a two-dimensional version of the Boole map, the Hénon-Devaney map we talked about a little bit in the introduction. He was studying the restricted three-body problem and using "truncated solutions" of the problem, the so called generating families.

A generalization to this map was object of study and S. Muñoz proposed and exhibited some properties of a two-parameter model of generalization in [M2]. The two-family is defined by

$$\begin{array}{rcl} f_{a,b}:\mathbb{R}^2\setminus\{y=0\} & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & \left(ax+\dfrac{1}{y}, by-\dfrac{b}{y}-abx\right) \end{array}$$

and here we will only consider $0<a\leq b\leq 1$. Observe that the Hénon-Devaney map is the "limit" in the space of parameters, that is, $f_{1,1}=f$.

This family will be the object of study throughout this thesis. At first, we will focus on giving some hyperbolic properties and finding an attracting region to these maps. It will become clear that this attracting region does not "exist" in the limit case.

It is an important remark that we will look at the $\mathbb{RP}^2$ as $\{(x,1); x \in \overline{\mathbb{R}}$ and $-\infty = \infty\}$. The idea behind it is that every point in the projective space will be the direction of the tangent of the unstable (respectively the stable) manifold.

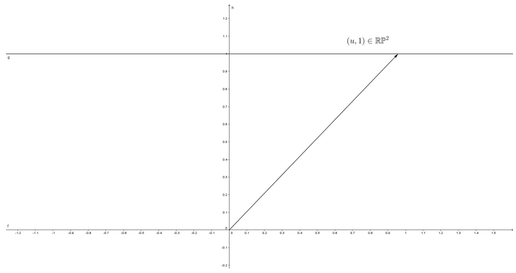

Figure 1.1: The projective space $\mathbb{RP}^2$.

Recall that the differential of each $f_{a,b}$ is given by

$$Df_{a,b}(x,y) = \begin{pmatrix} a & -\dfrac{1}{y^2} \\ -ab & b + \dfrac{b}{y^2} \end{pmatrix}$$

and when $|y| \to \infty$ we get that

$$Df_{a,b}(x,\infty) = \begin{pmatrix} a & 0 \\ -ab & b \end{pmatrix}$$

that have two invariant directions in the projective space, namely, $[(0,1)]$ and $\left[\left(\dfrac{(b-a)}{ab}, 1\right)\right]$.

Here we can see the first difference between the approach given to each one of the cases: when both parameters converge to their limit case 1, the second invariant direction collapses to the first one. In fact, this fact depends only on the a-parameter, he is the one playing the role of splitting those two directions.

In this chapter we want to prove theorem A

Theorem (A). *For every $a \le b \le 1$, there exists two invariant cone fields for $f_{a,b}$. These cone fields induce the directions E^{cu} and E^{cs}. For $a = b = 1$, the splitting is not uniformly dominated. For $a < b \le 1$, the splitting is dominated and E^{cs} is uniformly contractive. These that induces a C^1-foliation for the every map in the family, moreover if $a < b \le 1$ then these leaves have a C^2-differentiability.*

1.1 The Hénon-Devaney Map

Recall once again the Hénon map we defined in the introduction

$$\begin{aligned} f : \mathbb{R}^2 \setminus \{y = 0\} &\to \mathbb{R}^2 \\ (x,y) &\mapsto \left(x + \frac{1}{y}, y - \frac{1}{y} - x\right) \end{aligned}$$

and observe that its differential is given by

$$Df(x,y) = \begin{pmatrix} 1 & -\dfrac{1}{y^2} \\ -1 & 1+\dfrac{1}{y^2} \end{pmatrix}$$

The first thing we want to see is that f preservers the Lebesgue measure m and that is easy to see once we have that $\det(Df(x,y)) = 1$.

1.1.1 Invariant Cone Fields

Let us check that $Df(x,y)$ is hyperbolic for every finite $y \in \mathbb{R}^*$

$$Df(x,y)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda u \\ \lambda v \end{pmatrix}$$

in which we have

$$\begin{cases} u - \dfrac{1}{y^2}v & = \lambda u \\ -u + \left(1+\dfrac{1}{y^2}\right)v & = \lambda v \end{cases}$$

Replacing the first equation in the second we get

$$v = y^2(1-\lambda)u \tag{1.1}$$

Then

$$y^2 - y^2\lambda - \lambda = y^2\lambda - y^2\lambda^2$$

and follows that

$$y^2\lambda^2 - (2y^2+1)\lambda + y^2 = 0$$

Therefore

$$|\lambda_+| = \left|1 + \frac{1+\sqrt{1+4y^2}}{2y^2}\right| > 1$$

and also

$$|\lambda_-| = \left|1 + \frac{1-\sqrt{1+4y^2}}{2y^2}\right| < 1$$

From the equation 1.1, we can see that unstable eigenspace is located at the second and fourth quadrants and the stable one is the first and third. In terms of the projective space, the unstable eigenspace is a point $(u,1)$ for some negative u and the stable one is a point with some posite u.

Although $Df(x,y)$ is non-uniformly hyperbolic because both λ_+ and λ_- converge to 1 when $|y| \to \infty$, in fact, we have that

$$Df(x,\infty) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

and, using the notation we introduced before, we can see that $(0,1)$ is the invariant direction to $Df(x,\infty)$.

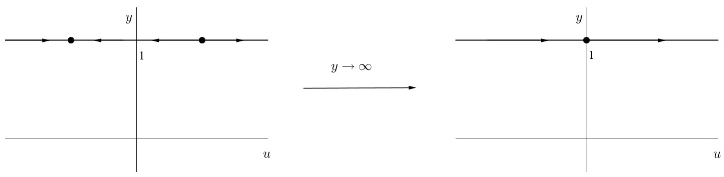

Figure 1.2: The transition from hyperbolic state to parabolic state.

In order to find the invariant direction we want to find an unstable cone field that does not depend on y that will be contracted by the differential on the projective space. Note that

$$Df(x,y)\begin{pmatrix} u \\ 1 \end{pmatrix} = \begin{pmatrix} u - \dfrac{1}{y^2} \\ -u + 1 + \dfrac{1}{y^2} \end{pmatrix}$$

and define

$$\begin{aligned} \mathcal{L}^f_{(x,y)} : \mathbb{RP}^2_{(x,y)} &\rightarrow \mathbb{RP}^2_{f(x,y)} \\ u &\mapsto \frac{u - \frac{1}{y^2}}{-u + 1 + \frac{1}{y^2}} = \frac{y^2 u - 1}{-y^2 u + y^2 + 1} \end{aligned}$$

Lemma 1.1. *$\mathcal{L}^f_{(x,y)}$ is a contraction in $(-\infty, 0)$.*

Proof. Let's compute $\frac{\partial \mathcal{L}^f_{(x,y)}}{\partial u}$ to see how it behaves

$$\left| \frac{\partial \mathcal{L}^f_{(x,y)}}{\partial u}(u) \right| = \left| \frac{y^2(-y^2u + y^2 + 1) + y^2(y^2 - 1)}{(-y^2u + y^2 + 1)^2} \right| = \frac{y^4}{(-y^2u + y^2 + 1)^2}$$

If $u \in (-\infty, 0)$ then

$$-y^2 u \geq 0 \Rightarrow (-y^2u + y^2 + 1)^2 \geq (y^2 + 1)^2$$

and hence

$$\left|\frac{\partial \mathcal{L}^f_{(x,y)}}{\partial u}(u)\right| \leq \frac{y^4}{y^4+2y^2+1} < 1 \qquad \forall u \in (-\infty, 0)$$

$\square$

Observe that $\lambda(y) := \frac{y^4}{y^4+2y^2+1}$ is uniform in $(u,1) \in \mathbb{RP}^2_{(x,y)}$, if $u<0$. Consider now the unstable cone in the tangent space

$$\mathcal{C}^u(x,y) = \{(u,v) \in \mathbb{R}^2 \ / \ uv < 0\}$$

and observe that the cone at the projective space is given by

$$[\mathcal{C}^u(x,y)] = \{(u,1) \in \mathbb{RP}^2_{(x,y)} \ / \ u < 0\}$$

Observe also that although we defined the cone as $\mathcal{C}^u(x,y)$, it does not actually depend on the base point (x,y).

The previous lemma tells us that $[\mathcal{C}^u(x,y)]$ is mapped by $\mathcal{L}^f_{(x,y)}$ inside the $[\mathcal{C}^u(f(x,y))]$, that is, if we look at the cone field in the tangent space we get

$$\overline{Df(x,y)(\mathcal{C}^u(x,y))} \subset \mathcal{C}^u(f(x,y)) \qquad \forall (x,y) \in \mathbb{R}^2 \setminus \{y=0\}$$

once we have that

$$\begin{aligned} \mathcal{L}^f_{(x,y)}(0) &= -\frac{1}{y^2+1} < 0 \\ \mathcal{L}^f_{(x,y)}(\infty) &= \mathcal{L}^f_{(x,y)}(-\infty) = -1 < 0 \end{aligned}$$

Let us check the same cone property for the inverse, we want to see what happens to the stable direction. Note that

$$\begin{aligned} f^{-1} : \mathbb{R}^2 \setminus \{x=-y\} &\to \mathbb{R}^2 \\ (x,y) &\mapsto \left(x - \frac{1}{x+y}, x+y\right) \end{aligned}$$

and observe that its differential is given by

$$Df^{-1}(x,y) = \begin{pmatrix} 1 + \frac{1}{(x+y)^2} & \frac{1}{(x+y)^2} \\ 1 & 1 \end{pmatrix}$$

Proceeding in the same way as before we can define

$$\begin{aligned} \mathcal{L}^{f^{-1}}_{(x,y)} : \mathbb{RP}^2_{(x,y)} &\to \mathbb{RP}^2_{f^{-1}(x,y)} \\ u &\mapsto \frac{u}{u+1} + \frac{1}{(x+y)^2} \end{aligned}$$

Lemma 1.2. $\mathcal{L}^{f^{-1}}_{(x,y)}$ *is a contraction in* $(0,\infty)$.

Proof. Like before, just do the computation

$$\frac{\partial \mathcal{L}^{f^{-1}}_{(x,y)}}{\partial u}(u) = \frac{1}{(u+1)^2} < 1 \qquad \forall u > 0$$

$\square$

Consider now the stable cone in the tangent space

$$\mathcal{C}^s(x,y) = \{(u,v) \in \mathbb{R}^2 \ / \ uv > 0\}$$

and observe that the cone at the projective space is given by

$$[\mathcal{C}^s(x,y)] = \{(u,1) \in \mathbb{RP}^2_{(x,y)} \ / \ u > 0\}$$

The same independence of the base point (x,y) we said before applies here.

Hence, as before, we get

$$\overline{Df^{-1}(x,y)(\mathcal{C}^s(x,y))} \subset \mathcal{C}^s(f^{-1}(x,y)) \qquad \forall (x,y) \in \mathbb{R}^2 \setminus \{x=-y\}$$

once we have that

$$\begin{aligned} \mathcal{L}^{f^{-1}}_{(x,y)}(0) &= \frac{1}{(x+y)^2} > 0 \\ \mathcal{L}^{f^{-1}}_{(x,y)}(\infty) &= \mathcal{L}^{f^{-1}}_{(x,y)}(-\infty) = 1 + \frac{1}{(x+y)^2} > 0 \end{aligned}$$

1.1.2 Invariant Manifolds

In this section we go a little further in the study we started before. We want to achieve the existence of invariant manifolds for almost every point of the plane. To conclude that we want to prove first that the intersection of the cone field is actually just one direction, that is, a dominated splitting given by

$$\mathfrak{C}^u(x,y) := \bigcap_{n\in\mathbb{N}} Df^n(f^{-n}(x,y))(\mathcal{C}^u(f^{-n}(x,y)))$$

Due the independence of each cone $\mathcal{C}^u(x,y)$ and the fact we discussed in the previous section, we have only two possibilities for $\mathfrak{C}^u(x,y)$: either it is a single point in the projective space or it is an interval inside $[\mathcal{C}^u] = [-\infty, 0]$.

Once we know that

$$Df(x,\infty) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

we can define in a similar way as we did before

$$\mathcal{L}^f_\infty(u) := \left[Df(x,\infty)\begin{pmatrix} u \\ 1 \end{pmatrix}\right] = \frac{u}{-u+1}$$

which is increasing for $u \in \mathcal{C}^u$ because the differential with respect to u is given by

$$\frac{\partial \mathcal{L}^f_\infty}{\partial u}(u) = \frac{1}{(-u+1)^2}$$

and is increasing when $u < 0$.

Recall that

$$\begin{cases} \mathcal{L}^f_{(x,y)}(0) & = \dfrac{-1}{y^2+1} < 0 \\ \mathcal{L}^f_{(x,y)}(-\infty) & = -1 \end{cases}$$

Then we get also that $\mathfrak{C}^u(x,y)$ is contained in an interval that is given by

$$Df(x_{-1},y_{-1})(\mathcal{C}^u) = \left[-1, \frac{-1}{y^2_{-1}+1}\right]$$

where $f^{-1}(x,y) = (x_{-1},y_{-1})$

Extrapolating the notation above, denote by $(x_{-n},y_{-n}) := f^{-n}(x,y)$, once we know that

$$\mathfrak{C}^u(x,y) \subset Df^n(x_{-n},y_{-n})(\mathcal{C}^u) \subset \left[-1, \frac{-1}{y^2_{-1}+1}\right] \qquad \forall n > 0$$

it is possible to improve the estimative we got earlier using the fact that we can control where the cone lies in every step and the fact that the $\frac{\partial \mathcal{L}^f}{\partial u}$ is increasing in u:

$$\left|\frac{\partial \mathcal{L}^f_{(x_{-n},y_{-n})}}{\partial u}\left(\frac{-1}{y^2_{-1}+1}\right)\right| = \frac{y^4_{-n}}{\left(y^2_{-n}\left(\frac{-1}{y^2_{-1}+1}\right)+y^2_{-n}+1\right)^2} = \frac{(y^2_{-1}+1)^2}{\left(y^2_{-1}+2+\frac{y^2_{-1}}{y^2_{-n}}+\frac{1}{y^2_{-n}}\right)^2}$$

Hence

$$\left|\left.\frac{\partial \mathcal{L}^f_{(x_{-n},y_{-n})}}{\partial u}\right|_{\left[-1,\frac{-1}{y^2_{-1}+1}\right]}\right| \le \frac{(y^2_{-1}+1)^2}{\left(y^2_{-1}+2\right)^2} < 1 \qquad \forall n > 0$$

The worst-case scenario is when "$y = \infty$", but even in that case we get the same contraction rate

$$\left|\left.\frac{\partial \mathcal{L}^f_\infty}{\partial u}\right|_{\left[-1,\frac{-1}{y^2_{-1}+1}\right]}\right| \le \frac{(y^2_{-1}+1)^2}{\left(y^2_{-1}+2\right)^2} < 1$$

and it only have one fixed point, namely $u = 0$.

To conclude that $[\mathfrak{C}^u(x,y)]$ is a single point in the projective space, all that is necessary is to have all the pre-images of (x,y) defined. This means we can only achieve this outside the set of the pre-images of the discontinuity of f^{-1}: $\cup_{n\geq 0} f^n(\{y=-x\})$.

The same idea will be used to get the stable direction. Consider now

$$\mathfrak{C}^s(x,y) := \bigcap_{n\in\mathbb{N}} Df^{-n}(f^n(x,y))(\mathcal{C}^s(f^n(x,y)))$$

Once again $\mathfrak{C}^s(x,y)$ is contained in the interval

$$Df^{-1}(x_1,y_1)(\mathcal{C}^s) = \left[\frac{1}{(x_1+y_1)^2}, 1 + \frac{1}{(x_1+y_1)^2}\right]$$

where $f(x,y) = (x_1,y_1)$. Using the same extrapolation of the notation and the fact that $\frac{\partial \mathcal{L}^{f^{-1}}}{\partial u}$ is decreasing in u

$$\left|\frac{\partial \mathcal{L}^{f^{-1}}_{(x_n,y_n)}}{\partial u}\left(\frac{1}{(x_1+y_1)^2}\right)\right| = \frac{1}{\left(\frac{1}{(x_1+y_1)^2}+1\right)^2} = \frac{(x_1+y_1)^2}{((x_1+y_1)^2+1)^2}$$

getting

$$\left|\left.\frac{\partial \mathcal{L}^{f^{-1}}_{(x_n,y_n)}}{\partial u}\right|_{\left[\frac{1}{(x_1+y_1)^2}, 1+\frac{1}{(x_1+y_1)^2}\right]}\right| \leq \frac{(x_1+y_1)^2}{((x_1+y_1)^2+1)^2} < 1 \qquad \forall n > 0$$

To finish this part of the proof it is necessary, once again, bypass the set that all the images of (x,y) are not defined. Now we have to rule out the pre-images of the discontinuity of f: $\cup_{n\geq 0} f^{-n}(\{y=0\})$. This set, alongside with the analogous of f^{-1}, will be explored in the next section, and it will help to understand how the lamination given by the discontinuities behave alongside of the invariant manifolds found here.

