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Graphs of Hecke Operators, Orthogonal Periods, and Prime Numbers in Short Intervals

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Contents

Introduction

The subject of this thesis are three mathematical problems in the theory of automorphic forms and in number theory. We describe below the three problems.

Graphs of Hecke operators and toroidal automorphic forms

The theory of toroidal automorphic forms begins with the paper [104] in which Zagier proposes an approach to the Riemann hypothesis using automorphic forms on the upper half plane. In this paper, inspired by classical formulas by Dirichlet and Hecke (cf. (2.1.2) and (2.1.3), Zagier defines a space $\mathcal{E}$ of functions on $\Gamma\backslash\mathbb{H}$ which annihilate some linear operators (cf. [104, Thm. pg. 286]), where $\mathbb{H}$ is the upper half plane and $\Gamma = PSL_2(\mathbb{Z})$. In § 4 of [104], Zagier shows that $\mathcal{E}$ is the set of K-fixed vectors of a certain G-invariant subspace $\mathcal{V}$ of the space of functions on $\Gamma\backslash G$ (where $G = PSL_2(\mathbb{R})$, $K = PSO(2)$) and that if $\mathcal{V}$ is an unitarizable representation of G, then the Riemann hypothesis follows. Using adelic language, we can translate these concepts to automorphic forms on $GL_2(\mathbb{A}_F)$ for F a global field. As noted by Zagier in [104, pp. 298-300], the analogue of the space $\mathcal{V}$ in adelic language is the space of toroidal automorphic forms (whose definition we recall below). These ideas were explored by Lachaud (cf. [72] and [73]) who connected them with Connes' approach to the Riemann Hypothesis (cf. [38]), by relating the space of toroidal automorphic forms with the construction of Pólya-Hilbert spaces. When F is a function field, Lorscheid studies in his thesis ([85]) the space of toroidal forms in $GL_2(\mathbb{A}_F)$ using the theory of graphs of Hecke operators. The graph of a Hecke operator Φ is an oriented graph with weighted edges that encodes the action of Φ on the space of unramified automorphic forms in $GL_n(\mathbb{A}_F)$. It is a computational device that allows us to make explicit calculations with automorphic forms and in particular with toroidal automorphic forms.

In this thesis we study toroidal automorphic forms on $GL_3(\mathbb{A}_F)$ when F is an elliptic function field. We do this by using an algorithm developed by Alvarenga in his thesis (cf. [2]) which allows us to compute the graphs of Hecke operators for elliptic function fields.

Let us start by recalling briefly the Hecke operators. Let F be a global function field over the finite field $\mathbb{F}_q$. We denote by $\mathbb{A}_F$ the adele ring of F, by $\mathcal{O}_{\mathbb{A}_F}$ the ring of adelic integers and by $\mathbb{A}_F^\times$ the group of ideles of F (cf. Section 1.1). We denote by G the general linear group GL_n and by Z the center of G. The group $G(\mathbb{A}_F)$ of adelic points of G with the adelic topology is a locally compact unimodular group (cf. Section 1.1). Let $K = G(\mathcal{O}_{\mathbb{A}_F})$ be the standard maximal compact open subgroup of $G(\mathbb{A}_F)$. We fix the Haar measure on $G(\mathbb{A}_F)$ for which $vol(K) = 1$.

The complex vector space $\mathcal{H}$ of all smooth compactly supported functions $\Phi : G(\mathbb{A}_F) \to \mathbb{C}$ together with the convolution product

$$\Phi_1 * \Phi_2 : g \longmapsto \int_{G(\mathbb{A}_F)} \Phi_1(gh^{-1})\Phi_2(h)dh$$

for $\Phi_1, \Phi_2 \in \mathcal{H}$ is called the Hecke algebra for $G(\mathbb{A}_F)$. Its elements are called Hecke operators. The zero element of $\mathcal{H}$ is the zero function, but there is no multiplicative unit. We define $\mathcal{H}_K$ to be the subalgebra of all bi-K-invariant elements. This subalgebra has a multiplicative unit, namely, the characteristic function $\epsilon_K := \mathrm{char}_K$ acts as the identity on $\mathcal{H}_K$ by convolution. We call $\mathcal{H}_K$ the unramified part of $\mathcal{H}$ and its elements are called unramified Hecke operators.

A Hecke operator $\Phi \in \mathcal{H}$ acts on the space $\mathcal{V} := C^0(G(F)Z(\mathbb{A}_F) \setminus G(\mathbb{A}_F))$ of continuous functions $f : G(F)Z(\mathbb{A}_F) \setminus G(\mathbb{A}_F) \to \mathbb{C}$ as follows

$$\Phi(f)(g) := \int_{G(\mathbb{A}_F)} \Phi(h)f(hg)dh.$$

The above action restricts to an action of $\mathcal{H}_K$ on $\mathcal{V}^K$, the space of right K-invariant functions. We denote by $\mathcal{A}$ the space of automorphic forms on $GL_n(\mathbb{A}_F)$ which is a subspace of $\mathcal{V}$ stable by the action of $\mathcal{H}$ (cf. section 1.5). We denote by $\mathcal{A}^K$ the space of *unramified automorphic forms*, i.e. automorphic forms which are invariant by the action of K.

The second concept that we need is that of toroidal automorphic forms. We start with the torus associated with a separable extension E/F of degree n. Choosing a basis of E over F gives a non-split maximal torus $T_E \subset G$, whose adelic points are described in the following way. The choice of basis gives an identification of $\mathbb{A}_E = \mathbb{A}_F \otimes_F E$ with $\mathbb{A}_F^{\oplus n}$. Given $g \in \mathbb{A}_E^\times$, the multiplication by g yields a map,

$$M(g) : \mathbb{A}_E \longrightarrow \mathbb{A}_E, \;\; M(g)(h) = gh.$$

The isomorphism $\mathbb{A}_E = \mathbb{A}_F \otimes_F E \simeq \mathbb{A}^{\oplus n}$ identifies $M(g)$ with a matrix $\Theta_E(g) \in G(\mathbb{A}_F)$. The map $\Theta_E : \mathbb{A}_E^\times \hookrightarrow G(\mathbb{A}_F)$ is a homomorphism and $T_E(\mathbb{A}_F) = \mathrm{Im}\Theta_E$. The *toroidal integral of an automorphic form f along T_E* is defined by

$$f_{T_E}(g) := \int\limits_{T_E(F)Z(\mathbb{A}_F)\backslash T_E(\mathbb{A}_F)} f(tg)\; dt,$$

where $g \in G(\mathbb{A}_F)$ and dt is a Haar measure on $T_E(F)Z(\mathbb{A}_F) \setminus T_E(\mathbb{A}_F)$. An automorphic form with vanishing toroidal integral along the torus T_E for all separable extension E/F of degree n is called a *toroidal automorphic forms.* Sections 2.1.1–2.1.4 discuss the arithmetic meaning of toroidal automorphic forms.

A theorem of Satake and Tamagawa shows that $\mathcal{H}_K$ is generated by the Hecke operators $\Phi_{x,r}$ with x a place of F and $1 \leq r \leq n$, where $\Phi_{x,r}$ is the characteristic function of

$$K \begin{pmatrix} \pi_x I_r & \\ & I_{n-r} \end{pmatrix} K$$

(cf. (2.2.1)). Therefore, to describe the action of $\mathcal{H}_K$ on $\mathcal{A}^K$ is sufficient to describe the action of the Hecke operators $\Phi_{x,r}$. By a theorem due to André Weil, there is a bijection between $G(F) \setminus G(\mathbb{A}_{\mathbb{F}})/K$ with the set $\mathrm{Bun}_n X$ of isomorphism classes of rank-n vector bundles on X, where X is the smooth curve over $\mathbb{F}_q$ associated with F. Similarly, there is a bijection between $G(F)Z(\mathbb{A}_F) \setminus G(\mathbb{A}_F)/K$ with the set $\mathbf{P}\mathrm{Bun}_n X$ of isomorphism classes of projective rank-n vector bundles on X (cf. section 2.2). This theorem allows us to see the unramified automorphic forms as functions on $\mathrm{Bun}_n X$ and to determine the action of an unramified Hecke operator $\Phi_{x,r}$ in terms of vector bundles, which we explain in the following.

For x a closed point on X, we denote by $\kappa(x)$ the residue field at x and by $\mathcal{K}_x^{\oplus r}$ the skyscraper sheaf on X whose stalk at x is $\kappa(x)^{\oplus r}$. Let $\mathcal{E}, \mathcal{E}' \in \mathrm{Bun}_n X$, we denote by $m_{x,r}(\mathcal{E}, \mathcal{E}')$ the number of subsheaves $\mathcal{E}''$ of $\mathcal{E}$ that are isomorphic to $\mathcal{E}'$ and such that the quotient $\mathcal{E}/\mathcal{E}''$ is isomorphic to $\mathcal{K}_x^{\oplus r}$. We denote by $\mathcal{V}_{x,r}(\mathcal{E})$ the set of $\big(\mathcal{E}, \mathcal{E}', m_{x,r}(\mathcal{E}, \mathcal{E}')\big)$ such that $m_{x,r}(\mathcal{E}, \mathcal{E}') \neq 0$. We define the graphs of Hecke operators $\mathcal{G}_{x,r}$ as follows.

Definition. *Let x be a closed point in X. The graph $\mathcal{G}_{x,r}$ is defined as*

$$\mathrm{Vert}\,\mathcal{G}_{x,r} = \mathrm{Bun}_n X \quad \textit{and} \quad \mathrm{Edge}\,\mathcal{G}_{x,r} = \coprod_{\mathcal{E} \in \mathrm{Bun}_n X} \mathcal{V}_{x,r}(\mathcal{E}).$$

With this definition, the action of $\Phi_{x,r}$ on unramified automorphic forms is as follows.

Theorem. *Let $\Phi_{x,r} \in \mathcal{H}_K$ and $f \in \mathcal{A}^K$. For all $\mathcal{E} \in \mathrm{Bun}_n X$, we have*

$$\Phi_{x,r}(f)(\mathcal{E}) = \sum_{(\mathcal{E}, \mathcal{E}', m) \in \mathcal{V}_{x,r}(\mathcal{E})} m f(\mathcal{E}').$$

Let $n = 3$ and X be an elliptic curve, we want to study the spaces

$$\mathcal{A}(x; \lambda_1, \lambda_2) = \left\{ f \in \mathcal{A}^K \;\middle|\; \Phi_{x,i}(f) = \lambda_i f,\ i = 1, 2 \right\}.$$

The first application of the graphs of Hecke operators will be to parametrize the spaces $\mathcal{A}(x; \lambda_1, \lambda_2)$. This is done in two steps. The first step is to define the nucleus of the graphs $\mathcal{G}_{x,r}$, $r = 1, 2$. This is done using the d-invariant which is defined on decomposable vector bundles in $\mathrm{Bun}_3 X$ as follows:

- If $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$ with $\mathcal{M} \in \mathrm{Bun}_2^{ind} X$ and $\mathcal{L} \in \mathrm{Pic}\, X$, we define $d(\mathcal{E}) := 2 \deg \mathcal{L} - \deg \mathcal{M}$.
- If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ is a sum of three line bundles with $\deg \mathcal{L}_1 \leq \deg \mathcal{L}_2 \leq \deg \mathcal{L}_3$, we define

$$d_1(\mathcal{E}) := \deg(\mathcal{L}_2) - \deg(\mathcal{L}_1), \qquad d_2(\mathcal{E}) := \deg(\mathcal{L}_3) - \deg(\mathcal{L}_2),$$

and

$$d(\mathcal{E}) := max\{2d_1(\mathcal{E}) + d_2(\mathcal{E}), 2d_2(\mathcal{E}) + d_1(\mathcal{E})\}.$$

We see that $|d(\mathcal{E})| = \delta(\mathcal{E})$ except for a finite number of vertices $\mathcal{E} \in \mathbf{P}\,\mathrm{Bun}_3 X$ (cf. Theorem 2.5.9), where δ is the invariant defined by Alvarenga in [2]. The nucleus $\mathcal{N}_x$ is a finite set of vertex of $\mathbf{P}\,\mathrm{Bun}_3 X$ with small d-invariant (cf. Definition 2.5.2) and it satisfy the following (cf. Theorem 2.5.13).

Theorem. *If $f, g \in \mathcal{A}(x; \lambda_1, \lambda_2)$ and $f|_{\mathcal{N}_x} = g|_{\mathcal{N}_x}$, then $f = g$.*

Using this theorem we can apply the algorithm from [2] to compute the equations satisfied by elements of $\mathcal{A}(x; \lambda_1, \lambda_2)$ on the nucleus and to obtain a parametrization of this space. We do this in Section 2.7 for a particular example. This gives us an explicit description of the eigenforms parametrized by their values at some particular vector bundles. We use this in Section 2.8 to study the unramifed eigenforms that are toroidal for a particular torus. We show in Section 2.8.1 how to prove the Riemann hypothesis for the elliptic function field in question, based on our parametrization of the space of eigenforms. These results are part of a project in progress with R. Alvarenga and O. Lorscheid.

Orthogonal period of a GL_4 Eisenstein series

The second problem is motivated by a conjecture of H. Jacquet in [62]. We denote by $\widetilde{GL}_n(\mathbb{A}_F)$ the metaplectic double cover of $GL_n(\mathbb{A}_F)$ over a global field F (cf. [68] for definition and properties of metaplectic groups). In [62], H. Jacquet conjectured a comparison between a relative trace formula on GL_n over a global field F with a relative trace formula on $\widetilde{GL}_n(\mathbb{A}_F)$. We give a sketch of what is the relative trace formula. We start with a triple (G, H_1, H_2) consisting of a reductive group G and two suitable subgroups H_1, H_2 which are algebraic groups over the global field F. Here the two subgroups H_1 and H_2 are possible the same. We associate to a test function $f \in \mathcal{C}_c^\infty(G(\mathbb{A}_F))$ the kernel function

$$K_f(x, y) := \sum_{\gamma \in G(F)} f(x^{-1}\gamma y),$$

where $x, y \in G(\mathbb{A}_F)$. We consider the linear functional on $\mathcal{C}_c^\infty(G(\mathbb{A}_F))$ defined by the double integral

$$I(f) = \int_{H_1(F)\backslash H_1(\mathbb{A}_F)} \int_{H_2(F)\backslash H_2(\mathbb{A}_F)} K_f(h_1, h_2) dh_1 dh_2.$$

For some groups G, we may multiply the kernel on the integrand by some weight factor: for example, a character of $H_i(F)\backslash H_i(\mathbb{A}_F)$. The relative trace formula attached to the triple (G, H_1, H_2) is an identity between two different expansions of $I(f)$, known as the "spectral expansion" and the "geometric expansion" of $I(f)$. The orthogonal period of an automorphic form $\phi : GL_n(F)\backslash GL_n(\mathbb{A}_F) \to \mathbb{C}$ is the integral

$$\mathcal{P}_H(\phi) := \int_{H(F)\backslash H(\mathbb{A}_F)} \phi(h)dh,$$

where H is the orthogonal group of some non-degenerate quadratic form in $F^{\oplus n}$. For the relative trace formula on GL_n appearing on Jacquet's paper, the spectral expansion uses the decomposition of $L^2(GL_n(F)\backslash GL_n(\mathbb{A}_F))$ obtained from automorphic forms. Orthogonal periods of automorphic forms appears in this expansion. On the other hand, in the spectral expansion of the relative trace formula on $\widetilde{GL}_n(\mathbb{A}_F)$, the Whittaker coefficients of the automorphic forms appear (for the definition of Whittaker coefficients see [18] and [19]). In the paper [62], Jacquet proves the case $n = 2$ of the comparison of the relative trace formulas and state the conjecture for general n. For a survey on works on the relative trace formula, cf. [63] and [77].

We can look at this problem from another perspective using the following fact. It is proven that the Whittaker coefficient of certain Eisenstein series on $\widetilde{GL}_n(\mathbb{A}_F)$ are Weyl group multiple Dirichlet series of type A_{n-1} (cf. [18] and [19]). If we put together this fact with the conjecture of H. Jacquet for orthogonal periods, we obtain the following conjecture.

Conjecture. *Orthogonal periods of Eisenstein series in GL_n induced from characters on the Levi subgroup can be expressed in terms of A_{n-1}-Weyl group multiple Dirichlet series.*

Chinta and Offen proved this conjecture when $n = 3$ in [33], thereby providing evidence in favor of Jacquet's conjecture. Our objective is to study the conjecture above for a GL_4-Eisenstein series by elementary number theoretic methods, inspired by the papers [34] and [33]. Let Q be a positive definite rational quadratic form in n variables and H_Q the associated orthogonal group. The paper [34] shows how to express the orthogonal period over H_Q of Eisenstein series induced from characters on the Levi subgroup of GL_n as a multiple Dirichlet series whose coefficients is a certain type of representation number associated with Q. In our case, we consider an Eisenstein series in GL_4 induced from a character on the parabolic subgroup of type $(2, 2)$ and we obtain a Dirichlet series whose coefficients $r(n)$ is given by

$$r(n) = 2\#\{L \subset \mathbb{Z}^4 \mid L \text{ is primitive of rank 2 and } disc(Q_L) = n\},$$

where we say that a lattice $L \subset \mathbb{Z}^n$ is *primitive* if $(\mathbb{Q} \cdot L) \cap \mathbb{Z}^n = L$. If Q_4 is the quadratic form $Q_4(x, y, z, w) = x^2 + y^2 + z^2 + w^2$, then Q_L denotes the restriction of Q_4 to L and $disc(Q_L)$ is the determinant of Q_L in a basis of L.

We show in Proposition 3.5.4 that $r(n)$ is equal to h_K^2 times some elementary factor (when $r(n) \neq 0$), where K is the imaginary quadratic field $\mathbb{Q}(\sqrt{-n})$ and h_K is the number of classes of ideals of the ring of integers $\mathcal{O}_K$ of K. This is the main ingredient which allows us to express the orthogonal period of our Eisenstein series in GL_4 in terms of A_3-Weyl group multiple Dirichlet series obtaining our main result Theorem 3.6.1. To prove Proposition 3.5.4, we use the Klein map defined in [1] to relate 2-dimensional primitive lattices inside $\mathbb{Z}^4$ with two triples of integers. This is done in section 3.5. Section 3.7 explores the connection between the Klein map and the Gauss map defined in [33]. These results are part of a joint project with G. Chinta.

Prime numbers in short intervals

Let $\pi(x)$ be the number of primes less than or equal to x. A classical theorem of Cramér [42] states that, assuming the Riemann hypothesis (RH), there are constants $c, \alpha > 0$ such that

$$\frac{\pi\big(x + c\sqrt{x}\log x\big) - \pi(x)}{\sqrt{x}} > \alpha$$

for all sufficiently large x. The order of magnitude in this estimate has never been improved, and the efforts have thus been concentrated in optimizing the values of the implicit constants. Recently, Carneiro, Milinovich and Soundararajan [26] used Fourier analysis to establish the best known values. This approach studies some Fourier optimization problems that are of the kind where one prescribes some constraints for a function and its Fourier transform, and then wants to optimize a certain quantity.

Let us denote by $\mathcal{A}^+$ the set of even and continuous functions $F\colon \mathbb{R} \to \mathbb{R}$ with $F \in L^1(\mathbb{R})$. For $1 \le A < \infty$, we write

$$\mathcal{C}^+(A) := \sup_{\substack{F \in \mathcal{A}^+ \\ F \neq 0}} \frac{1}{\|F\|_1}\left(F(0) - A\int_{[-1,1]^c} \big(\widehat{F}\big)^+(t)\,dt\right),$$

where we use the notation $f^+(x) = \max\{f(x), 0\}$, $[-1,1]^c = \mathbb{R}\setminus[-1,1]$, and

$$\widehat{F}(t) = \int_{-\infty}^{\infty} F(x)e^{-2\pi i x t}\,dx.$$

Assuming RH, [26, Theorem 1.3] establishes that for $\alpha \ge 0$ the estimate

$$\inf\left\{c > 0;\ \liminf_{x\to\infty} \frac{\pi(x + c\sqrt{x}\log x) - \pi(x)}{\sqrt{x}} > \alpha\right\} \le \frac{(1+2\alpha)}{\mathcal{C}^+(36/11)}.$$

The numerical example from [26, Eq (4.12)] given by

$$F(x) = -4.8\,x^2\,e^{-3.3x^2} + 1.5\,x^2\,e^{-7.4x^2} + 520\,x^{24}\,e^{-9.7x^2} + 1.3\,e^{-2.8x^2} + 0.18\,e^{-2x^2}$$

shows that

$$\mathcal{C}^+(36/11) > 1.1943... > \frac{25}{21}.$$

Therefore in (4.1.1) for $\alpha = 0$ and $\alpha = 1$, we can choose $c = 0.8374$ and $c = 2.512$, respectively. This improves the previous results established by Dudek [45], who shows that for $\alpha = 0$ and $\alpha = 1$, we can choose $c = 1 + \varepsilon$ and $c = 3 + \varepsilon$, respectively, for any $\varepsilon > 0$.

Next we consider prime numbers in arithmetic progressions. Let $q \ge 3$ and $b \ge 1$ be coprime integers. Denote by $\pi(x; q, b)$ the number of primes less than or equal to x that are congruent to b modulo q. Assuming the generalized Riemann hypothesis (GRH), Grenié, Molteni and Perelli [56, Theorem 1] state the equivalent of the result by Cramér (4.1.1) for primes in arithmetic progressions. They established that there are suitable constants $c_1, \alpha > 0$ such that

$$\frac{\pi\big(x + c_1\,\varphi(q)\sqrt{x}\log x; q, b\big) - \pi(x; q, b)}{\sqrt{x}} > \alpha$$

for all sufficiently large x. Our main goal in this paper is to establish good bounds for the constant $c_1 > 0$.

Theorem. *Assume the generalized Riemann hypothesis. Let $q \geq 3$ and $b \geq 1$ be coprime. Then we have for all $\alpha \geq 0$ that*

$$\inf\left\{c_1 > 0;\ \liminf_{x\to\infty} \frac{\pi\big(x + c_1\,\varphi(q)\sqrt{x}\log x; q, b\big) - \pi(x; q, b)}{\sqrt{x}} > \alpha\right\} \leq \frac{(1+2\alpha)}{\mathcal{C}^{+}(4)} < 0.8531\,(1+2\alpha)$$

where $\varphi(q)$ is Euler's totient function.

In particular, for all sufficiently large x, there is a prime p in $(x, x + 0.8531\,\varphi(q)\sqrt{x}\log x]$ that is congruent to b modulo q. Furthermore, there are at least $\sqrt{x}$ primes that are congruent to b modulo q in the interval $(x, x + 2.5591\,\varphi(q)\sqrt{x}\log x]$. This result improves asymptotically some results of a recent work by Dudek, Grenié, and Molteni [46, Theorem 1.1-1.3], which establish the constants $c_1 = 1$ and $c_1 = 3$ for $\alpha = 0$ and $\alpha = 1$ respectively. Our result establish the constants $c_1 = 0.8531$ and $c_1 = 2.5591$ for $\alpha = 0$ and $\alpha = 1$ respectively.

Theorem. *Assume the generalized Riemann hypothesis. Let $q \geq 3$ and $b \geq 1$ be coprime and denote by $p_{n,q,b}$ the n-th prime that is congruent to b modulo q. Then*

$$\limsup_{n\to\infty} \frac{p_{n+1,q,b} - p_{n,q,b}}{\sqrt{p_{n,q,b}}\ \log p_{n,q,b}} \leq 0.8531\varphi(q).$$

The construction of numerical examples via semidefinite programming also gives a slight improvement on [26, Theorem 1.3 and Corollary 1.4]: we get $\mathcal{C}^{+}(36/11) \geq 1.1961$. So assuming the Riemann hypothesis, we have for any $\alpha \geq 0$ in (4.1.3) that

$$\inf\left\{c > 0;\ \liminf_{x\to\infty} \frac{\pi(x + c\sqrt{x}\log x) - \pi(x)}{\sqrt{x}} > \alpha\right\} < 0.8358\,(1+2\alpha)$$

and

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{\sqrt{p_n}\ \log p_n} < 0.8358$$

where p_n denotes the n-th prime.

The proof of the first inequality in Theorem 4.1.1 follows the ideas developed in [26]. We need three main ingredients: the Guinand-Weil explicit formula for the Dirichlet characters modulo q, the Brun-Titchmarsh inequality for primes in arithmetic progressions and the derivation of an extremal problem in Fourier analysis. Since many of the computations to derive the extremal problem are similar to [26], we will highlight the principal differences.

For the second inequality in Theorem 4.1.1, we write the resulting optimization problem as a convex optimization problem over nonnegative functions. We then write these nonnegative functions as $f(x) = p(x^2)e^{-\pi x^2}$ for some polynomial p, as in the works of Cohn and Elkies [37] for the sphere packing problem, and use semidefinite programming to optimize over these nonnegative functions, which is an approach employed recently for problems involving the Riemann zeta-function and other $L-$functions in [35, 71]. These results are a joint work with A. Chirre and D. de Laat, it appears in the pre-print [36].

Content description

In what follows we summarize the content of this thesis.

The first chapter is dedicated to review classical results on automorphic forms as needed for chapter 2. Those who are familiar with the theory of automorphic forms over $GL_n(\mathbb{A}_F)$, where F is a global function field, may skip this chapter and start with chapter 2, coming back as needed for notations. The first four sections of chapter 1 review adelic topologies, admissible representations, Satake parameters and the tensor product theorem,respectively. Section 1.5 introduces automorphic forms over function fields. In section 1.6, we review the definition and basic properties of Eisenstein series. In section 1.7, we describe the action of the unramifed Hecke operators on Eisenstein series and we describe a spectral decomposition of the space of automorphic forms.

In the second chapter, we apply graphs of Hecke operators, as developed in [2], to study toroidal automorphic forms over an elliptic function field. In section 2.1, we review the concept of toroidal automorphic forms and discuss the arithmetical meaning of the toroidal condition for some classes of automorphic forms. In section 2.2, we review the graphs of Hecke operators and reformulate the study of unramified automorphic forms and Hecke operators in geometric terms. In sections 2.3 – 2.5, we introduce the concept of nucleus of the graphs for GL_3 and show how to reduce the computation of eigenforms to computations on the nucleus (see Theorem 2.5.13). In section 2.6, we use the algorithm developed in [2] to compute the graph of Hecke operators on the nucleus on GL_3 for an elliptic function field with only one point of degree 1. This computation is used in section 2.7 to parametrize the spaces of eigenforms for a particular example. In section 2.8 we apply this parametrization to the study of toroidal automorphic forms.

The third chapter is dedicated to the study of orthogonal periods and Weyl group multiple Dirichlet series. In sections 3.1 – 3.3 we review the theory of automorphic forms and express the orthogonal periods as a finite sum over a genus class. In section 3.4, we express the orthogonal periods of the Eisenstein series in GL_4 induced from the parabolic subgroup of type $(2, 2)$ as a Dirichlet series whose coefficients are a type of representation number of quadratic forms. In section 3.5, we use the results of [1] to express the representation number as a square of a class number times a elementary factor. This expression is used in section 3.6 to prove our main result Theorem 3.6.1. In section 3.7, we explore the connection between the Klein map and the Gauss map as defined in [33].

In the fourth chapter we study the problem of prime numbers in arithmetic progressions in short intervals. The proof of Theorem 4.1.1 essentially follows the ideas from [26]. The main ingredient are an extended version of the Guinand-Weil explicit formula, which we establish in section 4.2. We also need a version of the Brun-Titchmarsh inequality for primes in arithmetic progressions which we review in section 4.2.2. In section 4.4, we describe the algorithm which we use to compute a good lower bounds for the constant $\mathcal{C}^+(4)$.

Chapter 1

Background on Automorphic Forms

In the first chapter, we introduces our basic notation and review relevant notions and facts from the theory of automorphic forms, Hecke operators and representation theory, which will be useful throughout this thesis.

1.1 Notation

Let F be a global function field over $\mathbb{F}_q$, i.e. the function field of a geometrically irreducible smooth projective curve X over $\mathbb{F}_q$. Let g be the genus of X. Let $|X|$ be the set of closed points of X or, equivalently, the set of places of F. For $x \in X$, we denote by F_x the completion of F at x, by $\mathcal{O}_x$ its integers, by $\pi_x \in \mathcal{O}_x$ a uniformizer and by q_x the cardinality of the residue field $\kappa(x) := \mathcal{O}_x/(\pi_x) \cong \mathbb{F}_{q_x}$. With the choice of π_x, we can identify F_x with the field of Laurent series $\mathbb{F}_{q_x}((\pi_x))$ in π_x and $\mathcal{O}_x$ with the ring of formal power series $\mathbb{F}_{q_x}[[\pi_x]]$. Let $|x|$ be the degree of x. The field F_x comes with a valuation v_x that satisfies $v_x(\pi_x) = 1$ and with an absolute value $|\cdot|_x := q_x^{-v_x}$, which satisfies $|\pi_x|_x = q_x^{-1}$. The local field F_x is a locally compact ring, $\mathcal{O}_x$ is a compact neighborhood of 0 and the topology of F_x has the neighborhood basis $\{\pi_x^i \mathcal{O}_x\}_{i\geq 0}$ of 0. We denote by $|x|$ the degree of $\kappa(x)$ over $\mathbb{F}_q$.

To define the adele ring and the adelic topologies, we recall the definition of a *restrict direct product*. Let Σ be a set, $\{G_x\}_{x\in\Sigma}$ a family of groups and $\{K_x\}_{x\in\Sigma'}$ a family of subgroups $K_x \subset G_x$, with $\Sigma' \subset \Sigma$ and $\Sigma - \Sigma'$ a finite set. Then the restricted direct product of the G_x with respect to the K_x is

$$G = \left\{ (a_x)_{x\in\Sigma} \in \prod_x G_x \;\middle|\; a_x \in K_x \text{ for almost all } x \in \Sigma \right\}.$$

We use *almost all* when applied to Σ to mean all but finitely many. If the G_x are locally compact topological groups and the groups K_x are compact open subgroups, then we may give G the structure of a locally compact topological group as follows. We take as a base of neighborhoods of the identity the products $U = \prod_x U_x$, where each U_x is an open relatively compact neighborhood of the identity in G_x, and $U_x = K_x$ for almost all x. By the Tychonoff theorem, such a set U is relatively compact.

The first example of a restricted direct product is the adele ring $\mathbb{A}$ of the function field F. We take $|X|$ as our index set. The *adele ring* $\mathbb{A}$ of F is the restricted direct product of the F_x with respect to the $\mathcal{O}_x$. The group of *ideles* $\mathbb{A}^\times$ is the restricted direct product of the $F_x^\times$ with respect to the $\mathcal{O}_x^\times$. Observe that $\mathbb{A}^\times$ is the group of invertible elements of $\mathbb{A}$, but the idele topology is finer than the subspace topology induced from $\mathbb{A}$. The *idele norm* is the quasi-character $|\cdot| : \mathbb{A}^\times \longrightarrow \mathbb{C}^\times$ that sends an idele $(a_x) \in \mathbb{A}^\times$ to the product $\prod |a_x|_x$ over all local norms. By the product formula, this defines a quasi-character on the idele class group $\mathbb{A}^\times/F^\times$. We also define for $a \in \mathbb{A}^\times$, $\deg(a) \in \mathbb{Z}$ by

$$\deg(a) = \sum_{x \in |X|} v_x(a) \cdot |x|.$$

Therefore $|a| = q^{-\deg(a)}$ for all $a \in \mathbb{A}^\times$.

We easily check that the group $GL_n(\mathbb{A})$ is the restricted direct product of the groups $GL_n(F_x)$ with respect to the groups $GL_n(\mathcal{O}_x)$. So the restricted direct product construction gives to $GL_n(\mathbb{A})$ a natural topology. With this topology $GL_n(\mathbb{A})$ is a locally compact and totally disconnected topological group. If $H \subset GL_n$ is a closed algebraic subgroup, then $H(\mathbb{A})$ is a closed subgroup of $GL_n(\mathbb{A})$ and we give to it the topology induced from $GL_n(\mathbb{A})$. In general, if V is a linear algebraic group over F or an algebraic variety over F, there is a natural topology on the adelic points $V(\mathbb{A})$ of V, cf. [39] and [84].

By embedding an element $a \in F$ diagonally into $\mathbb{A}$ along the canonical inclusions $F \hookrightarrow F_x$, we may regard F as a subring of $\mathbb{A}$ and $F^\times$ as a subgroup of $\mathbb{A}^\times$. With these embeddings F and $F^\times$ are discrete in the respective spaces. Similarly, $GL_n(F)$ is a discrete subgroup of $GL_n(\mathbb{A})$. We denote by $\mathbb{A}_0^\times$ the ideles of degree 0. The product formula $\prod_{x\in|X|} |a|_x = 1$, $a \in F^\times$, can be reformulated as $F^\times \subset \mathbb{A}_0^\times$. We denote $\mathcal{O}_\mathbb{A} = \prod_{x\in|X|} \mathcal{O}_x$ and $\mathcal{O}_\mathbb{A}^\times = \prod_{x\in|X|} \mathcal{O}_x^\times$. Since $\mathcal{O}_\mathbb{A}^\times$ consists of the ideles $a = (a_x)$ with $v_x(a_x) = 0$ for all $x \in |X|$, this yields $\mathcal{O}_\mathbb{A}^\times \subset \mathbb{A}_0^\times$.

A *divisor* of F is an element

$$D = \sum D_x \cdot x \in \bigoplus_{x\in|X|} \mathbb{Z} \cdot x \cong \mathbb{A}^\times/\mathcal{O}_\mathbb{A}^\times$$

with $D_x \in \mathbb{Z}$ for all $x \in |X|$. The latter isomorphism is obtained by sending the divisor x to π_x, where π_x is interpreted as an idele via the inclusion $F_x^\times \subset \mathbb{A}^\times$, which sends an element $a \in F_x$ to the idele $(a_x) \in \mathbb{A}^\times$ with $a_x = a$ and $a_y = 1$ for $y \neq x$. We define the *idele class group* as $F^\times\backslash\mathbb{A}^\times$ and the *divisor class group* $\operatorname{Cl} F$ as $F^\times\backslash\mathbb{A}^\times/\mathcal{O}_\mathbb{A}^\times$. Since $F^\times \subset \mathbb{A}_0^\times$ and $\mathcal{O}_\mathbb{A}^\times \subset \mathbb{A}_0^\times$, we can define the degree of a divisor as the degree of a representative in $\mathbb{A}^\times$. The *class group* $\operatorname{Cl}^0 F = F^\times\backslash\mathbb{A}_0^\times/\mathcal{O}_\mathbb{A}^\times$ is a finite group (cf. [83, Thm. 7.13]), whose order h_F is called the *class number* of F.

These groups fit into an exact sequence

$$0 \longrightarrow \operatorname{Cl}^0 F \longrightarrow \operatorname{Cl} F \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0,$$

which splits non-canonically. For the surjectivity of the degree map cf. [83, Prop. 6.2]. In particular, there are always ideles of degree 1, even when F has no place of degree 1.

We now review some facts and establish notations concerning the structure of GL_n as a reductive group over $\mathbb{F}_q$. We denote by $\mathbb{G}_m$ the multiplicative group as an algebraic group over $\mathbb{F}_q$. Thus if R is an $\mathbb{F}_q$-algebra, $\mathbb{G}_m(R) = R^\times$.

Let B be the standard Borel subgroup of GL_n of upper triangular matrices and let $T \subset B$ be the maximal torus of diagonal matrices. If R is an $\mathbb{F}_q$-algebra, we will write the elements of $T(R)$ in the following way:

$$t = \mathrm{diag}(t_1, ..., t_n).$$

We have an identification of $\mathbb{Z}^n$ with $X(T) := Hom_{\mathbb{F}_q}(T, \mathbb{G}_m)$ as follows. If $\lambda \in \mathbb{Z}^n$, R is an $\mathbb{F}_q$-algebra and $t = \mathrm{diag}(t_1, ..., t_n) \in T(R)$, then

$$t^\lambda = t_1^{\lambda_1} \cdots t_n^{\lambda_n}.$$

If $(\varepsilon_i)_{i=1,\ldots,n}$ is the canonical basis of $\mathbb{Z}^n$, then the set

$$R \subset \mathbb{Z}^n$$

of roots of (G, T) is equal to

$$\{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n \textit{ and } i \neq j\},$$

and the set

$$\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n-1\} \subset R$$

of simple roots of (G, T, B) is identified with $\{1, \ldots, n-1\}$ by $i \mapsto \alpha_i$. Thus, by abuse of notation, we sometimes write $\Delta = \{1, \ldots, n-1\}$.

The Weyl group of (G, T) is $W = N(T)(\mathbb{F}_q)/T(\mathbb{F}_q)$, where $N(T)$ is the normalizer of T in GL_n. The Weyl group W can be identified with the group of permutation matrices in $GL_n(\mathbb{F}_q)$. A permutation matrix is a matrix in $GL_n(\mathbb{F}_q)$ such that every column has 1 entry equal to 1 and the others equal to 0. We identify the group of permutation matrices with the symmetric group on n letters, sending a permutation matrix M to σ, where $M\varepsilon_i = \varepsilon_{\sigma(i)}$. Thus we obtain an identification

$$W = \mathfrak{S}_n,$$

where $\mathfrak{S}_n$ is the symmetric group on n letters. With this identification,

$$S = \{s_1, \ldots, s_{n-1}\}, \quad \text{where} \quad s_i = (i, i+1),$$

is the set of simple reflections associated to Δ (s_i is associated to α_i). The Weyl group W acts on $\mathbb{Z}^n$ by

$$(w, \lambda) \mapsto (\lambda_{w^{-1}(1)}, \ldots, \lambda_{w^{-1}(n)}).$$

Thus

$$(\dot{w}t\dot{w}^{-1})^{\lambda} = t^{w^{-1}(\lambda)}$$

for all $w \in W$, $\lambda \in \mathbb{Z}^n$ and $t \in T(R)$, where R is an $\mathbb{F}_q$-algebra and $\dot{w} \in N(T)(\mathbb{F}_q)$ is a representative of w. We have

$$w(\varepsilon_i) = \varepsilon_{w(i)}$$

for $i = 1, \ldots, n$.

The standard parabolic subgroups are the algebraic subgroups of GL_n which contains B. They are parametrized by subsets $I \subset \Delta$ as follows. For each $I \subset \Delta$ we define the partition

$$d_I = (d_1, \ldots, d_{|\Delta - I|+1})$$

of n, with

$$\Delta - I = \{d_1, d_1 + d_2, \ldots, d_1 + \cdots + d_{|\Delta - I|}\}.$$

Let $P_I \subset G$ be the standard parabolic subgroup associated with I. The parabolic subgroup P_I has a standard Levi decomposition

$$P_I = M_I N_I,$$

with M_I isomorphic to $GL_{d_1} \times \cdots \times GL_{d_{|\Delta-I|+1}}$ embedded diagonally in GL_n, and N_I the unipotent radical of P_I. We sometimes say that P_I is the standard parabolic subgroup of GL_n of type $(d_1, \ldots, d_{|\Delta-I|+1})$.

Let G be a topological group and $d_l g$ a left invariant Haar measure on G. The *modulus character* of G, denoted by δ_G is characterized by the integration rule

$$\int_G f(g) d_l g = \delta_G(g_0) \int_G f(g g_0) d_l g$$

for every integrable function f on G and $g_0 \in G$. The modular character is a continuous homomorphism $\delta_G : G \longrightarrow \mathbb{R}^{\times}$. The group G is called *unimodular* in case $\delta_G = 1$.

If $x \in |X|$, then $GL_n(F_x)$ is a unimodular group and we consider from now on the Haar measure on $GL_n(F_x)$ such that the maximal compact open subgroup $GL_n(\mathcal{O}_x)$ has measure 1. Let $P = P_I$ and $\mathfrak{n}_I$ be the Lie algebra of N_I. Thus M_I acts on $\mathfrak{n}_I$ by the adjoint representation, and the modular character of $P(F_x)$ is given by $\delta_{P(F_x)}(mn) = |\det Ad_{\mathfrak{n}_I}(m)|_x$, where $m \in M_I(F_x)$ and $n \in N_I(F_x)$.

1.2 Hecke algebras and Admissible Representations

In this section, we will review some basic results of the theory of smooth representations of a totally disconnected group G. These results will be useful for the study of automorphic forms and of the action of Hecke operators on $GL_n(\mathbb{A}_F)$, where F is a global function field over $\mathbb{F}_q$.

Let G be a topological group. We say that G is totally disconnected if G is Hausdorff and every neighborhood of the unit element contains a compact open subgroup. The totally disconnected topological groups that will be important for us are of the form $H(\mathbb{A})$ and $H(F_x)$, where x is a place of the function field F, H is a linear algebraic group over F and $\mathbb{A}$ is the adele ring of F. By a representation of G, we mean a pair (π, V) where V is a complex vector space and π a homomorphism from G into the group of invertible linear maps in V. If H is a subgroup of G, we denote by V^H the space of vectors v in V such that $\pi(h)v = v$ for any h in H.

Definition 1.2.1. A representation (π, V) of G is *smooth* if the stabilizer of every vector in V is open, equivalently if $V = \bigcup_K V^K$ where K runs over the compact open subgroups of G. A smooth representation (π, V) is called *admissible* if V^K is finite dimensional for every compact open subgroup K of G.

Let dx be a fixed (left invariant) Haar measure on G. If K is any compact open subgroup of G, we denote by $\mathcal{H}(G, K)$ the complex vector space consisting of the complex valued functions f on G which satisfy the following two conditions:

(a) f is bi-invariant under K, that is $f(kgk') = f(g)$ for g in G and k, k' in K.

(b) f has compact support, or equivalently, f vanishes off a finite union of double cosets KgK.

The convolution product on $\mathcal{H}(G, K)$ is defined by,

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) dx.$$

This integral is well defined because the integrand is locally constant and compactly supported as a function of x. With respect to this multiplication, $\mathcal{H}(G, K)$ becomes an associative algebra over the complex field $\mathbb{C}$.

Let us choose a set of representatives $\{g_\alpha\}$ for the double coset space $K \setminus G/K$. If Ω is a measurable subset of G with $vol(\Omega) > 0$, let e_Ω be the characteristic function of Ω divided by $vol(\Omega)$. Then the family $\{e_{Kg_\alpha K}\}$ is a basis of the vector space $\mathcal{H}(G, K)$ and e_K is the unit element of this algebra.

When K' is a compact open subgroup of K, then $\mathcal{H}(G, K)$ is a subring of $\mathcal{H}(G, K')$ but with a different unit element if $K \neq K'$. Define $\mathcal{H}(G) = \bigcup_K \mathcal{H}(G, K)$ where K runs through a neighborhood basis of 1 consisting of compact open subgroups. Then $\mathcal{H}(G)$ is the space of locally constant and compactly supported functions on G. For the convolution product above, $\mathcal{H}(G)$ is an associative algebra. It has no unit unless G is discrete, and it is commutative if and only if G is commutative. Observe that $\mathcal{H}(G, K) = e_K \mathcal{H}(G) e_K$.

Definition 1.2.2. The algebra $\mathcal{H}(G)$ is called the *Hecke algebra* of G and $\mathcal{H}(G, K)$ is called the Hecke algebra of G with respect to K.

If (π, V) is a smooth representation of G, then the space V becomes an $\mathcal{H}(G)$-module by the formula

$$\pi(f)v = \int_G f(x)\pi(x)v dx,$$

where the right is computed as follows: Take K' a small compact open subgroup such that $v \in V^{K'}$ and $f \in \mathcal{H}(G, K')$. Let $\{g_i\}$ be a set of representatives of G/K', then $\pi(f)v$ is the finite sum $\sum_i vol(K')f(g_i)\pi(g_i)v$.

If K is a compact open subgroup of G we see that $e_K V = V^K$ and therefore V^K is a $\mathcal{H}(G, K)$-module.

Conversely, we want to know when a $\mathcal{H}(G)$-module comes from a smooth representation of G. This leads to the notion of an *idempotented algebra*, which is a pair (A, E) where A is a $\mathbb{C}$-algebra with a set of idempotents E such that $A = \bigcup_{e \in E} eAe$. For $e, f \in E$, we write $e \geq f$ if $ef = fe = f$. Then $\geq$ is a partial ordering on E. We assume that E is a directed set with this ordering. Usually we omit E and say only that A is an idempotented algebra. The Hecke algebra $\mathcal{H}(G)$ is an idempotented algebra if we take for E the set of e_K where K runs over the compact open subgroups of G.

Definition 1.2.3. Let (A, E) be an idempotented algebra. A *smooth module* W for (A, E) is an A-module W such that $AW = W$. An smooth module is called *admissible* if eW is finite dimensional for all $e \in E$.

If W is a smooth A-module and $e \in E$, we write $A[e] = eAe$ and $W[e] = eW$. Then $A[e]$ is a ring with unit and $W[e]$ is an $A[e]$-module. It is clear that if (π, V) is a smooth representation of G, then V is a smooth $\mathcal{H}(G)$-module. Reciprocally we have the following.

Theorem 1.2.4. *Let V be a smooth module over $\mathcal{H}(G)$. Then there exists a smooth representation $\pi : G \longrightarrow End_{\mathbb{C}}(V)$ such that $\phi \cdot x = \pi(\phi)x$ for $\phi \in \mathcal{H}(G)$ and $x \in V$.*

Proof. Let $x \in V$ and $g \in G$. Because V is a smooth module, there exists a compact open subgroup K_0 of G (depending on x) such that $x \in V[e_{K_0}]$. Then we define $\pi(g)x = e_{gK_0} \cdot x$. It is easy to see that this definition is independent of the choice of K_0. We choose subgroups K_0 and K_1 stabilizing x and $\pi(h)x$, respectively, so

$$\pi(g)\pi(h)x = \pi(e_{gK_1})\pi(e_{hK_0})x = \pi(e_{gK_1} * e_{hK_0})x.$$

We choose K_1 sufficiently small so that $h^{-1}K_1 h \subset K_0$. Then it is easy to verify that $e_{gK_1} * e_{hK_0} = e_{ghK_0}$, and so we deduce that $\pi(gh) = \pi(g)\pi(h)$. Thus π is a representation. It is easy to verify that π is smooth. □

If we denote by $Rep_s(G)$ and $Mod_s(\mathcal{H}(G))$ the abelian categories of smooth representations of G and smooth modules over $\mathcal{H}(G)$ respectively, then it follows from the theorem that the functor, which associates to an admissible representation (V, π) of G the $\mathcal{H}(G)$-module structure on V, is an equivalence of categories between $Rep_s(G)$ and $Mod_s(\mathcal{H}(G))$.

Regarding irreducible representations we have the following.

Theorem 1.2.5. *A nontrivial smooth representation V of G is irreducible if and only if for every compact open subgroups K of G, V^K is 0 or an irreducible $\mathcal{H}(G, K)$-module.*

Proof. This follows from the fact that if W is an $\mathcal{H}(G, K)$-submodule of V^K, then $(\mathcal{H}(G) \cdot W)^K = W$. □

Theorem 1.2.6. *Let (π_i, V_i) for $i = 1, 2, 3$, be smooth G-representations, K a compact open subgroup of G. If $V_1 \longrightarrow V_2 \longrightarrow V_3$ is an exact sequence of G-morphisms, then the sequence $V_1^K \longrightarrow V_2^K \longrightarrow V_3^K$ is exact as well.*

Proof. Given $v \in V_2^K$ whose image in V_3 is 0, choose $v_1 \in V_1$ with image $v \in V_2$. Then $e_K(v_1)$ lies in V_1^K and still has image v. □

Let H be a closed subgroup of G such that $H \backslash G$ is compact. If we have a smooth representation σ of G, restricting the representation to H gives us a smooth representation $Res_H^G \sigma$ of H. The *extension functor* from $Rep_s(H)$ to $Rep_s(G)$ is given as follows by induction.

Let H be a closed subgroup of G such that $H \backslash G$ is compact and (σ, U) a smooth representation of H. We call a function $f : G \longrightarrow U$ *smooth* if for some compact open subgroup K of G, $f(gk) = f(g)$ for all $g \in G$ and $k \in K$. We define the *induced representation* $Ind_H^G \sigma$ to be the space of functions $f : G \longrightarrow U$ such that

(i) $f(hg) = \sigma(h)f(g)$ for all $h \in H$, $g \in G$, and

(ii) f is a smooth function.

The group G acts on $Ind_H^G \sigma$ by the right regular representation: $(\pi(g)f)(h) := f(hg)$ for all $f \in Ind_H^G \sigma$ and $g, h \in G$.

Theorem 1.2.7. *Let H be a closed subgroup of G such that $H \backslash G$ is compact and (σ, U) a smooth representation of H.*

1. *The representation $Ind_H^G \sigma$ is a smooth representation.*
2. *If (σ, U) is an admissible representation, then $Ind_H^G \sigma$ is an admissible representation of G.*
3. *The functor $\sigma \longmapsto Ind_H^G \sigma$ is exact.*
4. *(Frobenius reciprocity) Let $\Lambda : Ind_H^G \sigma \longrightarrow U$ be the H-morphism $f \longmapsto f(1_G)$. If (π, V) is any smooth G-representation, then composition with Λ induces an isomorphism of $Hom_G(\pi, Ind_H^G \sigma)$ with $Hom_H(Res_H^G \pi, \sigma)$.*
5. *If χ is a smooth character of G, then*

$$Ind_H^G \sigma \otimes \chi \simeq Ind_H^G (\sigma \otimes \chi|_H).$$

Proof. For a proof, cf. [28, Thm. 2.4.1, Prop. 2.4.4]. □

Let $\mathbf{G}$ be a reductive group over F and $\mathbf{P}$ a parabolic subgroup defined over F with Levi factor $\mathbf{M}$ and $\mathbf{N}$ the unipotent radical of $\mathbf{P}$. Let x be a place of F. We fix R equal to F_x or $\mathbb{A}$, and define $G = \mathbf{G}(R)$, $P = \mathbf{P}(R)$, $M = \mathbf{M}(R)$ and $N = \mathbf{N}(R)$. We remember that δ_P denotes the modulus character of P. If (σ, U) is a smooth representation of M, it defines as well a smooth representation of P, since $P/N \cong M$. We define the *normalized induction* $i_P^G\sigma$ as $Ind_P^G(\sigma\delta_P^{1/2})$. The reason to add $\delta_P^{1/2}$ is to normalize $i_P^G\sigma$ so that if σ is unitary, then $i_P^G\sigma$ is unitary, cf. [28, Prop. 3.1.4]. This normalization will also be useful in connection with the Satake isomorphism.

Consider on the algebraic group GL_n over F the standard Levi subgroups $M_I \subset M_J$ corresponding to subsets $I \subset J$ of the set of simple roots Δ. Then $M_J \cap P_I$ is a parabolic subgroup of M_J with Levi subgroup M_I and if σ is an admissible representation of $M_I(F_x)$, we can consider the normalized induction $i_{M_I(F_x)}^{M_J(F_x)}\sigma$.

Theorem 1.2.8. *Consider $K \subset J \subset I$ subsets of Δ, $x \in |X|$ and π_x an admissible representation of $M_K(F_x)$. Then we have a canonical isomorphism*

$$i_{M_K(F_x)}^{M_I(F_x)}\pi_x \simeq i_{M_J(F_x)}^{M_I(F_x)}\left(i_{M_K(F_x)}^{M_J(F_x)}\pi_x\right).$$

Proof. For a proof, cf. [78, Prop. 7.1.3 (iv)]. □

If χ is any character of G and (π, V) a smooth representation of G, the *twisted representation* $(\pi \otimes \chi, V)$ acts on the same space as π via the formula

$$(\pi \otimes \chi)(h) = \chi(h)\pi(h) \quad \text{for } h \in G.$$

1.3 Spherical Hecke algebras and the Satake Isomorphism for $GL(n)$

In this section, we study the spherical representations of $G = GL_n$ over the local fields F_x for $x \in |X|$ and the Satake isomorphism. These results will be useful for the study of the unramified Hecke operators in this and the next chapter.

Let $T \subset GL_n$ be the diagonal torus, $x \in |X|$ a place and F_x the corresponding local field. Let $B \subset G$ be the standard Borel subgroup and $U \subset B$ the unipotent radical of B. Let $\delta_{B(F_x)} : B(F_x) \to \mathbb{C}^\times$ be the modulus character of $B(F_x)$. We have

$$\delta_{B(F_x)}(b) = \prod_{i<j} |b_{ii}/b_{jj}|_x.$$

Let χ be a character of $T(F_x)$. Using that $T = B/U$ we can extend χ to $B(F_x)$. The *principal series representation* is $I(\chi) = i_{B(F_x)}^{G(F_x)}\chi$. Recall that the normalized induction $I(\chi) = i_{B(F_x)}^{G(F_x)}\chi$ consists of all smooth functions $f : G(F_x) \longrightarrow \mathbb{C}$ such that

$$f(bg) = \delta_{B(F_x)}^{1/2}(b)\chi(b)f(g),$$

for all $b \in B(F_x)$ and $g \in G(F_x)$. The action is given by

$$(i(\chi)(g)f)(h) = f(hg),$$

for all $f \in I(\chi)$ and $g, h \in G(F_x)$.

