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On super curves with a fixed super volume form

by

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DISSERTATION

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To my family.

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Abstract

In this Ph.D. thesis, we focus on super curves with a trivial super volume form. The first part, focuses in giving a correct way to define $S(2)$-super curves, since is not enough just to give a super volume, we also have to consider an affine line bundle over the curve that should be trivial in order to obtain a $S(2)$-super curve. The second part, analyses family of $S(2)$-super curves over an purely even base, in order to proof that such families are ever split. In the last part, we study the moduli space of such curves.

Resumo

Nesta tese de doutorado, nos focamos em super curvas com uma forma de super volume trivial. A primeira parte, é focada em definir dum jeito correto $S(2)$-super curvas, pois não é suficiente dar uma super forma de volume, precisamos também considerar um fibrado de linhas afim sobre a curva que seja trivial em ordem de obter uma $S(2)$-super curva. A segunda parte, analisa famílias de $S(2)$-super curvas sobre uma base puramente par, para provar que tais famílias são sempre cindem. Na última parte, estudamos o spaço de moduli de ditas curvas.

Contents

Chapter 1

Introduction

1.1. In his famous work on classification of Lie super algebras [1], Kac introduced a list of infinite dimensional Lie super algebras generalizing the ordinary theory of Lie algebras of Cartan type. These are subalgebras of derivations on a super commutative algebra freely generated by n even variables and N odd variables. In the case $n = 1$ these are algebras of vector fields on a punctured super disc $\mathrm{Spec}\mathbb{C}((t))[\theta_1, \ldots, \theta_N]$ preserving some extra structure. In this case, the list consists of

1. $W(1|N)$, all vector fields.
2. $S(1|N)$, divergence free vector fields, that is, vector fields acting trivially on the Berezininan, or preserving the section $[dt|d\theta_1 \ldots d\theta_N]$ of the Berezinian bundle.
3. $CS(1|N)$, vector fields that preserve the Berezinian up to multiplication by a scalar function.
4. $K(1|N)$, vector fields preserving a *contact-like* form

   $$dt + \sum \theta_i d\theta_i,$$

   up to multiplication by a scalar function.

Some of these algebras are isomorphic, for example $W(1|1) \simeq K(1|2)$. While some others are not simple, for example $S(1|2)$ is not simple, but its derived algebra $S(2) := [S(1|2), S(1|2)]$ is.

For each such Lie super algebra, there is an associated class of algebraic super curves, with certain geometric structures preserved by these vector fields. That is, the class of super curves admitting an etále cover where infinitesimal changes of coordinates are given by vector fields in the corresponding Lie super algebra. For example, a $W(1|1)$-super curve is a general $1|1$-dimensional super curve, they consist of a smooth algebraic curve C together with a line bundle $\mathcal{L}$ over it. Another example are $K(1|1)$-super curves, called $SUSY$-super curves by Manin in [2] and $SUSY$-super Riemman surfaces in [3, 4], they consist of a smooth curve C and a choice of a square root of the canonical bundle Ω_C. Similarly, $K(1|2)$-super curves are called (oriented) $SUSY_2$-super curves by Manin. In [5] Vaintrob studied the geometry of all these super curves, obtaining a description of the corresponding moduli spaces in each case.

In this work we focus on one example that is missing in Vaintrob's list, these are the so-called $S(2)$-super curves. These are smooth $1|2$-dimensional super curves, endowed with a trivializing section of its Berezinian bundle and with the additional condition that the above mentioned changes of coordinates lie in Kac's $S(2)$-algebra as opposed to the full algebra $S(1|2)$.

1.2. Deligne exploited the isomorphism $W(1|1) \simeq K(1|2)$ in [6] to describe an involution in the moduli space of general smooth $1|1$-dimensional super curves, the fixed locus of which is the moduli space of $K(1|1)$-super curves. This involution is induced by an involution of the Lie super algebra $K(1|2)$, fixing its subalgebra $K(1|1)$. Geometrically, a $W(1|1)$-super curve, or a general $1|1$-dimensional super curve, over a purely even super scheme S (that is simply a scheme S) is given by a smooth curve C over S together with a line bundle $\mathcal{L}$ over it. Deligne's involution corresponds to taking the Serre dual of $\mathcal{L}$:

$$(C, \mathcal{L}) \leftrightarrow (C, \Omega_{C/S} \otimes_{\mathcal{O}_C} \mathcal{L}^*).$$

The fixed point set of this involution is parametrized by curves C together with a choice of a *theta-characteristic* (a square root of the canonical bundle). This is the moduli space of $K(1|1)$-super curves as shown in [5].

In [7], Donagi and Witten show that when the base S is an arbitrary (non-necessarily even) super scheme there exist non-split $W(1|1)$ super curves over S. In particular these curves are not given as the spectrum of the free super commutative algebra generated by a line bundle $\mathcal{L}$ as above. The description of Deligne's involution in this case is not so transparent.

1.3. Our main result is to generalize Deligne's involution to the case of $S(2)$ curves. In order to do so we consider the sequence of inclusions

$$K(1|2) \subset S(2) \subset S(1|2) \subset K(1|4). \tag{1.1}$$

This provides a sequence of embeddings of the corresponding moduli spaces: each $K(1|2)$-

super curve comes with a trivialization of its Berezinian bundle and the local changes of coordinates are in $S(2)$. Each $S(2)$-super curve is in particular a $1|2$-dimensional super curve with a trivialization of its Berezinian bundle Ber_C (an $S(1|2)$-super curve). Similarly if C is a $1|2$-dimensional super curve with a trivialization of Ber_C, consider its tangent bundle $\mathcal{T}_C$, a locally free $\mathcal{O}_C$ module of rank $1|2$. The Grassmanian $\widetilde{C}$ of rank $0|2$ subbundles of $\mathcal{T}_C$ is a $1|4$-dimensional super curve with a canonical $K(1|4)$-structure [8]. We show that there exists an involution of $K(1|4)$ that fixes pointwise its subalgebra $K(1|2)$ and preserves (but does not fix) $S(2)$. This involution implies

Theorem 1.4 (Theorem 5.1)**.** *There exists an involution μ of the moduli space $\mathcal{M}_{S(2)}$ of $S(2)$-super curves such that the fixed point set of μ consists of the moduli space $\mathcal{M}_{K(1|2)}$ of orientable $SUSY_2$-super curves.*

There are super curves with trivial Berezinian that are not $S(2)$-super curves. And the Lie super algebra $S(1|2)$ is not stable under the involution μ above. This shows that our generalization of Deligne's involution requires precisely the $S(2)$-structure as opposed to $S(1|2)$.

1.5. $S(1|2)$ curves admit a simple geometrical description: these are $1|2$-dimensional super curves together with a trivialization of its Berezinian bundle. In contrast, $S(2)$ curves do not admit this simple geometrical description. To any $S(1|2)$ curve C we attach an affine bundle, or a $\mathbb{G}_a$[1] torsor $\mathcal{A}_C$. The class of this bundle is an obstruction for the $S(1|2)$ curve to be an $S(2)$ curve, namely C is an $S(2)$ curve if and only if $\mathcal{A}_C$ is trivial (Proposition 5.2).

1.6. Given a $W(1|1)$-super curve C over a purely even super scheme S, it is split in the sense that there exits a smooth $1|0$-dimensional curve C_0 over S and a line bundle $\mathcal{L}$ over C_0 such that $C = \operatorname{Spec}\operatorname{Sym}_{\mathcal{O}_{C_0}} \mathcal{L}[-1]$. This allowed us to describe Deligne's involution as taking Serre's dual. A similar situation arises in the $S(2)$ case: for a purely even scheme S and an $S(2)$-super curve C over S, there exists a smooth $1|0$-dimensional curve C_0 over S, a rank two bundle $\mathcal{E}$ over C_0 satisfying $\det \mathcal{E} \xrightarrow{\sim} \Omega_{C_0/S}$ and such that $C = \operatorname{Spec}\operatorname{Sym}_{\mathcal{O}_{C_0}} \mathcal{E}[-1]$. That is we have

Theorem 1.7 (Theorem 5.2)**.** *Every $S(2)$-super curve over a purely even base S is split.*

In this situation, our involution above is given just as in the $W(1|1)$ case: it corresponds to

$$(C_0, \mathcal{E}) \leftrightarrow (C_0, \Omega_{C_0/S} \otimes_{\mathcal{O}_{C_0}} \mathcal{E}^*).$$

[1]The affine line with its additive group structure.

Observation 1.1. Theorem 1.7 is false for $S(1|2)$ super curves. Even over a purely even base S, there are $S(1|2)$-super curves that are not split (see Example 6.1), and therefore they are not $S(2)$-super curves.

The condition on the base S on Theorem 1.7 is necessary, that is, there exists families of $S(2)$-curves over super schemes that are not split (see Example 5.3).

From this point of view, the condition on a super curve with trivial Berezinian, of being an $S(2)$-super curve is the analog of the condition of being oriented for general $SUSY_2$-super curves as in [2].

1.8. Our second result is a description of the moduli spaces of $S(2)$ and $S(1|2)$ super curves. We first characterize the universal family of such curves over a purely even base S:

Proposition 1.1. *(See Proposition 6.1) The data of a family of $S(2)$-super curves $C \to S$ whose reduction coincides with a given family of curves $C_0 \to S$ over a purely even scheme S is equivalent to a rank 2 vector bundle $E \to C_0$ together with an isomorphism* $\det E \xrightarrow{\beta} \Omega_{C_0/S}$. *Two such super curves (E, β) and (E', β') are equivalent if and only if there exists a bundle isomorphism $\alpha : E \to E'$ such that $\beta' \circ \det \alpha = \beta$.*

Similarly for $S(1|2)$ super curves we have Proposition 6.2:

Proposition 1.2. *A family of $S(1|2)$ curves $C \to S$ with a given reduction $\pi : C_0 \to S$ over a purely even base S is determined by a rank 2 bundle $E \to C_0$ together with an isomorphism $\beta : \det E \to \Omega_{C_0/S}$ and a class $\Gamma \in H^1(C_0, \pi^*\mathcal{O}_S)$. Two such super curves (E, β, Γ) and (E', β', Γ') are equivalent if $\Gamma = \Gamma'$ and the pairs (E, β) and (E', β') are equivalent as in the previous proposition.*

The map $(E, \beta, \Gamma) \to (E, \beta)$ could be thought of as a fibration from the moduli space of $S(1|2)$ super curves to the moduli space of $S(2)$ curves over purely even bases. However, there are non-trivial odd deformations of $S(2)$ super curves.

We describe the full moduli space of $S(2)$ super curves under the assumption that the base super scheme is split. Given such a split super scheme S with purely even reduction S_{rd}. The datum of a family of $S(2)$ curves $C \to S$ with reduction $C_0 \to S_{\mathrm{rd}}$ is equivalent to a class in $H^1(C_0, G)$ where G is a sheaf of groups over C_0 described in 6.2 (see Proposition 6.3).

1.9. The organization of this thesis is as follows. In chapter 2 we recall the basic preliminaries on super commutative algebras, their modules, and their derivations. We introduce the relevant infinite dimensional Lie algebras in Kac's list and describe their associated infinite dimensional groups as groups of automorphisms of a super disc preserving certain geometric structure.

In chapter 3 we recall the basic preliminaries on super geometry. And in chapter 4 we define our curves of interest and construct a principal $\mathbb{G}_a$ bundle characterizing the obstruction of an $1|2$ dimensional super curve with trivial Berezinian being an $S(2)$ super curve. We show that every $S(2)$ curve over a purely even base is split (Theorem 5.2) and we finish that section attaching a SUSY $1|2N$ curve to any $1|N$ super curve.

In chapter 5 we describe the involution of the moduli space of $S(2)$ super curves generalizing that of Deligne for general $1|1$ super curves. In chapter 6 we give the above mentioned examples of families of super curves: $S(1|2)$ super curves that are not $S(2)$ super curves. Non-split $S(1|2)$ super curves over a purely even base. Non-split $S(2)$ super curves over a super scheme. Also, we give a description of the moduli spaces of $S(1|2)$ and $S(2)$ super curves. We describe the subspace of super curves over purely even schemes and then describe the possible deformations of such a curve in the *odd directions of the base* under the assumption that the base is a split superscheme. We identify the full automorphism group of such families of super curves for genus $g \geq 4$ and describe the corresponding orbifold quotient.

Finally, in chapter A, we give an explicit calculation of a non-trivial character for the group of automorphism with a trivial Berezinian.

Notation

Throughout this thesis, we consider the following agreements:

1. From now k will be a field algebraicly closed and $\mathrm{char}(k) = 0$.

2. Let $\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ be a field. We will consider the sign rule $(-1)^{\bar{i}} = (-1)^i$ for $i = 0, 1$.

3. Let R be a ring. The spectrum is the set of prime ideals, $\mathrm{Spec}(R)$, with the Zariski topology, specifying the closed sets by $V(\mathfrak{p}) := \{\mathfrak{q} \in \mathrm{Spec}(R) : \mathfrak{q} \subset \mathfrak{p}\}$.

4. For smooth varieties we consider the étale topology, an open set in M is an open map ϕ
$$U \to M$$
such that $T_pU \to T_{\phi(p)}M$ is an isomorphism.

5. Let $\{U_i\}_{i \in I}$ be a collection of sets. For a subset $J \subset I$ we will write $U_J = \cap_{i \in J} U_i$. Also, when J is explicitly showed we will write it without parenthesis, for example $U_{ij} = U_i \cap U_j$ or $U_{ijk} = U_i \cap U_j \cap U_k$.

6. For a matrix $A = (a_{ij})$ the i index will denote the row and j the column. For example, for $1 \leq i \leq n$, $1 \leq j \leq m$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}.$$

7. In a $\mathbb{Z}/2\mathbb{Z}$-graded space, we will write at the left the even elements and at the right the odd ones. Also, we will separate those classes by a vertical line: $|$.

Chapter 2

Preliminaries

2.1 Super algebras

We use classical references as [2].

Definition 2.1. Let G be an abelian group and k be a field. A *G-graded k-vector space* V is a k-vector space joint with a decomposition $V = \oplus_{g\in G} V_g$, where each V_g is a k-vector space. An element $v \in V$ is said to be homogeneous if $v \in V_g$ for some $g \in G$.

Example 2.1. Let V be a k-vector space and G be an abelian group with identity element $0 \in G$, then V is trivially G-graded by taking $V_0 = V$ and $V_g = \{0\}$ for $g \neq 0$.

Observation 2.1. Let V, W be two G-graded k-vector spaces, then the direct sum $V \oplus W$ is naturally a G-graded k-vector space by

$$(V \oplus W)_g = V_g \oplus W_g.$$

Similarly, the tensor product $V \otimes W$ has naturally a gradation given by

$$(V \otimes W)_g = \bigoplus_{g'g''=g} V_{g'} \otimes W_{g''}.$$

Definition 2.2. Let V, W be two G-graded k-vector spaces. A k-linear map $T : V \to W$ *preserves the gradation* if $T(V_g) \subset W_g$ for any $g \in G$.

Example 2.2. Let V be a G-graded k-vector space. For any $\lambda \in k$ the homothety T_λ : $V \to V$, $v \mapsto \lambda v$, is a preserving gradation k-linear map.

Example 2.3. Let V be a k-linear space, the tensor algebra

$$\mathrm{T}^\bullet V = \bigoplus_{n\geq 0} V^{\otimes n},$$

is naturally a $\mathbb{Z}$-graded k-vector space, with $(\mathrm{T}^\bullet V)_n = V^{\otimes n}$, for $n \geq 1$, $V^{\otimes 0} = k$, and $(\mathrm{T}^\bullet V)_n = \{0\}$, for $n < 0$.

Similarly, the symmetric algebra $\mathrm{Sym}^\bullet V$ and the exterior algebra $\bigwedge^\bullet V$ are $\mathbb{Z}$-graded k-vector space. There exists a second gradation on $\bigwedge^\bullet V$, given by

$$\bigwedge^\bullet V = \left(\bigoplus_{n\text{: even}} \bigwedge^n V\right) \oplus \left(\bigoplus_{n\text{: odd}} \bigwedge^n V\right),$$

defines a $\mathbb{Z}/2\mathbb{Z}$-gradation.

For any k-linear map $T : V \to W$, the induced map over $\mathrm{T}^\bullet V$, $\mathrm{Sym}^\bullet V$ or $\bigwedge^\bullet V$ preserves the gradation.

Definition 2.3. A *k-super commutative algebra R over a field k* is a $\mathbb{Z}/2\mathbb{Z}$-graded k-vector space, $R = R_{\bar{0}} \oplus R_{\bar{1}}$ with an unital multiplication $R \otimes_k R \to R$ that preserves the gradation, in other words $R_i \otimes R_j \to R_{i+j}$, such that for homogeneous elements $a \in R_i$, $b \in R_j$ we have the commutative rule: $ba = (-1)^{ij}ab$.

Let R, S be k-super algebras, a k-linear map $T : R \to S$ is said to be *even* if $T(R_i) \subset S_i$, $i = \bar{0}, \bar{1}$, and is said to be *odd* if $T(R_i) \subset S_{i+\bar{1}}$, $i = \bar{0}, \bar{1}$. An even k-linear map $T : R \to S$ is said to be a *homomorphism of super algebras* if $T(rr') = T(r)T(r')$ for any $r, r' \in R$ and $T(1) = 1$. The set of super algebras homomorphisms is going to be denoted by $\mathrm{Hom}_{\mathrm{SAlg}_k}(R, S)$. If there exists two homomorphisms $T : S \to R$ and $T' : R \to S$ with $T \circ T' = \mathrm{id}_R$ and $T' \circ T = \mathrm{id}_S$, we will say that R, S are *isomorphic* and that T, T' are *isomorphisms*. For a super algebra R, the space of isomorphisms $T : R \to R$ is going to be denoted by $\mathrm{Aut}_k(R)$ and any element is going to be called an *automorphism of R*.

Let R, S be super algebras over k, we say that R *is an S-algebra* if there exists a homomorphism of super algebras $\alpha_R : S \to R$. Let R, R' be two S-algebras, an *S-homomorphism* is an homomorphisms $T : R \to R'$ such that $T \circ \alpha_R = \alpha_{R'}$. For an S-algebra R, the space of automorphisms that are S-homomorphisms are denoted by $\mathrm{Aut}_S(R)$.