1.1.3 C^1-Foliation

To establish the C^1-variation of the foliation of the directions, all that is necessary is to study the differential of the graphic transformation, it is necessary to check that it is also a contraction for some special cone fields. And the cone fields are indeed special, in fact, they are the same cone fields used for $\mathcal{L}^f_{x,y}$ and $\mathcal{L}^{f^{-1}}_{x,y}$.

Observe we know that

- $\dfrac{\partial \mathcal{L}^f_{(x,y)}}{\partial u}(u) = \dfrac{y^4}{(-y^2u+y^2+1)^2}$

- $\dfrac{\partial \mathcal{L}^f_{(x,y)}}{\partial x}(u) = 0$
- $\dfrac{\partial \mathcal{L}^f_{(x,y)}}{\partial y}(u) = \dfrac{2y}{(-y^2u + y^2 + 1)^2}$

which implies that

$$\frac{\partial \mathcal{L}^f_{(x,y)}}{\partial u}(u) + \frac{\partial \mathcal{L}^f_{(x,y)}}{\partial x}(u) + \frac{\partial \mathcal{L}^f_{(x,y)}}{\partial y}(u) = \frac{y^4 + 2y}{(-y^2u + y^2 + 1)^2} < 1$$

for all $u \in \mathcal{C}^u$, just suppose the contrary and you will obtain a contradiction.

All the differentials of $\mathcal{L}^{f^{-1}}_{(x,y)}$

- $\dfrac{\partial \mathcal{L}^{f^{-1}}_{(x,y)}}{\partial u}(u) = \dfrac{1}{(u+1)^2}$
- $\dfrac{\partial \mathcal{L}^{f^{-1}}_{(x,y)}}{\partial x}(u) = \dfrac{-2}{(x+y)^3}$
- $\dfrac{\partial \mathcal{L}^{f^{-1}}_{(x,y)}}{\partial y}(u) = \dfrac{-2}{(x+y)^3}$

and, again we get analogously that

$$\frac{\partial \mathcal{L}^{f^{-1}}_{(x,y)}}{\partial u}(u) + \frac{\partial \mathcal{L}^{f^{-1}}_{(x,y)}}{\partial x}(u) + \frac{\partial \mathcal{L}^{f^{-1}}_{(x,y)}}{\partial y}(u) = \frac{(x+y)^3 - 4(u+1)^2}{(x+y)^3(u+1)^2} < 1$$

for all $u \in \mathcal{C}^s$.

To conclude, just proceed in the same way we did before.

1.2 Dominated Splitting for the Family

In this section we make the first moves towards the understanding and describing this two-parameter family. We want to find unstable and stable cones in order to determine a *dominated splitting* and invariant manifolds for $f_{a,b}$ in Lebesgue-almost every point.

To do so, let us compute

$$Df_{a,b}(x,y)\begin{pmatrix} u \\ 1 \end{pmatrix} = \begin{pmatrix} au - \dfrac{1}{y^2} \\ -abu + b + \dfrac{b}{y^2} \end{pmatrix}$$

and look at it in the projective space. This defines the map bellow

$$\begin{array}{rcl} \mathcal{L}^{f_{a,b}}_{(x,y)} : \mathbb{RP}^2_{(x,y)} & \to & \mathbb{RP}^2_{f(x,y)} \\ u & \mapsto & \dfrac{au - \frac{1}{y^2}}{-abu + b + \frac{b}{y^2}} = \dfrac{ay^2u - 1}{-aby^2u + by^2 + b} \end{array}$$

Lemma 1.3. *$\mathcal{L}^{f_{a,b}}_{(x,y)}$ is a contraction in $(-\infty, \frac{(b-a)}{ab})$.*

Proof. Computing $\frac{\partial \mathcal{L}^{f_{a,b}}_{(x,y)}}{\partial u}$ to see its behavior

$$\left| \frac{\partial \mathcal{L}^{f_{a,b}}_{(x,y)}}{\partial u}(u) \right| = \left| \frac{ay^2(-aby^2u + by^2 + b) + aby^2(ay^2 - 1)}{(-aby^2u + by^2 + b)^2} \right| = \frac{aby^4}{(-aby^2u + by^2 + b)^2}$$

For $u \in (-\infty, 0)$ we get

$$-aby^2u \geq 0 \Rightarrow (-aby^2u + by^2 + b)^2 \geq (by^2 + b)^2$$

and hence

$$\left| \frac{\partial \mathcal{L}^{f_{a,b}}_{(x,y)}}{\partial u}(u) \right| \leq \frac{a}{b} \cdot \frac{y^4}{y^4 + y^2 + 1} < 1 \qquad \forall u \in (-\infty, 0)$$

For $0 \leq u \leq \frac{(b-a)}{ab}$, we have that

$$\begin{array}{rcl} \mathcal{L}^{f_{a,b}}_{(x,y)}\left(\dfrac{(b-a)}{ab}\right) & = & -\dfrac{y^2(b-a) - 1}{aby^2 + b^2} \\ \mathcal{L}^{f_{a,b}}_{(x,y)}(0) & = & \dfrac{-1}{by^2 + b} < 0 \end{array}$$

Hence, subtracting one from the other we have

$$\mathcal{L}^{f_{a,b}}_{(x,y)}\left(\frac{(b-a)}{ab}\right) - \mathcal{L}^{f_{a,b}}_{(x,y)}(0) = \frac{by^4(b-a)}{(aby^2 + b^2)(by^2 + b)} > 0$$

and comparing the length of each one of the intervals we get

$$\frac{by^4(b-a)}{(aby^2 + b^2)(by^2 + b)} \cdot \frac{ab}{(b-a)} = \frac{ab^2y^4}{ab^2y^4 + ab^2y^2 + b^4y^2 + b^3} < 1$$

□

Observe that $\lambda(a, b, x, y) := \frac{a}{b} \cdot \frac{y^4}{y^4 + y^2 + 1} < 1$ is uniform in $(u, 1) \in \mathbb{RP}^2_{(x,y)}$, if $u < 0$. Even more, it is interesting to see also that for every $(x, y) \in \mathbb{R}^2 \setminus \{y = 0\}$ we get an *uniform control* of the contraction

$$y^4 + 2y^2 + 1 \geq y^4 \Rightarrow \lambda(a, b, x, y) \leq \frac{ay^4}{by^4} = \frac{a}{b} < 1$$

Also it is important to note that even though we denote λ as a function of 4 parameters, the contraction is once again linked directly to the parameter a. Once more, this differs deeply from the limit case $a = b = 1$, where we do not have an uniform bound for this interval. However, we do not have uniformity of control in the interval $(0, \frac{(b-a)}{ab})$.

We are in a good place of defining what will be the unstable cone in the tangent space. Indeed, define by

$$\mathcal{C}^u(x,y) = \left\{ (u,v) \in \mathbb{R}^2 \ /(u,v) \text{ lies outside the region delimited by } v = 0 \text{ and } u = \frac{(b-a)}{ab} v \right\}$$

and its representation in the projective space is given by

$$[\mathcal{C}^u(x,y)] = \left\{ (u,1) \in \mathbb{RP}^2_{(x,y)} \ / \ u < \frac{(b-a)}{ab} \right\}$$

Observe also that although we defined the cone as $\mathcal{C}^u(x,y)$, it does not actually depend on the base point (x,y).

Let us take a moment to stress something here that might be overlooked, the definition of the unstable cone *does not* depend on the parameters a and b. It is defined like that because, even if we lose the uniformity of the bound, the same cone is contracted for all $f_{a,b}$, for $a = b = 1$ included.

The previous lemma tells us that $[\mathcal{C}^u(x,y)]$ is mapped by $\mathcal{L}^{f_{a,b}}_{(x,y)}$ inside the $[\mathcal{C}^u(f_{a,b}(x,y))]$, that is, if we look at the cone field in the tangent space we get

$$\overline{Df_{a,b}(x,y)(\mathcal{C}^u(x,y))} \subset \mathcal{C}^u(f_{a,b}(x,y)) \qquad \forall (x,y) \in \mathbb{R}^2 \setminus \{y = 0\}$$

once we have that

$$\begin{aligned} \mathcal{L}^{f_{a,b}}_{(x,y)}\left(\frac{(b-a)}{ab}\right) &= -\frac{y^2(b-a)-1}{aby^2+b^2} < \frac{(b-a)}{ab} \\ \mathcal{L}^{f_{a,b}}_{(x,y)}(\infty) &= \mathcal{L}^{f_{a,b}}_{(x,y)}(-\infty) = -\frac{1}{b} < 0 \end{aligned}$$

and repeat here the same argument we used in the first section to get the existence of E^{cu}, that is, the intersection of the unstable cones is a single point.

Now it is time to check the cone property for the inverse, the objective is to draw the same consequences in order to get the stable direction. As the reader already knows

$$\begin{aligned} f^{-1}_{a,b} : \mathbb{R}^2 \setminus \{bx = -y\} &\to \mathbb{R}^2 \\ (x,y) &\mapsto \left(\frac{x}{a} - \frac{1}{a(x+\frac{y}{b})}, x + \frac{y}{b} \right) \end{aligned}$$

and observe that its differential is given by

$$Df^{-1}_{a,b}(x,y) = \begin{pmatrix} \frac{1}{a} + \frac{1}{a(x+\frac{y}{b})^2} & \frac{1}{ab(x+\frac{y}{b})^2} \\ 1 & \frac{1}{b} \end{pmatrix}$$

Define now the graphic operator for the inverse

$$\begin{aligned}\mathcal{L}^{f^{-1}_{a,b}}_{(x,y)} : \mathbb{RP}^2_{(x,y)} &\to \mathbb{RP}^2_{f^{-1}_{a,b}(x,y)} \\ u &\mapsto \frac{b(x+\frac{y}{b})^2 u + bu + 1}{ab(u+\frac{1}{b})(x+\frac{y}{b})^2}\end{aligned}$$

and at this point we have to split the study a bit more due a divergence of behavior depending on the value of b.This get clearer when looking at the differential with respect to the u-coordinate

$$\frac{\partial \mathcal{L}^{f^{-1}_{a,b}}_{(x,y)}}{\partial u}(u) = \frac{1}{ab(u+\frac{1}{b})^2} \qquad \forall u \in \mathbb{R}$$

exhibiting a stable cone that depends on a, corroborating and making explicit that the even the limit case we have separated directions E^{cu} and E^{cs} which does not happen when $a = b = 1$.

Lemma 1.4. $\mathcal{L}^{f^{-1}_{a,b}}_{(x,y)}$ *is a contraction in* $(\frac{(b-a)}{ab}, \infty)$.

Proof. Just like the previous lemma, start by computing the following differential

$$\frac{\partial \mathcal{L}^{f^{-1}_{a,b}}_{(x,y)}}{\partial u}(u) = \frac{1}{ab(u+\frac{1}{b})^2} < 1 \qquad \forall u > \frac{(b-a)}{ab}$$

□

Here we get an *uniform control* of the contraction because

$$\left(u+\frac{1}{b}\right)^2 \geq \left(\frac{(b-a)}{ab}+\frac{1}{b}\right)^2 \Rightarrow \frac{\partial \mathcal{L}^{f^{-1}_{a,b}}_{(x,y)}}{\partial u}(u) = \frac{1}{ab(u+1)^2} \leq \frac{a}{b} < 1 \qquad \forall u > \frac{(b-a)}{ab}$$

Consider now the stable cone in the tangent space

$$\mathcal{C}^s(x,y) = \left\{(u,v) \in \mathbb{R}^2 \ /(u,v) \text{ lies inside the region delimited by } v=0 \text{ and } u = \frac{(b-a)}{ab}v\right\}$$

and observe that the cone at the projective space is given by

$$[\mathcal{C}^s(x,y)] = \left\{(u,1) \in \mathbb{RP}^2_{(x,y)} \ / \ u > \frac{(b-a)}{ab}\right\}$$

The same independence of the base point (x,y) we said before applies here.

Hence, as before, we get

$$\overline{Df^{-1}_{a,b}(x,y)(\mathcal{C}^s(x,y))} \subset \mathcal{C}^s(f^{-1}_{a,b}(x,y)) \qquad \forall (x,y) \in \mathbb{R}^2 \setminus \{bx=-y\}$$

once we have that

$$\begin{aligned}
\mathcal{L}_{(x,y)}^{f_{a,1}^{-1}}\left(\frac{(b-a)}{ab}\right) &= \frac{(b-a)(x+\frac{y}{b})^2+b}{ab(x+\frac{y}{b})^2} > \frac{(b-a)}{ab} \\
\mathcal{L}_{(x,y)}^{f_{a,1}^{-1}}(\infty) &= \mathcal{L}_{(x,y)}^{f_{a,1}^{-1}}(-\infty) = \frac{1}{a} + \frac{1}{a(x+\frac{y}{b})^2} > \frac{(b-a)}{ab}
\end{aligned}$$

All the same arguments we gave before work here with no problem whatsoever. We still have the independence of the base point in the definition of the cone field and now we have some kind of uniformity depending on a and b.

Now we need to prove the differentiability of the invariant manifolds. To do that let us proceed just like the Hénon-Devaney map. Let us compute all the differentials of $\mathcal{L}_{(x,y)}^{f_{a,b}}$

- $\dfrac{\partial \mathcal{L}_{(x,y)}^{f_{a,b}}}{\partial u}(u) = \dfrac{aby^4}{(-aby^2u + by^2 + b)^2}$
- $\dfrac{\partial \mathcal{L}_{(x,y)}^{f_{a,b}}}{\partial x}(u) = 0$
- $\dfrac{\partial \mathcal{L}_{(x,y)}^{f_{a,b}}}{\partial y}(u) = \dfrac{2by}{(-aby^2u + by^2 + b)^2}$

and all the differentials of $\mathcal{L}_{(x,y)}^{f_{a,b}^{-1}}$

- $\dfrac{\partial \mathcal{L}_{(x,y)}^{f_{a,b}^{-1}}}{\partial u}(u) = \dfrac{1}{ab(u+\frac{1}{b})^2}$
- $\dfrac{\partial \mathcal{L}_{(x,y)}^{f_{a,b}^{-1}}}{\partial x}(u) = \dfrac{-2bu-2}{ab(x+\frac{y}{b})^3(u+\frac{1}{b})}$
- $\dfrac{\partial \mathcal{L}_{(x,y)}^{f_{a,b}^{-1}}}{\partial y}(u) = \dfrac{-2u-\frac{2}{b}}{ab(x+\frac{y}{b})^3(u+\frac{1}{b})}$

Proceeding in the same way as before we get the C^1-foliation and looking at the next differential we get the C^2. The fact we have the parameters a, b will give an extra control of the contraction rate, which gives us the C^2 differentiability.

Chapter 2

The Attractor $\Lambda_{a,b}$

In this chapter we want to find an attracting region in the plane, that is, we want to determine that, for every $a < b \leq 1$, there exists a subset $\mathfrak{R}_{a,b} \subset \mathbb{R}^2$ such that its images is contained inside itself. Due the symmetry, we will only focus on doing determining the attracting region for half of the plane and it will determined everywhere.

Our main objective in this chapter is to prove the following statement

Theorem (B). *For every $a < b \leq 1$, there exists a global attractor*

$$\Lambda_{a,b} := \bigcap_{n\in\mathbb{N}} f^n_{a,b}(\mathfrak{R}_{a,b}) \subset \mathfrak{R}_{a,b}$$

which is not compact. This induces an one dimensional dynamics via the projection along that stable manifold to $\{x = 0\}$:

$a < b = 1:$ *the one dimensional map is "Boole-like"*

$a < b < 1:$ *the induced map is a "Boole-like expanding" map.*

Let us take a moment just to recall the definitions of Boole-like and Boole-like expanding.

Definition. *We say that $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is Boole-like if*

(i) $g(x) < x$ *for* $x > 0$ *and* $g(x) > x$ *for* $x < 0$*. Also,* $\lim_{x\to\pm\infty} g(x) = \pm\infty$*;*

(ii) g *is increasing for every* $x \in \mathbb{R} \setminus \{0\}$*;*

(iii) $\lim_{|x|\to\infty} g'(x) = 1$ *and* $\lim_{|x|\to 0} g'(x) = \infty$*;*

(iv) $\lim_{|x|\to\infty} |g^{(k)}(x)|$ *bounded for every* $k \in \mathbb{N}$*,*

The idea behind this definition is trying to establish a notion of maps that have its graphic similar to the graphic of the Boole, a perturbation of it without creating fixed points.

Definition. *We say that* $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *is a Boole-like expanding if*

(i') *Conditions* (i), (ii) *and* (iv) *as before;*

(ii') $\lim_{|x|\to\infty} g'(x) \leq a < 1$ *and* $\lim_{|x|\to 0} g'(x) = \infty$*;*
And all of them imply, as we will see in section 2.3, that

(iii') *there exists an interval* I *that contains the discontinuity such that the return map* $g_I : I \to I$ *is expanding.*

For the second definition, as we said before, the objective is to think of something like the traditional Boole but instead of having the asymptote in the $\{x = y\}$ this map has an asymptote line that is below the identity.