Theorem 1.3.1. *The representation $I(\chi)$ is admissible and a $G(F_x)$-module of finite length.*

Proof. For a proof cf. [78, Thm. 7.3.3] and [27, Thm. 3.3] □

Let $K_x = G(\mathcal{O}_x)$ be the standard maximal compact subgroup of $G(F_x)$. A *spherical representation* or *unramified representation* of $G(F_x)$ is a smooth irreducible representation (V, π) of $G(F_x)$ such that

$$V^{K_x} \neq (0).$$

We say that a character $\chi : T(F_x) \longrightarrow \mathbb{C}^\times$ is *unramified* if χ is trivial on $T(\mathcal{O}_x)$. To $z = (z_1, ..., z_n) \in (\mathbb{C}^\times)^n$ we associate the unramified character $\chi_z : T(F_x) \to \mathbb{C}^\times$, given by $\chi_z(diag(t_1, ...t_n)) = \prod_{j=1}^n z_j^{\lambda_j}$, where $\lambda_j = v_x(t_j)$. These are all the unramified characters of $T(F_x)$. If χ is an unramified character of $T(F_x)$, then we have

$$\dim_{\mathbb{C}}(I(\chi)^{K_x}) = 1.$$

This follows easily from the Iwasawa decomposition

$$G(F_x) = B(F_x)K_x$$

and

$$T(F_x) \cap K_x = T(\mathcal{O}_x).$$

Theorem 1.3.2. *Let the notations be as above.*

(i) *Let χ be an unramified character of $T(F_x)$. If $0 = V_0 \subset V_1 \subset \cdots \subset V_{r-1} \subset V_r = I(\chi)$ is a Jordan-Hölder series of the $G(F_x)$-module $I(\chi)$, there exists a unique index j such that $1 \leq j \leq r$ and such that the representation of $G(F_x)$ in V_j/V_{j-1} is spherical. The isomorphism class of this spherical representation is well defined and we denote by $(V(\chi), \pi(\chi))$ this spherical subquotient of $I(\chi)$.*

(ii) *Any spherical representation (V, π) of $G(F_x)$ is isomorphic to $(V(\chi), \pi(\chi))$ for some unramified character χ of $T(F)$. In particular, we have*

$$\dim_{\mathbb{C}}(V^{K_x}) = 1$$

(iii) *Let χ and χ' be unramified characters of $G(F_x)$. Then the spherical representations $(V(\chi), \pi(\chi))$ and $(V(\chi'), \pi(\chi'))$ of $G(F_x)$ are isomorphic if and only if there exists $w \in W$ such that $\chi' = w(\chi)$, where W is the Weyl group of $G = GL_n$.*

Proof. For a proof cf. [78, Thm. 7.3.3 and Thm. 7.5.4] and [27, Sec. IV]. The proof that j is unique in (i) follows from Theorem 1.2.6. $\square$

Remark 1.3.3. Regarding the decomposition of $I(\chi)$ we have the following.

1. Let $z = (z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n$ and $\chi = \chi_z$. We say that χ is regular if $z_i \neq z_j$ for $i \neq j$. If χ is not regular, we say that it is irregular. If χ is regular, then $I(\chi)$ is irreducible if and only if $z_i/z_j \neq q_x^{\pm 1}$ for $i \neq j$. For a proof cf. [29, Prop. 3.5 (b)].

2. There are irregular characters χ such that $I(\chi)$ is irreducible. For example, if χ is unitary, then $I(\chi)$ is irreducible cf. [61].

3. In general, by [28, Cor. 7.2.3] the length of the representation $I(\chi)$ is less than or equal to $n!$, the cardinality of the Weyl group W of GL_n.

4. Let $I_x \subset K_x$ be the Iwahori subgroup, that is, the inverse image of $B(\kappa_x)$ under the homomorphism $GL_n(\mathcal{O}_x) \longrightarrow GL_n(\kappa(x))$, where B is the standard Borel subgroup of GL_n and $\kappa(x)$ is the residue field at x. The irreducible admissible representations V of $G(F_x)$ with $V^{I_x} \neq 0$ are precisely the irreducible admissible representations which appear in composition series of the unramified principal series, cf. [27, Thm. 3.8], [29, Prop. 2.6] and [12].

The spherical representation corresponding to the character χ_z is $(V(\chi_z), \pi(\chi_z))$. The Satake parameter of this unramified representation is the conjugacy class

$$\begin{pmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_n \end{pmatrix}$$

in $GL_n(\mathbb{C})$. The vector space $V(\chi_z)^K$ has dimension 1 and is composed of eigenvectors for the action of $\mathcal{H}(G(F_x), K_x)$.

Definition 1.3.4. The Hecke algebra $\mathcal{H}(G(F_x), K_x)$ is called the *spherical Hecke algebra* or the *unramified Hecke algebra*. Its elements are called *unramified Hecke operators*.

We now describe the structure of $\mathcal{H}(G(F_x), K_x)$ and its action on $V(\chi_z)^K$ in terms of the Satake transform.

Using the Iwasawa decomposition $G(F_x) = B(F_x)K_x$, we extend the character $\delta_{B(F_x)}^{1/2}\chi_z$ of $B(F_x)$ to a locally constant function ϕ_z on $G(F_x)$ defined by

$$\phi_z(g) = \delta_{B(F_x)}^{1/2}\chi_z(b),$$

if $g = bk$, with $b \in B(F_x)$ and $k \in K_x$. This defines a function

$$\phi_z : G(F_x) \longrightarrow \mathbb{C}[z_1, z_1^{-1}, ..., z_n, z_n^{-1}].$$

Observe that if we specialize z to $(z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n$, then $\phi_z \in I(\chi_z)$ and ϕ_z generates $I(\chi_z)^{K_x}$.

Definition 1.3.5. The *Satake transform* of $\Phi \in \mathcal{H}(G(F_x), K_x)$ is defined by

$$\Phi^\vee(z) = \int_{G(F_x)} \Phi(g)\phi_z(g)dg \quad \in \quad \mathbb{C}[z_1, z_1^{-1}, ..., z_n, z_n^{-1}].$$

Let $\Phi_{x,i} \in \mathcal{H}(G(F_x), K_x)$ be $\Phi_{x,i} = \mathrm{char}(K_x \operatorname{diag}(\pi_x, ..., \pi_x, 1, ..., 1)K_x)$, $1 \leq i \leq n$, where π_x appears i times. Let $\lambda_i = (1, ...1, 0, ..., 0)$, where 1 appears i times and

$$\theta = \left(\frac{n-1}{2}, \frac{n-3}{2}, ..., \frac{3-n}{2}, \frac{1-n}{2}\right).$$

For

$$\lambda = (\lambda_1, \ldots, \lambda_n), \delta = (\delta_1, \ldots, \delta_n) \in \mathbb{C}^n,$$

we define

$$\langle \delta, \lambda \rangle = \sum_{i=1}^{n} \delta_i \lambda_i,$$

The symmetric group $\mathfrak{S}_n$ acts on the $\mathbb{C}$-algebra $\mathbb{C}[z_1, z_1^{-1}, ..., z_n, z_n^{-1}]$ by permuting the variables $z_1, \ldots, z_n$ and we denoted by $\mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]^{\mathfrak{S}_n}$ the subring of elements fixed by this action.

Theorem 1.3.6. *Let the notations be as above.*

(i) *The Satake transform is an isomorphism of $\mathbb{C}$-algebras*

$$\mathcal{H}(G(F_x), K_x) \simeq \mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]^{\mathfrak{S}_n}.$$

(ii) *If $v \in V(\chi_z)$ (resp. $v \in I(\chi_z)$) generates $V(\chi_z)^{K_x}$ (resp. $I(\chi_z)^{K_x}$) and $\Phi \in \mathcal{H}(G(F_x), K_x)$, then*

$$\pi(\Phi)v = \Phi^\vee(z_1, \ldots, z_n)v.$$

(iii) *For $\Phi_{x,i} \in \mathcal{H}(G(F_x), K_x)$, $1 \leq i \leq n$, we have $\Phi^\vee_{x,i}(z) = q_x^{\langle \theta, \lambda_i \rangle} S_i(z)$, where $S_i(z)$ is the i-th symmetric polynomial in $z_1, \ldots, z_n$ and q_x is the cardinality of the residue field of the local field F_x.*

Proof. Part (i) follows from [78, Thm. 4.1.17]. Part (ii) follows from [78, Thm. 7.5.6]. Part (iii) follows from [79, claim 4.1.18], where we use that the only possibility for μ is $\mu = \lambda$. Applying [78, Cor. 4.1.16], yields

$$\Phi^\vee_{x,i}(z) = q_x^{\langle \theta, \lambda \rangle} S_i(z),$$

where S_i is the i-th symmetric polynomial in z_1,...,z_n, □

1.4 The tensor product theorem

One of the crucial results in the representation theory of $GL_n(\mathbb{A})$, is the tensor product theorem, which says that every irreducible admissible representation of $GL_n(\mathbb{A})$ factors into a restricted tensor product of irreducible representations of $GL_n(F_x)$, $x \in |X|$. In this section we review the definition of restricted tensor product and the Tensor product theorem.

Definition 1.4.1. Let $\{W_x \mid x \in \Sigma\}$ be a family of vector spaces. Let Σ_0 be a finite subset of Σ. For each $x \in \Sigma \setminus \Sigma_0$, let v_x° be a nonzero vector in W_x. For each finite subset S of Σ containing Σ_0, let $W_S = \otimes_{x \in S} W_x$; and if $S \subset S'$, let

$$f_S : W_S \to W_{S'} \quad \text{be defined by} \quad \bigotimes_{x \in S} w_x \mapsto \bigotimes_{x \in S} w_x \bigotimes_{x \in S' \setminus S} v_x^\circ.$$

Then $W = \otimes_{v_x^\circ} W_x$, the *restricted tensor product* of the W_x with respect to the v_x°, is defined by

$$W = \varinjlim_{S} W_S.$$

The space W is spanned by elements written in the form $w = \otimes w_x$, where $w_x = v_x^\circ$ for almost all $x \in \Sigma$.

The ordinary constructions with finite tensor products extend easily to restricted tensor products.

(1) Given linear maps $B_x : W_x \to W_x$ such that $B_x v_x^\circ = v_x^\circ$ for almost all $x \in \Sigma$ then one can define $B = \otimes B_x : W \to W$ by $B(\otimes w_x) = \otimes B_x w_x$.

(2) Given a family of algebras $\{A_x \mid x \in \Sigma\}$ and given nonzero idempotents $e_x \in A_x$ for almost all x, then $A = \otimes_{e_x} A_x$ is an algebra in the obvious way.

(3) If W_x is an A_x-module for each $x \in \Sigma$ such that $e_x \cdot v_x^\circ = v_x^\circ$ for almost all x, then $\otimes_{v_x^\circ} W_x$ is an A-module. The isomorphism class of W depends on $\{v_x^\circ\}$. However, if $\{v_x'^\circ\}$ is another collection of nonzero vectors such that v_x° and $v_x'^\circ$ lie on the same line in W_x for almost all x, then the A-module $\otimes_{v_x^\circ} W_x$ and $\otimes_{v_x'^\circ} W_x$ are isomorphic.

Let $G = \prod'_{K_x} G_x$ be the restricted product of locally compact totally disconnected groups G_x, with respect to compact open subgroups $K_x \subset G_x$. Then G itself is locally compact and totally disconnected, and $\mathcal{H}(G)$ is isomorphic to $\otimes_{e_{K_x}} \mathcal{H}(G_x)$.

For each $x \in \Sigma$, let W_x be an admissible G_x-module. Assume that $\dim W_x^{K_x} = 1$ for almost all x. Choosing for almost all x a nonzero vector $v_x^\circ \in W_x^{K_x}$, we may form the G-module $W = \otimes_{v_x^\circ} W_x$ (cf. Theorem 1.2.4). The isomorphism class of W is in fact independent of the choice of $v_x^\circ \in W_x^{K_x}$ and will be called the tensor product of the representations W_x. One sees that W is admissible, and that it is irreducible if and only if each W_x is irreducible (cf. [21, Thm. 3.4.4]). The admissible irreducible representations of G isomorphic to representations constructed in this way are said to be *factorizable*.

Theorem 1.4.2. *Suppose that $\mathcal{H}(G_x, K_x)$ is commutative for almost all x. Then every admissible irreducible representation W of G is factorizable, i.e. $W \simeq \otimes W_x$. The isomorphism classes of the factors W_x are determined by that of W. For almost all x, $\dim W_x^{K_x} = 1$.*

Proof. For a proof cf. [49, Thm. 2] and [21, Sec. 3.4]. □

In the following we apply this theorem to admissible representations of $GL_n(\mathbb{A})$, where $\mathbb{A}$ is the adele ring of the global function field F. The group $GL_n(\mathbb{A})$ is the restricted product of the groups $GL_n(F_x)$ with respect to the subgroups K_x, where $x \in |X|$.

Corollary 1.4.3. *Let σ be an irreducible admissible representation of $GL_n(\mathbb{A})$. Then σ has a factorization $\sigma \simeq \otimes_{x\in|X|}\sigma_x$ with uniquely determined factors σ_x, which are irreducible representation of $GL_n(F_x)$. For almost every $x \in |X|$, σ_x is a spherical representation.*

Proof. This follows from the theorem and Theorem 1.3.6. □

1.5 Automorphic Forms over Function Fields

We use the notation G for GL_n and Z for the centre of G. We will often write $G_{\mathbb{A}}$ instead of $G(\mathbb{A})$, Z_F instead of $Z(F)$, et cetera. We denote by K the subgroup $GL_n(\mathcal{O}_{\mathbb{A}})$ of $G_{\mathbb{A}}$, which is the standard maximal compact subgroup of $G_{\mathbb{A}}$ and $K_x = G(F_x)$ for $x \in |X|$. The topology of $G_{\mathbb{A}}$ has a neighbourhood basis $\mathcal{V}$ of the identity matrix that is given by all subgroups

$$K' = \prod_{x\in|X|} K'_x < \prod_{x\in|X|} K_x = K$$

such that for all $x \in |X|$ the subgroup K'_x of K_x is open and consequently of finite index and such that K'_x differs from K_x only for a finite number of places.

Consider the space $C^0(G_{\mathbb{A}})$ of continuous functions $f : G_{\mathbb{A}} \longrightarrow \mathbb{C}$. The group $G_{\mathbb{A}}$ acts on $C^0(G_{\mathbb{A}})$ through the *right regular representation* $\rho : G_{\mathbb{A}} \longrightarrow Aut(C^0(G_{\mathbb{A}}))$ that is defined by right translation of the argument: $(\rho(g)f)(h) = f(hg)$ for $g, h \in G_{\mathbb{A}}$ and $f \in C^0(G_{\mathbb{A}})$. We call $f \in C^0(G_{\mathbb{A}})$ *smooth* if it is invariant under the right regular representation by a compact open subgroup of $G_{\mathbb{A}}$. A function $f \in C^0(G_{\mathbb{A}})$ is called K-finite if the complex space generated by $\{\rho(k)f\}_{k\in K}$ is finite dimensional. It is easy to prove that $f \in C^0(G_{\mathbb{A}})$ is smooth if and only if it is K-finite.

A function f is called *left* or *right H-invariant* for a subgroup $H < G_{\mathbb{A}}$ if for all $h \in H$ and $g \in G_{\mathbb{A}}$, $f(hg) = f(g)$ or $f(gh) = f(g)$, respectively. If f is right and left H-invariant, it is called *bi-H-invariant*.

To define *moderate growth*, we need to define a *height function* $||g||$ on $G_{\mathbb{A}}$. We embed $G_{\mathbb{A}} \hookrightarrow \mathbb{A}^{n^2+1}$ via $g \mapsto (g, \det(g)^{-1})$. We define a local height $||g_v||_v$ on $G(F_v)$ for each place v by restricting the height function $(x_1, \ldots, x_{n^2+1}) \mapsto max_i|x_i|_v$ on $F_v^{n^2+1}$. We note that $||g_v||_v \geq 1$, and that $||g_v||_v = 1$ if and only if $g_v \in G(\mathcal{O}_v)$. We define the global height $||g||$ to be the product of the local heights. We say that f is of *moderate growth* if there exist constants C and N such that for all $g \in G_{\mathbb{A}}$,

$$|f(g)| \leq C||g||^N.$$

Definition 1.5.1. An *automorphic form* on $G_{\mathbb{A}}$ (with trivial central character) is a function $f \in C^0(G_{\mathbb{A}})$ that is K-finite, of moderate growth, left $G_F Z_{\mathbb{A}}$-invariant and such that the smooth representation $\rho(G_{\mathbb{A}})f$ of $G_{\mathbb{A}}$ is admissible. We denote by $\mathcal{A}$ the complex vector space of all automorphic forms on $G_{\mathbb{A}}$ (with trivial central character).

Definition 1.5.2. An *automorphic representation* (with trivial central character) is a representation of $G_{\mathbb{A}}$ that is isomorphic to a subquotient of the representation $\mathcal{A}$. That is, there exists $V \subset W \subset \mathcal{A}$, such that the representation is isomorphic to W/V.

For a subrepresentation $V \subset \mathcal{A}$ and an open compact subgroup K' of $G_{\mathbb{A}}$, let $V^{K'}$ be the subspace of all $f \in V$ that are K'-invariant. The functions in $\mathcal{A}^{K'}$ can be identified with certain functions on $G_F Z_{\mathbb{A}} \backslash G_{\mathbb{A}} / K'$ that are of moderate growth. By the definition of automorphic form, we have

$$V = \bigcup_{K' \in \mathcal{V}} V^{K'}$$

for every subrepresentation $V \subset \mathcal{A}$.

Definition 1.5.3. The subspace $\mathcal{A}^K$ of $\mathcal{A}$ is called the space of *unramified automorphic forms.*

Let $f \in C^0(G_{\mathbb{A}})$ and $P = P_I$ a standard parabolic subgroup of G with unipotent radical $N = N_I$. We define the constant term of f along P by

$$f^P(x) := \int_{N(F)\backslash N(\mathbb{A})} f(nx)dn,$$

where $x \in G(\mathbb{A})$. We call f *cuspidal* if $f^P(x) = 0$ for every $x \in G(\mathbb{A})$ and every proper standard parabolic subgroup P of G.

Definition 1.5.4. A *cusp form* is an automorphic form which is a cuspidal function. We denote by $\mathcal{A}_0$ the space of cusp forms. The space $\mathcal{A}_0$ is a subrepresentation of $\mathcal{A}$.

Theorem 1.5.5. *(G. Harder) Let K' be an compact open subgroup of $G_{\mathbb{A}}$. Then there exists a compact subset C of $G_{\mathbb{A}}$ such that every element of $\mathcal{A}_0^{K'}$ has support in $Z_{\mathbb{A}} \cdot G_F \cdot C$. In particular $\mathcal{A}_0^{K'}$ is finite dimensional.*

Proof. This follows from [57, Cor. 1.2.3] or [55, Lem. 10.9]. □

Theorem 1.5.6. *Let f be a function on $G_F Z_{\mathbb{A}} \backslash G_{\mathbb{A}}$. Then the following conditions are equivalent:*

1. *f is a cusp form.*

2. *f is cuspidal and right invariant under some compact open subgroup of $G_{\mathbb{A}}$.*

Proof. For a proof cf. [13, Prop. 5.9]. □

Theorem 1.5.7. *(Multiplicity One) The representation $\mathcal{A}_0$ is a direct sum of irreducible admissible $G_{\mathbb{A}}$-modules. The irreducible representations appearing in this decomposition have multiplicity one.*

Proof. By Theorem 1.5.5, the representation $\mathcal{A}_0$ is contained in $L^2(G_F Z_{\mathbb{A}} \backslash G_{\mathbb{A}})$, the Hilbert space of square-integrable functions on $G_F Z_{\mathbb{A}} \backslash G_{\mathbb{A}}$. Thus $\mathcal{A}_0$ is a unitarizable representation and therefore semisimple. The multiplicity one property follows from [95, Thm. 5.5]. □

Definition 1.5.8. We call an irreducible subrepresentation of $\mathcal{A}_0$ a *cuspidal representation.*

1.6 Eisenstein Series on $GL_n(\mathbb{A})$

One of the ways to construct automorphic forms on $GL_n(\mathbb{A})$ is by the formation of Eisenstein series. These series are formed using cusp forms in subgroups of smaller rank, namely the Levi subgroups of parabolic subgroups of GL_n. Originally Eisenstein series were defined as modular forms on the upper half plane, by the formula

$$E_{2k}(z) = \tfrac{1}{2} \sum_{\substack{c,d \in \mathbb{Z}^2 \\ gcd(c,d)=1}} \frac{1}{(cz+d)^{2k}}, \quad Im(z) > 0,$$

where $k \in \mathbb{Z}_{>1}$. With the generalization of automorphic forms to reductive Lie groups (cf. [11]), a general definition of Eisenstein series was given in this context. One of the first problems in the early stage of this theory was the analytic continuation of these Eisenstein series, which was solved by Langlands in [76]. With the advent of the conjectures of Langlands (the Langlands program), the concepts of automorphic form and of Eisenstein series were reformulated in the context of adelic groups, cf. [13], [87]. With this at hand, we can use the analogy between number fields and function fields to transfer the concepts of automorphic forms and Eisenstein series for adelic groups over function fields. In the function field case the analytic continuation of the Eisenstein series was obtained by Morris, cf. [89], [90], [91], cf. also [57]. In this section we review the definition and convergence of Eisenstein series and intertwining operators for $GL_n(\mathbb{A})$. The main reference for this section is [79, Appendice G].

Consider on the algebraic group $G = GL_n$ over F the standard Levi subgroup M_I corresponding to a subset I of the set of simple roots Δ. We associte with I a partition $d_I = (d_1, \ldots, d_s)$ of n as in section 1.1. We define a homomorphism

$$\deg_M : M(\mathbb{A}) \longrightarrow \bigoplus_{j=1}^{s} \tfrac{1}{d_j}\mathbb{Z} \subset \mathbb{Q}^s$$

by

$$\deg_M(m) = \left(\frac{\deg(\det(g_1))}{d_1}, \cdots, \frac{\deg(\det(g_s))}{d_s} \right)$$

for all $m = (g_1, \ldots, g_s) \in M(\mathbb{A}) \cong GL_{d_1}(\mathbb{A}) \times \cdots \times GL_{d_s}(\mathbb{A})$. Let $M(\mathbb{A})^1 \subset M(\mathbb{A})$ be the kernel of $\deg_M$. We have $M(F) \subset M(\mathbb{A})^1$ and $M(\mathcal{O}_{\mathbb{A}}) \subset M(\mathbb{A})^1$.

We define the homomorphism $H_I : P_I(\mathbb{A}) \longrightarrow \mathbb{Q}^s$ by $H_I(nm) = -\deg_{M_I}(m)$, where $n \in N_I(\mathbb{A})$ and $m \in M_I(\mathbb{A})$. We extend H_I to a locally constant function on $G(\mathbb{A})$ by using the Iwasawa decomposition $G(\mathbb{A}) = P_I(\mathbb{A})K$ and making H_I right invariant by K.

For each $I \subsetneq \Delta$ we will denote by Y_I or by Y_{M_I} the abelian group of quasi-characters of the discrete group

$$M_I(\mathbb{A})^1 \backslash M_I(\mathbb{A}).$$

The homomorphism $H_I|_{M_I(\mathbb{A})} = -\deg_{M_I}$ induces an isomorphism

$$M_I(\mathbb{A})^1 \backslash M_I(\mathbb{A}) \xrightarrow{\sim} \bigoplus_{j=1}^{s} \tfrac{1}{d_j}\mathbb{Z} \subset \mathbb{Q}^s.$$

For each $\lambda \in \bigoplus_{j=1}^{s}(\mathbb{C}/\frac{2\pi i}{\log q}d_j\mathbb{Z})$ and each $H \in \bigoplus_{j=1}^{s}\frac{1}{d_j}\mathbb{Z}$ we set

$$\langle \lambda, H\rangle = \lambda_1 H_1 + \cdots + \lambda_s H_s \in \mathbb{C}/\tfrac{2\pi i}{\log q}\mathbb{Z}.$$

This yields an isomorphism of groups

$$\begin{array}{ccc} \bigoplus_{j=1}^{s}(\mathbb{C}/\frac{2\pi i}{\log q}d_j\mathbb{Z}) & \xrightarrow{\sim} & Y_I \\ \lambda & \longmapsto & (m \mapsto q^{\langle \lambda, H_I(m)\rangle}) \end{array}.$$

We will denote by

$$X_I = X_{M_I} = \left\{ \lambda \in \bigoplus_{j=1}^{s} \left(\mathbb{C} \middle/ \tfrac{2\pi i}{\log q} d_j\,\mathbb{Z}\right) \;\middle|\; \sum_{j=1}^{s} \lambda_j \in \tfrac{2\pi i}{\log q}\mathbb{Z} \right\}$$

the subgroup of complex characters of $M_I(\mathbb{A})^1 \backslash M_I(\mathbb{A})$ which are trivial on $Z(\mathbb{A})$, where Z is the center of $G = GL_n$. So X_I is a complex manifold of dimension $s-1$ with a finite number of components.

The decomposition

$$\mathbb{C}^\times = \mathbb{R}_{>0} \cdot \{z \in \mathbb{C}^\times \mid |z| = 1\}$$

induces decompositions

$$Y_I = Re\ Y_I \oplus Im\ Y_I \quad \text{and} \quad X_I = Re\ X_I \oplus Im\ X_I$$

for every $I \subset \Delta$, where $Re\ Y_I$ (resp. $Im\ Y_I$) denotes the quasi-characters on Y_I with image contained in $\mathbb{R}^\times$ (resp. $\{z \in \mathbb{C}^\times \mid |z| = 1\}$), and similarly for X_I. By the above identification, we have

$$Re\ Y_I = \mathbb{R}^s \subset \bigoplus_{j=1}^{s} \left(\mathbb{C} \middle/ \tfrac{2\pi i}{\log q} d_j\,\mathbb{Z}\right)$$

and

$$Im\ Y_I = \bigoplus_{j=1}^{s} \left(i\mathbb{R} \Big/ \tfrac{2\pi i}{\log q} d_j\, \mathbb{Z} \right) \subset \bigoplus_{j=1}^{s} \left(\mathbb{C} \Big/ \tfrac{2\pi i}{\log q} d_j\, \mathbb{Z} \right).$$

We denote by

$$\rho_I \in X_I,$$

the element with coordinates

$$\rho_{I,j} = d_j(d_s + \cdots + d_{j+1} - d_{j-1} - \cdots - d_1)/2$$

for $j = 1, \ldots, s$.

Lemma 1.6.1. *The modular character of $P_I(\mathbb{A})$ is given by*

$$\delta^{1/2}_{P_I(\mathbb{A})}(nm) = q^{\langle \rho_I, H_I(m) \rangle},$$

where $m \in M_I(\mathbb{A})$ and $n \in N_I(\mathbb{A})$.

Proof. Let $\mathfrak{n}_I$ be the Lie Algebra of N_I and $\mathfrak{n}_I(\mathbb{A})$ be the adelic points of $\mathfrak{n}_I$, then we have $\delta_{P_I(\mathbb{A})}(nm) = |\det Ad_{\mathfrak{n}_I(\mathbb{A})}(m)|$ for $n \in N_I(\mathbb{A})$ and $m \in M_I(\mathbb{A})$.

For $1 \leq j < k \leq s$, let $\mathfrak{n}_{jk}$ be the space of matrices $n \in \mathfrak{n}_I$ with $n_{lm} = 0$ except if $d_1 + \cdots + d_{j-1} + 1 \leq l \leq d_1 + \cdots + d_j$ and $d_1 + \cdots + d_{k-1} + 1 \leq m \leq d_1 + \cdots + d_k$. So we have $\mathfrak{n}_I = \bigoplus_{j<k} \mathfrak{n}_{jk}$. We naturally identify $\mathfrak{n}_{jk}$ with $d_j \times d_k$ matrices. With this identification, if $m = (m_1, \cdots, m_s) \in M_I$ and $n \in \mathfrak{n}_{jk}$, then $Ad_{\mathfrak{n}_I}(m)n = m_j n m_k^{-1}$. Therefore $|\det Ad_{\mathfrak{n}_I}(m)| = |\det m_j|^{d_k} |\det m_k|^{-d_j}$. Multiplying the contributions of all spaces $\mathfrak{n}_{jk}$ in the decomposition, we obtain the result. ☐

Let π be a cuspidal representation of $M_I(\mathbb{A})$ with central character χ_π such that $\chi_\pi|_{Z(\mathbb{A})} = 1$. We call the pair (I, π) a *cuspidal pair.* There is a unique element

$$Re\ \pi \in Re\ X_I$$

such that

$$|\chi_\pi(z)| = q^{\langle Re\ \pi, H_I(z) \rangle}$$

for $z \in Z_I(\mathbb{A})$.

Let $\chi \in Y_I$ be a quasi-character of $M_I(\mathbb{A})$. We define $\pi(\chi)$ as the cuspidal representation with representation space $V(\pi(\chi)) = \{f \cdot \chi \mid f \in V(\pi)\}$. Therefore $\pi(\chi) \simeq \pi \otimes \chi$ (we defined twisted representation at the end of section 1.2). We define $I(\pi)$ as the set of functions

$$\phi : N_I(\mathbb{A}) M_I(F) \backslash G(\mathbb{A}) \longrightarrow \mathbb{C},$$

such that

1. ϕ is right K-finite; this means that there is a compact open subgroup K' of $G(\mathbb{A})$ such that $\phi(gk) = \phi(g)$ for every $g \in G(\mathbb{A})$ and every $k \in K'$.

2. For every $g \in G(\mathbb{A})$, the function $m \longmapsto \phi(mg)$, $m \in M_I(\mathbb{A})$, belongs to the space of $\pi(\delta_{P_I(\mathbb{A})}^{1/2})$.

In other words, $I(\pi)$ is the normalized induction of π from $M_I(\mathbb{A})$ to $G(\mathbb{A})$. For $\lambda \in X_I$ we define $I(\pi, \lambda) = I(\pi(\lambda))$. The map $\phi \longmapsto \phi_\lambda := \phi q^{\langle \lambda, H_I(.)\rangle}$ defines a bijection between the vector spaces $I(\pi)$ and $I(\pi, \lambda)$.

For $\varphi \in I(\pi)$ and $g \in G(\mathbb{A})$ we define the *Eisenstein series*

$$E(\varphi, \pi)(g) = \sum_{\gamma \in P_I(F)\backslash G(F)} \varphi(\gamma g)$$

and

$$E(\varphi, \lambda, \pi)(g) := E(\varphi_\lambda, \pi(\lambda))(g) = \sum_{\gamma \in P_I(F)\backslash G(F)} \varphi(\gamma g) q^{\langle \lambda, H_I(\gamma g)\rangle}$$

whenever these series converges.

Let

$$C_I = C_{M_I} = \left\{ \lambda \in X_I \;\middle|\; Re\left(\frac{\lambda_j}{d_j} - \frac{\lambda_{j+1}}{d_{j+1}}\right) > 0 \;,\; \forall j = 1, \ldots, s-1 \right\}$$

be the *open positive cone* in X_I.

Theorem 1.6.2. *Let π be a cuspidal representation of $M_I(\mathbb{A})$ and $\varphi \in I(\pi)$. The Eisenstein series*

$$E(\varphi, \lambda, \pi)(g) = \sum_{\gamma \in P_I(F)\backslash G(F)} \varphi(\gamma g) q^{\langle \lambda, H_I(\gamma g)\rangle}$$

viewed as series of functions of

$$(g, \lambda) \in G(\mathbb{A}) \times (\rho_I - Re\ \pi + C_I) \subset G(\mathbb{A}) \times X_I,$$

is normally convergent when g stays in a compact subset of $G(\mathbb{A})$ and $Re\ \lambda$ stays in a compact subset of $\rho_I - Re\ \pi + Re\ C_I$. In particular it depends holomorphically on $\lambda \in \rho_I - Re\ \pi + C_I$.

Proof. For a proof cf. [79, Cor. G.4.3]. □

Remark 1.6.3. Given a set S and functions $f_i : S \longrightarrow \mathbb{C}$, where $i \in I$ and I is a enumerable set, the series

$$\sum_{i \in I} f_i(x)$$

is called *normally convergent* if the series of uniform norms of the terms converges, i.e.

$$\sum_{i\in I} \|f_i\| < \infty,$$

where $\|f_i\| := \sup_{x\in S} |f_i(x)|$.

Let (I, π) be a cuspidal pair and let $w \in W$ be such that $w(I) \subset \Delta$. We denote by $\dot{w}$ a representative of w in $N(T)(\mathbb{F}_q)$, as in section 1.1. We set

$$(I', \pi') = (w(I), w(\pi)).$$

We consider in $N_{I'}(\mathbb{A})$ (resp. $N_{I'}(\mathbb{A}) \cap \dot{w}N_I(\mathbb{A})\dot{w}^{-1}$) the Haar measure such that the volume of $N_{I'}(F)\backslash N_{I'}(\mathbb{A})$ (resp. $N_{I'}(F) \cap \dot{w}N_I(F)\dot{w}^{-1}\backslash N_{I'}(\mathbb{A}) \cap \dot{w}N_I(\mathbb{A})\dot{w}^{-1}$) is equal to 1. We consider in $(N_{I'}(\mathbb{A}) \cap \dot{w}N_I(\mathbb{A})\dot{w}^{-1})\backslash N'_I(\mathbb{A})$ the measure dn which is the quotient of the two Haar measures.

For any $\varphi \in I(\pi)$ and $g \in G(\mathbb{A})$ we define

$$M(w, \pi)(\varphi)(g) = \int_{(N_{I'}(\mathbb{A})\cap\dot{w}N_I(\mathbb{A})\dot{w}^{-1})\backslash N'_I(\mathbb{A})} \varphi(\dot{w}^{-1}ng)\, dn$$

whenever this integral is convergent. If this integral converges for every $g \in G(\mathbb{A})$ and every $\varphi \in I(\pi)$, then $M(w, \pi)(\varphi) \in I(w(\pi))$ and the operator

$$M(w, \pi) : I(\pi) \longrightarrow I(w(\pi))$$

is a morphism of representations. This operator is called an *intertwining operator*. Regarding convergence of the intertwining operators we have the following.

Theorem 1.6.4. *Let (I, π) be a cuspidal pair and let $w \in W$ with $w(I) \subset \Delta$. For any $\varphi \in I(\pi)$ and any $\lambda \in \rho_I - Re\ \pi + C_I$ the integrals*

$$M(w, \pi(\lambda))(\varphi_\lambda)(g)$$

for $g \in G(\mathbb{A})$, are absolutely convergent. Moreover, if g stays in a compact subset of $G(\mathbb{A})$ the convergence is uniform and

$$M(w, \pi(\lambda))(\varphi_\lambda) \in I(w(\pi(\lambda))).$$

Proof. For a proof cf. [79, G.5.7] □

The intertwinig operators appears on the constant terms of Eisenstein series as follows.

Theorem 1.6.5. *Let $I, J \subset \Delta$ with $|I| = |J|$, $P = P_J$ and let $\varphi \in I(\pi)^K$. If π is a cuspidal representation of $M_I(\mathbb{A})$ with $Re\ \pi \in \rho_I + Re\ C_I$, then*

$$E^P(\varphi, \pi) = \sum_{\substack{w\in W\\ w(I)=J}} M(w, \pi)(\varphi).$$

Proof. For a proof cf. [79, Prop. G.6.4] □

1.6.1 Analytic Continuation and Functional Equation

In this section, we will describe the analytical continuation of Eisenstein series on GL_n over a global function field. For proofs of the theorems, see [79, Thm. G.9.1 and Thm. G.9.2], [87, Chap. IV] and [89].

Consider $G = GL_n$ as an algebraic group over F and Δ the set of simple roots of G as in section 1.1. For $I \subsetneq \Delta$, let $\mathcal{P}(X_I)$ be the $\mathbb{C}$-vector space of complex functions f on X_I which are polynomials in the variables

$$q^{\lambda_j/d_j} \quad \text{and} \quad q^{-\lambda_j/d_j} \qquad (j = 1, \ldots, s).$$

For a finite dimensional complex vector space V, we set

$$\mathcal{P}(X_I; V) = \mathcal{P}(X_I) \otimes_{\mathbb{C}} V.$$

We can view the elements of $\mathcal{P}(X_I; V)$ as polynomial functions on X_I with values in V. If Ω is an open subset of X_I that meets every component of X_I and if

$$F : \Omega \longrightarrow V$$

is a holomorphic function, then we say that F can be analytically continued as a rational function to X_I if there exists a $D(\lambda) \in \mathcal{P}(X_I)$ that does not vanish on any component of Ω and a $P(\lambda) \in \mathcal{P}(X_I, V)$ such that

$$D(\lambda)F(\lambda) = P(\lambda)$$

for $\lambda \in \Omega$. In particular F has a meromorphic continuation to X_I.

Let (I, π) be a cuspidal pair and let $d_I = (d_1, \ldots, d_s)$ be the partition of n corresponding to I. We define

$$Fix(\pi) = \{\lambda \in X_I \mid \pi \otimes \lambda \cong \pi\}.$$

By considering the central characters, we see that

$$Fix(\pi) \subset \sum_{j=1}^{s} \left(\tfrac{2\pi i}{\log q}\mathbb{Z} \Big/ \tfrac{2\pi i}{\log q} d_j \mathbb{Z} \right).$$

For each pair (j, k) with $j, k \in \{1, \ldots, s\}$ and $j < k$, we denote by $n_{jk}(\pi)$ the smallest positive integer n such that

$$n \left(\tfrac{\lambda_j}{d_j} - \tfrac{\lambda_k}{d_k} \right) \in \tfrac{2\pi i}{\log q}\mathbb{Z}$$

for every $\lambda \in Fix(\pi)$ and we denote by $h_{jk}(\pi)$ the polynomial function on X_I that is defined by

$$h_{jk}(\pi)(\lambda) = q^{n_{jk}(\pi)\left(\frac{\lambda_j}{d_j} - \frac{\lambda_k}{d_k}\right)} - 1.$$

A *radicial hyperplane* for (I,π) is a codimension 1 subvariety H of X_I of the form

$$H = \lambda_0 + \{\lambda \in X_I \mid h_{jk}(\pi)(\lambda) = 0\}$$

for some $\lambda_0 \in X_I$ and some pair (j,k) as above. Observe that the function

$$h_H(\lambda) = h_{jk}(\pi)(\lambda - \lambda_0)$$

defining H is uniquely determined by H.

Theorem 1.6.6. *Let us fix a compact open subgroup $K' \subset K$ of $G(\mathbb{A})$ and a cuspidal pair (I,π).*

1. *(Analytic continuation). The functions*

$$\begin{array}{ccc} \rho_I - Re\ \pi + C_I & \longrightarrow & Hom_{\mathbb{C}}(I(\pi)^{K'}, \mathbb{C}), \\ \lambda & \longmapsto & (\varphi \mapsto E(\varphi_\lambda, \pi(\lambda))(g)) \end{array}$$

   where $g \in G(\mathbb{A})$ and

$$\begin{array}{ccc} \rho_I - Re\ \pi + C_I & \longrightarrow & Hom_{\mathbb{C}}(I(\pi)^{K'}, I(w(\pi))^{K'}), \\ \lambda & \longmapsto & \big(\varphi \mapsto (M(w,\pi(\lambda))(\varphi_\lambda))_{-w(\lambda)}\big) \end{array}$$

   and $w \in W$ with $w(I) \subset \Delta$ can be analytically continued as rational functions to X_I.

2. *(Singularities). There exists a finite number of radicial hyperplanes $H_1, \ldots, H_k$ and positive integers $m_1, \ldots, m_k$ (depending on K' and (I,π)) with the following property: For $\lambda \in X_I$ we define $\Psi_\lambda \in Hom_{\mathbb{C}}(I(\pi)^{K'}, I(w(\pi))^{K'})$ by*

$$\Psi_\lambda(\varphi) = \prod_{i=1}^{k} h_{H_i}(\lambda)^{m_i} \cdot (M(w,\pi(\lambda))(\varphi_\lambda))_{-w(\lambda)},$$

   then $\lambda \mapsto \Psi_\lambda$ is in $\mathcal{P}\left(X_I; Hom_{\mathbb{C}}(I(\pi)^{K'}, I(w(\pi))^{K'})\right)$ for every $w \in W$ with $w(I) \subset \Delta$. Moreover if for $\lambda \in X_I$ and $g \in G(\mathbb{A})$ fixed we define $\Phi_\lambda(g) \in Hom_{\mathbb{C}}(I(\pi)^{K'}, \mathbb{C})$ by

$$\Phi_\lambda(g)(\varphi) = \prod_{i=1}^{k} h_{H_i}(\lambda)^{m_i} \cdot E(\varphi_\lambda, \pi(\lambda))(g),$$

   then $\lambda \mapsto \Phi_\lambda(g)$ is in $\mathcal{P}\left(X_I; Hom_{\mathbb{C}}(I(\pi)^{K'}, \mathbb{C})\right)$.

3. *(Functional equation for Eisenstein series). Let $w \in W$ with $w(I) \subset \Delta$. We have*

$$E(M(w,\pi(\lambda))(\varphi_\lambda), w(\pi(\lambda))) = E(\varphi_\lambda, \pi(\lambda))$$

   for every $\varphi \in I(\pi)^{K'}$.

4. *(Functional equation for intertwining operators). Let $w, w' \in W$ with $w(I) \subset \Delta$ and $w'(w(I)) \subset \Delta$. We have*

$$M(w', w(\pi(\lambda)))(M(w, \pi(\lambda))(\varphi_\lambda)) = M(w'w, \pi(\lambda))(\varphi_\lambda)$$

   for every $\varphi \in I(\pi)^{K'}$.

Proof. For a proof cf. [89], [87], [57] and [79]. □

Let I be a subset of Δ and $P = P_I$ be a standard parabolic subgroup of GL_n with Levi subgroup $M = M_I$. Let $d_I = (d_1, \ldots, d_s)$ be the partition of n associated with I. We put $K_M = M(\mathbb{A}) \cap K$, which is a maximal compact subgroup of $M(\mathbb{A})$. We call an irreducible admissible representation V of $M_I(\mathbb{A})$ *unramified* if $V^{K_M} \neq 0$. Let π be an irreducible cuspidal representation of $M_I(\mathbb{A})$. We have $\pi = \pi_1 \otimes \cdots \otimes \pi_s$, where π_j is a cuspidal representation of $GL_{d_j}(\mathbb{A})$. We see that π is unramified if and only if π_j is an unramified cuspidal representation of $GL_{d_j}(\mathbb{A})$ for $1 \leq j \leq s$. In this case, $V(\pi)^{K_M}$ is of dimension 1 by Theorem 1.2.5 and Theorem 1.3.6 (i).

Let $I(\pi)$ be the normalized induction of π from $P_I(\mathbb{A})$ to $G(\mathbb{A})$ as defined in section 1.6. We observe that $I(\pi)^K \neq 0$ if and only if $V(\pi)^{K_M} \neq 0$. In fact, if $V(\pi)^{K_M}$ is generated by a cusp form f, then there is a unique $\phi_f \in I(\pi)^K$ such that $\phi_f|_{M(\mathbb{A})} = f$ and we have that ϕ_f generates $I(\pi)^K$. This follows easily from the Iwasawa decomposition $G(\mathbb{A}) = P_I(\mathbb{A})K$. We call the Eisenstein series $E(\varphi, \lambda, \pi)$ with $\varphi \in I(\pi)^K$, *unramified Eisenstein series.*

For the applications in chapter 2 we need only the Ensenstein series in $GL_3(\mathbb{A})$ for a cuspidal pair (I, π) with π an unramified cuspidal representation. Below we make some remarks on the action of the intertwining operator on $I(\pi(\lambda))^K$ when P_I is the Borel subgroup or a maximal proper parabolic subgroup. We can use these results to obtain more precise description of the radicial hyperplanes on Theorem 1.6.6 and of the functional equation of the unramified Eisenstein series on $GL_3(\mathbb{A})$.

We consider the case $I = \emptyset$ in more detail. Then the Levi subgroup $M_\emptyset$ is the maximal torus of diagonal matrices and will be denoted by T. We put $U = N_\emptyset$. Let π be an unramified cuspidal representation of $T(\mathbb{A})$. The cuspidal representation π is an unramified character of $T(\mathbb{A})$, which we will denote by χ instead of π. Let $E = T(\mathcal{O}_\mathbb{A})$ be the maximal compact subgroup of $T(\mathbb{A})$. We denote the group of unramified characters by $\Lambda(T) = Hom(T(\mathbb{A})/Z(\mathbb{A})ET(F), \mathbb{C}^\times)$.

Let $\overline{T} = T/Z$ be the maximal torus of PGL_n. We denote by $\overline{E}$ the image of E under the natural map $T(\mathbb{A}) \longrightarrow \overline{T}(\mathbb{A})$. Observe that $X_\emptyset = X_T = Hom(T(\mathbb{A})/Z(\mathbb{A})T(\mathbb{A})^1, \mathbb{C}^\times)$ is a subgroup of $\Lambda(T)$ and that we have an isomorphism of groups

$$\Lambda(T)/X_T \xrightarrow{\sim} Hom(\overline{T}(\mathbb{A})^1/\overline{T}(F)\overline{E}, \mathbb{C}^\times).$$

The group $\overline{T}(\mathbb{A})^1/\overline{T}(F)\overline{E}$ is finite and isomorphic to the product of $n-1$ copies of the class group of F.

Indeed, if we choose an idele ξ with $\deg(\xi) = 1$, then we obtain a (non-canonical) isomorphism

$$\mathbb{A}^\times/\mathcal{O}_\mathbb{A}^\times F^\times \simeq \mathbb{A}^\times/\mathbb{A}_0^\times \times \mathbb{A}_0^\times/\mathcal{O}_\mathbb{A}^\times F^\times,$$

and using that $\overline{T} \simeq \mathbb{G}_m^{n-1}$, we obtain an isomorphism of $\overline{T}(\mathbb{A})^1/\overline{T}(F)\overline{E}$ with the product of $n-1$ copies of $\mathbb{A}_0^\times/\mathcal{O}_\mathbb{A}^\times F^\times$.

Therefore we can give to $\Lambda(T)$ the structure of a complex Lie group such that X_T is an open subgroup of finite index.

For $\chi \in \Lambda(T)$, we define the function

$$\begin{array}{rccc} \phi_\chi: & G_\mathbb{A} & \longrightarrow & \mathbb{C}^\times \\ & g & \longmapsto & \chi \cdot \delta^{1/2}(t) \end{array},$$

where $g = utk$ with $u \in U(\mathbb{A})$, $t \in T(\mathbb{A})$ and $k \in K$. The one dimensional space $I(\lambda)^K$ is generated by ϕ_λ. Thus we have

$$M(w,\chi)(\phi_\chi) = c(w,\chi)\phi_{w(\chi)}$$

for some constant $c(w,\lambda) \in \mathbb{C}$. The value of $c(w,\lambda)$ has been computed explicitly in terms of L-functions of Hecke characters, cf. [57, Equation (1.6.3), pg. 276]. This gives us a more accurate description of the radicial hyperplanes appearing in Theorem 1.6.6, see [57, Thm. 1.6.6].

Let $\alpha \in \Delta$, $I = \Delta - \{\alpha\}$ and consider a cuspidal pair (I,π) with π an unramified cuspidal representation of $M_I(\mathbb{A})$, then $I(\pi)^K$ is of dimension 1. This follows easily from the Iwasawa decomposition $G(\mathbb{A}) = P_I(\mathbb{A})$. There exists a unique $w_0 \in W$ such that $w_0(I) \subset \Delta$ and $w_0 \neq 1$ (cf. [79, Lem. G.5.2]). If $\phi \in I(\pi)$ generates $I(\pi)^K$, we can compute explicitly $M(w_0,\pi(\lambda))(\phi_\lambda)$ in terms of a generator of $I(w_0(\pi))^K$ and automorphic L-functions, cf. [82, Sec. 3.4].

1.7 The Space of Unramified Automorphic Forms

The automorphic forms on $GL_n(\mathbb{A})$ that are invariant by the action of K are called *unramified automorphic forms*. We denote by $\mathcal{A}^K$ the space of unramified automorphic forms. In this section, we describe the action of the unramified Hecke algebra $\mathcal{H}^K$ on the unramified Eisenstein series and we explain the decomposition of the space of eigenforms in terms of cuspidal data.

Let $G = GL_n$, $K = GL_n(\mathcal{O}_\mathbb{A})$ and $K_x = GL_n(\mathcal{O}_x)$. We use the notation $\mathcal{H} = \mathcal{H}(G(\mathbb{A}))$, $\mathcal{H}_K = \mathcal{H}(G(\mathbb{A}),K)$ and $\mathcal{H}_{K_x} = \mathcal{H}(G(F_x),K_x)$. Thus we have a restricted tensor product decomposition

$$\mathcal{H}_K = \otimes_{e_{K_x}}\mathcal{H}_{K_x},$$

which allow us to see $\mathcal{H}_{K_x}$ as a subalgebra of $\mathcal{H}_K$.

Definition 1.7.1. We call a $f \in \mathcal{A}$ an *eigenform with eigencharacter* λ_f if it is an eigenvector for every $\Phi \in \mathcal{H}_K$ with eigenvalue $\lambda_f(\Phi)$. Note that $\lambda_f : \mathcal{H}_K \longrightarrow \mathbb{C}$ is a homomorphism of $\mathbb{C}$-algebras.

The action of unramified Hecke operators on Eisenstein series induced from unramified cuspidal representations is as follows.

Theorem 1.7.2. *Let $P = P_I$ be a standard parabolic subgroup of type $d_I = (d_1, ..., d_s)$ with standard Levi subgroup $M = M_I$ and $\pi = \pi_1 \otimes \cdots \otimes \pi_k$ an unramified cuspidal representation of $M(\mathbb{A})$ with $\chi_\pi|_{Z(\mathbb{A})} = 1$, where π_i is an unramified cuspidal representation of $GL_{d_i}(\mathbb{A})$. Let $x \in |X|$ and $\operatorname{diag}(z_{d_1+\ldots+d_{i-1}+1}, \ldots, z_{d_1+\ldots+d_i})$, be the Satake parameters of π_i at x. Let $\nu = (\nu_1, \ldots, \nu_s) \in X_I$. If $\Phi \in \mathcal{H}_{K_x}$ and $\phi \in I(\pi)^K$, then*

$$\Phi \cdot E(\phi, \nu, \pi) = \Phi^\vee(z_1', \ldots, z_n')E(\phi, \nu, \pi),$$

where $z_j' = z_j \cdot q_x^{-\frac{\nu_k}{d_k}}$ for $d_1 + \cdots + d_{k-1} < j \leq d_1 + \cdots + d_k$ and $\Phi^\vee$ is the Satake transform of Φ defined in section 1.3.

Proof. Let $\pi = \otimes_x \pi_x$ be the tensor product decomposition of π. By the description of the induction in terms of tensor products in [27, Thm. 1.4], it follows that

$$I(\pi) = \bigotimes_{x \in |X|} i_{M_I(F_x)}^{G(F_x)} \pi_x.$$

By our hypothesis, π_x is a unramified representation for every $x \in |X|$.

Fix $x \in |X|$. The Satake parameters of π_x are $z = \operatorname{diag}(z_1, \ldots, z_n)$. Therefore π_x is isomorphic to $\pi(\chi_z)$, the unique spherical subquotient of $i_{M_\emptyset(F_x)}^{M_I(F_x)} \chi_z$.

By the exactness of $i_{M_I(F_x)}^{G(F_x)}$ and the isomorphism $i_{M_I(F_x)}^{G(F_x)} \left(i_{M_\emptyset(F_x)}^{M_I(F_x)} \chi_z \right) \simeq i_{M_\emptyset(F_x)}^{G(F_x)} \chi_z$ given by Theorem 1.2.8, we conclude that $i_{M_I(F_x)}^{G(F_x)} \pi_x$ is a subquotient of the principal series $i_{M_\emptyset(F_x)}^{G(F_x)} \chi_z$.

It follows from $\dim(i_{M_I(F_x)}^{G(F_x)} \pi_x)^{K_x} = 1$, Theorem 1.3.2 (i) and Theorem 1.3.6 (ii) that if v is such that $(i_{M_I(F_x)}^{G(F_x)} \pi_x)^{K_x} = \langle v \rangle$ and $\Phi \in \mathcal{H}_{K_x}$, then

$$\pi(\Phi)v = \Phi^\vee(z_1, \ldots, z_n)v.$$

The character χ' corresponding to ν is given by

$$\chi'(m) = q^{\langle \nu, H_I(m) \rangle}$$

for $m \in M_I(\mathbb{A})$.

We apply these conclusions to $I(\pi \otimes \chi')$. We remember that $M_I \simeq GL_{d_1} \times \cdots \times GL_{d_s}$, $H_I|_{M_I(\mathbb{A})} = -\deg_{M_I}$,

$$\deg_{M_I}(m) = \left(\tfrac{\deg(\det(g_1))}{d_1}, \cdots, \tfrac{\deg(\det(g_s))}{d_s} \right),$$

and

$$\deg(a) = \sum_{x \in |X|} v_x(a) \cdot |x| \quad (\forall a \in \mathbb{A}^\times).$$

From what we saw above and theorem 1.2.7, $i_{M_I(F_x)}^{G(F_x)} \pi_x \otimes \chi'_x$ is a subquotient of $i_{M_\emptyset(F_x)}^{G(F_x)}(\chi_z \cdot \chi'_x|_{T(F_x)})$, and we have

$$\begin{aligned}\chi_z \cdot \chi'_x(diag(t_1, \ldots, t_n)) &= \prod_{j=1}^{n} z_j^{v_x(t_j)} \cdot \prod_{k=1}^{s} q_x^{-\frac{\nu_k}{d_k} v_x(t_{d_1+\cdots+d_{k-1}+1} \cdot \ldots \cdot t_{d_1+\cdots+d_k})} \\ &= \chi_{z'}(diag(t_1, \ldots, t_n)),\end{aligned}$$

where

$$z' = (z'_1, \ldots, z'_n), \quad z'_j = z_j \cdot q_x^{-\frac{\nu_k}{d_k}} \quad \text{if } d_1 + \cdots + d_{k-1} < j \leq d_1 + \cdots + d_k.$$

For $\Phi \in \mathcal{H}_{K_x}$, the action of $\pi(\Phi)$ in $(i_{M_I(F_x)}^{G(F_x)} \pi_x \otimes \chi'_x)^{K_x}$ is given by multiplication by $\Phi^\vee(z'_1, \ldots, z'_n)$. The theorem follows because the map

$$\begin{array}{ccc} I(\pi \otimes \chi') & \longrightarrow & \mathcal{A} \\ \varphi & \longmapsto & E(\varphi, \nu, \pi) \end{array}$$

is a morphism of representations. □

Remark 1.7.3. If $n = 2$ we obtain another proof of Lemma 3.3.2 in [85].