An element $a \in R$ is called *even* if $a \in R_{\bar{0}}$ and is said to be *odd* if $a \in R_{\bar{1}}$. Also, we say that $a \in R$ has *parity j* if $a \in R_j$.

For a non-nilpotent even element $f \in R_{\bar{0}}$ we denote by $R_{(f)}$ the super algebra given by the localization of R with respect to the multiplicative set $\{1, f, f^2, \dots\}$.

We are going to say that $R = R_{\bar{0}} \oplus R_{\bar{1}}$ is a *purely even super algebra* if $R_1 = 0$.

Observation 2.2. For any super algebra R, $R_{\bar{0}}$ is a commutative ring and $R_{\bar{1}}$ is an $R_{\bar{0}}$-module.

Example 2.4. A commutative ring R over k can be seen as a purely even super algebra with $R_{\bar{0}} = R$ and $R_{\bar{1}} = 0$.

Example 2.5. Given a super algebra R, the super algebra of polynomials $R[t]$, with t an even variable, is defined as the usual algebra of polynomials with the $\mathbb{Z}/2\mathbb{Z}$-gradation:

$$R[t]_j := \{a_0 + a_1 t + \cdots + a_n t^n : n \in \mathbb{N}, a_i \in R_j\},$$

then, $R[t]$ is a super algebra. Recursively, we will consider the super algebra $R[t_1, \ldots, t_n] := R[t_1, \ldots, t_{n-1}][t_n]$, for the even variables $t_1, \ldots, t_n$.

For a super algebra R we can construct the *Grassmann algebra* $R[\theta]$, with θ an odd variable, defined as the usual algebra of polynomials with the $\mathbb{Z}/2\mathbb{Z}$-gradation:

$$R[\theta]_j := \{a_0 + a_1\theta : a_0 \in R_j, a_1 \in R_{j+\bar{1}}\},$$

then, $R[\theta]$ is a super algebra with $\theta \in R[\theta]_{\bar{1}}$. Recursively, we will consider the Grassmann algebra of rank n, $R[\theta^1, \ldots, \theta^n] := R[\theta^1, \ldots, \theta^{n-1}][\theta^n]$, for the odd variables $\theta^1, \ldots, \theta^n$.

We write $R[m|n] := R[t_1, \ldots, t_m][\theta^1, \ldots, \theta^n]$, where $t_1, \ldots, t_m$ are even and $\theta^1, \ldots, \theta^n$ are odd variables.

Similarly, we define the Laurent series $R[[t]] = \{a_0 + a_1 t + a_2 t^2 + \cdots : a_i \in R\}$, $R[[t_1, \ldots, t_m]] := R[[t_1, \ldots, t_{m-1}]][[t_m]]$ and $R[[m|n]] := R[[t_1, \ldots, t_m]][\theta^1, \ldots, \theta^n]$, for the even variables $t_1, \ldots, t_m$ and odd variables $\theta^1, \ldots, \theta^n$.

Also, observe that we have a projection $R[m|n] \to R$ joint with a section $R \to R[m|n]$. Equally, we have the projection $R[[m|n]] \to R$ with its respective section.

Observation 2.3. Let R, S super algebras, an homomorphism $T : R \to S$ and $m, n \in \mathbb{N}$, we get homomorphism given by

$$R[m|n] \to S[m|n]$$

extended by $\widehat{T}(rt_i^n\theta^j) = T(r)s_i^n\rho^j$.

Definition 2.4. Let R be a super algebra and $\mathcal{S} \subset R_{\bar{0}}$ a multiplicative set. We denote by $R_{\mathcal{S}}$ the super algebra given by the localization of R with respect to the multiplicative set $\mathcal{S}$. In this case, as $\mathbb{Z}/2\mathbb{Z}$-graded k-vector space we have: $(R_{\mathcal{S}})_0 = (R_{\bar{0}})_{\mathcal{S}}$ and $(R_{\mathcal{S}})_1 = (R_{\bar{1}})_{\mathcal{S}}$, here recall that $R_{\bar{1}}$ is an $R_{\bar{0}}$-module. For an even non-nilpotent element $f \in R$, we denote by $R_{(f)}$ the localization of R through the multiplicative set $(f) = \{1, f, f^2, \ldots\}$. Also, for a prime ideal $\mathfrak{p} \subset R$, the set $R - \mathfrak{p}$ is a multiplicative set and we will denote by $R_{(\mathfrak{p})}$ by the localization $R_{R-\mathfrak{p}}$.

Example 2.6. Let R be a super algebra and $R[[m|n]] := R[[t_1, \ldots, t_m]][\theta^1, \ldots, \theta^n]$. We have the maximal ideal $\mathfrak{m} := \langle t_1, \ldots, t_m|\theta^1, \ldots, \theta^n\rangle$, and the localization $R[[m|n]]_{(\mathfrak{m})}$ is called *$R[[m|n]]$-punctured disk.*

Definition 2.5. Let R be a super algebra and its ideal $J := R_{\bar{1}} + R_{\bar{1}}^2$. We define the *reduced ring of* R as the quotient $R_{\mathrm{rd}} := \frac{R}{J}$, this is a ring endowed with the projection $R \to R_{\mathrm{rd}}$.

Observation 2.4. The ring R_{rd} is not necessarily reduced, since it may contain nilpotent even elements.

Observation 2.5. The projection $R \to R_{\mathrm{rd}}$ does not necessarily have a section. Take for example the super algebra $R_0 = k[t|\theta^1, \theta^2]$ with its homogeneous ideal $I = \langle t^2 - \theta^1\theta^2\rangle$, then the quotient $R = R_0/I$ is a super algebra. In this case, $R_{\mathrm{rd}} = k[t]/(t^2)$ and suppose that the projection $R \to R_{\mathrm{rd}}$ does has a section, then there exists an element $\phi(t) = t + a(t)\theta^1\theta^2 \in R_0$ with $\phi(t)^2 \in I$. Finally, we have the equation:

$$(t + a(t)\theta^1\theta^2) = (t^2 - \theta^1\theta^2)(p(t) + q(t)\theta^1\theta^2)$$
$$t^2 + 2ta(t)\theta^1\theta^2 = t^2p(t) + (-p(t) + t^2q(t))\theta^1\theta^2$$

then $p(t) = 1$ and $t| - p(t) + t^2q(t)$, what is impossible.

Example 2.7. For a commutative ring R and the Grassmann algebra $R[\theta^1, \ldots, \theta^n]$, observe that

$$R[\theta^1, \ldots, \theta^n] \to (R[\theta^1, \ldots, \theta^n])_{\mathrm{rd}} \simeq R,$$

and there exists a section $R \to R[\theta^1, \ldots, \theta^n]$.

Observation 2.6. Let $T : R \to S$ be a homomorphism of super algebras. Since $T(R_{\bar{1}}) \subset S_{\bar{1}}$, then $T(R_{\bar{1}} + R_{\bar{1}}^2) \subset S_{\bar{1}} + S_{\bar{1}}^2$. That is, T induces a homomorphism of commutative rings $T_{rd} : R_{\mathrm{rd}} \to S_{\mathrm{rd}}$, and we can write the map:

$$\mathrm{Hom}_{\mathrm{SAlg}}(R, S) \to \mathrm{Hom}_{\mathrm{CRings}}(R_{\mathrm{rd}}, S_{\mathrm{rd}}). \tag{2.1}$$

Observe that in general, the map (2.1) is not surjective. Take the example given in Observation 2.5 and observe that for $R = R_{\mathrm{rd}}, S = R$ the morphism $\mathrm{id}_{R_{\mathrm{rd}}} \in \mathrm{Hom}_{\mathrm{CRings}}(R_{\mathrm{rd}}, R_{\mathrm{rd}})$ is not in the image of (2.1), since the projection $R \to R_{\mathrm{rd}}$ does not have a section.

Let $R = R_{\bar{0}} \oplus R_{\bar{1}}$ be a super algebra, and consider a commutative ring S, observe that any morphism $\phi : R \to S$ vanishes over $R_{\bar{1}}$, then we can factorize the map through the projection $R \to R_{\mathrm{rd}}$:

$$\begin{array}{ccc} R & \xrightarrow{\phi} & S \\ \downarrow & \nearrow_{\overline{\phi}} & \\ R_{\mathrm{rd}} & & \end{array}$$

so, for any super algebra R and a commutative ring S, we get a natural identification:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{SAlg}}(R, S) &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CRings}}(R_{\mathrm{rd}}, S) \\ \phi &\mapsto \overline{\phi}. \end{aligned} \tag{2.2}$$

On the other hand, for $R = R_{\bar{0}} \oplus R_{\bar{1}}$ a super algebra, S a commutative ring, then any morphism $S \to R$ is a morphism $S \to R_0$, so we get the identification

$$\begin{aligned} \mathrm{Hom}_{\mathrm{SAlg}}(S, R) &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CRings}}(S, R_0) \\ \phi &\mapsto \widehat{\phi}. \end{aligned}$$

2.2 Super modules

Definition 2.6. Let R be a super algebra and consider M be $\mathbb{Z}/2\mathbb{Z}$-graded k-vector space, we will say that M is an *R-super module* if is endowed with a k-bilinear homogeneous product

$$\sigma_M : R \otimes_k M \to M,$$

that is $R_i \otimes M_j \to M_{i+j}$, and for any $a, b \in R$ and $m \in M$:

$$\begin{aligned} \sigma_M(1 \otimes m) &= m, \\ \sigma_M(a \otimes \sigma_M(b \otimes m)) &= \sigma_M((ab) \otimes m). \end{aligned}$$

In case there is no confusion we will only write $\sigma_M(a \otimes m) = a \cdot m = am$.

For a super module M we can construct the super module ΠM as the same set M with $\Pi M_i := M_{i+\bar{1}}$, and $\sigma_{\Pi M} = \sigma_M$.

Example 2.8. For a super algebra R the super algebra $R[m|n]$ is a R-super module. In particular, R is a R-super module. Also, observe that ΠR is an R-module.

Observation 2.7. Let M_1, M_2 be two R-super module, then the direct sum $M_1 \oplus M_2$ is an R-super module by $a(m_1 \oplus m_2) = (am_1) \oplus (am_2)$. Similarly, the $\mathbb{Z}/2\mathbb{Z}$-graded tensor product

$M_1 \otimes M_2$ is an R-super module with the action $a(m_1 \otimes m_2) = (am_1) \otimes m_2 + (-1)^{ij} m_1 \otimes (am_2)$, for $a \in R_i$, $m_1 \in M_j$.

Let R be an S-super algebra and M be an S-super module, we construct the R-super module $R \otimes_S M$ with the action $a(b \otimes m) = (ab) \otimes m$.

Example 2.9. The direct sum $R^{m|n} := R^m \oplus (\Pi R)^n$ is an R-super module.

Definition 2.7. For two super modules M_1, M_2 over R and a k-linear map $T : M_1 \to M_2$, we say that T is *even* if $T(M_{1,i}) \subset M_{2,i}$, $i = 0, 1$, and *odd* if $T(M_{1,i}) \subset M_{2,i+\bar{1}}$, $i = \bar{0}, \bar{1}$. We say that T is *an R-homomorphism of R-super modules* if T has parity j and for any $a \in R_i$ we have $T(am) = (-1)^{ij} aT(m)$.

Over the space of R-homomorphism of R modules, denoted by $\underline{\mathrm{Hom}}_R(M_1, M_2)$, we get a $\mathbb{Z}/2\mathbb{Z}$-gradation of even and odd R-homomorphism:

$$\underline{\mathrm{Hom}}_R(M_1, M_2) = \mathrm{Hom}_R(M_1, M_2)_{\bar{0}} \oplus \mathrm{Hom}_R(M_1, M_2)_{\bar{1}}.$$

An element $T \in \mathrm{Hom}_R(M_1, M_2)$ is said to be *invertible* if there exists an homomorphism $S \in \mathrm{Hom}_R(M_2, M_1)_{\bar{0}}$ such that $T \circ S = \mathrm{id}_{M_2}$ and $S \circ T = \mathrm{id}_{M_1}$. When T is even an invertible $\in \mathrm{Hom}_R(M_1, M_2)_{\bar{0}}$ we will say that T is an *isomorphism.* In this case, we say that T has *inverse* S and that M_1 and M_2 are *isomorphic.* When $M_1 = M_2$, an isomorphism $T : M_1 \to M_1$ is called *automorphism* instead of isomorphism.

An R-module M is *free, finitely generated and that has rank $m|n$* if M is isomorphic to the super module $R^{m|n}$ given in Example 2.9.

Observation 2.8. Let R be an S-super algebra and M be an S-super module, then the $\mathbb{Z}/2\mathbb{Z}$-graded tensor product $R \otimes_S M$ is an R-super module. Also, if M is a free, finitely generated and that has rank $m|n$ S-super module, then $R \otimes_S M$ is a free, finitely generated and that has rank $m|n$ R-super module.

In particular, if M is a rank $m|n$ free R-super module, then $M_{\mathrm{rd}} := R_{\mathrm{rd}} \otimes_R M$ is a rank $m|n$ free R_{rd}-super module. Such M_{rd} is called *the reduced module.*

Observation 2.9. From the odd k-linear map $M \to \Pi M$, $m \to m$, we see that is bijective but not an isomorphism since this morphism is odd.

Example 2.10. Let R be a super algebra, for a super module M, an element $a \in R_i$ induce an R-homomorphism with parity i by multiplication:

$$\begin{aligned} T_a : M &\to M \\ m &\mapsto am. \end{aligned}$$

Observe that any invertible element in $R_{\bar{0}}$ induce an automorphism in M.

Observation 2.10. Let M be a super module, and take

$$\mathrm{End}_R(M) := \underline{\mathrm{Hom}}_R(M, M),$$

that is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra with the composition as product. The subset of invertible elements in $\mathrm{End}_R(M)$ is denoted by $\mathrm{Inv}_R(M)$.

Example 2.11. Let M be a R-super module, the *dual module* is given by the module $M^* := \underline{\mathrm{Hom}}_R(M, R)$. Observe that if M is a free module, then M^* is also free.

Observation 2.11. Let S be a R-super algebra and M, N be R-super modules. We extend a morphism $f \in \underline{\mathrm{Hom}}_S(S \otimes M, S \otimes N)_{\bar{0}}$ by $\widehat{f}(r \otimes m) = r \otimes f(m)$. More generally, any element $f \in \underline{\mathrm{Hom}}_S(S \otimes M, S \otimes N)_j$ is extended by $\widehat{f}(r \otimes m) = (-1)^{ij} r \otimes f(m)$, for any $r \in R_i$.

Example 2.12. Let M be a free super module of rank $m|n$ and choose generators $\{t_1, \ldots, t_m| \theta^1, \ldots, \theta^n\}$, with t_i even and θ^j odd, the construction given in (2.5) gives us a super algebra, $\bigwedge_R(M) := R[t_1, \ldots, t_m|\theta^1, \ldots, \theta^n]$. Observe, that this algebra is independent on the choice of generators.

Now suppose that R is a commutative ring and M a rank n free R-module. Consider the super algebra $R[0|n] = R[\Pi M]$. Similarly, take the super algebra $S[0|1] = S[\Pi N]$, where S is a commutative ring and N is a rank 1 free S-module N. Any morphism of super algebras $\phi : R[0|n] \to S[0|1]$ is given by a morphism $R \to S$ and a morphism of R-modules $M \to N$. In particular, we get the following lemma:

Lemma 2.1. *Let R be a commutative ring, M be a rank $(0|n)$ free R-super module, then there is a natural identification $Hom_{R-SAlg}(R[M], R[0|1]) \simeq \Pi M^*$.*

Proof. Let $\phi \in Hom_{R-SAlg}(R[M], R[0|1])$, then the restriction to $M \to N$ determines ϕ. Such morphism of R-super modules correspond to an odd morphism $M \to R$. Then we get the isomorphism:

$$\begin{aligned} Hom_{R-SAlg}(R[M], R[0|1]) &\to \Pi M^* \\ \phi &\mapsto \phi|_M. \end{aligned}$$

Since any morphism in M^* is odd, then ΠM^* is purely even, as $Hom_{R-SAlg}(R[M], R[0|1])$. □

Observe that the previous lemma is not true if M is not free.

The following lemma is going to be useful for doing computations:

Lemma 2.2. *Let W be an $n \times n$ matrix with entries in R. Suppose that every entry is nilpotent, then W is nilpotent.*

Proof. Suppose that $W = (w_{ij})_{ij}$, with $w_{ij} \in R$ nilpotent. Then, $W^k = (P_{k,rs}(w_{ij}))_{rs}$ where $P_{k,rs}$ are homogeneous polynomials of degree k. Since, there exists an integer $N \in \mathbb{N}$ such that $w_{ij}^N = 0$, for any $i, j = 1, \ldots, n$. Finally, $W^{n^2 N} = 0$. □

Observation 2.12. Let $T : R^{m|n} \to R^{m|n}$ be an invertible morphism represented by the matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.1}$$

where A, B, C, D is a $m \times m$, $m \times n$, $n \times m$ and $n \times n$ matrix, respectively, with inverse

$$S = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix},$$

then $AA' + BC' = \mathrm{id}$, then $AA' = \mathrm{id} - BC'$. Since, B, C' has odd entries follows that BC' has nilpotent entries, from Lemma (2.2) BC' is nilpotent. Finally, $AA' = \mathrm{id} - BC'$ is invertible and A, A' are invertible too. Similarly, D is invertible.

Definition 2.8. Let M be a rank $m|n$ free super module and $T \in \mathrm{Inv}(M)$ represented in some basis by the matrix given by (2.1). We define the *Berezinian of T* by

$$\mathrm{Ber}(T) = \det(A - BD^{-1}C)\det(D)^{-1}. \tag{2.2}$$

From the previous observation, $\det(D)$ is invertible, so (2.2) is well defined.

