The "Smoothest" Average and New Uncertainty Principles for the Fourier Transform

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Two years ago, a colleague from economics asked me for the "best" way to compute the average income over the last year. At first I didn't understand but then he explained it to me: suppose you are given a real-valued function $f(x)$ and want to compute a local average at a certain scale. What we usually do is to pick a nice probability measure u, centered at 0 and having standard deviation at the desired scale, and convolve $f * u$. Classical candidates for u are the characteristic function or the Gaussian. This got me interested in finding the "best" function u – this problem comes in two parts: (1) describing what one considers to be desirable properties of the convolution $f * u$ and (2) understanding which functions u satisfy these properties. I tried a basic notion for the first part, "the convolution should be as smooth as the scale allows", and ran into lots of really funky classical Fourier Analysis that seems to be new: (a) new uncertainty principles for the Fourier transform, (b) that potentially have the characteristic function as an extremizer, (c) which leads to strange new patterns in hypergeometric functions and (d) produces curious local stability inequalities. Noah Kravitz and I managed to solve two specific instances on the discrete lattice completely, this results in some sharp weighted estimates for polynomials on the unit interval – both the Dirichlet and the Fejer kernel make an appearance. The entire talk will be completely classical Harmonic Analysis, there are lots and lots of open problems and I will discuss several.