Let π be a cuspidal representation of $M_I(\mathbb{A})$ and $\nu' \in X_I$. Given $\nu_1, \ldots, \nu_{s-1} \in \mathbb{C}$, put $\nu_s = -\nu_1 - \cdots - \nu_{s-1}$. Let K' be a compact open subgroup of $G(\mathbb{A})$ and $D(\lambda) \in \mathcal{P}(X_I)$ be a polynomial in X_I such that for $\varphi \in I(\pi)^{K'}$, $D(\nu)E(\varphi, \nu, \pi)(g)$ is holomorphic in a neighborhood of $\nu' \in X_I$, for every $g \in G(\mathbb{A})$. For $\alpha = (\alpha_1, \ldots, \alpha_{s-1}), \beta = (\beta_1, \ldots, \beta_{s-1}) \in \mathbb{N}^{s-1}$, we define $|\alpha| = \alpha_1 + \ldots + \alpha_{s-1}$ and $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for $i = 1, \ldots, s-1$. When $\beta \leq \alpha$ we define

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_{s-1}}{\beta_{s-1}},$$

and

$$\widetilde{E}^\alpha(\varphi, \nu', \pi)(g) = \frac{\partial^{|\alpha|}}{\partial \nu_1^{\alpha_1} \cdots \partial \nu_{s-1}^{\alpha_{s-1}}} D(\nu' + \nu) E(\varphi, \nu' + \nu, \pi)(g)|_{\nu=0}.$$

The automorphic forms $\widetilde{E}^\alpha(\varphi, \nu', \pi)$ are called the *derivatives of Eisenstein series*.

We denote by $\mathcal{E}(\pi, \nu')$ the space of automorphic forms generated by $\widetilde{E}^\alpha(\varphi, \nu', \pi)$, where $\alpha \in \mathbb{N}^{s-1}$, $\varphi \in I(\pi)^{K'}$, and we vary K' and $D(\lambda)$.

We also define $\mathcal{E}(\pi(q^{\langle \nu', H_I(\cdot) \rangle})) = \mathcal{E}(\pi, \nu')$, which is easily seen to not depend on the choices of π and ν'. So we can write $\mathcal{E}(\pi)$ for every cuspidal representation π of a proper Levi subgroup without reference to the auxiliary element of X_I that we used in the definition.

Given a cuspidal pair (I, π) as above, we define the equivalence classes of (I, π), denoted by $[(I, \pi)]$ as the set of pairs $(w(I), w(\pi))$ with $w \in W$ such that $w(I) \subset \Delta$. We denote by Ξ the set of equivalence classes $[(I, \pi)]$ with $I \subsetneq \Delta$.

By the functional equation of Eisenstein series (cf. Theorem 1.6.6, $\mathcal{E}(\pi, \nu') = \mathcal{E}(w(\pi), w(\nu'))$ for $w \in W$. So it makes sense to define $\mathcal{E}(\pi)$ for an equivalence class $[(I, \pi)]$.

Theorem 1.7.4. *We have a direct sum decomposition of the space of automorphic forms*

$$\mathcal{A} = \mathcal{A}_0 \oplus \bigoplus_{[(I,\pi)]\in\Xi} \mathcal{E}(\pi).$$

Proof. By the arguments in [50, Thm. 1.4], the above sum is a direct sum, and by [87, Appendix II], the space of automorphic forms $\mathcal{A}$ is generated by the cusp forms and the derivatives of Eisenstein series. □

Our next aim is to obtain a decomposition of a certain subspace of the space of unramified automorphic forms. We start by analysing when a derivative of Eisenstein series is unramified. Let $\widetilde{E}^{\alpha}(\varphi, \nu', \pi)$ be a derivative of Eisenstein series as above. We observe that if K'' is a compact open subgroup with $K'' \subset K$, then $e_{K''}(\widetilde{E}^{\alpha}(\varphi, \nu', \pi)) = \widetilde{E}^{\alpha}(e_{K''}(\varphi), \nu', \pi)$. From this it follows that for $\mathcal{E}(\pi, \nu')^K \neq 0$ we need $I(\pi)^K \neq 0$. Therefore the derivatives of Eisenstein series which are unramified are induced from unramified cuspidal representations.

Let π be an unramified cuspidal representation of $M_I(\mathbb{A})$ with $I \subset \Delta$. Let the Satake parameters of π at the place $x \in |X|$ be $z = \mathrm{diag}(z_1, \ldots, z_n)$. Given $\nu_1, \ldots, \nu_{s-1} \in \mathbb{C}$, put $\nu_s = -\nu_1 - \cdots - \nu_{s-1}$. Let $\nu' = (\nu'_1, \ldots, \nu'_s) \in X_I$, we define $z'_j = z_j \cdot q_x^{-\frac{\nu'_k}{d_k}}$ for $d_1 + \cdots + d_{k-1} < j \leq d_1 + \cdots + d_k$ as in Theorem 1.7.2. We also define $z''_j = z_j \cdot q_x^{-\frac{(\nu'_k+\nu_k)}{d_k}}$ for $d_1 + \cdots + d_{k-1} < j \leq d_1 + \cdots + d_k$. For $\Phi \in \mathcal{H}_{K_x}$ we put $\Lambda_\Phi(\pi; z'_1, \ldots, z'_n) = \Phi^\vee(z'_1, \ldots, z'_n)$ and

$$\Lambda_\Phi^\alpha(\pi; z'_1, \ldots, z'_n) = \frac{\partial^{|\alpha|}}{\partial \nu_1^{\alpha_1} \cdots \partial \nu_{s-1}^{\alpha_{s-1}}} \Lambda_\Phi(\pi; z''_1, \ldots, z''_n)|_{\nu=0}.$$

Lemma 1.7.5. *If $\Phi \in \mathcal{H}_{K_x}$, π is an unramified cuspidal representation of $M_I(\mathbb{A})$ and $\varphi \in I(\pi)^K$, then*

$$\Phi \cdot \widetilde{E}^{\alpha}(\varphi, \nu', \pi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \Lambda_\Phi^{\alpha-\beta}(\pi; z'_1, \ldots, z'_n) \widetilde{E}^{\beta}(\varphi, \nu', \pi)$$

Proof. This follows by multiplying the equation in Theorem 1.7.2 by $D(\nu)$ and taking derivatives. □

Definition 1.7.6. Let $x \in |X|$ and $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{n-1}) \in (\mathbb{C})^{n-1}$. The space of $\mathcal{H}_{K_x}$-*eigenforms with eigenvalues* $\underline{\lambda}$, denoted by $\mathcal{A}(x; \underline{\lambda})$, is the set of $f \in \mathcal{A}^K$ such that $\Phi_{x,i}(f) = \lambda_i f$ for $i = 1, \ldots, n-1$.

We observe that for $f \in \mathcal{A}^K$, the invariance by $Z(\mathbb{A})$ on the right implies that $\Phi_{x,n}(f) = f$.

Theorem 1.7.7. *Let π be an unramified cuspidal representation of $M_I(\mathbb{A})$ and $\nu' \in X_I$. The space $\mathcal{E}(\pi, \nu')$ contains nontrivial $\mathcal{H}_{K_x}$-eigenforms and the eigenvalue of an $\mathcal{H}_{K_x}$-eigenform for the Hecke operator $\Phi \in \mathcal{H}_{K_x}$ is $\Phi^\vee(z'_1, \dots, z'_n)$.*

Proof. For $\alpha \in \mathbb{N}^{n-1}$ we define $\mathcal{E}^{\leq \alpha}(\pi, \nu')$ as the vector space generated by the automorphic forms $\widetilde{E}^\beta(\varphi, \nu', \pi)$ with $\beta \leq \alpha$. This is a finite dimensional vector space, which by Lemma 1.7.5, is stable by the action of $\mathcal{H}_K$. We construct a basis $\{f_1, \dots, f_k\}$ of $\mathcal{E}^{\leq \alpha}(\pi, \nu')$ of the form $f_i = \widetilde{E}^{\beta_i}(\varphi, \nu', \pi)$ such that if $\beta_i \leq \beta_j$, then $i \leq j$. On this basis, by Lemma 1.7.5 the action of $\Phi \in \mathcal{H}_{K_x}$ is given by an upper triangular matrix with diagonal entries $\Phi^\vee(z'_1, \dots, z'_n)$. Therefore every $\mathcal{H}_{K_x}$-eigenform on this space has eigenvalue $\Phi^\vee(z'_1, \dots, z'_n)$ for the Hecke operator $\Phi \in \mathcal{H}_{K_x}$. For some α, we have $\mathcal{E}^{\leq \alpha}(\pi, \nu') \neq 0$ because Eisenstein series are not equal to 0 everywhere. Thus $\mathcal{E}^{\leq \alpha}(\pi, \nu')$ has a nontrivial $\mathcal{H}_{K_x}$-eigenform. □

If $\underline{\lambda} = (\lambda_1, \dots, \lambda_{n-1}) \in (\mathbb{C})^{n-1}$, there is a $z = (z_1, \dots, z_n) \in (\mathbb{C}^\times)^n$ such that $z_1 \cdot \ldots \cdot z_n = 1$ and $q_x^{\langle \theta, \lambda_i \rangle} S_i(z) = \lambda_i$ for $i = 1, \dots, n-1$, where θ and λ_i is as in Theorem 1.3.6. The conjugacy class of $diag(z_1, \dots, z_n)$ is well defined in $GL_n(\mathbb{C})$ and we denote by $\Pi(x; \underline{\lambda})$ the set of equivalence classes of pairs (I, π) where $I \subsetneq \Delta$ and π is an unramified cuspidal representation of $M_I(\mathbb{A})$ such that the conjugacy class of its Satake parameter at x in $GL_n(\mathbb{C})$ is $diag(z_1, \dots, z_n)$.

Definition 1.7.8. Let $x \in |X|$ and $\underline{\lambda} \in (\mathbb{C})^{n-1}$. We define the space of $\mathcal{H}_{K_x}$-*cusp eigenforms with eigenvalues* $\underline{\lambda}$ as follows

$$\mathcal{A}_0(x; \underline{\lambda}) = \mathcal{A}_0^K \cap \mathcal{A}(x; \underline{\lambda})$$

For $[(I, \pi)] \in \Pi(x; \underline{\lambda})$, we define

$$\mathcal{E}(\pi, x; \underline{\lambda}) = \mathcal{E}(\pi)^K \cap \mathcal{A}(x; \underline{\lambda}).$$

Theorem 1.7.9. *Let $x \in |X|$ and $\underline{\lambda} \in (\mathbb{C})^{n-1}$. The space of $\mathcal{H}_{K_x}$-eigenforms with eigenvalues $\underline{\lambda}$ admits the direct sum decomposition*

$$\mathcal{A}(x; \underline{\lambda}) = \mathcal{A}_0(x; \underline{\lambda}) \oplus \bigoplus_{[(I,\pi)] \in \Pi(x; \underline{\lambda})} \mathcal{E}(\pi, x; \underline{\lambda}).$$

Proof. It follows from Theorem 1.7.4 that

$$\mathcal{A}^K = \mathcal{A}_0^K \oplus \bigoplus_{[(I,\pi)] \in \Xi} \mathcal{E}(\pi)^K.$$

Each factor of the direct sum is invariant under the action of $\mathcal{H}_{K_x}$, which yields the desired direct sum decomposition. □

Chapter 2

Toroidal Automorphic forms over function fields

The space of toroidal automorphic forms was introduced by Zagier in [104]. Let F be a global field. An automorphic form on GL_2 is toroidal if the periods along all embedded non-split tori are zero. The interest in this space stems from the fact (among others) that an Eisenstein series of weight s is toroidal if s is a non-trivial zero of the zeta function of F, and thus a connection with the Riemann hypothesis is stablished. In fact, the paper by Zagier proposes an approach to the Riemann hypothesis based on this space of automorphic forms.

In this chapter we study the generalization of this concept to GL_n, $n \geq 3$ and, for a global function field F, we study the space of unramified toroidal automorphic forms in GL_3 using the theory of graphs of Hecke operators developed by Alvarenga in [2].

2.1 Introduction

We start with the Eisenstein series $E(z,s)$ on the upper half plane $\mathbb{H}$. It is defined for $z = x + iy \in \mathbb{H}$ and $s \in \mathbb{C}$ with $Re(s) > 1$ by

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} Im(\gamma z)^s = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{y^s}{|cz+d|^{2s}} \tag{2.1.1}$$

where $\Gamma = PSL_2(\mathbb{Z})$ and

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z} \right\}.$$

We put $E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s)$, where $\zeta(s) = \sum_{n \geq 1} n^{-s}$ is the Riemann zeta function. Let $E = \mathbb{Q}[\sqrt{D}]$ be an imaginary quadratic field of discriminant $D < 0$. With each positive binary quadratic form $Q(m,n) = am^2 + bmn + cn^2$ of discriminant $b^2 - 4ac = D$, we associate the root $z_Q = \frac{-b+\sqrt{D}}{2a} \in \mathbb{H}$. Let $\{z_1, \ldots, z_r\}$ be the Γ-equivalence classes in $\mathbb{H}$ of points of the form z_Q for a positive binary quadratic form of discriminant D. The following formula is a classical theorem of Dirichlet (cf. [104, Example 1, pg. 280-281]):

$$\sum_{i=1}^{r} E^*(z_i, s) = \frac{w}{2}|D|^{s/2}(2\pi)^{-s}\Gamma(s)\zeta_E(s) \tag{2.1.2}$$

where w is the number of roots of unity contained in E and $\zeta_E(s)$ is the Dedekind zeta function of E.

For quadratic fields of positive discriminant there is a formula of Hecke, which we will describe in the following. Let $E = \mathbb{Q}[\sqrt{D}]$ be a quadratic field of discriminant $D > 0$ and $Q_1, \ldots, Q_{h(D)}$ representatives for the Γ-equivalence classes of quadratic forms of discriminant D. To each Q_i we associate, not a point $z_{Q_i} \in \Gamma\backslash\mathbb{H}$ as before, but a closed geodesic $C_{Q_i} \subset \Gamma\backslash\mathbb{H}$. We have the following formula of Hecke (cf. [60, p. 201] and [104, p. 281-283] for details.):

$$\sum_{i=1}^{h(D)} \int_{C_{Q_i}} E^*(z, s)|d_{Q_i}z| = \pi^{-s}D^{s/2}\Gamma\left(\frac{s}{2}\right)^2 \zeta_E(s). \tag{2.1.3}$$

Since $\zeta_E(s) = \zeta(s)L(\chi_E, s)$, where χ_E is the quadratic Hecke character associated to E by class field theory, it follows that if ρ is a zero of $\zeta(s)$ of order n, then it is also a zero of $\zeta_E(s)$ of order at least n. If we take derivatives in s in equations (2.1.2) and (2.1.3) and put $s = \rho$ we obtain the equations

$$\sum_{i=1}^{r} \frac{d^{(k)}}{ds^{(k)}} E^*(z_i, \rho) = 0 \tag{2.1.4}$$

and

$$\sum_{i=1}^{h(D)} \int_{C_{Q_i}} \frac{d^{(k)}}{ds^{(k)}} E^*(z, \rho)|d_{Q_i}z| = 0, \tag{2.1.5}$$

for $0 \leq k < n$.

Inspired by these formulas of Dirichlet and Hecke and other identities, Zagier defines a space $\mathcal{E}$ of functions on $\Gamma\backslash\mathbb{H}$ which annihilate certain linear operators (cf. [104, Thm. pg. 286]). In § 4 of [104], Zagier shows that $\mathcal{E}$ is the set of K-fixed vectors of a certain G-invariant subspace $\mathcal{V}$ of the space of functions on $\Gamma\backslash G$ (where $G = PSL_2(\mathbb{R})$, $K = PSO(2)$). If one can show that $\mathcal{V}$ is unitarizable as a representation of G, i.e. if one can construct a positive definite G-invariant scalar product on $\mathcal{V}$, then the Riemann hypothesis follows. These ideas were explored by Lachaud (cf. [72] and [73]) who connected them with Connes' approach to the Riemann Hypothesis (cf. [38]), by relating the space of toroidal automorphic forms with the construction of Pólya-Hilbert spaces.

In this chapter, we will work with the analog of these concepts for automorphic forms in $GL_n(\mathbb{A}_F)$, where F is a global field. As noted by Zagier in [104, pp. 298-300], the analogue of the space $\mathcal{V}$ in adelic language is the space of toroidal forms, which we define (and generalize) in section 2.1.1. Below we discuss the cases of integrals against tori for which we know closed formulas. For $n \geq 3$ only for a limited class of automorphic forms such closed formulas are known, which makes the arithmetic meaning of the space of toroidal forms still an open problem. Lorscheid studies in his thesis ([85]) the space of toroidal forms in $GL_2(\mathbb{A}_F)$ when F is a function field over a finite field. In this case, Weil's theorem allows us to describe geometrically automorphic forms as functions on vector bundles, which enables the use of algebraic geometry. The theory of graphs of Hecke operators, as developed in Lorscheid's thesis, is a computational device that allows us to make explicit calculations with automorphic forms and in particular with toroidal automorphic forms. The graph of a Hecke operator Φ is an oriented graph with weighted edges that encodes the action of Φ on the space of unramified automorphic forms in $GL_n(\mathbb{A})$. Alvarenga developes in his thesis the theory of graphs of Hecke operators for GL_n and an algorithm to calculate the weights of the edges of the graph in the case where F is an elliptic function field (cf. [2]). In the remainder of this chapter we apply Alvarenga's results to the study of toroidal forms in $GL_3(\mathbb{A}_F)$ where F is an elliptic function field.

2.1.1 Definitions

Let F be a global field. We denote by G the algebraic group GL_n over F and by $\mathbb{A} = \mathbb{A}_F$ the ring of adeles of F. If F is a function field over a finite field, then we write $\mathcal{O}_\mathbb{A} = \prod_{x \in |X|} \mathcal{O}_x$ and $K = G(\mathcal{O}_\mathbb{A})$ for the standard maximal compact subgroup of $G(\mathbb{A})$. Let Z be the center of G. We denote by $\mathcal{A}$ the space of automorphic forms on $G(\mathbb{A})$ with trivial central character.

Let E/F be a separable field extension of degree n. Choosing a basis of E over F gives an embedding of $E^\times$ in $G(F)$ and a non-split maximal torus $T \subset G$ with $T(F) = E^\times$ and $T(\mathbb{A}_F) = \mathbb{A}_E^\times$. We say that T is *associated with* E/F. We have $\mathbb{A}_E^\times \simeq (\mathbb{A}_E^\times)^1 \times V$, where $(\mathbb{A}_E^\times)^1$ is the kernel of the idele norm and V is its image; and $E^\times \backslash (\mathbb{A}_E^\times)^1$ is a compact group. This implies that $T_F Z_\mathbb{A} \setminus T_\mathbb{A}$ is compact.

Definition 2.1.1. Let T be a maximal torus of GL_n over F associated with a separable field extension E/F of degree n. Endow $T_\mathbb{A}$ and $T_F Z_\mathbb{A}$ with Haar measures and $T_F Z_\mathbb{A} \setminus T_\mathbb{A}$ with the quotient measure. Let $f \in \mathcal{A}$, we call

$$f_T(g) := \int\limits_{T_F Z_\mathbb{A} \backslash T_\mathbb{A}} f(tg)\; dt$$

the *toroidal integral* of f along T (evaluated at g).

Definition 2.1.2. Let T be a maximal torus of G associated with a separable field extension E/F of degree n. We define

$$\mathcal{A}_{tor}(E) = \{f \in \mathcal{A} \mid \forall g \in G_\mathbb{A},\; f_T(g) = 0\},$$

the space of *E-toroidal automorphic forms*, and

$$\mathcal{A}_{tor} = \bigcap_{\substack{E/F \text{ separable} \\ \text{extension, } [E:F]=n}} \mathcal{A}_{tor}(E),$$

the space of *toroidal automorphic forms*.

Remark 2.1.3. The spaces $\mathcal{A}_{tor}(E)$ do not depend on the choice of basis of E over F, which defines the embedding $T \subset G$. In fact, if $\gamma \in G(F)$ corresponds to another choice of basis of E over F, the correspondent torus is $T_\gamma = \gamma^{-1}T\gamma$ and we have $f_{T_\gamma}(g) = f_T(\gamma g)$ for an appropriate choice of Haar measure on $T_\gamma(\mathbb{A})$.

Proposition 2.1.4. *For all T and E as above,*

$$\mathcal{A}_{tor}(E) = \{f \in \mathcal{A} \mid \forall \Phi \in \mathcal{H},\ \Phi(f)_T(e) = 0\},$$

where $\mathcal{H}$ is the Hecke algebra of $G(\mathbb{A})$.

Proof. This follows from the fact that if $f \in \mathcal{A}$, then $G(\mathbb{A}) \cdot f = \mathcal{H} \cdot f$. □

Let Ξ be the set of equivalence classes $[(I, \pi)]$ with (I, π) a cuspidal pair and $I \subsetneq \Delta$, as in Section 1.7, where $\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\}$ is the set of simple roots of GL_n. If E/F is a separable field extension of degree n and $[(I, \pi)] \in \Xi$, we define

$$\mathcal{A}_{0,tor}(E) := \mathcal{A}_0 \cap \mathcal{A}_{tor}(E),\ \mathcal{A}_{0,tor} := \mathcal{A}_0 \cap \mathcal{A}_{tor},$$

$$\mathcal{E}(\pi)_{tor}(E) := \mathcal{E}(\pi) \cap \mathcal{A}_{tor}(E) \ \text{ and } \ \mathcal{E}(\pi)_{tor} := \mathcal{E}(\pi) \cap \mathcal{A}_{tor}.$$

Theorem 2.1.5. *We have the direct sum decomposition*

$$\mathcal{A}_{tor} = \mathcal{A}_{0,tor} \oplus \bigoplus_{[(I,\pi)] \in \Xi} \mathcal{E}(\pi)_{tor}.$$

If E/F is a separable field extension of degree n, then

$$\mathcal{A}_{tor}(E) = \mathcal{A}_{0,tor}(E) \oplus \bigoplus_{[(I,\pi)] \in \Xi} \mathcal{E}(\pi)_{tor}(E).$$

Proof. By the Strong Multiplicity One theorem (cf. [64, Thm. 4.4]), the $G(\mathbb{A})$-modules $\mathcal{A}_0$ and $\mathcal{E}(\pi)$ for $[(I, \pi)] \in \Xi$ don't have irreducible constituent in common. This implies that every subrepresentation V of $\mathcal{A}$ is the direct sum of the subrepresentations $V \cap \mathcal{A}_0$ and $V \cap \mathcal{E}(\pi)$ for $[(I, \pi)] \in \Xi$, which proves the theorem. □

Let $n = 2$. We will consider the adelic analogue of the Eisenstein series $E(z, s)$. Fix a Haar measure dz on $Z(\mathbb{A})$ and let φ be a Schwartz-Bruhat function on $\mathbb{A}_F^2$ (a Schwartz-Bruhat function in $\mathbb{A}_F^2$ is a locally constant function with compact support). The Eisenstein series $E(g, \varphi, s)$ is defined by

$$E(g,\varphi,s) = \int\limits_{Z(F)\backslash Z(\mathbb{A})} \sum_{\xi\in F^2\backslash\{0\}} \varphi(\xi z g)|\det zg|^s dz,$$

for $g \in G(\mathbb{A})$ and $s \in \mathbb{C}$ with sufficiently large real part. Switching the order of summation and integration in this formula, we obtain a formula for $E(g,\varphi,s)$ analogous to $E(z,s)$, given by

$$E(g,\varphi,s) = \sum_{\gamma\in P(F)\backslash G(F)} f(\gamma g,\varphi,s),$$

where $P \subset G$ is the subgroup of upper triangular matrices, and

$$f(g,\varphi,s) = |\det g|^s \int_{\mathbb{A}^\times} \varphi((0,a)g)|a|^{2s} d^\times a.$$

Let E/F be a separable quadratic extension. Choose a basis of E over F, which we use to identify $F^2\backslash\{0\}$ with $E^\times$ and to construct the maximal torus $T \subset G$. Observe that the F-idele norm of $\det t$, for $t \in T(\mathbb{A})$, equals the E-idele norm of t under the identification of $T(\mathbb{A})$ with $\mathbb{A}_E^\times$. We find

$$\int\limits_{T(F)Z(\mathbb{A})\backslash T(\mathbb{A})} E(tg,\varphi,s)dt = |\det g|^s \int_{\mathbb{A}_E^\times} \varphi(eg)|e|_E^s d^\times e, \tag{2.1.6}$$

(cf. [104, p. 299]), where $\varphi_g : e \mapsto \varphi(eg)$ is a Schwartz-Bruhat function on $\mathbb{A}_E$. If Φ is a Schwartz-Bruhat function on $\mathbb{A}_E$, the Tate integral of Φ is defined as

$$\zeta(\Phi,s) = \int_{\mathbb{A}_E^\times} \Phi(t)|t|^s dt.$$

By Tate's thesis, $\zeta(\Phi,s)$ equals an elementary holomorphic function of s times $\zeta_E(s)$, (cf. [97] and [101, Chap. VII]). In particular, if ρ is a non-trivial zero of $\zeta_E(s)$ of multiplicity $\geq i+1$ and

$$f(g) = \frac{d^i}{ds^i} E(g,\varphi,s)|_{s=\rho},$$

then f is E-toroidal.

Equation (2.1.6) is the analogue of the formulas (2.1.2) and (2.1.3) of Dirichlet and Hecke. In fact we can deduce (2.1.2) and (2.1.3) from (2.1.6), cf. [102, Chap. VI].

Observe that the definition of $\mathcal{A}_{tor}$ is formally similar to the condition

$$\int\limits_{N(F)\backslash N(\mathbb{A})} f(ng) = 0$$

that defines cusp forms, where N is the unipotent radical of a parabolic subgroup of G. In this sense, the space $\mathcal{A}_{tor}$ can be thought of as an analogue of the space $\mathcal{A}_0$ of cusp forms, which is a unitarizable representation of $G(\mathbb{A})$ (cf. proof of Theorem 1.5.7). But if the zeta function ζ_F has a multiple zero, then $\mathcal{A}_{tor}$ contain the derivative of an Eisenstein series and $\mathcal{A}_{tor}$ is not a semisimple $G(\mathbb{A})$-module. In fact $\mathcal{A}_{tor}$ is unitarizable if and only if the zeros of ζ_F are simple (cf. [104, pp. 21-22]). But we can state a conjecture which implies the Riemann hypothesis (when $n = 2$), even if ζ_F has multiple zeroes as follows (cf. also [85, Sec. 6.5-6.7]).

Conjecture 1. *Let $V = \otimes_x V_x$ be an automorphic representation, which is a subquotient of $\mathcal{A}_{tor}$. Then V_x is a tempered representation for every $x \in |X|$.*

Remark 2.1.6. For the definition and basic properties of tempered representations see [93, Sec. VII.2], [14] and [96]. We observe that a spherical representations $(V(\chi_z), \pi(\chi_z))$ with $z = (z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n$ is tempered if and only if $|z_1| = \ldots = |z_n| = 1$.

We observe that if $\pi = \otimes_x \pi_x$ is a cuspidal representation of $G(\mathbb{A})$, the fact that each π_x is a tempered representation is a deep theorem, proved by Drinfel'd for $n = 2$ in [44] and by Lafforgue for general n in [74]. When F is a function field over a finite field, the structure of the zeroes of $\zeta_F(s)$ is simpler and the Riemann hypothesis was proven by Hasse and Weil (cf. [59], [100]). In this case we make a second conjecture.

Conjecture 2. *If F is a function field over a finite field, then $\mathcal{A}_{tor}$ is an admissible representation.*

For $n = 2$ this conjecture is proved in [86, Theorem 10.2].

2.1.2 Waldspurger's theorem

Let F be a global field of characteristic different from 2 and (π, V) a cuspidal representation of $GL_2(\mathbb{A}_F)$ with trivial central character. The representation V has a restricted tensor product decomposition $V = \otimes_v V_v$ an we take $e = (\otimes e_v) \in V$ to be a pure vector of L^2-norm equal to 1. Let T be a maximal torus in GL_2 over F associated with a separable quadratic extension E of F. We denote by Π the change of base of π to $GL_2(\mathbb{A}_E)$ and χ_E the quadratic character associated to E by class field theory. Consider $g = (g_v) \in GL_2(\mathbb{A}_F)$ with $g_v = 1$ for every archimedean place v (if *char* $F = 0$). We denote by S_F the set of places of F. Waldspurger's fundamental formula [99, proposition 7, p. 222] and [31, theorem C.1] is

$$\left| \int_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})} e(tg)dt \right|^2 = C(\pi)L(\Pi, \tfrac{1}{2}) \prod_{v \in S_F} \alpha(e_v, g_v, T_v)$$

where

$$\alpha(e_v, g_v, T_v) = \frac{L(\chi_{T_v}, 1)L_2(\pi_v, 1)}{\zeta_v(2)L(\prod_v, \frac{1}{2})} \int_{Z_v\backslash T_v} \frac{\langle \pi_v(tg_v)e_v, \pi_v(g_v)e_v\rangle}{\langle e_v, e_v\rangle} dt,$$

and the term $C(\pi)$ is a non-zero constant depending only on (π, V) and the Haar measure on $T(\mathbb{A})$. In these formulas, ζ_v denotes the local ζ-function of the field F_v, the functions $L(\chi_{T_v}, \cdot)$,

$L_2(\pi_v, \cdot)$ and $L(\Pi_v, \cdot)$ are the local factors in v of the corresponding automorphic L-function. The function $L_2(\pi, \cdot)$ denotes the L-function of the lift of π to PGL_3 ([54]). For every place $v \in S_F$, we fix an π_v-invariant inner product $\langle \ , \ \rangle$ on V_v; as the inner product is unique up to multiplication by scalar, the local integral defining $\alpha(e_v, g_v, T_v)$ does not depend on the choice of $\langle \ , \ \rangle$. Waldspurger shows in [99] that all these integrals converge, that for almost all $v \in S_F$, $\alpha(e_v, g_v, T_v) = 1$ and that for every v, there is an $e_v \in V_v$ with $\alpha(e_v, g_v, T_v) \neq 0$. Therefore e is E-toroidal if and only if $L(\Pi, \frac{1}{2}) = 0$. We have $L(\Pi, \frac{1}{2}) = L(\pi, \frac{1}{2})L(\pi \otimes \chi_E, \frac{1}{2})$, and there is a separable quadratic extension E/F such that $L(\pi \otimes \chi_E, \frac{1}{2}) \neq 0$ (cf. [41, Thm. 6.1]). Thus e is a toroidal automorphic form if and only if $L(\pi, \frac{1}{2}) = 0$.

By the spectral decomposition of the space of automorphic forms and multiplicity one, we see that Waldspurger's and Zagier's theorems are sufficient to describe the space of toroidal forms in terms of zeroes of automorphic L-functions on $GL_2(\mathbb{A}_F)$ (cf. [41, Remark 2.6]). In the next sections we describe what is known for $n \geq 3$. We will see that only in a few cases we know how to compute closed formulas for toroidal periods and relate them with central values of automorphic L-functions.

2.1.3 Wielonsky's theorem

Wielonsky obtained a generalization of Zagier's theorem for $G = GL_n$ that we describe in this section. Let g be a matrix in $G(\mathbb{A})$, V the affine space of dimension n, φ a Bruhat-Schwartz function on $V(\mathbb{A})$, $e = (0, \ldots, 0, 1) \in V(F)$ and ω a quasi-character of $F^\times \backslash \mathbb{A}^\times$. We denote by σ the real number such that

$$|\omega(t)| = |t|^\sigma \quad \text{for} \quad t \in \mathbb{A}^\times.$$

We fix dt a Haar measure on $\mathbb{A}^\times$ and define

$$N(\varphi, g, \omega) = \int_{\mathbb{A}^\times} \varphi(etg)\omega(\det tg)dt,$$

which converges if $\sigma > 1/n$. We define the Eisenstein series $E(\varphi, g, \omega)$ by

$$E(\varphi, g, \omega) = \sum_{\gamma \in P(F) \backslash G(F)} N(\varphi, \gamma g, \omega)$$

where P is the standard parabolic subgroup of G of type $(n-1, 1)$. This series converges if $\sigma > 1$. Let $T \subset G$ be a maximal torus of G associated with a separable extension E/F of F. The choice of basis that we used to define the embedding $T \subset G$ also gives us an identification of E with $V(F)$ and of $V(\mathbb{A}_F)$ with $\mathbb{A}_E$, which we use implicitly below. Let $\zeta(\varphi, \omega)$ be the Tate integral

$$\zeta(\varphi, \omega) = \int_{\mathbb{A}_E^\times} \varphi(t)\omega(t)dt,$$

and $\varphi_g(t) = \varphi(tg)$ for $g \in G(\mathbb{A})$, $t \in V(\mathbb{A})$. Wielonsky's generalization of Zagier's theorem is the formula

$$\int_{T(F)Z(\mathbb{A})\backslash T(\mathbb{A})} E(\varphi, tg, \omega)dt = \omega(\det g)\zeta(\varphi_g, \omega \circ N_{E/F}). \qquad (2.1.7)$$

Using this toroidal integral, Wielonsky also obtains a generalization of Hecke's formula, cf. [102, Chap. VI].

2.1.4 The Bump-Goldfeld Theorem and its generalizations

An important application of Dirichlet and Hecke's formulas lies in the proof of the Kronecker limit formula for quadratic fields, which evaluates the constant term in the Laurent expansion of the zeta function of a quadratic field in $s = 1$. Bump and Goldfeld obtain in [22] a Kronecker limit formula for real cubic fields. This formula is obtained from an equality between a toroidal integral in GL_3 and a renormalization of a product of Eisenstein series, as defined by Zagier in [105]. Note that in this case, the identity does not provide a closed formula in terms of a central value of a L-function. Kudla exhibits in [70] the identity of Bump-Goldfeld as a particular case of the theory of see-saw dual reductive pairs [70, Example 7, p. 264]. These ideas were developed in more detail in the thesis of James Wodson [103, Theorem 3.2.1, p. 24] who generalizes the formula of Bump-Goldfeld to any totally real field, which we enunciate below.

Let K be a totally real field of degree n and let $\alpha_1, \ldots, \alpha_n$ be a $\mathbb{Z}$ basis for the ring $\mathfrak{o}$ of integers of K. We denote the Galois conjugates of $\alpha \in K$ by $\alpha^{(i)}$, $1 \leq i \leq n$. Let A be the matrix

$$A = \begin{pmatrix} \alpha_1^{(1)} & \alpha_1^{(2)} & \ldots & \alpha_1^{(n)} \\ \alpha_2^{(1)} & \alpha_2^{(2)} & \ldots & \alpha_2^{(n)} \\ \vdots & \vdots & & \vdots \\ \alpha_n^{(1)} & \alpha_n^{(2)} & \ldots & \alpha_n^{(n)} \end{pmatrix}.$$

For the definition of the GL_n-Eisenstein series $G_{(\nu_1,\nu_2)}(G)$ and the Hilbert Eisenstein series $E_K^*(z, \nu, \overline{\chi})$ of the theorem below, cf. [103].

Theorem 2.1.7. *Let $s = (n\nu_1 + n\nu_2 + 2 - n)/2$ and $\nu = (\nu_1 - \nu_2 + 1)/2$. For $(t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}$, let $t_n = (t_1 \ldots t_{n-1})^{-1}$, and let T be the diagonal matrix with diagonal entries $t_1, \ldots, t_n$. Let A be the matrix defined above. Then we have*

$$\begin{aligned} &\int_{(\mathbb{R}^\times)^{n-1}/\mathfrak{o}^\times} G_{(\nu_1,\nu_2)}(AT)\overline{\psi}(t_1, \ldots, t_n)\frac{dt_1}{|t_1|}\cdots\frac{dt_{n-1}}{|t_{n-1}|} \\ &= \frac{1}{n}D^{\nu_2/2-\nu_1/2}\lambda(2s)R.N. \int\int_{\Gamma\backslash\mathbb{H}} E_K^*(z, \nu, \overline{\chi})E(z, s)\frac{dxdy}{y^2}, \end{aligned} \qquad (2.1.8)$$

where $\lambda(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, D is the discriminant of K and $\varepsilon \in \mathfrak{o}^\times$ acts on the components of $(r_1, \ldots, r_{n-1}) \in (\mathbb{R}^\times)^{n-1}$ by taking $r_j \mapsto |\varepsilon_j|r_j$.

Beineke and Bump study an analogue of the formula of Bump-Goldfeld in [22] for the diagonal torus. In this case the toroidal period in GL_3 is defined by a process of renormalization, cf. [20], [8]. The formula in Theorem 2.1.7 for the toroidal integral of the GL_n-Eisenstein series $G_{(\nu_1,\nu_2)}(G)$ is not related with the central value of some L-function. However, the theorem implies, on the right-hand side, additional functional equations beyond those arising from the functional equations of $E^*_K(z,\nu,\overline{\chi})$ and $E(z,s)$; because on the left-hand side, the GL_n-Eisenstein series satisfies the functional equations of Langlands. For the applications of identities like (2.1.8) to "additional functional equations" see [22], [20] and [80].

2.2 Graphs of Hecke operators

The results from section 2.1 illustrate the arithmetic meaning of the space of toroidal forms. In the following we will study toroidal forms over a global field F of positive characteristic p. Our goal is to apply the theory of graphs of Hecke operators, as developed in [2], to generalize the results in [85] from GL_2 to GL_3. In this section, we review the definition of the graph of a Hecke operator and its geometric translation in terms of exact sequences of vector bundles. The definition of the graph of a Hecke operator is based on the following proposition.

Proposition 2.2.1 ([2, Prop. 1.3.5]). *Let $K' \subset GL_n(\mathbb{A})$ be an open compact subgroup. Fix $\Phi \in \mathcal{H}_{K'}$. For all $[g] \in GL_n(F)\backslash GL_n(\mathbb{A})/K'$, there is a unique set of pairwise distinct classes $[g_1], \dots, [g_r] \in GL_n(F)\backslash GL_n(\mathbb{A})/K'$ and numbers $m_1, \dots, m_r \in \mathbb{C}^\times$ such that*

$$\Phi(f)(g) = \sum_{i=1}^{r} m_i f(g_i),$$

for all $f \in \mathcal{A}^{K'}$.

For $[g], [g_1], \dots, [g_r] \in GL_n(F)\backslash GL_n(\mathbb{A})/K'$ and $m_1, \dots, m_r \in \mathbb{C}^\times$ as in the above proposition, we denote $\mathcal{V}_{\Phi,K'}([g]) := \{([g],[g_i],m_i)\}_{i=1,\dots,r}$.

Definition 2.2.2. We define the graph of Φ relative to K' as the graph $\mathcal{G}_{\Phi,K'}$ whose vertices are

$$Vert\mathcal{G}_{\Phi,K'} = GL_n(F)\backslash GL_n(\mathbb{A})/K'$$

and whose oriented weighted edges are

$$Edge\mathcal{G}_{\Phi,K'} = \bigcup_{[g]\in Vert\mathcal{G}_{\Phi,K'}} \mathcal{V}_{\Phi,K'}([g]).$$

The classes $[g_i]$ are called the Φ-neighbours of $[g]$ (relative to K').

We are interested in the unramified Hecke algebra, i.e. the case $K' = K = GL_n(\mathcal{O}_\mathbb{A})$.

For a place x of F, let $\Phi_{x,r}$ be the characteristic function of

$$K \begin{pmatrix} \pi_x I_r & \\ & I_{n-r} \end{pmatrix} K$$

where I_k is the $(k \times k)$-identity matrix. We note that $\Phi_{x,n}$ is invertible and its inverse is given by the characteristic function of $K(\pi_x I_n)^{-1}K$. We note that the above definition of $\Phi_{x,r}$ coincides with that from section 1.3 by means of the inclusion $\mathcal{H}_{K_x} \subset \mathcal{H}_K$. The Satake isomorphism (cf. Theorem 1.3.6) and the restricted tensor product decomposition

$$\mathcal{H}_K = \otimes_{e_{K_x}} \mathcal{H}_{K_x}$$

describes the algebraic structure of $\mathcal{H}_K$ as follows,

$$\mathcal{H}_K \cong \mathbb{C}[\Phi_{x,1}, \ldots, \Phi_{x,n}, \Phi_{x,n}^{-1}]_{x \in |X|}. \tag{2.2.1}$$

In particular, $\mathcal{H}_K$ is commutative.

Remark 2.2.3. By Proposition 2.1.3 in [2], it is enough to determine the graphs for the generators $\Phi_{x,1}, \ldots, \Phi_{x,n}, \Phi_{x,n}^{-1}$ of $\mathcal{H}_K$, where $x \in |X|$, in order to understand the graph of any unramified Hecke operator. We use the shorthand notation $\mathcal{G}_{x,r}$ for the graph $\mathcal{G}_{\Phi_{x,r},K}$ and $\mathcal{V}_{x,r}([g])$ for the $\Phi_{x,r}$-neighborhood $\mathcal{V}_{\Phi_{x,r},K}([g])$ of $[g]$, where $x \in |X|$ and $r = 1, \ldots, n$.

Let $\kappa(x)$ be the residual field of the place $x \in |X|$. Using the *Schubert Cell decomposition* of the Grassmanian $Gr(k,n)(\kappa(x))$ of k-dimensional spaces of $\kappa(x)^n$, we obtain a more explicit description of the neighbours of a vertex in $\mathcal{G}_{x,r}$. In the following, we identify a uniformizer $\pi_x \in F_x$ with an idele by sending π_x to the idele $a = (a_y)_{y \in |X|}$ with $a_x = \pi_x$ and $a_y = 1$ for $y \neq x$. If $b \in \kappa(x)$, we send b to the adele $a = (a_y)_{y \in |X|}$ with $a_x = b$ and $a_y = 0$ for $y \neq x$, where we consider $\kappa(x)$ as a subfield of F_x. We start with two lemmas describing a decomposition of the support of $\Phi_{x,r}$.

Lemma 2.2.4 ([2, Lem. 2.2.3])**.** *There is a bijection, denoted by $w \mapsto \xi_w$, of $Gr(k,n)(\kappa(x))$ with the set*

$$\left\{ \begin{pmatrix} \epsilon_1 & b_{12} & \cdots & & b_{1n} \\ & \ddots & & & \vdots \\ & & & \epsilon_{n-1} & b_{n-1\ n} \\ & & & & \epsilon_n \end{pmatrix} \middle| \begin{array}{c} \epsilon_i \in \{1, \pi_x\}, \\ \#\{i | \epsilon_i = 1\} = k \textit{ and } b_{ij} \in \kappa(x) \\ \textit{with } b_{ij} = 0 \textit{ if either} \\ \epsilon_j = \pi_x \textit{ or } \epsilon_i = 1 \end{array} \right\}$$

Lemma 2.2.5 ([2, Lem. 2.2.6])**.** *There is a decomposition of sets*

$$K \begin{pmatrix} \pi_x I_r & \\ & I_{n-r} \end{pmatrix} K = \coprod_{w \in Gr(n-r,n)(\kappa(x))} \xi_w K.$$

This provides us with a description of the neighbours of $[g] \in Vert \mathcal{G}_{x,r}$.

Theorem 2.2.6 ([2, Thm. 2.2.7]). *The $\Phi_{x,r}$-neighbours of $[g]$ are the classes $[g\xi_w]$, where ξ_w is as in the previous lemma, and the multiplicity of an edge from $[g]$ to $[g']$ equals the number of $w \in Gr(n-r,n)(\kappa(x))$ such that $[g\xi_w] = [g']$. The multiplicities of the edges originating in $[g]$ sum up to $\#Gr(n-r,n)(\kappa(x))$.*

If $F = \mathbb{F}_q(T)$ is a rational function field, it is easy to calculate explicitly the graphs of the Hecke operators $\mathcal{G}_{x,r}$ using this theorem, cf. [2, Sec. 2.5]. When the genus of F is positive, matrix calculations become more difficult and it is more convenient to translate these concepts into the language of algebraic geometry where other tools are at our disposal. Let F be a function field over a finite field with constant field $\mathbb{F}_q$. Let X be the geometrically irreducible smooth projective curve over $\mathbb{F}_q$ whose function field is F. A well-known theorem by Weil states that $G(F)\backslash G(\mathbb{A})/K$ stays in bijection with the set $\operatorname{Bun}_n X$ of isomorphism classes of rank n vector bundles on X. This allows us to give an interpretation of $\mathcal{G}_{x,r}$ in geometric terms. We begin by reviewing Weil's theorem.

The bijection

$$\begin{array}{cccc} F^\times\backslash\mathbb{A}^\times/\mathcal{O}_\mathbb{A}^\times & = Cl F & \stackrel{1:1}{\longleftrightarrow} & \operatorname{Pic} X = \operatorname{Bun}_1 X, \\ [a] & & \longmapsto & \mathcal{L}_a \end{array}$$

where $\mathcal{L}_a = \mathcal{L}_D$ if D is the divisor determined by a, generalises to all vector bundles as follows; also cf.
[52, Lem. 3.1]. A rank n-bundle $\mathcal{E}$ can be described by choosing bases

$$\mathcal{E}_\eta \cong \mathcal{O}_{X,\eta}^n = F^n \text{ and } \mathcal{E}_x \cong \mathcal{O}_{X,x}^n = (\mathcal{O}_x \cap F)^n$$

for all stalks, where η is the generic point of X, and the inclusion maps $\mathcal{E}_x \hookrightarrow \mathcal{E}_\eta$ for all $x \in |X|$. After tensoring with F_x, we obtain for every $x \in |X|$

$$F_x^n = \mathcal{O}_x^n \otimes_{\mathcal{O}_x} F_x \cong \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_x \otimes_{\mathcal{O}_x} F_x = \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} F \otimes_F F_x = \mathcal{E}_\eta \otimes_F F_x \cong F_x^n$$

which yields an element g of $G(\mathbb{A})$ (where "$=$" stands for a canonical isomorphism and "$\cong$" stands for an isomorphism that depends on our choice of basis).

A change of bases for $\mathcal{E}_\eta$ and $\mathcal{E}_x$ corresponds to multiplying g from the left by an element of $G(F)$ and from the right by an element of K, respectively.

Since the inclusion $F \subset F_x$ is dense for every place x, and $G(\mathcal{O}_\mathbb{A})$ is open in $G(\mathbb{A})$, every class in $G(F)\backslash G(\mathbb{A})/K$ is represented by a $g \in G(\mathbb{A})$ such that $g_x \in G(F)$ for all places x. This means that the above construction can be reversed. Weil's theorem asserts the following.

Theorem 2.2.7 ([52, Lem. 3.1]). *For every $n \geq 1$, the above construction yields a bijection*

$$\begin{array}{ccc} G(F)\backslash G(\mathbb{A})/K & \stackrel{1:1}{\longleftrightarrow} & \operatorname{Bun}_n X. \\ [g] & \longmapsto & \mathcal{E}_g \end{array}$$

There is a natural action

$$\begin{array}{ccc} \operatorname{Bun}_n X \times \operatorname{Pic} X & \longrightarrow & \operatorname{Bun}_n X. \\ (\mathcal{E}, \mathcal{L}) & \longmapsto & \mathcal{E} \otimes \mathcal{L} \end{array}$$

Let $\mathbf{P}\operatorname{Bun}_n X$ be the orbit set $\operatorname{Bun}_n X / \operatorname{Pic} X$, which is nothing else but the set of isomorphism classes of $\mathbb{P}^{n-1}$-bundles over X ([58, Ex. II.7.10]). Accordingly we will call elements of $\mathbf{P}\operatorname{Bun}_n X$ *projective space bundles*. If two vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ are in the same orbit of the action of $\operatorname{Pic} X$, we write

$$\mathcal{E}_1 \sim \mathcal{E}_2,$$

and say that $\mathcal{E}_1$ and $\mathcal{E}_2$ are *projectively equivalent*. By $[\mathcal{E}] \in \mathbf{P}\operatorname{Bun}_n X$ we mean the class that is represented by the rank n-bundle $\mathcal{E}$. Let $\mathcal{E} \in \operatorname{Bun}_n X$ and $\mathcal{L} \in \operatorname{Pic} X$. Since $(\mathcal{E} \otimes \mathcal{L})^\vee \simeq \mathcal{E}^\vee \otimes \mathcal{L}^\vee$, it makes sense to define $[\mathcal{E}]^\vee := [\mathcal{E}^\vee]$.

It is easy to see in the proof of Weil's theorem that $\mathcal{E}_g \otimes \mathcal{L}_a = \mathcal{E}_{ag}$ for $a \in \mathbb{A}^\times$. Thus we get the following result.

Theorem 2.2.8. *We have a bijection*

$$G(F)Z(\mathbb{A})\backslash G(\mathbb{A})/K \overset{1:1}{\longleftrightarrow} \mathbf{P}\operatorname{Bun}_n X.$$

By Weil's theorem, the vertices of $\mathcal{G}_{x,r}$ are identified with the geometric objects $\operatorname{Bun}_n X$. The next task is to describe the edges of $\mathcal{G}_{x,r}$ in geometric terms. We say that two exact sequences of sheaves

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathcal{F}'_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'_2 \longrightarrow 0$$

are *isomorphic with fixed* $\mathcal{F}$ if there are isomorphisms $\mathcal{F}_1 \to \mathcal{F}'_1$ and $\mathcal{F}_2 \to \mathcal{F}'_2$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_2 & \longrightarrow & 0 \\ & & \downarrow \cong & & \| & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathcal{F}'_1 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'_2 & \longrightarrow & 0 \end{array}$$

commutes. Let $\mathcal{K}_x$ be the torsion sheaf that is supported at x and has stalk $\kappa(x)$ at x. Fix $\mathcal{E} \in \operatorname{Bun}_n X$. For $r \in \{1, \dots, n\}$, and $\mathcal{E}' \in \operatorname{Bun}_n X$ we define $m_{x,r}(\mathcal{E}, \mathcal{E}')$ as the number of isomorphism classes of exact sequences

$$0 \longrightarrow \mathcal{E}'' \longrightarrow \mathcal{E} \longrightarrow \mathcal{K}_x^{\oplus r} \longrightarrow 0$$

with fixed $\mathcal{E}$ and with $\mathcal{E}'' \cong \mathcal{E}'$.

Definition 2.2.9. Let $x \in |X|$. For a vector bundle $\mathcal{E} \in \operatorname{Bun}_n X$, we define

$$\mathcal{V}_{x,r}(\mathcal{E}) := \{(\mathcal{E}, \mathcal{E}', m) | m = m_{x,r}(\mathcal{E}, \mathcal{E}') \neq 0\}.$$

We call $\mathcal{E}'$ a $\Phi_{x,r}$-*neighbour* of $\mathcal{E}$ if $m_{x,r}(\mathcal{E}, \mathcal{E}') \neq 0$ and we call $m_{x,r}(\mathcal{E}, \mathcal{E}')$ its *multiplicity*.

The following theorem gives a geometric description of $\mathcal{G}_{x,r}$.

Theorem 2.2.10 ([2, Lem. 2.3.3]). *For every $x \in |X|$, the map*

$$\begin{aligned} \mathcal{V}_{x,r}([g]) &\longrightarrow \mathcal{V}_{x,r}([\mathcal{E}_g]) \\ ([g],[g'],m) &\longmapsto (\mathcal{E}_g, \mathcal{E}_{g'}, m) \end{aligned}$$

is a well-defined bijection.

We summarize Theorems 2.2.7 and 2.2.10 in the following theorem.

Theorem 2.2.11. *Let $x \in |X|$. The graph $\mathcal{G}_{x,r}$ of $\Phi_{x,r}$ is described in geometric terms as*

$$Vert\ \mathcal{G}_{x,r} = \operatorname{Bun}_n X \qquad and \qquad Edge\ \mathcal{G}_{x,r} = \coprod_{\mathcal{E} \in \operatorname{Bun}_n X} \mathcal{V}_{x,r}(\mathcal{E}).$$

2.3 Geometric classification of the vertices

In this section, our aim is to show how to classify geometrically the vector bundles on the curve X. In particular we will review the classification of vector bundles on an elliptic curve by Atiyah. This will be important for the explicit calculations of eigenforms in the upcoming sections.

For two vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ over X, the $\mathbb{F}_q$-vector space of sheaf morphisms

$$\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}_2) \simeq \Gamma(X, \mathcal{E}_1^\vee \otimes \mathcal{E}_2)$$

is finite dimensional.

Let $\mathcal{E}$ be a locally free sheaf and $\mathcal{E}'$ a locally free subsheaf. Note that the quotient $\mathcal{E}/\mathcal{E}'$ is not necessarily locally free. We will call $\mathcal{E}'$ a *subbundle* if the quotient $\mathcal{E}/\mathcal{E}'$ is still a vector bundle i.e. a locally free sheaf.

For every $d \geq 1$ we define $X_d := X \otimes \mathbb{F}_{q^d}$, which is the curve over $\mathbb{F}_{q^d}$ obtained from X by extension of scalars.

We call a vector bundle indecomposable if for every decomposition

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$$

into two subbundles $\mathcal{E}_1$ and $\mathcal{E}_2$, one factor is trivial and the other is isomorphic to $\mathcal{E}$. The *Krull-Schmidt theorem* holds for the category of vector bundles over X, i.e. every vector bundle $\mathcal{E}$ on X has, up to permutation of factors, a unique decomposition into a direct sum of indecomposable subbundles, see [6].