Observation 2.13. The Berezinian verifies the following conditions:

1. If $T = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ or $T = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$, then $\mathrm{Ber}(T) = \det(A)\det(D)^{-1}$. In particular, for the identity matrix, id_M, $\mathrm{Ber}(\mathrm{id}_M) = 1$

2. Let T, S be two automorphisms, then $\mathrm{Ber}(TS) = \mathrm{Ber}(T)\mathrm{Ber}(S)$. In particular, $\mathrm{Ber}(T)$ does not depend on the basis chosen. Also, $\mathrm{Ber}(T)$ is invertible for any $T \in \mathrm{Inv}(M)$.

3. Suppose that $k = \mathbb{C}$, and that M is finitely generated, so for $T \in \mathrm{End}_k(M)$ we can define $\exp(T) = \sum_{i \geq 0} \frac{T^i}{i!}$. In this case we have

$$\mathrm{Ber}(\exp(T)) = \exp(\mathrm{str}(T)), \tag{2.3}$$

where $\mathrm{str}(T) := \mathrm{tr}(A) - \mathrm{tr}(D)$ is called the *super trace.*

4. To define the Berezinian, we just need that D in (2.2) is invertible, and the observations above still hold even when T is not necessarily invertible.

Observation 2.14. For the free super module M with generators $\{t_1, \ldots, t_n|\theta^1, \ldots, \theta^m\}$, we can construct the free module $\mathrm{Ber}(M)$ generated by the formal element $[t_1, \ldots, t_n|\theta^1, \ldots, \theta^m]$ with parity $m \mod 2$. Then $\mathrm{Ber}(M)$ has rank $1|0$ if m is even and rank $0|1$ if m is odd. An invertible homomorphism $T : M \to M$, induce the automorphism $\mathrm{Ber}(T) : \mathrm{Ber}(M) \to \mathrm{Ber}(M)$.

For a morphism of super algebras $\phi : R \to S$ and a free R-module M. We obtain the commutative diagram

$$\begin{array}{ccc} \mathrm{Inv}(M) & \xrightarrow{\mathrm{Ber}} & \mathrm{Inv}(\mathrm{Ber}(M)) \\ \downarrow & & \downarrow \\ \mathrm{Inv}(M_S) & \xrightarrow{\mathrm{Ber}} & \mathrm{Inv}(\mathrm{Ber}(M_S)) \end{array} \tag{2.4}$$

where $M_S := S \otimes_R M$.

Example 2.13. Given the super algebra R, we get $R[m|n]$ with the projection $R[m|n] \to R$. For a free $R[m|n]$-module M and $T \in \mathrm{Inv}(M)$ represented by a matrix (f_{ij}), with $f_{ij} \in R[m|n]$ with expression

$$f_{ij} = a_{ij} + \text{higher degree terms},$$

where $a_{ij} \in R$, then in (2.4) we get $\mathrm{Ber}(T_R) = \mathrm{Ber}(a_{ij})$.

A similar result is obtained by taking the projection $R[[m|n]] \to R$.

2.3 Super derivations

Definition 2.9. Let R be a S-super algebra and let $D \in \mathrm{Hom}_S(R, R)$ with parity i. We say that D is *an S-derivation* if $D(s) = 0$ for any $s \in S$, and for any $a \in R_j$ and $b \in R$ we have:

$$D(ab) = D(a)b + (-1)^{ij} aD(b).$$

The vector space of derivations has a structure of R super module given by $(aD)(b) = aD(b)$, for any $a, b \in R$. We denote by $\mathrm{Der}_{R/S}$ the *super module of derivations.*

For two derivations $D_1 \in \mathrm{Der}_{R/S,i}$ and $D_2 \in \mathrm{Der}_{R/S,j}$ we define *the bracket* by:

$$[D_1, D_2] = D_1 D_2 - (-1)^{ij} D_2 D_1,$$

with this structure $\mathrm{Der}_{R/S}$ is a super Lie algebra.

The dual module $(\mathrm{Der}_{R/S})^* = \Omega_{R/S}$ is called *the space of 1-forms.*

Let R be a k-super algebra, we are going to say that *is smooth* if Ω_R is a free R-super module.

Let R be a commutative ring, we know that $Hom_{R-CRing}(Sym^\bullet_R(Der_{R/k}), R[\epsilon]) \simeq Der_{R/k}$. Suppose that R is a super algebra such that $Der_{R/k}$ is a rank $m|n$ free module. In this case, similar to commutative rings, we get the identification

$$Hom_{R-SAlg}(Sym^\bullet_R(Der_{R/k}), R[\epsilon_0, \epsilon_1]) \simeq Der_{R/k}$$

where ϵ_i has parity i, $\epsilon_0^2 = 0$ and $\epsilon_0 \epsilon_1 = 0$.

Observation 2.15. Consider a super algebra R, and the following recipe:

1. First, try to find R_{rd}.
2. Second, suppose that $R = \bigwedge^\bullet_{R_{\mathrm{rd}}}(M)$ for some rank $0|n$ free R_{rd}-super module. In order to find M we could consider the exact sequence of R_{rd}-super modules:

$$0 \to M \to R_{\mathrm{rd}} \otimes_R Der_{R/k} \to Der_{R_{\mathrm{rd}}/k} \to 0,$$

in this case $M \simeq (R_{\mathrm{rd}} \otimes_R Der_{R/k})_1$.

This recipe does not work if R does not have the form $R = \bigwedge^\bullet_{R_{\mathrm{rd}}}(M)$, and this could happen, for example, if the projection $R \to R_{\mathrm{rd}}$ does not have a section.

Example 2.14. Let $R[m|n] := R[t_1, \ldots, t_m][\theta^1, \ldots, \theta^n]$, be the super algebra of polynomials associated to the super algebra R, then the set of derivations of $R[m|n]$ over R is a $R[m|n]$ free module with even part generated by $\{\partial_{t_1}, \ldots, \partial_{t_m}\}$ and odd part generated by $\{\partial_{\theta^1}, \ldots, \partial_{\theta^n}\}$.

Similarly, the generators of $\mathrm{Der}_R(R[[m|n]])$ are given by $\{\partial_{t_1}, \ldots, \partial_{t_m} | \partial_{\theta^1}, \ldots, \partial_{\theta^n}\}$.

Definition 2.10. Let $X = A_1 \partial_{t_1} + \cdots + A_m \partial_{t_m} + B_1 \partial_{\theta^1} + \cdots + B_n \partial_{\theta^n} \in \mathrm{Der}_R(R[[m|n]])$ be a vector field, we define *the super divergence operator*:

$$\mathrm{sdiv}(X) := \partial_{t_1} A_1 + \cdots + \partial_{t_m} A_m + (-1)^{b_1} \partial_{\theta^1} B_1 + \cdots + (-1)^{b_n} \partial_{\theta^n} B_n, \tag{2.1}$$

where B_i has parity b_i. Denote by $S(m|n)$ the *space of divergence free vector fields*. Observe that $S(m|n) \subset \mathrm{Der}_R(R[[m|n]])$ is a sub-super Lie algebra.

Observation 2.16. In a complete analogy with the usual definition of divergence, we define the *super Lie derivative* by taking the module $\mathrm{Ber}(\Omega_{R[m|n]})$ with generator $\Delta = [dt_1 \dots dt_m | d\theta^1 \dots d\theta^n]$ and the action

$$L_X(f\Delta) = X(f)\Delta + (-1)^{ij} f \mathrm{sdiv} X \Delta,$$

where $f \in R_i$ and $X \in \mathrm{Der}_R(R[[m|n]])_j$.

With this description, $S(m|n) = \{X \in \mathrm{Der}_R(R[[m|n]]) : L_X\Delta = 0\}$.

Observation 2.17. There exists a more general description given in [5], Proposition 2.4, for $S(1|n)$ super algebras, given by

$$S(1|n, \lambda) := \{X \in \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[[1|n]]) : L_X(t^\lambda \Delta) = 0\},$$

where $\mathbb{C}[[1|n]] = \mathbb{C}[[t]][\theta^1, \dots, \theta^n]$ and $\lambda \in \mathbb{C}$. Also, there are non-trivial isomorphisms given by $S(1|n, \lambda) \simeq S(1|n, \mu)$ for $\lambda - \mu \in \mathbb{Z}$.

These algebras are simple for $n \geq 2$ and $\lambda \notin \mathbb{Z}$

Proposition 2.1. *The super Lie algebra $S(1|n)$ is not simple.*

Proof. Firstly, observe that $\partial_{\theta^i}\partial_{\theta^i} = 0$, and also for $X = A_0\partial_t + A_1\partial_{\theta^1} + \cdots + A_n\partial_{\theta^n} \in S(1|n)_i$ we have that $\partial_t A_0 = (-1)^i(\partial_{\theta^1}A_1 + \cdots + \partial_{\theta^n}A_n)$. Let $X, Y \in S(1|n)$ vector fields with $X = A_0\partial_t + A_1\partial_{\theta^1} + \cdots + A_n\partial_{\theta^n} \in S(1|n)_i$ and $Y = B_0\partial_t + B_1\partial_{\theta^1} + \cdots + B_n\partial_{\theta^n} \in S(1|n)_j$. In order to calculate $\partial_{\theta^1} \cdots \partial_{\theta^n}([X, Y] \cdot t)$, observe that

$$\begin{aligned}
[X, Y] \cdot t &= X(B_0) - (-1)^{ij} Y(A_0) \\
&= A_0\partial_t B_0 + \sum_{k=1}^{n} A_k \partial_{\theta^k} B_0 - (-1)^{ij} \left(B_0 \partial_t A_0 + \sum_{k=1}^{n} B_k \partial_{\theta^k} A_0 \right) \\
&= A_0 \left((-1)^j \sum_{i=1}^{n} \partial_{\theta^k} B_k \right) + \sum_{k=1}^{n} A_k \partial_{\theta^k} B_0 \\
&\qquad - (-1)^{ij} \left(B_0 \left((-1)^i \sum_{i=1}^{n} \partial_{\theta^k} A_k \right) + \sum_{k=1}^{n} B_k \partial_{\theta^k} A_0 \right).
\end{aligned}$$

Here, collecting terms:

$$\begin{aligned}\partial_{\theta^k}((-1)^j A_0\partial_{\theta^k}B_k - (-1)^{ij}B_k\partial_{\theta^k}A_0) &= (-1)^j\partial_{\theta^k}A_0\partial_{\theta^k}B_k - (-1)^{ij}\partial_{\theta^k}B_k\partial_{\theta^k}A_0\\ &= (-1)^j\partial_{\theta^k}A_0\partial_{\theta^k}B_k - (-1)^{ij+j(i+1)}\partial_{\theta^k}A_0\partial_{\theta^k}B_k\\ &= 0.\end{aligned}$$

In the same way, $\partial_{\theta^k}(A_k\partial_{\theta^k}B_0 - (-1)^{ij+i}B_0\partial_{\theta^k}A_k) = 0$.

Applying the operator $\partial_{\theta^1}\cdots\partial_{\theta^n}$, we get:

$$\partial_{\theta^1}\cdots\partial_{\theta^n}([X,Y]\cdot t) = 0.$$

Finally, since $\partial_{\theta^1}\cdots\partial_{\theta^n}(X_0\cdot t) = 1$, then $X_0 \notin [S(1|n), S(1|n)]$. That is, $S(1|n)$ is not simple. □

Then $S(1|n,\lambda)$ is not simple for $\lambda\in\mathbb{Z}$, since the vector field $X_0 = \theta^1\cdots\theta^n\partial_t \in S(1|n,\lambda)$ is not in $S(n,\lambda) := [S(1|n,\lambda), S(1|n,\lambda)]$. Actually, $S(n) := S(n,0)$ is simple for $n\geq 2$.

Example 2.15. Let us consider the maximal ideal $\mathfrak{m} = \langle t|\theta^1,\theta^2\rangle \subset \mathbb{C}[[1|2]]$. The space of divergence free vector fields in $\mathrm{Der}_{\mathbb{C}}(\mathbb{C}[[1|2]]_{\mathfrak{m}})$ has even generators:

$$\begin{aligned}L_m =& - t^{m+1}\partial_t - \frac{m+1}{2}t^m\sum_{i=1}^N \theta^i\partial_{\theta^i},\ \text{for } m\in\mathbb{Z},\\ J_m^0 =& t^m(\theta^1\partial_{\theta^1} - \theta^2\partial_{\theta^2}),\ \text{for } m\in\mathbb{Z},\\ J_m^1 =& t^m\theta^1\partial_{\theta^2},\ \text{for } m\in\mathbb{Z},\\ J_m^2 =& t^m\theta^2\partial_{\theta^1},\ \text{for } m\in\mathbb{Z},\\ K =& \theta^1\theta^2\partial_t,\end{aligned}$$

and odd part with generators:

$$\begin{aligned}G_m^i =& - t^{m+1/2}\partial_{\theta^i},\ \text{for } m\in\mathbb{Z}+\frac{1}{2},\ \text{for } i=1,2.\\ H_m^i =& t^{m+1/2}\theta^i\partial_t - (m+1/2)\,t^{m-1/2}\theta^i\sum_{j=1}^N\theta^j\partial_{\theta^j},\ \text{for } i=1,2 \text{ and } m\in\mathbb{Z}+\frac{1}{2}.\end{aligned}$$

The map $S(1|2)\to S(1|2)/S(2)$ has its image generated by $K=\theta^1\theta^2\partial_t$. For $X\in S(1|2)$, then $X\in S(2)$ if and only if $\partial_{\theta^1}\partial_{\theta^2}(X\cdot t) = 0$.

2.4 Automorphisms of Super algebras

Definition 2.11. Let S be an R-super algebras, we will consider the group of even R-homomorphisms $T : S \to S$ with $T(ab) = T(a)T(b)$. Such group is denoted by $\mathrm{Aut}_R(S)$ and its elements are called *automorphisms of R-super algebras.*

Example 2.16. Given the super algebra R, consider the R-algebra $S = R[t]/(t^{n+1})$, the space of automorphism of R-algebras, $\mathrm{Aut}_R(S)$, is a super group. Any automorphism is given by

$$F(t) = a_0 + a_1 t + \cdots + a_n t^n,$$

where a_0 is nilpotent and a_1 is invertible. Composing with an affine automorphism, we are concerned by automorphism given by

$$F(t) = t + a_2 t^2 \cdots + a_n t^n.$$

Let $X = p(t)\partial_t \in \mathrm{Der}_R(S)$ be a nilpotent vector field, with $p(t) = b_2 t^2 + \cdots + b_n t^n$, then observe that:

$$\exp(X)t = t + p(t) + \frac{1}{2} p(t)p'(t) + \cdots ,$$

the set of nilpotent elements in $\mathrm{Der}_R(S)$ is denoted by $\mathrm{Der}_{R,+}(S)$, and the automorphisms generated by elements in $\mathrm{Der}_{R,+}(S)$ is denoted by $\mathrm{Aut}_{R,+}(S)$.

Finally, we get an isomorphism

$$\mathrm{Der}_{R,+}(S) \to \mathrm{Aut}_{R,+}(S). \tag{2.1}$$

Considering the affine automorphism, denoted by $\mathrm{Aut}_{R,0}(S)$, we obtain the surjection

$$\mathrm{Aut}_{R,0}(S) \ltimes \mathrm{Aut}_{R,+}(S) \to \mathrm{Aut}_R(S). \tag{2.2}$$

In particular, for any $\Phi \in \mathrm{Aut}_R(S)$ there exists an affine automorphism $\phi \in \mathrm{Aut}_{R,0}(S)$ and a vector field $X \in \mathrm{Der}_{R,+}(S)$ such that

$$\Phi = \exp(X) \circ \phi.$$

Definition 2.12. Let R be a super algebra, for $\Phi = (\phi_1, \ldots, \phi_m | \rho^1, \ldots, \rho^n) \in \mathrm{Aut}_R(R[[m|n]])$

we define the *Jacobian* as:

$$\mathrm{Jac}\Phi := \begin{pmatrix} \partial_{t_1}\phi_1 & \cdots & \partial_{t_1}\phi_m & \partial_{t_1}\rho^1 & \cdots & \partial_{t_1}\rho^n \\ \vdots & & \vdots & \vdots & & \vdots \\ \partial_{t_m}\phi_1 & \cdots & \partial_{t_m}\phi_m & \partial_{t_m}\rho^1 & \cdots & \partial_{t_m}\rho^n \\ \partial_{\theta^1}\phi_1 & \cdots & \partial_{\theta^1}\phi_m & \partial_{\theta^1}\rho^1 & \cdots & \partial_{\theta^1}\rho^n \\ \vdots & & \vdots & \vdots & & \vdots \\ \partial_{\theta^n}\phi_1 & \cdots & \partial_{\theta^n}\phi_m & \partial_{\theta^n}\rho^1 & \cdots & \partial_{\theta^n}\rho^n \end{pmatrix}.$$

From the chain rule, we obtain

$$\mathrm{Jac}(\Psi \circ \Phi) = \Phi(\mathrm{Jac}(\Psi))\mathrm{Jac}(\Phi).$$

Considering the $1|n$-free super module $\Omega_{R[m|n]} = (\mathrm{Der}_R(R[[1|n]]))^*$, we obtain an homomorphism of groups:

$$\begin{aligned} \mathrm{Aut}_R(R[[1|n]]) &\to \mathrm{Inv}_R(\Omega_{R[m|n]}) \\ \Phi &\mapsto \mathrm{Jac}\Phi \end{aligned}$$

In Observation (2.14), we constructed the formal element $\Delta_0 := [dt|d\theta^1 \cdots d\theta^n]$, for the generators $\{dt|d\theta^1 \cdots d\theta^n\}$ over $\mathrm{Ber}(\Omega_{R[1|n]})$, called *super volume form*. There is a group homomorphism given by

$$\begin{aligned} \mathrm{Aut}_R(R[[1|n]]) &\to \mathrm{Inv}_R(\mathrm{Ber}(\Omega_{R[m|n]})) \\ \Phi &\mapsto \mathrm{Ber}(\mathrm{Jac}\Phi). \end{aligned} \tag{2.3}$$

This homomorphism depends on the basis chosen. Also, $\Phi^*\Delta_0 = \mathrm{Ber}(\mathrm{Jac}\Phi)\Delta_0$.