2.1 Attracting Region

The first step we need to take is to find where the attractor lies, that consists in determining a region that is mapped inside itself by $f_{a,b}$. This region will be the attracting basin of our attractor.

We will make a few assumptions, splitting the steps into different regions in the domain of $f_{a,b}$ depending on where the image of that set is mapped to. Recall once again that, for every a, b we have a symmetry that allow us only to determine the region only for "half" of the plane, which will eliminate a few of those splits. We will denote by $f_{a,b}(\cdot)_x$ the x-coordinate of $f_{a,b}$ and $f_{a,b}(\cdot)_y$ the y-coordinate.

At first, let us consider the points (x, y) such that $y > 0$, $x + \frac{y}{b} > 0$ and $f_{a,b}(x, y)_y > 0$. We want to find x_0 such that if $x \leq x_0$ then $-\frac{f_{a,b}(x,y)_y}{b} < f_{a,b}(x, y)_x \leq x_0$

It is easy to see that $-\frac{f_{a,b}(x,y)_y}{b} < f_{a,b}(x, y)_x$ because it is a direct consequence of the fact that $y > 0$. It is enough to determine x_0 such the inequalities we said before is satisfied due the continuity of $f_{a,b}$ in that region and the fact that $ax + \frac{1}{y}$ is increasing in terms of x for any fixed y.

Observe that

$$f_{a,b}(x_0, y)_x \leq x_0 \Rightarrow \frac{1}{y} \leq (1 - a)x_0$$

We also know that $ax_0 + \frac{1}{y}$ is decreasing in terms of y, which implies that it is enough to satisfy the previous condition for y_0 such that

$$by_0 - \frac{b}{y_0} - abx_0 = 0$$

which implies that

$$\frac{1}{(1 - a)x_0} \leq y_0$$

For the (x, y) such that $y > 0$, $x + \frac{y}{b} > 0$ and $f_{a,b}(x, y)_y < 0$, we want to find x_0 such that the image of $f_{a,b}$ is contained inside the area delimited by $y = -bt$ and $y = -b(t - \frac{x_0}{b})$. Asking that the last condition holds is the same as asking that

$$by - \frac{b}{y} - abx = f_{a,b}(x, y)_y \leq -b\left((f_{a,b}(x, y)_x) - \frac{x_0}{b}\right) = -b\left(\left(ax + \frac{1}{y}\right) - \frac{x_0}{b}\right)$$

which implies that

$$by \leq x_0$$

Now using the fact that $by - \frac{b}{y} - abx_0$ is increasing in terms of y we get that it is enough to get condition above for y_0, that is

$$y_0 \leq \frac{x_0}{b}$$

Hence, these two conditions together we get that

$$\frac{1}{(1-a)x_0} \leq \frac{x_0}{b} \Rightarrow x_0^2 \geq \frac{b}{(1-a)}$$

We have two remaining cases: (x, y) such that $y < 0$, $x + \frac{y}{b} > 0$, (x, y) is contained inside area delimited by $y = -bt$ and $y = -b(t - \frac{x_0}{b})$ and $f_{a,b}(x, y)_y < 0$, we want that the image of $f_{a,b}$ is such that $f_{a,b}(x, y)_x > -x_0$ and (x, y) such that $y < 0$, $x + \frac{y}{b} > 0$, (x, y) is contained inside area delimited by $y = -bt$ and $y = -b(t + \frac{x_0}{b})$ and $f_{a,b}(x, y)_y > 0$, we want that the image of $f_{a,b}$ is contained inside the area delimited by $y = -bt$ and $y = -b(t + \frac{x_0}{b})$. If we proceed in the same way as before, we are going to get the same condition.

Taking x_0 such the condition above hold, it will give us the attracting region $\mathfrak{R}_{a,b}$ once we get that $f_{a,b}(\mathfrak{R}_{a,b}) \subset \mathfrak{R}_{a,b}$ a direct consequence of our choice of x_0.

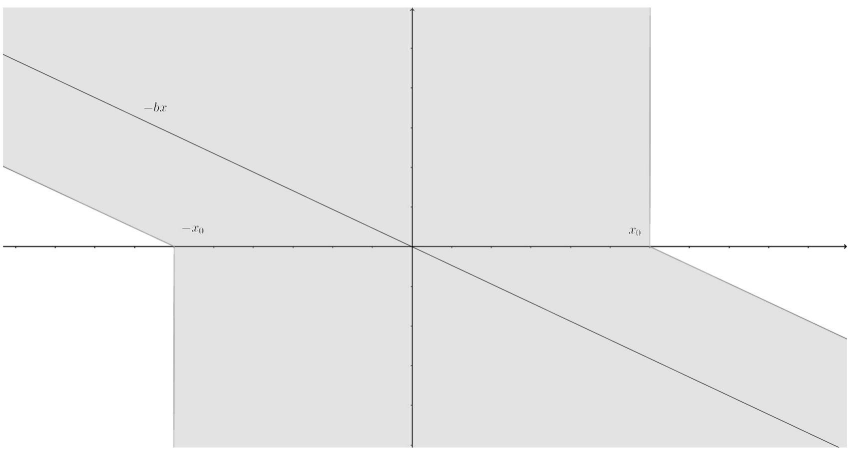

Figure 2.1: The region $\mathfrak{R}_{a,b}$.

Remark: Here we have an interesting thing to point out. When $a \to 1$, the attracting region becomes the whole plane and there is no point on finding the attractor. We lose this region for the Hénon-Devaney map.

2.2 Topological Consequences of the Attractor

In this section we want to conclude the proof of the Theorem B. We already have all the tools to conclude the proof. In everything that follows, for obvious reasons, we only will deal with $a < b \leq 1$.

Let us recall that we have the graphic transformation introduced in the last chapter given by

$$\mathcal{L}_{(x,y)}^{f_{a,b}^{-1}}(u) = \frac{b(x+\frac{y}{b})^2 u + bu + 1}{ab(u+\frac{1}{b})(x+\frac{y}{b})^2}$$

and if $(x+\frac{y}{b})^2 \to \infty$ we have the inverse of $\mathcal{L}_{\infty}^{f_{a,b}}$ we defined before that is given by

$$\mathcal{L}_{\infty}^{f_{a,b}^{-1}}(u) = \frac{bu}{a(bu+1)}$$

that is, if the operator that is defined by the matrix $Df_{a,b}^{-1}$ when we take any curve going to infinite which is not asymptotic to the discontinuity of the inverse at infinity.

Recall that, as we stated in the introduction of the Chapter 2, the matrix $Df_{a,b}^{-1}$ have two invariant directions, that is, $\mathcal{L}_{\infty}^{f_{a,b}^{-1}}$ have two fixed points, namely $u = 0$ and $= \frac{(b-a)}{ab} > 0$. Our goal is to use the foliation given by $\mathcal{L}_{\infty}^{f_{a,b}^{-1}}$, that is, straight lines with slope $\frac{(b-a)}{ab}$ to approach the holonomy given by $\mathcal{L}_{(x,y)}^{f_{a,b}^{-1}}$ if $(x+\frac{y}{b})^2$ is big enough.

But this arouses the first problem with this approach. The region $\mathfrak{R}_{a,b}$ *contains* the line $\{y = -bx\}$, which could imply that the attractor given by the intersection of all forward iterates of this region could be "close to $\{y = -bx\}$ at infinity". This can be easily solved just by looking at the second iterate of the region $\mathfrak{R}_{a,b}$.

Observe that the attractor is contained in the region delimited by two pairs of hyperboles, the image of $\{(x_0, y), y \in \mathbb{R}\}$ and the image of the $\{y = -bx\}$ itself: just look at $f_{a,b}(t, -bt) = (at - \frac{1}{bt}, -b(b+a)t + \frac{1}{t})$ and $f_{a,b}(x_0, y)$.

Let us elaborate this a little better, just observe that

$$\bigtriangledown f_{a,b}(t, -bt) = \left(a + \frac{1}{bt^2}, -b(b+a) - \frac{1}{t^2}\right)$$

which implies that

$$[\bigtriangledown f_{a,b}(t, -bt)] = \left(\frac{a + \frac{1}{bt^2}}{-b(b+a) - \frac{1}{t^2}}, 1\right)$$

and this tells us that when $|t| \to \infty$ g, the image of $\{y = -bx\}$ is asymptotic to the direction $(\frac{a}{-b(b+a)}, 1)$ and to $(1, -b)$ when $|t| \to 0$.

One could ask about what happens to the $f_{a,b}(t,-bt)$ when $|x| \to 0$ once we get that it is asymptotic to the line $\{y=-bx\}$, but it can be reduce to the previous case just by iterating once and it will enter the region that is away from the discontinuity of $f_{a,b}^{-1}$.

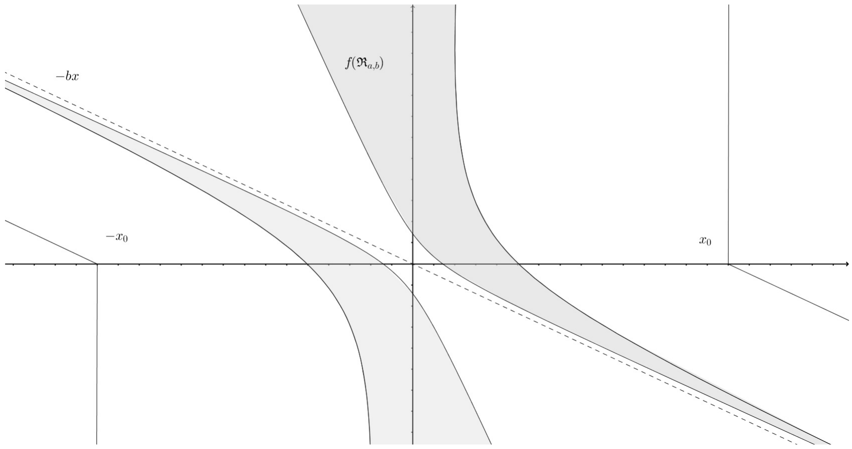

Figure 2.2: The new attracting region.

Next, we want to achieve the C^2-convergence of the operators $\mathcal{L}_{(x,y)}^{f_{a,b}^{-1}} \to \mathcal{L}_{\infty}^{f_{a,b}^{-1}}$ when $(x+\frac{y}{b})^2 \to \infty$ and $\theta_{(x,y)} \to \frac{(b-a)}{ab}$, where $\theta_{(x,y)}$ is the fixed point of $\mathcal{L}_{(x,y)}^{f_{a,b}^{-1}}$ related to the stable direction.

Once we know that all points in the new attracting region lies above the limiting line $\{(\frac{a}{b(a+b)}t, t\}$, follows that

$$x \geq -\frac{a}{b(a+b)}y \Rightarrow \left(x+\frac{y}{b}\right)^2 \geq \left(\frac{1}{b}-\frac{a}{b(a+b)}\right)^2 y^2$$

and also

$$x \leq x_0 \Rightarrow \left(x+\frac{y}{b}\right)^2 \leq \left(x_0+\frac{y}{b}\right)^2$$

which tells us about the order of $(x+\frac{y}{b})^2$.

Take orbit $f_{a,b}^{-n}(x,y)$ such that $\theta_n := \theta_{f_{a,b}^{-n}(x,y)}$ and which $f_{a,b}^{-n}(x,y)_y \to \infty$. This is the orbit which will give the convergence of the stable direction. Observe that, using the uniformity of the contraction rate of the differential, we get

$$|\mathcal{L}_{f_{a,b}^{-n}(x,y)}^{f_{a,b}^{-1}} \circ \cdots \circ \mathcal{L}_{(x,y)}^{f_{a,b}^{-1}}(I)| \leq \frac{1}{b}\left(\frac{a}{b}\right)^n \overset{n\to\infty}{\longrightarrow} 0$$

where $I = \left[\frac{(b-a)}{ab}, \infty\right]$, implying that the limit Θ of the stable direction θ_n exists and is greater or equal than $\frac{(b-a)}{ab}$.

However we also have that $\mathcal{L}^{f_{a,b}^{-1}}_{(x,y)} \xrightarrow{C^0} \mathcal{L}^{f_{a,b}^{-1}}_{\infty}$ once we know that

$$\mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(u) - \mathcal{L}^{f_{a,b}^{-1}}_{\infty}(u) = \frac{1}{a(x+\frac{y}{b})^2} \longrightarrow 0 \qquad \forall u \in \mathbb{R}$$

when $y \to \infty$. Given $\varepsilon > 0$, there is n_0 such that for $n \geq n_0$,

$$|\mathcal{L}^{f_{a,b}^{-1}}_{f_{a,b}^{-n}(x,y)}(u) - \mathcal{L}^{f_{a,b}^{-1}}_{\infty}(u)| < \varepsilon$$

This implies that

$$\begin{aligned}\left|\Theta - \frac{(b-a)}{ab}\right| \leq & \left|\mathcal{L}^{f_{a,b}^{-1}}_{f_{a,b}^{-n}(x,y)} \circ \cdots \circ \mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(\theta_n) - \Theta\right| \\ & + \left|\mathcal{L}^{f_{a,b}^{-1}}_{f_{a,b}^{-n}(x,y)} \circ \cdots \circ \mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(\theta_n) - \mathcal{L}^{f_{a,b}^{-1}}_{\infty} \circ \cdots \circ \mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(\theta_n)\right| \\ & + \left|\mathcal{L}^{f_{a,b}^{-1}}_{\infty} \circ \mathcal{L}^{f_{a,b}^{-1}}_{f_{a,b}^{-(n-1)}(x,y)} \circ \cdots \circ \mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(\theta_n) - \frac{(b-a)}{ab}\right|\end{aligned}$$

Which tells us that

$$\left|\Theta - \frac{(b-a)}{ab}\right| \leq 2\varepsilon + \left|\mathcal{L}^{f_{a,b}^{-1}}_{\infty} \circ \cdots \circ \mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(\theta_n) - \frac{(b-a)}{ab}\right|$$

But we have that

$$\begin{aligned}\left|\mathcal{L}^{f_{a,b}^{-1}}_{\infty} \circ \cdots \circ \mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(\theta_n) - \frac{(b-a)}{ab}\right| &= \left|\mathcal{L}^{f_{a,b}^{-1}}_{\infty} \circ \cdots \circ \mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(\theta_n) - \mathcal{L}^{f_{a,b}^{-1}}_{\infty}\left(\frac{(b-a)}{ab}\right)\right| \\ &\leq \frac{a}{b}\left|\mathcal{L}^{f_{a,b}^{-1}}_{f_{a,b}^{-(n-1)}(x,y)} \circ \cdots \circ \mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(\theta_n) - \frac{(b-a)}{ab}\right|\end{aligned}$$

Again, the last one is less or equal than

$$\left|\mathcal{L}^{f_{a,b}^{-1}}_{f_{a,b}^{-(n-1)}(x,y)} \circ \cdots \circ \mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(\theta_n) - \mathcal{L}^{f_{a,b}^{-1}}_{f_{a,b}^{-(n-1)}(x,y)} \circ \cdots \circ \mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(\theta_{n-1})\right| + \left|\mathcal{L}^{f_{a,b}^{-1}}_{f_{a,b}^{-(n-1)}(x,y)} \circ \cdots \circ \mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(\theta_{n-1}) - \frac{(b-a)}{ab}\right|$$

For the first term we have

$$\left|\mathcal{L}^{f_{a,b}^{-1}}_{f_{a,b}^{-(n-1)}(x,y)} \circ \cdots \circ \mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(\theta_n) - \mathcal{L}^{f_{a,b}^{-1}}_{f_{a,b}^{-(n-1)}(x,y)} \circ \cdots \circ \mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(\theta_{n-1})\right| \leq \left(\frac{a}{b}\right)^{n-1} |\theta_n - \theta_{n-1}|$$

and for the second

$$\begin{aligned}\left|\mathcal{L}^{f_{a,b}^{-1}}_{f_{a,b}^{-(n-1)}(x,y)} \circ \cdots \circ \mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(\theta_{n-1}) - \frac{(b-a)}{ab}\right| &\leq \left|\mathcal{L}^{f_{a,b}^{-1}}_{f_{a,b}^{-(n-1)}(x,y)} \circ \cdots \circ \mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(\theta_{n-1}) - \Theta\right| \\ &\quad + \left|\Theta - \frac{(b-a)}{ab}\right| \\ &\leq \varepsilon + \left|\Theta - \frac{(b-a)}{ab}\right|\end{aligned}$$

Putting this all together we have that

$$\begin{aligned}\left|\Theta - \frac{(b-a)}{ab}\right| &\leq 2\varepsilon + \frac{a}{b}\left(\left(\frac{a}{b}\right)^{n-1}|\theta_n - \theta_{n-1}| + \varepsilon + \left|\Theta - \frac{(b-a)}{ab}\right|\right)\\ &= C_1\varepsilon + \left(\frac{a}{b}\right)^{n}|\theta_n - \theta_{n-1}| + \frac{a}{b}\left|\Theta - \frac{(b-a)}{ab}\right|\end{aligned}$$

which will give us

$$\left(1 - \frac{a}{b}\right)\left|\Theta - \frac{(b-a)}{ab}\right| \leq C_2\varepsilon$$

and finally

$$\left|\Theta - \frac{(b-a)}{ab}\right| \leq C_3\varepsilon$$

that is $\lim \theta_n = \Theta = \frac{(b-a)}{ab}$.