Let $p : X_d = X \otimes \mathbb{F}_{q^d} \to X$ be the canonical projection. The *inverse image* or the *constant extension* of vector bundles from X to X_d is defined as

$$\begin{array}{rccc} p^* : & \operatorname{Bun}_n X & \to & \operatorname{Bun}_n X_d \\ & \mathcal{E} & \mapsto & p^*\mathcal{E} \end{array}.$$

The isomorphism classes of the rank n-bundles whose extension to $\mathbb{F}_{q^d}$ is isomorphic to $p^*\mathcal{E}$ are classified by $H^1(\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q), \mathrm{Aut}(\mathcal{E} \otimes \mathbb{F}_{q^d}))$, cf. [5, Sec. 1]. The algebraic group $\mathrm{Aut}(\mathcal{E} \otimes \mathbb{F}_{q^d})$ is an open subvariety of the irreducible affine variety $\mathrm{End}(\mathcal{E} \otimes \mathbb{F}_{q^d})$, and thus a connected algebraic group. By Lang's theorem [75, Cor. to Thm. 1], we have $H^1(\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q), \mathrm{Aut}(\mathcal{E} \otimes \mathbb{F}_{q^d})) = 0$, which implies that p^* is injective. In particular, one can consider the constant field extension to the geometric curve $\overline{X} = X \otimes \overline{\mathbb{F}}_q$ over an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. It follows that two vector bundles are isomorphic if and only if they are geometrically isomorphic, i.e. if their constant field extensions to $\overline{X}$ are isomorphic.

On the other hand, $p : X_d \to X$ defines the *direct image* or the *trace* of vector bundles

$$p_* : \begin{array}{ccc} \mathrm{Bun}_n X_d & \to & \mathrm{Bun}_{nd} X \\ \mathcal{E} & \mapsto & p_*\mathcal{E} \end{array} ,$$

and we have

$$p_* p^* \mathcal{E} \simeq \mathcal{E}^d$$

for $\mathcal{E} \in \mathrm{Bun}_n X$. An element $\sigma \in \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ induces a morphism $X \otimes \mathbb{F}_{q^d} \to X \otimes \mathbb{F}_{q^d}$. For a bundle $\mathcal{E}$ on $X \otimes \mathbb{F}_{q^d}$, we shall denote by $\mathcal{E}^\sigma$ the pullback of $\mathcal{E}$ to $X \otimes \mathbb{F}_{q^d}$ by this morphism. This pullback is called a *conjugate* of $\mathcal{E}$. This gives an action of $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ on $\mathrm{Bun}_n X_d$. For $\mathcal{E} \in \mathrm{Bun}_n X_d$, we have

$$p^* p_* \mathcal{E} \simeq \bigoplus_{\sigma \in \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)} \mathcal{E}^\sigma .$$

This is a decomposition of $p^*(p_*\mathcal{E})$ over X_d. It is defined over X only if the factors are defined over X, i.e. $\mathcal{E} = \mathcal{E}^\sigma$ for all $\sigma \in \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$. If $\mathcal{E}$ is not defined over X, $p_*\mathcal{E}$ can be indecomposable. This shows that indecomposable vector bundle might decompose in a constant field extension. We call a vector bundle *geometrically indecomposable* if its extension to $\overline{X}$ is indecomposable. In [5, Thm. 1.8], it is shown that every indecomposable vector bundle over X is the trace of a geometrically indecomposable bundle over some extension X_d of X.

There are certain compatibilities of constant extensions and traces with the action of $\mathrm{Pic}\, X$ on $\mathrm{Bun}_n X$. Namely, for a vector bundle $\mathcal{E}$ and a line bundle $\mathcal{L}$ over X, we have

$$p^*(\mathcal{E} \otimes \mathcal{L}) \simeq p^*\mathcal{E} \otimes p^*\mathcal{L},$$

and for a vector bundle $\mathcal{E}_1$ over X_d,

$$p_*\mathcal{E}_1 \otimes \mathcal{L} \simeq p_*(\mathcal{E}_1 \otimes p^*\mathcal{L})$$

by the projection formula (cf. [58, Exercise II 5.1(d)]. Using that $(\mathcal{E}_1^\sigma)^\vee \simeq (\mathcal{E}_1^\vee)^\sigma$ for $\sigma \in \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$, we obtain

$$p^*(p_*\mathcal{E}_1)^\vee \simeq \bigoplus_{\sigma\in Gal(\mathbb{F}_{q^d}/\mathbb{F}_q)} (\mathcal{E}_1^\sigma)^\vee \simeq \bigoplus_{\sigma\in Gal(\mathbb{F}_{q^d}/\mathbb{F}_q)} (\mathcal{E}_1^\vee)^\sigma \simeq p^*p_*(\mathcal{E}_1^\vee),$$

which implies that $(p_*\mathcal{E}_1)^\vee \simeq p_*(\mathcal{E}_1^\vee)$ by the injectivity of p^*.

In the following, let X be an elliptic curve. Atiyah classifies in [7] all indecomposable vector bundles over X. Let $\mathrm{Coh}(X)$ be the category of coherent sheaves on X. We denote by $\deg(\mathcal{F})$ and $\mathrm{rk}(\mathcal{F})$ the degree and the rank, respectively, of the coherent sheaf $\mathcal{F}$.

Let$\mathcal{F} \in \mathrm{Coh}(X)$. We observe that if $\mathrm{rk}(\mathcal{F}) = 0$ and $\mathcal{F} \neq 0$, then $\deg(\mathcal{F}) > 0$. The *slope* of $\mathcal{F}$ is defined as

$$\mu(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} \in \mathbb{Q} \cup \{\infty\},$$

where $\mu(\mathcal{F}) = \infty$ if $\mathrm{rk}(\mathcal{F}) = 0$ and $\mathcal{F} \neq 0$, and $\mu(0) = 0$. A sheaf $\mathcal{F}$ is called *semistable* if $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$ for all nonzero subsheaves $\mathcal{G} \subset \mathcal{F}$. It is called *stable* if $\mu(\mathcal{G}) < \mu(\mathcal{F})$ for all proper nonzero subsheaves $\mathcal{G} \subset \mathcal{F}$. The full subcategory C_μ of $\mathrm{Coh}(X)$ consisting of all semistable sheaves of a fixed slope $\mu \in \mathbb{Q} \cup \{\infty\}$ is abelian and closed under extension. Moreover if $\mathcal{F}, \mathcal{G}$ are semistable with $\mu(\mathcal{F}) < \mu(\mathcal{G})$, then $\mathrm{Hom}(\mathcal{G}, \mathcal{F}) = \mathrm{Ext}(\mathcal{F}, \mathcal{G}) = 0$. Any sheaf $\mathcal{F}$ possesses a unique filtration (the *Harder-Narasinham filtration*, or *HN-filtration*)

$$0 = \mathcal{F}^{r+1} \subset \mathcal{F}^r \subset \cdots \subset \mathcal{F}^1 = \mathcal{F}$$

for which $\mathcal{F}^i/\mathcal{F}^{i+1}$ is semistable of slope μ_i and $\mu_1 < \cdots < \mu_r$. Moreover, recall that $\mathbf{C}_\infty$ is the category of torsion sheaves and hence equivalent to the product category $\prod_x \mathrm{Tor}_x$ where x runs through the set of closed points of X and Tor_x denotes the category of torsion sheaves supported at x.

Theorem 2.3.1 ([7, Thm. 7]). *Let X be an elliptic curve. We have the following.*

(i) *The HN-filtration of any coherent sheaf splits (noncanonically). In particular, any indecomposable coherent sheaf is semistable.*

(ii) *The set of stable sheaves of slope μ is the class of simple objects of $\mathbf{C}_\mu$.*

(iii) *The choice of any rational point $x_0 \in X(\mathbb{F}_q)$, induces an exact equivalence of abelian categories $\epsilon_{\nu,\mu} : \mathbf{C}_\mu \to \mathbf{C}_\nu$, for any $\mu, \nu \in \mathbb{Q} \cup \{\infty\}$.*

Notation. Let $l > 0$ and $x \in |X|$. We denote by $\mathcal{K}_x^{(l)}$ the torsion sheaf with support at x and stalk at x equal to $\mathcal{O}_{X,x}/\mathfrak{m}_x^l$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$. The torsion sheaves $\mathcal{K}_x^{(l)}$ are the indecomposable objects of the category $\mathbf{C}_\infty$. Let $\mathcal{E}$ be an indecomposable vector bundle on X. By Theorem 2.3.1 (i), $\mathcal{E}$ is semistable. Let μ be its slope and choose $x_0 \in X(\mathbb{F}_q)$. It follows that $\epsilon_{\infty,\mu}(\mathcal{E})$ is an indecomposable torsion sheaf of the form $\mathcal{K}_x^{(l)}$. Conversely, $\mathcal{K}_x^{(l)}$ corresponds to an indecomposable vector bundle $\mathcal{E}$ such that $\epsilon_{\infty,\mu}(\mathcal{E}) = \mathcal{K}_x^{(l)}$. We conclude that an indecomposable vector bundle $\mathcal{E}$ is completely determined by

(i) $(\mathrm{rk}(\mathcal{E}), \deg(\mathcal{E})) = (n, d)$;

(ii) the closed point $x \in |X|$ that is the support of $\epsilon_{\infty,\mu}(\mathcal{E})$;

(iii) and the weight l that determines the irreducible torsion sheaf $\epsilon_{\infty,\mu}(\mathcal{E})$ as $\mathcal{K}_x^{(l)}$.

We denote $\mathcal{E}$ by $\mathcal{E}_{(x,l)}^{(n,d)}$. By Atiyah's theorem, $gcd(n,d) = \deg \mathcal{K}_x^{(l)} = |x| \cdot l$. Moreover we have:

- if $|x| = 1$, then $\mathcal{E}_{(x,l)}^{(n,d)}$ is geometrically indecomposable;
- if $|x| = i > 1$, then $\mathcal{E}_{(x,l)}^{(n,d)}$ is a trace of a vector bundle on X_i.

2.4 Multiplicities on the cusps

In this section, we fix $n = 3$ and an elliptic curve X over $\mathbb{F}_q$ with a distinguished rational point x_0. We describe the simplest cases where we can compute the multiplicities of edges on the graphs $\mathcal{G}_{x,r}$. This is the case for $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$ with $\mathcal{L} \in \operatorname{Pic} X$ and $\mathcal{M} \in \operatorname{Bun}_2 X$, where $\deg \mathcal{L}$ is big or small in relation to $\deg \mathcal{M}$. These are intuitively the vertices at the "cusps" of $\mathcal{G}_{x,i}$. Lorscheid defines in [85, Defn. 5.4.5] the concepts of a nucleus and cusps for graphs of Hecke operators in PGL_2. For GL_3, we only define the nucleus in section 2.5 and omit a rigorous definition of cusps. Intuitively, the nucleus carries the most important arithmetic information about the automorphic forms and the cusps are vertices away from the nucleus, as we will see from section 2.5 onwards. The results of this section is part of a joint project with Roberto Alvarenga and Oliver Lorscheid.

Lemma 2.4.1. *Let $x \in |X|$, $\mathcal{M} \in \operatorname{Bun}_2 X$, $\mathcal{L} \in \operatorname{Pic} X$ and $\mathcal{M}' \in \operatorname{Bun}_2 X$ with $m_{x,1}(\mathcal{M}, \mathcal{M}') \neq 0$.*

(i) *If either $\mathcal{M}$ is indecomposable with $2\deg \mathcal{L} - \deg \mathcal{M} > 0$ or $\mathcal{M} = \mathcal{L}_1 \oplus \mathcal{L}_2$ is decomposable with $\mathcal{L}_i \in \operatorname{Pic} X$ and $\deg \mathcal{L} > \deg \mathcal{L}_i$, $i = 1, 2$, then $\operatorname{Ext}(\mathcal{M}, \mathcal{L}) = 0$ and $\operatorname{Ext}(\mathcal{M}', \mathcal{L}) = 0$.*

(ii) *If either $\mathcal{M}$ is indecomposable with $2\deg \mathcal{L} - \deg \mathcal{M} > 2|x|$ or $\mathcal{M} = \mathcal{L}_1 \oplus \mathcal{L}_2$ is decomposable with $\mathcal{L}_i \in \operatorname{Pic} X$ and $\deg \mathcal{L} > \deg \mathcal{L}_i + |x|$, $i = 1, 2$, then $\operatorname{Ext}(\mathcal{M}, \mathcal{L}) = 0$ and $\operatorname{Ext}(\mathcal{M}, \mathcal{L}(-x)) = 0$.*

(iii) *If either $\mathcal{M}$ is indecomposable with $2\deg \mathcal{L} - \deg \mathcal{M} < -2|x|$ or $\mathcal{M} = \mathcal{L}_1 \oplus \mathcal{L}_2$ is decomposable with $\mathcal{L}_i \in \operatorname{Pic} X$ and $\deg \mathcal{L} < \deg \mathcal{L}_i - |x|$, $i = 1, 2$, then $\operatorname{Ext}(\mathcal{L}, \mathcal{M}) = 0$ and $\operatorname{Ext}(\mathcal{L}, \mathcal{M}') = 0$.*

(iv) *If either $\mathcal{M}$ is indecomposable with $2\deg \mathcal{L} - \deg \mathcal{M} < 0$ or $\mathcal{M} = \mathcal{L}_1 \oplus \mathcal{L}_2$ is decomposable with $\mathcal{L}_i \in \operatorname{Pic} X$ and $\deg \mathcal{L} < \deg \mathcal{L}_i$, $i = 1, 2$, then $\operatorname{Ext}(\mathcal{L}, \mathcal{M}) = 0$ and $\operatorname{Ext}(\mathcal{L}(-x), \mathcal{M}) = 0$.*

Proof. **(i)** If $\mathcal{M}$ is indecomposable, then $\mathcal{M}$ is semistable and $\mu(\mathcal{M}) = \deg \mathcal{M}/2 < \deg \mathcal{L} = \mu(\mathcal{L})$, which implies using Serre duality that $\mathrm{Ext}(\mathcal{M}, \mathcal{L}) = \mathrm{Hom}(\mathcal{L}, \mathcal{M}) = 0$. If $\mathcal{M}$ is decomposable, then $\mathrm{Ext}(\mathcal{M}, \mathcal{L}) = \mathrm{Hom}(\mathcal{L}, \mathcal{M}) = H^0(X, \mathcal{M} \otimes \mathcal{L}^\vee) = 0$.

If $\mathcal{M}'$ is indecomposable, then $\mu(\mathcal{M}') = (\deg \mathcal{M} - |x|)/2 < \deg \mathcal{L} = \mu(\mathcal{L})$, which implies $\mathrm{Ext}(\mathcal{M}', \mathcal{L}) = 0$.

If $\mathcal{M}'$ is decomposable and $\mathcal{M}$ is indecomposable, then we write $\mathcal{M}' = \mathcal{L}'_1 \oplus \mathcal{L}'_2$ with $\deg \mathcal{L}'_2 - \deg \mathcal{L}'_1 = \delta(\mathcal{M}') \geq 0$ (cf. [85, Prop. 5.3.7]). We have $\delta(\mathcal{M}') \leq |x|$ because $\delta(\mathcal{M}) \leq 0$ (cf. [85, Lemma 5.4.2]), and $\deg \mathcal{L}'_2 + \deg \mathcal{L}'_1 = \deg \mathcal{M} - |x|$. Thus

$$\deg \mathcal{L}'_1 \leq \deg \mathcal{L}'_2 = \frac{\deg \mathcal{M} - |x| + \delta(\mathcal{M}')}{2} < \deg \mathcal{L},$$

which implies that $\mathrm{Ext}(\mathcal{M}', \mathcal{L}) = H^0(X, \mathcal{M}' \otimes \mathcal{L}^\vee) = 0$.

If $\mathcal{M}'$ is decomposable and $\mathcal{M}$ is decomposable, then we can write $\mathcal{M}' = \mathcal{L}'_1 \oplus \mathcal{L}'_2$ with $\deg \mathcal{L}'_i \leq \deg \mathcal{L}_i$. Therefore, $\mathrm{Ext}(\mathcal{M}', \mathcal{L}) = H^0(X, \mathcal{M}' \otimes \mathcal{L}^\vee) = 0$.

(ii) If $\mathcal{M}$ is indecomposable, then $\mu(\mathcal{M}) = \deg \mathcal{M}/2 < \deg \mathcal{L} - |x| = \mu(\mathcal{L}(-x)) < \mu(\mathcal{L})$, which implies $\mathrm{Ext}(\mathcal{M}, \mathcal{L}) = 0$ and $\mathrm{Ext}(\mathcal{M}, \mathcal{L}(-x)) = 0$.

If $\mathcal{M}$ is decomposable, then $\mathrm{Ext}(\mathcal{M}, \mathcal{L}) = H^0(X, \mathcal{M} \otimes \mathcal{L}^\vee) = 0$, and similarly $\mathrm{Ext}(\mathcal{M}, \mathcal{L}(-x)) = 0$.

(iii) If $\mathcal{M}$ is indecomposable, then $\mathcal{M}$ is semistable and $\mu(\mathcal{M}) = \deg \mathcal{M}/2 > \deg \mathcal{L} = \mu(\mathcal{L})$, which implies $\mathrm{Ext}(\mathcal{L}, \mathcal{M}) = \mathrm{Hom}(\mathcal{M}, \mathcal{L}) = 0$. If $\mathcal{M}$ is decomposable, then $\mathrm{Ext}(\mathcal{L}, \mathcal{M}) = \mathrm{Hom}(\mathcal{M}, \mathcal{L}) = H^0(X, \mathcal{L} \otimes \mathcal{M}^\vee) = 0$.

If $\mathcal{M}'$ is indecomposable, then $\mu(\mathcal{M}') = (\deg \mathcal{M} - |x|)/2 > \deg \mathcal{L} = \mu(\mathcal{L})$, which implies $\mathrm{Ext}(\mathcal{L}, \mathcal{M}') = 0$.

If $\mathcal{M}'$ is decomposable and $\mathcal{M}$ is indecomposable, then we write $\mathcal{M}' = \mathcal{L}'_1 \oplus \mathcal{L}'_2$ with $\deg \mathcal{L}'_2 - \deg \mathcal{L}'_1 = \delta(\mathcal{M}') \geq 0$. We have $\delta(\mathcal{M}') \leq |x|$ because $\delta(\mathcal{M}) \leq 0$, and $\deg \mathcal{L}'_2 + \deg \mathcal{L}'_1 = \deg \mathcal{M} - |x|$. Thus

$$\deg \mathcal{L}'_2 \geq \deg \mathcal{L}'_1 = \frac{\deg \mathcal{M} - |x| - \delta(\mathcal{M}')}{2} > \deg \mathcal{L},$$

which implies $\mathrm{Ext}(\mathcal{L}, \mathcal{M}') = 0$.

If $\mathcal{M}'$ is decomposable and $\mathcal{M}$ is decomposable, then we can write $\mathcal{M}' = \mathcal{L}'_1 \oplus \mathcal{L}'_2$ with $\deg \mathcal{L}'_i \leq \deg \mathcal{L}_i$ and $\deg \mathcal{M}' = \deg \mathcal{M} - |x|$. Therefore, $\deg \mathcal{L}'_i \geq \deg \mathcal{L}_i - |x| > \deg \mathcal{L}$, which implies $\mathrm{Ext}(\mathcal{L}, \mathcal{M}') = H^0(X, \mathcal{M}'^\vee \otimes \mathcal{L}) = 0$.

(iv) If $\mathcal{M}$ is indecomposable, then $\deg \mathcal{L} - |x| = \mu(\mathcal{L}(-x)) < \mu(\mathcal{L}) < \mu(\mathcal{M}) = \deg \mathcal{M}/2$, which implies $\mathrm{Ext}(\mathcal{L}, \mathcal{M}) = 0$ and $\mathrm{Ext}(\mathcal{L}(-x), \mathcal{M}) = 0$.

If $\mathcal{M}$ is decomposable, then $\mathrm{Ext}(\mathcal{L}, \mathcal{M}) = H^0(X, \mathcal{M}^\vee \otimes \mathcal{L}) = 0$, and similarly $\mathrm{Ext}(\mathcal{L}(-x), \mathcal{M}) = 0$

□

In the proof of the next theorem, we will use the Hall algebra $\mathbf{H}_X$ of the elliptic curve X (cf.[2] Section 3.3 and Lemma 4.2.1). Let Y be a smooth projective curve over $\mathbb{F}_q$ of genus g. Let $\mathrm{Coh}(Y)$ be the category of classes of isomorphism of coherent sheaves on Y. The Hall algebra $\mathbf{H}_Y$ of coherent sheaves on Y, as introduced by Kapranov in [66], encodes the extensions of coherent sheaves on Y. Let v be a square root of q^{-1}. The Hall algebra of Y is defined to be the $\mathbb{C}$-vector space

$$\mathbf{H}_Y := \bigoplus_{\mathcal{F}\in \mathrm{Coh}(Y)} \mathbb{C}\mathcal{F}$$

with the product

$$\mathcal{F}\cdot\mathcal{G} = v^{-\langle\mathcal{F},\mathcal{G}\rangle}\sum_{\mathcal{H}} h^{\mathcal{H}}_{\mathcal{F},\mathcal{G}}\mathcal{H}$$

where $\langle\mathcal{F},\mathcal{G}\rangle := \dim_{\mathbb{F}_q}\mathrm{Ext}^0(\mathcal{F},\mathcal{G}) - \dim_{\mathbb{F}_q}\mathrm{Ext}^1(\mathcal{F},\mathcal{G})$ and

$$h^{\mathcal{H}}_{\mathcal{F},\mathcal{G}} := \frac{\left|\{0\longrightarrow\mathcal{G}\longrightarrow\mathcal{H}\longrightarrow\mathcal{F}\longrightarrow 0\}\right|}{|\mathrm{Aut}(\mathcal{F})|\,|\mathrm{Aut}(\mathcal{G})|}.$$

The Riemann-Roch theorem yields

$$\langle\mathcal{F},\mathcal{G}\rangle = (1-g)\mathrm{rk}(\mathcal{F})\mathrm{rk}(\mathcal{G}) + \mathrm{rk}(\mathcal{F})\deg(\mathcal{G}) - \mathrm{rk}(\mathcal{G})\deg(\mathcal{F}).$$

We link the theory of Hall algebras with graphs of Hecke operators as follows. From [2, Lemma 2.1] the quantities $m_{x,r}(\mathcal{E},\mathcal{E}')$ and $h^{\mathcal{E}}_{\mathcal{K}_x^r,\mathcal{E}'}$ are equals, thus we can recover the multiplicities $m_{x,r}(\mathcal{E},\mathcal{E}')$ from the product $\mathcal{K}_x^{\oplus r}\cdot\mathcal{E}'$ in the Hall algebra of Y. Therefore, for a fixed n, the graphs of Hecke operators can be described by calculating explicitly the products $\mathcal{K}_x^{\oplus r}\mathcal{E}'$ where $\mathcal{E}'$ runs through the set of vector bundles of rank n on Y.

Theorem 2.4.2. *Let $x\in|X|$, $\mathcal{M}\in\mathrm{Bun}_2 X$, $\mathcal{L}\in\mathrm{Pic}\, X$ and $\mathcal{M}'\in Bun_2X$ with $m_{x,1}(\mathcal{M},\mathcal{M}') = m\neq 0$. In (i)-(iv) below, we consider the corresponding hypotheses from lemma 2.4.1. We have:*

(i) $m_{x,1}(\mathcal{M}\oplus\mathcal{L},\mathcal{M}'\oplus\mathcal{L})\geq m$,

(ii) $m_{x,1}(\mathcal{M}\oplus\mathcal{L},\mathcal{M}\oplus\mathcal{L}(-x))\geq q_x^2$,

(iii) $m_{x,1}(\mathcal{M}\oplus\mathcal{L},\mathcal{M}'\oplus\mathcal{L})\geq m\cdot q_x$,

(iv) $m_{x,1}(\mathcal{M}\oplus\mathcal{L},\mathcal{M}\oplus\mathcal{L}(-x))\geq 1$.

Proof. Let $\mathcal{K}_x$ be the skyscraper sheaf at x. If $\mathcal{M}_1=\mathcal{M},\ldots,\mathcal{M}_r\in\mathrm{Coh}(X)$ are the extensions of $\mathcal{M}'$ by $\mathcal{K}_x$, $m_i = h^{\mathcal{M}_i}_{\mathcal{K}_x,\mathcal{M}'}$, then $m=m_1$ and

$$\mathcal{K}_x\cdot\mathcal{M}' = v^{2|x|}(m_1\mathcal{M}_1+\cdots+m_r\mathcal{M}_r),$$

$$\mathcal{K}_x\cdot\mathcal{L} = v^{|x|}(\mathcal{L}(x)+\mathcal{L}\oplus\mathcal{K}_x),$$

and

$$\mathcal{K}_x \cdot \mathcal{L}(-x) = v^{|x|}(\mathcal{L} + \mathcal{L}(-x) \oplus \mathcal{K}_x)$$

We denote by $\pi^{vec}(-)$ the vector bundle part in the product (cf. [2, Sec. 3.3]). We have for $\mathcal{E} \in \mathrm{Bun}_n X$,

$$\pi^{vec}(\mathcal{K}_x \cdot \mathcal{E}) = [\mathcal{K}_x, \mathcal{E}]$$

(cf. [2, Lem. 4.2.5]), where the commutator is taken in the Hall algebra $\mathbf{H}_X$.

(i) In $\mathbf{H}_X$ we have $\mathcal{M}' \oplus \mathcal{L} = v^{2\deg\mathcal{L} - \deg\mathcal{M} + |x|}\mathcal{M}' \cdot \mathcal{L}$, since $\mathrm{Ext}(\mathcal{M}', \mathcal{L}) = 0$. Thus,

$$\begin{aligned}
\mathcal{K}_x \cdot (\mathcal{M}' \oplus \mathcal{L}) &= v^{2\deg\mathcal{L} - \deg\mathcal{M} + |x|} \cdot (\mathcal{K}_x \cdot \mathcal{M}') \cdot \mathcal{L} \\
&= v^{2\deg\mathcal{L} - \deg\mathcal{M} + |x|} v^{2|x|}(m_1\mathcal{M}_1 + \cdots + m_r\mathcal{M}_r) \cdot \mathcal{L} \\
&= m v^{2\deg\mathcal{L} - \deg\mathcal{M} + 3|x|}\mathcal{M} \cdot \mathcal{L} + v^{2\deg\mathcal{L} - \deg\mathcal{M} + 3|x|}\left(\sum_{i=2}^{r} m_i\mathcal{M}_i\right) \cdot \mathcal{L} \\
&= m v^{2\deg\mathcal{L} - \deg\mathcal{M} + 3|x|} v^{-(2\deg\mathcal{L} - \deg\mathcal{M})}\mathcal{M} \oplus \mathcal{L} \\
&\quad + v^{2\deg\mathcal{L} - \deg\mathcal{M} + 3|x|}\left(\sum_{i=2}^{r} m_i\mathcal{M}_i\right) \cdot \mathcal{L} \\
&= m v^{3|x|}\mathcal{M} \oplus \mathcal{L} + v^{2\deg\mathcal{L} - \deg\mathcal{M} + 3|x|}(m_2\mathcal{M}_1 + \cdots + m_r\mathcal{M}_r) \cdot \mathcal{L}.
\end{aligned}$$

Therefore, $m_{x,1}(\mathcal{M} \oplus \mathcal{L}, \mathcal{M}' \oplus \mathcal{L}) \geq v^{-3|x|} m v^{3|x|} = m$.

(ii) Since $\mathrm{Ext}(\mathcal{M}, \mathcal{L}(-x)) = 0$, $\mathcal{M} \oplus \mathcal{L}(-x) = v^{2(\deg\mathcal{L} - |x|) - \deg\mathcal{M}}\mathcal{M} \cdot \mathcal{L}(-x)$ in $\mathbf{H}_X$. Thus,

$$\begin{aligned}
\mathcal{K}_x \cdot (\mathcal{M} \oplus \mathcal{L}(-x)) &= v^{2(\deg\mathcal{L} - |x|) - \deg\mathcal{M}}\mathcal{K}_x \cdot \mathcal{M} \cdot \mathcal{L}(-x) \\
&= v^{2(\deg\mathcal{L} - |x|) - \deg\mathcal{M}}(\mathcal{M} \cdot \mathcal{K}_x + [\mathcal{K}_x, \mathcal{M}]) \cdot \mathcal{L}(-x) \\
&= v^{2(\deg\mathcal{L} - |x|) - \deg\mathcal{M}}(\mathcal{M} \cdot \mathcal{K}_x \cdot \mathcal{L}(-x) + [\mathcal{K}_x, \mathcal{M}] \cdot \mathcal{L}(-x)) \\
&= v^{2(\deg\mathcal{L} - |x|) - \deg\mathcal{M}}(\mathcal{M} \cdot v^{|x|}\mathcal{L} + \mathcal{M} \cdot v^{|x|}(\mathcal{K}_x \oplus \mathcal{L}(-x)) \\
&\quad + \pi^{vec}(\mathcal{K}_x \cdot \mathcal{M}) \cdot \mathcal{L}(-x)) \\
&= v^{-|x|}\mathcal{M} \oplus \mathcal{L} + v^{2(\deg\mathcal{L} - |x|) - \deg\mathcal{M}}(\mathcal{M} \cdot v^{|x|}(\mathcal{K}_x \oplus \mathcal{L}(-x)) \\
&\quad + \pi^{vec}(\mathcal{K}_x \cdot \mathcal{M}) \cdot \mathcal{L}(-x)).
\end{aligned}$$

Therefore, $m_{x,1}(\mathcal{M} \oplus \mathcal{L}, \mathcal{M} \oplus \mathcal{L}(-x)) \geq v^{-3|x|}v^{-|x|} = q^{2|x|} = q_x^2$.

(iii) Since $\mathrm{Ext}(\mathcal{L}, \mathcal{M}') = 0$, we have $\mathcal{M}' \oplus \mathcal{L} = v^{\deg\mathcal{M}' - 2\deg\mathcal{L}}\mathcal{L} \cdot \mathcal{M}'$, thus

$$\begin{aligned}
\mathcal{K}_x \cdot (\mathcal{M}' \oplus \mathcal{L}) &= v^{\deg \mathcal{M}' - 2\deg \mathcal{L}} \mathcal{K}_x \cdot \mathcal{L} \cdot \mathcal{M}' \\
&= v^{\deg \mathcal{M} - |x| - 2\deg \mathcal{L}} (\mathcal{L} \cdot \mathcal{K}_x \cdot \mathcal{M}' + [\mathcal{K}_x, \mathcal{L}] \cdot \mathcal{M}') \\
&= v^{\deg \mathcal{M} - |x| - 2\deg \mathcal{L}} (\mathcal{L} \cdot v^{2|x|} \Big(\sum_{i=1}^{r} m_i \mathcal{M}_i \Big) + \pi^{vec}(\mathcal{K}_x \cdot \mathcal{L}) \cdot \mathcal{M}') \\
&= m v^{|x|} \mathcal{L} \oplus \mathcal{M} + v^{\deg \mathcal{M} - |x| - 2\deg \mathcal{L}} (\mathcal{L} \cdot v^{2|x|} \Big(\sum_{i=2}^{r} m_i \mathcal{M}_i \Big) + \pi^{vec}(\mathcal{K}_x \cdot \mathcal{L}) \cdot \mathcal{M}').
\end{aligned}$$

Therefore, $m_{x,1}(\mathcal{M} \oplus \mathcal{L}, \mathcal{M}' \oplus \mathcal{L}) \geq v^{-3|x|} v^{|x|} m = m q_x$.

(iv) Since $\mathrm{Ext}(\mathcal{L}(-x), \mathcal{M}) = 0$, we have $\mathcal{L}(-x) \oplus \mathcal{M} = v^{\deg \mathcal{M} - 2(\deg \mathcal{L} - |x|)} \mathcal{L}(-x) \cdot \mathcal{M}$, and thus in $\mathbf{H}_X$,

$$\begin{aligned}
\mathcal{K}_x \cdot (\mathcal{L}(-x) \oplus \mathcal{M}) &= v^{\deg \mathcal{M} - 2(\deg \mathcal{L} - |x|)} \mathcal{K}_x \cdot \mathcal{L}(-x) \cdot \mathcal{M} \\
&= v^{\deg \mathcal{M} - 2(\deg \mathcal{L} - |x|)} v^{|x|} v^{-(\deg \mathcal{M} - 2\deg \mathcal{L})} \mathcal{L} \oplus \mathcal{M} + \\
&\quad + v^{\deg \mathcal{M} - 2(\deg \mathcal{L} - |x|)} v^{|x|} (\mathcal{K}_x \oplus \mathcal{L}(-x)) \cdot \mathcal{M}
\end{aligned}$$

Therefore, $m_{x,1}(\mathcal{L} \oplus \mathcal{M}, \mathcal{L}(-x) \oplus \mathcal{M}) \geq v^{-3|x|} v^{3|x|} = 1$.

□

2.5 The space of eigenforms

Let $\mathcal{A}(x; \lambda_1, \lambda_2)$ be the space of unramified automorphic forms in $GL_3(\mathbb{A})$ with trivial central character and that are eigenfunctions of $\Phi_{x,1}$ and $\Phi_{x,2}$, with respective eigenvalues λ_1 and λ_2. By Weil's theorem, we can view the automorphic forms in $\mathcal{A}(x; \lambda_1, \lambda_2)$ as functions in $\mathbf{P}\,\mathrm{Bun}_3 X$. In this section, we prove that an $\mathcal{H}_{K_x}$-eigenform in $\mathbf{P}\,\mathrm{Bun}_3 X$ is determined by its values on a finite set of vertices, called the nucleus of the graphs $\mathcal{G}_{x,i}$, $i = 1, 2$. We recall that $\mathcal{G}_{x,i}$ is a shorthand notation for the graph $\mathcal{G}_{\Phi_{x,r},K}$ (cf. Remark 2.2.3).

Definition 2.5.1. We define the invariant d on decomposable vector bundles in $\mathrm{Bun}_3 X$ as follows:

- If $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$ with $\mathcal{M} \in \mathrm{Bun}_2^{ind} X$ and $\mathcal{L} \in \mathrm{Pic}\, X$, we define $d(\mathcal{E}) := 2\deg \mathcal{L} - \deg \mathcal{M}$.
- If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ is a sum of three line bundles with $\deg \mathcal{L}_1 \leq \deg \mathcal{L}_2 \leq \deg \mathcal{L}_3$, we

define

$$\begin{aligned} d_+(\mathcal{E}) &:= 2\deg \mathcal{L}_3 - \deg(\mathcal{L}_1 \oplus \mathcal{L}_2), \\ d_-(\mathcal{E}) &:= 2\deg \mathcal{L}_1 - \deg(\mathcal{L}_2 \oplus \mathcal{L}_3), \\ d_1(\mathcal{E}) &:= \deg(\mathcal{L}_2) - \deg(\mathcal{L}_1), \\ d_2(\mathcal{E}) &:= \deg(\mathcal{L}_3) - \deg(\mathcal{L}_2), \end{aligned}$$

and $d(\mathcal{E}) := max\{d_+(\mathcal{E}), -d_-(\mathcal{E})\}$.

Observe that $d(\mathcal{E})$, $d_+(\mathcal{E})$, $d_-(\mathcal{E})$, $d_1(\mathcal{E})$ and $d_2(\mathcal{E})$ are well defined as functions of $\mathbf{P}\,\mathrm{Bun}_3 X$.

Definition 2.5.2. Let $x \in |X|$. We define the *nucleus* $\mathcal{N}_x$ of $\mathcal{G}_{x,i}$ for $i = 1, 2$, as the set of vertices $\mathcal{E}$ such that

- $\mathcal{E}$ is indecomposable or,
- $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$, $\mathcal{M} \in \mathrm{Bun}_2^{ind} X$, $\mathcal{L} \in \mathrm{Pic}\, X$, $\delta(\mathcal{M}) < 0$ and

$$\delta(\mathcal{M}) + 1 - 2|x| \leq d(\mathcal{E}) \leq -\delta(\mathcal{M}) - 1 + 2|x| \text{ or,}$$

- $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$, $\mathcal{M} \in \mathrm{Bun}_2^{ind} X$, $\mathcal{L} \in \mathrm{Pic}\, X$, $\delta(\mathcal{M}) = 0$ and $-2|x| \leq d(\mathcal{E}) \leq 2|x|$ or,
- $\mathcal{E}$ is a sum of 3 line bundles and $d_1(\mathcal{E}), d_2(\mathcal{E}) \leq |x|$ or ($|x| < d_1(\mathcal{E}) \leq 2|x|$ and $d_1(\mathcal{E}) - |x| < d_2(\mathcal{E}) \leq d_1(\mathcal{E})$) or ($|x| < d_2(\mathcal{E}) \leq 2|x|$ and $d_2(\mathcal{E}) - |x| < d_1(\mathcal{E}) \leq d_2(\mathcal{E})$).

Observe that $\mathcal{N}_x$ is invariant under the involution $\mathcal{E} \mapsto \mathcal{E}^\vee$.

Theorem 2.4.2 allows us to describe the neighborhoods of vertices that are not in the nucleus.

Theorem 2.5.3. *Let $\mathcal{E} \in \mathrm{Bun}_3 X$.*

1. *If $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$ with $\mathcal{M} \in \mathrm{Bun}_2 X$ and $\mathcal{L} \in \mathrm{Pic}\, X$ such that either $\mathcal{M} \in \mathrm{Bun}_2^{ind} X$ and $d(\mathcal{E}) > 2|x|$ or $\mathcal{M} = \mathcal{L}_1 \oplus \mathcal{L}_2$ for $\mathcal{L}_i \in \mathrm{Pic}\, X$ with $\deg \mathcal{L}_1 \leq \deg \mathcal{L}_2 \leq \deg \mathcal{L}$ and $d_2(\mathcal{E}) > |x|$, then*

$$\mathcal{V}_{x,1}(\mathcal{E}) = \{(\mathcal{E}, \mathcal{M} \oplus \mathcal{L}(-x), q_x^2)\} \cup \{(\mathcal{E}, \mathcal{M}' \oplus \mathcal{L}, m) \mid (\mathcal{M}, \mathcal{M}', m) \in \mathcal{V}_{x,1}(\mathcal{M})\}.$$

2. *If $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$ with $\mathcal{M} \in \mathrm{Bun}_2 X$ and $\mathcal{L} \in \mathrm{Pic}\, X$ such that either $\mathcal{M} \in \mathrm{Bun}_2^{ind} X$ and $d(\mathcal{E}) < -2|x|$ or $\mathcal{M} = \mathcal{L}_1 \oplus \mathcal{L}_2$ for $\mathcal{L}_i \in \mathrm{Pic}\, X$ with $\deg \mathcal{L} \leq \deg \mathcal{L}_1 \leq \deg \mathcal{L}_2$ and $d_1(\mathcal{E}) > |x|$, then*

$$\mathcal{V}_{x,1}(\mathcal{E}) = \{(\mathcal{E}, \mathcal{M} \oplus \mathcal{L}(-x), 1)\} \cup \{(\mathcal{E}, \mathcal{M}' \oplus \mathcal{L}, mq_x) \mid (\mathcal{M}, \mathcal{M}', m) \in \mathcal{V}_{x,1}(\mathcal{M})\}.$$

To describe the neighborhoods of a vertex outside of the nucleus of the graph $\mathcal{G}_{x,2}$ we need a duality theorem.

Theorem 2.5.4. *Let $\mathcal{E}, \mathcal{E}' \in \operatorname{Bun}_n X$, then:*

(i) $m_{x,r}(\mathcal{E}, \mathcal{E}') \neq 0 \Longleftrightarrow m_{x,r}(\mathcal{E}'^{\vee}, \mathcal{E}^{\vee}) \neq 0,$

(ii) $m_{x,r}(\mathcal{E}, \mathcal{E}') \neq 0 \Longleftrightarrow m_{x,n-r}(\mathcal{E}', \mathcal{E}(-x)) \neq 0,$

(iii) $m_{x,r}(\mathcal{E}, \mathcal{E}') = m_{x,n-r}(\mathcal{E}^{\vee}, \mathcal{E}'^{\vee}(-x)).$

Proof. We begin with the proof of part (i). Consider an exact sequence

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{K}_x^r \longrightarrow 0.$$

Using the long exact sequence in cohomology, we obtain

$$0 \longrightarrow \mathcal{H}om(\mathcal{K}_x^r, \mathcal{O}_X) \longrightarrow \mathcal{E}^{\vee} \longrightarrow \mathcal{E}'^{\vee} \longrightarrow \mathcal{E}xt^1(\mathcal{K}_x^r, \mathcal{O}_X) \longrightarrow \cdots$$

We have $\mathcal{H}om(\mathcal{K}_x^r, \mathcal{O}_X) = 0$, $\mathcal{E}xt^1(\mathcal{K}_x^r, \mathcal{O}_X) = \mathcal{K}_x^r$ and $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_X) = 0$, thus the connecting homomorphism is surjective. So we have constructed an exact sequence

$$0 \longrightarrow \mathcal{E}^{\vee} \longrightarrow \mathcal{E}'^{\vee} \longrightarrow \mathcal{K}_x^r \longrightarrow 0,$$

and therefore $m_{x,r}(\mathcal{E}'^{\vee}, \mathcal{E}^{\vee}) \neq 0$.

Next we prove part (ii). Consider an exact sequence

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{K}_x^r \longrightarrow 0.$$

We denote by φ the map from $\mathcal{E}'$ to $\mathcal{E}$ and by $\mathcal{I}_x$ the ideal sheaf of the point $x \in X$, i.e. $\mathcal{I}_x = Ker(\mathcal{O}_X \longrightarrow \mathcal{K}_x)$. We see that $\mathcal{I}_x\mathcal{E} \subset Im(\varphi)$, and if we denote by $\varphi^* : \mathcal{I}_x\mathcal{E} \longrightarrow \mathcal{E}'$ the inverse of φ, then $Coker(\varphi^*) \simeq \mathcal{K}_x^{n-r}$. Indeed, let U be an affine neighborhood of x such that $A = \mathcal{O}_X(U)$ is a principal ideal domain. The point x corresponds to a prime element $\pi_x \in A$ and $\mathcal{I}_x$ corresponds to the ideal (π_x). The exact sequence restricted to U corresponds to an exact sequence of modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow (A/\pi_x)^r \longrightarrow 0,$$

with $M \simeq A^n$. By the structure of modules over a principal domain, we see that there exists a basis $\{m_1, \ldots, m_n\}$ of M such that $\{\pi_x m_1, \ldots, \pi_x m_r, m_{r+1} \ldots, m_n\}$ is a basis of M'. It follows that $(\pi_x)M \subset M'$ and that the map φ^* corresponds to the inclusion of $(\pi_x)M$ in M'. Therefore $M'/(\pi_x)M \simeq (A/\pi_x)^{n-r}$, which proves that $Coker(\varphi^*) \simeq \mathcal{K}_x^{n-r}$.

We also have $\mathcal{O}_X(-x) = \mathcal{I}_x$ and $\mathcal{I}_x\mathcal{E} = \mathcal{E}(-x)$. Thus we obtain an exact sequence

$$0 \longrightarrow \mathcal{E}(-x) \longrightarrow \mathcal{E}' \longrightarrow \mathcal{K}_x^{n-r} \longrightarrow 0,$$

which proves that $m_{x,n-r}(\mathcal{E}', \mathcal{E}(-x)) \neq 0$.

We denote the functor from part (i) by $(-)^\vee$ and the functor from part (ii) by $(-)^*$. Composition yields a functor $((-)^\vee)^*$, which is well defined on isomorphism classes of short exact sequences with fixed middle term. This gives us a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes} \\ 0 \longrightarrow \mathcal{E}'' \longrightarrow \mathcal{E} \longrightarrow \mathcal{K}_x^r \longrightarrow 0 \\ \text{with fixed } \mathcal{E} \text{ and } \mathcal{E}'' \simeq \mathcal{E}' \end{array} \right\} \stackrel{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{isomorphism classes} \\ 0 \longrightarrow \mathcal{E}''' \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{K}_x^{n-r} \longrightarrow 0 \\ \text{with fixed } \mathcal{E}^\vee \text{ and } \mathcal{E}''' \simeq \mathcal{E}'^\vee(-x) \end{array} \right\}$$

This proves that $m_{x,r}(\mathcal{E},\mathcal{E}') = m_{x,n-r}(\mathcal{E}^\vee, \mathcal{E}'^\vee(-x))$. □

Using the theorem, we obtain

$$m_{x,2}(\mathcal{E},\mathcal{E}') = m_{x,1}(\mathcal{E}^\vee, \mathcal{E}'^\vee(-x)), \tag{2.5.1}$$

and using the calculation of the multiplicities of $\mathcal{G}_{x,1}$ in Theorem 2.5.3, we obtain the following

Theorem 2.5.5. *Let $\mathcal{E} \in \operatorname{Bun}_3 X$.*

1. *If $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$ with $\mathcal{M} \in \operatorname{Bun}_2 X$ and $\mathcal{L} \in \operatorname{Pic} X$ such that either $\mathcal{M} \in \operatorname{Bun}_2^{ind} X$ with $d(\mathcal{E}) > 2|x|$ or $\mathcal{M} = \mathcal{L}_1 \oplus \mathcal{L}_2$ with $\mathcal{L}_i \in \operatorname{Pic} X$, $\deg \mathcal{L}_1 \leq \deg \mathcal{L}_2 \leq \deg \mathcal{L}$ and $d_2(\mathcal{E}) > |x|$, then*

$$\mathcal{V}_{x,2}(\mathcal{E}) = \{(\mathcal{E}, \mathcal{M}(-x) \oplus \mathcal{L}, 1)\} \cup \{(\mathcal{E}, (\mathcal{M}'^\vee \oplus \mathcal{L})(-x), m q_x) \mid (\mathcal{M}^\vee, \mathcal{M}', m) \in \mathcal{V}_{x,1}(\mathcal{M}^\vee)\}.$$

2. *If $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$ with $\mathcal{M} \in \operatorname{Bun}_2 X$ and $\mathcal{L} \in \operatorname{Pic} X$ such that either $\mathcal{M} \in \operatorname{Bun}_2^{ind} X$ with $d(\mathcal{E}) < -2|x|$ or $\mathcal{M} = \mathcal{L}_1 \oplus \mathcal{L}_2$ with $\mathcal{L}_i \in \operatorname{Pic} X$, $\deg \mathcal{L} \leq \deg \mathcal{L}_1 \leq \deg \mathcal{L}_2$ and $d_1(\mathcal{E}) > |x|$, then*

$$\mathcal{V}_{x,2}(\mathcal{E}) = \{(\mathcal{E}, \mathcal{M}(-x) \oplus \mathcal{L}, q_x^2)\} \cup \{(\mathcal{E}, (\mathcal{M}'^\vee \oplus \mathcal{L})(-x), m) \mid (\mathcal{M}^\vee, \mathcal{M}', m) \in \mathcal{V}_{x,1}(\mathcal{M}^\vee)\}.$$

In what follows, k-(sub)bundle means a (sub)bundle of rank k. We review the definition of the δ invariant given by Alvarenga in [2]. For a subbundle $\mathcal{M}$ of a n-bundle $\mathcal{E}$, we define

$$\delta(\mathcal{M},\mathcal{E}) := rk(\mathcal{E}) \deg(\mathcal{M}) - rk(\mathcal{M}) \deg(\mathcal{E})$$

and for $k = 1, \ldots, n-1$,

$$\delta_k(\mathcal{E}) := \sup_{\substack{\mathcal{M} \hookrightarrow \mathcal{E} \\ k-\text{subbundle}}} \delta(\mathcal{M},\mathcal{E}).$$

The δ-*invariant* is defined by

$$\delta(\mathcal{E}) := \max\{\delta_1(\mathcal{E}), \ldots, \delta_{n-1}(\mathcal{E})\}.$$

By Proposition 2.4.4 in [2],

$$-ng \leq \delta(\mathcal{E}) < \infty,$$

for every n-bundle $\mathcal{E}$, where g is the genus of the curve X. Let $\mathcal{M}$ be a k-subbundle of $\mathcal{E}$, we say that $\mathcal{M}$ is *maximal* if $\delta_k(\mathcal{E}) = \delta(\mathcal{M}, \mathcal{E})$.

We can use (2.5.1) to improve [2, Thm. 2.4.15] as follows.

Theorem 2.5.6. *Let $\mathcal{E}'$ be a neighbor of $\mathcal{E}$ in $\mathcal{G}_{x,1}$, then:*

- $\delta_1(\mathcal{E}') \in \{\delta_1(\mathcal{E}) - 2|x|, \ldots, \delta_1(\mathcal{E}) + |x|\}$ *and* $\delta_1(\mathcal{E}') - \delta_1(\mathcal{E}) \equiv |x| \pmod 3$,
- $\delta_2(\mathcal{E}') \in \{\delta_2(\mathcal{E}) - |x|, \ldots, \delta_2(\mathcal{E}) + 2|x|\}$, *and* $\delta_2(\mathcal{E}') - \delta_2(\mathcal{E}) \equiv 2|x| \pmod 3$.

Let $\mathcal{E}'$ be a neighbor of $\mathcal{E}$ in $\mathcal{G}_{x,2}$, then:

- $\delta_1(\mathcal{E}') \in \{\delta_1(\mathcal{E}) - |x|, \ldots, \delta_1(\mathcal{E}) + 2|x|\}$, *and* $\delta_1(\mathcal{E}') - \delta_1(\mathcal{E}) \equiv 2|x| \pmod 3$,
- $\delta_2(\mathcal{E}') \in \{\delta_2(\mathcal{E}) - 2|x|, \ldots, \delta_2(\mathcal{E}) + |x|\}$, *and* $\delta_2(\mathcal{E}') - \delta_2(\mathcal{E}) \equiv |x| \pmod 3$.

Proof. Let $\mathcal{M}$ be a 2-subbundle of $\mathcal{E}$ and put $\mathcal{L} = \mathcal{E}/\mathcal{M}$. Then $\mathcal{L}$ is a line bundle and, by duality, $\mathcal{L}^\vee$ is a line subbundle of $\mathcal{E}^\vee$ with quotient $\mathcal{M}^\vee$. We see that to maximize $\deg \mathcal{M}$ is equivalent with maximizing $\deg \mathcal{L}^\vee$. Let $\mathcal{M}$ as above such that $\deg \mathcal{L}^\vee$ attains the maximum, then $\deg \mathcal{L}^\vee = (\deg \mathcal{E}^\vee + \delta_1(\mathcal{E}^\vee))/3$ and $\deg \mathcal{M} = \deg \mathcal{E} + \deg \mathcal{L}^\vee$, which implies

$$\delta_2(\mathcal{E}) = 3 \deg \mathcal{M} - 2 \deg \mathcal{E} = \delta_1(\mathcal{E}^\vee).$$

If $\mathcal{E}'$ is a neighbor of $\mathcal{E}$ in $\mathcal{G}_{x,2}$, then [2, Thm. 2.4.15] and (2.5.1) implies that

$$\delta_2(\mathcal{E}') = \delta_1(\mathcal{E}'^\vee) = \delta_1(\mathcal{E}'^\vee(-x)) \in \{\delta_2(\mathcal{E}) - 2|x|, \ldots, \delta_2(\mathcal{E}) + |x|\}.$$

Analogously we prove that $\delta_2(\mathcal{E}') \in \{\delta_2(\mathcal{E}) - |x|, \ldots, \delta_2(\mathcal{E}) + 2|x|\}$ if $\mathcal{E}'$ is a neighbor of $\mathcal{E}$ in $\mathcal{G}_{x,1}$. This together with [2, Thm. 2.4.15] finishes the proof. □

Next we want to compare the invariant d for decomposable vector bundles with the δ-invariant.

Convention: Until the end of this section, if $\mathcal{E}$ is a sum of line bundles and we write a decomposition as a sum of line bundles, $\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_k$, we assume that

$$\deg \mathcal{L}_1 \leq \cdots \leq \deg \mathcal{L}_k.$$

Let $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ be a sum of three line bundles. If $\mathcal{L}$ is a line subbundle of $\mathcal{E}$, then there is a factor $\mathcal{L}_i$ of $\mathcal{E}$ such that $\mathcal{L} \to \mathcal{E} \to \mathcal{L}_i$ is not zero. Thus we get an immersion $\mathcal{L} \hookrightarrow \mathcal{L}_i$, which implies $\deg \mathcal{L} \leq \deg \mathcal{L}_i$. Therefore

$$\delta_1(\mathcal{E}) = 3 \deg \mathcal{L}_3 - \deg \mathcal{E} = 2 \deg \mathcal{L}_3 - \deg(\mathcal{L}_1 \oplus \mathcal{L}_2) = d_+(\mathcal{E}).$$

If $\mathcal{M}$ is a 2-subbundle of $\mathcal{E}$ and $\mathcal{L} = \mathcal{E}/\mathcal{M}$, then we obtain an immersion $\mathcal{L}^\vee \hookrightarrow \mathcal{E}^\vee$, which implies $\deg \mathcal{L}^\vee \leq \deg \mathcal{L}_i^\vee$ for some i. Therefore $\deg \mathcal{M}$ will be maximal if and only if $\deg \mathcal{L}^\vee$ is maximal. We conclude that $\mathcal{M} = \mathcal{L}_2 \oplus \mathcal{L}_3$ is maximal and thus

$$\delta_2(\mathcal{E}) = 3\deg(\mathcal{L}_2 \oplus \mathcal{L}_3) - 2\deg \mathcal{E} = \deg(\mathcal{L}_2 \oplus \mathcal{L}_3) - 2\deg \mathcal{L}_1 = -d_-(\mathcal{E}).$$

Therefore $\delta(\mathcal{E}) = d(\mathcal{E})$.

For $\mathcal{M} \in \operatorname{Bun}_2 X$ we define,

$$m(\mathcal{M}) := \frac{\deg \mathcal{M} + \delta(\mathcal{M})}{2},$$

which is the degree of a maximal line bundle of $\mathcal{M}$.

Let $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$ with $\mathcal{M} \in \operatorname{Bun}_2^{ind} X$, $\mathcal{L} \in \operatorname{Pic} X$ and $\deg \mathcal{L} \geq \deg \mathcal{M} - m(\mathcal{M})$. It follows from $\delta(\mathcal{M}) \leq 0$ that $m(\mathcal{M}) + \delta(\mathcal{M}) \leq m(\mathcal{M}) \leq \deg \mathcal{M} - m(\mathcal{M})$. So $\deg \mathcal{L} \geq m(\mathcal{M})$, which implies

$$\delta_1(\mathcal{E}) = 3\deg \mathcal{L} - \deg \mathcal{E} = 2\deg \mathcal{L} - \deg \mathcal{M} = d(\mathcal{E}).$$

Remark 2.5.7. If $\deg \mathcal{L} > m(\mathcal{M})$, then there is a unique maximal line bundle $\mathcal{L}' \hookrightarrow \mathcal{E}$, namely $\mathcal{L}' = \mathcal{L}$.