We will denote by $\mathrm{Aut}^{\delta}_R(R[[1|n]])$ as the kernel of (2.3). When there is no confusion we just write $\mathrm{Aut}^{\omega}[[1|n]]$ and we say that such automorphisms preserves the Berezinian.

Example 2.17. Let $\Phi = (F|\rho^1, \ldots, \rho^n) \in \mathrm{Aut}_R(R[[1|n]])$, and suppose that Φ verifies:

$$\begin{aligned} F(t|\theta^1, \ldots, \theta^n) &= F(t) \\ \rho^i(t|\theta^1, \ldots, \theta^n) &= \theta^1 g_{1i}(t) + \cdots + \theta^n g_{ni}(t), \text{ for } i = 1, \ldots, n. \end{aligned}$$

then the Jacobian is given by

$$\mathrm{Jac}(\Phi) = \begin{pmatrix} \partial_t F(t) & \partial_t \rho^1 & \cdots & \partial_t \rho^n \\ 0 & g_{11}(t) & \cdots & g_{1n}(t) \\ \vdots & \vdots & & \vdots \\ 0 & g_{n1}(t) & \cdots & g_{nn}(t) \end{pmatrix},$$

then $\Phi \in \mathrm{Aut}_R^\omega(R[[1|n]])$ if and only if $\partial_t F(t) \det^{-1} G(t) = 1$, where $G(t) = (g_{ij}(t))_{ij}$.

Using the projection $R[[t|\theta^1, \ldots, \theta^n]] \to R[[t]]$, observe that for the diagram

$$\begin{array}{ccc} \mathrm{Inv}(\Omega_{R[[1|n]]}) & \xrightarrow{\mathrm{Ber}} & \mathrm{Inv}(\mathrm{Ber}(\Omega_{R[[1|n]]})) \\ \downarrow & & \downarrow \\ \mathrm{Inv}(\Omega_{R[[t]]}) & \xrightarrow{\mathrm{Ber}} & \mathrm{Inv}(\mathrm{Ber}(\Omega_{R[[t]]})) \end{array}$$

Then $\Phi \in \mathrm{Aut}_R^\delta(R[[1|n]])$ if and only if $\widehat{\Phi} \in \mathrm{Aut}_R^\delta(R[[t]])$.

2.5 Super Symplectic Forms

Definition 2.13. The even non-degenerate form

$$\omega = dt + \theta^1 d\theta^1 + \cdots + \theta^n d\theta^n \tag{2.1}$$

is called *super symplectic form.*

When an element $\Phi \in \mathrm{Aut}_R(R[[1|n]])$ *preserves* (2.1) *up to multiplication* if $\Phi^*\omega = f\omega$, for some function $f \in R[[1|n]]$, in such case we write $\Phi \in \mathrm{Aut}_R^\omega(R[[1|n]])$. When there is no confusion we write $\mathrm{Aut}^\omega[[1|n]]$. Observe that $\mathrm{Aut}^\omega[[1|n]]$ is a group with the composition as multiplication.

Observation 2.18. In [5] is described another super Lie group that looks similar to $\mathrm{Aut}^\omega[[1|n]]$. First, define the *twisted contact form*:

$$\omega_+ = dt + \theta^1 d\theta^1 + \cdots + \theta^{n-1} d\theta^{n-1} + t\theta^n d\theta^n.$$

Second, we define $\mathrm{Aut}_+^\omega[[1|n]] = \{\Phi \in \mathrm{Aut}[[1|n]] : \Phi^*\omega_+ = f\omega_+, \text{ for some } f \in R[[1|n]]\}$.

Observation 2.19. Let $\Phi = (F|\rho^1, \ldots, \rho^n) \in \mathrm{Aut}_R(R[[1|n]])$, taking the pullback of (2.1) through Φ we obtain:

$$\begin{aligned}\Phi^*(dt+\theta^1 d\theta^1+\cdots+\theta^n d\theta^n) =&\partial_t F dt-\partial_{\theta^1} F d\theta^1-\cdots-\partial_{\theta^n} F d\theta^n\\ &+\rho^1(\partial_t\rho^1 dt+\partial_{\theta^1}\rho^1 d\theta^1+\cdots+\partial_{\theta^n}\rho^1 d\theta^n)+\cdots\\ &+\rho^n(\partial_t\rho^n dt+\partial_{\theta^1}\rho^n d\theta^1+\cdots+\partial_{\theta^n}\rho^n d\theta^n)\\ =&(\partial_t F+\rho^1\partial_t\rho^1+\cdots+\rho^n\partial_t\rho^n)dt\\ &+(-\partial_{\theta^1}F+\rho^1\partial_t\rho^1+\cdots+\rho^n\partial_t\phi^n)d\theta^1+\cdots\\ &+(-\partial_{\theta^n}F+\rho^1\partial_t\rho^1+\cdots+\rho^n\partial_t\rho^n)d\theta^n.\end{aligned}$$

Defining the operators $D^i=\theta^i\partial_t+\partial_{\theta^i}$, $i=1,\ldots,n$, we obtain that $\Phi\in\mathrm{Aut}^\omega[[1|n]]$ if and only if the following equations are verified:

$$D^iF=\rho^1D^i\rho^1+\cdots+\rho^nD^i\rho^n,\text{ for all } i=1,\ldots,n. \tag{2.2}$$

Observation 2.20. The super Lie group $\mathrm{Aut}^\omega_R(R[[1|2]])$ has two connected components, one of them defined by the one containing the identity and the other one containing the element $\Phi_0\in\mathrm{Aut}^\omega_R(R[[1|2]])$ given by

$$\Phi_0(t|\theta^1,\theta^2):=(t|\theta^2,\theta^1). \tag{2.3}$$

Later, we will characterize the two components of $\mathrm{Aut}^\omega_R(R[[1|2]])$.

Definition 2.14. Let $X\in\mathrm{Der}_R(R[[1|n]])$ be a vector field, if $L_X\omega=f\omega$, with ω as (2.1) for some function $f\in R[[1|n]]$, we say that X is a *super conformal vector field* and write $X\in K(1|n)$.

Observation 2.21. The super conformal vector fields are given by:

$$D^f=f\partial_t+\frac{1}{2}(-1)^j\sum_{i=1}^n(D^if)D^i,$$

where $D^i=\theta^i\partial_t+\partial_{\theta^i}$, for any $f\in R[[1|n]]_j$.

For $n=2$, observe that for some homogeneous $f\in R[[1|n]]_j$ and the vector field D^f we get

$$\begin{aligned}\mathrm{sdiv}_{\Delta_0}(D^f)&=\partial_t f+(-1)^{j+1}(-1)^j\frac{1}{2}(D^1D^1f)+(-1)^{j+1}(-1)^j\frac{1}{2}(D^2D^2f)\\ &=\partial_t f-\frac{1}{2}\partial_t f-\frac{1}{2}\partial_t f\\ &=0.\end{aligned}$$

Also, we remark that the condition $K(1|n) \subset S(1|n)$ only holds for $n = 2$.

Proposition 2.2. *The super Lie algebra $K(1|2)$ is contained in $S(2)$.*

Proof. Taking $D^f t = f + \frac{1}{2}(-1)^j\left(D^1 f\theta^1 + D^2 f\theta^2\right) = f + \frac{1}{2}(-1)^j\left((\partial_{\theta^1} f)\theta^1 + (\partial_{\theta^2} f)\theta^2\right)$, then

$$\begin{aligned}\partial_{\theta^1}\partial_{\theta^2}(D^f t) &= \partial_{\theta^1}\partial_{\theta^2}\left(f + \frac{1}{2}(-1)^j\left((\partial_{\theta^1} f)\theta^1 + (\partial_{\theta^2} f)\theta^2\right)\right)\\ &= \partial_{\theta^1}\partial_{\theta^2} f + \frac{1}{2}(-1)^j\left((-1)^j\partial_{\theta^2}\partial_{\theta^1} f + (-1)^{j+1}\partial_{\theta^1}\partial_{\theta^2} f\right)\\ &= 0.\end{aligned}$$

Finally, we get the inclusion $K(1|2) \subset S(2)$. □

Observation 2.22. Suppose that $N = 2n$, over $R[[1|2n]]$ consider the change of coordinates

$$\begin{aligned}s =& t + i\left(\theta^1\theta^2 + \cdots + \theta^{2n-1}\theta^{2n}\right),\\ \rho^j =& -i(-\theta^{2j-1} + i\theta^{2j}),\ j = 1\ldots, n,\\ \eta^j =& -i(\theta^{2j-1} + i\theta^{2j}),\ j = 1\ldots, n,\end{aligned}$$

we obtain that $\omega = dt + \theta^1 d\theta^1 + \cdots + \theta^n d\theta^n$ changes as

$$\widetilde{\omega} = ds + \rho^1 d\eta^1 + \cdots + \rho^n d\eta^n \tag{2.4}$$

so, the group of automorphisms of $R[[1|2n]]$ that preserve ω up to multiplication by a function coincides with the group of automorphisms of $R[[s|\rho^1, \cdots, \rho^n, \eta^1, \cdots, \eta^n]]$ that preserve (2.4) up to multiplication by a function.

Let us consider $R[[1|2n]] = R[[1|n]][\rho^1, \ldots, \rho^n]$ and the inclusion $R[[1|n]] \hookrightarrow R[[1|2n]]$. Let $\Phi = (F|\phi^1, \ldots, \phi^n) \in \mathrm{Aut}[[1|n]]$ and consider the pullback of (2.4) through the super function $\widetilde{\Phi} = (F|\phi^1, \ldots, \phi^n, \eta^1, \ldots, \eta^n)$, is given by

$$\begin{aligned}\widetilde{\Phi}^*(dt + \rho^1 d\theta^1 + \cdots + \rho^n d\theta^n) =& \partial_t F dt - \partial_{\theta^1} F d\theta^1 - \cdots - \partial_{\theta^n} F d\theta^n\\ &+ \eta^1(\partial_t\phi^1 dt + \partial_{\theta^1}\phi^1 d\theta^1 + \cdots + \partial_{\theta^n}\phi^1 d\theta^n) + \cdots\\ &+ \eta^n(\partial_t\phi^n dt + \partial_{\theta^1}\phi^n d\theta^1 + \cdots + \partial_{\theta^n}\phi^n d\theta^n)\\ =& (\partial_t F + \eta^1\partial_t\phi^1 + \cdots + \eta^n\partial_t\phi^n)dt\\ &+ (-\partial_{\theta^1} F + \eta^1\partial_t\phi^1 + \cdots + \eta^n\partial_t\phi^n)d\theta^1 + \cdots\\ &+ (-\partial_{\theta^n} F + \eta^1\partial_t\phi^1 + \cdots + \eta^n\partial_t\phi^n)d\theta^n.\end{aligned}$$

Then, defining the differential operators $D_j = \rho^j \partial_t + \partial_{\theta^j}$, the function $\widetilde{\Phi} \in \mathrm{Aut}^\omega[[1|2n]]$ if and only if

$$D_i F = \eta^1 D_i \phi^1 + \cdots + \eta^n D_i \phi^n, \ i = 1, \ldots, n. \tag{2.5}$$

Since the matrix $(D_i \phi^j)_{ij}$ is invertible, we get:

$$\begin{pmatrix} D_1\phi^1 & \cdots & D_1\phi^n \\ \vdots & & \vdots \\ D_n\phi^1 & \cdots & D_n\phi^n \end{pmatrix}^{-1} \begin{pmatrix} D_1 F \\ \vdots \\ D_n F \end{pmatrix} = \begin{pmatrix} \eta^1 \\ \vdots \\ \eta^n \end{pmatrix}. \tag{2.6}$$

With these coordinates, $\widetilde{\Phi} = (F|\phi^1, \ldots, \phi^n, \eta^1, \ldots, \eta^n)$ makes the following diagram commutative:

$$\begin{array}{ccc} R[[1|n]] & \xrightarrow{\Phi} & R[[1|n]] \\ \downarrow & & \downarrow \\ R[[1|2n]] & \xrightarrow{\widetilde{\Phi}} & R[[1|2n]] \end{array} \tag{2.7}$$

and we obtain the inclusion of groups:

$$\mathrm{Aut}[[1|n]] \overset{j}{\hookrightarrow} \mathrm{Aut}^\omega[[1|2n]]. \tag{2.8}$$

Also, we obtain an inclusion of Lie algebras $\mathrm{Der}_R(R[[1|n]]) \overset{\widehat{j}}{\hookrightarrow} K(1|2n)$.

Chapter 3

Super Geometry

3.1 Super schemes

Definition 3.1. Let X be a topological space. A *sheaf of super algebras over* X, is a sheaf $\mathcal{F}$ such that for any open subset $U \subset X$ the set of sections $\mathcal{F}(U)$ is a super algebra. A *local super space* is a pair $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ is a sheaf of super algebras and for any closed point $p \in X$ the stalk $\mathcal{F}_{X,p}$ is a local super algebra.

Observation 3.1. Let X, Y be two topological spaces and $\mathcal{F}$ be a sheaf of super algebras over X. Then any continuous function $\phi : X \to Y$ define the *push-forward sheaf of super algebras* $\mathcal{F}_*$ *over* Y given by $\phi_*\mathcal{F}(V) := \mathcal{F}(\phi^{-1}(V))$. The sheaf $\phi_*\mathcal{F}$ is also a sheaf of super algebras (over Y).

Definition 3.2. Let R be a super algebra, we define the spectrum $\mathrm{Spec}(R)$ as the set of prime ideals with the Zariski topology.

Example 3.1. Let R be a super algebra and its ideal $J = R_{\bar{1}} + R_{\bar{1}}^2$. Let $\mathfrak{p} \subset R$ be a prime ideal. Since any nilpotent element is inside to any prime ideal, we have that $J \subset \mathfrak{p}$. Then, using the projection $\pi : R \to R_{\mathrm{rd}}$, we get the homeomorphism:

$$\begin{aligned} \mathrm{Spec}(R) &\to \mathrm{Spec}(R_{\mathrm{rd}}) \\ \mathfrak{p} &\mapsto \pi(\mathfrak{p}) \\ \pi^{-1}(\mathfrak{q}) &\leftarrow\!\shortmid q. \end{aligned}$$

In particular, for a commutative ring R, the super algebra $R[0|n] = R[\theta^1, \ldots, \theta^n]$, the projection $\pi : R[0|n] \to R$ induces a homeomorphism $\mathrm{Spec}(R[0|n]) \xrightarrow{\sim} \mathrm{Spec}(R)$.

Observation 3.2. Let R be a super algebra. Recall that for the Zariski topology a subset

$U \subset \operatorname{Spec}(R)$ is open if is empty or there exists a proper ideal $I \subset R$ with

$$U = U_I := \{\mathfrak{p} \in \operatorname{Spec}(R) : I \not\subseteq \mathfrak{p}\}.$$

More precisely, we have a basis of the Zariski topology given by non-nilpotent elements $f \in R_{\bar{0}}$:

$$U_{(f)} = \{\mathfrak{p} \in \operatorname{Spec}(R) : f \notin \mathfrak{p}\}.$$

Observe that U_f could be identified with $\operatorname{Spec}(R_{(f)})$ and $U_{(fg)} = U_{(f)} \cap U_{(g)}$. Also, we notice that if $I \subset I'$, then $U_{I'} \subset U_I$.

Example 3.2. Let R be a super algebra and $f \in R$ be an even element. To the open set $U_{(f)}$ we assign the super algebra R_f. Using this correspondence, we define the *sheaf of super algebras* $\mathcal{O}_{\operatorname{Spec}(R)}$ by

$$\mathcal{O}_{\operatorname{Spec}(R)}(U_I) = \{s : U_I \to \prod_{I \subset \mathfrak{p}} R_{(\mathfrak{p})} : \text{for any non-nilpotent } f \in I \text{ the section } s|_{U_{(f)}} \in R_f\}.$$

The pair $(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$ is called *super affine scheme*. The super affine scheme is a local super space.

Definition 3.3. Let $\mathcal{F}, \mathcal{F}'$ be two super algebras over the topological space X. A *a morphism between* $\phi : \mathcal{F} \to \mathcal{F}'$ is a family of homomorphisms of super algebras $\{\phi(U) : \mathcal{F}(U) \to \mathcal{F}'(U) : U \subset X \text{ open subset}\}$ such that for any pair of open subsets $V \subset U \subset X$ the following diagrams commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{F}'(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{F}'(V) \end{array}$$

where the vertical lines are the restrictions maps of sheaves. Clearly, could be defined the composition of two morphisms and there exists the identity morphism.

Let $\phi : \mathcal{F} \to \mathcal{F}'$ be a morphism, we can define $\ker(\phi)$, $\operatorname{ima}(\phi)$ and $\operatorname{coker}(\phi)$ as sheaves of super algebras over X. A morphism $\phi : \mathcal{F} \to \mathcal{F}'$ is said to be injective, respectively surjective, if $\ker(\phi) = \{0\}$, respectively $\operatorname{coker}(\phi) = \{0\}$.

Two sheaves of super algebras are isomorphic if there exist $\phi : \mathcal{F} \to \mathcal{F}'$ and $\phi' : \mathcal{F}' \to \mathcal{F}$ such that $\phi' \circ \phi = \operatorname{id}_{\mathcal{F}}$ and $\phi \circ \phi' = \operatorname{id}_{\mathcal{F}'}$.

Let $(X, \mathcal{F}), (Y, \mathcal{G})$ be two locally super spaces a *morphism between them* is a pair $(F, F^{\#}) : (X, \mathcal{F}) \to (Y, \mathcal{G})$, where $F : X \to Y$ is a continuous map and a morphism of sheaves $F^{\#} : \mathcal{F}' \to F_*(\mathcal{F})$. Naturally, we can define a composition of morphisms and the identity morphism. If there is no confusion, we just write $F : X \to Y$.

A morphism $(F, F^{\#}) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is said to be an *immersion*, respectively *submersion*, if $F^{\#}$ is surjective, respectively injective. In case $F : X \to Y$ is a closed map and $(F, F^{\#})$ an immersion, we will say that $(F, F^{\#})$ is a closed embedding.