Using the estimative we did before and determining all the differentials of $\mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}$ and the fact that the order of convergence is $o(y^2)$, it is easy to see that the C^2 convergence follows in a very natural way because we have that

$$\mathcal{L}^{f_{a,b}^{-1}}_{(x,y)}(u) = \mathcal{L}^{f_{a,b}^{-1}}_{\infty}(u) + \frac{1}{a(x+\frac{y}{b})^2}$$

The $\mathcal{L}^{f_{a,b}^{-1}}_{\infty}$ induces an constant holonomy, hence once we know that $\mathcal{L}^{f_{a,b}^{-1}}_{x,y}$ is C^2-close to it, the holonomies for points close enough to the infinity can be looked as straight lines with slope $\frac{(b-a)}{ab}$. Therefore, define let $\pi^s_{(x,y),\mathcal{W}^s((x,y))\cap\{x=0\}}$ be the stable holonomy for the attractor $\Lambda_{a,b}$ that takes the point (x,y) and maps it to its projection along the stable manifold to the set $\{x=0\}$. Just as an abuse of notation we will refer to this holonomy as simply π^s and will omit the starting point and the finishing point of the map.

With this definition in hand, the first thing we want to see is that the C^2-convergence of the operators $\mathcal{L}^{f_{a,b}^{-1}}_{(x,y)} \to \mathcal{L}^{f_{a,b}^{-1}}_{\infty}$, will give some control for the differential of the holonomy π^s. All we need to observe that $\left|\dfrac{\partial \pi^s}{\partial x}\right|$ is bounded for $(x+\frac{y}{b}) \to \infty$ and $\dfrac{\partial \pi^s}{\partial y} \to 1$ when $y \to \infty$.

For both cases, we need to be *away* from the discontinuity because when we are close to $\{y=-bx\}$, but once we are at the refined $\mathfrak{R}_{a,b}$ this is automatically established. Even more, we have a control of "how far" we are from $\{y=-bx\}$, because we know that the asymptote for $\mathfrak{R}_{a,b}$ is the direction $(\frac{a}{-b(b+a)}, 1)$.

Hence, if $(x+\frac{y}{b})$ is big enough this will follow immediately that the holonomy π^s is close to the π^s_∞, the holonomy induced by $\mathcal{L}^{f_{a,b}^{-1}}_{\infty}$. Once the foliation are straight lines with slope $\frac{(b-a)}{ab}$, follows that the induced holonomy π^s_∞ that maps a line given by

$\{(\frac{-a}{b(b+a)}t, t)\}$ into the $\{x = 0\}$ is a translation. This implies that $\left|\dfrac{\partial \pi^s}{\partial x}\right|$ is bounded and that $\dfrac{\partial \pi^s}{\partial y} \to 1$.

At last, we are in a good to define the induced map for the attractor

$$\begin{array}{rcl} \pi^s \circ f_{a,b} \circ \mathrm{inc} : \mathbb{R} & \to & \mathbb{R} \\ y & \mapsto & \pi^s_{f_{a,b}(x,y), \mathcal{W}^s(f_{a,b}(x,y)) \cap \{x=0\}} \circ f_{a,b}(0, y) \end{array}$$

where inc denotes the natural inclusion

$$\begin{array}{rcl} \mathrm{inc} : \mathbb{R} & \to & \mathbb{R}^2 \\ y & \mapsto & (0, y) \end{array}$$

Observe that

$$f_{a,b} \circ \mathrm{inc}(y) = f_{a,b}(0, y) = \left(\frac{1}{y}, by - \frac{b}{y}\right)$$

and also that, computing the differential

$$\begin{aligned} \frac{\mathrm{d}(\pi^s \circ f_{a,b} \circ \mathrm{inc})}{\mathrm{d}\, y}(y) &= \bigtriangledown \pi^s (f_{a,b} \circ \mathrm{inc}) \cdot (D f_{a,b}(0, y) \cdot (0, 1) \\ &= \left(\frac{\partial \pi^s}{\partial x}, \frac{\partial \pi^s}{\partial y}\right) \cdot \left(-\frac{1}{y^2}, b + \frac{b}{y^2}\right) \\ &= -\frac{1}{y^2} \cdot \frac{\partial \pi^s}{\partial x} + b\left(1 + \frac{1}{y^2}\right) \cdot \frac{\partial \pi^s}{\partial y} \end{aligned}$$

which implies

$$\frac{\mathrm{d}(\pi^s \circ f_{a,b} \circ \mathrm{inc})}{\mathrm{d}\, y}(y) \to b$$

when $|y| \to \infty$ because $\frac{\partial \pi^s}{\partial x}$ is limited near infinity and $\frac{\partial \pi^s}{\partial y} \to 1$.

2.3 Ergodic Consequences for $f_{a,b}$

Here we want to explore the ergodicity of the two-parameter family. We want to use the attractor we got in the previous section and the induced map to obtain some ergodic information.

Theorem (C). *For $a < b \leq 1$, there exists a SRB measure $\mu_{a,b}$ supported in an attractor $\Lambda_{a,b}$ such that it can be disintegrated over the unstable manifold $\mathcal{W}^u(\Lambda_{a,b})$ of the attractor inside $\mathfrak{R}_{a,b}$ and we also have that $\mu^u_{a,b}$ is absolute continuous with respect to the one-dimensional Lebesgue measure. Also*

$a < b < 1:$ *The measure $\mu_{a,b}$ is finite and the attractor $\Lambda_{a,b}$ lies inside a non-compact region;*

$a < b = 1$: *The measure* $\mu_{a,b}$ *is infinite and the attractor* $\Lambda_{a,b}$ *lies in a non-compact region.*

The proof consists into just understanding the induced map and using classical arguments to define the measure for each one of the maps. Both are based in the same idea, looking at the one-dimensional map we introduced in the proof of theorem B and its compactification. However, the reason why this works for each one of the cases differs deeply.

First we consider both cases together and will look only to the induced map. From now on we will refer to the induced map by h_b and will explain briefly how to get the invariant measure for Boole-like and Boole-like expanding using first a general idea and the conclusion for the ergodicity differs, as we already said. This first part of the argument follows the ideas established on [AW].

The first step is to see that the induced map has a return set, namely an interval consisting of the first two pre-images of 0 by the positive and the negative branch of h_b, $I_0 := [x_1^-, x_1^+]$. We will also use the following notation $A^+ := [0, x_1^+]$ and $A^- := [x_1^- 1, 0]$.

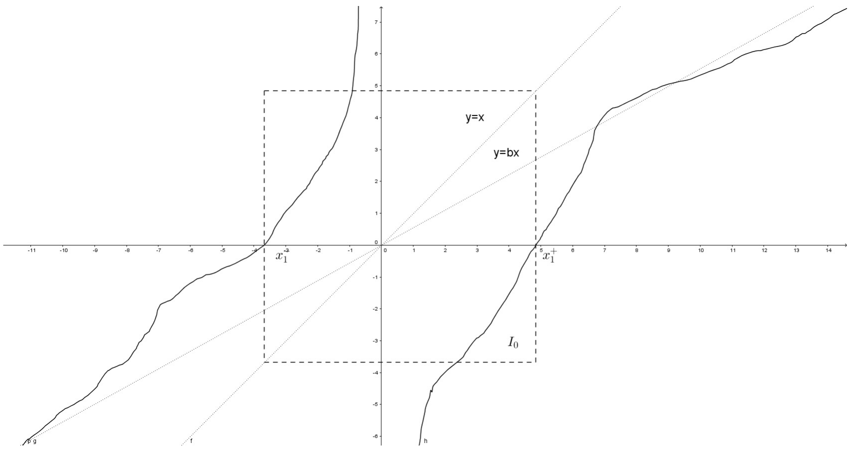

Figure 2.3: The return interval I_0.

Just like Boole, each one of the pre-images of each one of the x_1's have a pre-image of the same sign that is bigger than x_1^+ and smaller than x_1^-, and a pre-image of the opposite sign that lies inside I_0. This actually happen to every point in the real line, depending on the sign of it.

We can define now all the other sets which will make clear the return time for each one the sets in the real line. Define

$$\begin{array}{l} x_i^+ = h_b^{-1}(x_{i-1}^+) > 0 \\ x_i^- = h_b^{-1}(x_{i-1}^-) < 0 \end{array} \qquad i > 1$$

and

$$\begin{array}{l} u_i^- = h_b^{-1}(x_{i-1}^+) < 0 \\ u_i^+ = h_b^{-1}(x_{i-1}^-) > 0 \end{array} \qquad i > 1$$

where $u_1^+ = x_1^+$ and $u_1^- = x_1^+$.

Once each one of the branches is increasing, the sequence of $(x_i^+)_{i\in\mathbb{N}}$ and $(u_i^-)_{i\in\mathbb{N}}$ form two increasing sequences and the opposite for $(x_i^-)_{i\in\mathbb{N}}$ and $(u_i^+)_{i\in\mathbb{N}}$. The picture bellow will help elucidate how the pre-images organize in along the real line

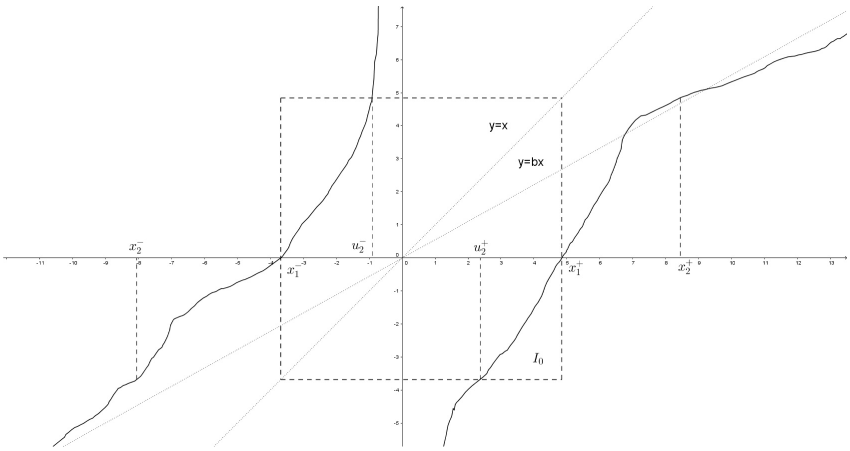

Figure 2.4: How the x_i's and u_i's are placed.

Just observe that $x_i^+ \to \infty$ when $i \to \infty$. Suppose the contrary, that is, there exists $L = \sup_{i\in\mathbb{N}} x_i^+$ and we have that $h_b^{-1}(L)^+ > L$ because h_b lies bellow the diagonal and is increasing. And taking $\varepsilon = h_b^{-1}(L)^+ - L$ would have a contraction with the fact that $L = \sup_{i\in\mathbb{N}} x_i^+$. Analogously $x_i^- \to -\infty$ and, once we know that $u_{i+1}^{\pm}$ is a solution of $x_i^{\mp} = h_b(u_i^{\pm})$, we have that $u_i \to 0$.

Using the correspondent notation used in [AW], let us consider $B_i^+ = [x_i^+, x_{i+1}^+]$ and $A_i^+ = [u_{i+1}^+, u_i^+]$, the analogous for B_i^- and A_i^-. Follows from our definitions that $h_b(B_{i+1}^+) = B_i^+$ and $h_b(B_{i+1}^-) = B_i^-$. Also $h_b(B_1^+) = A^+$, $h_b(B_1^-) = A^-$, $h_b(A_1^-) = A^+$ and $h_b(A_1^+) = A^-$.

We can say a bit more about the image of the A_i's, because due the choice of them we have that $h_b(A_i^+) = B_i^-$ and $h_b(A_i^-) = B_i^+$.

With this information, every point in the real line enters I_0 in finite time, which allows us to define the the induced transformation

$$\begin{aligned} h_b^{\rho} : I_0 &\rightarrow I_0 \\ x &\mapsto h_b^{\rho(x)}(x) \end{aligned}$$

where $\rho(x) = \min\{k \in \mathbb{N}; h_b^k(x) \in I_0\}$, that is, the first return time of x.

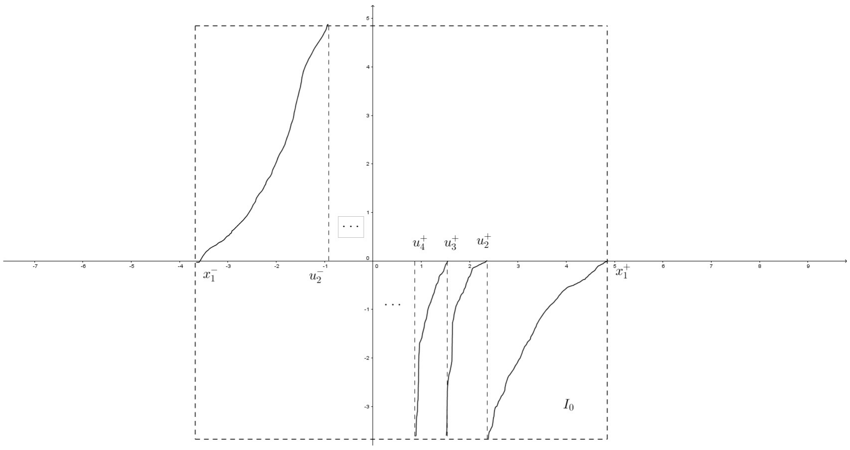

Figure 2.5: The return map.

Once $h_b'(x) \to b$ when $x \to \infty$, there exists $m_0 > 0$, $c < 1$ such that

$$0 < c \leq \inf\{h_b'(x)\} \leq \sup\{h_b'(x)\} \leq b' \leq 1$$

for all $|x| \geq x_{m_0}$ and there exists n_0 such that for $n > n_0$, we have that

$$h_b'(x) \geq \frac{1}{c^{m_0+n_0}} \qquad \forall x \in A_n$$

because we have the the differential explodes near 0. So we can't control the differential in the intervals given by $[u_{n_0}, u_1] \cup [u_1, x_{m_0}]$. The continuity of the differential and compactness of the set where we cannot control the differential, tells us that even though we cannot control the differential every where, we have a lower bound for it that we will still refer as the previous c and an upper bound b'.

With this, we can conclude the item (iii'), that is

Lemma 2.1. *The return map is expanding for $f_{a,b}$.*

Once we know that the return map is expanding we have that, for every $b \leq 1$ there exists $\mu_{a,b}$ that is ergodic and is absolute continuous with respect to the Lebesgue measure (check [V], chapters 1 and 11).

To conclude if the measure $\mu_{a,b}$ is either finite or not, here we have to split the cases and look at the return map ρ. The measure $\mu_{a,b}$ is obtained by "spreading" the measure restricted to the interval $[x_1^-, x_1^+]$ to all the real line using the return time and the measure is finite if, and only if, the return map is integrable with respect to the measure $\mu_{a,b}$ (once again, check [V] for this).

If $b < 1$, then the b' we found in the previous step still less than 1. In that case, for n_0 large enough we have that

$$\int_{\mathbb{R}} \rho \, d\text{Leb} = \int_{[x_{n_0}^-, x_{n_0}^+]} \rho \, d\text{Leb} + \int_{[x_{n_0}^+, \infty)} \rho \, d\text{Leb} + \int_{(-\infty, x_{n_0}^-]} \rho \, d\text{Leb}$$

and will deal with $\int_{[x_{n_0}^+,\infty)} \rho \, d\mathrm{Leb}$ because the other case is completely analogous.

$$\begin{aligned}
\int_{[x_{n_0}^+,\infty)} \rho \, d\mathrm{Leb} &= \sum_{j\in\mathbb{N}} \int_{[x_{n_0+j},x_{n_0+j+1}]} \rho \, d\mathrm{Leb} \\
&= \sum_{j\in\mathbb{N}} \int_{[x_{n_0+j},x_{n_0+j+1}]} n_0 + j \; d\mathrm{Leb} \\
&= \sum_{j\in\mathbb{N}} (n_0+j)(x_{n_0+j+1} - x_{n_0+j}) \\
&\leq \sum_{j\in\mathbb{N}} b'^j (n_0+j)(x_{n_0+1} - x_{n_0})
\end{aligned}$$

and the last series converges by the ratio test. If $b = 1$, by the same arguments we have that this integral diverges. Once we know that the measure $\mu_{a,b}$ behaves, we get that $\mu_{a,b}$ is finite for $b < 1$ and $\mu_{a,1}$ is infinite.

To conclude the proof, just look at the pullback of $\mu_{a,b}$ via the projection that we defined before, that is, just look at the

$$(\pi^s \circ f_{a,b} \circ \mathrm{inc})^* \mu_{a,b} = \mu_{a,b} \circ \pi^s \circ f_{a,b} \circ \mathrm{inc}$$

It defines a measure supported in the attractor $\Lambda_{a,b}$ and that gives full measure for the set $f_{a,b}(\mathfrak{R}_{a,b})$.