Observe that $m(\mathcal{M}^\vee) = m(\mathcal{M}) - \deg \mathcal{M}$. It follows from $\deg \mathcal{L} \geq \deg \mathcal{M} - m(\mathcal{M})$ that $\deg \mathcal{L}^\vee \leq m(\mathcal{M}^\vee)$, and we conclude that if $\mathcal{L}'$ is a maximal line subbundle of $\mathcal{M}^\vee$, then $\mathcal{L}'$ is a maximal line subbundle of $\mathcal{E}^\vee$. By duality, we conclude that $\mathcal{E}$ has a maximal 2-subbundle of the form $\mathcal{M} = \mathcal{L}'' \oplus \mathcal{L}$ with $\mathcal{L}''$ a maximal line bundle in $\mathcal{M}$. Therefore

$$\delta_2(\mathcal{E}) = 3\deg \mathcal{M} - 2\deg \mathcal{E} = \deg \mathcal{L} + 3m(\mathcal{M}) - 2\deg \mathcal{M}.$$

We have

$\delta_1(\mathcal{E}) - \delta_2(\mathcal{E}) = \deg \mathcal{L} + \deg \mathcal{M} - 3m(\mathcal{M}) \geq 2(\deg \mathcal{M} - 2m(\mathcal{M})) = -2\delta(\mathcal{M}) \geq 0.$
Therefore $\delta(\mathcal{E}) = \delta_1(\mathcal{E}) = d(\mathcal{E})$.

Let $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$ with $\mathcal{M} \in \operatorname{Bun}_2^{ind} X$, $\mathcal{L} \in \operatorname{Pic}(X)$ and $\deg \mathcal{L} \leq m(\mathcal{M})$. As $\deg \mathcal{L} \leq m(\mathcal{M})$, we obtain

$$\delta_1(\mathcal{E}) = 3m(\mathcal{M}) - \deg \mathcal{E} = 3m(\mathcal{M}) - \deg \mathcal{M} - \deg \mathcal{L}.$$

Also $\deg \mathcal{L} \leq \deg \mathcal{M} - m(\mathcal{M})$, which implies $\deg \mathcal{L}^\vee \geq m(\mathcal{M}^\vee)$. Thus $\mathcal{L}^\vee$ is a maximal line subbundle of $\mathcal{E}^\vee$. By duality, we conclude that $\mathcal{M}$ is a maximal 2-subbundle in $\mathcal{E}$, which implies

$$\delta_2(\mathcal{E}) = 3\deg \mathcal{M} - 2\deg \mathcal{E} = \deg \mathcal{M} - 2\deg \mathcal{L} = |d(\mathcal{E})|.$$

We have

$\delta_2(\mathcal{E}) - \delta_1(\mathcal{E}) = 2\deg \mathcal{M} - \deg \mathcal{L} - 3m(\mathcal{M}) \geq 2(\deg \mathcal{M} - 2m(\mathcal{M})) = -2\delta(M) \geq 0.$
Therefore $\delta(\mathcal{E}) = \delta_2(\mathcal{E}) = |d(\mathcal{E})|$.

Remark 2.5.8. If $\deg \mathcal{L} < \deg \mathcal{M} - m(\mathcal{M})$, then there is a unique maximal 2-subbundle of $\mathcal{E}$, which is $\mathcal{M}$.

So we have proved the following.

Theorem 2.5.9. *Let X be a curve corresponding to the global function field F and $\mathcal{E}$ be a decomposable vector bundle.*

1. *We have $d(\mathcal{E}) = \delta(\mathcal{E})$ if $\mathcal{E}$ is a sum of three line bundles.*
2. *We have $\delta(\mathcal{E}) \geq |d(\mathcal{E})|$.*
3. *Let X be an elliptic curve and $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$ with $\mathcal{M} \in \operatorname{Bun}_2^{ind} X$ and $\mathcal{L} \in \operatorname{Pic} X$. If $d(\mathcal{E}) \leq \delta(\mathcal{M})$ or $d(\mathcal{E}) \geq -\delta(\mathcal{M})$, then $\delta(\mathcal{E}) = |d(\mathcal{E})|$.*

Corollary 2.5.10. *Let $k \in \mathbb{Z}$ and X be an elliptic curve. There is only a finite number of $\mathcal{E} \in \mathbf{P}\operatorname{Bun}_3 X$ with $\delta(\mathcal{E}) \leq k$.*

Proof. From the inequality $|d(\mathcal{E})| \leq \delta(\mathcal{E})$ for $\mathcal{E}$ decomposable, we conclude that it is sufficient to prove that the action of $\operatorname{Pic} X$ in $\operatorname{Bun}_n^{ind} X$ has a finite number of orbits for $n \leq 3$. And in fact, by Atiyah's theorem, every orbit has a representative of the form $\mathcal{E}_{(x,l)}^{(n,d)}$ with $0 \leq d < n$ and $|x| \leq n$ (for every $n \geq 1$). As the number of these representatives is finite, it follows that the number of orbits is finite. □

Remarks 2.5.7 and 2.5.8 describe cases of 3-bundles $\mathcal{E}$ that have a unique maximal line bundle or 2-subbundle. In the following we relate this property with the neighbours of $\mathcal{E}$, which will be useful for the proof of the main theorem of this section.

Let $\mathcal{E} \in \operatorname{Bun}_n X$ be a fixed vector bundle. We write $\overline{\mathcal{E}} := \mathcal{E}/\mathcal{I}_x\mathcal{E}$, which is a torsion sheaf with support on x and stalk isomorphic to $\kappa(x)^n$. For $\mathcal{F} \subset \mathcal{E}$ a subbundle or neighbour of $\mathcal{E}$ in $\mathcal{G}_{x,i}$, we denote by $\overline{\mathcal{F}}$ the image of $\mathcal{F}$ in $\overline{\mathcal{E}}$. If $\mathcal{F}$ is a k-subbundle, then $\overline{\mathcal{F}} \simeq \kappa(x)^k$. If $\mathcal{E}'$ is a neighbour of $\mathcal{E}$ in $\mathcal{G}_{x,i}$, then $\mathcal{I}_x\mathcal{E} \subset \mathcal{E}'$, $\overline{\mathcal{E}'} \simeq \kappa(x)^{n-i}$ and if $p : \mathcal{E} \longrightarrow \overline{\mathcal{E}}$ is the projection, then $\mathcal{E}' = p^{-1}(\overline{\mathcal{E}'})$. Therefore if $\mathcal{F}$ is a $(n-i)$-subbundle of $\mathcal{E}$, then $\mathcal{F}$ is a subsheaf of $\mathcal{E}'$ if and only if $\overline{\mathcal{F}} = \overline{\mathcal{E}'}$.

Remark 2.5.11. Consider either $\mathcal{E}$ to be a sum of 3 line bundles with $d_1(\mathcal{E}) > 0$ and $d_1(\mathcal{E}) \geq d_2(\mathcal{E})$ or $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$ with $\mathcal{M} \in \operatorname{Bun}_2^{ind} X$, $\mathcal{L} \in \operatorname{Pic} X$, $\deg \mathcal{L} \leq m(\mathcal{M})$ and $\deg \mathcal{L} < \deg \mathcal{M} - m(\mathcal{M})$. By the above discussion and Remark 2.5.8, $\delta(\mathcal{E}) = \delta_2(\mathcal{E})$ and there is a unique neighbour $\mathcal{E}'$ of $\mathcal{E}$ in $\mathcal{G}_{x,1}$ with $\delta_2(\mathcal{E}') = \delta(\mathcal{E}') = \delta(\mathcal{E}) + 2|x|$. For the other neighbours $\mathcal{E}''$, we have $\delta(\mathcal{E}'') < \delta(\mathcal{E}')$. If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$, then $\mathcal{E}' = \mathcal{L}_1(-x) \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$; and if $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$ with $\mathcal{M} \in \operatorname{Bun}_2^{ind} X$ and $\mathcal{L} \in \operatorname{Pic} X$, then $\mathcal{E}' = \mathcal{M} \oplus \mathcal{L}(-x)$.

Remark 2.5.12. Consider either $\mathcal{E}$ to be a sum of 3 line bundles with $d_2(\mathcal{E}) > 0$ and $d_2(\mathcal{E}) \geq d_1(\mathcal{E})$ or $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$ with $\mathcal{M} \in \operatorname{Bun}_2^{ind} X$, $\mathcal{L} \in \operatorname{Pic} X$, $\deg \mathcal{L} \geq \deg \mathcal{M} - m(\mathcal{M})$ and $\deg \mathcal{L} > m(\mathcal{M})$. By the above discussion and Remark 2.5.7, $\delta(\mathcal{E}) = \delta_1(\mathcal{E})$ and there is a unique neighbour $\mathcal{E}'$ of $\mathcal{E}$ in $\mathcal{G}_{x,2}$ with $\delta_1(\mathcal{E}') = \delta(\mathcal{E}') = \delta(\mathcal{E}) + 2|x|$. For the other neighbours $\mathcal{E}''$ of $\mathcal{E}$ we have $\delta(\mathcal{E}'') < \delta(\mathcal{E}')$. If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$, then $\mathcal{E}' = \mathcal{L}_1(-x) \oplus \mathcal{L}_2(-x) \oplus \mathcal{L}_3$; and if $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$ with $\mathcal{M} \in \operatorname{Bun}_2^{ind} X$ and $\mathcal{L} \in \operatorname{Pic} X$, then $\mathcal{E}' = \mathcal{M}(-x) \oplus \mathcal{L}$.

We are prepared to prove the main theorem of this section.

Theorem 2.5.13. *If $f, g \in \mathcal{A}(x; \lambda_1, \lambda_2)$ and $f|_{\mathcal{N}_x} = g|_{\mathcal{N}_x}$, then $f = g$.*

Proof. We describe 6 relations between values of f on vertices $\mathcal{E} \in \mathbf{P}\operatorname{Bun}_3 X$ away from the nucleus, i.e. $|d(\mathcal{E})|$ large, that allows us to express $f(\mathcal{E})$ as a linear combination of $f(\mathcal{E}')$ on vertices $\mathcal{E}'$ closest to the nucleus.

Relation (i): If $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$ with $\mathcal{M} \in \operatorname{Bun}_2^{ind} X$, $\mathcal{L} \in \operatorname{Pic} X$, $d(\mathcal{E}) \leq \delta(\mathcal{M}) - 2|x|$ and $d(\mathcal{E}) < -\delta(\mathcal{M}) - 2|x|$, then we apply $\mathcal{G}_{x,1}$ to the vertex $\mathcal{M} \oplus \mathcal{L}(x)$, which implies

$$\lambda_1 f(\mathcal{M} \oplus \mathcal{L}(x)) = m_1 f(\mathcal{M} \oplus \mathcal{L}) + \sum_{\mathcal{E}'} m' f(\mathcal{E}')$$

where $(\mathcal{M}\oplus\mathcal{L}(x), \mathcal{E}', m') \in \mathcal{V}_{x,1}(\mathcal{M}\oplus\mathcal{L}(x))$ and $\mathcal{E}' \neq \mathcal{M}\oplus\mathcal{L}$. By Remark 2.5.11, $\delta(\mathcal{E}') < \delta(\mathcal{E})$. This allows us to express $f(\mathcal{E})$ as a linear combination of $f(\mathcal{E}')$ with $\delta(\mathcal{E}') < \delta(\mathcal{E})$.

Relation (ii): If $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$ with $\mathcal{M} \in \operatorname{Bun}_2^{ind} X$, $\mathcal{L} \in \operatorname{Pic} X$, $d(\mathcal{E}) \geq -\delta(\mathcal{M}) + 2|x|$ and $d(\mathcal{E}) > \delta(\mathcal{M}) + 2|x|$, then we apply $\mathcal{G}_{x,2}$ to the vertex $\mathcal{M}(x) \oplus \mathcal{L}$, which implies

$$\lambda_2 f(\mathcal{M}(x) \oplus \mathcal{L}) = m_1 f(\mathcal{M} \oplus \mathcal{L}) + \sum_{\mathcal{E}'} m' f(\mathcal{E}')$$

where $(\mathcal{M}(x)\oplus\mathcal{L}, \mathcal{E}', m') \in \mathcal{V}_{x,2}(\mathcal{M}(x)\oplus\mathcal{L})$ and $\mathcal{E}' \neq \mathcal{M}\oplus\mathcal{L}$. By Remark 2.5.12, $\delta(\mathcal{E}') < \delta(\mathcal{E})$. This allows us to express $f(\mathcal{E})$ as a linear combination of $f(\mathcal{E}')$ with $\delta(\mathcal{E}') < \delta(\mathcal{E})$.

Relation (iii): If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ with $\mathcal{L}_i \in \operatorname{Pic} X$, $d_1(\mathcal{E}) > |x|$ and $d_1(\mathcal{E}) - |x| \geq d_2(\mathcal{E})$, we apply $\mathcal{G}_{x,1}$ to the vertex $\mathcal{L}_1(x) \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$, which implies

$$\lambda_1 f(\mathcal{L}_1(x) \oplus \mathcal{L}_2 \oplus \mathcal{L}_3) = m_1 f(\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3) + \sum_{\mathcal{E}'} m' f(\mathcal{E}')$$

where $(\mathcal{L}_1(x) \oplus \mathcal{L}_2 \oplus \mathcal{L}_3, \mathcal{E}', m') \in \mathcal{V}_{x,1}(\mathcal{L}_1(x) \oplus \mathcal{L}_2 \oplus \mathcal{L}_3)$ and $\mathcal{E}' \neq \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$. By Remark 2.5.11, $\delta(\mathcal{E}') < \delta(\mathcal{E})$. This allows us to express $f(\mathcal{E})$ as a linear combination of $f(\mathcal{E}')$ with $\delta(\mathcal{E}') < \delta(\mathcal{E})$.

Relation (iv): If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ with $\mathcal{L}_i \in \operatorname{Pic} X$, $d_2(\mathcal{E}) > |x|$ and $d_2(\mathcal{E}) - |x| \geq d_1(\mathcal{E})$, then we apply $\mathcal{G}_{x,2}$ to the vertex $\mathcal{L}_1(x) \oplus \mathcal{L}_2(x) \oplus \mathcal{L}_3$, which implies

$$\lambda_2 f(\mathcal{L}_1(x) \oplus \mathcal{L}_2(x) \oplus \mathcal{L}_3) = m_1 f(\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3) + \sum_{\mathcal{E}'} m' f(\mathcal{E}')$$

where $(\mathcal{L}_1(x) \oplus \mathcal{L}_2(x) \oplus \mathcal{L}_3, \mathcal{E}', m') \in \mathcal{V}_{x,2}(\mathcal{L}_1(x) \oplus \mathcal{L}_2(x) \oplus \mathcal{L}_3)$ and $\mathcal{E}' \neq \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$. By Remark 2.5.12, $\delta(\mathcal{E}') < \delta(\mathcal{E})$. This allows us to express $f(\mathcal{E})$ as a linear combination of $f(\mathcal{E}')$ with $\delta(\mathcal{E}') < \delta(\mathcal{E})$.

Relation (v): If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ with $\mathcal{L}_i \in \operatorname{Pic} X$, $d_2(\mathcal{E}) > 2|x|$ and $d_2(\mathcal{E}) \geq d_1(\mathcal{E})$, then we apply $\mathcal{G}_{x,2}$ to the vertex $\mathcal{M} \oplus \mathcal{L}(-x)$, where $\mathcal{M} = \mathcal{L}_1 \oplus \mathcal{L}_2$ and $\mathcal{L} = \mathcal{L}_3$. We have $d_2(\mathcal{M} \oplus \mathcal{L}(-x)) > |x|$ and $d(\mathcal{E}) = d_+(\mathcal{E})$. By Theorem 2.5.5, we obtain

$$\lambda_2 f(\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3(-x)) = f(\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3) + \sum_{\mathcal{M}'} m q_x f(\mathcal{M}'^{\vee} \oplus \mathcal{L}_3(-x))$$

where $(\mathcal{M}^{\vee}, \mathcal{M}', m) \in \mathcal{V}_{x,2}(\mathcal{M}^{\vee})$. We have $d(\mathcal{M} \oplus \mathcal{L}(-x)) < d(\mathcal{E})$ by the definition of the invariant d. Let $\mathcal{M}'$ be a neighbor of $\mathcal{M}^{\vee}$ in $\mathcal{G}_x$. If $\mathcal{M}' \in \operatorname{Bun}_2^{ind} X$, then $d(\mathcal{M}'^{\vee} \oplus \mathcal{L}) =$

$d(\mathcal{E}) - |x|$. If $\mathcal{M}'$ is a sum of two line bundles, then the situation is more complex. In this case, we can write $\mathcal{M}'^{\vee} = \mathcal{L}'_1 \oplus \mathcal{L}'_2$ with $\deg \mathcal{L}'_i \geq \deg \mathcal{L}_i$ for $i = 1, 2$ and we have $\deg \mathcal{M}'^{\vee} = \deg \mathcal{M} + |x|$. For $\mathcal{E}' = \mathcal{L}'_1 \oplus \mathcal{L}'_2 \oplus \mathcal{L}(-x)$, we have $d_1(\mathcal{E}') + d_2(\mathcal{E}') < d_1(\mathcal{E}) + d_2(\mathcal{E})$ and $d_2(\mathcal{E}') < d_2(\mathcal{E})$, which implies $d_+(\mathcal{E}') < d(\mathcal{E})$. We differentiate two cases:

- If $\deg \mathcal{L}(-x) \geq \deg \mathcal{L}'_i$, $i = 1, 2$, then $d_1(\mathcal{E}') + d_2(\mathcal{E}') \leq d_1(\mathcal{E}) + d_2(\mathcal{E}) - |x|$ and $d_1(\mathcal{E}') \leq d_1(\mathcal{E}) + |x|$, which implies

$$-d_-(\mathcal{E}') = 2d_1(\mathcal{E}') + d_2(\mathcal{E}') \leq d(\mathcal{E}).$$

- If $\deg \mathcal{L}(-x)$ lies between $\deg \mathcal{L}'_1$ and $\deg \mathcal{L}'_2$, then $d_1(\mathcal{E}') + d_2(\mathcal{E}') \leq d_1(\mathcal{E}) + |x|$ and $d_1(\mathcal{E}') \leq d_1(\mathcal{E}) + d_2(\mathcal{E}) - |x|$, which implies

$$-d_-(\mathcal{E}') = 2d_1(\mathcal{E}') + d_2(\mathcal{E}') \leq d(\mathcal{E}).$$

This allows us to express $f(\mathcal{E})$ as a linear combination of f on the vertices $\mathcal{E}'$ with $|d(\mathcal{E}')| \leq d(\mathcal{E})$. If $|d(\mathcal{E}')| = d(\mathcal{E})$, then $\mathcal{E}'$ is a sum of 3 line bundles and $d_1(\mathcal{E}') + d_2(\mathcal{E}') < d_1(\mathcal{E}) + d_2(\mathcal{E})$.

Relation (vi): If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ with $\mathcal{L}_i \in \operatorname{Pic} X$, $d_1(\mathcal{E}) > 2|x|$ and $d_1(\mathcal{E}) \geq d_2(\mathcal{E})$, then we apply $\mathcal{G}_{x,1}$ to the vertex $\mathcal{M} \oplus \mathcal{L}(x)$, where $\mathcal{M} = \mathcal{L}_2 \oplus \mathcal{L}_3$, $\mathcal{L} = \mathcal{L}_1$. We have $d_1(\mathcal{M} \oplus \mathcal{L}(x)) > |x|$ and $d(\mathcal{E}) = -d_-(\mathcal{E})$. By Theorem 2.5.3, we obtain

$$\lambda_1 f(\mathcal{L}_1(x) \oplus \mathcal{L}_2 \oplus \mathcal{L}_3) = f(\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3) + \sum_{\mathcal{M}'} m q_x f(\mathcal{M}' \oplus \mathcal{L}(x))$$

where $(\mathcal{M}, \mathcal{M}', m) \in \mathcal{V}_{x,1}(\mathcal{M})$. We have $d(\mathcal{M} \oplus \mathcal{L}(x)) < d(\mathcal{E})$ by the definition of the invariant d. Let $\mathcal{M}'$ be a neighbor of $\mathcal{M}^{\vee}$ in $\mathcal{G}_x$. If $\mathcal{M}' \in \operatorname{Bun}_2^{ind} X$, then $d(\mathcal{M}' \oplus \mathcal{L}(x)) = -d(\mathcal{E}) + |x|$. If $\mathcal{M}'$ is a sum of two line bundles, then the situation is more complex. In this case we can write $\mathcal{M}' = \mathcal{L}'_2 \oplus \mathcal{L}'_3$ with $\deg \mathcal{L}'_i \leq \deg \mathcal{L}_i$, $i = 2, 3$. The proof that we have $-d_-(\mathcal{E}') < d(\mathcal{E})$ and $d_+(\mathcal{E}') \leq d(\mathcal{E})$ for $\mathcal{E}' = \mathcal{L}(x) \oplus \mathcal{L}'_2 \oplus \mathcal{L}'_3$ is analogous to the case (v). This allows us to express $f(\mathcal{E})$ as a linear combination of values of f in vertices $\mathcal{E}'$ that satisfy $|d(\mathcal{E}')| \leq d(\mathcal{E})$ and if $|d(\mathcal{E}')| = d(\mathcal{E})$, then $\mathcal{E}'$ is a sum of 3 line bundles and $d_1(\mathcal{E}') + d_2(\mathcal{E}') < d_1(\mathcal{E}) + d_2(\mathcal{E})$.

We can apply these relations successively to a vertex $\mathcal{E}$ outside the nucleus to express $f(\mathcal{E})$ as a linear combination of values of f in vertices of the nucleus. □

2.6 Multiplicities for the nucleus

By the results of the last section, the unramified eigenforms are completely described by the restriction to the nucleus, which is a finite set of $\mathbf{P} \operatorname{Bun}_3 X$. We will use this property to compute $\mathcal{A}(x; \lambda_1, \lambda_2)$ in an explicit example in the next section. In the theorems of this section, we describe the neighborhoods of some vertices that will be relevant for these computations. These theorems for a general elliptic curve is part of a joint project with Oliver Lorscheid and Roberto Alvarenga in which we study the questions of this thesis for an arbitrary elliptic curve (cf. [3] for a complete proof of the theorems of this section). For simplicity, we state the theorem only in the case of an elliptic curve with a unique rational point, which suffices for our applications.

The proof of the following theorem is based on an algorithm developed by Roberto Alvarenga in [2], which determines the neighbors of a given vertex in $\mathcal{G}_{x,1}$. Simplifying matters, this algorithm is based on $\mathbf{H}_X$ the elliptic Hall algebra of X and can be described as follows: the goal is to calculate in $\mathbf{H}_X$ the product of the skyscraper sheaf at x by a vector bundle. The first step consists in a coordinate change from the basis $\mathrm{Coh}(X)$ given by the isomorphism classes of coherent sheaves to the twisted spherical Hall basis of $\mathbf{H}_X$, considered by Burban-Schiffmann [23] and Fratila [51]. Next one can use the explicit description of the twisted spherical Hall subalgebras in [51, Thm. 5.2] to reorder the elements that appear in the above product in increasing order of slopes. Another base change yields the desired result.

For an accurate description of the algorithm see [2, Section 4.4]. For examples of the algorithm see [2, Thm. 4.5.1, Thm. 4.5.2, Thm. 4.6.3].

Let X be an elliptic curve over $\mathbb{F}_q$ with only one point of degree 1. The zeta function of X is defined by the formal power series

$$Z_X(T) := \exp\left(\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} T^n\right).$$

Hasse and Weil proved that this series is actually a rational function and satisfy a functional equation (cf. [83, Chap. VIII, Thm. 6.1 and Thm. 7.1]). From this it follows that

$$Z_X(T) = \frac{(1-\omega_1 T)(1-\omega_2 T)}{(1-T)(1-qT)} = \frac{1-a_1T+qT^2}{(1-T)(1-qT)},$$

where $a_1 = q+1-\#X(\mathbb{F}_q) = q$. Using this formula, we obtain $\#X(\mathbb{F}_{q^2}) = 2q+1$ and $\#X(\mathbb{F}_{q^3}) = 3q^2+1$ (cf. [83, Chap. VIII, 5.8]). Therefore X has q points of degree 2 and q^2 points of degree 3. We denote by $y_1, \ldots, y_q$ the points of X degree 2 and by $z_1, \ldots, z_{q^2}$ the points of degree 3. We adopt the following notation for the vertices of $\mathbf{P}\,\mathrm{Bun}_3 X$:

Notation:

- $\mathcal{S}_0 = [\mathcal{E}^{(3,0)}_{(x,3)}], \quad \mathcal{S}_1 = [\mathcal{E}^{(3,1)}_{(x,1)}], \quad \mathcal{S}_2 = [\mathcal{E}^{(3,2)}_{(x,1)}], \quad \mathcal{T}_i = [\mathcal{E}^{(3,0)}_{(z_i,1)}],$
- $\mathcal{T}_i[k] = [\mathcal{E}^{(2,0)}_{(y_i,1)} \oplus \mathcal{O}(kx)], \quad \mathcal{S}_0[k] = [\mathcal{E}^{(2,0)}_{(x,2)} \oplus \mathcal{O}(kx)], \quad \mathcal{S}_x[k] = [\mathcal{E}^{(2,1)}_{(x,1)} \oplus \mathcal{O}(kx)],$
- $\mathcal{O}(j:k) = [\mathcal{O} \oplus \mathcal{O}(jx) \oplus \mathcal{O}(kx)].$

As X has only 1 point of degree 1, we have the dualities

$$\mathcal{S}_0^\vee = \mathcal{S}_0, \quad \mathcal{S}_1^\vee = \mathcal{S}_2, \quad \mathcal{S}_2^\vee = \mathcal{S}_1, \quad \mathcal{S}_0[k]^\vee = \mathcal{S}_0[-k] \quad \text{and} \quad \mathcal{S}_x[k]^\vee = \mathcal{S}_x[-k+1].$$

For $0 \leq j \leq k$, we have

$$\mathcal{O}(j:k)^\vee = \mathcal{O}(k-j:k),$$

and for the vertices that involve traces of line bundles, we have

$$\mathcal{T}_i^\vee = \mathcal{T}_j \text{ for some } j; \quad \mathcal{T}_i[k]^\vee = \mathcal{T}_j[-k] \text{ for some } j.$$

For the calculations in the next section, we do not need to explicitly determine j in terms of i.

Theorem 2.6.1. *Let X be an elliptic curve over a finite field with only one rational point x. Then we have the following multiplicities for the graph $\mathcal{G}_{x,1}$ for PGL_3:*

(i) *For $\mathcal{E} = \mathcal{T}_i$, we have*

$$\mathcal{V}_{x,1}(\mathcal{T}_i) = \{(\mathcal{T}_i, \mathcal{S}_2, q^2+q+1)\}.$$

(ii) *For a geometrically indecomposible bundle $\mathcal{E} = \mathcal{E}_{(x,l)}^{(3,d)}$ with $l = gcd(3,d)$, we have:*

- *If $d = 0$, then*

$$\mathcal{V}_{x,1}(\mathcal{S}_0) = \{(\mathcal{S}_0, \mathcal{S}_2, q^2), (\mathcal{S}_0, \mathcal{S}_x[1], q), (\mathcal{S}_0, \mathcal{S}_0[-1], 1)\}.$$

- *If $d = 1$, then*

$$\mathcal{V}_{x,1}(\mathcal{S}_1) = \{(\mathcal{S}_1, \mathcal{S}_0, 1), (\mathcal{S}_1, \mathcal{T}_i, 1), (\mathcal{S}_1, \mathcal{T}_j[0], 1) \mid |y_j| = 2,\ |z_i| = 3\}.$$

- *If $d = 2$, then*

$$\mathcal{V}_{x,1}(\mathcal{S}_2) = \{(\mathcal{S}_2, \mathcal{S}_1, q^2+q), (\mathcal{S}_2, \mathcal{S}_x[0], 1)\}.$$

(iii) *For $\mathcal{E} = \mathcal{S}_x[0]$, we have*

$$\begin{aligned}\mathcal{V}_{x,1}(\mathcal{S}_x[0]) &= \{(\mathcal{S}_x[0], \mathcal{S}_0[0], 1), (\mathcal{S}_x[0], \mathcal{T}_i[0], q) \mid |y_i| = 2\} \\ &\quad \cup \{(\mathcal{S}_x[0], \mathcal{S}_0, q-1), (\mathcal{S}_x[0], \mathcal{S}_x[-1], 1)\}.\end{aligned}$$

(iv) *For $\mathcal{E} = \mathcal{S}_x[1]$, we have*

$$\begin{aligned}\mathcal{V}_{x,1}(\mathcal{S}_x[1]) &= \{(\mathcal{S}_x[1], \mathcal{S}_x[0], q), (\mathcal{S}_x[1], \mathcal{S}_1, q^2-q)\} \\ &\quad \cup \{(\mathcal{S}_x[1], \mathcal{S}_0[1], 1), (\mathcal{S}_x[1], \mathcal{T}_i[1], 1) \mid |y_i| = 2\}.\end{aligned}$$

(v) *For $\mathcal{O}(0:0)$, we have*

$$\mathcal{V}_{x,1}(\mathcal{O}(0:0)) = \{(\mathcal{O}(0:0), \mathcal{O}(1:1), q^2+q+1)\}.$$

(vi) *For* $\mathcal{E} = \mathcal{S}_0[0]$*, we have*

$$\mathcal{V}_{x,1}(\mathcal{S}_0[0]) = \{(\mathcal{S}_0[0], \mathcal{S}_x[1], q^2), (\mathcal{S}_0[0], \mathcal{S}_0[-1], q), (\mathcal{S}_0[0], \mathcal{O}(1:1), 1)\}.$$

(vii) *For* $\mathcal{E} = \mathcal{T}_i[0]$*, we have*

$$\mathcal{V}_{x,1}(\mathcal{T}_i[0]) = \{(\mathcal{T}_i[0], \mathcal{S}_2, q^2-1), (\mathcal{T}_i[0], \mathcal{T}_i[-1], 1), (\mathcal{T}_i[0], \mathcal{S}_x[1], q+1)\}.$$

(viii) *For* $\mathcal{E} = \mathcal{T}_i[-1]$*, we have*

$$\mathcal{V}_{x,1}(\mathcal{T}_i[-1]) = \{(\mathcal{T}_i[-1], \mathcal{S}_x[0], q^2+q), (\mathcal{T}_i[-1], \mathcal{T}_i[-2], 1)\}.$$

(ix) *For* $\mathcal{E} = \mathcal{S}_0[-1]$*, we have*

$$\begin{aligned}\mathcal{V}_{x,1}(\mathcal{S}_0[-1]) &= \{(\mathcal{S}_0[-1], \mathcal{S}_x[0], q^2), (\mathcal{S}_0[-1], \mathcal{S}_0[1], q-1)\} \\ &\cup \{(\mathcal{S}_0[-1], \mathcal{S}_0[-2], 1), (\mathcal{S}_0[-1], \mathcal{O}(0:1), 1)\}.\end{aligned}$$

(x) *For* $\mathcal{E} = \mathcal{O}(1:1)$*, we have*

$$\begin{aligned}\mathcal{V}_{x,1}(\mathcal{O}(1:1)) &= \{(\mathcal{O}(1:1), \mathcal{O}(2:2), 1), (\mathcal{O}(1:1), \mathcal{O}(0:1), q+1)\} \\ &\cup \{(\mathcal{O}(1:1), \mathcal{S}_0[1], q^2-1)\}.\end{aligned}$$

(xi) *For* $\mathcal{E} = \mathcal{T}_i[1]$*, we have*

$$\mathcal{V}_{x,1}(\mathcal{T}_i[1]) = \{(\mathcal{T}_i[1], \mathcal{T}_i[0], q^2), (\mathcal{T}_i[1], \mathcal{S}_x[2], q+1)\}.$$

(xii) *For* $\mathcal{E} = \mathcal{S}_x[-1]$*, we have*

$$\begin{aligned}\mathcal{V}_{x,1}(\mathcal{S}_x[-1]) &= \{(\mathcal{S}_x[-1], \mathcal{S}_0[-1], q), (\mathcal{S}_x[-1], \mathcal{S}_x[-2], 1)\} \\ &\cup \{(\mathcal{S}_x[-1], \mathcal{T}_i[-1], q) \mid |y_i| = 2\}.\end{aligned}$$

(xiii) *For* $\mathcal{E} = \mathcal{S}_0[1]$*, we have*

$$\begin{aligned}\mathcal{V}_{x,1}(\mathcal{S}_0[1]) &= \{(\mathcal{S}_0[1], \mathcal{S}_0, q^2-q), (\mathcal{S}_0[1], \mathcal{S}_x[2], q)\} \\ &\cup \{(\mathcal{S}_0[1], \mathcal{S}_0[0], q), (\mathcal{S}_0[1], \mathcal{O}(1:2), 1)\}.\end{aligned}$$

(xiv) *For $\mathcal{E} = \mathcal{O}(0:1)$, we have*

$$\mathcal{V}_{x,1}(\mathcal{O}(0:1)) = \{(\mathcal{O}(0:1), \mathcal{S}_0[0], q^2-1), (\mathcal{O}(0:1), \mathcal{O}(1:2), q+1), (\mathcal{O}(0:1), \mathcal{O}(0:0), 1)\}.$$

(xv) *For $\mathcal{E} = \mathcal{O}(1:3)$, we have*

$$\begin{aligned}\mathcal{V}_{x,1}(\mathcal{O}(1:3)) &= \{(\mathcal{O}(1:3), \mathcal{S}_0[3], q-1), (\mathcal{O}(1:3), \mathcal{O}(2:4), 1)\} \\ &\cup \{(\mathcal{O}(1:3), \mathcal{O}(0:3), 1), (\mathcal{O}(1:3), \mathcal{O}(1:2), q^2)\}.\end{aligned}$$

We will also use graphs of Hecke operators for places of degree 2 of X. We summarize the necessary results in the two theorems below. The proof is also based on Alvarenga's algorithm and omitted.

Theorem 2.6.2. *Let X be the elliptic curve over $\mathbb{F}_2$ with Weierstrass equation $y^2 + y = x^3 + x + 1$. Let y and y' be the two places of degree 2 of X. We consider the graph $\mathcal{G}_y$ of the Hecke operator $\Phi_{y,1}$ in PGL_2. We define $\mathcal{M}_z := [\mathcal{E}^{(2,0)}_{(z,1)}]$ for $|z| = 2$, $\mathcal{S}^2_0 := [\mathcal{E}^{(2,0)}_{(x,2)}]$ and $\mathcal{C}_k := [\mathcal{O}_X \oplus \mathcal{O}_X(kx)]$. We have in $\mathcal{G}_y$:*

$$\mathcal{V}_y(\mathcal{C}_0) = \{(\mathcal{C}_0, \mathcal{C}_2, 3), (\mathcal{C}_0, \mathcal{M}_y, 2)\},$$
$$\mathcal{V}_y(\mathcal{M}_y) = \{(\mathcal{M}_y, \mathcal{M}_{y'}, 1), (\mathcal{M}_y, \mathcal{M}_y, 3), (\mathcal{M}_y, \mathcal{C}_0, 1)\}.$$

Theorem 2.6.3. *Let X be an elliptic curve over a finite field $\mathbb{F}_q$ with only 1 rational point and y_i a place of degree 2, then for the graph $\mathcal{G}_{y_i,1}$ in PGL_3 on the vertex $\mathcal{E} = \mathcal{O}(0:0)$, we have*

$$\mathcal{V}_{y,1}(\mathcal{O}(0:0)) = \{(\mathcal{O}(0:0), \mathcal{O}(2:2), q^2+q+1), (\mathcal{O}(0:0), \mathcal{T}_i[1], q^4-q)\}.$$

2.7 An Explicit Example

We apply the results from the previous sections to calculate the spaces $\mathcal{A}(x; \lambda_1, \lambda_2)$ and to determine the toroidal automorphic forms in a specific example.

Let X be the elliptic curve over $\mathbb{F}_2$ with Weierstrass equation $y^2 + y = x^3 + x + 1$. This elliptic curve has has only 1 point of degree 1, which we denote by x. Let y_1 and y_2 be the two points of X of degree 2 and z_1, z_2, z_3, z_4 the 4 points of degree 3. In this section, we obtain a parametrization of the space $\mathcal{A}(x; \lambda_1, \lambda_2)$ of unramified automorphic forms f with trivial central character, such that $\Phi_{x,i}(f) = \lambda_i f$ for $i = 1, 2$. Let $f \in \mathcal{A}(x; \lambda_1, \lambda_2)$. We consider f as a function in $\mathbf{P}\operatorname{Bun}_3 X$.

The nucleus of $\mathcal{G}_{x,1}$ and $\mathcal{G}_{x,2}$ is composed of the following vertices:

- $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{T}_i$, for $i = 1, \ldots, 4$;

- $\mathcal{T}_i[k]$, $\mathcal{S}_0[k]$, for $i = 1, 2$ and $k = -1, 0, 1$; $\mathcal{S}_x[k]$, for $k = 0, 1$;
- $\mathcal{O}(0:0)$, $\mathcal{O}(0:1)$, $\mathcal{O}(1:1)$, $\mathcal{O}(1:2)$ and $\mathcal{O}(2:4)$.

Fix $f \in \mathcal{A}(x; \lambda_1, \lambda_2)$. To ease the notation we use the following conventions.
Notation:

- $S_0 = f(\mathcal{S}_0)$, $S_1 = f(\mathcal{S}_1)$, $S_2 = f(\mathcal{S}_2)$, $T_i = f(\mathcal{T}_i)$,
- $t_{i,k} = f(\mathcal{T}_i[k])$, $s_{0,k} = f(\mathcal{S}_0[k])$, $s_{x,k} = f(\mathcal{S}_x[k])$,
- $D_{j,k} = f(\mathcal{O}(j:k))$.

By Theorems 2.6.1 and (2.5.1), f must satisfies the followig equations.
Eigenvalue equations on the nucleus:

$$
\begin{aligned}
(\mathcal{T}_i) \quad & \lambda_1 T_i = 7S_2 \\
& \lambda_2 T_i = 7S_1 \\
(\mathcal{S}_1) \quad & \lambda_1 S_1 = \sum T_i + \sum t_{i,0} + S_0 \\
& \lambda_2 S_1 = 6S_2 + s_{x,1} \\
(\mathcal{S}_2) \quad & \lambda_1 S_2 = 6S_1 + s_{x,0} \\
& \lambda_2 S_2 = \sum T_i + \sum t_{i,0} + S_0 \\
(\mathcal{S}_0) \quad & \lambda_1 S_0 = 4S_2 + 2s_{x,1} + s_{0,-1} \\
& \lambda_2 S_0 = 4S_1 + 2s_{x,0} + s_{0,1} \\
(\mathcal{T}_i[0]) \quad & \lambda_1 t_{i,0} = 3S_2 + 3s_{x,1} + t_{i,-1} \\
& \lambda_2 t_{i,0} = 3S_1 + 3s_{x,0} + t_{i,1} \\
(\mathcal{T}_i[1]) \quad & \lambda_1 t_{i,1} = 4t_{i,0} + 3s_{x,2} \\
& \lambda_2 t_{i,1} = 6s_{x,1} + t_{i,2} \\
(\mathcal{T}_i[-1]) \quad & \lambda_1 t_{i,-1} = 6s_{x,0} + t_{i,-2} \\
& \lambda_2 t_{i,-1} = 4t_{i,0} + 3s_{x,-1} \\
(\mathcal{S}_x[0]) \quad & \lambda_1 s_{x,0} = 2\sum t_{i,0} + S_0 + s_{x,-1} + s_{0,0} \\
& \lambda_2 s_{x,0} = 2S_2 + 2s_{x,1} + \sum t_{i,-1} + s_{0,-1} \\
(\mathcal{S}_x[1]) \quad & \lambda_1 s_{x,1} = 2S_1 + 2s_{x,0} + \sum t_{i,1} + s_{0,1} \\
& \lambda_2 s_{x,1} = 2\sum t_{i,0} + S_0 + s_{x,2} + s_{0,0} \\
(\mathcal{S}_x[-1]) \quad & \lambda_1 s_{x,-1} = 2\sum t_{i,-1} + 2s_{0,-1} + s_{x,-2} \\
& \lambda_2 s_{x,-1} = ?
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{S}_x[2]) \quad & \lambda_1 s_{x,2} = ? \\
& \lambda_2 s_{x,2} = 2\sum t_{i,1} + 2s_{0,1} + s_{x,3} \\
(\mathcal{S}_0[0]) \quad & \lambda_1 s_{0,0} = 4s_{x,1} + 2s_{0,-1} + D_{1,1} \\
& \lambda_2 s_{0,0} = 4s_{x,0} + 2s_{0,1} + D_{1,0} \\
(\mathcal{S}_0[1]) \quad & \lambda_1 s_{0,1} = 2S_0 + 2s_{x,2} + 2s_{0,0} + D_{2,1} \\
& \lambda_2 s_{0,1} = 4s_{x,1} + s_{0,-1} + s_{0,2} + D_{1,1} \\
(\mathcal{S}_0[-1]) \quad & \lambda_1 s_{0,-1} = 4s_{x,0} + s_{0,1} + s_{0,-2} + D_{1,0} \\
& \lambda_2 s_{0,-1} = 2S_0 + 2s_{x,-1} + 2s_{0,0} + D_{2,1} \\
(\mathcal{O}(0:0)) \quad & \lambda_1 D_{0,0} = 7D_{1,1} \\
& \lambda_2 D_{0,0} = 7D_{1,0} \\
(\mathcal{O}(1:0)) \quad & \lambda_1 D_{1,0} = 3s_{0,0} + 3D_{2,1} + D_{0,0} \\
& \lambda_2 D_{1,0} = ? \\
(\mathcal{O}(1:1)) \quad & \lambda_1 D_{1,1} = 3D_{1,0} + 3s_{0,1} + D_{2,2} \\
& \lambda_2 D_{1,1} = 3s_{0,0} + 3D_{2,1} + D_{0,0} \\
(\mathcal{O}(1:3)) \quad & \lambda_1 D_{1,3} = s_{0,3} + D_{2,4} + D_{0,3} + 4D_{1,2}
\end{aligned}
$$

The equations with "?" will not be necessary to parameterize $\mathcal{A}(x;\lambda_1,\lambda_2)$.

We split the parameterization of $\mathcal{A}(x;\lambda_1,\lambda_2)$ into the following two cases.

First case: $\lambda_1 = \lambda_2 = 0$.

Successive use of the eigenvalue equations leads to the following identities where we indicate on the left hand side which equation we apply for each deduction.

$(\mathcal{T}_i) \implies S_1 = 0$ and $S_2 = 0$

$(\mathcal{S}_1) \implies s_{x,1} = 0$

$(\mathcal{S}_2) \implies s_{x,0} = 0$

$(\mathcal{S}_0) \implies s_{0,-1} = 0$ and $s_{0,1} = 0$

$(\mathcal{T}_i[0]) \implies t_{i,-1} = 0$ and $t_{i,1} = 0$

$(\mathcal{T}_i[1]) \implies t_{i,2} = 0$

$(\mathcal{T}_i[-1]) \implies t_{i,-2} = 0$

$(\mathcal{S}_x[-1]) \implies s_{x,-2} = 0$

$(\mathcal{S}_x[2]) \implies s_{x,3} = 0$

$(\mathcal{S}_0[0]) \implies D_{1,1} = 0$ and $D_{1,0} = 0$

$(\mathcal{S}_0[1]) \implies s_{0,2} = 0$

$(\mathcal{S}_0[-1]) \implies s_{0,-2} = 0$

$(\mathcal{O}(1:1)) \implies D_{2,2} = 0$

$(\mathcal{S}_x[0])$ **and** $(s_{x,1}) \implies s_{x,2} = s_{x,-1}$

$(\mathcal{T}_i[1]) \implies t_{1,0} = t_{2,0}$

$(\mathcal{T}_i[1]) \implies s_{x,2} = -\frac{4}{3}t_{i,0}$

$(\mathcal{S}_x[0]) \implies S_0 - 2s_{x,-1} + s_{0,0} = 0$

$(\mathcal{S}_0[1]) \implies 2S_0 + 2s_{x,2} + 2s_{0,0} + D_{2,1} = 0$

$(\mathcal{O}(1:0)) \implies 3s_{0,0} + 3D_{2,1} + D_{0,0} = 0$

$(\mathcal{S}_1) \implies \sum T_i + \sum t_{i,0} + S_0 = 0$

To prove that $D_{2,4}$ can be expressed as a linear combination of the previously calculated values, see the argument at the end of the second case below. Therefore f is determined by the values $s_{x,2}, S_0, T_1, T_2, T_3$, and we have

$$\dim \mathcal{A}(x; 0, 0) \leq 5.$$

In the next section, we prove that the space of unramified cusp forms is a subspace of dimension 3 of $\mathcal{A}(x; 0, 0)$.

Second case: $(\lambda_1, \lambda_2) \neq (0, 0)$.

The equations $(\mathcal{T}_i)$ implies that $T_i = T_j$ for every i and j. We use below the notation $T = T_i$, $t = \frac{T_i}{7}$ and $t_i = t_{i,0}$.

$$(\mathcal{T}_i) \implies S_1 = \lambda_2 t$$
$$S_2 = \lambda_1 t$$

$$(\mathcal{S}_1) \implies S_0 = \lambda_1 S_1 - 4T - (t_1 + t_2) = (\lambda_1\lambda_2 - 28)t - (t_1 + t_2)$$
$$s_{x,1} = \lambda_2 S_1 - 6S_2 = (\lambda_2^2 - 6\lambda_1)t$$

$$(\mathcal{S}_2) \implies s_{x,0} = \lambda_1 S_2 - 6S_1 = (\lambda_1^2 - 6\lambda_2)t$$

$$(\mathcal{S}_0) \implies s_{0,-1} = \lambda_1 S_0 - 4S_2 - 2s_{x,1}$$
$$= (\lambda_1^2\lambda_2 - 28\lambda_1 - 4\lambda_1 - 2\lambda_2^2 + 12\lambda_1)t - \lambda_1(t_1 + t_2)$$
$$= (\lambda_1^2\lambda_2 - 2\lambda_2^2 - 20\lambda_1)t - \lambda_1(t_1 + t_2)$$
$$s_{0,1} = \lambda_2 S_0 - 4S_1 - 2s_{x,0}$$
$$= (\lambda_1\lambda_2^2 - 28\lambda_2 - 4\lambda_2 - 2\lambda_1^2 + 12\lambda_2)t - \lambda_2(t_1 + t_2)$$
$$= (\lambda_1\lambda_2^2 - 2\lambda_1^2 - 20\lambda_2)t - \lambda_2(t_1 + t_2)$$

$$(\mathcal{T}_i[0]) \implies t_{i,-1} = \lambda_1 t_i - 3S_2 - 3s_{x,1} = (-3\lambda_2^2 + 15\lambda_1)t + \lambda_1 t_i$$

$$t_{i,1} = \lambda_2 t_i - 3S_1 - 3s_{x,0} = (-3\lambda_1^2 + 15\lambda_2)t + \lambda_2 t_i$$

$$(\mathcal{T}_i[-1]) \implies s_{x,-1} = \tfrac{1}{3}(\lambda_2 t_{i,-1} - 4t_i) = (-\lambda_2^3 + 5\lambda_1\lambda_2)t + \tfrac{1}{3}(\lambda_1\lambda_2 - 4)t_i$$

This implies that $\lambda_1\lambda_2 = 4$ or $t_1 = t_2$.

$$\begin{aligned}(\mathcal{S}_x[0]) \implies s_{0,0} &= \lambda_1 s_{x,0} - 2(t_1 + t_2) - S_0 - s_{x,-1} \\ &= (\lambda_1^3 - 6\lambda_1\lambda_2 - \lambda_1\lambda_2 + 28 + \lambda_2^3 - 5\lambda_1\lambda_2)t + \tfrac{1}{6}(-6 - \lambda_1\lambda_2 + 4)(t_1 + t_2) \\ &= (\lambda_1^3 + \lambda_2^3 - 12\lambda_1\lambda_2 + 28)t + \tfrac{1}{6}(-\lambda_1\lambda_2 - 2)(t_1 + t_2)\end{aligned}$$

$$\begin{aligned}(\mathcal{S}_0[0]) \implies D_{1,1} &= \lambda_1 s_{0,0} - 4s_{x,1} - 2s_{0,-1} \\ &= (\lambda_1^4 + \lambda_1\lambda_2^3 - 12\lambda_1^2\lambda_2 + 28\lambda_1 - 4\lambda_2^2 + 24\lambda_1 - 2\lambda_1^2\lambda_2 + 4\lambda_2^2 + 40\lambda_1)t \\ &\quad + \tfrac{1}{6}(-\lambda_1^2\lambda_2 - 2\lambda_1 + 12\lambda_1)(t_1 + t_2) \\ &= (\lambda_1^4 + \lambda_1\lambda_2^3 - 14\lambda_1^2\lambda_2 + 92\lambda_1)t + \tfrac{1}{6}(-\lambda_1^2\lambda_2 + 10\lambda_1)(t_1 + t_2)\end{aligned}$$

$$\begin{aligned}(\mathcal{S}_0[0]) \implies D_{1,0} &= \lambda_2 s_{0,0} - 4s_{x,0} - 2s_{0,1} \\ &= (\lambda_2^4 + \lambda_2\lambda_1^3 - 12\lambda_1\lambda_2^2 + 28\lambda_2 - 4\lambda_1^2 + 24\lambda_2 - 2\lambda_1\lambda_2^2 + 4\lambda_1^2 + 40\lambda_2)t \\ &\quad + \tfrac{1}{6}(-\lambda_1\lambda_2^2 - 2\lambda_2 + 12\lambda_2)(t_1 + t_2) \\ &= (\lambda_2^4 + \lambda_2\lambda_1^3 - 14\lambda_1\lambda_2^2 + 92\lambda_2)t + \tfrac{1}{6}(-\lambda_1\lambda_2^2 + 10\lambda_2)(t_1 + t_2)\end{aligned}$$

$$\begin{aligned}(\mathcal{S}_0[-1]) \implies D_{2,1} &= \lambda_2 s_{0,-1} - 2S_0 - 2s_{x,-1} - 2s_{0,0} \\ &= (\lambda_1^2\lambda_2^2 - 2\lambda_2^3 - 20\lambda_1\lambda_2 - 2\lambda_1\lambda_2 + 56 + 2\lambda_2^3 - 10\lambda_1\lambda_2 - 2\lambda_1^3 - 2\lambda_2^3 \\ &\quad + 24\lambda_1\lambda_2 - 56)t + \tfrac{1}{3}(-3\lambda_1\lambda_2 + 6 - \lambda_1\lambda_2 + 4 + \lambda_1\lambda_2 + 2)(t_1 + t_2) \\ &= (\lambda_1^2\lambda_2^2 - 2\lambda_1^3 - 2\lambda_2^3 - 8\lambda_1\lambda_2)t + (-\lambda_1\lambda_2 + 4)(t_1 + t_2)\end{aligned}$$

$$\begin{aligned}(\mathcal{O}(1:0)) \implies D_{0,0} &= \lambda_1 D_{1,0} - 3s_{0,0} - 3D_{2,1} \\ &= (\lambda_1\lambda_2^4 + \lambda_2\lambda_1^4 - 14\lambda_1^2\lambda_2^2 + 92\lambda_1\lambda_2 - 3\lambda_1^3 - 3\lambda_2^3 + 36\lambda_1\lambda_2 \\ &\quad - 84 - 3\lambda_1^2\lambda_2^2 + 6\lambda_2^3 + 6\lambda_1^3 + 24\lambda_1\lambda_2)t \\ &\quad + \tfrac{1}{6}(-\lambda_1^2\lambda_2^2 + 10\lambda_1\lambda_2 + 3\lambda_1\lambda_2 + 6 + 18\lambda_1\lambda_2 - 72)(t_1 + t_2) \\ &= (\lambda_1\lambda_2^4 + \lambda_2\lambda_1^4 - 17\lambda_1^2\lambda_2^2 + 3\lambda_1^3 + 3\lambda_2^3 + 152\lambda_1\lambda_2 - 84)t \\ &\quad + \tfrac{1}{6}(-\lambda_1^2\lambda_2^2 + 31\lambda_1\lambda_2 - 66)(t_1 + t_2)\end{aligned}$$

$$\textstyle\sum(\mathcal{T}_i[1]) \implies s_{x,2} = \tfrac{1}{6}(\lambda_1 \sum t_{i,1} - 4(t_1 + t_2)) = (-\lambda_1^3 + 5\lambda_1\lambda_2)t + \tfrac{1}{6}(\lambda_2 - 4)(t_1 + t_2)$$

$$(\mathcal{T}_i[1]) \implies t_{i,2} = \lambda_2 t_{i,1} - 6s_{x,1} = (-3\lambda_2\lambda_1^2 + 9\lambda_2^2 + 36\lambda_1)t + \lambda_2^2 t_i$$

$$(\mathcal{T}_i[-1]) \implies t_{i,-2} = \lambda_1 t_{i,-1} - 6s_{x,0} = (-3\lambda_1\lambda_2^2 + 9\lambda_1^2 + 36\lambda_2)t + \lambda_1^2 t_i$$

$$(\mathcal{S}_x[-1]) \implies s_{x,-2} = \lambda_1 s_{x,-1} - 2\sum t_{i,-1} - 2s_{0,-1}$$
$$= (-\lambda_1\lambda_2^3 + 5\lambda_1^2\lambda_2 + 12\lambda_2^2 - 60\lambda_1 - 2\lambda_1^2\lambda_2 + 4\lambda_2^2 + 40\lambda_1)t$$
$$+\tfrac{1}{6}(\lambda_1^2\lambda_2 - 4\lambda_1 - 12\lambda_1 + 12\lambda_1)(t_1 + t_2)$$
$$= (-\lambda_1\lambda_2^3 + 3\lambda_1^2\lambda_2 + 16\lambda_2^2 - 20\lambda_1)t + \tfrac{1}{6}(\lambda_1^2\lambda_2 - 4\lambda_1)(t_1 + t_2)$$

$$(\mathcal{S}_x[2]) \implies s_{x,3} = \lambda_2 s_{x,2} - 2\sum t_{i,1} - 2s_{0,1}$$
$$= (-\lambda_2\lambda_1^3 + 5\lambda_1\lambda_2^2 + 12\lambda_1^2 - 60\lambda_2 - 2\lambda_1\lambda_2^2 + 4\lambda_1^2 + 40\lambda_2)t$$
$$+\tfrac{1}{6}(\lambda_2^2 - 4\lambda_2 - 12\lambda_2 + 12\lambda_2)(t_1 + t_2)$$
$$= (-\lambda_2\lambda_1^3 + 3\lambda_1\lambda_2^2 + 16\lambda_1^2 - 20\lambda_2)t + \tfrac{1}{6}(\lambda_2^2 - 4\lambda_2)(t_1 + t_2)$$

$$(\mathcal{O}(0:0)) \implies D_{0,0} = 7(\lambda_1^3 + \lambda_2^3 - 14\lambda_1\lambda_2 + 92)t + \tfrac{7}{6}(-\lambda_1\lambda_2 + 10)(t_1 + t_2),$$

where we used that $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$.

$$(\mathcal{O}(1:1)) \implies D_{2,2} = \lambda_1 D_{1,1} - 3D_{0,1} - 3s_{0,1}$$
$$= (\lambda_1^5 + \lambda_1^2\lambda_2^3 - 14\lambda_1^3\lambda_2 + 92\lambda_1^2 - 3\lambda_2^4 - 3\lambda_2\lambda_1^3 + 42\lambda_1\lambda_2^2$$
$$-276\lambda_2 - 3\lambda_1\lambda_2^2 + 6\lambda_1^2 + 60\lambda_2)t$$
$$+\tfrac{1}{6}(-\lambda_1^3\lambda_2 + 10\lambda_1^2 + 3\lambda_1\lambda_2^2 - 30\lambda_2 + 18\lambda_2)(t_1 + t_2)$$
$$= (\lambda_1^5 + \lambda_1^2\lambda_2^3 - 17\lambda_1^3\lambda_2 - 3\lambda_2^4 + 39\lambda_1\lambda_2^2 + 98\lambda_1^2 - 216\lambda_2)t$$
$$+\tfrac{1}{6}(-\lambda_1^3\lambda_2 + 3\lambda_1\lambda_2^2 + 10\lambda_1^2 - 12\lambda_2)(t_1 + t_2)$$

It remains to show that we can express $D_{2,4}$ as a linear combination of t, t_1 and t_2. We are not giving an explicit expression, but just justifying why it happens. We start by proving this for $D_{0,2}$, $D_{1,3}$, $D_{0,3}$ and $D_{2,3}$.