We say that $(X, \mathcal{F}), (Y, \mathcal{G})$ are isomorphic if there exists two morphisms $(F, F^{\#}) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ and $(G, G^{\#}) : (Y, \mathcal{G}) \to (X, \mathcal{F})$ such that $(F, F^{\#}) \circ (G, G^{\#}) = (\mathrm{id}_Y, \mathrm{id}_{\mathcal{G}})$ and $(G, G^{\#}) \circ (F, F^{\#}) = (\mathrm{id}_X, \mathrm{id}_{\mathcal{F}})$.

Example 3.3. Any even morphism of super algebras $F : S \to R$ defines the continuous applications

$$\begin{aligned} F^* : \mathrm{Spec}(R) \to & \mathrm{Spec}(S) \\ \mathfrak{p} \mapsto & F^{-1}(\mathfrak{p}). \end{aligned}$$

Over $\mathrm{Spec}(S)$ we define the sheaf $F_*(\mathcal{O}_{\mathrm{Spec}(R)})(U) := \mathcal{O}_{\mathrm{Spec}(R)}((F^*)^{-1}(U))$, and a morphism of sheaves given by $F^{\#}(U_f) : \mathcal{O}_{\mathrm{Spec}(S)}(U_f) = S_f \to F_*(\mathcal{O}_{\mathrm{Spec}(R)})(U_f) = R_{F(f)}$ the localization of $F : S \to R$. Then the pair $(F^*, F^{\#})$ is a morphism.

A pair $(X, \mathcal{O}_X)$ is called super affine scheme if is isomorphic to $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$, for some super algebra R.

Definition 3.4. A *super scheme* is a local super space $(M, \mathcal{O}_M)$ if there exists an open covering of M, $\{U_i\}_{i \in I}$, such that

$$(U_i, \mathcal{O}_M|_{U_i}) = (\mathrm{Spec} R_i, \mathcal{O}_{R_i})$$

for some super algebra R_i. A morphism $M \to N$ is also called a *family of super schemes.*

Observation 3.3. For any super algebra R, we have the projection $R \to R_{\mathrm{rd}}$, then for an affine super scheme we get the closed embedding $\mathrm{Spec}(R_{\mathrm{rd}}) \to \mathrm{Spec}(R)$. More generally, we get a projection $\mathcal{O}_M \to \mathcal{O}_{M_{\mathrm{rd}}}$ and a closed embedding

$$M_{\mathrm{rd}} \hookrightarrow M. \tag{3.1}$$

We will say that M_{rd} is the *reduced scheme of* M.

For a super scheme M we say that is *projected* if there exists a left inverse for (3.1). In this case, we have an inclusion $\mathcal{O}_{M_{\mathrm{rd}}} \to \mathcal{O}_M$.

Following, (2.2) we obtain

Lemma 3.1. *Let M be a super scheme and N be a scheme, then we have the natural identification* $\mathrm{Hom}_{\mathrm{SSch}}(N, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sch}}(N, M_{\mathrm{rd}})$.

Observation 3.4. Let $(M.\mathcal{O}_M)$ be a super scheme. From the gradation $\mathcal{O}_M = (\mathcal{O}_M)_{\bar{0}} \oplus (\mathcal{O}_M)_{\bar{1}}$ we get a sheaf of algebras given by $(\mathcal{O}_M)_{\bar{0}}$. The pair $M_0 = (M, (\mathcal{O}_M)_{\bar{0}})$ is a super scheme.

Similar, to Lemma 3.1, we get:

Lemma 3.2. *Let M be a super scheme and N be a scheme, then we have the natural identification* $\mathrm{Hom}_{\mathrm{SSch}}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sch}}(M_0, N)$.

Example 3.4. Considering the Example (2.4), a scheme $(M, \mathcal{O}_M)$ defines naturally a super scheme $(M, \mathcal{O}_M)$. Also, any locally free $\mathcal{O}_M$-sheaf with rank n, $\mathcal{F}$, through (2.12) for $m|n = 0|n$, we obtain a super manifold. Such super scheme is going to be denoted by $(M, \bigwedge^\bullet \mathcal{F})$.

Let M be a super scheme. We say that M is *split* if there exists a locally free sheaf $\mathcal{F}$ over M_{rd} with $M \simeq (M_{\mathrm{rd}}, \bigwedge^\bullet \mathcal{F})$.

Observation 3.5. Let R be a super algebra, the reduction $R \to R_{\mathrm{rd}}$ induce the closed embedding $\mathrm{Spec}(R_{\mathrm{rd}}) \to \mathrm{Spec}(R)$. In this case, $\mathrm{Spec}(R)$ is split if there exists an R_{rd}-module, M, such that $\bigwedge^\bullet_{R_{\mathrm{rd}}}(M) = R$, in particular we have a section $R_{\mathrm{rd}} \to \bigwedge^\bullet_{R_{\mathrm{rd}}}(M) = R$ of the reduction. As we already see in Observation 2.5, this not happens ever. An easy test to see if $\mathrm{Spec}(R)$ is split or not is to check if $\Omega_{R/k}$ is a free super module over R.

Definition 3.5. Let $(M, \mathcal{O}_M)$ be a super scheme, an *$\mathcal{O}_M$-module*, $\mathcal{F}$, is a sheaf such that for any open subset $U \subset M$ the space of sections $\mathcal{F}(U)$ is an $\mathcal{O}_M(U)$-super module such that for any pair of open sets $V \subset U \subset M$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_M(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_M(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

where the vertical lines are the restrictions maps.

We will say that $\mathcal{F}$ is a *locally free $\mathcal{O}_M$-module* if there exists an open covering $\{U_i\}_{i \in I}$ of M such that $\mathcal{F}(U_i)$ is a free $\mathcal{O}_M(U_i)$-super module, for any $i \in I$.

Given two $\mathcal{O}_M$-modules $\mathcal{F}, \mathcal{G}$, we can construct the *sheaf of homomorphisms $\mathcal{O}_M$-modules* $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ by

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G})(U) := \underline{\mathrm{Hom}}_{\mathcal{O}_M(U)}(\mathcal{F}(U), \mathcal{G}(U)).$$

In particular, for a $\mathcal{O}_M$-module $\mathcal{F}$ we can construct the dual $\mathcal{O}_M$-module $\mathcal{F}^*$ given by $\mathcal{F}^* := \mathrm{Hom}(\mathcal{F}, \mathcal{O}_M)$.

Observation 3.6. Let R be a super algebra and N be an R-super module. For the affine scheme $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ we define the $\mathcal{O}_{\mathrm{Spec}(R)}$-module as the sheaf $\widetilde{N}$ defined over the open set U_f as $\mathcal{O}_{\mathrm{Spec}(R)}(U_f) = N_f$. With this definition, $\widetilde{N}$ is an $\mathcal{O}_{\mathrm{Spec}(R)}$-module. Reciprocally, any $\mathcal{O}_{\mathrm{Spec}(R)}$-module is defined by an R-super module.

Similarly, a free sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$-module is given by free R-super modules.

Example 3.5. Let R be a k-super algebra and S an R-super algebra, the super module of derivations $\mathrm{Der}_k(S)$ induces an $\mathcal{O}_{\mathrm{Spec}(R)}$-module called *module of derivations over* S. For $S = R$ we will denote this sheaf as $\mathrm{Der}(\mathcal{O}_{\mathrm{Spec}(R)}) =: \mathcal{T}_{\mathrm{Spec}(R)}$. More generally, for a super scheme $(M, \mathcal{O}_M)$ we define the $\mathcal{O}_M$*-module of derivations* $\mathcal{T}_M$ by $\mathcal{T}_M|_{U_i} = \mathcal{T}_{U_i}|_{U_i}$, for the open affine covering $\{U_i\}_{i\in I}$.

For a morphism of algebras $S \to R$ we could define over $\mathrm{Spec}(R)$ the *sheaf of* S*-relative differentials* given by the R-super module $\mathrm{Der}_S(R)$. More generally, for a family $M \to N$ we define the $\mathcal{O}_M$*-module of relative differentials* $\mathcal{T}_{M/N}$ by $\mathcal{T}_{M/N}|_{U_i} = \mathcal{T}_{U_i/V_i}|_{U_i}$, for the open affine covering $\{U_i\}_{i\in I}$ such that $U_i \to V_i$, with $V_i \subset N$ affine open subset.

On the other side, the dual $\mathcal{O}_M$-module $\Omega_M := (\mathcal{T}_M)^*$ is called *cotangent bundle*. Similarly, for a family $M \to N$ we define $\Omega_{M/N} := (\mathcal{T}_{M/N})^*$ is called *relative cotangent bundle*.

Definition 3.6. A *super manifold* is a super scheme $(M, \mathcal{O}_M)$, such that the sheaf of $\mathcal{O}_M$-modules given by $\mathcal{T}_M$ is a locally free sheaf of modules. If $\mathcal{T}_M$ has rank $m|n$ we say that M has *dimension* $m|n$.

For a closed point, $p \in M$, there exists an open set U such that $\mathcal{O}_M(U) = \bigwedge^{\bullet}_{\mathcal{O}_{M_{\mathrm{rd}}}}(E)$, for some $O_{M_{\mathrm{rd}}}(U)$ free module E.

For a morphism $M \to S$ we say that is a *family of super manifolds* when $\mathcal{T}_{M/S}$ is a locally free sheaf of modules. If $\mathcal{T}_{M/S}$ has rank $m|n$ we say that $M \to S$ is a family of $m|n$ dimensional super manifolds.

Example 3.6. Let $\mathbb{C}^{m|n} = \mathrm{Spec}\mathbb{C}[t_1, \ldots, t_m|\theta^1, \ldots, \theta^n]$ be a super manifold. Let us consider the space of $r|s$ vector subspaces on $\mathbb{C}^{m|n}$, we call this space *the rank* $r|s$ *super Grassmann space over* $\mathbb{C}^{m|n}$, and denote it by $\mathrm{Gr}(r|s, \mathbb{C}^{m|n})$. It was proved in [9], Chapter 4, Section 8, that this space is a super manifold. The reduced space, $\mathrm{Gr}(r|s, \mathbb{C}^{m|n})_{\mathrm{rd}}$, is isomorphic to the Grassmann space $\mathrm{Gr}(r, \mathbb{C}^m) \times \mathrm{Gr}(s, \mathbb{C}^n)$.

More generally, for a locally free sheaf $\mathcal{F}$ of rank $m|n$ over a super manifold M we can construct the super manifold $\mathrm{Gr}(r|s, \mathcal{F})$ that parametrizes the subspaces of rank $r|s$ over each fibre in $\mathcal{F}$. In this case, we have a natural projection $\mathrm{Gr}(r|s, \mathcal{F}) \to M$ with fibre $\mathrm{Gr}(r|s, \mathbb{C}^{m|n})$ over any closed point.

3.2 Geometric structures

3.2.1 Splitting super manifolds

Recall that a split super scheme is given by a pair $M = (M_{\mathrm{rd}}, \bigwedge^\bullet E)$ where M_{rd} is a scheme and E is a locally free $\mathcal{O}_{M_{\mathrm{rd}}}$-module. Observe that this is not true even locally, as we already see in Observation 3.5. When this happens locally, we are going to say that M is *locally split.*

Proposition 3.1. *Let M be a super manifold, then M is locally split.*

Proof. Let $p \in M$ be a closed point. Since $\mathcal{T}_M$ is locally free, then there exists an open set $U \subset M$ and local generators $\langle \partial_{z_1}, \dots, \partial_{z_m} | \partial_{\theta^1}, \dots, \partial_{\theta^n} \rangle$. Using the coordinates $(z_1, \dots, z_m)$ we get the scheme U_{rd} and the local coordinates $(\theta^1, \dots, \theta^n)$ give us the rank n free $\mathcal{O}_{U_{\mathrm{rd}}}$-module E such that $U \simeq (U_{\mathrm{rd}}, \bigwedge^\bullet E)$. □

From now on we are going to consider that any super scheme is locally split.

Observation 3.7. Let M be a super scheme, let us consider $J_M := \ker(\mathcal{O}_M \to \mathcal{O}_{M_{\mathrm{rd}}})$ an $\mathcal{O}_M$-module. Then we have the exact sequence:

$$0 \to J_M \to \mathcal{O}_M \to \mathcal{O}_{M_{\mathrm{rd}}} \to 0. \tag{3.1}$$

Since $\mathcal{O}_{M_{\mathrm{rd}}} = \mathcal{O}_M / J_M$, then J_M / J_M^2 is an $\mathcal{O}_{M_{\mathrm{rd}}}$-module.

Let $i \in \mathbb{N}$, we can construct the local super space $M^{(i)} := (M_{\mathrm{rd}}, \mathcal{O}_M / J^{i+1})$ joint with the sequence of inclusions:

$$M^{(0)} \hookrightarrow M^{(1)} \hookrightarrow \cdots \hookrightarrow M,$$

where, in particular, $M^{(0)} = M_{\mathrm{rd}}$. Also, if M is a manifold and J_M / J_M^2 has rank r then $J_M^{r+1} = 0$.

Observation 3.8. For M locally split, for an open subset small enough $U \subset M$ we have that $U \simeq (U_{\mathrm{rd}}, \bigwedge^\bullet (J_M / J_M^2))$. We can suspect that $M \simeq (M_{\mathrm{rd}}, \bigwedge^\bullet (J_M / J_M^2))$. In order to check that consider a covering $\{U_i\}_{i \in I}$ and local splits $\{\pi_i : \mathcal{O}_{M_{\mathrm{rd}}}|_{U_i} \xrightarrow{\sim} \mathcal{O}_M|_{U_i}\}$ of (3.1) we obtain:

$$\pi_{ij} = \pi_i|_{U_{ij}} - \pi_j|_{U_{ij}} \in J_M(U_{ij}) \tag{3.2}$$

observe that

$$\begin{aligned} \pi_{ij}(fg) =& \pi_i(fg) - \pi_j(fg) \\ =& \pi_i(f)\pi_i(g) - \pi_j(f)\pi_i(g) \\ =& \pi_i(f)\pi_{ij}(g) + \pi_{ij}(f)\pi_i(g) \end{aligned}$$

Then, we can identify π_{ij} with an element in $\mathcal{T}_{M_0} \otimes J_M(U_{ij})$. Additionally, π_{ij} verifies the cocicle condition:

$$\begin{aligned}\pi_{ij} + \pi_{jk} + \pi_{ki} =& (\pi_i|_{U_{ijk}} - \pi_j|_{U_{ijk}}) + (\pi_j|_{U_{ijk}} - \pi_k|_{U_{ijk}}) + (\pi_k|_{U_{ijk}} - \pi_i|_{U_{ijk}}) \\ =& 0\end{aligned}$$

so $\{\pi_{ij}\} \in H^1(M_0, T_{M_0} \otimes J_M)$.This class vanishes when $M \to M_{\text{rd}}$ is projected.

In some cases we can refine the sheaf $\mathcal{T}_{M_0} \otimes J_M$. For example, if $M \to S$ is a family with S a scheme, then we use the sheaf $\mathcal{T}_{M_0} \otimes J_M^2$.

Example 3.7. Let M be an $m|1$-super manifold family over a point. For J_M recall the sequence of sheaves of super algebras (3.1).

Observe that J_M is an $\mathcal{O}_{M_{\text{rd}}}$-module. Also, the $\mathcal{O}_{M_{\text{rd}}}$-module J_M has rank 1. Since $J_M^2 = 0$ the sheaf $\mathcal{T}_{M_0} \otimes J_M^2$ is null. Finally, we get that $M \simeq M_{\text{rd}}(J)$. In other words, any $m|1$-super manifold over a point is split. In general we proof:

Proposition 3.2. *Let S be a purely even scheme. Then any $m|1$-dimensional family $M \to S$ is split.*

Observation 3.9. For a family of $m|1$, $\pi : M \to S$ with S a super scheme that is not a purely even scheme, in the sequence (3.1) the first problem is that not necessarily J is an $\mathcal{O}_{M_{\text{rd}}}$-module, since $\pi^*\mathcal{O}_S$ could have nilpotent elements. For example, consider the $1|1$-dimensional family $\mathbb{C}^{2|2} \to \text{Spec}(\mathbb{C}[\rho])$, with ρ an odd variable, over it we have the two automorphism given by

$$\begin{aligned}A(t|\theta) &:= (t+1|\theta), \\ B(t|\theta) &:= (t+\tau+\theta\rho|\theta),\end{aligned}$$

with $\tau \in \mathbb{C}$, $\Im(\tau) > 0$. The quotient $\mathbb{T}_\tau := \mathbb{C}^{2|2} \to \text{Spec}(\mathbb{C}[\rho])/\langle A, B\rangle$ is an analytical family of super torus. In order to see that this quotient is algebraic, let us recall the Weierstrass function $\wp$ given by the parameter τ. Then we obtain the closed immersion:

$$\begin{aligned}\mathbb{T}_\tau &\to \mathbb{P}^2(L) \\ (t|\theta) &\mapsto (\wp(t;\tau+\theta\rho), \partial_t\wp(t;\tau+\theta\rho), 1|\theta),\end{aligned}$$

where L is the trivial bundle over $\mathbb{P}^2$. The image of this immersion is given by the equation:

$$y^2 = 4x^3 - g_2(\tau+\phi\rho)x - g_3(\tau+\phi\rho), \tag{3.3}$$

with $(x, y, 1|\phi) \in \mathbb{P}^2(L)$. Since (3.3) is even, then $\mathbb{T}_\tau$ is a $1|1$-dimensional family.