Chapter 3

Coding the Hénon Map

Here we introduce a system of coordinates that is induced using the pre-images and images of the discontinuities of the Hénon-Devaney map. The coordinates will help us to describe the dynamics in a symbolic way, which will help us to understand exactly the itineraries of each point relative to the position within the curves.

We want to find a conjugacy between the Hénon-Devaney map and a symbolic model that tells us where exactly we are in the plane at each moment. We want to give a description of how the orbits visit a determined region in the plane via the conjugacy of the map and a product of two two-sided subshifts of finite type. In other words

Theorem (D). *There exists $\Sigma_i, \Sigma_j \subset \Sigma := \{-2,-1,0,1,2\}^{\mathbb{Z}}$ and $h : \mathbb{R}^2 \to \Sigma_i \times \Sigma_j$ an homeomorphism such that the following diagram commutes*

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "f"] \arrow[d, "h"'] & \mathbb{R}^2 \arrow[d, "h"] \\
\Sigma_i \times \Sigma_j \arrow[r, "\sigma"'] & \Sigma_i \times \Sigma_j
\end{tikzcd}$$

where

$$\begin{array}{rcl}
\sigma : \Sigma_i \times \Sigma_j & \to & \Sigma_i \times \Sigma_j \\
((s_n), (s_m)) & \mapsto & (\sigma_i(s_n), \sigma_j(s_m))
\end{array}$$

and σ_i, σ_j are the usual shift maps restricted to each one of its respective spaces.

3.1 The Discontinuities

Let us investigate the images and pre-images of the discontinuities of f and f^{-1}, namely the exceptional curves.

3.1.1 Pre-images of $\{y = 0\}$

Let's take the first step in that direction

$$f^{-1}(\{y=0\}) = \left\{f^{-1}(t,0); t \in \mathbb{R}\right\} = \left\{\left(t - \frac{1}{t}, t\right); t \in \mathbb{R}\right\}$$

which is the Boole's graph with inverted axis.

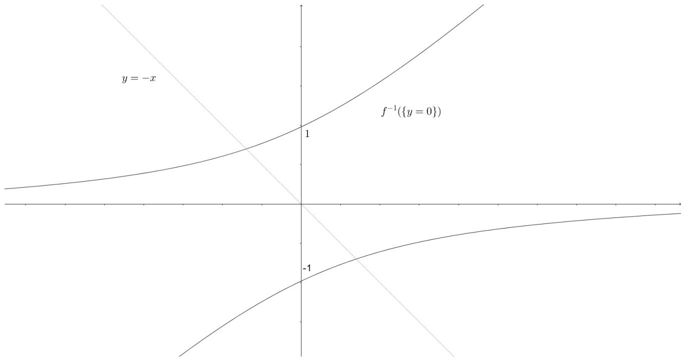

Figure 3.1: $f^{-1}(\{y=0\})$.

Before analysing the other cases, let us first understand a few general aspects of $f^{-n}\{y=0\}$ which do not depend on the values of x. Denoting $f^{-2}(x,y) = (f_x^{-2}(x,y), f_y^{-k}(x,y))$, we have that each coordinate in $f^{-2}(t,0)$ is an increasing function of the parameter t

$$\begin{aligned}
(f_x^{-2})' &= (f_x^{-1})' + \frac{(f_x^{-1})' + (f_y^{-1})'}{(f_x^{-1}+f_y^{-1})^2} \\
&= 1 + \frac{1}{t^2} + \frac{\left(2+\dfrac{1}{t^2}\right)}{(f_x^{-1}+f_y^{-1})^2} > 0
\end{aligned}$$

and it holds for all $t \in \mathbb{R}^* \setminus \{t; f_y^{-1}(t,0) = -f_x^{-1}(t,0)\}$ and, analogously, we have the same for $f_y^{-2}(t,0)$.

Also it is easy to see that $f^{-2}(\{y=0\}) \cap f^{-1}(\{y=0\}) = \emptyset$ because if there exists $x_1 \in \mathbb{R}^*$ and $x_2 \in \mathbb{R}^* \setminus f^{-1}(\{y=-x\})$ such that

$$\begin{cases}
t_1 - \dfrac{1}{t_1} = t_2 - \dfrac{1}{t_2} - \dfrac{1}{2t_2 - \dfrac{1}{t_2}} \\
t_1 = 2t_2 - \dfrac{1}{t_2}
\end{cases}$$

that implies that $t_2 = 0$, which is absurd.

Lemma 3.1. *Using the notation $f^{-n}(t,0) = (f_x^{-n}(t,0), f_y^{-n}(t,0))$*

(a) Each coordinate of the pre-image $f^{-n}(t,0)$ is an increasing function of the parameter t in each connected component of $\mathbb{R} \setminus \left(\cup_{j=0}^{n-1} f^{-j}(\{y=-x\} \cap \{y=0\})\right)$;

(b) $f^{-(n-1)}(t,0) \cap f^{-n}(t,0) = \emptyset$.

Proof. Both proofs are made by induction. We already did the induction step before stating the lemma and the proofs resemble the previous cases.

(a) Assume that it holds for n, that is, $(f_x^{-n})'(t,0)$ and $(f_y^{-n})'(t,0)$ are positive. Then

$$(f_y^{-(n+1)})'(t,0) = (f_x^{-n}(t,0) + f_y^{-n}(t,0))' = (f_x^{-n})'(t,0) + (f_y^{-n})'(t,0) > 0$$

and

$$(f_x^{-(n+1)})'(t,0) = (f_x^{-n})'(t,0) + \frac{((f_x^{-n})'(t,0) + (f_y^{-n})'(t,0))}{(f_x^{-n}(t,0) + f_y^{-n}(t,0))^2} > 0$$

(b) If there is t_1 and t_2 such that

$$\begin{cases} f_x^{-n}(t_1,0) &= f_x^{-(n+1)}(t_2,0) = f_x^{-n}(t_2,0) - \dfrac{1}{f_x^{-n}(t_2,0) + f_y^{-n}(t_2,0)} \\ f_y^{-n}(t_1,0) &= f_y^{-(n+1)}(t_2,0) = f_x^{-n}(t_2,0) + f_y^{-n}(t_2,0) \end{cases}$$

replacing

$$f_x^{-n}(t_2,0) = f_y^{-n}(t_1,0) - f_y^{-n}(t_2,0)$$

from the second equation in the first one, we get

$$\begin{aligned} f_x^{-n}(t_2,0) &= f_x^{-n}(t_1,0) + \frac{1}{f_y^{-n}(t_1,0)} \\ &= f_x^{-(n-1)}(t_1,0) - \frac{1}{f_y^{-n}(t_1,0)} + \frac{1}{f_y^{-n}(t_1,0)} = f_x^{-(n-1)}(t_1,0) \end{aligned}$$

Back to the second equation

$$f_x^{-(n-1)}(t_1,0) + f_y^{-(n-1)}(t_1,0) =: f_y^{-n}(t_1,0) = f_x^{-(n-1)}(t_1,0) + f_y^{-n}(t_2,0)$$

and putting the information together follows that exists x_1 and x_2 such that

$$\begin{cases} f_x^{-(n-1)}(t_1,0) &= f_x^{-n}(t_2,0) \\ f_y^{-(n-1)}(t_1,0) &= f_y^{-n}(t_2,0) \end{cases}$$

that is $f^{-(n-1)}(t,0) \cap f^{-n}(t,0) \neq \emptyset$, a contradiction.

□

Remark 3.1. *Once we know that*

$$f^{-n}(t,0) = \left(f_x^{-(n-1)}(t,0) - \frac{1}{f_x^{-(n-1)}(t,0) + f_y^{-(n-1)}(t,0)}, f_x^{-(n-1)}(t,0) + f_y^{-(n-1)}(t,0) \right)$$

it is easy to see that

(i) $\lim_{t\to\infty} f_x^{-n}(t,0) = \infty$;

(ii) $\lim_{t\to\infty} f_y^{-n}(t,0) = \infty$.

In the next step, $f^{-2}(\{y=0\})$ will have 4 curves, because of the discontinuities of f^{-1}, that is, the pre-images below

(i) $f^{-1}\left(\left\{\left(t-\frac{1}{t},t\right);t>0 \text{ and } t>-\left(t-\frac{1}{t}\right)\right\}\right)$

(ii) $f^{-1}\left(\left\{\left(t-\frac{1}{t},t\right);t>0 \text{ and } t<-\left(t-\frac{1}{t}\right)\right\}\right)$

and the other two cases follow from the previous two because we have that $-f^{-1}(x,y) = f^{-1}(-x,-y)$.

In (i), we get that

$$f^{-2}(\{y=0\}) = \left\{\left(t-\frac{1}{t}-\frac{1}{2t-\frac{1}{t}}, 2t-\frac{1}{t}\right);t>0 \text{ and } t>-\left(t-\frac{1}{t}\right)\right\}$$

Let $t_1 > 0$ and $t_2 \in \{t\in\mathbb{R}^*; t>0 \text{ and } f_y^{-1}(t,0) > -f_x^{-1}(t,0)\}$ such that $f_x^{-1}(t_1,0) = 0 = f_x^{-2}(t_2,0)$ then

$$f^{-2}(t_1,0) = \left(f_x^{-1}(t_1,0) - \frac{1}{f_x^{-1}(t_1,0)+f_y^{-1}(t_1,0)}, f_x^{-1}(t_1,0)+f_y^{-1}(t_1,0)\right)$$

and recalling the choice of t_1 we have that

$$f^{-2}(t_1,0) = \left(-\frac{1}{f_y^{-1}(t_1,0)}, f_y^{-1}(t_1,0)\right)$$

Once $f_y^{-1}(t_1,0) > 0 = f_x^{-1}(t_1,0)$ and t_1 and t_2 are in the same connected component due the choice of t_2, we conclude from the previous lemma that $t_1 < t_2$ and also that

$$f_y^{-1}(t_1,0) < f_y^{-1}(t_2,0) + f_x^{-1}(t_2,0) = f_y^{-2}(t_2,0)$$

because $f_y^{-2}(t_2,0) > 0$. This means, using (b) from the previous lemma, that the pre-image of the curve above the discontinuity of the inverse is another curve that is above $f^{-1}(\{y=0\})$ with $f_y^{-1}(t,0) > -f_x^{-1}(t,0)$.

For (ii) we will prove something similar but now the curve is below 0 and above $f^{-1}(\{y=0\})$ with $t<0$. Indeed, first note that for every $t\in\{t\in\mathbb{R}^*; t > 0 \text{ and } f_y^{-1}(t,0) < -f_x^{-1}(t,0)\}$

$$f_y^{-2}(t,0) = f_y^{-1}(t,0) + f_x^{-1}(t,0) < 0$$

Here we need to see that this is also above the first pre-image. The easiest way to check directly, take two parameters that have the same x coordinate and see where the curves are. The induction step is very different from the general proof we will see a bit later.

Let us find t_0 one of the parameters of (ii) such that $f_x^{-2}(t_0, 0) = 0$ and examine what happens in the y coordinate. All we have to do is to find the right solution to

$$t - \frac{1}{t} - \frac{1}{2t - \dfrac{1}{t}} = 0$$

or equivalently

$$2t^4 - 4t^2 + 1 = 0$$

where we get that $t_0 = \sqrt{1 - \dfrac{\sqrt{2}}{2}}$. Hence

$$0 > f_y^{-2}(t_0, 0) = 2t_0 - \frac{1}{t_0} = \sqrt{4 - \sqrt{2}} - \sqrt{2 + \sqrt{2}} > -1$$

that is, this curve is between $\{y = 0\}$ and its negative pre-image.

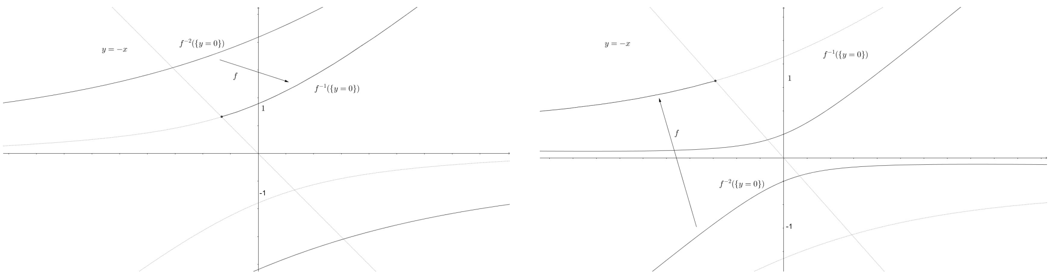

Figure 3.2: The cases (i) and (ii).

Using the same ideas we introduced before, we will prove that the $f^{-(n+1)}(\{y = 0; f_y^{-j}(t,0) > -f_x^{-j}(t,0),\ 0 \leq j \leq n\})$ is a family of curves that "covers" the upper part of $\mathbb{R}^2$ in the sense that, for each point in this region, there is a curve of the mentioned family above and bellow it. The previous idea will be the induction step and the strategy is the same. Indeed, let us proceed in the same way, let $t_{n+1} \in \{t \in \mathbb{R}; f_y^{-j}(t,0) > -f_x^{-j}(t,0),\ 0 \leq j \leq n\}$ such that

$$f_x^{-(n+1)}(t_{n+1}, 0) = 0$$

We want to see that each curve containing $f^{-n}(t_n, 0)$ is an increasing sequence of curves, to do that we use the induction step. Let us assume that it holds for n, that is

$$f_y^{-n}(t_n, 0) > f_y^{-(n-1)}(t_{n-1}, 0) > 0 = f_x^{-(n-1)}(t_{n-1}, 0)$$

and then $t_n \in \{t \in \mathbb{R}; f_y^{-j}(t,0) > -f_x^{-j}(t,0),\ 0 \leq j \leq n\}$. Therefore we can compare t_n and t_{n+1} because they are in the same connected component of f^{n+1}. Hence

$$f^{-(n+1)}(t_n, 0) = f^{-1}(f^{-n}(t_n, 0)) = \left(-\frac{1}{f_y^{-n}(t_n, 0)}, f_y^{-n}(t_n, 0)\right)$$

and once

$$f_x^{-(n+1)}(t_n, 0) = -\frac{1}{f_y^{-n}(t_n, 0)} < 0 = f_x^{-(n+1)}(t_{n+1}, 0)$$

we get, by lemma 3.1, that $t_n < t_{n+1}$ which implies

$$f_y^{-(n+1)}(t_{n+1}, 0) > f_y^{-n}(t_n, 0)$$

The conclusion here is the same as described previously in (i), that is to say the $(n+1)$-th curve is above the n-th curve and also that $(f_y^{-n}(t_n, 0))_{n\in\mathbb{N}}$ is increasing.

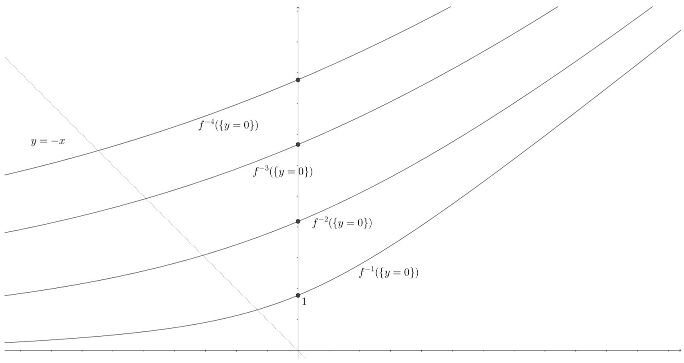

Figure 3.3: The curves moving up.

Remark 3.2. *Notice that the area between two subsequent pre-images of $\{y = 0\}$ and in the same side of the anti-diagonal is mapped inside the pre-images of those said curves.*

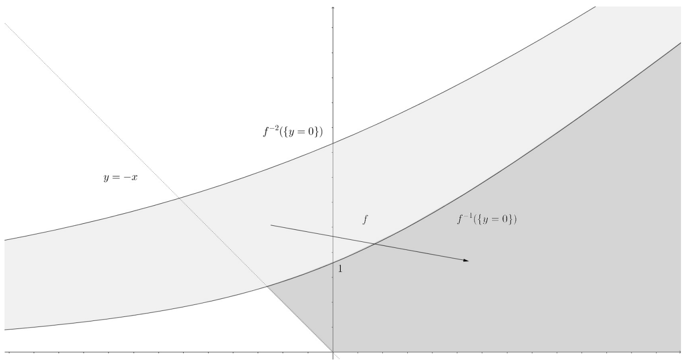

Figure 3.4: How the areas between curves are mapped.