For $D_{0,2}$, applying the relation (iv) of Theorem 2.5.13, we can express $D_{0,2}$ as a linear combination of the values $f(\mathcal{E})$ with $\delta(\mathcal{E}) < 4$, which have already been expressed in terms of t, t_1 and t_2. For $D_{1,3}$, $D_{2,3}$ and $D_{0,3}$, we apply analogous argument. For $D_{2,4}$, we have by Theorem 2.6.1 the equation

$$\lambda_1 D_{1,3} = s_{0,3} + D_{2,4} + D_{0,3} + 4D_{1,2},$$

which we use to express $D_{2,4}$ as a function of t, t_1 and t_2 using the previous expressions.

Matching the two expressions for $D_{0,0}$, we obtain

$$P(\lambda_1, \lambda_2)t + Q(\lambda_1, \lambda_2)(t_1 + t_2) = 0, \tag{2.7.1}$$

where

$$P(\lambda_1, \lambda_2) = (\lambda_1\lambda_2 - 4)(\lambda_1^3 + \lambda_2^3 - 17\lambda_1\lambda_2 + 182),$$
$$Q(\lambda_1, \lambda_2) = -\frac{1}{6}(\lambda_1\lambda_2 - 4)(\lambda_1\lambda_2 - 34).$$

This yields the dimension formulas:

$$\dim \mathcal{A}(x;\lambda_1,\lambda_2) \le \begin{cases} 3 & \text{if} \quad \lambda_1\lambda_2 = 4, \\ 2 & \text{if} \quad \lambda_1\lambda_2 \neq 4 \text{ and } (P(\lambda_1,\lambda_2),Q(\lambda_1,\lambda_2)) = (0,0), \\ 1 & \text{if} \quad \lambda_1\lambda_2 \neq 4 \text{ and } (P(\lambda_1,\lambda_2),Q(\lambda_1,\lambda_2)) \neq (0,0). \end{cases}$$

2.7.1 The space of Cusp forms

Let $\mathcal{A}_0(x;\lambda_1,\lambda_2)$ be the space of cusp forms in $\mathcal{A}(x;\lambda_1,\lambda_2)$. In this section, we prove that $\mathcal{A}_0(x;\lambda_1,\lambda_2) = 0$ if $(\lambda_1,\lambda_2) \neq (0,0)$ and that $\mathcal{A}_0(x;0,0) = 3$. We begin with the expression of the constant terms along the parabolic subgroups for an unramified automorphic form.

We denote by $P_{1,2}$ and $P_{2,1}$ the standard parabolic subgroups of GL_3 of type $(1,2)$ and $(2,1)$ respectively, where we see GL_3 as an algebraic group over F. We denote by U the unipotent radical of the standard Borel subgroup, and by $U_{1,2}$ and $U_{2,1}$ the unipotent radicals of the parabolic subgroups $P_{1,2}$ and $P_{2,1}$, respectively.

Theorem 2.7.1. *Let X be an elliptic curve over the finite field $\mathbb{F}_q$ with only one rational point and $f : GL_3(F)Z(\mathbb{A}) \setminus GL_3(\mathbb{A})/K \longrightarrow \mathbb{C}$ a function, then we have*

$$\int_{U(F)\backslash U(\mathbb{A})} f(u)du = c_1\Big(f(\mathcal{O}(0:0)) + (2q+1)(q-1)f(\mathcal{S}_0[0]) + q(q-1)^2 f(\mathcal{S}_0)\Big),$$

$$\int_{U_{1,2}(F)\backslash U_{1,2}(\mathbb{A})} f(u)du = c_2\Big(f(\mathcal{O}(0:0)) + 2(q-1)f(\mathcal{S}_0[0]) + (q-1)^2 f(\mathcal{S}_0)\Big),$$

$$\int_{U_{2,1}(F)\backslash U_{2,1}(\mathbb{A})} f(u)du = c_3\Big(f(\mathcal{O}(0:0)) + (q^2-1)f(\mathcal{S}_0[0])\Big),$$

for nonzero constants c_1, c_2, c_3 that depend on the choice of Haar measure on U, $U_{1,2}$ and $U_{2,1}$, respectively.

Proof. Let $x \in |X|$ be the place of degree 1. By the strong approximation for unipotent groups, if N is one of U, $U_{1,2}$ and $U_{2,1}$, then $N(F)N(F_x) \longrightarrow N(\mathbb{A})$ has dense image. Thus we find for every $n \in N(\mathbb{A})$ a $\gamma \in N(F)$ and a $k \in N(\mathcal{O}_\mathbb{A})$ such that $\gamma n k \in N(F_x) \subset N(\mathbb{A})$. In other words, the natural map

$$\Gamma_N \backslash N(F_x)/K'_x \longrightarrow N(F)\backslash N(\mathbb{A})/K'$$

is surjective, where $K' = N(\mathcal{O}_\mathbb{A})$, $K'_x = N(\mathcal{O}_x)$, $\Gamma_N = N(\mathcal{O}_X(X - \{x\})) = N(F) \cap N(\mathcal{O}^x_\mathbb{A})$ and $\mathcal{O}^x_\mathbb{A} = \prod_{y\neq x} \mathcal{O}_y$. It is also easy to prove that the map is injective.

Next we consider the unipotent radical of the Borel subgroup $N = U$. We will obtain explicit representatives for $\Gamma_N \backslash N(F_x)/K'_x$. Let π be a uniformizer at x. We define

$$N(a,b,c) = \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}$$

Claim: Every double coset in $\Gamma_N \backslash N(F_x)/K'_x$ has a unique representative of one of the forms

$$n(a,c) := N(a,0,c),\ n(b) := N(0,b,0),$$

where

$$a = a_{-1}\pi^{-1}, \quad b = b_{-1}\pi^{-1}, \quad c = c_{-1}\pi^{-1},$$

with $a_{-1}, b_{-1}, c_{-1} \in \mathbb{F}_q$ and $(a_{-1}, c_{-1}) \neq (0,0)$.

The decomposition $F + \pi^{-1}\mathcal{O}_{\mathbb{A}} = \mathbb{A}$ implies that $\mathcal{O}_X(X - \{x\}) + \pi^{-1}\mathcal{O}_x = F_x$. Let $N(a,b,c) \in N(F_x)$. If we multiply $N(a,b,c)$ from the left by an appropriate element of Γ_N, we can assume that $a,b,c \in \pi^{-1}\mathcal{O}_x$. Multiplying $N(a,b,c)$ from the right by an element of K'_x yields a representative of the forms $n(a,c)$ or $n(b)$. From $\mathcal{O}_X(X - \{x\}) \cap \pi^{-1}\mathcal{O}_x = \mathbb{F}_q$, we obtain the uniqueness of the representative.

For $S \subset N(\mathbb{A})$, we denote by $\overline{S}$ the image of S in $N(F)\backslash N(\mathbb{A})$. By the above claim,

$$N(F)\backslash N(\mathbb{A}) = \bigsqcup_{(a,c)\neq(0,0)} \overline{n(a,c)K'} \sqcup \bigsqcup_{b} \overline{n(b)K'},$$

where a, b and c are as in the claim. Next we compute $vol(\overline{n(a,c)K'})$ and $vol(\overline{n(b)K'})$. Since $n(b)$ is in the center of $N(\mathbb{A})$,

$$vol(\overline{n(b)K'}) = vol(\overline{K'n(b)}) = vol(\overline{K'}).$$

For $n(a,c)$, we put $K_1 = n(a,c)K'n(a,c)^{-1} \cap K'$. We see that K_1 is the kernel of the surjective homomorphism

$$\begin{array}{ccc} K' & \longrightarrow & \pi^{-1}\mathcal{O}_x/\mathcal{O}_x, \\ N(l,m,n) & \longmapsto & cl_x - an_x \end{array}$$

thus $[K' : K_1] = q$. The restriction of the above map to $N(F) \cap K'$ remains surjective, whence $K_1 \cdot (N(F) \cap K') = K'$, which implies $N(F)K_1 = N(F)K'$. Therefore $vol(\overline{K_1}) = vol(\overline{K'})$.

We have $[n(a,c)K'n(a,c)^{-1} : K_1] = q$, thus we can write $n(a,c)K'n(a,c)^{-1} = \bigsqcup_{i=1}^{q} K_1 n_i$. As $N(F) \cap (n(a,c)K'n(a,c)^{-1}) \subset K'$, it follows that $N(F)K_1 n_i$ is disjoint from $N(F)K_1 n_j$ if $i \neq j$. Thus $\overline{n(a,c)K'n(a,c)^{-1}} = \bigsqcup_{i=1}^{q} \overline{K_1 n_i}$, which implies $vol(\overline{n(a,b)K'}) = vol(\overline{n(a,c)K'n(a,c)^{-1}}) = q \cdot vol(\overline{K_1}) = q \cdot vol(\overline{K'})$. We conclude that

$$\int_{N(F)\backslash N(\mathbb{A})} f(n)dn = vol(\overline{K'}) \left(q \sum_{(a,c)\neq(0,0)} f(n(a,c)) + \sum_{b} f(n(b)) \right).$$

We can adopt the same strategy to find representatives for $N(F)\backslash N(\mathbb{A})/K'$ when $N = U_{1,2}$ or $N = U_{2,1}$. For $N = U_{1,2}$, we get representatives of the form

$$l(a,b) := N(a,b,0),$$

where

$$a = a_{-1}\pi^{-1}, \quad b = b_{-1}\pi^{-1},$$

with $a_{-1}, b_{-1} \in \mathbb{F}_q$; and for $U_{2,1}$ of the form

$$m(b,c) := N(0,b,c),$$

where

$$b = b_{-1}\pi^{-1}, \quad c = c_{-1}\pi^{-1},$$

with $b_{-1}, c_{-1} \in \mathbb{F}_q$. Observe that $U_{1,2}$ and $U_{2,1}$ are commutative, which implies

$$\int_{U_{1,2}(F)\backslash U_{1,2}(\mathbb{A})} f(n)dn = vol(\overline{K}) \left(\sum_{(a,b)} f(l(a,b)) \right),$$

$$\int_{U_{2,1}(F)\backslash U_{2,1}(\mathbb{A})} f(n)dn = vol(\overline{K}) \left(\sum_{(a,b)} f(m(a,b)) \right).$$

We denote by Θ the bijection from $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})/K$ to $\mathbf{P}Bun_3X$. We consider a, b and c as in the description of the representatives. To compute the images of the representatives by Θ, we use the following facts:

1. The 3-bundle $\mathcal{E}_{(x,2)}^{(2,0)} \oplus \mathcal{O}_X$ is the unique nontrivial extension of $\mathcal{O}_X$ by $\mathcal{O}_X \otimes \mathcal{O}_X$.
2. The 3-bundle $\mathcal{E}_{(x,3)}^{(3,0)}$ is the unique nontrivial extension of $\mathcal{O}_X$ by $\mathcal{E}_{(x,2)}^{(2,0)}$.

It follows that

$$\begin{array}{ll}
\Theta(N(0,0,0)) = \mathcal{O}(0:0). & \\
\Theta(N(a,0,0)) = \mathcal{S}_0[0], & \text{if } a \neq 0,\ b = c = 0. \\
\Theta(N(0,0,c)) = \mathcal{S}_0[0], & \text{if } c \neq 0,\ a = b = 0. \\
\Theta(N(a,0,c)) = \mathcal{S}_0, & \text{if } b = 0,\ ac \neq 0. \\
\Theta(N(0,b,0)) = \mathcal{S}_0[0], & \text{if } b \neq 0,\ a = c = 0. \\
\Theta(N(a,b,0)) = \mathcal{S}_0, & \text{if } c = 0,\ ab \neq 0. \\
\Theta(N(0,b,c)) = \mathcal{S}_0[0], & \text{if } a = 0,\ bc \neq 0.
\end{array}$$

Using this in the integrals above, the theorem is proven. □

Theorem 2.7.2. *Let X be the elliptic curve over $\mathbb{F}_2$ with Weierstrass equation $y^2 + y = z^3 + z + 1$. If x is the unique place of degree 1, then the cusp eigenforms on $PGL_3(\mathbb{A})$ satisfy $\lambda_{x,1} = \lambda_{x,2} = 0$. We have $\dim \mathcal{A}_0(x;0,0) = 3$, and if $f \in \mathcal{A}_0(x;0,0)$, then $Supp(f) \subset \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4\}$ and*

$$\sum_{i=1}^{4} f(\mathcal{T}_i) = 0.$$

Proof. Fix $f \in \mathcal{A}_0(x;\lambda_1,\lambda_2)$. As the constant terms of f over the unipotent radicals of the parabolic subgroups are 0, by Theorem 2.7.1 we obtain:

$$\begin{cases} D_{0,0} + 5s_{0,0} + 2S_0 & = & 0 \\ D_{0,0} + 2s_{0,0} + S_0 & = & 0 \\ D_{0,0} + 3s_{0,0} & = & 0 \end{cases}.$$

The only solution of this system of equations is $D_{0,0} = s_{0,0} = S_0 = 0$. Suppose that the eigenvalues of the cusp eigenform f are not both zero. Using the expressions of $D_{0,0}$, $s_{0,0}$ and S_0 obtained above, we obtain,

$$(\lambda_1\lambda_2 - 28)t - (t_1 + t_2) = 0, \tag{2.7.2}$$

$$(\lambda_1^3 + \lambda_2^3 - 12\lambda_1\lambda_2 + 28)t + \frac{1}{6}(-\lambda_1\lambda_2 - 2)(t_1 + t_2) = 0, \tag{2.7.3}$$

$$(\lambda_1^3 + \lambda_2^3 - 14\lambda_1\lambda_2 + 92)t + \frac{1}{6}(-\lambda_1\lambda_2 + 10)(t_1 + t_2) = 0. \tag{2.7.4}$$

Subtracting equation (2.7.4) from equation (2.7.3) yields $(-\lambda_1\lambda_2 + 32)t + (t_1 + t_2) = 0$. Adding equation (2.7.2) we obtain $t = 0$, which implies $t_1 + t_2 = 0$. Since f is not zero, we must have $t_2 = -t_1 \neq 0$.

Claim:

$$t_{i,n} = \lambda_2^n t_i \text{ for } n > 0 \text{ and } t_{i,-n} = \lambda_1^n t_i \text{ for } n > 0.$$

We have

$$\begin{aligned} \lambda_2 t_{i,n} &= 6s_{x,n} + t_{i,n+1}, \quad n \geq 1, \\ \lambda_1 t_{i,-n} &= 6s_{x,-n+1} + t_{i,-n-1}, \quad n \geq 1. \end{aligned} \tag{2.7.5}$$

We have already proved that for every vertex $\mathcal{E}$ in the nucleus that is not of the form $\mathcal{T}_i[n]$, $f(\mathcal{E})$ is a linear combination of t and $t_1 + t_2$. If $\mathcal{E}$ is not in the nucleus, then by Theorem 2.4.2, using the multiplicities of the graph $\mathcal{G}_{x,1}$ in GL_2 (cf. [85, Example 7.3.1]), $m_{x,k}(\mathcal{E}, \mathcal{T}_1[n]) = m_{x,k}(\mathcal{E}, \mathcal{T}_2[n])$. Applying the relations of Theorem 2.5.13 to $\mathcal{E}$, we express $f(\mathcal{E})$ as a linear combination of t and $t_1 + t_2$. Thus $f(\mathcal{E}) = 0$.

Using the equations (2.7.5), we conclude that $t_{i,n} = \lambda_2^n t_i$, $t_{i,-n} = \lambda_1^n t_i$, $n \geq 1$. But this contradicts the fact that f has compact support in $GL_3(F)Z(\mathbb{A}) \setminus GL_3(\mathbb{A})$. So we must have $\lambda_1 = \lambda_2 = 0$.

Let $f \in \mathcal{A}_0(x; 0, 0)$. By the cuspidal conditions, we have $S_0 = D_{0,0} = s_{0,0} = 0$. We use this on the equations of the parametrization of eigenforms with $\lambda_1 = \lambda_2 = 0$, obtaining successively,

$$D_{2,1} = 0, \quad s_{x,2} = 0, \quad s_{x,-1} = 0, \quad t_{i,0} = 0 \quad \text{and} \quad \sum T_i = 0.$$

This proves that $f(\mathcal{E}) = 0$ if $\mathcal{E} \in \mathcal{N}_x \backslash \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4\}$. Let $\mathcal{E} \in \mathbf{P}\,\mathrm{Bun}_3\, X$ be a vertex such that $\mathcal{T}_i$ is a neighbour of $\mathcal{E}$ in the graph $\mathcal{G}_{x,j}$. As $\delta(\mathcal{T}_i) = -3$, by Theorem 2.5.6, we conclude that $\delta(\mathcal{E}) < 0$, i.e. $\mathcal{E} = \mathcal{S}_2$ or $\mathcal{E} = \mathcal{S}_1$. Therefore $f(\mathcal{E}) = 0$. We conclude by induction as in Theorem 2.5.13 that $f(\mathcal{E}) = 0$ for every vertex $\mathcal{E}$ that is not a trace. We also have by Theorem 1.5.6 that a cuspidal function in $\mathbf{P}\operatorname{Bun}_3 X$ generates an admissible $GL_3(\mathbb{A})$-module. This finishes the proof of the theorem. □

2.7.2 Eisenstein series induced from the Borel subgroup

In this section, we will study the eigenforms coming from unramified Eisenstein series from the Borel subgroup. For this we will use the parametrization of $\mathcal{A}(x; \lambda_1, \lambda_2)$ obtained above. We start with the description of the eigenvalues of the Eisenstein series by means of the Satake parameters.

Let χ be an unramified character of $T(\mathbb{A})$. Because $h_F = 1$, we see that, in the notation of chapter 1, $\Lambda(T) = X_T$. Thus there exists $s = (s_1, s_2, s_3) \in \bigoplus_{i=1}^{3}(\mathbb{C}/\frac{2\pi i}{\log 2}\mathbb{Z})$ with $s_1+s_2+s_3 = 0$ in $\mathbb{C}/\frac{2\pi i}{\log 2}\mathbb{Z}$ and such that

$$\chi(\operatorname{diag}(t_1, t_2, t_3)) = q^{-\sum_{i=1}^{3} s_i \deg(t_i)},$$

for every $t = \operatorname{diag}(t_1, t_2, t_3) \in T(\mathbb{A})$. We put

- $\lambda_1 = \lambda_{x,1} = 2(2^{-s_1} + 2^{-s_2} + 2^{s_1+s_2})$, $\lambda_2 = \lambda_{x,2} = 2(2^{-s_1-s_2} + 2^{s_1} + 2^{s_2})$,
- $\lambda_{y,1} = 4(4^{-s_1} + 4^{-s_2} + 4^{s_1+s_2}) = \lambda_{x,1}^2 - 4\lambda_{x,2}$ for $|y| = 2$,
- $\lambda_{y,2} = 4(4^{-s_1-s_2} + 4^{s_1} + 4^{s_2}) = \lambda_{x,2}^2 - 4\lambda_{x,1}$ for $|y| = 2$.

By Theorem 1.7.7, every eigenform E in $\mathcal{E}(\chi)$ satisfy $\Phi_{x,1}(E) = \lambda_1 E$, $\Phi_{x,2}(E) = \lambda_2 E$ $\Phi_{y,1}(E) = \lambda_{y,1} E$ and $\Phi_{y,2}(E) = \lambda_{y,2} E$ for $|y| = 2$.

For general ν (outside a finite union of radicial hyperplanes), the Eisenstein series $E(\phi_\chi, \nu, \chi)$ is defined and its eigenvalue λ_1 and λ_2 for the action of $\Phi_{x,1}$ and $\Phi_{x,2}$ respectively, are not both 0 (cf. sections 1.6.1 and 1.7 for notation). Therefore, by the parametrization of the spaces $\mathcal{A}(x; \lambda_1, \lambda_2)$, $E(\phi_\chi, \nu, \chi)(\mathcal{T}_i) = E(\phi_\chi, \nu, \chi)(\mathcal{T}_j)$, which implies, by analytic continuation, that $\widetilde{E}^\alpha(\phi_\chi, \nu, \chi)(\mathcal{T}_i) = \widetilde{E}^\alpha(\phi_\chi, \nu, \chi)(\mathcal{T}_j)$ for every $\nu \in X_I$. The same argument also proves that $\widetilde{E}^\alpha(\phi_\chi, \nu, \chi)(\mathcal{T}_i[0]) = \widetilde{E}^\alpha(\phi_\chi, \nu, \chi)(\mathcal{T}_j[0])$. Therefore we can use in our calculations below the notation used in the parametrization of $\mathcal{A}(x; \lambda_1, \lambda_2)$ even if $\lambda_1 = \lambda_2 = 0$.

Let $E \in \mathcal{E}(x, \chi; \lambda_1, \lambda_2)$ be an eigenform. By Theorem 2.6.3, E satisfies the eigenvalue equations

$$\lambda_{y_i,1} D_{0,0} = 7D_{2,2} + 14t_{i,1}, \quad i = 1, 2. \tag{2.7.6}$$

It follows from $\lambda_{y_1,1} = \lambda_{y_2,1}$ that $t_{1,1} = t_{2,1}$. Using the formula for $t_{i,1}$, this implies that $t_1 = t_2$ if $\lambda_{x,2} \neq 0$. And in our parameterizations of $\mathcal{A}(x; \lambda_1, \lambda_2)$, we see that this is also true when $\lambda_{x,2} = 0$. Let $\lambda_1 = \lambda_{x,1}$ and $\lambda_2 = \lambda_{x,2}$. From the eigenvalue equation (2.7.6), we obtain

$$\begin{aligned} D_{2,2} &= \tfrac{\lambda_{y,1}}{7} D_{0,0} - \textstyle\sum t_{i,1} \\ &= [\lambda_{y,1}(\lambda_1^3 + \lambda_2^3 - 14\lambda_1\lambda_2 + 92) + 6\lambda_1^2 - 30\lambda_2]t + [\tfrac{\lambda_{y,1}}{6}(-\lambda_1\lambda_2 + 10) - \lambda_2](t_1 + t_2), \end{aligned}$$

and using the expression for $\lambda_{y,1}$,

$$\begin{aligned}D_{2,2} &= [(\lambda_1^2-4\lambda_2)(\lambda_1^3+\lambda_2^3-14\lambda_1\lambda_2+92)+6\lambda_1^2-30\lambda_2]t\\ &\quad +[\tfrac{\lambda_1^2-4\lambda_2}{6}(-\lambda_1\lambda_2+10)-\lambda_2](t_1+t_2)\\ &= (\lambda_1^5+\lambda_1^2\lambda_2^3-18\lambda_1^3\lambda_2-4\lambda_2^4+56\lambda_1\lambda_2^2+98\lambda_1^2-398\lambda_2)t\\ &\quad +\tfrac{1}{6}(-\lambda_1^3\lambda_2+10\lambda_1^2+4\lambda_1\lambda_2^2-46\lambda_2)(t_1+t_2).\end{aligned}$$

Subtracting this expression from the expression of $D_{2,2}$ from before, we obtain

$$\lambda_2[(\lambda_1^3+\lambda_2^3-17\lambda_1\lambda_2+182)t+\frac{1}{6}(-\lambda_1\lambda_2+34)(t_1+t_2)]=0.$$

If $(\lambda_1,\lambda_2)\neq(0,0)$, then equation (2.7.1) implies that E satisfies

$$(\lambda_1^3+\lambda_2^3-17\lambda_1\lambda_2+182)t+\frac{1}{6}(-\lambda_1\lambda_2+34)(t_1+t_2)=0.$$

This proves the following theorem.

Theorem 2.7.3. *Let X be the elliptic curve over $\mathbb{F}_2$ with Weierstrass equation $y^2+y=z^3+z+1$, which has a unique place of degree 1 denoted by x. Let $\chi\in X_T$ such that $[(\emptyset,\chi)]\in\Pi(x;\lambda_1,\lambda_2)$ (cf. section 1.7 for notation). If $E\in\mathcal{E}(x,\chi;\lambda_1,\lambda_2)$ is an eigenform, then in the parametrization of $\mathcal{A}(x;\lambda_1,\lambda_2)$, E satisfies the relations:*

$$E(\mathcal{T}_i)=E(\mathcal{T}_j)\ \textit{for}\ i,j\in\{1,2,3,4\}\ \textit{and}\ E(\mathcal{T}_1[0])=E(\mathcal{T}_2[0]).$$

If $(\lambda_1,\lambda_2)\neq(0,0)$, then E satisfies

$$(\lambda_1^3+\lambda_2^3-17\lambda_1\lambda_2+182)t+\frac{1}{6}(-\lambda_1\lambda_2+34)(t_1+t_2)=0.$$

2.7.3 Cuspidal Eisenstein series

In this section, we will study the unramified cuspidal Eisenstein series induced from the parabolic subgroup $P_I=P_{2,1}$ contained in $\mathcal{A}(x;\lambda_1,\lambda_2)$. We start by describing the cuspidal representations of the Levi subgroup.

Let $\pi=\pi_1\otimes\chi$ be an unramified cuspidal representation of $M_I(\mathbb{A})$ where π_1 is an unramified cuspidal representation of $GL_2(\mathbb{A})$ and χ is an unramified idele class character. From $h_F=1$, it follows that the central character of π belongs to X_I, and twisting by its inverse, we can assume that π_1 has trivial central character, i.e. is a cuspidal representation of $PGL_2(\mathbb{A})$, and $\chi=\chi_0$ is the trivial idele class character. Thus it is sufficient to study the space of unramified cusp forms of $PGL_2(\mathbb{A})$. From [85, Theorem 8.2.1], the dimension of the space of unramified cusp forms in $PGL_2(\mathbb{A})$ is 2 and $\Phi_{x,1}(f)=0$ for every unramified cusp form f in $PGL_2(\mathbb{A})$. We decompose this space of dimension 2 using the Hecke operators $\Phi_{y,1}$ for $y\in|X|$ with $|y|=2$.

Let g be a cusp eigenform in $PGL_2(\mathbb{A})$. We use the notation for elements of $\mathbf{P}Bun_2X$ introduced in Theorem 2.6.2. The correspondence with the notation in [85, Example 7.3.1] is as follows (cf. [2, Section 4.6]):

- $\mathcal{M}_{y_i}$ corresponds to t_i,

- $\mathcal{S}_0^2$ corresponds to s_0,
- $\mathcal{C}_k$ corresponds to c_k.

From [85, Thm. 8.2.1 and Example 7.3.1], we have

$$g(\mathcal{C}_0) = -g(\mathcal{S}_0^2) = g(\mathcal{M}_y) + g(\mathcal{M}_{y'}) \text{ and } Supp(g) \subset \{\mathcal{S}_0^2, \mathcal{C}_0, \mathcal{M}_y, \mathcal{M}_{y'}\}.$$

From Theorem 2.6.2, we get the eigenvalue equations

- $\lambda_y g(\mathcal{C}_0) = 3g(\mathcal{C}_2) + 2g(\mathcal{M}_y)$
- $\lambda_y g(\mathcal{M}_y) = g(\mathcal{C}_0) + g(\mathcal{M}_{y'}) + 3g(\mathcal{M}_y)$,

which implies

$$(\lambda_y - 2)g(\mathcal{M}_y) + \lambda_y g(\mathcal{M}_{y'}) = 0 \text{ and } 2g(\mathcal{M}_{y'}) = (\lambda_y - 4)g(\mathcal{M}_y)$$

If $g(\mathcal{M}_y) = 0$, then $g(\mathcal{M}_{y'}) = 0$ and thus $g(\mathcal{C}_0) = g(\mathcal{S}_0^2) = 0$, so we conclude that g is identically 0, which is a contradiction. Thus $g(\mathcal{M}_y) \neq 0$. Using the eigenvalue equations at the place y, we obtain $[2(\lambda_y - 2) + \lambda_y(\lambda_y - 4)]g(\mathcal{M}_y) = 0$. Therefore $\lambda_y = 1 \pm \sqrt{5}$.

From [85, Theorem 8.2.1], the dimension of the space of unramified cusp forms in $PGL_2(\mathbb{A})$ is 2. So there are 2 cuspidal eigenforms g_1 and g_2 whose respective eigenvalues for Φ_{y_1} and Φ_{y_2} are

$$\begin{array}{ll} g_1: & \lambda_{y_1}^{(1)} = 1 + \sqrt{5},\ \lambda_{y_2}^{(1)} = 1 - \sqrt{5}, \\ g_2: & \lambda_{y_1}^{(2)} = 1 - \sqrt{5},\ \lambda_{y_2}^{(2)} = 1 + \sqrt{5}, \end{array}$$

Each g_i generates a cuspidal representation π_i' and we denote by $\pi_i = \pi_i' \otimes \chi_0$ the corresponding cuspidal representation of $M_I(\mathbb{A})$. Let $(\alpha_j^{(i)}, \beta_j^{(i)})$ be the Satake parameters associated with π_i' at the place y_j, then

$$\lambda_{y_j}^{(i)} = 2(\alpha_j^{(i)} + \beta_j^{(i)}) \quad \text{and} \quad \alpha_j^{(i)}\beta_j^{(i)} = 1.$$

Next we study the unramified Eisenstein series induced by π_i. Let $s \in X_I$ and write $s = (s_1, s_2) \in (\mathbb{C}/\frac{4\pi i}{\log 2}\mathbb{Z}) \oplus (\mathbb{C}/\frac{2\pi i}{\log 2}\mathbb{Z})$ with $s_1 + s_2 = 0$ in $\mathbb{C}/\frac{2\pi i}{\log 2}\mathbb{Z}$. Let $E \in \mathcal{E}(\pi_i, s)$ be an eigenform. The Satake parameters associated with $\pi_i(s)$ are:

- $\mathrm{diag}(i \cdot 2^{-s_1/2}, -i \cdot 2^{-s_1/2}, 2^{s_2})$ at x,
- $\mathrm{diag}(\alpha_j^{(i)} 4^{-s_1/2}, \beta_j^{(i)} 4^{-s_1/2}, 4^{s_2})$ at y_j.

Thus the eigenvalues of E for the action of the Hecke operators $\Phi_{x,i}$ and $\Phi_{y,i}$ with $|y| = 2$ and $i = 1, 2$ are:

- $\lambda_{x,1} = 2(i \cdot 2^{-s_1/2} - i \cdot 2^{-s_1/2} + 2^{s_2}) = 2^{1+s_2}, \quad \lambda_{x,2} = 2^{1-s_2}$,
- $\lambda_{y_j,1} = 4(\alpha_j^{(i)} + \beta_j^{(i)})4^{-s_1/2} + 4^{1+s_2} = \lambda_{y_j}^{(i)}\lambda_{x,2} + \lambda_{x,1}^2$,

- $\lambda_{y_j,2} = 4(\alpha_j^{(i)} + \beta_j^{(i)})4^{s_1/2} + 4^{1-s_2} = \lambda_{y_j}^{(i)}\lambda_{x,1} + \lambda_{x,2}^2$.

In particular, $\lambda_{x,1}\lambda_{x,2} = 4$. In our calculations below, we use our parametrization of $\mathcal{A}(x;\lambda_1,\lambda_2)$.

According to the eigenvalue equations from Theorem 2.6.3, we have

$$\lambda_{y_j,1}D_{0,0} = 7D_{2,2} + 14t_{j,1}.$$

We define $\lambda_1 = \lambda_{x,1}$, $\lambda_2 = \lambda_{x,2}$, $\Lambda = \lambda_{y_1}$, $\Lambda' = \lambda_{y_2}$. Using the formulas for $D_{0,0}$, $D_{2,2}$ and $t_{i,1}$, we obtain

$$(\lambda_2\Lambda + \lambda_1^2)[(\lambda_1^3 + \lambda_2^3 + 36)t + (t_1 + t_2)] = (\lambda_1^5 - 3\lambda_2^4 + 24\lambda_1^2 - 14\lambda_2)t + \lambda_1^2(t_1 + t_2) + 2\lambda_2 t_1$$

and

$$(\lambda_2\Lambda' + \lambda_1^2)[(\lambda_1^3 + \lambda_2^3 + 36)t + (t_1 + t_2)] = (\lambda_1^5 - 3\lambda_2^4 + 24\lambda_1^2 - 14\lambda_2)t + \lambda_1^2(t_1 + t_2) + 2\lambda_2 t_2$$

Adding the two equations and using $\lambda_1\lambda_2 = 4$, we obtain

$$(2\lambda_2^6 + 11\lambda_2^3 + 32\lambda_2 + 128)t = 0.$$

Thus $t = 0$. Subtracting the second equation from the first and using that $\Lambda - \Lambda' = (-1)^{i-1}2\sqrt{5}$, we obtain

$$\lambda_2(\Lambda - \Lambda')(t_1 + t_2) = 2\lambda_2(t_1 - t_2) \Longrightarrow t_1 - t_2 = (-1)^{i-1}\sqrt{5}(t_1 + t_2)$$

which implies $t_2 = \frac{\sqrt{5}-3}{2}t_1$ if $i = 1$ and $t_1 = \frac{\sqrt{5}-3}{2}t_2$ if $i = 2$.
This proves the following theorem.

Theorem 2.7.4. *Let X be the elliptic curve over $\mathbb{F}_2$ with Weierstrass equation $y^2 + y = z^3 + z + 1$. Denote by x the unique point of degree 1 of X. Let π_1, π_2 be the unramified cuspidal representations of $M_I(\mathbb{A})$ as above. Let λ_1, λ_2 be complex numbers with $\lambda_1\lambda_2 = 4$. Let $i \in \{1, 2\}$ and $s \in X_I$ such that $[(I, \pi_i(s))] \in \Pi(x;\lambda_1,\lambda_2)$ (cf. section 1.7 for notation). The space $\mathcal{E}(x, \pi_i(s), \lambda_1, \lambda_2)$ is of dimension 1. If E_i is a generator of $\mathcal{E}(x, \pi_i(s), \lambda_1, \lambda_2)$, then using the parametrization of $\mathcal{A}(x;\lambda_1,\lambda_2)$, we obtain that E_i satisfies the relations:*

- *E_1: $t = 0$ and $t_2 = \frac{\sqrt{5}-3}{2}t_1$,*

- *E_2: $t = 0$ and $t_1 = \frac{\sqrt{5}-3}{2}t_2$.*

These relations determine E_i up to multiplication by a constant.

2.8 The space of Toroidal automorphic forms

Let T' be the 3-dimensional torus associated with the constant field extension $E = \mathbb{F}_{q^3}F$ of F, with respect to a basis of E over F that is contained in $\mathbb{F}_{q^3}$. We denote by $\Theta_E : \mathbb{A}_E^\times \longrightarrow G_{\mathbb{A}_F}$ the corresponding morphism. In this section, we will study the unramified E-toroidal forms.

Theorem 2.8.1. *If T' is the torus above and $c_{T'} = vol(T'_F Z_\mathbb{A} \setminus T'_\mathbb{A})/\#(\operatorname{Pic} X_3/p^*(\operatorname{Pic} X))$, then for $f : G(F)\backslash G(\mathbb{A}_F)/K \longrightarrow \mathbb{C}$,*

$$\int_{T'_F Z_\mathbb{A}\backslash T'_\mathbb{A}} f(t)dt = c_{T'} \cdot \sum_{[\mathcal{L}]\in \operatorname{Pic} X_3/p^*(\operatorname{Pic} X)} f([p_*\mathcal{L}]).$$

Proof. We introduce the following notation. For an $x \in |X|$ that is inert in E/F, we define $\mathcal{O}_{E,x} := \mathcal{O}_{E,y}$, where y is the unique place that lies over x. For an $x \in |X|$ that is split in E/F, we define $\mathcal{O}_{E,x} := \mathcal{O}_{E,y_1} \oplus \mathcal{O}_{E,y_2} \oplus \mathcal{O}_{E,y_3}$, where y_1, y_2 and y_3 are the 3 places that lie over x. Note that there is no place that ramifies. Let $\mathcal{O}_{E_x}$ denote the completion of $\mathcal{O}_{E,x}$. Then $\mathcal{O}_{E_x}$ is a free module of rank 3 over $\mathcal{O}_{F_x} = \mathcal{O}_x$ for every $x \in |X|$.

The basis of E over F that defines T' is contained in $\mathbb{F}_{q^3}$. It is thus a basis of $\mathcal{O}_{E_x}$ over $\mathcal{O}_{F_x}$ for every $x \in |X|$, and therefore $\Theta_E(\mathcal{O}_{\mathbb{A}_E}^\times) \subset K$. This gives us a commutative diagram

$$\begin{array}{ccc} E^\times\backslash\mathbb{A}_E^\times/\mathcal{O}_{\mathbb{A}_E}^\times & \xrightarrow{1:1} & \operatorname{Pic} X_3 \\ \downarrow{\scriptstyle \Theta_E} & & \downarrow{\scriptstyle p_*} \\ G_F\backslash G_{\mathbb{A}_F}/K & \xrightarrow{1:1} & \operatorname{Bun}_3 X \end{array}$$

where the horizontal arrows are the bijections from Weil's theorem.

The action of $\mathbb{A}_F^\times$ on $E^\times\backslash\mathbb{A}_E^\times/\mathcal{O}_{\mathbb{A}_E}^\times$ and $G_F\backslash G_{\mathbb{A}_F}/K$ by scalar multiplication is compatible with the action of $\operatorname{Pic} X$ on $\operatorname{Pic} X_3$ and $\operatorname{Bun}_3 X$ by tensoring in the sense that all maps in the above diagram become equivariant if we identify $\operatorname{Pic} X$ with $F^\times \setminus \mathbb{A}_F^\times/\mathcal{O}_{\mathbb{A}_F}^\times$. Taking orbits under these compatible actions yields the commutative diagram

$$\begin{array}{ccc} E^\times\mathbb{A}_F^\times\backslash\mathbb{A}_E^\times/\mathcal{O}_{\mathbb{A}_E}^\times & \xrightarrow{1:1} & \operatorname{Pic} X_3/p^* \operatorname{Pic} X \\ \downarrow{\scriptstyle \Theta_E} & & \downarrow{\scriptstyle p_*} \\ G_F Z_{\mathbb{A}_F}\backslash G_{\mathbb{A}_F}/K & \xrightarrow{1:1} & \mathbf{P}\operatorname{Bun}_3 X \end{array}$$

Since f is right K-invariant, this commutative diagram yields the assertion of the theorem, whose constant c equal to the volume of the projection of $\mathcal{O}_{\mathbb{A}_E}^\times$ in $E^\times\mathbb{A}_F^\times\backslash\mathbb{A}_E^\times$. Evaluation at a constant function f yields $c = c_{T'}$. □

For our fixed elliptic curve X over $\mathbb{F}_2$, we next determine the T'-toroidal automorphic forms in $\mathcal{A}(x; \lambda_1, \lambda_2)$. The toroidal condition for the 3-dimensional torus T' reads

$$D_{0,0} + 3\sum T_i = 0. \tag{2.8.1}$$

In particular, we see that the cusp eigenforms are toroidal for this torus. In the case $(\lambda_1, \lambda_2) \neq (0,0)$, we see that equation (2.8.1) is equivalent to

$$D_{0,0} = -3\sum T_i = -12T = -84t.$$

Using the expression of $D_{0,0}$, we obtain

$$(\lambda_1^3 + \lambda_2^3 - 14\lambda_1\lambda_2 + 104)t + \frac{1}{6}(-\lambda_1\lambda_2 + 10)(t_1 + t_2) = 0 \tag{2.8.2}$$

Let λ_1, λ_2 and χ as in Theorem 2.7.3 and $E \in \mathcal{E}(x, \chi; \lambda_1, \lambda_2)$ an eigenform. Suppose that $(\lambda_1, \lambda_2) \neq (0,0)$. Next we analyze when E is T'-toroidal. By Theorem 2.7.3, E satisfies

$$(\lambda_1^3 + \lambda_2^3 - 17\lambda_1\lambda_2 + 182)t + \frac{1}{6}(-\lambda_1\lambda_2 + 34)(t_1 + t_2) = 0 \tag{2.8.3}$$

Suppose that E is T'-toroidal. Subtracting Equation (2.8.3) from Equation (2.8.2), we obtain

$$t_1 + t_2 = \frac{3}{4}(\lambda_1\lambda_2 - 26)t. \tag{2.8.4}$$

In particular we must have $t \neq 0$, because $E \neq 0$. Replacing Equation (2.8.4) in Equation (2.8.2) yields

$$\lambda_1^3 + \lambda_2^3 - \frac{1}{8}(\lambda_1\lambda_2)^2 - \frac{19}{2}\lambda_1\lambda_2 + \frac{143}{2} = 0.$$

Let $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1\lambda_2 = 4$. Let π_i, s be as in Theorem 2.7.4 and let $E_i \in \mathcal{E}(x, \pi_i(s), \lambda_1, \lambda_2)$ be a generator of this space. We prove that E_i is not a T'-toroidal automorphic form. In fact, the toroidal condition $D_{0,0} + 3\sum T_i = 0$ is equivalent to

$$(\lambda_1^3 + \lambda_2^3 - 14\lambda_1\lambda_2 + 104)t + \frac{1}{6}(-\lambda_1\lambda_2 + 10)(t_1 + t_2) = 0.$$

If E_i is a T'-toroidal automorphic form, then $t_1 + t_2 = 0$, because by Theorem 2.7.4 we have $t = 0$ and $\lambda_1\lambda_2 = 4$. But by Theorem 2.7.4, this implies that $t_1 = t_2 = 0$, which contradicts $E_i \neq 0$.

This proves the following theorem.

Theorem 2.8.2. *Let X be the elliptic curve over $\mathbb{F}_2$ with Weierstrass equation $y^2 + y = z^3 + z + 1$. Let $T' \subset GL_3$ be the maximal torus corresponding to the constant field extension $\mathbb{F}_8F/F$. Regarding the T'-toroidal automorphic forms we have:*

1. *Unramified cusp forms are T'-toroidal automorphic forms.*

2. *Let $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1\lambda_2 = 4$. Let M_I be the standard Levi subgroup of the standard parabolic subgroup P_I of type $(2,1)$. Let π_1, π_2 be the unramified cuspidal representations of $M_I(\mathbb{A})$ as in section 2.7.2, and $s \in X_I$ such that $[(I, \pi_i(s))] \in \Pi(x; \lambda_1, \lambda_2)$. The automorphic forms in $\mathcal{E}(x, \pi_i(s), \lambda_1, \lambda_2) \backslash \{0\}$ are not T'-toroidal.*

3. *Let $(\lambda_1, \lambda_2) \neq (0,0)$ and χ be an unramified character of $T(\mathbb{A})$ such that $[(\emptyset, \chi)] \in \Pi(x; \lambda_1, \lambda_2)$. If $E \in \mathcal{E}(x, \chi; \lambda_1, \lambda_2)$ is a T'-toroidal eigenform, then*

$$\lambda_1^3+\lambda_2^3-\frac{1}{8}(\lambda_1\lambda_2)^2-\frac{19}{2}\lambda_1\lambda_2+\frac{143}{2}=0.$$

2.8.1 An application to the Riemann hypothesis over finite fields

Let F be the function field of the elliptic curve over $\mathbb{F}_2$ with Weierstrass equation $y^2+y=z^3+z+1$ and $E=\mathbb{F}_8F$ the constant field extension to $\mathbb{F}_8$. We denote by x the unique place of F of degree 1. In this section, we show how to prove the Riemann hypothesis for the zeta function of E using the parametrization of the spaces $\mathcal{A}(x;\lambda_1,\lambda_2)$ of section 2.7.

We put $G=GL_3$ and denote by $T\subset GL_3$ the torus associated with the extension E/F, where the embedding is obtained by chosing a basis of $\mathbb{F}_8$ over $\mathbb{F}_2$ as a basis of E over F. Let V be the affine space of dimension 3. We remember that, as in section 2.1.3, the chosen basis to construct the embedding $T\subset G$ also gives us an identification of E with $V(F)$ and of $V(\mathbb{A}_F)$ with $\mathbb{A}_E$, which we use implicitly below.

We recall some facts about L-functions. If χ is a quasi-character of $F^\times\backslash\mathbb{A}^\times$ and $S=\{x\in|X|\mid\exists\, a_x\in\mathcal{O}_x^\times,\ \chi(a_x)\neq 1\}$, The *L-function of* χ is defined by

$$L_F(\chi,s)=\prod_{x\in|X|-S}\frac{1}{1-\chi(\pi_x)|\pi_x|^s}.$$

The *zeta function of* F is defined by $\zeta_F(s):=L_F(1,s)$. For the analytic properties of the L-function of the quasi-character χ, see [85, Thm. 2.2.2]. We have the following factorization of the zeta function of E,

$$\zeta_E(s)=\prod_{\omega\in\mathrm{Hom}(\mathrm{Gal}(E/F),\mathbb{C}^\times)}L_F(\widetilde{\omega},s),$$

where $\widetilde{\omega}$ corresponds to ω under the reciprocity homomorphism from class field theory (cf. [85, Sec. 2.2.9, Lem. 2.2.10]). If φ is a Schwartz-Bruhat function on $\mathbb{A}_E$, ω is a quasi-character of $E^\times\backslash\mathbb{A}^\times$, the *Tate integral* is defined by

$$\zeta(\varphi,\omega)=\int_{\mathbb{A}_E^\times}\varphi(t)\omega(t)dt.$$

The Tate integral $\zeta(\varphi,\omega|\cdot|^s)$ is a holomorphic multiple of $L_E(\omega,s)$ (cf. [85, Thm. 2.2.7]). We denote by ψ_0 the Schwartz-Bruhat function

$$\psi_0=h_F(q-1)^{-1}(vol\mathcal{O}_{\mathbb{A}_E})^{-1}\,\mathrm{char}_{\mathcal{O}_{\mathbb{A}_E}}.$$

If ω is unramified, then $\zeta(\psi_0,\omega|\cdot|^s)=L_F(\omega,s)$ (cf. [85, Thm. 2.2.7]).

Next we recall Wielonsky's formula (2.1.7) from section 2.1.3:

$$\int_{T(F)Z(\mathbb{A})\backslash T(\mathbb{A})}E(\varphi,tg,\omega)dt=\omega(\det g)\zeta(\varphi_g,\omega\circ N_{E/F})$$

where φ is a Schwartz-Bruhat function on $V(\mathbb{A})$, ω is a quasi-character of $F^\times\backslash\mathbb{A}^\times$ and

$E(\varphi, g, \omega)$ is the Eisenstein series from section 2.1.3.

To apply the theory from section 2.7 we need to compute the action of the Hecke operators $\Phi_{x,1}$ and $\Phi_{x,2}$ on $E(\varphi, g, \omega)$ when this Eisenstein series is an unramified automorphic form. First we identify this Eisenstein series as an Eisenstein series induced from a parabolic subgroup.

For $I = \{\alpha_1\} \subset \Delta$ we have $d_I = (2, 1)$. Consider a quasi-character ω of $F^\times \backslash \mathbb{A}^\times$. With ω we associate the quasi-character of $M_I(\mathbb{A})$ defined by

$$\sigma_\omega(m) = \omega(\det m_1)\omega(m_2)^{-2},$$

where $m = (m_1, m_2) \in M_I(\mathbb{A}) \simeq GL_2(\mathbb{A}) \times GL_1(\mathbb{A})$. Since $M_I(\mathbb{A}) = P_I(\mathbb{A})/N_I(\mathbb{A})$, we can extend σ_ω to $P_I(\mathbb{A})$. Let φ be a Schwartz-Bruhat function on $V(\mathbb{A}_F)$, $e = (0, 0, 1) \in V(F)$ and $\sigma \in \mathbb{R}$ such that $|\omega(t)| = |t|^\sigma$ for $t \in \mathbb{A}_F^\times$. We define

$$N(\varphi, g, \omega) = \int_{\mathbb{A}^\times} \varphi(etg)\omega(\det tg)dt,$$

which converges if $\sigma > 1/n$. We define the Eisenstein series $E(\varphi, g, \omega)$ by

$$E(\varphi, g, \omega) = \sum_{\gamma \in P_I(F)\backslash G(F)} N(\varphi, \gamma g, \omega).$$

This series converges if $\sigma > 1$. The group of quasi-characters of $F^\times \backslash \mathbb{A}_F^\times$ has a structure of complex Lie Group of dimension 1 with connected components of the form $\{\omega|\cdot|^s \mid s \in \mathbb{C}\}$, and the map $s \mapsto \omega|\cdot|^s$ is an open map. The Eisenstein series $E(\varphi, g, \omega)$ has a meromorphic continuation to the group of quasi-characters of $F^\times \backslash \mathbb{A}_F^\times$ (cf. [102, Prop. 9]). In particular, as $h_F = 1$, we see that in [102, Prop. 9] the unramified characters that appears as poles of $E(\varphi, g, \omega)$ are $\omega = \omega_0$ the trivial quasi-character and $\omega = |\cdot|$.

By a change of variables, we check that $N(\varphi, pg, \omega) = \sigma_\omega(p)N(\varphi, g, \omega)$ for every $p \in P_I(\mathbb{A})$ and every $g \in G(\mathbb{A})$. Therefore

$$N(\varphi, g, \omega) \in Ind_{P_I(\mathbb{A})}^{G(\mathbb{A})}\sigma_\omega.$$

From this it follows that the Eisenstein series $E(\varphi, g, \omega)$ is an unramified automorphic form when ω is unramified and φ is a multiple of ψ_0, where we use the identification of $V(\mathbb{A}_F)$ with $\mathbb{A}_E$ to see ψ_0 as a Schwartz-Bruhat function of $V(\mathbb{A}_F)$.

Theorem 2.8.3. *Let F be the function field of the elliptic curve X over $\mathbb{F}_2$ with Weierstrass equation $y^2 + y = z^3 + z + 1$. Let $y \in |X|$ be a place of F. Let ω be an unramified quasi-character of F, ψ_0 the Schwartz-Bruhat function defined above and $E(\psi_0, g, \omega)$ the Eisenstein series defined by Wielonsky for $GL_3(\mathbb{A})$. The action of the Hecke operator $\Phi \in \mathcal{H}_{K_x}$ on $E(\psi_0, g, \omega)$ is given by*

$$\Phi \cdot E(\psi_0, g, \omega) = \Phi^\vee(z_1, z_2, z_3)E(\psi_0, g, \omega),$$

where

$$z_1 = \omega(\pi_y)q_y, \quad z_2 = \omega(\pi_y), \quad z_3 = \omega(\pi_y)^{-2}q_y^{-1},$$

and $\Phi^\vee$ is the Satake transform of Φ as defined in section 1.3.