To see that this family is not split, suppose that there exists a $1|0$-dimensional family $M \to \mathrm{Spec}(\mathbb{C}[\rho])$ and a line bundle L_0 over the family such that $M(L_0) = \mathbb{T}_\tau$. Observe that such family should be a family of torus, also the change of coordinates over any torus should have the form $\Phi(t|\theta) = (\phi(t)|\theta\lambda(t))$ and in this case $\lambda(t)$ corresponds to the cocicle of L_0. Over $\mathcal{T}_{\mathbb{T}_\tau/S}$, $S = \mathrm{Spec}([\rho])$, we have the global section given by ∂_t, so we have the exact sequence

$$0 \to \langle \partial_t \rangle \to \mathcal{T}_{\mathbb{T}_\tau/S} \to \mathcal{T}_{\mathbb{T}_\tau/S}/\langle \partial_t \rangle \to 0.$$

Given a change of coordinates $\Phi(t|\theta) = (\phi(t)|\theta\lambda(t))$ then the change of coordinates of $\mathcal{T}_{\mathbb{T}_\tau/S}/\langle \partial_t \rangle$ are given by $\lambda(t)$. On the other side, for coordinates (t, θ) the vector field ∂_θ is a well defined global section in $\mathcal{T}_{\mathbb{T}_\tau/S}/\langle \partial_t \rangle$, that is $\mathcal{T}_{\mathbb{T}_\tau/S}/\langle \partial_t \rangle$ is a trivial bundle, in particular L_0 is also a trivial bundle. From this, the tangent bundle over $M(L_0)$ should be trivial, since the tangent bundle over the torus is trivial and we can define a global section $\partial_{\theta'}$ for θ' a global section in L_0. If this happens, then the space of global sections has dimension $1|1$. Let us take a section s of $\mathcal{T}_{\mathbb{T}_\tau/S}$. With respect to the étale topology, from the projection $\mathbb{T}_\tau := \mathbb{C}^{2|2} \to \mathrm{Spec}(\mathbb{C}[\rho]) \to \mathbb{T}_\tau := \mathbb{C}^{2|2} \to \mathrm{Spec}(\mathbb{C}[\rho])/\langle A, B \rangle$, we obtain a section of the tangent bundle $\mathcal{T}_{\mathbb{C}/S}$, such section should have the form

$$\begin{aligned} s(t|\theta) &= s(t+1|\theta), \\ s(t|\theta) &= s(t+\tau+\theta\rho|\theta). \end{aligned} \tag{3.4}$$

Using the decomposition $s(t|\theta) = a(t|\theta)\partial_t + b(t|\theta)\partial_\theta$. From the relations (3.4), we obtain that b should satisfy

$$\begin{aligned} b(t|\theta) &= b(t+1|\theta), \\ b(t|\theta) &= b(t+\tau+\theta\rho|\theta). \end{aligned} \tag{3.5}$$

from this b should be constant, then we get:

$$\begin{aligned} a(t|\theta) &= a(t+1|\theta), \\ a(t|\theta) &= a(t+\tau+\theta\rho|\theta) - b\rho. \end{aligned} \tag{3.6}$$

Similarly to (3.5), taking derivative on (3.6) we obtain that a is constant and $b = 0$. That is, the vector space of sections has dimension $1|0$, and this contradicts that such space of sections has dimension $1|1$. Then the family of torus $\mathbb{T}_\tau$ is not split.

Example 3.8. For a family of $m|2$-super manifold $M \to S$, for S an even super scheme. In the same sequence (3.1) with $J_M = \ker(\mathcal{O}_M \to \mathcal{O}_{M_{\mathrm{rd}}})$ is not true that $J_M^2 = 0$. Instead, consider $\mathcal{F} := J_M/J_M^2$, a rank two bundle over M_{rd} and construct the exact sequence:

$$0 \to \det \mathcal{F} \to (\mathcal{O}_M)_0 \to \mathcal{O}_{M_{\mathrm{rd}}} \to 0, \tag{3.7}$$

where $\mathcal{O}_M = (\mathcal{O}_M)_0 \oplus (\mathcal{O}_M)_1$. For the manifold $M_0 := (M, (\mathcal{O}_M)_0)$ we get the inclusions $M_{\mathrm{rd}} \hookrightarrow M_0 \xrightarrow{j} M$.

Repeating the arguments given in (3.2) we take local splits $\pi_i : \mathcal{O}_{M_0}(U_i) \to \mathcal{O}_{M,0}(U_i)$ in (3.7) to define

$$\omega_{ij} = \pi_i|_{U_{ij}} - \pi_j|_{U_{ij}}. \tag{3.8}$$

In this case we can identify ω_{ij} with an element in $\mathcal{T}_{M_0} \otimes \det \mathcal{F}(U_{ij})$ and $\{\omega_{ij}\} \in H^1(M_0, T_{M_0} \otimes \det \mathcal{F})$. It was proved in [10] the following

Proposition 3.3. *Let S be a scheme and $M \to S$ be a $m|2$ dimensional super manifold. The class given by* (3.8) *vanishes if and only if $M \to S$ is split.*

For an $m|n$ super manifold with n bigger than two, there is an obstruction constructed in [11].

3.2.2 $S(1|n)$-super curves

Definition 3.7. Let M be a super manifold M and $\mathcal{F}$ be a locally free sheaf of rank $m|n$, consider a cover $\{U_i\}_{i\in I}$ with trivializations $\phi_i : \mathcal{F}(U_i) \to \mathcal{O}_{U_i}^{m|n}$. Then, for $i, j \in I$ over $U_{ij} = U_{ij}$ we have

$$\mathcal{O}_{U_{ij}}^{m|n} \xrightarrow{\phi_i^{-1}} \mathcal{F}(U_{ij}) \xrightarrow{\phi_j} \mathcal{O}_{U_{ij}}^{m|n} \tag{3.9}$$

so $\phi_{ij} : \mathcal{O}_{U_{ij}}^{m|n} \to \mathcal{O}_{U_{ij}}^{m|n}$ an invertible homomorphism, then from the homomorphism of groups (2.3), we obtain

$$\mathrm{Ber}(\phi_{ij}) \in \mathrm{Inv}(\mathcal{O}_X(U_{ij})).$$

With this we construct a bundle of rank $1|0$ if n is even or rank $0|1$ if n is odd.

This bundle $\mathrm{Ber}\mathcal{F}$ is called *Berezinian bundle of $\mathcal{F}$*. Set $\mathrm{Ber}_M := \mathrm{Ber}(\Omega_M)$, and for a family $M \to S$, we will write $\mathrm{Ber}_{M/S} := \mathrm{Ber}(\Omega_{M/S})$.

Example 3.9. Let $(M, \mathcal{O}_M)$ be an $m|n$ a super manifold, the tangent bundle is a rank $m|n$ locally free sheaf. If over an open set U we consider local coordinates $(z_1, \dots, z_m|\theta^1, \dots, \theta^n)$, then the tangent space is locally trivialized, by $\langle \partial_{z_1}, \dots, \partial_{z_m}|\partial_{\theta^1}, \dots, \partial_{\theta^n}\rangle$. Similarly, the cotangent bundle Ω_M is locally trivialized, by $\langle dz_1, \dots, dz_m|d\theta^1, \dots, d\theta^n\rangle$ over the open set $U \subset M$.

On the other hand, we get a local generator of $\mathrm{Ber}_M(U)$ given by $[dz_1 \dots dz_m|d\theta^1 \dots d\theta^n]$.

Let $M = (M_{\mathrm{rd}}, \bigwedge^\bullet E)$ be a split super manifold, then we can take the local coordinates $(z_1, \dots, z_m|\theta^1, \dots, \theta^n)$ over an open set $U \subset M_{\mathrm{rd}}$. For another coordinates $\Phi = (w_1, \dots, w_m|\rho^1, \dots, \rho^n)$ with

$$\begin{aligned} w_i &= \phi_i(z_1, \dots, z_m),\ i = 1, \dots, m \\ \rho^j &= \theta^1 a_{1j}(z_1, \dots, z_m) + \cdots + \theta^n a_{nj}(z_1, \dots, z_m),\ j = 1, \dots, n. \end{aligned}$$

The change of coordinates for the cotangent bundle is given by

$$\mathrm{Jac}(\Phi) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where $A = (\partial_{z_k}\phi_l), B = (\partial_{z_i}\rho^j)_{ij}, C = (a_{ij})$. Then the change of coordinates for the Berezinian of the cotangent bundle is given by

$$\mathrm{Ber}(\mathrm{Jac}(\Phi)) = \det A \det C^{-1} = \det(\partial_{z_k}\phi_l)\det(a_{ij})^{-1}.$$

Using the closed embedding, $j : M_{\mathrm{rd}} \to M$, we get the isomorphism

$$j^*\mathrm{Ber}_M \simeq \Omega^m_{M_{\mathrm{rd}}} \otimes \det E^*.$$

Observe that in this case Ber_M is a trivial bundle when $\Omega^m_{M_{\mathrm{rd}}} \simeq \det E$ as line bundles over M_{rd}.

Definition 3.8. A $1|n$-super curve $(C, \mathcal{O}_C)$ is a connected super manifold of dimension $1|n$. An $S(1|n)$-*super curve* is a pair $(C \to S, \Delta)$, where $C \to S$ is a super curve joint with a nonvanishing section $\Delta \in H^0(C, \mathrm{Ber}_{C/S})$.

Let $C \to S$ be a super curve, for a section $\Delta \in H^0(C, \mathrm{Ber}_{C/S})$ and coordinate patch $\{(U_i, \Phi_i)\}$. There exists a family of functions $f_i \in H^0(U_i, \mathcal{O}_C)$ such that

$$\Delta|_{U_i} = f_i[dz_i|d\theta_i^1 \cdots d\theta_i^N]. \tag{3.10}$$

Observation 3.10. Let us consider a $1|n$-super curve $C \to S$, a coordinate patch $U \subset C$ with a trivialization Φ, and a nonvanishing section $\Delta \in H^0(U, \mathrm{Ber}_{C/S})$ with $f \in H^0(U, \mathcal{O}_C)$ as in (3.10). Taking an even function $F(z|\theta^1, \cdots, \theta^n)$, and shrinking U if is necessary, with

$$\partial_z F(z|\theta^1, \cdots, \theta^n) = f(z|\theta^1, \cdots, \theta^n),$$

then the system of coordinates $\Psi = (w|\rho^1, \cdots, \rho^n)$, given by

$$w = F(z|\theta^1, \cdots, \theta^n),$$
$$\rho^i = \theta^i, \; i = 1 \dots, n;$$

verifies $\Delta = [dw|d\rho^1 \cdots d\rho^n]$. We will say that such coordinate system Ψ is *compatible* with the section Δ.

Finally, for a nonvanishing section $\Delta \in H^0(C, \mathrm{Ber}_{C/S})$ there exists $\{(U_i, \Phi_i)\}_i$ an atlas for $C \to S$ such that

$$\Delta|_{U_i} = [dz_i|d\theta_i^1 \cdots d\theta_i^n].$$

For any pair of coordinates Φ, Ψ defined over the same open set U both compatible with $\Delta|_U$, then the change of coordinates $\Phi \circ \Psi^{-1}$ preserves the Berezinian.

For a fixed curve $C \to S$ and a nonvanishing section $\Delta \in H^0(C, \mathrm{Ber}_{C/S})$ we will only consider coordinates compatible with Δ.

Observation 3.11. Let C be a super curve, from the inclusion $j : C_{\mathrm{rd}} \to C$ a section $\Delta \in H^0(C, \mathrm{Ber}_C)$ induces a global section $j^*\Delta \in H^0(C_{\mathrm{rd}}, j^*\mathrm{Ber}_C)$. Actually, if Δ does not vanish then $j^*\Delta$ is a non-vanishing section.

Example 3.10. Let $C \to S$ be a split super curve associated to the bundle E and the curve C_{rd}, then $C \to S$ has a trivial Berezinian if and only if $\Omega_{C_{\mathrm{rd}}} \xrightarrow{\sim} \det E$.

Over a curve $C \to S$ with a nonvanishing section $\Delta \in H^0(C, \mathrm{Ber}_{C/S})$, over an open set we can define the space of vector fields:

$$\begin{aligned} S(1|2)(U) &:= \{X \in \mathcal{T}_{C/S}(U) : \mathrm{sdiv}_\Delta X = 0\}, \\ S(2)(U) &:= [S(1|2)(U), S(1|2)(U)]. \end{aligned} \tag{3.11}$$

Observe that these sheaves are not $\mathcal{O}_C$-modules. For the family $\pi : C \to S$, they are sheaves of $\pi^*\mathcal{O}_S$-modules.

Finally, we get a $\pi^*\mathcal{O}_S$-module $\mathcal{A}_C$ defined over an open set $U \subset C$ by:

$$\mathcal{A}_C(U) := \frac{S(1|2)(U)}{S(2)(U)}.$$

Observation 3.12. Let $C = (C_{\mathrm{rd}}, E)$ be a split $1|2$ dimensional super curve over a point. We already see that C is an $S(1|2)$-super curve if and only if $\det E \xrightarrow{\sim} \Omega_{C_{\mathrm{rd}}}$. Let $\phi = (z|\theta^1, \theta^2)$ be local coordinates in C. The local section $\theta^1\theta^2\partial_z$ is well defined globally, since for any change of coordinates $\Phi = (F|\phi^1, \phi^2) = (\widetilde{z}|\widetilde{\theta}^1, \widetilde{\theta}^2)$, where $F(z|\theta^1, \theta^2) = F(z)$ and

$\phi^i = \theta^1 a_{1i} + \theta^2 a_{2i}$, for $i = 1, 2$. Here, $a_{11}a_{22} - a_{12}a_{12} = F'$, then the local section transform $\widetilde{\theta}^1\widetilde{\theta}^2\partial_{\widetilde{z}}$ as

$$\begin{aligned}\widetilde{\theta}^1\widetilde{\theta}^2\partial_{\widetilde{z}} &= ((a_{11}a_{22} - a_{12}a_{12})\theta^1\theta^2\partial_z)(F'(z))^{-1}\partial_z \\ &= \theta^1\theta^2\partial_z.\end{aligned}$$

Then we obtain that the local section $\theta^1\theta^2\partial_z$ is globally defined. Finally, $\mathcal{A}_C$ is a trivial $\pi^*\mathcal{O}_S$-module.

More generally, we have:

Proposition 3.4. *Let S be an scheme and $C \to S$ be a split $S(1|2)$-super curve, then the bundle $\mathcal{A}_C$ is trivial.*

3.2.3 $SUSY$-super curves

Definition 3.9. A $1|n$-super curve $C \to S$ with a covering and atlas $\{\Phi_i\}_i$ such that any change of coordinates verifies $\Phi_{ij} = \Phi_i \circ \Phi_j^{-1} \in \mathrm{Aut}^\omega[[1|n]]$ we will say that $C \to S$ is a *$SUSY_n$-super curve.*

Observe that the local form (2.1) in coordinates $\phi_i = (z_i|\theta_i^1, \dots, \theta_i^n)$

$$\omega_i = dz_i + \theta_i^1 d\theta_i^1 + \cdots + \theta_i^n d\theta_i^n \tag{3.12}$$

is well defined, up to multiplication by a function, over $C \to S$. Then for a $SUSY_n$-super curve we can define the line bundle $\mathcal{D}$ locally generated by the section ω_i given in (3.12). For any coordinate system $\Phi = (z|\theta^1, \dots, \theta^n)$ we are going to say that Φ is *compatible to the $SUSY$-super structure* if $dz + \theta^1 d\theta^1 + \cdots + \theta^n d\theta^n$ generates $\mathcal{D}$ locally.

Also, we can define a $SUSY_n$-super structure over the $1|n$-super curve $C \to S$ as a locally free subsheaf $E \subset \mathcal{T}_C$ of rank $0|n$, for which the Frobenius form

$$E \otimes E \to \mathcal{T}_C/E$$

is nondegenerate and split, i.e., it locally has an isotropic direct subsheaf of maximal possible rank k for $n = 2k$ or $2k + 1$ (cf. 2).

For a system of coordinates $\Phi = (z|\theta^1, \dots, \theta^n)$ compatible to the $SUSY$-super structure, then $\theta^1\partial_z + \partial_{\theta^1}, \cdots, \theta^n\partial_z + \partial_{\theta^n}$ generates E.

Let us recall some properties about $SUSY_2$-super curve. First, in this special case, there exists an exterior automorphism $\Phi \in \mathrm{Aut}^{\omega}[[1|2]]$ given by

$$\Phi : (z|\theta^1, \theta^2) \mapsto (z|\theta^2, \theta^1).$$

For some coordinates $(z|\theta^1, \theta^2)$ over C and considering the change of coordinates $\Phi = (F|\phi^1, \phi^2)$, we obtain the equations

$$D^i F = \phi^1 D^i \phi^1 + \phi^2 D^i \phi^2,\ i = 1, 2,$$

where $D^i = \theta^i \partial_z + \partial_{\theta_i}$. Then, taking $\{D^i, D^j\} = 2\delta_{ij}\partial_z$, we get the relations:

$$\{D^i, D^j\}F = D^i\phi^1 D^j\phi^1 + D^i\phi^2 D^j\phi^2 - \phi^1\{D^i, D^j\}\phi^1 - \phi^2\{D^i, D^j\}\phi^2,$$

that reducing terms, we obtain:

$$\begin{aligned} \partial_z F + \phi^1\partial_z\phi^1 + \phi^2\partial_z\phi^2 &= D^i\phi^1 D^i\phi^1 + D^i\phi^2 D^i\phi^2,\ i = 1, 2, \\ 0 &= D^1\phi^1 D^2\phi^1 + D^1\phi^2 D^2\phi^2. \end{aligned} \tag{3.13}$$

A simpler way to write these equations uses the matrix $A = A(\Phi) = (D^i\phi^j)$ and the expressions (3.13) are write as:

$$AA^t = (\partial_z F + \phi^1\partial_z\phi^1 + \phi^2\partial_z\phi^2)\mathrm{id}. \tag{3.14}$$

In particular,

$$\det A = \pm(\partial_z F + \phi^1\partial_z\phi^1 + \phi^2\partial_z\phi^2). \tag{3.15}$$

Actually, Φ is an inner automorphism if and only if $\det A(\Phi) = \partial_z F + \phi^1\partial_z\phi^1 + \phi^2\partial_z\phi^2$, otherwise we will say that Φ is an outer automorphism. With this, we have the description:

Proposition 3.5. *The elements in the connected component of* $\mathrm{Aut}^{\omega}[[1|2]]$ *containing the identity are the inner automorphisms. The other component are the outer automorphisms.*

Geometrically, the condition that $\{\Phi_i\}_{i\in I}$ is an atlas with $\Phi_{ij} = \Phi_i \circ \Phi_i^{-1}$ an inner automorphism, for any $i, j \in I$, is expressed as a split on $E = L_1 \oplus L_2$, where E is the bundle defining the $SUSY_2$-super structure. We are going to say that such curves are *orientable* $SUSY_2$ *super curves.*

Suppose that C is a curve over a point with a $SUSY_2$-super structure. Over the reduced curve C_{rd} we could consider the Čech class $\gamma \in H^1(C_{\mathrm{rd}}, \{\pm 1\})$ that choose ± 1 depending on the sign took in equation (3.15). This class γ vanishes if and only if C can be endowed with an orientable $SUSY_2$ super structure.