But we cannot conclude yet what we stated before because we do not know if the curves "diverge", we need to understand how the sequence we found behaves:

Lemma 3.2. *The sequence $(f_y^{-n}(t_n, 0))_{n\in\mathbb{N}}$ diverges.*

Proof. Suppose that there exists

$$\lim_n f_y^{-n}(t_n, 0) = \sup_n f_y^{-n}(t_n, 0) = L > 0$$

and because $t_1 = 1$ we know that $L > 1$. Observe now that

$$f(0, L) = \left(\frac{1}{L}, L - \frac{1}{L}\right)$$

with $L - \dfrac{1}{L} > 0$. Also, there exists $n_0 \in \mathbb{N}$ such that

$$f_y^{-n}(t_n, 0) \geq L - \frac{1}{L} \qquad \forall n \geq n_0$$

For all $n \geq n_0$, let t_n^L be the parameter in which

$$f_x^{-n}(t_n^L, 0) = \frac{1}{L} > 0$$

in the same connected component of t_n. Thus

$$f_y^{-n}(t_n^L, 0) > f_y^{-n}(t_n, 0) \geq L - \frac{1}{L}$$

This tells us that $f(0, L)$ is in the first quadrant, also it cannot be one of the discontinuities of f^{-n} and it is bellow the curves containing $f^{-n}(t_n, 0)$ for all $n \geq n_0$.

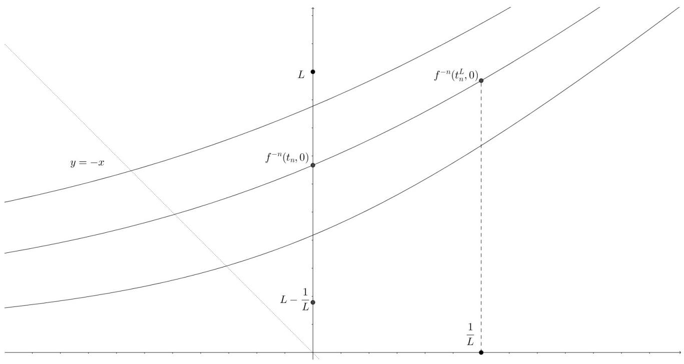

Figure 3.5: The point $(0, L)$ goes under the curves.

Now we know that $f(0, L)$ is in the area delimited by the coordinates axis and bounded above by the the curve containing $f^{-n_0}(t_{n_0}, 0)$. Recalling the previous remark, we see that $(0, L) = f^{-1}\left(\dfrac{1}{L}, L - \dfrac{1}{L}\right)$ is in the region limited above by $f^{-1}(f^{-n_0}(t_{n_0}, 0))$ although it is also above it, a contradiction. $\square$

Now we want to understand of the generalization of (ii), that is, the points that in the n-th pre-image changes sign with respect the anti-diagonal:

$$D_n := \{t \in \mathbb{R}; f_y^{-j}(t,0) > -f_x^{-j}(t,0),\ 0 \leq j < n \text{ and the opposite for } n\}$$

Here the induction hypothesis is that for n is under the curve $\{y = 0\}$ and above the pre-image of D_{n-1}. Our objective here is to see that the $(n+1)$-th pre-image is trapped between $f^{-1}(D_n)$ and $\{y = 0\}$. The idea here is purely geometric: consider a straight line l that comes from the y-axis and that touches D_{n+1} as illustrated below. All it is left is to see what happens to the pre-image of this line.

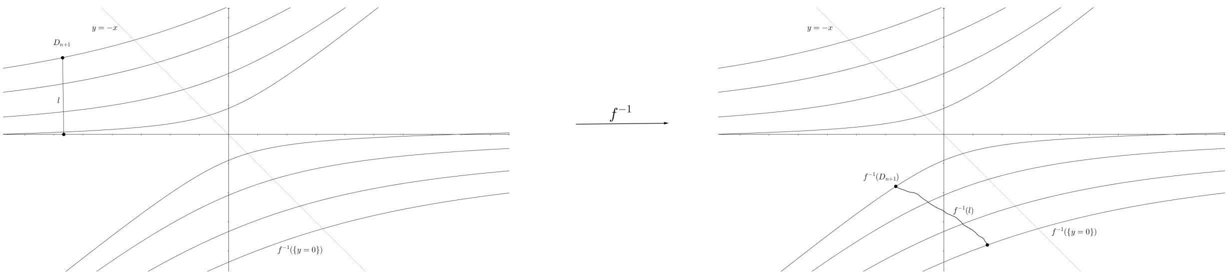

Figure 3.6: The pre-images.

When we look at the pre-image of this line and use the induction hypothesis, also remember that that D_j are sets that are moving up according to the previous case, we observe that the pre-image of l touches each one of the curves only once. This implies that $f^{-1}(D_{n+1})$ is either between D_n and D_1, under D_1 or it is above D_n as we wanted.

If the first one occurs, then there would be a line joining $f^{-1}(D_{n+1})$ and $f^{-1}(D_1)$ that does not touch $f^{-1}(D_n)$ although its would intersect D_n which is a contradiction.

If the second one holds, something similar to the previous case would happen: the line connecting $f^{-1}(D_{n+1})$ and $f^{-1}(D_1)$ would go trough $f^{-1}(D_1)$ twice, implying that the image of the curves crosses D_1 also twice, which is absurd. Therefore, the third case holds as we wanted.

Corollary 3.1. *The curves given by $f^{-1}(D_n)$ converges pointwise to 0.*

Proof. To establish this just observe that, in case it does not, each limit would be a limiting point to the direct image of it, that is, a limit for the curves we already proved that diverge in Lemma 3.2. □

Remark 3.3. *It is interesting to see that these points that are in the curves defined by $f^{-1}(D_n)$ will be mapped "away" from the $\{y = 0\}$, more precisely, as a consequence of Lemma 3.2 the curves that contain each D_n covers the lower part of the plane. We will explore a bit more of this in the Section 3.*

3.1.2 Images of $\{y = -x\}$

We want to make the same study for the images of the inverse map's discontinuity

$$f(\{x = -y\}) = \{f(t, -t); t \in \mathbb{R}\} = \left\{ \left(t - \frac{1}{t}, -2t + \frac{1}{t}\right); t \in \mathbb{R} \right\}$$

Observe that, using the previous notation, one can write

$$f(t, -t) = (f_x^{-1}(t, 0), -f_y^{-2}(t, 0))$$

and that is in fact a little more general as describe in the next remark.

Remark 3.4. *We have the following identification*

$$f^n(t, -t) = (f_x^{-n}(t, 0), -f_y^{-(n+1)}(t, 0))$$

Proof. All we need to do is to check it by induction. Assume it holds for n, then

$$f^{n+1}(t, -t) = f(f^n(t, -t)) = f(f_x^{-n}(t, 0), -f_y^{-(n+1)}(t, 0))$$

which implies that

$$f^{n+1}(t, -t) = \left(f_x^{-n}(t, 0) - \frac{1}{f_y^{-(n+1)}(t, 0)}, -f_y^{-(n+1)}(t, 0) + \frac{1}{f_y^{-(n+1)}(t, 0)} - f_x^{-n}(t, 0) \right)$$

Recalling the definitions of $f_x^{-n}(t, 0)$ and $f_y^{-n}(t, 0)$,

$$f_x^{-(n+1)}(t, 0) = f_x^{-n}(t, 0) - \frac{1}{f_x^{-n}(t, 0) + f_y^{-n}(t, 0)} = f_x^{-n}(t, 0) - \frac{1}{f_y^{-(n+1)}(t, 0)}$$

and it is straightforward to see that

$$f^{n+1}(t, -t) = \left(f_x^{-(n+1)}(t, 0), -f_y^{-(n+2)}(t, 0)\right)$$

□

To understand the behaviour of $f^n(t, -t)$ we need to make a refinement of the remark 3.1. We started this analysis looking only at the "infinity", but we need to take it a bit further.

Remark 3.5 (Maybe there is more information here than we need but just in case...)**.** *Recalling that the pre-images of* $\{y = 0\}$ *are given by*

$$f^{-n}(t, 0) = \left(f_x^{-(n-1)}(t, 0) - \frac{1}{f_x^{-(n-1)}(t, 0) + f_y^{-(n-1)}(t, 0)}, f_x^{-(n-1)}(t, 0) + f_y^{-(n-1)}(t, 0) \right)$$

we can understand the full behaviour of $f^{-n}(t, 0)$ *near the discontinuities as described bellow* *(i)* $\lim_{t\to t_d^-} f_x^{-n}(t,0) = \infty$, *where* $t_d \in \left(\cup_{j=0}^{n-1} f^{-j}(\{y=-x\}\cap\{y=0\})\right)\cup\{\infty\}$*;*

(ii) $\lim_{t\to t_d^+} f_x^{-n}(t,0) = -\infty$, *where* $t_d \in \left(\cup_{j=0}^{n-1} f^{-j}(\{y=-x\})\cap\{y=0\}\right)\cup\{-\infty\}$*;*

(iii) $\lim_{t\to t_d^-} f_y^{-n}(t,0) = \infty$, *where* $t_d \in f^{-(n-1)}(\{y=-x\}\cap\{y=0\})\cup\{\infty\}$*;*

(iv) $\lim_{t\to t_d^+} f_y^{-n}(t,0) = 0$, *where* $t_d \in f^{-(n-1)}(\{y=-x\}\cap\{y=0\})$*;*

(v) $\lim_{t\to t_d^-} f_y^{-n}(t,0) = 0$, *where* $t_d \in \left(\cup_{j=0}^{n-2} f^{-j}(\{y=-x\}\cap\{y=0\})\right)$*;*

(vi) $\lim_{t\to t_d^+} f_y^{-n}(t,0) = -\infty$, *where* $t_d \in \left(\cup_{j=0}^{n-2} f^{-j}(\{y=-x\}\cap\{y=0\})\right)\cup\{-\infty\}$*;*

Proof. Using the recurrence formula we already know, one can deduce by induction that

$$f^{-n}(t,0) = \left(t - \sum_{j=1}^{n} \frac{1}{f_y^{-j}(t,0)}, f_x^{-(n-1)}(t,0) + f_y^{-(n-1)}(t,0)\right)$$

To get items (i) and (ii), all we need to do is understand what the first coordinate of the previous relation is telling us. Just notice that for each t_d only one of the $f_y^{-(j+1)}(t_d,0) = f_x^{-j}(t_d,0) + f_y^{-j}(t_d,0)$ vanishes per time and the sign comes from which side it approaches 0.

The items (iii) and (iv) come directly from the formula and observing that the second coordinate vanishes. For the last two just replace $f_x^{-(n-1)}(x,0)$ by the induction formula we first stated here, that is,

$$f_y^{-n}(t,0) = f_x^{-(n-1)}(t,0) + f_y^{-(n-1)}(t,0) = f_y^{-(n-1)}(t,0) + x - \sum_{j=1}^{n} \frac{1}{f_y^{-j}(t,0)}$$

and repeat the analysis we did in (i). $\square$

It is an interesting observation that for $f(t,-t) = (f_x^{-n}(t,0), -f_y^{-(n+1)}(t,0))$, the discontinuities of $f_x^{-n}(t,0)$ and $-f_y^{-(n+1)}(t,0)$ are the same, that is, $-f_y^{-(n+1)}(t,0))$ is an continuous function in each connected component of $\left(\cup_{j=0}^{n-1} f^{-j}(\{y=-x\}\cap\{y=0\})\right)$ that is onto $\mathbb{R}$.

Putting this all together, we can see that what is happening in this case is something very similar to the case of $f^{-1}(t,0)$. It is actually pretty much the same idea but the exceptional curve here that we have to avoid is $\{y=0\}$, that is, when the n-th image touchs this curve its $(n+1)$-th image will split into two different curves just like happened before.

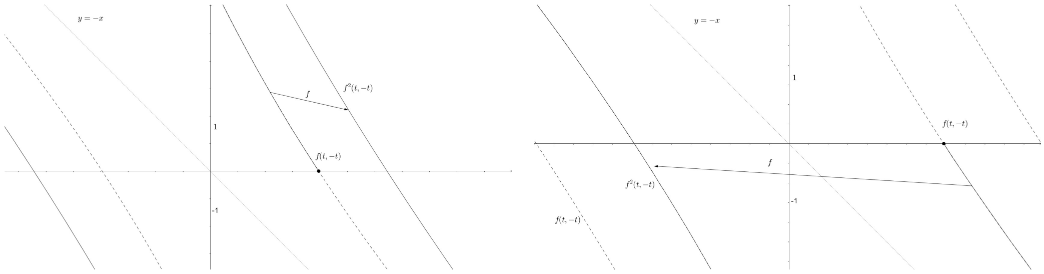

Figure 3.7: How the images reorganize.

The relation of the discontinuities tells us exactly which are the points that nullify $f_y^n(t,-t)$: the discontinuities of $f_x^{-(n+1)}(t,0)$. Hence it is expected to have something similar to lemma 3.1 and the other results that follow from it.

Lemma 3.3. *With the notation* $f^n(t,-t) = (f_x^n(t,-t), f_y^n(t,-t))$

(a) $f_x^n(t,-t)$ *is increasing and* $f_y^n(t,-t)$ *is decreasing with respect the parameter* t *in each connected component of* $\mathbb{R} \setminus \left(\cup_{j=0}^{n-1} f^{-j}(\{y=-x\} \cap \{y=0\})\right)$*;*

(b) $f^{(n-1)}(t,-t) \cap f^n(t,-t) = \emptyset$.

Proof. The proof becomes very easy when we use the remark 3.4. The first item follows directly from 3.1. The proof of the second one follows the same idea, all we have to see is that we can reduce this to the case (b) of the original lemma.

Suppose there exists t_1 and t_2 such that

$$\begin{cases} f_x^{(n-1)}(t_1,-t_1) & = & f_x^n(t_2,-t_2) \\ f_y^{(n-1)}(t_1,-t_1) & = & f_y^n(t_2,-t_2) \end{cases}$$

implying that

$$\begin{cases} f_x^{-(n-1)}(t_1,0) & = & f_x^{-n}(t_2,0) \\ f_y^{-n}(t_1,0) & = & f_y^{-(n+1)}(t_2,0) \end{cases}$$

Just using the first equation in the second we get

$$f_y^{-(n-1)}(t_1,0) = f_y^{-n}(t_2,0)$$

in other words

$$f^{-(n-1)}(t,0) \cap f^{-n}(t,0) \neq \emptyset$$

which is a contradiction $\square$

We have to check again the reorganizing pattern of the curves under the action of f. The good news is that, disconsidering the change of sign, it is like the previous case: the curves that are the n-th images of $\{t \in \mathbb{R}; f_y^{-j}(t,0) > -f_x^{-j}(t,0),\ 0 \leq j \leq n\}$ covers the whole part under the anti-diagonal and the images of the other positive parameters go inside the area delimited by $f(\{y=-x\})$, just like happened in the last picture. But that is just to look at what we already did in the last cases and it will follow directly from the relation between that image and the pre-image.

Just obverse that we already know that, in this set

$$f_y^{-n}(t,0) < f_y^{-(n+1)}(t,0) \qquad \forall n \in \mathbb{N}$$

or in other words

$$f_y^{(n-1)}(t,-t) > f_y^{n}(t,-t)$$

the curves given by these sets are moving down in each step. Using the same analysis one can see that D_n is a curve that is between the curves given by $f(t,-t)$, just like what happens to the pre-images.

3.2 New Coordinates

This construction just uses all the information we got so far: how the images and pre-images cover the whole plane. We will use the symmetries of the map to make easier the understanding of the proof. At this first instance we will only consider the upper part of the plane and, once $f(-x,-y) = -f(x,y)$, everything can be mirrored to the lower part of $\mathbb{R}^2$.

Let $\{R_i\}_{i\geq 0}$ and $\{L_j\}_{j\geq 0}$ be the families of curves given respectively by the lamination of the the highest i-pre-image of $\{y=0\}$ and the highest j-image of $\{y=-x\}$ with respect to the anti-diagonal, as described in the figure bellow. Denote the mirrored curves by $\{R_i\}_{i<0}$ and $\{L_j\}_{j<0}$

The key here is to use the only information we have: the boundaries of each intersection are curves that we know exactly how it moves. Denote by the pair (i,j) the region delimited by $(R_i, R_{i-1}, L_j, L_{j-1})$. Observe now that once the curves that delimit each (i,j) are related by the image and the pre-image of f, that is, R_{i+1} is the pre-image of one part of R_i and the same holds for L_{j+1}, because it is the image of a part of f.

We need the additional information that the Corollary 3.1 gives us: we have to add the curves inside each (i,j). The curves determined in 3.1 are curves inside R_1 that are induced by the pre-images of the $\{R_i\}_{i<0}$, that is, it is a family of curves $R_{1\oplus i_1}$ contained in R_1 such that

$$f(R_{1\oplus i_1}) \subset R_{i_1} \qquad i_1 \in \mathbb{Z}^-$$

where $R_{1\oplus i_1} = f^{-1}(D_{i_1})$. Using the analogous definition, we can define $\{L_{1\oplus j_1}\}_{j_1\in\mathbb{Z}^-}$.

However, this is not restricted to R_1 and L_1: it is a consequence of the choice of the pre-images and how they distribute above the plane that we can "extend" the curves inside of R_1 and L_1 to any R_i and L_j, for $i,j>0$.