Proof. By Lemma 1.6.1, the square root of the modular character of $P_I(\mathbb{A})$ is given by

$$\delta_{P_I(\mathbb{A})}^{1/2}(nm) = |\det m_1|^{1/2}|m_2|^{-1} = \sigma_{|\cdot|^{1/2}}(nm)$$

where $n \in N_I(\mathbb{A})$ and $m = (m_1, m_2) \in M_I(\mathbb{A})$. Therefore

$$Ind_{P_I(\mathbb{A})}^{G(\mathbb{A})}\sigma_\omega = i_{M_I(\mathbb{A})}^{G(\mathbb{A})}\sigma_{\omega|\cdot|^{-1/2}}.$$

For $y \in |X|$, we denote by ω_y the restriction of ω to $F_y^\times \subset \mathbb{A}^\times$. It follows from the description of the induction in terms of tensor products in [27, Thm. 1.4], that

$$i_{M_I(\mathbb{A})}^{G(\mathbb{A})}\sigma_{\omega|\cdot|^{-1/2}} = \bigotimes_{x\in|X|} i_{M_I(F_x)}^{G(F_x)}\sigma_{\omega_x|\cdot|_x^{-1/2}}.$$

If χ is the quasi-character of $M_\emptyset(F_x)$ given by

$$\chi(\mathrm{diag}(t_1, t_2, t_3)) = \omega(t_1t_2)|t_1|^{-1}\omega(t_3)^{-2}|t_3|,$$

then $\sigma_{\omega_x|\cdot|_x^{-1/2}}$ is isomorphic to the unique spherical subquotient of $i_{M_\emptyset(F_x)}^{M_I(F_x)}\chi$.

By the exactness of $i_{M_I(F_x)}^{G(F_x)}$ and the isomorphism $i_{M_I(F_x)}^{G(F_x)}\left(i_{M_\emptyset(F_x)}^{M_I(F_x)}\chi\right) \simeq i_{M_\emptyset(F_x)}^{G(F_x)}\chi$ given by Theorem 1.2.8, we conclude that $i_{M_I(F_x)}^{G(F_x)}\sigma_{\omega_x|\cdot|_x^{1/2}}$ is a subquotient of the principal series $i_{M_\emptyset(F_x)}^{G(F_x)}\chi$.

It follows from $\dim(i_{M_I(F_x)}^{G(F_x)}\sigma_{\omega_x|\cdot|_x^{-1/2}})^{K_x} = 1$, Theorem 1.3.2 (i) and Theorem 1.3.6 (ii) that if v is such that $(i_{M_I(F_x)}^{G(F_x)}\sigma_{\omega_x|\cdot|_x^{-1/2}})^{K_x} = \langle v\rangle$ and $\Phi \in \mathcal{H}_{K_x}$, then

$$\Phi \cdot v = \Phi^\vee(z_1, z_2, z_3)v.$$

If $N \in Ind_{P_I(\mathbb{A})}^{G(\mathbb{A})}\sigma_\omega$, we denote by $E(N, g, \omega)$ the Eisenstein series

$$E(N, g, \omega) := \sum_{\gamma\in P_I(F)\backslash G(F)} N(\gamma g),$$

which converges if $\sigma > 1$ (The proof of convergence of the Eisenstein series in Theorem 1.6.2 also works for the Eisenstein series $E(N, g, \omega)$). The theorem follows because the map

$$\begin{array}{rcl} Ind_{P_I(\mathbb{A})}^{G(\mathbb{A})}\sigma_\omega & \longrightarrow & \mathcal{A} \\ M & \longmapsto & E(M, g, \omega) \end{array}$$

is a morphism of representations. □

The function field F has class number 1, therefore the unramified quasi-characters of $F^\times\backslash\mathbb{A}^\times$ are of the form $|\cdot|^s$, and we have

$$\int_{T(F)Z(\mathbb{A})\backslash T(\mathbb{A})} E(\psi_0, t, |\cdot|^s)dt = \zeta_E(s).$$

Suppose that $E(\psi_0, g, |\cdot|^s)$ is a T-toroidal automorphic form. If $Y = 2^s$, then by Theorem 2.8.3 and Theorem 1.3.6 (iii), we conclude that the Hecke operators $\Phi_{x,1}$ and $\Phi_{x,2}$ acts on the unramified Eisenstein series $E(\psi_0, g, \omega)$ with respective eigenvalues

$$\lambda_1 = 2\left(\frac{3}{Y} + \frac{Y^2}{2}\right) \qquad \text{and} \qquad \lambda_2 = 2\left(\frac{2}{Y^2} + \frac{3Y}{2}\right).$$

If for $i = 1, 2$ $g_i \in G(\mathbb{A})$ maps to $\mathcal{T}_i[0] \in \mathbf{P}\operatorname{Bun}_3 X$ by the bijection of Weil's theorem, then we conclude as in section 2.7.2 that

$$E(\varphi, g_1, |\cdot|^s) = E(\varphi, g_2, |\cdot|^s)$$

and that λ_1 and λ_2 satisfy

$$\lambda_1^3 + \lambda_2^3 - \frac{1}{8}(\lambda_1\lambda_2)^2 - \frac{19}{2}\lambda_1\lambda_2 + \frac{143}{2} = 0.$$

Using that $\lambda_1 = 2(\frac{3}{Y} + \frac{Y^2}{2})$ and $\lambda_2 = 2(\frac{2}{Y^2} + \frac{3Y}{2})$, we see that this equation is equivalent to

$$Y^{12} + 64 = 0.$$

Since $Y = 2^s$, we have $Re(s) = \frac{1}{2}$, which proves the Riemann hypothesis for the zeta function ζ_E. We have proven the following.

Theorem 2.8.4. *The Riemann hypothesis holds for $\zeta_E(s)$, where $E = \mathbb{F}_8F$ and F is the function field of the elliptic curve over $\mathbb{F}_2$ with Weierstrass equation $y^2 + y = z^3 + z + 1$.*

Remark 2.8.5. Observe that the representation σ_ω is not a cuspidal representation of $M_I(\mathbb{A})$, but a residual representation. In the decomposition of the space $\mathcal{A}$ of automorphic forms in Theorem 1.7.4, the Eisenstein series of Wielonsky belongs to the spaces $\mathcal{E}(\chi)$, χ a quasi-character of $M_\emptyset(\mathbb{A})$. To prove this we use the theory of residual Eisenstein series (cf. [34, Equation 3.3] for the analogue over number fields and also [90], [87, Chap. V and VI] and [76, Chap. VII and App. II]).

Chapter 3

Orthogonal period of a GL_4 Eisenstein series

3.1 Introduction

Jacquet conjectured that the orthogonal period of a GL_n Eisenstein series is related to the Whittaker coefficient of some metaplectic Eisenstein series in the double cover of GL_n (cf. [62]). In the papers [34], [33], Chinta and Offen solved the case $n = 3$ for the Borel subgroup and reformulate the problem in terms of a kind of representation numbers. In this chapter we study the case $n = 4$ for the parabolic subgroup of type $(2, 2)$. This chapter is part of a joint project with Gautam Chinta. $(2, 2)$.

3.2 An orthogonal period as a finite sum over a genus class

Let $S = \{\infty\} \cup \{p \mid p \text{ is a prime number}\}$ be the set of places of $\mathbb{Q}$ and $S_f = S - \{\infty\}$ the set of finite places. We denote by $\mathbb{Q}_\infty = \mathbb{R}$ the completion of $\mathbb{Q}$ with respect to the archimedean norm of $\mathbb{Q}$, and for p a prime number, the field of p-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic norm. The ring of p-adic integers $\mathbb{Z}_p$ is the maximal compact subring of $\mathbb{Q}_p$. For $v \in S$, we write $v < \infty$ if v is a prime number. We denote by $\mathbb{A}$ the ring of adeles of $\mathbb{Q}$, i.e. the restricted direct product of $\mathbb{Q}_v$, $v \in S$, with respect to $\mathbb{Z}_p$, $p \in S_f$. For an algebraic variety G defined over $\mathbb{Q}$ and a place v of $\mathbb{Q}$ we denote $G_v = G(\mathbb{Q}_v)$ and $G_\mathbb{A} = G(\mathbb{A})$. We consider $G = GL_4$ as an algebraic group over $\mathbb{Q}$. We denote by K the standard maximal compact subgroup of $G_\mathbb{A}$, i.e.

$$K = O(4) \prod_p GL_4(\mathbb{Z}_p),$$

where $K_\infty = O(4)$ is the orthogonal group in $GL_4(\mathbb{R})$. Let $Y = \prod'_v Y_v$ be the restricted product of sets Y_v with respect to subsets $Z_v \subset Y_v$ ($v \in S$). We define $Y_f = \prod'_{v<\infty} Y_v =$

$\prod_p' Y_p$. Let

$$X = \{g \in G : {}^t g = g\}$$

be the space of symmetric matrices in G. There is an action of G on X given by $g.x = gx^tg$. For $x \in X$, we let

$$H^x = \{g \in G : g.x = x\}$$

be the orthogonal group associated with x. For $x \in X_{\mathbb{Q}}$, we define the class of x to be

$$[x] = GL_4(\mathbb{Z}).x$$

and denote $x \sim y$ if $y \in [x]$. The genus class of x is defined as

$$[[x]] = X_{\mathbb{Q}} \cap [(G_\infty K_f).x],$$

and we denote by $[[x]]/\sim$ the set of classes in the genus class of x. Let X_∞^+ be the set of positive definite orthogonal matrices in X_∞. It is well known that if $x \in X_{\mathbb{Q}} \cap X_\infty^+$ then $[[x]]/\sim$ is a finite set (cf. [10, Prop. 2.3 and Thm. 5.1]). Let $x \in X_{\mathbb{Q}} \cap X_\infty^+$ and let $\theta \in G_\infty$ be such that

$$\theta.e = x.$$

We define

$$Stab_{G(\mathbb{Z})}(x) = \{g \in G(\mathbb{Z}) \mid g.x = x\} \text{ and } \epsilon(x) = \#Stab_{G(\mathbb{Z})}(x).$$

By [10, Prop. 2.2], we have $G_{\mathbb{A}} = G_{\mathbb{Q}} G_\infty K_f$, from which follows that the embedding of G_∞ in $G_{\mathbb{A}}$ defines a bijection

$$G_{\mathbb{Q}} \backslash G_{\mathbb{A}} / K \simeq G_{\mathbb{Z}} \backslash G_\infty / K_\infty = GL_4(\mathbb{Z}) \backslash GL_4(\mathbb{R}) / O(4).$$

The symmetric space $GL_4(\mathbb{R})/O(4)$ is identified with X_∞^+ via $g \mapsto g.e$. Thus a function ϕ on $G_{\mathbb{Q}} \backslash G_{\mathbb{A}} / K$ can be regarded as a function ϕ^+ on $G_{\mathbb{Z}} \backslash X_\infty^+$ by setting $\phi^+(g.e) = \phi(g)$ for $g \in G_\infty$.

Lemma 3.2.1. *Let ϕ be a complex valued function on $G_{\mathbb{Q}} \backslash G_{\mathbb{A}} / K$ then for all $x \in X_{\mathbb{Q}} \cap X_\infty^+$ we have*

$$\int_{H_{\mathbb{Q}}^x \backslash H_{\mathbb{A}}^x} \phi(h\theta) dh = vol((H_{\mathbb{A}_f}^x \cap K_f) H_\infty^x) \sum_{[y] \in [[x]]/\sim} \epsilon(y)^{-1} \phi^+(y).$$

Proof. For a proof cf. [34, Lem. 2.1]. □

3.3 Classical and adelic Eisenstein series

In this chapter, we consider only Eisenstein series in GL_4 induced by characters on the standard parabolic subgroup of type $(2,2)$. We write $P = MV$, where V is the unipotent radical and M is the standard Levi subgroup of P. For $\mu = (\mu_1, \mu_2) \in \mathbb{C}^2$, such that $\mu_1 + \mu_2 = 0$, we associate the character of $M_{\mathbb{A}} = P_{\mathbb{A}}/V_{\mathbb{A}}$

$$diag(m_1, m_2) \mapsto |\det m_1|_{\mathbb{A}}^{\mu_1} |\det m_2|_{\mathbb{A}}^{\mu_2},$$

which we denote also by μ. We denote by $I_P^G(\mu) = Ind_{P_{\mathbb{A}}}^{G_{\mathbb{A}}}(\mu \cdot \delta_{P_{\mathbb{A}}}^{1/2})$ the normalized induction of μ from $M_{\mathbb{A}}$ to $G_{\mathbb{A}}$. For $\varphi \in I_P^G(\mu)$, we consider the Eisenstein series

$$E_P(g, \varphi, \mu) = \sum_{\gamma \in P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}} \varphi(\gamma g).$$

Let $\varphi_\mu(vmk) = |\det m_1|_{\mathbb{A}}^{\mu_1+1} |\det m_2|_{\mathbb{A}}^{\mu_2-1}$, where $m = diag(m_1, m_2) \in M_{\mathbb{A}}$, $v \in V_{\mathbb{A}}$ and $k \in K$, be the K-invariant element of $I_P^G(\mu)$, normalized such that $\varphi_\mu(e) = 1$. We define $E_P(g;\mu) = E_P(g, \varphi_\mu, \mu)$ and $E_P^+(g \cdot e; \mu) = E_P(g;\mu)$ the associated function on $G_{\mathbb{Z}} \backslash X_\infty^+$. Next we find an expression of $E_P^+(x;\mu)$ in terms of $x \in X_\infty^+$.

For $x \in X_\infty^+$, we denote by $d_2(x)$ the determinant of the lower right 2×2 block of x. Observe that $d_2(x) > 0$ because x is positive definite. If $g \in G_\infty$, by the Iwasawa decomposition we can write $g = vmk$ with $v \in V_\infty$, $m \in M_\infty$ and $k \in K_\infty$. From this it follows that

$$\det(g.e) = |\det m_1|^2 |\det m_2|^2 \qquad \text{and} \qquad d_2(g.e) = |\det m_2|^2,$$

which implies

$$\det(g.e)^{(\mu_1+1)/2} \cdot d_2(g.e)^{-(\mu_1-\mu_2+2)/2} = |\det m_1|^{\mu_1+1} |\det m_2|^{\mu_2-1} = \varphi_\mu(g).$$

Using this and the natural bijection $P_{\mathbb{Z}} \backslash G_{\mathbb{Z}} \simeq P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}$, we express $E_P^+(x, \mu)$ as a function in $G_{\mathbb{Z}} \backslash X_\infty^+$ in the following way,

$$E_P^+(x, \mu) = \det x^{(\mu_1+1)/2} \sum_{\delta \in P_{\mathbb{Z}} \backslash G_{\mathbb{Z}}} d_2(\delta \cdot x)^{-(\mu_1-\mu_2+2)/2}. \tag{3.3.1}$$

3.4 Eisenstein series and representation numbers

For $x \in X_{\mathbb{Q}}$, we let Q_x denote the quadratic form associated with the matrix x, i.e. $Q_x(\xi) = {}^t\xi x \xi$ for $\xi \in \mathbb{R}^4$. We let $x \in X_{\mathbb{Q}} \cap X_\infty^+$ be integral, i.e. $Q_x(\xi) \in \mathbb{Z}$ for all $\xi \in \mathbb{Z}^4$. We show that for such x, the Eisenstein series $E_P^+(x;\mu)$ is a Dirichlet series in $\mu_1 - \mu_2$. We interpret the coefficients in terms of a type of representation number, which counts certain points on the (partial) flag variety $P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}$. To define the representation numbers we will use the Plücker coordinates of the flag variety. To any $g \in G_{\mathbb{Q}}$, we associate $v_2(g) \in \mathbb{Q}^6$, the vector of all

2×2 minors in the botton rows of g. For a vector $v \in \mathbb{Q}^6$, we denote by $[v]$ the associated point in the projective space $\mathbb{P}^5_{\mathbb{Q}}$. The map

$$P_{\mathbb{Q}} g \mapsto [v_2(g)]$$

is an embedding

$$P_{\mathbb{Q}} \backslash G_{\mathbb{Q}} \hookrightarrow \mathbb{P}^5_{\mathbb{Q}},$$

and if $(a : b : c : d : e : f)$ are the projective coordinates in $\mathbb{P}^5_{\mathbb{Q}}$, the image is the set of $(a : b : c : d : e : f) \in \mathbb{P}^5_{\mathbb{Q}}$ such that $af - be + cd = 0$ (cf. [94, Example 1.24]). It will be more convenient for us to use the identification $P_{\mathbb{Z}} \backslash G_{\mathbb{Z}} \simeq P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}$ and work with integral coordinates. The map

$$g \mapsto [v_2(g)]$$

also defines an embedding

$$P_{\mathbb{Z}} \backslash G_{\mathbb{Z}} \hookrightarrow \mathbb{Z}^6 / \{\pm 1\}. \tag{3.4.1}$$

We define

$$\mathcal{I}(P) = \{v \in \mathbb{Z}^6 \mid \exists g \in G_{\mathbb{Z}}, v_2(g) = v\}.$$

Therefore $v = (a, b, c, d, e, f) \in \mathcal{I}(P)$ if and only if $af - be + cd = 0$ and $gcd(a, b, c, d, e, f) = 1$.

We identify $\bigwedge^2 \mathbb{Z}^4 \simeq \mathbb{Z}^6$ by mapping the basis $\{e_i \wedge e_j\}_{1 \le i < j \le 4}$ in lexicographic order to the basis $\{e_i\}_{1 \le i \le 6}$. Let $Q_{\wedge^2 x}$ be the quadratic form corresponding to the operator $\wedge^2 x$ in $\bigwedge^2 \mathbb{Z}^4$. The operator $\wedge^2 x$ in $\bigwedge^2 \mathbb{Z}^4$ is given on the basis $\{e_i \wedge e_j\}_{1 \le i < j \le 4}$ by the matrix of 2×2 minors of x. Therefore if we consider $Q_{\wedge^2 x}$ as a quadratic form in $\mathbb{Z}^6$ through the isomorphism $\bigwedge^2 \mathbb{Z}^4 \simeq \mathbb{Z}^6$, the matrix of $Q_{\wedge^2 x}$ on the canonical basis of $\mathbb{Z}^6$ is given by the matrix of 2×2 minors of x. The representation numbers that are important to us are

$$r_P(x; k) = \#\{v \in \mathcal{I}(P) \ : \ Q_{\wedge^2 x}(v) = k\}$$

where k is a positive integer.

We define the Dirichlet series

$$Z_P(x; s) = \sum_{k \ge 1} \frac{r_P(x; k)}{k^s},$$

the genus representation numbers

$$r_P(gen(x); k) = \sum_{y \in [[x]]/\sim} \epsilon(y)^{-1} r_P(y; k)$$

and the associated Dirichlet series

$$Z_P(gen(x);s) = \sum_{k\geq 1} \frac{r_P(gen(x);k)}{k^s}.$$

By [34, Lem 3.2] we have the identity

$$d_2(\delta.x) = Q_{\wedge^2 x}(v_2(\delta)).$$

This together with Equation (3.3.1) implies the following.

Proposition 3.4.1. *Consider $x \in X_{\mathbb{Q}} \cap X_\infty^+$ and Q_x integral. Then*

$$E_P^+(x;\mu) = \frac{1}{2}\det x^{(\mu_1+1)/2} Z_P(x;(\mu_1-\mu_2+2)/2).$$

If $\theta \in G_\infty$ is such that $\theta.e = x$, then

$$\int_{H^x_{\mathbb{Q}}\backslash H^x_{\mathbb{A}}} E_P(h\theta;\mu)dh = \frac{1}{2} vol((H^x_{\mathbb{A}_f} \cap K_f)H^x_\infty)\det x^{(\mu_1+1)/2} Z_P(gen(x);(\mu_1-\mu_2+2)/2).$$

Therefore the problem of comparing the orthogonal period of $E_P(g;\mu)$ with Weyl group multiple Dirichlet series is reduced to the study of the representation numbers $r_P(gen(x);k)$, which we do in the next section for $x = I_4$.

3.5 Explicit formula for the representation numbers

In this section we solve the representation number problem of the previous section when $x = I_4$. In this case the genus $[[I_4]]$ has only one class, cf. [30, Chap. 9 Sec. 4 Cor. 2]. Inspired by [33] we compute the numbers $r(d) := r_P(I_4;d)$ in terms of class numbers of imaginary quadratic fields.

Using the identification $\bigwedge^2 \mathbb{Z}^4 \simeq \mathbb{Z}^6$ of the previous section, we obtain $Q_{\wedge^2 I_4}(v) = v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2 + v_6^2$ for $v = (v_1,v_2,v_3,v_4,v_5,v_6) \in \mathbb{Z}^6$. Therefore $r(n)$ is the number of 6-tuples $(v_1,v_2,v_3,v_4,v_5,v_6) \in \mathbb{Z}^6$ such that

- $v_1v_6 - v_2v_5 + v_3v_4 = 0$,
- $v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2 + v_6^2 = n$, and
- $gcd(v_1,v_2,v_3,v_4,v_5,v_6) = 1$.

We say that a lattice $L \subset \mathbb{Z}^n$ is primitive if $(\mathbb{Q}\cdot L) \cap \mathbb{Z}^n = L$, i.e. if L is generated by some vectors of a basis of $\mathbb{Z}^n$. Let $Gr_{2,4}(\mathbb{Z})$ denote the set of 2-dimensional primitive lattices of $\mathbb{Z}^4$. We have a natural embedding

$$\begin{array}{rccc} \Psi: & Gr_{2,4}(\mathbb{Z}) & \longrightarrow & \bigwedge^2 \mathbb{Z}^4/\{\pm 1\}, \\ & \langle v,w\rangle & \longmapsto & v\wedge w \end{array}$$

which corresponds to the embedding (3.4.1). Using the identification $\bigwedge^2 \mathbb{Z}^4 \simeq \mathbb{Z}^6$, we denote by $(a:b:c:d:e:f) \in \mathbb{Z}^6/\{\pm 1\}$ the equivalence class of $(a:b:c:d:e:f) \in \mathbb{Z}^6 \backslash \{0\}$.

We denote by $\mathbf{B}(\mathbb{Q})$ the $\mathbb{Q}$-algebra of Hamilton quaternions, by $\overline{x}$ the conjugate of any element $x \in \mathbf{B}(\mathbb{Q})$ and by $Tr(x) = x + \overline{x}$ the (reduced) trace. The (reduced) norm on $\mathbf{B}(\mathbb{Q})$ is given by

$$Nr(x) = x\overline{x} = \overline{x}x = x_0^2 + x_1^2 + x_2^2 + x_3^2,$$

where $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbf{B}(\mathbb{Q})$. We identify $\mathbb{Q}^4$ with $\mathbf{B}(\mathbb{Q})$ via the map $(a, b, c, d) \mapsto a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. We denote by $\mathbf{B}(\mathbb{Z})$ the subring of $\mathbf{B}(\mathbb{Q})$ of quaternions with integral coefficients. Let $\mathbf{B}_0(\mathbb{Q})$ denote the subset of trace zero quaternions and $\mathbf{B}_0(\mathbb{Z})$ the trace zero quaternions with integral coefficients. The identification of $\mathbb{Q}^4$ with $\mathbf{B}(\mathbb{Q})$ gives an identification of $\mathbb{Q}^3$ with $\mathbf{B}_0(\mathbb{Q})$ and of $\mathbb{Z}^3$ with $\mathbf{B}_0(\mathbb{Z})$. For $L \in Gr_{2,4}(\mathbb{Z})$, we denote by Q_L the restriction of the quadratic form

$$Q_4(x, y, z, w) = x^2 + y^2 + z^2 + w^2 = Nr(x + y\mathbf{i} + z\mathbf{j} + w\mathbf{k})$$

to L, and by $disc(Q_L)$ the determinant of the matrix of Q_L with respect to some basis of L.

Lemma 3.5.1. *If $L \in Gr_{2,4}(\mathbb{Z})$ with $L = \langle v, w \rangle$, then*

$$disc(Q_L) = Q_{\wedge I_4}(v \wedge w).$$

Proof. If $v = (a, b, c, d)$ and $w = (a', b', c', d')$, we have

$$\begin{aligned} v \wedge w =& (ab' - a'b)e_1 \wedge e_2 + (ac' - a'c)e_1 \wedge e_3 + (ad' - a'd)e_1 \wedge e_4 \\ &+ (bc' - b'c)e_2 \wedge e_3 + (bd' - b'd)e_2 \wedge e_4 + (cd' - c'd)e_3 \wedge e_4 \end{aligned}$$

$$\begin{aligned} Q_L(x, y) &= (ax + a'y)^2 + (bx + b'y)^2 + (cx + c'y)^2 + (dx + d'y)^2 \\ &= (a^2 + b^2 + c^2 + d^2)x^2 + 2(aa' + bb' + cc' + dd')xy + (a'^2 + b'^2 + c'^2 + d'^2)y^2. \end{aligned}$$

From these expressions the identity follows easily. □

Proposition 3.5.2. *If $L \in Gr_{2,4}(\mathbb{Z})$ and $\Psi(L) = (a:b:c:d:e:f)$, then*

$$\Psi(L^\perp) = (f : -e : d : c : -b : a).$$

Proof. We begin by noting that the primitive 6-tuple $(f : -e : d : c : -b : a)$ are coordinates of a pure tensor in $\bigwedge^2 \mathbb{Z}^4$ because $fa - (-e)(-b) + dc = 0$. Therefore there is a primitive 2-dimensional lattice L' in $\mathbb{Z}^4$ with $\Psi(L') = (f : -e : d : c : -b : a)$. We have to show that $L' = L^\perp$. For this we can change the language from 2-dimensional lattices to 2-dimensional planes in $\mathbb{Q}^4$ and work in the Grassmaniann variety $Gr_{2,4}$. The functions

$$(a:b:c:d:e:f) \mapsto (f:-e:d:c:-b:a) \qquad \text{and} \qquad L \mapsto L^{\perp}$$

define morphisms of the algebraic variety $Gr(2,4)$. To prove that they are equal, it is enough to show that they coincide in an Zariski open set. Consider the open set where $F \neq 0$. Multiplying by a scalar, we can assume that $f = 1$. Let $v_1 = (x_1, y_1, z_1, w_1)$ and $v_2 = (x_2, y_2, z_2, w_2)$ be two generators of the plane L and L' be the 2-dimensional plane with $\Psi(L') = (f : -e : d : c : -b : a)$. In the article [69], we have a way to construct w_1 and w_2 vectors generating L', given by $w_1 = (f, 0, -c, b)$ and $w_2 = (0, f, -e, d)$. From $\Psi(L) = (a : b : c : d : e : f)$ it follows that

$$\begin{aligned}
a =& x_1y_2 - x_2y_1, \\
b =& x_1z_2 - x_2z_1, \\
c =& x_1w_2 - x_2w_1, \\
d =& y_1z_2 - y_2z_1, \\
e =& y_1w_2 - y_2w_1, \\
f =& z_1w_2 - z_2w_1.
\end{aligned}$$

Next we compute the inner products of the v_i's with the w_j's

$$\begin{aligned}
\langle v_1, w_1\rangle =& x_1(z_1w_2 - z_2w_1) - z_1(x_1w_2 - x_2w_1) + w_1(x_1z_2 - x_2z_1) = 0, \\
\langle v_2, w_1\rangle =& x_2(z_1w_2 - z_2w_1) - z_2(x_1w_2 - x_2w_1) + w_2(x_1z_2 - x_2z_1) = 0, \\
\langle v_1, w_2\rangle =& y_1(z_1w_2 - z_2w_1) - z_1(y_1w_2 - y_2w_1) + w_1(y_1z_2 - y_2z_1) = 0, \\
\langle v_2, w_2\rangle =& y_2(z_1w_2 - z_2w_1) - z_2(y_1w_2 - y_2w_1) + w_2(y_1z_2 - y_2z_1) = 0.
\end{aligned}$$

which proves that $L' = L^{\perp}$. □

In order to express $r(d)$ in terms of squares of class numbers we need another parametrization of $Gr_{2,4}(\mathbb{Z})$ as described in [1]. We define

$$\mathbf{K}(\mathbb{Z}) = \{(a_1, a_2) \mid a_1, a_2 \in \mathbf{B}_0(\mathbb{Z})\backslash\{0\} \text{ and } Nr(a_1) = Nr(a_2)\}/\sim$$

where $(a_1, a_2) \sim (a_1', a_2')$ if there is $\lambda \in \{\pm 1\}$ with $(a_1, a_2) = (\lambda a_1', \lambda a_2')$. We denote by $[a_1, a_2]$ the equivalence class of (a_1, a_2) in $\mathbf{K}(\mathbb{Z})$. If $L \in Gr_{2,4}(\mathbb{Z})$ with $L = \langle v_1, v_2\rangle$, we put

$$\begin{aligned}
a_1(L) &:= v_1\overline{v_2} - \frac{1}{2}Tr(v_1\overline{v_2}), \\
a_2(L) &:= \overline{v_2}v_1 - \frac{1}{2}Tr(\overline{v_2}v_1),
\end{aligned}$$

and define the *Klein map*

$$\begin{array}{rccc}
\Phi : & Gr_{2,4}(\mathbb{Z}) & \longrightarrow & \mathbf{K}(\mathbb{Z}). \\
& L & \longmapsto & [a_1(L), a_2(L)]
\end{array}$$

We say that a pair of vectors $(w_1, w_2) \in \mathbb{Z}^3 \times \mathbb{Z}^3$ is pair-primitive if $\frac{1}{p}w_1 \notin \mathbb{Z}^3$ or $\frac{1}{p}w_2 \notin \mathbb{Z}^3$ for all odd primes p and if $\frac{1}{4}(w_1 + w_2) \notin \mathbb{Z}^3$ or $\frac{1}{4}(w_1 - w_2) \notin \mathbb{Z}^3$. The following result is proven in [1]. For convenience, we include a proof of this result.

Proposition 3.5.3 ([1, Lem. 2.3 and Prop. 2.5])**.** *The Klein map Φ is a well-defined bijection between $Gr_{2,4}(\mathbb{Z})$ and the set of $[a_1, a_2] \in \mathbf{K}(\mathbb{Z})$ such that (a_1, a_2) is pair-primitive and $a_1 \equiv a_2 \pmod 2$. Moreover, we have*

1. *$disc(Q_L) = Nr(a_1(L)) = Nr(a_2(L))$.*
2. *$\Phi(L^\perp) = [a_1(L), -a_2(L)]$.*

Proof. We write $L = \langle v_1, v_2 \rangle$ and define $a_1(L)$ and $a_2(L)$ as before using v_1, v_2. We have

$$\begin{aligned} Nr(a_1(L)) = a_1(L)\overline{a_1(L)} &= (v_1\overline{v_2} - \tfrac{1}{2}Tr(v_1\overline{v_2}))(v_2\overline{v_1} - \tfrac{1}{2}Tr(v_1\overline{v_2})) \\ &= Nr(v_1)Nr(v_2) + \tfrac{1}{4}Tr(v_1\overline{v_2})^2 - \tfrac{1}{2}Tr(v_1\overline{v_2})(v_2\overline{v_1} + v_1\overline{v_2}) \\ &= Nr(v_1)Nr(v_2) - \tfrac{1}{4}Tr(v_1\overline{v_2})^2 \end{aligned}$$

and analogously

$$Nr(a_2(L)) = Nr(v_1)Nr(v_2) - \tfrac{1}{4}Tr(\overline{v_1}v_2)^2 = Q(a_1(L)).$$

This proves that $disc(Q_L) = Nr(a_1(L)) = Nr(a_2(L))$. We observe that the maps

$$\begin{aligned} (u, v) &\mapsto u\overline{v} - \tfrac{1}{2}Tr(u\overline{v}), \\ (u, v) &\mapsto \overline{v}u - \tfrac{1}{2}Tr(\overline{v}u) \end{aligned}$$

are bilinear and antisymmetric. From this it follows that $[a_1(L), a_2(L)]$ does not depend on the choice of the basis v_1, v_2 of L.

We identify $\bigwedge^2 \mathbb{Z}^4$ with $\mathbb{Z}^6$ via the standard basis $1 \wedge \mathbf{i}$, $1 \wedge \mathbf{j}$, $1 \wedge \mathbf{k}$, $\mathbf{i} \wedge \mathbf{j}$, $\mathbf{i} \wedge \mathbf{k}$, $\mathbf{j} \wedge \mathbf{k}$. A direct calculation of wedge product shows that

$$v_1 \wedge v_2 = \tfrac{1}{2}(-(a_1 + a_2)_1, -(a_1 + a_2)_2, -(a_1 + a_2)_3, (a_2 - a_1)_3, (a_1 - a_2)_2, (a_2 - a_1)_1) \quad (3.5.1)$$

where $a_i = a_i(L)$ for $i = 1, 2$ and for $v \in \mathbb{Z}^3 = \mathbf{B}_0(\mathbb{Z})$ we denote the coordinates of v by v_1, v_2 and v_3. Formula (3.5.1) shows that the Plucker embedding Ψ factor through the Klein map. It follows that $a_1 \equiv a_2 \pmod 2$ because $v_1 \wedge v_2$ has integral coordinates and we see that the pair-primitive condition on (a_1, a_2) is equivalent with the vector above being primitive in $\mathbb{Z}^6$. Observe that for every pair-primitive pair (a_1, a_2) the coordinates of Equation (3.5.1) satisfies the identity $af - be + cd = 0$, which describes pure tensors in $\bigwedge^6 \mathbb{Z}^4$. Therefore Φ is a well defined bijection with image the set of $[a_1, a_2] \in \mathbf{K}(\mathbb{Z})$ such that (a_1, a_2) is pair-primitive and $a_1 \equiv a_2 \pmod 2$. Equation (3.5.1) and Proposition 3.5.2 implies that $\Phi(L^\perp) = [a_1(L), -a_2(L)]$. □

By Lemma 3.5.1 we have $r(d) = 2|\mathcal{R}_d|$, where $\mathcal{R}_d$ is the set of $L \in Gr_{2,4}(\mathbb{Z})$ such that $disc(Q_L) = d$. Therefore $r(d)$ is the number of pair primitive pairs (a_1, a_2) with $a_1 \equiv a_2 \pmod 2$ and $Nr(a_1) = Nr(a_2) = d$. We can compute $r(d)$ using ideas from [1], in particular, their Lemma 2.2 and the arguments of Corollary 2.6 and Corollary 2.7. We define

$$\mathbb{D} := \{D \in \mathbb{N} \mid D \not\equiv 0, 7, 12, 15 \pmod{16}\}.$$

Proposition 3.5.4. *Let d be a positive integer. We have $r(d) > 0$ if and only if $d \in \mathbb{D}$. Let $d \in \mathbb{D}$ and write $d = d_0 4^e f^2$ with d_0 squarefree, f odd and $e \in \{0, 1\}$. Then we have:*

$$r(d) = c_d r_3(d_0)^2 f^2 \sum_{c|f} \frac{2^{\omega(c)}}{c} \prod_{p|f} \left(1 - p^{-1}\left(\frac{-d_0}{p}\right)\right)^{e_p(f/c)}, \tag{3.5.2}$$

where

$$c_d = \begin{cases} 1 & \text{if } d \equiv 3 \pmod 4, \\ 1/3 & \text{if } d \equiv 1, 2 \pmod 4, \\ 2/3 & \text{if } d \equiv 0 \pmod 4, \end{cases}$$

$$e_p(n) = \begin{cases} 2 & \text{if } p|n, \\ 1 & \text{if } p \nmid n, \end{cases}$$

$\left(\frac{-d_0}{p}\right)$ is the Legendre symbol and $\omega(c) = \sum_{p|c} 1$.

Proof. If d is a positive integer, we denote by $r_3'(d)$ the number of triples $(a, b, c) \in \mathbb{Z}^3$ such that $a^2 + b^2 + c^2 = d$ and $gcd(a, b, c) = 1$. We recall Legendre's theorem, which says that $r_3'(d) > 0$ if and only if $d \not\equiv 0, 4, 7 \pmod 8$. Therefore $r(d) = 0$ if $d \equiv 7 \pmod 8$.

Let $d \equiv 3 \pmod 4$. As in the proof of Corollary 2.6 in [1], we see that $r(d)$ is the number of pair-primitive tuples (v, v') such that $Nr(v) = Nr(v') = d$, the condition $v \equiv v' \pmod 2$ being automatically satisfied. The pair primitive tuples (v, v') are precisely of the form $(cw, c'w')$ with $Nr(w) = \frac{d}{c^2}$, $Nr(w') = \frac{d}{c'^2}$, w and w' primitive vectors and $gcd(c, c') = 1$. So writing $d = d_0 f^2$ with d_0 squarefree, we obtain

$$r(d) = \sum_{\substack{c,c'|f \\ gcd(c,c')=1}} r_3'\left(\frac{d}{c^2}\right) r_3'\left(\frac{d}{c'^2}\right). \tag{3.5.3}$$

Let $d \equiv 1, 2 \pmod 4$. As in the previous case, $r(d)$ is the number of pair-primitive tuples (v, v') such that $Nr(v) = Nr(v') = d$, and $v \equiv v' \pmod 2$. The pair-primitive tuples (v, v') are of the form $(cw, c'w')$ with $Nr(w) = \frac{d}{c^2}$, $Nr(w') = \frac{d}{c'^2}$, w and w' primitive vectors,

$gcd(c,c')=1$ and $w\equiv w' \pmod 2$. This reduces the choices of w' to a third of the options. So writing $d=d_0f^2$ with d_0 squarefree, we obtain

$$r(d)=\frac{1}{3}\sum_{\substack{c,c'|f\\ gcd(c,c')=1}} r_3'\left(\frac{d}{c^2}\right)r_3'\left(\frac{d}{c'^2}\right). \tag{3.5.4}$$

Let $d\equiv 0 \pmod 4$. In this case the pair-primitives tuples (w,w') with $Nr(w)=Nr(w')=d$ and $w\equiv w' \pmod 2$ are of the form $(2v,2v')$ with (v,v') pair-primitive, $Nr(v)=Nr(v')=\frac{d}{4}$ and $v\not\equiv v' \pmod 2$. Therefore $r(d)$ is the number of such pair-primitive tuples (v,v'). In particular we have $r(d)=0$ if $\frac{d}{4}\equiv 0,3 \pmod 4$. Next we suppose that $\frac{d}{4}\equiv 1,2 \pmod 4$. The pair-primitive tuples (v,v') are of the form $(cw,c'w')$ with $Nr(w)=\frac{d}{4c^2}$, $Nr(w')=\frac{d}{4c'^2}$, w and w' primitive vectors, $gcd(c,c')=1$ and $w\not\equiv w' \pmod 2$. This reduces the choices of w' to two thirds of the options. So writing $\frac{d}{4}=d_0f^2$ with d_0 squarefree and f odd, we obtain

$$r(d)=\frac{2}{3}\sum_{\substack{c,c'|f\\ gcd(c,c')=1}} r_3'\left(\frac{d}{4c^2}\right)r_3'\left(\frac{d}{4c'^2}\right). \tag{3.5.5}$$

Let n be a positive integer and write $n=n_0m^2$ with n_0 the squarefree part of n. We suppose that m is odd. We have the following formula for $r_3'(n)$ (cf. [40]):

$$r_3'(n)=r_3(n_0)m\prod_{p|m}\left(1-p^{-1}\left(\frac{-n_0}{p}\right)\right)$$

Thus

$$r_3'\left(\frac{d}{c^2}\right)r_3'\left(\frac{d}{c'^2}\right)=r_3\left(d_0\right)^2\frac{f^2}{cc'}\prod_{p|f}\left(1-p^{-1}\left(\frac{-d_0}{p}\right)\right)^{e_p},$$

where

$$e_p=e_p\left(\tfrac{f}{cc'}\right)=\begin{cases}2 & \text{if } p|\frac{f}{cc'},\\ 1 & \text{if } p\nmid\frac{f}{cc'}.\end{cases}$$

Using this expression in (3.5.3), (3.5.4) and (3.5.5) we obtain (3.5.2). □

If $d_0>3$, we can also write Equation (3.5.2) in terms of class numbers using the formulas

$$r_3(d_0)=\begin{cases}24h_K & \text{when } d_0\equiv 3 \pmod 8,\\ 12h_K & \text{when } d_0\equiv 1,2 \pmod 4,\end{cases} \tag{3.5.6}$$

where $K=\mathbb{Q}(\sqrt{-d_0})$ and h_K is the class number of K (see [47, Prop. 2.3]).

Proposition 3.5.4 also allows us to obtain expressions for the Dirichlet series

$$Z_P(w):=Z_P(I_4;w)=\sum_{d=1}^{\infty}\frac{r(d)}{d^w}. \tag{3.5.7}$$

As these formulas depend on d mod 4 we split our sum into distinct congruence classes mod 4. For simplicity, we focus on the case $d \equiv 3 \pmod 4$. The other case are handled similarly. Let

$$Z_P^{(3)}(w) = \sum_{d \equiv 3 \,(\mathrm{mod}\, 4)} \frac{r(d)}{d^w} = \sum_{\substack{d \equiv 3 \,(\mathrm{mod}\, 4) \\ \square\text{-free}}} \sum_{f \geq 1, \mathrm{odd}} \frac{r(d_0 f^2)}{(d_0 f^2)^w}. \tag{3.5.8}$$

In fact, since $r(d) = 0$ for $d \equiv 7 \pmod 8$, the above sum is only over $d \equiv 3 \pmod 8$.

Theorem 3.5.5. *For $re(w)$ sufficiently large,*

$$Z_P^{(3)}(w) = \sum_{\substack{d \equiv 3 \,(\mathrm{mod}\, 4) \\ \square\text{-free}}} \frac{r_3(d_0)^2 P_{d_0}(w)}{d_0^w} \tag{3.5.9}$$

where P_{d_0} is given by the Euler product

$$P_{d_0}(w) = \prod_p P\left(p^{-w}, \left(\tfrac{-d_0}{p}\right)\right),$$

and

$$P(y, \epsilon) = \frac{1 + (\epsilon^2 - 2\epsilon + p - 2\epsilon p)y^2 + \epsilon^2 p y^4}{(1 - py^2)(1 - p^2 y^2)}. \tag{3.5.10}$$

Proof. By (3.5.7) and Proposition 3.5.4, we have

$$Z_P^{(3)}(w) = \sum_{\substack{d \equiv 3 \,(\mathrm{mod}\, 4) \\ \square\text{-free}}} \frac{r_3(d_0)^2}{d_0^w} \left[\sum_{f \geq 1, \mathrm{odd}} f^{2-2w} \sum_{c | f} \frac{2^{\omega(c)}}{c} \prod_{p | f} \left(1 - p^{-1} \left(\frac{-d_0}{p}\right)\right)^{e_p(f/c)} \right].$$

The inner sum over f is an Euler product, and setting $y = p^{-w}$, $A = 1 - p^{-1}\left(\frac{-d_0}{p}\right)$, its p-part is

$$P(y, \epsilon) = 1 + p^2 y^2 A \left(A + \frac{2}{p}\right) + \sum_{k=2}^{\infty} A p^{2k} y^{2k} \left[A + \frac{2A}{p} \frac{1 - p^{1-k}}{1 - p^{-1}} + \frac{2}{p^k}\right] \tag{3.5.11}$$

for $\epsilon = \left(\frac{-d_0}{p}\right)$. Summing the geometric series above and combining, we arrive at (3.5.10). $\square$

3.6 Weyl group multiple Dirichlet series

In this section, we show that the Dirichlet series constructed from the coefficients counting planes in $\mathbb{Z}^4$ coincides with a specialization of a multiple Dirichlet series arising in the Fourier expansion of the minimal parabolic Eisenstein series on a metaplectic double cover of $SL(4)$. In a more general context Brubaker, Bump and Friedberg [18] have expressed the Fourier coefficients of the Eisenstein series on the n-fold cover of $SL(r)$ in terms of crystal bases. We will instead use formulas of Chinta and Gunnells which Brubaker, Bump, Friedberg and Hoffstein [19] have shown to be equal to the ones in [18]. Actually [18] works over a number field containing a 4^{th} root of unity; the formulas over $\mathbb{Q}$ require a modification at the prime 2, which fortunately plays no role in the present work. We refer the reader to Karasiewicz [67] for the analogous formulas on the double cover of $SL(3)$ over $\mathbb{Q}$.

We now define the multiple Dirichlet series to which we must compare $Z_P(w)$. This is the A_3 quadratic Weyl group multiple Dirichlet series and arises in the Fourier expansion of the Borel Eisenstein series on a metaplectic double cover of $SL(4)$. There are various ways to define this series, but we follow the presentation of Chinta-Gunnells [32]. Let ψ_1, ψ_2, ψ_3 be three primitive, quadratic Dirichlet characters unramified away from 2. Thus each of the ψ_i is either trivial or one of $\chi_{-4}, \chi_8, \chi_{-8}$. Define

$$Z_{A_3}(s_1, s_2, w; \psi_1, \psi_2, \psi_3) = \sum_{\substack{d,n_1,n_2>0 \\ odd}} \frac{\chi_{d'}(\hat{n}_1)\chi_{d'}(\hat{n}_2)}{n_1^{s_1} n_2^{s_2} d^w} a(n_1, n_2, d)\psi_1(n_1)\psi_2(n_2)\psi_3(d), \tag{3.6.1}$$

where

- $d' = (-1)^{(d-1)/2} d$ and $\chi_{d'}$ is the Kronecker symbol associated to the squarefree part of d'
- $\hat{n}$ is the part of n relatively prime to the squarefree part of d
- the coefficients $a(n_1, n_2, d)$ are weakly multiplicative in all entries and are defined on prime powers by

$$H(x_1, x_2, y) = \sum_{k,l,m} a(p^k, p^l, p^m) x_1^k x_2^l y^m \tag{3.6.2}$$

$$= \frac{1 - x_1 y - x_2 y + x_1 x_2 y + p x_1 x_2 y^2 - p x_1 x_2^2 y^2 - p x_1^2 x_2 y^2 - p x_1^2 x_2^2 y^3}{(1 - x_1)(1 - x_2)(1 - y)(1 - p x_1^2 y^2)(1 - p x_2^2 y^2)(1 - p^2 x_1^2 x_2^2 y^2)}. \tag{3.6.3}$$

As shown in [32], we can write this as

$$Z_{A_3}(s_1, s_2, w; \psi_1, \psi_2, \psi_3) = \sum_{\substack{d_0>0 \\ odd, \square\text{-}free}} \frac{L_2(s_1, \chi_{d_0'}\psi_1) L_2(s_2, \chi_{d_0'}\psi_2)\psi_3(d_0)}{d_0^w} Q_{d_0}(s_1, s_2, w; \psi_1, \psi_2), \tag{3.6.4}$$

say, where Q_{d_0} is the Euler product

$$Q_{d_0}(s_1, s_2, w; \psi_1, \psi_2) = \prod_{p \ odd} Q_{d_0,p}(\epsilon_{1,p} p^{-s_1}, \epsilon_{2,p} p^{-s_2}, p^{-w}) \tag{3.6.5}$$

with $\epsilon_{1,p} = \chi_{d_0'}(\hat{p})\psi_1(p), \epsilon_{2,p} = \chi_{d_0'}(\hat{p})\psi_2(p)$ and

$$Q_{d_0,p}(x_1, x_2, y) = \begin{cases} \frac{H(x_1,x_2,y)+H(x_1,x_2,-y)}{2}(1 - x_1)(1 - x_2) & \text{if } p \nmid d_0, \\ \frac{H(x_1,x_2,y)-H(x_1,x_2,-y)}{2} & \text{if } p | d_0. \end{cases} \tag{3.6.6}$$

To go further it is convenient to divide the sum over d into congruence classes mod 8. As in the computation of $Z_P^{(3)}(s)$ above, we will concentrate on the case $d \equiv 3 \pmod 8$. Define

$$\begin{aligned} Z_{A_3}^{(3)}(s_1, s_2, w) &= \tfrac{1}{4}\left[Z_{A_3}(s_1, s_2, w; 1, 1, 1) - Z_{A_3}(s_1, s_2, w; 1, 1, \chi_{-4})\right. \\ &\qquad \left. - Z_{A_3}(s_1, s_2, w; 1, 1, \chi_8) + Z_{A_3}(s_1, s_2, w; 1, 1, \chi_{-8})\right] \\ &= \sum_{\substack{0<d_0\equiv 3 \,(\mathrm{mod}\, 8)\\ \square\text{-free}}} \frac{L_2(s_1, \chi_{-d_0})L_2(s_2, \chi_{-d_0})}{d_0^w} Q_{d_0}(s_1, s_2, w; 1, 1). \end{aligned}$$

If $s_1 = s_2 = 1$, then we have

$$\begin{aligned} Z_{A_3}^{(3)}(1, 1, w) &= \sum_{\substack{0<d_0\equiv 3 \,(\mathrm{mod}\, 8)\\ \square\text{-free}}} \frac{L_2(1, \chi_{-d_0})^2}{d^w} Q_{d_0}(1, 1, w; 1, 1) && (3.6.7) \\ &= \frac{9}{4} \sum_{\substack{0<d_0\equiv 3 \,(\mathrm{mod}\, 8)\\ \square\text{-free}}} \frac{L(1, \chi_{-d_0})^2}{d^w} Q_{d_0}(1, 1, w; 1, 1) && (3.6.8) \\ &= \frac{9}{4} \cdot \left(\frac{\pi^2}{576}\right) \sum_{\substack{0<d_0\equiv 3 \,(\mathrm{mod}\, 8)\\ \square\text{-free}}} \frac{r_3(d_0)^2}{d^{w+1}} Q_{d_0}(1, 1, w; 1, 1) && (3.6.9) \end{aligned}$$

where we have used that for squarefree $d_0 \equiv 3 \pmod 8$. We have

$$1 - \frac{\chi_{-d_0}(2)}{2} = \frac{3}{2}, \tag{3.6.10}$$

and by [15, Chap. 5, 1.1 Thm. 2] and (3.5.6), it follows that

$$L(1, \chi_{-d_0}) = \frac{\pi r_3(d_0)}{24\sqrt{d_0}}. \tag{3.6.11}$$

Let us write $Q_{d_0}(w)$ for $Q_{d_0}(1, 1, w; 1, 1)$.

Theorem 3.6.1. *We have*

$$\frac{\pi^2}{256}\zeta_2(2w)\zeta_2(2w-1)Z_P^{(3)}(w) = Z_{A_3}^{(3)}(1, 1, w-1).$$

Proof. Comparing (3.5.9) with the last line of (3.6.7) we see that we need to prove

$$\zeta_2(2w)\zeta_2(2w-1)P_{d_0}(w) = Q_{d_0}(w-1) \tag{3.6.12}$$

As both sides are Euler products it suffices to show that the p-parts match, for all odd primes p. Let $\epsilon = \chi_{d_0}(p)$. From (3.5.10) of Theorem 3.5.5 the p-part of the lefthand side of (3.6.12) is

$$\frac{P(y,\epsilon)}{(1-y^2)(1-py^2)} = \begin{cases} \dfrac{1}{(1-py^2)(1-p^2y^2)} & \text{if } \epsilon = 1, \\ \dfrac{1+3(p+1)y^2+py^4}{(1-y^2)(1-py^2)^2(1-p^2y^2)} & \text{if } \epsilon = -1, \\ \dfrac{1+py^2}{(1-y^2)(1-py^2)^2(1-p^2y^2)} & \text{if } \epsilon = 0. \end{cases} \tag{3.6.13}$$

On the other hand, the p-part of the righthand side is given in (3.6.5), (3.6.6) to be

$$Q_{d_0,p}(\tfrac{1}{p},\tfrac{1}{p},py) = \begin{cases} \frac{H(\frac{1}{p},\frac{1}{p},py)+H(\frac{1}{p},\frac{1}{p},-py)}{2}(1-\frac{1}{p})^2 & \text{if } \epsilon = 1, \\ \frac{H(-\frac{1}{p},-\frac{1}{p},py)+H(-\frac{1}{p},-\frac{1}{p},-py)}{2}(1+\frac{1}{p})^2 & \text{if } \epsilon = 1, \\ \frac{H(\frac{1}{p},\frac{1}{p},py)-H(\frac{1}{p},\frac{1}{p},-py)}{2} & \text{if } \epsilon = 0. \end{cases} \tag{3.6.14}$$

Using the definition of H in (3.6.2) we readily verify that p-parts of (3.6.13) and (3.6.14) match up in each of the 3 cases. □

3.7 Relation between Q_L and $Q_{L^\perp}$

In this section we study more closely the sets $\mathcal{R}_d$, when d is squarefree. We use the results from [47] to obtain a natural $(Cl_K)^2$ torsor structure in a certain quotient related with $\mathcal{R}_d$, where Cl_K is the ideal class group of $K=\mathbb{Q}(\sqrt{-d})$. Theorems 3.7.4 and 3.7.5 below exhibits relations between the Klein map and the Gauss map defined in [33]. We recall that a *torsor* for a group G is a set X with an action of G, which is transitive and with trivial stabilizers.

Let $d \in \mathbb{D}$ be a squarefree integer and put $K=\mathbb{Q}(\sqrt{-d})$. We define for n a positive integer,

$$\mathcal{R}_3(n) = \{(x,y,z) \in \mathbb{Z}^3 \mid x^2+y^2+z^2 = n \text{ and } gcd(x,y,z)=1\},$$

and for $d \in \mathbb{D}$ be a squarefree integer

$$\widetilde{\mathcal{R}_3(d)} = \begin{cases} SO_3(\mathbb{Z}) \setminus \mathcal{R}_3(d) & \text{if } d_0 \equiv 3 \pmod 8, \\ SO_3(\mathbb{Z})^+ \setminus \mathcal{R}_3(d) & \text{if } d_0 \equiv 1,2 \pmod 4, \end{cases}$$

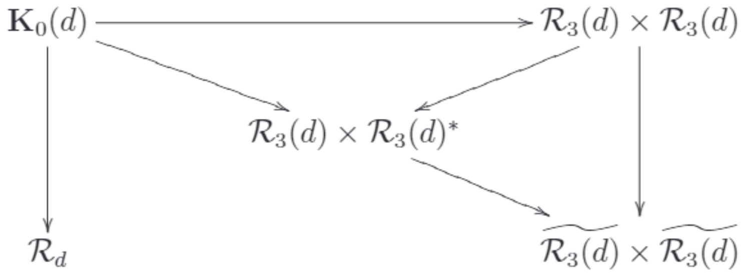
where $SO_3(\mathbb{Z})^+$ is the index-2 subgroup of $SO_3(\mathbb{Z})$ that consists of the matrices that act on the coordinate lines via even permutations. By [47, Prop. 3.5], the quotient $\widetilde{\mathcal{R}_3(d)}$ has a natural Cl_K-torsor structure. This induces a $(Cl_K)^2$-torsor structure on $\widetilde{\mathcal{R}_3(d)} \times \widetilde{\mathcal{R}_3(d)}$.