Now, let study the Berezinian of $\Phi = (F|\phi^1, \phi^2)$. Since $D^iF = \phi^1 D^i\phi^1 + \phi^2 D^i\phi^2$, for $i = 1, 2$ or equivalently

$$\begin{pmatrix} D^1F \\ D^2F \end{pmatrix} = A \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix},$$

then we get:

$$\begin{aligned}
\operatorname{Ber}(\Phi) &= \operatorname{Ber}\begin{pmatrix} \partial_z F & \partial_z\phi^1 & \partial_z\phi^2 \\ D^1F & D^1\phi^1 & D^1\phi^2 \\ D^2F & D^2\phi^1 & D^2\phi^2 \end{pmatrix} \\
&= \left(\partial_z F - [\partial_z\phi^1 \quad \partial_z\phi^2]A^{-1}\begin{pmatrix} D^1F \\ D^2F \end{pmatrix}\right) \det A^{-1}, \\
&= \left(\partial_z F - [\partial_z\phi^1 \quad \partial_z\phi^2]\begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}\right) \det A^{-1}, \\
&= \left(\partial_z F + \phi^1\partial_z\phi^1 + \phi^2\partial_z\phi^2\right) \det A^{-1}, \\
&= \pm 1,
\end{aligned}$$

where the sign depends on equation (3.15). The inclusion $\{\pm 1\} \subset \mathcal{O}_{C_{\mathrm{rd}}}$ give us that any orientable $SUSY_2$-super curve has a trivial Berezinian. Actually, we obtain

Proposition 3.6. *Let C be a $SUSY_2$-super curve, then the coordinates compatibles with the $SUSY$-structure define a global section on the Berezinian if and only if C is orientable.*

Also, from (3.14), we get the following relationships for an inner automorphism $\Phi = (F|\phi^1, \phi^2)$:

$$\begin{aligned} D^1\phi^1 &= D^2\phi^2, \\ D^1\phi^2 &= -D^2\phi^1, \end{aligned} \tag{3.16}$$

that looks very similar to the Cauchy conditions for complex structures. Then, we can take the following holomorphic coordinates:

$$\begin{aligned} w &= z + i\theta^1\theta^2, \\ \theta &= \frac{1}{2}(\theta^1 - i\theta^2), \\ \rho &= \frac{1}{2}(\theta^1 + i\theta^2), \end{aligned}$$

with its respective change of coordinates $\left(F + i\phi^1\phi^2 | \frac{1}{2}(\phi^1 - i\phi^2), \frac{1}{2}(\phi^1 - i\phi^2)\right)$, we obtain that our initial curve C is endowed with a projection to another $1|1$ super curve C' given by

$C \to C'$, $(w|\theta,\rho) \mapsto (w|\theta)$. Reciprocally, if we start with a $1|1$ super curve we can construct an associated $SUSY_2$-super curve, such curve is going to be orientable. We will explain this with more detail later.

Another fact that we can observe for oriented $SUSY_2$-super curves over points, is that in the projection $C \to C'$, we know that C' is an split curve, defined by its reduction C_{rd} and a line bundle L, then we obtain:

Proposition 3.7. *Let C be an oriented $SUSY_2$-super curves over a point, then C is split.*

Proof. We already see that C has a system of coordinates (z,θ,ρ) with a change of coordinates $\Phi = (G|\phi,\eta)$, where F,ϕ does not depend on ρ. Since C' is split we can modify the $(G|\phi)$ by $(\widetilde{G}|\widetilde{\phi})$ such that $\widetilde{G}$ does not depend on the odd coordinate. Finally, we can modify Φ by $\widetilde{\Phi}$ such that C is split. □

Observe that the process described above was not canonical. Later, we are going to show a most natural way to split such curves.

Example 3.11. We are going to see that not any $SUSY_2$-super curve is an $S(1|2)$-super curve.

Let C be a split $1|2$ super curve associated to the reduced genus $g \geq 1$ curve C_{rd} and the rank two bundle $E = A \oplus B$, where A, B are line bundles over C_{rd} with $A^{\otimes 2} = B^{\otimes 2} = \Omega_{C_{\mathrm{rd}}}$ and $A \not\simeq B$. First, observe that local coordinates over C given by $(z|\theta^1,\theta^2)$, where z is a local coordinate in C_{rd}, θ^1 and θ^2 are local sections of A and B, respectively, define (locally) the form $dz + \theta^1 d\theta^1 + \theta^2 d\theta^2$, up to multiplication by a function. Finally, C is a $SUSY_2$-super curve.

On the other hand, observe that for the Berezinian bundle $\mathrm{Ber}(C)$ and the inclusion $j : C_{\mathrm{rd}} \to C$ we obtain that $j^*\mathrm{Ber}(C) = \Omega_{C_{\mathrm{rd}}} \otimes (A \otimes B)^* = A \otimes B^* \not\simeq \mathcal{O}_{C_{\mathrm{rd}}}$. Finally, C is not an $S(1|2)$-super curve.

Chapter 4

Ind-Schemes

We are going to set here the basis to work with infinite dimensional schemes since we want to deal later with them. For example, for rings as $R[[t]]$ is not clear the correspondence between super schemes and $\mathrm{Spec}(R[[t]])$.

4.1 Introduction

Recall that a category $\mathcal{C}$ is a pair $(\mathrm{Obj}(\mathcal{C}), \mathrm{Hom}_{\mathcal{C}})$, where $\mathrm{Obj}(\mathcal{C})$ is the collection of objects of $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is the collection of morphism between A, B. From now on, we are going to consider locally small categories, that is the collection $\mathrm{Hom}(A, B)$ is a set for any pair of objects A, B.

Example 4.1. Let $\mathcal{C} = \mathrm{Sets}$ the category of sets with objects the collection of sets and for any pair of sets A, B, $\mathrm{Hom}_{\mathrm{Sets}}(A, B)$ is the collection of functions between A, B.

Example 4.2. Let X be a topological space, we define the category $\mathcal{C} = \mathcal{U}_X$ with objects the collection of open subsets of X and for any pair of open sets A, B, $\mathrm{Hom}_{\mathrm{Sets}}(A, B)$ is the set with one element $\{A \hookrightarrow B : \text{inclusion}\}$ if $A \subset B$ or empty otherwise.

Example 4.3. Let k be a field. We define the category CRings_k of commutative rings with objects the k-commutative rings and for any pair of objects A, B we denote by $\mathrm{Hom}_{\mathrm{CRings}_k}(A, B)$ the collection of homomorphism of k-commutative rings. Here we are going to consider finitely generated k-commutative rings. In the same direction, we define Sch_k the category of schemes over a field k.

Example 4.4. Let k be a field. We define the category $\mathcal{C} = \mathrm{SAlg}_k$ of super algebras with objects the collection of super algebras with morphisms the collection of homomorphisms of super algebras. If there is no confusion we will write SAlg. Here we are going to consider finitely generated k-super algebras. Also, we define SSch_k the category of super schemes over k.

Definition 4.1. Let $\mathcal{C}, \mathcal{C}'$ be two categories, a *functor* $\mathcal{F}$ is an asignation between objects, such that $\mathcal{F}_{\mathrm{Obj}}(A)$ is an object of $\mathcal{C}'$ for any object A of $\mathcal{C}$, for morphisms, the asignation is such that $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$, for any f a morphism between A, B and g a morphism between B, C. We are going to denote this by $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$, and similarly we will denote the asignation of objects and morphisms by $\mathcal{F}_{\mathrm{Obj}} : \mathrm{Obj}_{\mathcal{C}} \to \mathrm{Obj}_{\mathcal{C}'}$ and for any pair of objects of $\mathcal{C}$, A, B, we write $\mathcal{F}_{\mathrm{Hom}} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(\mathcal{F}(A), \mathcal{F}(B))$, respectively.

Example 4.5. Let X be a super scheme, then the structure sheaf $\mathcal{O}_X$ is a contravariant functor between $\mathcal{U}_X$ and SAlg_k.

Example 4.6. Let SSch_k be the category of super schemes over the field k. Fix a pair of non-negative integers m, n, then the asignation:

$$S \mapsto \{\phi : X \to S : \text{where } \phi \text{ is a flat morphism of codimension } m|n.\}$$

is a functor denoted by $\mathrm{SSch}_k(m|n)$ and any $\phi : X \to S$ is called a *family of super schemes $m|n$-dimensional*. Observe that for any morphism $f : S \to S'$ and a family $\phi : X \to S'$ we can define the pullback family $\phi_f : X \times_{S'} S \to S$.

Example 4.7. Let $\mathcal{C}$ be a category and an object A, then we can define the contravariant functor

$$\begin{aligned} h_A : \mathcal{C} &\to \mathrm{Sets} \\ B &\mapsto h_A(B) := \mathrm{Hom}_{\mathcal{C}}(A, B). \end{aligned}$$

Similarly, the asignation $h^A(B) := \mathrm{Hom}_{\mathcal{C}}(B, A)$ defines a covariant functor.

Definition 4.2. Let $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ be a functor $\mathcal{F}$. We are going to say that $\mathcal{F}$ is faithful if for any pair of objects A, B the map $\mathcal{F}_{\mathrm{Hom}}(A, B) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(\mathcal{F}(A), \mathcal{F}(B))$ is injective. Similarly, we are going to say that $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ is fully faithful if $\mathcal{F}_{\mathrm{Hom}}(A, B) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(\mathcal{F}(A), \mathcal{F}(B))$ is bijective.

Example 4.8. We have the reverse parity functor $\Pi : \mathrm{SMod} \to \mathrm{SMod}$, with $\Pi \circ \Pi = \mathrm{id}$. If M has rank $m|n$, then ΠM has rank $n|m$.

Example 4.9. Let m, n natural numbers. Then, we obtain a functor:

$$\begin{aligned} \mathrm{SAlg} &\to \mathrm{SAlg} \\ R &\mapsto R[m|n] = R[t_1, \ldots, t_m|\theta^1, \ldots, \theta^n], \end{aligned}$$

that also, give us a functor

$$\begin{aligned} \text{CRings} &\hookrightarrow \text{SAlg} \\ R &\mapsto R[m|n]. \end{aligned} \tag{4.1}$$

In particular, we obtain a natural inclusion functor $j : \text{CRings} \to \text{SAlg}$.

Example 4.10. Let SAlg be the category of super algebras, then the reduction $R \mapsto R_{\text{rd}}$ is a functor. This functor is a left inverse of the inclusion (4.1).

Definition 4.3. Let $\mathcal{F}, \mathcal{F}'$ be two functors. A *natural transformation* η is an asignation between the objects $\eta(A) : \mathcal{F}(A) \to \mathcal{F}'(A)$, such that for any morphism $f : A \to B$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}[B] \\ \eta(A)\downarrow & & \downarrow\eta(B) \\ \mathcal{F}'(A) & \xrightarrow{\mathcal{F}'(f)} & \mathcal{F}[B] \end{array}$$

We are going to denote this by $\eta : \mathcal{F} \to \mathcal{F}'$. For two natural transformations $\eta : \mathcal{F} \to \mathcal{F}', \eta' : \mathcal{F}' \to \mathcal{F}''$, the *composition* $\eta' \circ \eta : \mathcal{F} \to \mathcal{F}''$ is naturally defined. Also, the identity $\text{id}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$ is a natural transformation.

Two functors are said to be *isomorphic* if there exists two natural transformations $\eta : \mathcal{F} \to \mathcal{F}'$ and $\eta' : \mathcal{F}' \to \mathcal{F}$ such that $\eta \circ \eta' = \text{id}'_{\mathcal{F}}$ and $\eta' \circ \eta = \text{id}_{\mathcal{F}}$.

We are going to say that a functor $\mathcal{F}$ is *representable* if is isomorphic to h_A, for some object A.

Observation 4.1. Let R be a super algebra and S be a commutative ring, recall the natural identification (2.2). That means, considering the composition

$$\text{CRings}_k \overset{j}{\hookrightarrow} \text{SAlg}_k \overset{h_R}{\to} \text{Sets},$$

we get that: $h_R|_{\text{CRings}_k} = h_{R_{\text{rd}}}$.

Using this, we gain the following lemma

Lemma 4.1. *Let M be a super scheme, then the functor $h_M|_{\text{Sch}}$ is representable by the scheme M_{rd}.*

Observation 4.2. Suppose that M is a $m|n$ dimensional super manifold, recall that M_{rd} is given as the manifold that represents the functor $h_M|_{\text{Sch}}$. Since $\mathcal{T}_M$ is an $\mathcal{O}_M$-module with rank $m|n$, then for the inclusion $j : M_{\text{rd}} \to M$ the pullback $j^*\mathcal{T}_M$ is an $\mathcal{O}_{M_{\text{rd}}}$-module with rank $m|n$. Locally, for an open set $U \subset M_{\mathrm{rd}}$ the super manifold M is isomorphic to $(U, \bigwedge^\bullet (j^*\mathcal{T}_M)_1)$. Finally, in order to understand a super manifold through functoriality we have to understand first the functor $h_M|_{\mathrm{Sch}}$ and the families $N \to M[\epsilon_0, \epsilon_1]$, where ϵ_i with parity i and $\epsilon_0^2 = \epsilon_1^2 = \epsilon_0\epsilon_1 = 0$, with a fixed family $N_0 \to M$. This recipe is similar to observation (2.15).

Example 4.11. In this example, we will see how to use the tools given by Lemma 4.1 and Observation 4.2.

Suppose that we try to understand the space of super manifolds with dimension $1|1$ and fixed genus g, that is the functor that asigns to any super scheme Y flat families $X \to Y$ with relative dimension $1|1$ such that for any closed point $y \in Y$ the reduced fiber $(X_y)_{\mathrm{rd}}$ is a projective curve with genus g. Let first consider the restriction to schemes, then we are looking for families $X \to S$, where S is a scheme. It was proved in [12, Proposition 2.9] that any such family is canonically split, and that $\mathcal{O}_X = \mathcal{O}_{X_{\mathrm{rd}}} \oplus \Pi L$, where L is a line bundle over X_{rd}. Then, the family over a scheme is given by choosing $X_{\mathrm{rd}} \to S$ a family of genus g curves and a line bundle L over it. We will denote by $\mathcal{M}_{g,1}$ to the functor of genus g curves joint with a (stable) line bundle over it.

Let S be a scheme and a fixed family of $1|1$ genus g super manifolds $X \to S$, for E a trivial rank 1 free $\mathcal{O}_S$-module, if we try to extend the family $X \to S$ to $(S, \bigwedge^\bullet E)$ we have to understand the diagram:

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ S & \longrightarrow & (S, \bigwedge^\bullet E) \end{array}$$

Now, fix local coordinates over $X \to S$, that is, a covering of the $1|1$ super curve $X_y \to \{y\}$ for a closed point $y \hookrightarrow S$ given by $\{U_i\}_{i\in I}$ and a family of isomorphisms $\phi_i : U_i \times S \simeq V_i$, where $V_i \subset X$ is open and the diagram commutes:

$$\begin{array}{ccc} U_i \times S & \longrightarrow & V_i \\ & \searrow & \downarrow \\ & & S \end{array}$$

Extending these coordinates over $(S, \bigwedge^\bullet E)$ by Φ_i, shrinking U_i if is necessary, we obtain that the change of coordinates $\Phi_{ij} := \Phi_i \circ \Phi_j^{-1} \in \mathrm{Aut}(U_i \times (S, \bigwedge^\bullet E))$ has reduction $\Phi_{ij,\mathrm{rd}} = \phi_{ij} \in \mathrm{Aut}(U_i \times S)$, where $\phi_{ij} := \phi_i \circ \phi_j^{-1}$. Then

$$\Phi_{ij} = \phi_{ij} + \epsilon X_{ij},$$

where X_{ij} is a derivation over X_y and ϵ generates E. Actually, what we achieve is

Proposition 4.1. *Any extension of the family $X \to S$ with fixed fibre $X_y \to \{y\}$ relative to the closed point over $(S, \bigwedge^\bullet E)$ is parametrized by $(H^1(X_y, \mathcal{T}_{X_y}))_1$.*

Then, we can see this extension as a section of the bundle $R^1 p_* \mathcal{T}_X$.

Finally, the functor of $1|1$ genus g super curves is described with the reduced scheme $\mathcal{M}_{(g,1)}$ and odd part described by $R^1 p_* \mathcal{T}$.

4.2 Ind-Schemes

Definition 4.4. Let $\mathcal{C}, \mathcal{D}$ be two categories, an *ind-family over* $\mathcal{C}$ is a collection of functors $\{\mathcal{F}_l : \mathcal{C} \to \mathcal{D}\}_{l \in \mathbb{N}}$ joint with natural transformations

$$i_l : \mathcal{F}_l \to \mathcal{F}_{l+1}.$$

We define the *limit* $\mathcal{F}$, as a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, joint with natural transformations $j_l : \mathcal{F}_l \to \mathcal{F}$, such that the diagrams are commutative:

$$\begin{array}{ccc} \mathcal{F}_l & \xrightarrow{j_l} & \mathcal{F} \\ {\scriptstyle i_l}\downarrow & \nearrow_{j_{l+1}} & \\ \mathcal{F}_{l+1} & & \end{array}$$

for any $k \in \mathbb{N}$, and that is universal.

From now on we will consider that $\mathcal{D}$ is the category of sets Sets.