In order to do so, the study of how the pre-images of $f^{-n}(\{y = 0\})$ distribute over the plane proved that

$$f(i,j) = \begin{cases} (i-1, j+1) & i > 1 \text{ and } j > 0; \\ (i-1, 1 \oplus j) & i > 1 \text{ and } j < 0 \end{cases}$$

which takes the the subdivisions of R_1 and L_1 to all the previously mentioned sets. Therefore we can define the sets $\{R_{i_0 \oplus i_1}\}$ and $\{L_{j_0 \oplus j_1}\}$, for integers of alternating signs and $R_{i_0 \oplus 0} := R_{i_0}$ and $L_{j_0 \oplus 0} := L_{j_0}$.

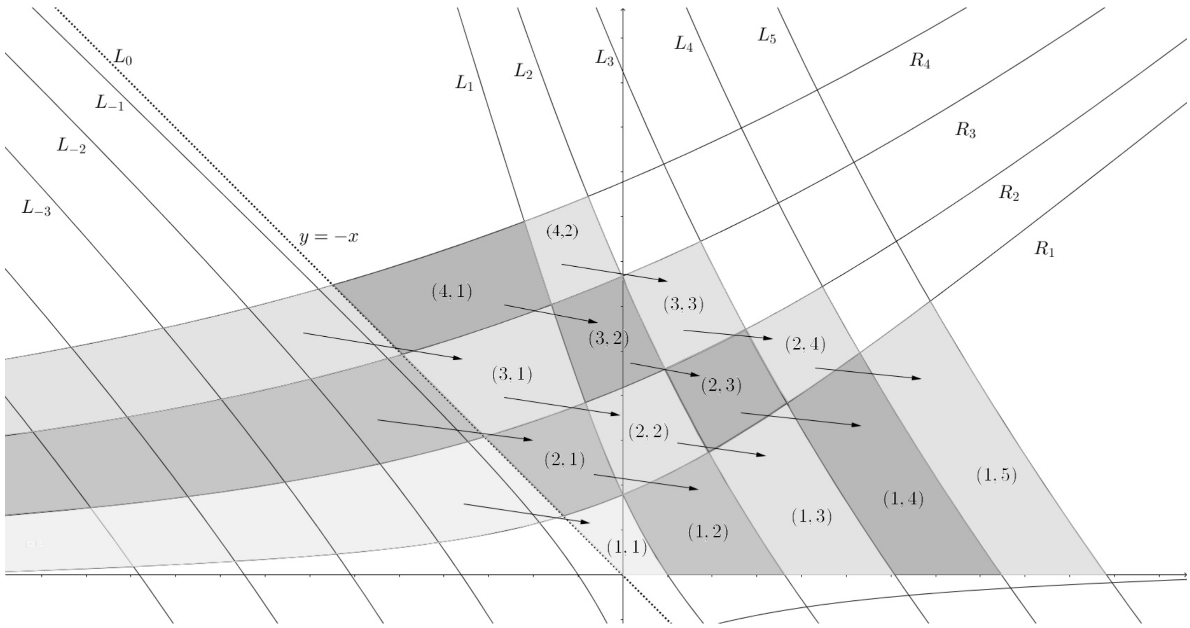

Figure 3.8: The dynamics of the part above R_1.

Once more, the pair $(i_0 \oplus i_1, j_0 \oplus j_1)$ will denote the region delimited by the curves $(R_{i_0 \oplus i_1}, R_{i_0 \oplus i_1 - 1}, L_{j_0 \oplus j_1}, L_{j_0 \oplus j_1 - 1})$.

Again, we know that $f(R_{1 \oplus i_1}) \subset R_{i_1}$. This shows us that we can also induce an lamination within the region between $R_{1 \oplus i_1}$ and $R_{1 \oplus i_1 - 1}$ that comes from what we defined in the previous step, i.e., the one we already have inside R_{i_1}, namely $\{R_{i_1 \oplus i_2}\}_{i_2 \in \mathbb{Z}^+}$. With this in hand, we can define the curves $\{R_{1 \oplus i_1 \oplus i_2}\}$ and then extend it to the curves

$$\{R_{i_0 \oplus i_1 \oplus i_2};\ \text{where } i_0, i_1, i_2 \in \mathbb{Z} \text{ of alternating signs}\}$$

that lay inside the region delimited by $R_{i_0 \oplus i_1}$ and $R_{i_0 \oplus i_1 - 1}$. Using the same argument we can define the curves inside each region delimited by $L_{j_0 \oplus j_1}$ and $L_{j_0 \oplus j_1 - 1}$: the set of curves $\{L_{j_0 \oplus j_1 \oplus j_2}\}$.

Using the same argument as before, it is possible to subdivide the region delimited by each $R_{1 \oplus i_1 \oplus i_2}$ and $R_{1 \oplus i_1 \oplus i_2 - 1}$ once we know that $f^2(R_{1 \oplus i_1 \oplus i_2}) \subset R_{i_2}$, we can continue to subdivide each region we got in the previous step. Proceeding in the same way we stated before we got, by construction, two dense sets of curves that are transversal:

$$\{R_{i_0 \oplus_{n\in\mathbb{N}} i_n};\ \text{where } i_n \in \mathbb{Z} \text{ of alternating signs}\}$$

and the correspondent in the other direction

$$\{L_{j_0 \oplus_{m\in\mathbb{N}} j_m};\ \text{where } j_m \in \mathbb{Z} \text{ of alternating signs}\}$$

Due the density of the curves, if a point does not lie over any of these curves, it may be represented by the new coordinates gives regarding these curves: the intersection of all regions that we introduced, that is, we may identify each point by the coordinates $(i_0 \oplus_{n\in\mathbb{N}} i_n, j_0 \oplus_{m\in\mathbb{N}} j_m)$. If a point lies over a R or a L curve, then it means that it has a finite representation in that coordinate, e.g., if we have a point that lies over the $R_{i_0 \oplus i_1 \cdots \oplus i_k}$, it will have a finite i-coordinate: $(i_0 \oplus i_1 \oplus \cdots \oplus i_k, j_0 \oplus_{m\in\mathbb{N}} j_m)$. The density playing along the transversality give the unique representation of each point in the plane.

Although the coordinates look a bit terrifying, it is a very useful way to describe de dynamics because of the way we constructed them

$$f\left(i_0 \oplus_{n\in\mathbb{N}} i_n, j_0 \oplus_{m\in\mathbb{N}} j_m\right) = \begin{cases} (i_0 - 1 \oplus_{n\in\mathbb{N}} i_n, j_0 + 1 \oplus_{m\in\mathbb{N}} j_m) & i_0 > 1 \text{ and } j_0 > 0; \\ (i_1 \oplus_{n\geq 2} i_n, j_0 + 1 \oplus_{m\in\mathbb{N}} j_m) & i_0 = 1 \text{ and } j_0 > 0; \\ (i_0 - 1 \oplus_{n\in\mathbb{N}} i_n, 1 \oplus j_0 \oplus_{m\in\mathbb{N}} j_m) & i_0 > 1 \text{ and } j_0 < 0; \\ (i_1 \oplus_{n\geq 2} i_n, 1 \oplus j_0 \oplus_{m\in\mathbb{N}} j_m) & i_0 = 1 \text{ and } j_0 < 0; \end{cases}$$

and using the mirroring property that the Hénon-Devaney has, one can see what happens with the signs changed. With this in hand, we will proceed to give an complete symbolic description of the map.

3.3 The Subshift

In this section we will explain how to encrypt the Hénon-Devaney map into a subshift of finite type. Let $\mathcal{A} = \{-2, -1, 0, 1, 2\}$ be the alphabet of symbols we will use but, however, it will not be a complete shift. We want to find a conjugacy between the original map and a product of two subshifts, one for each coordinate we introduced before.

At first we will consider points which have the complete description on terms of the (i, j) coordinates. We will consider them first not only because they will give us the idea behind the coding but also because they form the set in $\mathbb{R}^2$ that the dynamics is defined for all iterations backwards and forwards. Also, in terms of the Lebesgue measure in the plane, the complement, that is, the point which have finite orbit backward or forward, have zero measure. This comes from the fact that these points lay all on the set given by

$$\left(\bigcup_{n\in\mathbb{N}} f^{-n}(\{y=0\})\right) \cup \left(\bigcup_{n\in\mathbb{N}} f^{n}(\{y=-x\})\right)$$

that has zero Lebesgue measure.

We will have a region of interest for each coordinate, and it is defined by when $|i_0| = 1$ for the i-coordinate and $|j_0| = 1$ for the j-coordinate. This particular region has to be highlighted because it is exactly where the dynamics change. The symbol that will be attributed to the point in each instant and for each coordinate is:

$$i_0 \oplus_{n\in\mathbb{N}} i_n \overset{s_i}{\mapsto} \begin{cases} 2 & i_0 > 1 \\ 1 & i_0 = 1 \\ -1 & i_0 = -1 \\ -2 & i_0 < -2 \end{cases}$$

and the same for the j-coordinate. Given any point p with full orbit defined, the sequence we will associate is linked to the itinerary of the point:

$$(\dots \underbrace{s_{i_{-2}}}_{f^{-2}(p)} \underbrace{s_{i_{-1}}}_{f^{-1}(p)} ; \underbrace{s_{i_0}}_{p} \underbrace{s_{i_1}}_{f(p)} \underbrace{s_{i_2}}_{f^2(p)} \dots , \dots \underbrace{s_{j_{-2}}}_{f^{-2}(p)} \underbrace{s_{j_{-1}}}_{f^{-1}(p)} ; \underbrace{s_{j_0}}_{p} \underbrace{s_{j_1}}_{f(p)} \underbrace{s_{j_2}}_{f^2(p)} \dots)$$

where s_{i_n} represents the symbol that has to be associated to the i-coordinate at the instant $f^{i_n}(p)$ and s_{j_m} represents the symbol that has to be associated to the j-coordinate at the instant $f^{j_m}(p)$.

Example 3.1. *Let us take a moment to understand how the coding will take place with some examples.*

	Initial Point p	$f(p)$	$f^2(p)$
coordinates	$(3 \oplus -2, 1 \oplus -4)$	$(2 \oplus -2, 2 \oplus -4)$	$(1 \oplus -2, 3 \oplus -4)$
coding	(2 , 1)	(22 , 12)	(221 , 122)
coordinates	$(-1 \oplus 3 \oplus -1, -1)$	$(3 \oplus -1, -2)$	$(2 \oplus -1, 1 \oplus -2)$
coding	$(-1\ ,\ -1)$	$(-12\ ,\ -1-2)$	(-122 , -1-21)
coordinates	$(-1 \oplus 1 \oplus -1, -1)$	$(1 \oplus -1, -2)$	$(-1, 1 \oplus -2)$
coding	$(-1\ ,\ -1)$	(-11 , -1-2)	(-11-1 , -1-21)
coordinates	$(3 \oplus -2, 3 \oplus -4)$	$(2 \oplus -2, 4 \oplus -4)$	$(1 \oplus -2, 5 \oplus -4)$
coding	(2 , 2)	(22 , 22)	(221 , 222)
coordinates	$(3 \oplus -2, -1 \oplus 4)$	$(2 \oplus -2, 1 \oplus -1 \oplus -4)$	$(1 \oplus -2, 2 \oplus -1 \oplus -4)$
coding	(2 , -1)	(22 , -11)	(221 , -112)

To completely understand how the orbits behave under the iteration of f, keep in mind the description we introduced before using the coordinates. It makes easier to see how f acts in the coordinates and just compute which R and L-stripe you are.

We will deal with each one of the coordinates separately, first defining the coding in the i-coordinate and then proving some lemmas about it. The j-coordinate will be dealt latter on but the the idea is pretty much the same. Even tough they "see" different things, they have an intrinsic relation that will become very clear once we clarify the coding.

3.3.1 Coding the i-coordinate

To code the i-coordinate, let us define for $p = (i_0 \oplus_{n \in \mathbb{N}} i_n, j_0 \oplus_{m \in \mathbb{N}} j_m)$, the definition of $h_i(p)$ is split depending on the sign of i_0 and j_0:

$$\dots \underbrace{\pm 2 \cdots \pm 2 \pm 1}_{|j_2|} \underbrace{\mp 2 \cdots \mp 2 \mp 1}_{|j_1|} \underbrace{\pm 2 \cdots \pm 2}_{|j_0|} ; \underbrace{\pm 2 \cdots \pm 2}_{|i_0|-1} \underbrace{\pm 1 \mp 2 \cdots \mp 2}_{|i_1|} \underbrace{\mp 1 \pm 2 \cdots \pm 2}_{|i_2|} \dots$$

if $\mathrm{sign}(i_0) = \mathrm{sign}(j_0)$ and

$$\dots \underbrace{\pm 2 \cdots \pm 2 \pm 1}_{|j_2|} \underbrace{\mp 2 \cdots \mp 2 \mp 1}_{|j_1|} \underbrace{\pm 2 \cdots \pm 1}_{|j_0|} ; \underbrace{\mp 2 \cdots \mp 2}_{|i_0|-1} \underbrace{\mp 1 \pm 2 \cdots \pm 2}_{|i_1|} \underbrace{\pm 1 \mp 2 \cdots \mp 2}_{|i_2|} \dots$$

if $\mathrm{sign}(i_0) \neq \mathrm{sign}(j_0)$, where the sign of each block is the same of the sign of i_n and j_m. If any j_m or i_n has module 1, then the block associated to it will only be the respective 1, and if it has higher module you start "adding" 2's. To clarify the idea, let us check some examples

Example 3.2. *Here we are going to code some examples just to help understand exactly how h_i codes the i-coordinate.*

(i) $(3 \oplus -2 \oplus \dots, 1 \oplus -4 \oplus \dots)$*: The sign of i_0 and j_0 are equal then*

$$h_i(3 \oplus -2 \oplus \dots, 1 \oplus -4 \oplus \dots) = \dots \underbrace{\text{-2-2-2-1}}_{|-4|} \underbrace{2}_{|1|} ; \underbrace{22}_{|3|-1} \underbrace{\text{1-2}}_{|-2|} \dots$$

(ii) $(-1 \oplus 3 \oplus -1 \dots, -1 \oplus 2 \oplus \dots)$*: Once again they have the same sign*

$$h_i(-1 \oplus 3 \oplus -1 \dots, -1 \oplus 2 \oplus \dots) = \dots \underbrace{21}_{|2|} \underbrace{\text{-2}}_{|-1|} ; \underbrace{\quad}_{|-1|-1} \underbrace{\text{-122}}_{|3|} \underbrace{1}_{|-1|} \dots$$

(iii) $(3 \oplus -2 \oplus \dots, -1 \oplus 4 \oplus \dots)$*: Now i_0 and j_0 have different signs*

$$h_i(3 \oplus -2 \oplus \dots, -1 \oplus 4 \oplus \dots) = \dots \underbrace{2221}_{|4|} \underbrace{\text{-1}}_{|-1|} ; \underbrace{22}_{|3|-1} \underbrace{\text{1-2}}_{|-2|} \dots$$

Let $\sigma_i : \Sigma \to \Sigma$ be the shift map on the space of the sequences over the alphabet $\mathcal{A}$. Then

Lemma 3.4. $\sigma_i \circ h_i = h_i \circ f$

Proof. We will do the proof only looking at the upper plane of $\mathbb{R}^2$ due the symmetry of f. Hence

- $i_0 > 0 \quad j_0 > 0$:
  Let $p = (i_0 \oplus_{n \in \mathbb{N}} i_n, j_0 \oplus_{m \in \mathbb{N}} j_m)$, then we know that

$$f(p) = \begin{cases} (i_0 - 1 \oplus_{n \in \mathbb{N}} i_n, j_0 + 1 \oplus_{m \in \mathbb{N}} j_m) & i_0 > 1; \\ (i_1 \oplus_{n \geq 2} i_n, j_0 + 1 \oplus_{m \in \mathbb{N}} j_m) & i_0 = 1; \end{cases}$$

which implies that

$$h_i \circ f(p) = \begin{cases} \cdots \underbrace{-2\cdots-2-1}_{|j_1|}\underbrace{2\dots2}_{|j_0+1|};\underbrace{2\dots2}_{|i_0-1|-1}\underbrace{1-2\cdots-2}_{|i_1|}\cdots & i_0 > 1; \\ \cdots \underbrace{-2\cdots-2-1}_{|j_1|}\underbrace{2\dots21}_{|j_0+1|};\underbrace{-2\cdots-2}_{|i_1|-1}\underbrace{-12\dots2}_{|i_2|}\cdots & i_0 = 1; \end{cases}$$

once we have alternating signs for i_0 and i_1. Also, we know that

$$h_i(p) = \begin{cases} \cdots \underbrace{-2\cdots-2-1}_{|j_1|}\underbrace{2\dots2}_{|j_0|};\underbrace{2\dots2}_{|i_0|-1}\underbrace{1-2\cdots-2}_{|i_1|}\cdots & i_0 > 1; \\ \cdots \underbrace{-2\cdots-2-1}_{|j_1|}\underbrace{2\dots2}_{|j_0|};\underbrace{1-2\cdots-2}_{|i_1|}\underbrace{-12\dots2}_{|i_2|}\cdots & i_0 = 1; \end{cases}$$

Now applying the shift we get

$$\sigma_i \circ h_i(p) = \begin{cases} \cdots \underbrace{-2\cdots-2-1}_{|j_1|}\underbrace{2\dots2}_{|j_0|+1};\underbrace{2\dots2}_{|i_0|-2}\underbrace{1-2\cdots-2}_{|i_1|}\cdots & i_0 > 1; \\ \cdots \underbrace{-2\cdots-2-1}_{|j_1|}\underbrace{2\dots21}_{|j_0|+1};\underbrace{-2\cdots-2}_{|i_1|-1}\underbrace{-12\dots2}_{|i_2|}\cdots & i_0 = 1; \end{cases}$$

and proves the statement in these cases.