We define

$$\mathbf{K}_0(d) = \{(a_1,a_2) \in \mathbb{Z}^3 \times \mathbb{Z}^3 \mid Nr(a_1) = Nr(a_2) = d, (a_1,a_2) \text{ pair-primitive}, \ a_1 \equiv a_2 \mod 2\}.$$

By the Klein map we have a $2:1$-map $\mathbf{K}_0(d) \to \mathcal{R}_d$. We define $\mathcal{G}$ as the trivial group if $d \equiv 3 \pmod 4$ and as the group A_3 of even permutations if $d \equiv 1,2 \pmod 4$. The group

$\mathcal{G}$ acts on $\mathcal{R}_3(d)$ by permutation of coordinates. We define $\mathcal{R}_3^*(d) = \mathcal{G} \setminus \mathcal{R}_3(d)$. By the argument of the proof of Proposition 3.5.4, it follows that the map $\mathbf{K}_0(d) \to \mathcal{R}_3(d) \times \mathcal{R}_3^*(d)$ is a bijection. So using the bijection $\mathbf{K}_0(d) \to \mathcal{R}_3(d) \times \mathcal{R}_3^*(d)$ and the natural projection $\mathcal{R}_3(d) \times \mathcal{R}_3^*(d) \to \widetilde{\mathcal{R}_3(d)} \times \widetilde{\mathcal{R}_3(d)}$, we can consider $\widetilde{\mathcal{R}_3(d)} \times \widetilde{\mathcal{R}_3(d)}$ as a quotient of $\mathbf{K}_0(d)$. This gives us the desired quotient related with $\mathcal{R}_d$ and with a natural $(Cl_K)^2$-torsor structure. We express this by the following diagram.



Observe that the map $\mathbf{K}_0(d) \to \widetilde{\mathcal{R}_3(d)} \times \widetilde{\mathcal{R}_3(d)}$ don't factorize through $\mathbf{K}_0(d) \to \mathcal{R}_d$.

Let $d \in \mathbb{D}$ be a squarefree integer and $L \in \mathcal{R}_d$. It follows from Lemma 3.5.1 and Proposition 3.5.2 that $disc(Q_L) = disc(Q_{L^\perp})$. Consider the pairs (L, Q_L) and $(L^\perp, Q_{L^\perp})$. As $(L^\perp, Q_{L^\perp})$ is determined by (L, Q_L), we want to find an intrinsic relation between the two pairs. We prove below that (L, Q_L) determines the genus class of the quadratic form $Q_{L^\perp}$. For this we need the concepts of Legendre composition introduced in [98].

Definition 3.7.1. A *quadratic lattice* is a pair (L, Q_L) where L is a lattice and Q_L is a integer valued quadratic form on L. We say that two quadratic lattices are isomorphic if there is a linear isomorphism between them which preserves the quadratic forms. If $\mathrm{rk} L = 2$ we call (L, Q) a *binary quadratic lattice.*

Definition 3.7.2 ([98, Defn. 2.1])**.** We say that the binary quadratic lattice (M, Q_M) is a *Legendre composition* of the binary quadratic lattices (L, Q_L) and $(L', Q_{L'})$ if there is a linear and surjective homomorphism $\mu : L \otimes L' \to M$ such that

$$Q_L(u)Q_{L'}(v) = Q_M(\mu(u \otimes v)).$$

Next we review some results about binary quadratic over forms $\mathbb{Z}$ and a theorem of Gauss. For proofs of the results of this and the next paragraph cf. [33, Sec. 4] and [16]. We denote the quadratic form $q(x, y) = ax^2 + bxy + cy^2$ with $a, b, c \in \mathbb{Z}$ by $[a, b, c]$. The discriminant of q is the number $b^2 - 4ac$. Observe that this discriminant is equal to $-4d'$, where d' is the determinant of a matrix that represents the quadratic form q. We call two binary quadratic forms q and q' over $\mathbb{Z}$ *properly equivalent* or $SL_2(\mathbb{Z})$-equivalent if there is a $g \in SL_2(\mathbb{Z})$ such that

$$q'(x, y) = q((x, y)g).$$

We say that q and q' are in the same *genus* if for every prime number p there is a $g_p \in GL_2(\mathbb{Z}_p)$ such that

$$q'(x, y) = q((x, y)g_p)$$

and there is $g_\infty \in GL_2(\mathbb{R})$ such that

$$q'(x, y) = q((x, y)g_\infty).$$

Let D be a negative discriminant. Write $D = df^2$ where d is a fundamental discriminant (i.e. either $d \equiv 1 \pmod 4$ and squarefree or $d = 4d'$ with $d' \not\equiv 1 \pmod 4$ and d' squarefree). We will assume f is odd. Let $Cl(D)$ be the group of $SL_2(\mathbb{Z})$-equivalence classes of primitive integral binary quadratic forms of discriminant D (for the description of the group structure on $Cl(D)$ cf. [16] and [30, Chap. 14]). We call e a prime discriminant if $e = -4, 8, -8$ or $p' = (-1)^{(p-1)/2}p$ for an odd prime p. Note that e is a fundamental discriminant. Write $D = D_1D_2$ where D_1 is an even fundamental discriminant and D_2 is an odd discriminant. Let D_0 be D_1 times the product of the prime discriminants dividing D_2.

For each odd prime p dividing D we define a character $\chi^{(p)}$ on $Cl(D)$ by

$$\chi^{(p)}([a, b, c]) = \begin{cases} \chi_{p'}(a) & \text{if } \ gcd(p, a) = 1, \\ \chi_{p'}(c) & \text{if } \ gcd(p, c) = 1. \end{cases}$$

The primitivity of $[a, b, c]$ ensures that at least one of these two conditions occur. These characters generate a group $\mathcal{X}(D)$, called the group of genus class characters of $Cl(D)$. The order of $\mathcal{X}(D)$ is $2^{\omega(D)-1}$, where $\omega(D)$ is the number of distinct prime divisors of D. For each squarefree odd number e_1 dividing D we define the genus class character

$$\chi_{e_1,e_2} = \prod_{p|e_1} \chi^{(p)},$$

where $e_1e_2 = D_0$. Then as e_1 ranges over the squarefree positive odd divisors of D, χ_{e_1,e_2} range over the genus characters exactly once (if D is even) or twice (if D is odd). Two forms are in the same genus if and only if $\chi(q_1) = \chi(q_2)$ for all $\chi \in \mathcal{X}(D)$.

We define the Gauss map as follows. Let n be a positive integer which is not divisible by 4. Let

$$D = \begin{cases} -4n & \text{if } \ n \equiv 1 \text{ or } 2 \pmod 4, \\ -n & \text{if } \ n \equiv 3 \pmod 4. \end{cases} \tag{3.7.1}$$

Consider $\mathbb{Q}^3$ equipped with the quadratic form $Q(x, y, z) = x^2 + y^2 + z^2$. We have a map from $\mathcal{R}_3(n)$ to equivalence classes of primitive binary quadratic forms of discriminant D defined as follows. Let $v \in \mathcal{R}_3(n)$. Let W be the orthogonal complement of v in $\mathbb{Q}^3$. We take $M(v) = \mathbb{Z}^3 \cap W$ if $n \equiv 1, 2 \pmod 4$ and $M(v) = \frac{1}{2}\mathbb{Z}^3 \cap W$ if $n \equiv 3 \pmod 4$. The quadratic form $Q|_{M(v)}$ is primitive of discriminant D. We call a basis (u, u') of $M(v)$ oriented if (u, u', v) is an oriented basis of $\mathbb{Q}^3$. We define the $SL_2(\mathbb{Z})$-class of the quadratic form $Q|_{M(v)}$ using an oriented basis of $M(v)$. We have $Q|_{M(v)} \in Cl(D)$, and the map $\Phi : \mathcal{R}_3(n) \to Cl(D)$ defined by $\Phi(v) = Q|_{M(v)}$ is called the *Gauss map*.

Theorem 3.7.3. *(Gauss) Let n be a positive integer which is not divisible by 4, D as in (3.7.1). A quadratic form $q \in Cl(D)$ is in the image of the Gauss map Φ if and only if for any genus character χ_{e_1,e_2} of $Cl(D)$ with e_1 odd, we have.*

$$\chi_{e_1,e_2}(q) = \begin{cases} \chi_{-8}(|e_1|) & \text{if } n \equiv 3 \pmod 4, \\ \chi_{-4}(|e_1|) & \text{if } n \equiv 1 \text{ or } 2 \pmod 4. \end{cases} \tag{3.7.2}$$

If $n \equiv 1, 2 \pmod 4$, we call $\mathcal{G}_n$ the genus class of quadratic forms which satisfy Equations (3.7.2). If $n \equiv 3 \pmod 4$, we call $\mathcal{G}_n$ the genus class of the quadratic forms $2q$ with q satisfying Equations (3.7.2).

After this digression we come back to the 2-dimensional lattices $L \subset \mathbb{Z}^4$. Let $L \in \mathcal{R}_d$ with $L^\perp$ the orthogonal to L inside $\mathbb{Z}^4$. We consider the maps

$$\begin{array}{rccc} \mu_1 : & L \otimes L^\perp & \longrightarrow & \mathbf{B}_0(\mathbb{Z}), \\ & v \otimes w & \longmapsto & v\overline{w} \end{array}$$

and

$$\begin{array}{rccc} \mu_2 : & L \otimes L^\perp & \longrightarrow & \mathbf{B}_0(\mathbb{Z}). \\ & v \otimes w & \longmapsto & \overline{v}w \end{array}$$

Theorem 3.7.4. *Let $L \in \mathcal{R}_d$ with $d \in \mathbb{D}$ squarefree, and for $i = 1, 2$, we define $M_i = (a_i(L))^\perp$, where we take the orthogonal space inside $\mathbb{Z}^3$. We consider in M_i the quadratic form Q_{M_i} which is the restriction of the quadratic form Nr on $\mathbb{Z}^3 = \mathbf{B}_0(\mathbb{Z})$. Then the 2-dimensional lattice M_i is the image of μ_i and (M_i, Q_{M_i}) is a Legendre composition of (L, Q_L) and $(L^\perp, Q_{L^\perp})$.*

Proof. We write $L = \langle v_1, v_2 \rangle$. We denote by M_1' the image of μ_1. Observe that the map $w \mapsto \mu_1(v_2 \otimes w)$ from $L^\perp$ to $\mathbf{B}_0(\mathbb{Z})$ is injective, therefore $\mathrm{rk} M_1'$ is equal to 2 or 3. On the other hand, if $w \in L^\perp$, then

$$\begin{aligned} \langle a_1(L), v_2\overline{w} \rangle &= -\frac{1}{2} Tr(v_1\overline{v_2}v_2\overline{w} - \frac{1}{2} Tr(v_1\overline{v_2})v_2\overline{w}) \\ &= -\frac{1}{2} Nr(v_2) Tr(v_1\overline{w}) + \frac{1}{4} Tr(v_1\overline{v_2}) Tr(v_2\overline{w}) = 0 \end{aligned}$$

Using that $-a_1(L) = v_2\overline{v_1} - \frac{1}{2} Tr(v_1\overline{v_2})$ we obtain $\langle a_1(L), v_1\overline{w} \rangle = 0$. Therefore $M_1' \subset M_1$ and $\mathrm{rk} M_1' = 2$. If $v \in L$ and $w \in L^\perp$, then

$$Q_L(v) Q_{L^\perp}(w) = Nr(v\overline{w}) = Nr(\mu_1(v \otimes w)).$$

Therefore M_1' is a Legendre composition of (L, Q_L) and $(L^\perp, Q_{L^\perp})$. By Gauss's theorem 3.7.3, $disc(Q_{M_1})$ is equal to d if $d \not\equiv 3 \pmod 4$ and to $4d$ if $d \equiv 3 \pmod 4$. By First Conclusion in art. 235 of [53], $disc(Q_{M_1'})$ divide d if $d \not\equiv 3 \pmod 4$ and divide $4d$ if $d \equiv 3 \pmod 4$. From this and the inclusion $M_1' \subset M_1$ it follows that $M_1 = M_1'$. Analogously we prove that the image of μ_2 is M_2. □

In [98] it is proven that for binary quadratic forms over $\mathbb{Z}$, the Gaussian composition is well defined for properly equivalent forms, but Legendre composition is not. Furthermore it is shown that two $GL_2(\mathbb{Z})$-classes of binary quadratic forms with the same discriminant, have at most 2 Legendre compositions; cf. [98, pp. 43-44]. If we adapt these results to binary quadratic forms over $\mathbb{Z}_p$ and use the fact that every binary quadratic form over $\mathbb{Z}_p$ has an automorphism of determinant -1, then we can conclude that over $\mathbb{Z}_p$ there is a single Legendre composition of two classes of binary quadratic forms of the same discriminant. This implies the following:

- Let (M, Q_M) be a Legendre composition of the binary quadratic lattices (L, Q_L) and $(L', Q_{L'})$. Choose a basis (v, w) of M and identify Q_M with a binary quadratic form q_M. Then the genus class of q_M is independent of the chosen Legendre composition (M, Q_M). We call it the *genus class of Legendre compositions* of (L, Q_L) and $(L', Q_{L'})$.

Let $L \in \mathcal{R}_d$. By Theorems 3.7.3 and 3.7.4 the genus class of Legendre compositions of (L, Q_L) and $(L', Q_{L'})$ is $\mathcal{G}_d$. This proves that the genus class of $(L', Q_{L'})$ is determined by the genus class of (L, Q_L). Conversely, if we use the formula proved in [4], and the fact that the genus $[[I_4]]$ has only one class, we conclude that if (L', Q') is in the genus of (L, Q_L) and (L'', Q'') is in the genus of $(L^\perp, Q_{L^\perp})$, then there is $M \subset \mathbb{Z}^4$ 2-dimensional lattice such that (M, Q_M) is in the same class of (L', Q') and $(L^\perp, Q_{M^\perp})$ is in the same class of (L'', Q''). We can prove by the local-global principle that for $D \in \mathbb{D}$ squarefree every integral positive binary quadratic lattice of discriminant D can be represented by $x^2 + y^2 + z^2 + w^2$. So we have proven the following.

Theorem 3.7.5. *Let $d \in \mathbb{D}$ be a squarefree integer. Consider (M, Q_M) and $(M', Q_{M'})$ two positive binary quadratic lattices of discriminant d. Then there exists $L \in \mathcal{R}_d$ with (L, Q_L) isomorphic with (M, Q_M) and $(L^\perp, Q_{L^\perp})$ isomorphic with $(M', Q_{M'})$ if and only if the genus class of Legendre compositions of (M, Q_M) and $(M', Q_{M'})$ is the genus class $\mathcal{G}_n$ defined after Theorem 3.7.3.*

Chapter 4

Primes in arithmetic progressions and semidefinite programming

Assuming the generalized Riemann hypothesis, we give asymptotic bounds on the size of intervals that contain primes from a given arithmetic progression using the approach developed by Carneiro, Milinovich and Soundararajan, cf. [26]. For this we extend the Guinand-Weil explicit formula over all Dirichlet characters modulo $q \geq 3$, and we reduce the associated extremal problems to convex optimization problems that can be solved numerically via semidefinite programming. The content of this chapter appear in the pre-print [36].

4.1 Introduction

4.1.1 Prime numbers

Denote by $\pi(x)$ the number of primes less than or equal to x. A classical theorem of Cramér [42] states that, assuming the Riemann hypothesis (RH), there are constants $c, \alpha > 0$ such that

$$\frac{\pi\big(x + c\sqrt{x}\log x\big) - \pi(x)}{\sqrt{x}} > \alpha \tag{4.1.1}$$

for all sufficiently large x. The order of magnitude in this estimate has never been improved, and the efforts have thus been concentrated in optimizing the values of the implicit constants. Recently, Carneiro, Milinovich and Soundararajan [26] used Fourier analysis to establish the best known values. This approach studies some Fourier optimization problems that are of the kind where one prescribes some constraints for a function and its Fourier transform, and then wants to optimize a certain quantity.

Let us denote by $\mathcal{A}^+$ the set of even and continuous functions $F\colon \mathbb{R} \to \mathbb{R}$ with $F \in L^1(\mathbb{R})$.

For $1 \leq A < \infty$, we write

$$\mathcal{C}^+(A) := \sup_{\substack{F \in \mathcal{A}^+ \\ F \neq 0}} \frac{1}{\|F\|_1} \left(F(0) - A \int_{[-1,1]^c} \big(\widehat{F}\big)^+(t)\, dt \right), \tag{4.1.2}$$

where we use the notation $f^+(x) = \max\{f(x), 0\}$, $[-1,1]^c = \mathbb{R} \setminus [-1,1]$, and

$$\widehat{F}(t) = \int_{-\infty}^{\infty} F(x) e^{-2\pi i x t}\, dx.$$

Assuming RH, [26, Theorem 1.3] establishes that for $\alpha \geq 0$,

$$\inf \left\{ c > 0;\ \liminf_{x \to \infty} \frac{\pi(x + c\sqrt{x}\log x) - \pi(x)}{\sqrt{x}} > \alpha \right\} \leq \frac{(1 + 2\alpha)}{\mathcal{C}^+(36/11)}. \tag{4.1.3}$$

The numerical example from [26, Eq (4.12)] given by

$$F(x) = -4.8\, x^2\, e^{-3.3x^2} + 1.5\, x^2\, e^{-7.4x^2} + 520\, x^{24}\, e^{-9.7x^2} + 1.3\, e^{-2.8x^2} + 0.18\, e^{-2x^2}$$

shows that

$$\mathcal{C}^+(36/11) > 1.1943... > \frac{25}{21}.$$

Therefore in (4.1.1) for $\alpha = 0$ and $\alpha = 1$, we can choose $c = 0.8374$ and $c = 2.512$, respectively. This improves the previous results established by Dudek [45], who shows that for $\alpha = 0$ and $\alpha = 1$, we can choose $c = 1 + \varepsilon$ and $c = 3 + \varepsilon$, respectively, for any $\varepsilon > 0$.

4.1.2 Prime numbers in arithmetic progressions

Let $q \geq 3$ and $b \geq 1$ be coprime integers. Denote by $\pi(x; q, b)$ the number of primes less than or equal to x that are congruent to b modulo q. Assuming the generalized Riemann hypothesis (GRH), Grenié, Molteni and Perelli [56, Theorem 1] state the equivalent of the result by Cramér (4.1.1) for primes in arithmetic progressions. They established that there are suitable constants $c_1, \alpha > 0$ such that

$$\frac{\pi\big(x + c_1\, \varphi(q)\sqrt{x}\log x; q, b\big) - \pi(x; q, b)}{\sqrt{x}} > \alpha,$$

for all sufficiently large x. Our main goal in this paper is to establish good bounds for the constant $c_1 > 0$.

Theorem 4.1.1. *Assume the generalized Riemann hypothesis. Let $q \geq 3$ and $b \geq 1$ be coprime. Then, for any $\alpha \geq 0$, we have*

$$\inf \left\{ c_1 > 0;\ \liminf_{x \to \infty} \frac{\pi\big(x + c_1\, \varphi(q)\sqrt{x}\log x; q, b\big) - \pi(x; q, b)}{\sqrt{x}} > \alpha \right\} \leq \frac{(1 + 2\alpha)}{\mathcal{C}^+(4)} < 0.8531\,(1 + 2\alpha). \tag{4.1.4}$$

where $\varphi(q)$ is Euler's totient function.

In particular, for all sufficiently large x there is a prime p that is congruent to b modulo q in the interval $(x, x + 0.8531\,\varphi(q)\sqrt{x}\log x]$. Furthermore, there are at least $\sqrt{x}$ primes that are congruent to b modulo q in the interval $(x, x + 2.5591\,\varphi(q)\sqrt{x}\log x]$. This result improves asymptotically some results of a recent work by Dudek, Grenié, and Molteni [46, Theorem 1.1-1.3], which establish the constants $c_1 = 1$ and $c_1 = 3$ for $\alpha = 0$ and $\alpha = 1$ respectively. Our result establish the constants $c_1 = 0.8531$ and $c_1 = 2.5591$ for $\alpha = 0$ and $\alpha = 1$ respectively.

Corollary 4.1.2. *Assume the generalized Riemann hypothesis. Let $q \geq 3$ and $b \geq 1$ be coprime and denote by $p_{n,q,b}$ the n-th prime that is congruent to b modulo q. Then*

$$\limsup_{n\to\infty} \frac{p_{n+1,q,b} - p_{n,q,b}}{\sqrt{p_{n,q,b}}\,\log p_{n,q,b}} \leq 0.8531\varphi(q).$$

4.1.3 Optimized bounds

The construction of numerical examples via semidefinite programming also gives a slight improvement on [26, Theorem 1.3 and Corollary 1.4]: we get $\mathcal{C}^+(36/11) \geq 1.1961$. So assuming the Riemann hypothesis, we have for any $\alpha \geq 0$ in (4.1.3) that

$$\inf\left\{c > 0;\ \liminf_{x\to\infty} \frac{\pi(x + c\sqrt{x}\log x) - \pi(x)}{\sqrt{x}} > \alpha\right\} < 0.8358\,(1 + 2\alpha) \tag{4.1.5}$$

and

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{\sqrt{p_n}\,\log p_n} < 0.8358$$

where p_n denotes the n-th prime.

4.1.4 Strategy outline

The proof of the first inequality in Theorem 4.1.1 follows the ideas developed in [26]. We need three main ingredients: the Guinand-Weil explicit formula for the Dirichlet characters modulo q, the Brun-Titchmarsh inequality for primes in arithmetic progressions and the derivation of an extremal problem in Fourier analysis. We start establishing an extended version of the classical Guinand-Weil explicit formula, that contains certain sums that run over all Dirichlet characters modulo q. In particular, one of these sums allows us to count primes in an arithmetic progression, and we can bound many of these primes using the Brun-Titchmarsh inequality for primes in arithmetic progressions. Since many of the computations to derive the extremal problem are similar to [26], we will highlight the principal differences.

For the second inequality in Theorem 4.1.1, we write the resulting optimization problem as a convex optimization problem over nonnegative functions. We then write these non-negative functions as $f(x) = p(x^2)e^{-\pi x^2}$ for some polynomial p, as in the works of Cohn and Elkies [37] for the sphere packing problem, and use semidefinite programming to optimize over these nonnegative functions, which is an approach employed recently for problems involving the Riemann zeta-function and other $L-$functions in [35, 71].

4.2 Guinand-Weil explicit formula and Brun-Titchmarsh inequality

4.2.1 Guinand-Weil explicit formula

The classical Guinand-Weil explicit formula [25, Lemma 5] establishes the relation between the zeros of a primitive Dirichlet character modulo q and the primes that are coprime to q. The following lemma states a version of this explicit formula that contains the sum over primitive and imprimitive Dirichlet characters modulo q.

Lemma 4.2.1. *Let $q \geq 3$ and $b \geq 1$ be coprime. Let $h(s)$ be analytic in the strip $|\mathrm{Im}\, s| \leq \frac{1}{2}+\varepsilon$ for some $\varepsilon > 0$, and assume that $|h(s)| \ll (1+|s|)^{-(1+\delta)}$ as $|\mathrm{Re}\, s| \to \infty$, for some $\delta > 0$.*[1] *Then*

$$\sum_{\chi} \overline{\chi(b)} \sum_{\rho_\chi} h\left(\frac{\rho - \frac{1}{2}}{i}\right) = h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right) + \frac{1}{2\pi} \sum_{\chi} \overline{\chi(b)} \int_{-\infty}^{\infty} h(u)\, \mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{\mu_\chi}{2} + \frac{iu}{2}\right) du$$
$$- \frac{\varphi(q)}{2\pi} \sum_{n \equiv b \,(\mathrm{mod}\, q)} \frac{\Lambda(n)}{\sqrt{n}}\, \widehat{h}\left(\frac{\log n}{2\pi}\right) - \frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left(\sum_{\chi} \overline{\chi(bn)}\right) \widehat{h}\left(\frac{-\log n}{2\pi}\right) + O(\|\widehat{h}\|_\infty),$$

where χ runs over the Dirichlet characters modulo q, ρ_χ are the non-trivial zeros of the Dirichlet L-function $L(s,\chi)$, Γ'/Γ is the logarithmic derivative of the Gamma function, $\mu_\chi \in \{0,1\}$ and $\Lambda(n)$ is the Von-Mangoldt function that is given by $\Lambda(n) = \log p$ for prime powers $n = p^m$, $m \geq 1$ and zero otherwise. The error term in the above expression depends on q and b.

Proof. Let χ be a primitive Dirichlet character modulo q. The Guinand-Weil explicit formula for χ (see [25, Lemma 5]) states that

$$\sum_{\rho_\chi} h\left(\frac{\rho - \frac{1}{2}}{i}\right) = \left\{\frac{\log q}{2\pi}\,\widehat{h}(0) - \frac{\log \pi}{2\pi}\,\widehat{h}(0)\right\} + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\, \mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{\mu_\chi}{2} + \frac{iu}{2}\right) du$$
$$- \frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left\{\chi(n)\, \widehat{h}\left(\frac{\log n}{2\pi}\right) + \overline{\chi(n)}\, \widehat{h}\left(\frac{-\log n}{2\pi}\right)\right\}, \tag{4.2.1}$$

where the sum runs over all non-trivial zeros ρ_χ of $L(s,\chi)$, $\mu_\chi = 0$ if $\chi(-1) = 1$ and $\mu_\chi = 1$

[1] We use the notation $f = O(g)$ ($f \ll g$) to mean that there is a constant $C > 0$ such that $|f(t)| \leq C\, g(t)$.

if $\chi(-1) = -1$. Note that

$$\left| \frac{\log q}{2\pi} \widehat{h}(0) - \frac{\log \pi}{2\pi} \widehat{h}(0) \right| \ll \|\widehat{h}\|_\infty.$$

We want to establish a similar formula as (4.2.1) for an imprimitive Dirichlet character modulo q. We know that each imprimitive character χ modulo q is induced by a unique primitive character χ^* modulo f, with $f|q$ and $f < q$. This implies that $\chi(n) = \chi_0(n)\chi^*(n)$ for all $n \in \mathbb{Z}$, where $\chi_0(n)$ is the principal character modulo q, and

$$L(s, \chi) = L(s, \chi^*) \prod_{p|q} \left(1 - \frac{\chi^*(p)}{p^s}\right). \tag{4.2.2}$$

If we write $\widetilde{\chi}_0(n) = 1 - \chi_0(n)$, then $\chi^*(n) = \chi(n) + \chi^*(n)\widetilde{\chi}_0(n)$. Let χ be a non-principal imprimitive character modulo q. Therefore, using the Guinand-Weil explicit formula for χ^*, we get that

$$\begin{aligned} \sum_{\rho_{\chi^*}} h\left(\frac{\rho - \frac{1}{2}}{i}\right) &= \left\{\frac{\log f}{2\pi} \widehat{h}(0) - \frac{\log \pi}{2\pi} \widehat{h}(0)\right\} + \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) \operatorname{Re} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{\mu_{\chi^*}}{2} + \frac{iu}{2}\right) du \\ &\quad - \frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left\{\chi(n)\, \widehat{h}\left(\frac{\log n}{2\pi}\right) + \overline{\chi(n)}\, \widehat{h}\left(\frac{-\log n}{2\pi}\right)\right\} \\ &\quad - \frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left\{\chi^*(n)\widetilde{\chi}_0(n)\, \widehat{h}\left(\frac{\log n}{2\pi}\right) + \overline{\chi^*(n)}\widetilde{\chi}_0(n)\, \widehat{h}\left(\frac{-\log n}{2\pi}\right)\right\}. \end{aligned} \tag{4.2.3}$$

Note that $\widetilde{\chi}_0(n) = 0$ when n and q are coprime. Therefore the last sum can be bounded in the following form

$$\left| \frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \chi^*(n)\widetilde{\chi}_0(n)\, \widehat{h}\left(\frac{\log n}{2\pi}\right) \right| \ll \sum_{p|q, k\geq 1} \frac{\log p}{p^{k/2}} \|\widehat{h}\|_\infty \ll \|\widehat{h}\|_\infty. \tag{4.2.4}$$

On the other hand, we have that $L(s, \chi)$ and $L(s, \chi^*)$ have the same set of non-trivial zeros by (4.2.2). Therefore we conclude in (4.2.3) that for each imprimitive character χ modulo q, we have

$$\begin{aligned} \sum_{\rho_\chi} h\left(\frac{\rho - \frac{1}{2}}{i}\right) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) \operatorname{Re} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{\mu_\chi}{2} + \frac{iu}{2}\right) du + O(\|\widehat{h}\|_\infty) \\ &\quad - \frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left\{\chi(n)\, \widehat{h}\left(\frac{\log n}{2\pi}\right) + \overline{\chi(n)}\, \widehat{h}\left(\frac{-\log n}{2\pi}\right)\right\} \end{aligned} \tag{4.2.5}$$

where $\mu_\chi \in \{0, 1\}$. Finally, in the case of a principal character $\chi_0(n)$, we use the Guinand-

Weil explicit formula for the Riemann zeta function (see [24, Lemma 8]), which states that

$$\sum_{\rho} h\left(\frac{\rho-\frac{1}{2}}{i}\right) = h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right) - \frac{\log\pi}{2\pi}\widehat{h}(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)du$$
$$- \frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\left(\chi_0(n)\,\widehat{h}\left(\frac{\log n}{2\pi}\right) + \overline{\chi_0(n)}\,\widehat{h}\left(\frac{-\log n}{2\pi}\right)\right)$$
$$- \frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\left(\widetilde{\chi}_0(n)\,\widehat{h}\left(\frac{\log n}{2\pi}\right) + \overline{\widetilde{\chi}_0(n)}\,\widehat{h}\left(\frac{-\log n}{2\pi}\right)\right), \tag{4.2.6}$$

where the sum runs over all non-trivial zeros ρ of $\zeta(s)$. Note that the last sum in (4.2.6) can be bounded as (4.2.4). Therefore, multiplying (4.2.1), (4.2.5) and (4.2.6) by $\overline{\chi(b)}$ (note that in the last case $\overline{\chi_0(b)} = 1$) and summing these results to obtain the final sum over all character modulo q, we get (inserting the respective error terms)

$$\sum_{\chi}\overline{\chi(b)}\sum_{\rho_\chi} h\left(\frac{\rho-\frac{1}{2}}{i}\right) = h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right) + \frac{1}{2\pi}\sum_{\chi}\overline{\chi(b)}\int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{\mu_\chi}{2}+\frac{iu}{2}\right)du$$
$$- \frac{1}{2\pi}\sum_{\chi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left\{\overline{\chi(b)}\chi(n)\,\widehat{h}\left(\frac{\log n}{2\pi}\right) + \overline{\chi(b)\chi(n)}\,\widehat{h}\left(\frac{-\log n}{2\pi}\right)\right\}$$
$$+ O\big(\|\widehat{h}\|_\infty\big)$$

where the sums run over all Dirichlet characters modulo q, and $\mu_{\chi_0} = 0$. Using Fubini's theorem and the fact that

$$\sum_{\chi}\overline{\chi(b)}\chi(n) = \begin{cases}\varphi(q) & \text{if } n \equiv b \pmod q,\\ 0 & \text{if } n \not\equiv b \pmod q,\end{cases}$$

we obtain the desired result. □

4.2.2 Brun-Titchmarsh inequality

We will use the following version of the Brun-Titchmarsh inequality due to Montgomery and Vaughan [88, Theorem 2]:

$$\pi(x+y;q,b) - \pi(x;q,b) < \frac{2y}{\varphi(q)\log(y/q)}$$

for all $x \geq 1$ and $y > q$. In our case we use this inequality in the following form: for every sufficiently small $\varepsilon > 0$ and $x \geq q^{1/\varepsilon}$, we have

$$\pi(x+\sqrt{x};q,b) - \pi(x;q,b) < \frac{4}{(1-2\varepsilon)\,\varphi(q)}\frac{\sqrt{x}}{\log x}. \tag{4.2.7}$$

4.3 Proof of Theorem 4.1.1: First part

We follow the idea developed in [26, Section 5]. We start by fixing coprime integers $q \geq 3$ and $b \geq 1$ and assuming GRH. Also, we fix an even and bandlimited Schwartz function $F : \mathbb{R} \to \mathbb{R}$ such that $F(0) > 0$ and $\operatorname{supp}(\widehat{F}) \subset [-N, N]$ for some parameter $N \geq 1$. Therefore, F extends to an entire function, and using the Phragmén-Lindelöf principle, the hypotheses of Lemma 4.2.1 are satisfied. Throughout this proof, the error terms can depend on q, b, and F. Let $0 < \Delta \leq 1$ and $1 < a$ be free parameters (to be chosen later) such that

$$2\pi\Delta N \leq \log a. \tag{4.3.1}$$

We need to have in mind that $a \to \infty$ and $\Delta \to 0$. Considering the function $f(z) = \Delta F(\Delta z)$, we have $\operatorname{supp}(\widehat{f}) \subset [-\Delta N, \Delta N]$. Applying Lemma 4.2.1 to the function $h(z) = f(z)a^{iz}$, we obtain

$$\begin{aligned}\sum_{\chi} \overline{\chi(b)} \sum_{\gamma_\chi} h(\gamma_\chi) = \left\{ h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right) \right\} + \frac{1}{2\pi} \sum_{\chi} \overline{\chi(b)} \int_{-\infty}^{\infty} h(u) \operatorname{Re} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{\mu_\chi}{2} + \frac{iu}{2}\right) du \\ - \frac{\varphi(q)}{2\pi} \sum_{n \equiv b \,(\mathrm{mod}\, q)} \frac{\Lambda(n)}{\sqrt{n}} \widehat{h}\left(\frac{\log n}{2\pi}\right) - \frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left(\sum_{\chi} \overline{\chi(bn)}\right) \widehat{h}\left(\frac{-\log n}{2\pi}\right) \\ + O\big(\|\widehat{h}\|_\infty\big),\end{aligned} \tag{4.3.2}$$

where γ_χ is the imaginary parts of a non-trivial zero ρ_χ of $L(s, \chi)$. We start by estimating some terms on the right-hand side of (4.3.2). Using the estimate from [26, Pag. 553], we get

$$h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right) = \Delta F(0)\big(\sqrt{a} + \sqrt{a^{-1}}\big) + O\big(\Delta^2 \sqrt{a}\big). \tag{4.3.3}$$

Using Stirling's formula and the estimate from [26, Pag. 554], we have

$$\int_{-\infty}^{\infty} h(u) \operatorname{Re} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{\mu_\chi}{2} + \frac{iu}{2}\right) du = O(1).$$

Therefore,

$$\frac{1}{2\pi} \sum_{\chi} \overline{\chi(b)} \int_{-\infty}^{\infty} h(u) \operatorname{Re} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{\mu_\chi}{2} + \frac{iu}{2}\right) du = O(1). \tag{4.3.4}$$

By (4.3.1), we have

$$\widehat{h}\left(\frac{-\log n}{2\pi}\right) = 0$$

for $n \geq 2$. This implies that

$$\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left(\sum_{\chi}\overline{\chi(bn)}\right)\widehat{h}\left(\frac{-\log n}{2\pi}\right)=0. \tag{4.3.5}$$

Inserting (4.3.3), (4.3.4) and (4.3.5) in (4.3.2) yields

$$\sum_{\chi}\overline{\chi(b)}\sum_{\gamma_\chi}h(\gamma_\chi)=\Delta F(0)\big(\sqrt{a}+\sqrt{a^{-1}}\big)+O\big(\Delta^2\sqrt{a}\big)-\frac{\varphi(q)}{2\pi}\sum_{n\equiv b\,(\mathrm{mod}\,q)}\frac{\Lambda(n)}{\sqrt{n}}\widehat{h}\left(\frac{\log n}{2\pi}\right)+O(1).$$

Therefore

$$\Delta F(0)\sqrt{a}\leq\sum_{\chi}\sum_{\gamma_\chi}\big|h(\gamma_\chi)\big|+\frac{\varphi(q)}{2\pi}\sum_{n\equiv b\,(\mathrm{mod}\,q)}\frac{\Lambda(n)}{\sqrt{n}}\,(\widehat{h})^{+}\left(\frac{\log n}{2\pi}\right)+O\big(\Delta^2\sqrt{a}\big)+O(1). \tag{4.3.6}$$

Next we estimate the terms on the right-hand side of (4.3.6). We recall that for each primitive Dirichlet character modulo q, we have the formula [43, Chapter 16]

$$N(T,\chi)=\frac{T}{\pi}\log\left(\frac{qT}{2\pi}\right)-\frac{T}{\pi}+O(\log T+\log q),$$

where $N(T,\chi)$ denotes the number of zeros of $L(s,\chi)$ in the rectangle $0<\sigma<1$ and $|\gamma|\leq T$. Using integration by parts as in [26, Eq. (5.4)], we obtain for each primitive Dirichlet character modulo q that

$$\sum_{\gamma_\chi}|h(\gamma_\chi)|=\frac{\log(1/2\pi\Delta)}{2\pi}\,\|F\|_1+O(1). \tag{4.3.7}$$

Note that the main term in the above expression is independent of q. By (4.2.2), Equation (4.3.7) holds for each non-principal imprimitive character modulo q. In the case of the principal character χ_0, we use the estimate for the zeros of the Riemann zeta-function from [26, Eq. (5.4)]. Therefore, considering that the number of Dirichlet characters modulo q is $\varphi(q)$, we conclude that

$$\sum_{\chi}\sum_{\gamma_\chi}\big|h(\gamma_\chi)\big|=\varphi(q)\,\frac{\log(1/2\pi\Delta)}{2\pi}\,\|F\|_1+O(1). \tag{4.3.8}$$

Next we bound the second sum on the right-hand side of (4.3.6). Using the relation between the functions h and F, this sum equate

$$\sum_{n\equiv b\,(\mathrm{mod}\,q)}\frac{\Lambda(n)}{\sqrt{n}}\,(\widehat{F})^{+}\left(\frac{\log(n/a)}{2\pi\Delta}\right). \tag{4.3.9}$$

Fix $\alpha \geq 0$ and assume that $c_1 > 0$ is a fixed constant such that

$$\liminf_{x\to\infty} \frac{\pi\big(x + c_1\,\varphi(q)\sqrt{x}\log x; q, a\big) - \pi(x; q, a)}{\sqrt{x}} \leq \alpha.$$

This implies that for $\varepsilon > 0$, there exists a sequence of $x \to \infty$ such that

$$\frac{\pi\big(x + c_1\,\varphi(q)\sqrt{x}\log x; q, a\big) - \pi(x; q, a)}{\sqrt{x}} < \alpha + \varepsilon.$$

For each x in the sequence, we choose a and Δ such that

$$\big[x, x + c_1\,\varphi(q)\sqrt{x}\log x\big] = \Big[ae^{-2\pi\Delta}, ae^{2\pi\Delta}\Big].$$

This implies that

$$4\pi\Delta = c_1\,\varphi(q)\frac{\log x}{\sqrt{x}} + O\bigg(\frac{\log^2 x}{x}\bigg)$$

(see [26, Eq. (5.7)-(5.8)]), and

$$a = x + O(\sqrt{x}\log x).$$

Since $\operatorname{supp}(\widehat{F}) \subset [-N, N]$, the sum in (4.3.9) runs over $ae^{-2\pi\Delta N} \leq n \leq ae^{2\pi\Delta N}$ with $n \equiv b\,(\operatorname{mod} q)$. Thus the contribution of the prime powers $n = p^k$ with $n \equiv b \pmod q$ in that interval is $O(1)$. The contribution of the (at most) $(\alpha+\varepsilon)\sqrt{x}$ primes in the interval $(x, x + c_1\,\varphi(q)\sqrt{x}\log x] = (ae^{-2\pi\Delta}, ae^{2\pi\Delta}]$ to the sum (4.3.9) is bounded above by

$$\|F\|_1 \sum_{p\in(ae^{-2\pi\Delta}, ae^{2\pi\Delta}]} \frac{\log p}{\sqrt{p}} \leq \|F\|_1(\alpha+\varepsilon)\sqrt{x}\,\frac{\log x}{\sqrt{x}} = \|F\|_1(\alpha+\varepsilon)\log x$$

(using that $(\widehat{F})^+(t) \leq \|F\|_1$). Finally, to estimate the contribution of the primes in the intervals $[ae^{-2\pi\Delta N}, ae^{-2\pi\Delta}]$ and $[ae^{2\pi\Delta}, ae^{2\pi\Delta N}]$, we use the Brun-Titchmarsh inequality (4.2.7). We also need the following estimate: for $g \in C^1([a,b])$, we have

$$0 \leq S(g^+, P) - \int_a^b g^+(t)\,dt \leq \delta(b-a)\sup_{x\in[a,b]}|g'(x)| \tag{4.3.10}$$

where P is a partition of $[a,b]$ of norm at most δ and $S(g^+, P)$ is the upper Riemann sum of the function g^+ and the partition P. We apply (4.3.10) to the function

$$g(t) = \widehat{F}\bigg(\frac{\log(t/a)}{2\pi\Delta}\bigg)$$

and the partition $P = \{x_0 < ... < x_J\}$ that covers the interval $[ae^{2\pi\Delta}, ae^{2\pi\Delta N}] \subset \cup_{j=0}^{J-1}[x_j, x_{j+1}]$, with $x_0 = ae^{2\pi\Delta}$, $x_{j+1} = x_j + \sqrt{x_j}$. If we define $M_j = \sup\{g^+(x) : x \in [x_j.x_{j+1}]\}$, then $S(g^+, P) = \sum_{j=0}^{J-1} M_j\sqrt{x_j}$. Therefore, it follows from (4.2.7) and (4.3.10) that

$$\begin{aligned}
&\sum_{\substack{1\leq\frac{\log p/a}{2\pi\Delta}\leq N\\ p\equiv b\,(\mathrm{mod}\,q)}} \frac{\log p}{\sqrt{p}}\,(\widehat{F})^+\left(\frac{\log(p/a)}{2\pi\Delta}\right)\\
&\quad\leq \sum_{j=0}^{J-1}\left(\frac{\log x_j}{\sqrt{x_j}}M_j\right)\frac{4\sqrt{x_j}}{(1-2\varepsilon)\,\varphi(q)\log x_j} \leq \frac{4}{(1-2\varepsilon)\,\varphi(q)}\left(\frac{1}{\sqrt{a}}\sum_{j=0}^{J-1} M_j\sqrt{x_j}\right)\\
&\quad= \frac{4}{(1-2\varepsilon)\,\varphi(q)}\left(\frac{1}{\sqrt{a}}\int_{x_0}^{x_J}(\widehat{F})^+\left(\frac{\log(t/a)}{2\pi\Delta}\right)dt + \frac{1}{\sqrt{a}}\left(S(g^+,P) - \int_{x_0}^{x_J}(\widehat{F})^+\left(\frac{\log(t/a)}{2\pi\Delta}\right)dt\right)\right)\\
&\quad\leq \frac{4}{(1-2\varepsilon)\,\varphi(q)}\left(\sqrt{a}(2\pi\Delta)\int_1^N(\widehat{F})^+(t)\,e^{2\pi\Delta t}\,dt + O\left(\frac{1}{2\pi\Delta}(e^{2\pi\Delta N} - e^{2\pi\Delta})\right)\right)\\
&\quad\leq \frac{4\sqrt{a}\,(2\pi\Delta)}{(1-2\varepsilon)\,\varphi(q)}\int_1^N(\widehat{F})^+(t)\,dt + O(1), \qquad (4.3.11)
\end{aligned}$$

where we use that $0 \leq e^x - 1 \leq 2x$ for $0 \leq x \leq 1$. We treat the other interval in a similar way, obtaining

$$\sum_{\substack{-N\leq\frac{\log p/a}{2\pi\Delta}\leq -1\\ p\equiv b\,(\mathrm{mod}\,q)}} \frac{\log p}{\sqrt{p}}\,(\widehat{F})^+\left(\frac{\log(p/a)}{2\pi\Delta}\right) \leq \frac{4\sqrt{a}\,(2\pi\Delta)}{(1-2\varepsilon)\,\varphi(q)}\int_{-N}^{-1}(\widehat{F})^+(t)\,dt + O(1). \qquad (4.3.12)$$

Combining (4.3.11) and (4.3.12), we obtain

$$\sum_{\substack{1<\left|\frac{\log(p/a)}{2\pi\Delta}\right|\leq N\\ p\equiv b\,(\mathrm{mod}\,q)}} \frac{\log p}{\sqrt{p}}\,(\widehat{F})^+\left(\frac{\log(p/a)}{2\pi\Delta}\right) \leq \frac{4\sqrt{a}(2\pi\Delta)}{(1-2\varepsilon)\varphi(q)}\int_{[-1,1]^c}(\widehat{F})^+(t)\,dt + O(1).$$

Therefore, joining the previous estimates, we conclude that

$$\sum_{n\equiv b\,(\mathrm{mod}\,q)} \frac{\Lambda(n)}{\sqrt{n}}\,(\widehat{h})^+\left(\frac{\log n}{2\pi}\right) \leq \|F\|_1\,(\alpha+\varepsilon)\log x + \frac{4\sqrt{a}\,(2\pi\Delta)}{(1-2\varepsilon)\varphi(q)}\int_{[-1,1]^c}(\widehat{F})^+(t)\,dt + O(1). \qquad (4.3.13)$$

Inserting the estimates (4.3.8) and (4.3.13) in (4.3.6) yields that

$$\Delta\sqrt{a}\left(F(0) - \frac{4}{(1-2\varepsilon)}\int_{[-1,1]^c}(\widehat{F})^+(t)\,dt\right) \leq \frac{\varphi(q)}{2\pi}\,\|F\|_1\,(\alpha+\varepsilon)\log x + \varphi(q)\frac{\log(1/2\pi\Delta)}{2\pi}\,\|F\|_1 + O(1).$$

Sending $x \to \infty$ along the sequence yields that

$$c_1 \le (1+2\alpha+2\varepsilon)\frac{\|F\|_1}{F(0) - \frac{4}{(1-2\varepsilon)}\int_{[-1,1]^c}(\widehat{F})^+(t)\,dt},$$

and with $\varepsilon \to 0$, we get

$$c_1 \le (1+2\alpha)\frac{\|F\|_1}{F(0) - 4\int_{[-1,1]^c}(\widehat{F})^+(t)\,dt}.$$

Finally we recall that in [26, Subsection 4.1] it is shown that when searching for the sharp constants $\mathcal{C}^+(A)$ in (4.1.2), we can restrict to the subset of $\mathcal{A}^+$ such that $\widehat{F} \in C_c^\infty(\mathbb{R})$ without loss of generality. We conclude that

$$\inf\left\{c_1 > 0;\ \liminf_{x\to\infty} \frac{\pi\big(x + c_1\,\varphi(q)\sqrt{x}\log x; q, b\big) - \pi(x; q, b)}{\sqrt{x}} > \alpha\right\} \le \frac{(1+2\alpha)}{\mathcal{C}^+(4)}.$$

4.4 Numerically optimizing the bounds

We first reformulate (4.1.2) as a convex optimization problem.

Lemma 4.4.1. *Let $\mathcal{F}$ be the set of tuples $(f_1,\dots,f_4)$ of even, nonnegative, continuous functions $f_1,\dots,f_4 \in L^1(\mathbb{R})$ such that $\widehat{f_1}(0)+\widehat{f_2}(0) = 1$ and $f_1 - f_2 = \widehat{f_3} - \widehat{f_4}$. Then we have*

$$\mathcal{C}^+(A) = \sup_{(f_1,f_2,f_3,f_4)\in\mathcal{F}}\left(f_1(0) - f_2(0) - A\int_{[-1,1]^c} f_3(t)\,dt\right)$$

for $A \ge 1$.

Proof. For $(f_1,f_2,f_3,f_4) \in \mathcal{F}$ we set $F := f_1 - f_2 = \widehat{f_3} - \widehat{f_4}$. Then

$$\|F\|_1 = \|f_1 - f_2\|_1 \le \|f_1\|_1 + \|f_2\|_1 = \widehat{f_1}(0) + \widehat{f_2}(0) = 1.$$

Further $F(0) = f_1(0) - f_2(0)$ and $(\widehat{F})^+(x) = (f_3 - f_4)^+(x) \le f_3(x)$. This shows that

$$\mathcal{C}^+(A) \ge \sup_{(f_1,f_2,f_3,f_4)\in\mathcal{F}}\left(f_1(0) - f_2(0) - A\int_{[-1,1]^c} f_3(t)\,dt\right).$$

On the other hand, for $F \in \mathcal{A}^+$ with $F \ne 0$, we define $f_1 := F^+$, $f_2 := F^-$, $f_3 := (\widehat{F})^+$, and $f_4 := (\widehat{F})^-$, where $f_i^-(x) = \min\{f_i(x), 0\}$. Then $\|F\|_1 = \widehat{f_1}(0) + \widehat{f_2}(0)$, $F(0) = f_1(0) - f_2(0)$, and $(\widehat{F})^+ = f_3$, which shows the other inequality. □

To find good functions, we restrict to functions f_i of the form $f_i(x) = p_i(x^2)e^{-\pi x^2}$ where p_i is a polynomial. We then use the sum-of-squares characterization

$$p_i(u) = v_d(u)^{\mathsf{T}} Q_i v_d(u) + u\, v_{d-1}(u)^{\mathsf{T}} R_i v_{d-1}(u),$$

where Q_i and R_i are positive semidefinite matrices, and $v_k(u)$ is a vector whose entries form a basis for the polynomials of degree at most k. This enforces that p_i is nonnegative on $[0,\infty)$, and each polynomial that is nonnegative on $[0,\infty)$ is of this form. For the numerical conditioning, a good choice for the basis is

$$v_k(u) = \big(L_0^{-1/2}(\pi u), \ldots, L_k^{-1/2}(\pi u)\big),$$

where $L_i^{-1/2}$ is the Laguerre polynomial of degree i with parameter $-1/2$.

The conditions $\widehat{f}_1(0) + \widehat{f}_2(0) = 1$ and $f_1 - f_2 = \widehat{f}_3 - \widehat{f}_4$ are linear in the entries of the matrices Q_i and R_i. A numerically stable way of enforcing these constraints is to first compute $\widehat{f}_3 - \widehat{f}_4$ by using that the Fourier transform of $|x|^{2k}e^{-\pi|x|^2}$ is $k!/\pi^k L_k^{-1/2}(\pi|x|^2)e^{-\pi|x|^2}$, and then express $f_1 - f_2 - (\widehat{f}_3 - \widehat{f}_4)$ in the Laguerre basis and equating all coefficients to zero. The linear objective

$$f_1(0) - f_2(0) - A\int_{[-1,1]^c} f_3(t)\,dt$$

can be expressed explicitly as a linear functional in the entries of Q_i and R_i, for $i = 1, 2, 3$, using the identity

$$\int x^m e^{-\pi x^2}\,dx = -\frac{1}{2\pi^{m/2+1/2}}\Gamma\left(\frac{m+1}{2}, \pi x^2\right),$$

where Γ is the upper incomplete gamma function. This reduces the optimization problem to a semidefinite program that can be optimized with a numerical semidefinite programming solver.

The main issue at this point is to get a rigorous bound from the numerical output, for which we adapt the approach from [81]. Instead of optimizing over Q_i and R_i, we fix $\varepsilon = 10^{-20}$ and set $Q_i = \widetilde{Q}_i + \varepsilon I$ and $R_i = \widetilde{R}_i + \varepsilon I$. We then optimize over the positive semidefinite matrices $\widetilde{Q}_i$ and $\widetilde{R}_i$. In this way the matrices Q_i and R_i will be positive semidefinite even if the matrices $\widetilde{Q}_i$ and $\widetilde{R}_i$ computed by the solver have slightly negative eigenvalues. Next we use ball arithmetic (using the Arb library [65]) to verify rigorously that all matrices Q_i and R_i are indeed positive semidefinite, and compute a rigorous lower bound b on the smallest eigenvalue of Q_4. We also compute a rigorous upper bound B on the largest (in absolute value) coefficient of $(\widehat{f}_1 - \widehat{f}_2) - (f_3 - f_4)$ in the basis given by the diagonal and upper diagonal of the matrix $v_d(x)v_d(x)^{\mathsf{T}}$. Let Q_4' be the symmetric tridiagonal matrix such that

$$(\widehat{f}_1 - \widehat{f}_2) - (f_3 - f_4) = v_d(x)^{\mathsf{T}} Q_4'\, v_d(x).$$

Since $\lambda_{\min}(Q_4 + Q_4') \geq \lambda_{\min}(Q_4) + \lambda_{\min}(Q_4')$ and $\lambda_{\min}(Q_4') \geq -2B$ by the Gershgorin circle theorem, the matrix $Q_4 + Q_4'$ is positive semidefinite if $b \geq 2B$. We verify this inequality and replace Q_4 by $Q_4 + Q_4'$. Then the identity $(\widehat{f}_1 - \widehat{f}_2) - (f_3 - f_4) = 0$, and thus the identity

$f_1 - f_2 = (\widehat{f_3} - \widehat{f_4})$, holds exactly. Since

$$\frac{1}{\widehat{f_1}(0) + \widehat{f_2}(0)} \left(f_1(0) - f_2(0) - A \int_{[-1,1]^c} f_3(t)\, dt \right) \tag{4.4.1}$$

does not depend on Q_4, an upper bound on (4.4.1), again computed using ball arithmetic, gives a rigorous upper bound on $\mathcal{C}^+(A)$. The arXiv version of this paper (cf. [36]) contains the matrices Q_i and R_i (computed with degree $d = 90$) and a simple script using Julia/Nemo/Arb [9, 48, 65] to perform this verification procedure (as well as a script to setup and solve the semidefinite programs using sdpa-gmp [92]).

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