- $i_0 > 0 \quad j_0 < 0$:

  The proof here is basically the same, we only chance how we apply f

$$f(p) = \begin{cases} (i_0 - 1 \oplus_{n\in\mathbb{N}} i_n, 1 \oplus j_0 \oplus_{m\in\mathbb{N}} j_m) & i_0 > 1 \\ (i_1 \oplus_{n\geq 2} i_n, 1 \oplus j_0 \oplus_{m\in\mathbb{N}} j_m) & i_0 = 1; \end{cases}$$

and then

$$h_i \circ f(p) = \begin{cases} \cdots \underbrace{2\dots21}_{|j_1|}\underbrace{-2\cdots-2-1}_{|j_0|}\underbrace{2}_{1};\underbrace{2\dots2}_{|i_0-1|-1}\underbrace{1-2\cdots-2}_{|i_1|}\cdots & i_0 > 1; \\ \cdots \underbrace{2\dots21}_{|j_1|}\underbrace{-2\cdots-2-1}_{|j_0|}\underbrace{1}_{1};\underbrace{-2\cdots-2}_{|i_1|-1}\underbrace{-12\dots2}_{|i_2|}\cdots & i_0 = 1; \end{cases}$$

Also we have that

$$h_i(p) = \begin{cases} \cdots \underbrace{2\dots21}_{|j_1|}\underbrace{-2\cdots-2-1}_{|j_0|};\underbrace{2\dots2}_{|i_0|-1}\underbrace{1-2\cdots-2}_{|i_1|}\cdots & i_0 > 1; \\ \cdots \underbrace{2\dots21}_{|j_1|}\underbrace{-2\cdots-2-1}_{|j_0|};\underbrace{1-2\cdots-2}_{|i_1|}\underbrace{-12\dots2}_{|i_2|}\cdots & i_0 = 1; \end{cases}$$

and applying the shift

$$\sigma_i \circ h_i(p) = \begin{cases} \cdots \underbrace{2\dots21}_{|j_1|}\underbrace{-2\cdots-2-1}_{|j_0|}\underbrace{2}_{1};\underbrace{2\dots2}_{|i_0|-2}\underbrace{1-2\cdots-2}_{|i_1|}\cdots & i_0 > 1; \\ \cdots \underbrace{2\dots21}_{|j_1|}\underbrace{-2\cdots-2-1}_{|j_0|}\underbrace{1}_{1};\underbrace{-2\cdots-2}_{|i_1|-1}\underbrace{-12\dots2}_{|i_2|}\cdots & i_0 = 1; \end{cases}$$

Therefore putting together both items above, we conclude the Lemma's proof. □

The last thing regarding the i-coordinate coding that is needed to be discussed is how to code the point that have finite orbit foreword or backward. As we discussed before, this happens if you are on a pre-image of $\{y = 0\}$ or on an image of the $\{y = -x\}$, which implies that you have a finite i of j-coordinate. In this case, you just use the h_i defined and when you reach the "final" number you just put 0 in the next step and cease to code. These point will have a finite coding backward or forward.

Example 3.3. *Each one of the examples below has different type of finite orbit. To fully understand what is happening here, try to visualise the geometric interpretation of the finite orbit.*

(i) $(3, 1 \oplus -4 \oplus \dots)$*: This point lies over the* $f^{-3}(\{y = 0\})$ *but it is not over any image of* $\{y = -x\}$

$$h_i(3, 1 \oplus -4 \oplus \dots) = \dots \underbrace{\text{-2-2-2-1}}_{|-4|}\underbrace{2}_{|1|};\underbrace{22}_{|3|-1}0$$

(ii) $(3 \oplus -2 \oplus \dots, 1 \oplus -4)$*: The point here is over* $f^4(\{y = -x\})$ *but it is not over any pre-images of* $\{y = 0\}$

$$h_i(3 \oplus -2 \oplus \dots, 1 \oplus -4) = 0\underbrace{\text{-2-2-2-1}}_{|-4|}\underbrace{2}_{|1|};\underbrace{22}_{|3|-1}\underbrace{\text{1-2}}_{|-2|}\dots$$

(iii) $(3, 1 \oplus -4)$*: This one lies over one of the intersections between* $f^{-3}(\{y = 0\})$ *and* $f^4(\{y = -x\})$

$$h_i(3, 1 \oplus -4) = 0\underbrace{\text{-2-2-2-1}}_{|-4|}\underbrace{2}_{|1|};\underbrace{22}_{|3|-1}0$$

3.3.2 Coding the j-coordinate

The j-coordinate will have the same kind of coding and, in fact, it is possible to see a direct relation between both coordinates. They have an strict relation and it will become very clear once we define the other map.

To code the j-coordinate, let $p = (i_0 \oplus_{n\in\mathbb{N}} i_n, j_0 \oplus_{m\in\mathbb{N}} j_m)$ be a point with full orbit, the definition of $h_j(p)$ is once again split depending on the sign of i_0 and j_0:

$$\dots\underbrace{\pm1\pm2\cdots\pm2}_{|j_2|}\underbrace{\mp1\mp2\cdots\mp2}_{|j_1|}\underbrace{\pm1\pm2\dots;\pm2}_{|j_0|}\underbrace{\pm2\cdots\pm2}_{|i_0|}\underbrace{\mp1\mp2\cdots\mp2}_{|i_1|}\underbrace{\pm1\pm2\cdots\pm2}_{|i_2|}\dots$$

if $\text{sign}(i_0) = \text{sign}(j_0)$ and

$$\dots\underbrace{\pm1\pm2\cdots\pm2}_{|j_2|}\underbrace{\mp1\mp2\cdots\mp2}_{|j_1|}\underbrace{\pm1\pm2\dots;\pm2}_{|j_0|}\underbrace{\pm1\pm2\cdots\pm2}_{|i_0|}\underbrace{\mp1\mp2\cdots\mp2}_{|i_1|}\underbrace{\pm1\pm2\cdots\pm2}_{|i_2|}\dots$$

if $\operatorname{sign}(i_0) \neq \operatorname{sign}(j_0)$, where the sign of each block is the same of the sign of i_n and j_m.Keep in mind that if any j_m or i_n has module 1, then the block associated to it will only be the respective 1. Lets look once more to the examples we presented before, but now under the j-perspective:

Example 3.4. *Here we are going to code some examples just to help understand exactly how h_j codes the j-coordinate.*

(i) $(3 \oplus -2 \oplus \dots, 1 \oplus -4 \oplus \dots)$*: The sign of i_0 and j_0 are equal then*

$$h_j(3 \oplus -2 \oplus \dots, 1 \oplus -4 \oplus \dots) = \dots \underbrace{\text{-1-2-2-2}}_{|-4|}\,;\underbrace{\;1\;}_{|1|}\underbrace{222}_{|3|}\underbrace{\text{-1-2}}_{|-2|}\dots$$

(ii) $(-1 \oplus 3 \oplus -1 \dots, -1 \oplus 2 \oplus \dots)$*: Once again they have the same sign*

$$h_j(-1 \oplus 3 \oplus -1 \dots, -1 \oplus 2 \oplus \dots) = \dots \underbrace{\;12\;}_{|2|}\,;\underbrace{\;\text{-1}\;}_{|-1|}\underbrace{\;-2\;}_{|-1|}\underbrace{122}_{|3|}\underbrace{\;\text{-1}\;}_{|-1|}\dots$$

(iii) $(3 \oplus -2 \oplus \dots, -1 \oplus 4 \oplus \dots)$*: Now i_0 and j_0 have different signs*

$$h_j(3 \oplus -2 \oplus \dots, -1 \oplus 4 \oplus \dots) = \dots \underbrace{1222}_{|4|}\,;\underbrace{\;\text{-1}\;}_{|-1|}\underbrace{122}_{|3|-1}\underbrace{\text{-1-2}}_{|-2|}\dots$$

Let $\sigma_j : \Sigma \to \Sigma$ be the shift map on the space of the sequences over the alphabet $\mathcal{A}$. Then

Lemma 3.5. $\sigma_j \circ h_j = h_j \circ f$

Proof. We will do the proof only looking at the upper plane of $\mathbb{R}^2$ due the symmetry of f. Hence

- $i_0 > 0 \quad j_0 > 0$:

  Let $p = (i_0 \oplus_{n\in\mathbb{N}} i_n, j_0 \oplus_{m\in\mathbb{N}} j_m)$, then we know that

$$f(p) = \begin{cases} (i_0 - 1 \oplus_{n\in\mathbb{N}} i_n, j_0 + 1 \oplus_{m\in\mathbb{N}} j_m) & i_0 > 1; \\ (i_1 \oplus_{n\geq 2} i_n, j_0 + 1 \oplus_{m\in\mathbb{N}} j_m) & i_0 = 1; \end{cases}$$

  which implies that

$$h_j \circ f(p) = \begin{cases} \dots \underbrace{\text{-1-2}\cdots\text{-2}}_{|j_1|}\underbrace{12\dots;2}_{|j_0+1|}\underbrace{2\dots2}_{|i_0-1|}\underbrace{\text{-1-2}\cdots\text{-2}}_{|i_1|}\dots & i_0 > 1; \\ \dots \underbrace{\text{-1-2}\cdots\text{-2}}_{|j_1|}\underbrace{12\dots2}_{|j_0+1|};\underbrace{\text{-1-2}\cdots\text{-2}}_{|i_1|}\underbrace{12\dots2}_{|i_2|}\dots & i_0 = 1; \end{cases}$$

  once we have alternating signs for i_0 and i_1. Also, we know that

$$h_j(p) = \begin{cases} \dots \underbrace{\text{-1-2}\cdots\text{-2}}_{|j_1|}\underbrace{12\dots;2}_{|j_0|}\underbrace{2\dots2}_{|i_0|}\underbrace{\text{-1-2}\cdots\text{-2}}_{|i_1|}\dots & i_0 > 1; \\ \dots \underbrace{\text{-1-2}\cdots\text{-2}}_{|j_1|}\underbrace{12\dots;2}_{|j_0|}\underbrace{\;2\;}_{|i_0|}\underbrace{\text{-1-2}\cdots\text{-2}}_{|i_1|}\underbrace{12\dots2}_{|i_2|}\dots & i_0 = 1; \end{cases}$$

Now applying the shift we get

$$\sigma_j \circ h_j(p) = \begin{cases} \cdots \underbrace{\text{-1-2}\cdots\text{-2}}_{|j_1|}\underbrace{12\ldots;2}_{|j_0|+1}\underbrace{2\ldots 2}_{|i_0|-1}\underbrace{\text{-1-2}\cdots\text{-2}}_{|i_1|}\cdots & i_0 > 1; \\ \cdots \underbrace{\text{-1-2}\cdots\text{-2}}_{|j_1|}\underbrace{12\ldots;2}_{|j_0|+1}\underbrace{\text{-1-2}\cdots\text{-2}}_{|i_1|-1}\underbrace{12\ldots 2}_{|i_2|}\cdots & i_0 = 1; \end{cases}$$

and proves the statement in these cases.

- $i_0 > 0 \quad j_0 < 0$:

  The proof here is basically the same, we only chance how we apply f

$$f(p) = \begin{cases} (i_0 - 1 \oplus_{n\in\mathbb{N}} i_n, 1 \oplus j_0 \oplus_{m\in\mathbb{N}} j_m) & i_0 > 1 \\ (i_1 \oplus_{n\geq 2} i_n, 1 \oplus j_0 \oplus_{m\in\mathbb{N}} j_m) & i_0 = 1; \end{cases}$$

  and then

$$h_j \circ f(p) = \begin{cases} \cdots \underbrace{12\ldots 2}_{|j_1|}\underbrace{\text{-1-2}\cdots\text{-2}}_{|j_0|};\underbrace{1}_{1}\underbrace{2\ldots 2}_{|i_0-1|}\underbrace{\text{-1-2}\cdots\text{-2}}_{|i_1|}\cdots & i_0 > 1; \\ \cdots \underbrace{12\ldots 2}_{|j_1|}\underbrace{\text{-1-2}\cdots\text{-2}}_{|j_0|};\underbrace{1}_{1}\underbrace{\text{-1-2}\cdots\text{-2}}_{|i_1|}\underbrace{12\ldots 2}_{|i_2|}\cdots & i_0 = 1; \end{cases}$$

  Also we have that

$$h_j(p) = \begin{cases} \cdots \underbrace{12\ldots 2}_{|j_1|}\underbrace{\text{-1-2}\ldots;\text{-2}}_{|j_0|}\underbrace{12\ldots 2}_{|i_0|}\underbrace{\text{-1-2}\cdots\text{-2}}_{|i_1|}\cdots & i_0 > 1; \\ \cdots \underbrace{12\ldots 2}_{|j_1|}\underbrace{\text{-1-2}\ldots;\text{-2}}_{|j_0|}\underbrace{1}_{|i_0|}\underbrace{\text{-1-2}\cdots\text{-2}}_{|i_1|}\underbrace{112\ldots 2}_{|i_2|}\cdots & i_0 = 1; \end{cases}$$

  and applying the shift

$$\sigma_j \circ h_j(p) = \begin{cases} \cdots \underbrace{12\ldots 2}_{|j_1|}\underbrace{\text{-1-2}\cdots\text{-2}}_{|j_0|};\underbrace{1}_{1}\underbrace{2\ldots 2}_{|i_0|-1}\underbrace{\text{-1-2}\cdots\text{-2}}_{|i_1|}\cdots & i_0 > 1; \\ \cdots \underbrace{12\ldots 2}_{|j_1|}\underbrace{\text{-1-2}\cdots\text{-2}}_{|j_0|};\underbrace{1}_{1}\underbrace{\text{-1-2}\cdots\text{-2}}_{|i_1|}\underbrace{12\ldots 2}_{|i_2|}\cdots & i_0 = 1; \end{cases}$$

therefore putting together both items above, we conclude the Lemma's proof. □

As before, we define here the image of the point with finite orbit in the exact same way as before: just add zero after using all the available i_n's and j_m's.

3.4 Conjugacy and its consequences

Each one of the coordinates identifies every time the point enters the zone of interest and how long it takes to get there. The length of each block between each ± 1 is how long it will take to return the region delimited by R_1 and R_{-1} in the i-coordinate and L_1 and L_{-1} in the j-coordinate.

We can restate the theorem by being a bit more precise about each one of the subshifts we mentioned and also the precise map

Theorem (D'). *Let Σ_i and Σ_j be the image of $h_i(\mathbb{R}^2)$ and $h_i(\mathbb{R}^2)$, respectively. Define the map*

$$\begin{array}{rcl} h:\mathbb{R}^2 & \to & \Sigma_i \times \Sigma_j \\ p & \mapsto & (h_i(p), h_j(p)) \end{array}$$

We know that this map is an homeomorphism between these spaces, due the construction of each h_i. Therefore we have the following commuting diagram

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "f"] \arrow[d, "h"'] & \mathbb{R}^2 \arrow[d, "h"] \\ \Sigma_i \times \Sigma_j \arrow[r, "\sigma"'] & \Sigma_i \times \Sigma_j \end{tikzcd}$$

where

$$\begin{array}{rcl} \sigma:\Sigma_i \times \Sigma_j & \to & \Sigma_i \times \Sigma_j \\ ((s_n),(s_m)) & \mapsto & (\sigma_i(s_n), \sigma_j(s_m)) \end{array}$$

Which gives this immediate consequence.

Corollary. *The Hénon-Devaney map has a density of hyperbolic periodic points.*

Our initial goal with this coding were trying to get some tools walking towards the recurrence of the Hénon-Devaney map, however we managed to get something a bit weaker than that. With this coding we can only get "density of recurrence" in the sense that, given an open set R in the plane we can find a dense of orbits that enters R in finite time, even more, we can determine in which time we want the point enters the region. As a consequence of this fact, we also get that there exists a orbit which is dense in the plane.

We wanted to prove the conjecture

Conjecture 3.1. *The Hénon-Devaney map is ergodic.*

If we were able to prove that f is recurrent then we should be able to achieve the ergodicity, using the classical Hopf's argument for the first return map in a neighborhood of the origin in the plane. However... while the recurrence remains unproven, this is just pure philosophy...

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