Lemma 4.2. *The limit $\mathcal{F}$, joint with these natural transformation exists and are unique up to isomorphism.*

Example 4.12. Let $\{\mathcal{F}_l\}_{l \in \mathbb{N}}$ a family as above such that there exists an $L \in \mathbb{N}$ with $\mathcal{F}_l = \mathcal{F}_L$ and $i_l = \mathrm{id}$, for any $l \geq L$. Then $\lim \mathcal{F}_l = \mathcal{F}_L$.

Example 4.13. Let R be a k-super algebra and consider $h_l^R : \mathrm{SAlg}_k \to \mathrm{Sets}$ given by $h_l^R(S) = \mathrm{Hom}(R[t]/(t^l), S)$ is a representable functor such that the limit is represented by the super algebra $R[[t]]$ that is not finitely generated.

Observation 4.3. Suppose that we have two families $\{\mathcal{F}_l\}_{l \in \mathbb{N}}$ and $\{\mathcal{G}_l\}_{l \in \mathbb{N}}$ such for any

$l \in \mathbb{N}$ there exists a natural transformation $j_l : \mathcal{F}_l \to \mathcal{G}_l$ with commutative diagram

$$\begin{array}{ccc} \mathcal{F}_l & \xrightarrow{j_l} & \mathcal{G}_l \\ \downarrow & & \downarrow \\ \mathcal{F}_{l+1} & \xrightarrow[j_{l+1}]{} & \mathcal{G}_{l+1} \end{array}$$

then, for the limits $\mathcal{F}, \mathcal{G}$ there exists a unique natural transformation $\mathcal{F} \to \mathcal{G}$.

In particular, if any $\mathcal{F}_l, \mathcal{G}_l$ are groups with $\mathcal{F}_l \to \mathcal{G}_l$ homomorphism, we obtain the limit $\mathcal{F} \to \mathcal{G}$ and also $\mathcal{G}/\mathcal{F}$ is the limit of $\mathcal{G}_l/\mathcal{F}_l$.

Definition 4.5. Let $\{\mathcal{F}_l\}_{l\in\mathbb{N}}$ be a ind-family over SSch_k. We will say that such family is an *ind-(super) scheme* if $\mathcal{F}_l \simeq h_{M_l}$ is a (super) scheme for any $l \in \mathbb{N}$ and for the induced maps $M_l \to M_{l+1}$ are closed immersions.

An ind-scheme $\{\mathcal{F}_l\}_{l\in\mathbb{N}}$ is said to be *smooth* if each M_l is smooth, for any $l \in \mathbb{N}$. Equivalently we will say that $\{\mathcal{F}_l\}_{l\in\mathbb{N}}$ is an *ind-super manifold*.

Given a morphism between ind-schemes $\{\phi_l : \mathcal{F}_l \to \mathcal{G}_l\}$ is smooth if ϕ_l is smooth for any $l \in \mathbb{N}$.

When each M_l is an algebraic (super) group and each map $M_l \to M_{l+1}$ is an homomorphism of groups, we will say that the ind-scheme $\{\mathcal{F}_l\}_{l\in\mathbb{N}}$ is an *ind-(super) group*.

Example 4.14. Given a family of ind-scheme group $\mathcal{F}_l \to \mathcal{G}_l$, the projection $\mathcal{G}_l \to \mathcal{G}_l/\mathcal{F}_l$ is smooth. In particular, we have a $\mathcal{F}$-principal bundle

$$\mathcal{G} \to \mathcal{G}/\mathcal{F}$$

Example 4.15. Let R be a super algebra. Following Example 4.13 we define the ind-group $\mathrm{Aut}_R(R[[t]]) = \lim \mathrm{Aut}_R(R[t]/(t^l))$.

4.2.1 The group $\mathrm{Aut}_R(R[[m|n]])$

In this section, we are interested in the super group $\mathrm{Aut}\mathcal{O} := \mathrm{Aut}(R[[1|N]])$, where the product is formed by the composition.

Let R be a super algebra, for the super algebra $R[[m|n]]$ we will consider the collection of even automorphisms that are also homomorphism of R-super algebras and denote this group as $\mathrm{Aut}_R(R[[m|n]])$. If there is no confusion, we will use $\mathrm{Aut}(R[[m|n]])$ or $\mathrm{Aut}[[m|n]]$.

We are interested in the group of automorphism $\mathrm{Aut}(R[[1|n]])$ and its group structure given by $\Phi * \Psi = \Psi \circ \Phi$.

Example 4.16. Let R be a k super algebra, for a nilpotent $X \in \mathrm{Der}_R(R[[1|n]])_0$, we define its exponential by:

$$\exp(X) = \mathrm{id} + \frac{X}{1!} + \frac{X^2}{2!} + \cdots . \tag{4.1}$$

For two nilpotent $N, L \in \mathrm{Der}_R(R[[1|n]])_0$ with $[N, L] = 0$ we get

$$\exp(N + L) = \exp(N)\exp(L).$$

In particular, $\exp(N)$ has an inverse $\exp(-N)$. Finally, $\exp(X) \in \mathrm{Aut}_R(R[[m|n]])$.

Let $l \in \mathbb{N}$, and the R-super algebra $R_l := R[t|\theta^1, \cdots, \theta^n]/\mathfrak{m}^l$, where $\mathfrak{m} := \langle t|\theta^1, \ldots, \theta^n\rangle$. Similar to Example 2.1, an element $X \in \mathrm{Der}_R(R[t|\theta^1, \cdots, \theta^n]/\mathfrak{m}^l)$ is nilpotent, so we can define the exponential as (4.1). For the ind-family of R-algebras $\{R[t|\theta^1, \cdots, \theta^n]/\mathfrak{m}^l\}_{l\in\mathbb{N}}$, we get the pronilpotent Lie algebra:

$$\mathrm{Der}_{R,+}(R[[1|n]]) := \lim_{l\to\infty} \mathrm{Der}_R(R[t|\theta^1, \cdots, \theta^n]/\mathfrak{m}^l)$$

and a well defined *exponential*:

$$\exp : \mathrm{Der}_{R,+}(R[[1|n]]) \to \mathrm{Aut}_R(R[[1|n]]). \tag{4.2}$$

Its image is prounipotent ind-group denoted by $\mathrm{Aut}_{R,+}(R[[1|n]])$. An automorphism Φ is said to be generated by a vector field $X \in \mathrm{Der}_{R,+}(R[[1|n]])$ if $\exp(X) = \Phi$ in (4.2).

Denote by $\mathrm{Aut}_0(R[[1|n]])$ the quotient group $\mathrm{Aut}_R(R[[1|n]])/\mathrm{Aut}_{R,+}(R[[1|n]])$. It was proven in [13, Lemma 6.2.1]:

Proposition 4.2. *The group* $\mathrm{Aut}_R(R[[1|n]])$ *is a semi-direct product of* $\mathrm{Aut}_0(R[[1|n]])$ *and* $\mathrm{Aut}_+(R[[1|n]])$.

4.2.2 The group $\mathrm{Aut}^{\delta}_R(R[[1|n]])$

Recall the definition $\mathrm{Aut}^{\delta}_R(R[[1|n]])$ for the elements in $\mathrm{Aut}_R(R[[1|n]])$ that preserves the Berezinian.

Consider the finitely generated R-super algebra $R_l := R[t|\theta^1, \ldots, \theta^n]/\langle t|\theta^1, \ldots, \theta^n\rangle^l$. For a vector field $X \in \mathrm{Der}_R(R_l)$ observe that $\exp(X)$ is well defined and following (2.3) we obtain $\mathrm{Ber}(\exp(X)) = \exp(\mathrm{sdiv}_{\Delta_0}(X))$. In particular, for a vector field $X \in \mathrm{Der}_{R,+}(R[[1|n]])$ we get the relation

$$\mathrm{Ber}(\exp(X)) = \exp(\mathrm{sdiv}_{\Delta_0}(X)),$$

then, $\Phi = \exp(X)$ preserves the Berezinian if and only if $\mathrm{sdiv}(X) = 0$. This define the subalgebra

$$S(1|N)_+ := \mathrm{Der}_{R,+}(R[[1|n]]) \cap S(1|N).$$

For the group $\mathrm{Aut}^\delta(R[[1|n]])$, we will denote by

$$\mathrm{Aut}^\delta_+(R[[1|n]]) := \mathrm{Aut}_+(R[[1|n]]) \cap \mathrm{Aut}^\delta(R[[1|n]]),$$

and

$$\mathrm{Aut}^\delta_0(R[[1|n]]) := \mathrm{Aut}^\delta(R[[1|n]])/\mathrm{Aut}^\delta_+(R[[1|n]])$$

so we get the surjection

$$S(1|N)_+ \xrightarrow{\exp} \mathrm{Aut}^\delta_+(R[[1|n]]).$$

From Lemma 4.2 we get:

Lemma 4.3. *The group* $\mathrm{Aut}^\delta_R(R[[1|n]])$ *is a semi-direct product of* $\mathrm{Aut}^\delta_0(R[[1|n]])$ *and the prounipotent group* $\mathrm{Aut}^\delta_+(R[[1|n]])$.

Definition 4.6. Recall the super Lie algebra $S(n) = [S(1|n), S(1|n)]$. The automorphisms generated by fields inside $S(n)$ and the group $\mathrm{Aut}^\delta_0(R[[1|n]])$. The sets of such elements is going to be denoted by $\mathrm{Aut}^\Delta(R[[1|n]]) \subset \mathrm{Aut}^\delta(R[[1|n]])$. The group $\mathrm{Aut}^\Delta(R[[1|n]])$ is simple for $n \geq 2$.

When there is no confusion we simply write $\mathrm{Aut}^\Delta[[1|2]]$.

From the Lemma 4.3, for any change of coordinates $\Phi \in \mathrm{Aut}^\delta(R[[1|2]])$ there exists a divergence free field $X \in S(1|2)_+$ and $T \in \mathrm{Aut}^\delta_0(R[[1|2]])$ with $\Phi(z|\theta^1, \theta^2) = \exp(X)(T(z|\theta^1, \theta^2))$. Our interest is to study such automorphisms where $X \in S(2)_+ := S(2) \cap \mathrm{Der}_{R,+}(R[[1|2]])$, when this happens we write $\Phi \in \mathrm{Aut}^\Delta(R[[1|2]])$ and observe that $\mathrm{Aut}^\Delta(R[[1|2]])$ is a subgroup of $\mathrm{Aut}^\delta(R[[1|2]])$.

On the other hand, we have the isomorphisms

$$\exp : S(1|2)_+ \to \mathrm{Aut}^\delta_+(R[[1|n]]),$$
$$\exp : S(2)_+ \to \mathrm{Aut}^\Delta_+(R[[1|n]]),$$

and $\mathrm{Aut}^\delta_0(R[[1|n]]) = \mathrm{Aut}^\Delta_0(R[[1|n]])$, so, we have the isomorphism:

$$\exp : \frac{S(1|2)}{S(2)} \to \frac{\mathrm{Aut}^\delta(R[[1|n]])}{\mathrm{Aut}^\Delta(R[[1|n]])} \simeq \mathbb{G}_a. \tag{4.3}$$

4.2.3 The group $\mathrm{Aut}_R^\omega(R[[1|n]])$

Recall the definition of $\mathrm{Aut}_R^\omega(R[[1|n]])$, as the automorphisms preserving the even nondegenerate form (2.1) up to multiplication by a function. Observe that $\mathrm{Aut}^\omega[[1|n]]$ is an ind-group with the composition as multiplication.

Also, recall that a vector field $X \in K(1|n)$ if $L_X\omega = f\omega$, with ω as (2.1) for some function $f \in R[[1|n]]$ and write $K(1|n)_+ := \mathrm{Der}_{R,+}(R[[1|n]]) \cap K(1|n)$. We can notice that for a vector field $X \in K(1|n)_+$ we have $\exp(X) \in \mathrm{Aut}^\omega[[1|n]]$. The group generated by automorphisms $\phi = \exp(X)$, $X \in K(1|n)_+$, is denoted by $\mathrm{Aut}_+^\omega[[1|n]]$. In other words, we have

$$\mathrm{Aut}_+^\omega[[1|n]] = \mathrm{Aut}^\omega[[1|n]] \cap \mathrm{Aut}_+[[1|n]].$$

Also, define $\mathrm{Aut}_0^\omega[[1|n]] = \mathrm{Aut}^\omega[[1|n]]/\mathrm{Aut}_+^\omega[[1|n]]$.

In this case, the Lemma 4.2 is write as:

Lemma 4.4. *The group* $\mathrm{Aut}_R^\omega(R[[1|n]])$ *is a semi-direct product of* $\mathrm{Aut}_0^\omega(R[[1|n]])$ *and the group* $\mathrm{Aut}_+^\delta(R[[1|n]])$.

Observation 4.4. Recall the exterior automorphisms $\Phi_0(z|\theta^1, \theta^2) = (z|\theta^2, \theta^1)$ defined in (2.3). This automorphism is not generated by a vector field. Actually, $\mathrm{Aut}_0^\omega(R[[1|n]])$ has two components one containing the id, $\mathrm{Aut}_0^{\omega,+}(R[[1|n]])$, and the other one containing Φ_0. The group $\mathrm{Aut}^{\omega,+}(R[[1|n]])$ is a semi-direct product of $\mathrm{Aut}_0^{\omega,+}(R[[1|n]])$ and $\mathrm{Aut}_+^\omega(R[[1|n]])$.

4.3 The bundle Aut_X

In this section, given a super curve X we are going to define the space of pairs $\{(x, t_x) : x \in X, t_x : \mathcal{O}_{X,x} \xrightarrow{\sim} \mathcal{O}\}$. Clearly this space is not a smooth manifold. Instead, we are going to give a structure of ind-scheme.

Definition 4.7. Let $R = k[t_1, \ldots, t_m|\theta^1, \ldots, \theta^n]/\langle P_1, \ldots, P_l\rangle$ be a finitely generated super algebra over k, for some polinomials $P_i \in k[m|n]$. For each $p \in \mathbb{N}$ we will define the p-Jet ring by the even variables $t_j^{(r)}$ and odd variables $\theta^{k,(r)}$, where $r = 0, \ldots, p$, $j = 1, \ldots, m$, $k = 1, \ldots, n$, such that they verify the equations $P_{i,s}$ that appear as the coefficient of t^s on $P_l(t_j(t), \theta^k(t))$, with $t^{(r)}(t) = \sum t_j^{(r)} t^r$ and similar for $\theta^k(t)$. Such quotient, $k[t_1, \ldots, t_m|\theta^1, \ldots, \theta^n]/\langle P_{1,s}, \ldots, P_{l,s}\rangle =: J_pR$ is called the *p-Jet algebra of R*.

Proposition 4.3. *The super algebra* J_pR *represents the functor:*

$$\begin{aligned} \mathrm{SAlg}_k &\to \mathrm{Sets} \\ S &\mapsto \mathrm{Hom}_{\mathrm{SAlg}}(R, S \otimes k[t]/(t^{p+1})). \end{aligned}$$

Proof. We will recall [14], Proposition 2.2. Let us consider the finitely generated super algebra $R = k[t_1, \ldots, t_m|\theta^1, \ldots, \theta^n]/\langle P_1, \ldots, P_l\rangle$. An homomorphism $\phi : R \to S \otimes k[t]/(t^{p+1})$ is defined by the images $\phi(t_1), \ldots, \phi(t_m), \phi(\theta^1), \ldots, \phi(\theta^n)$ such that verifies the relations induced by $P_1, \ldots, P_l$. Considering

$$\phi(t_i) = a_{0i} + a_{1i}t + \cdots + a_{pi}t^p$$
$$\phi(\theta^j) = b_{0j} + b_{1j}t + \cdots + b_{pj}t^p$$

Observe that the equations, independent on S, obtained by

$$P_l(\phi(t_1), \ldots, \phi(t_m)|\phi(\theta^1), \ldots, \phi(\theta^n)) = \sum P_{i,s}(a_{00}, \ldots, a_{mp}, b_{00}, \ldots, b_{np})t^s.$$

This, give us the relations over variables $\{a_{si}, b_{rj}\}$ that define the scheme J_pR. □

Example 4.17. Let R be a k-super algebra, then J_1R is canonically isomorphic to the super algebra of derivations $\mathrm{Der}_k(R)$.

Following the Proposition 4.3 we get the family of functors $\{h_{J_pR}\}_{p\in\mathbb{N}}$ joint with a family of projections

$$\mathrm{Hom}_{\mathrm{SAlg}}(R, S \otimes k[t]/(t^{p+2})) \to \mathrm{Hom}_{\mathrm{SAlg}}(R, S \otimes k[t]/(t^{p+1}))$$
$$f \mapsto \overline{f},$$

that induce a family of closed immersions $J_pR \to J_{p+1}R$. Finally, we obtain the ind-algebra *Jet-algebra of* R, denoted by JR. This ind-algebra represents the functor $S \to \mathrm{Hom}_{\mathrm{SAlg}}(R, S[[t]])$.

Observation 4.5. We say that JR is an ind-super algebra instead of a super algebra, since we are considering finitely generated k-super algebras.

Proposition 4.4. *Let R be a smooth k-super algebra, then JR is smooth.*

Proof. This follos directly from the construction. □

Observation 4.6. for any $p \in \mathbb{N}$ we get an inclusion $R \to J_pR$ given by identifying $t_j^{(0)}$ and $\theta^{k,(0)}$ with t_j, θ_k, respectively. Since, this inclusion respect the diagrams 4.3, then this inclusion is well defined $R \to JR$.

Dualizing the process, from Proposition 4.3 we obtain:

Proposition 4.5. *Let $X = \mathrm{Spec}R$ be an affine super scheme. The super scheme $J_pX := \mathrm{Spec}J_pR$ represents the functor:*

$$\begin{aligned} \mathrm{SSch}_k &\to \mathrm{Sets} \\ Y &\mapsto \mathrm{Hom}_{\mathrm{SAlg}}(Y \times \mathrm{Spec}(k[t]/(t^{p+1}), X)). \end{aligned}$$

Definition 4.8. From the family $\{J_pX\}_{p\in\mathbb{N}}$, we obtain the ind-scheme *Jet-scheme of* X, denoted by JX. This jet-scheme is endowed with a projection $JX \to X$ defined in Observation 4.6.

Lemma 4.5. *The projection $JX \to X$ defined in Observation 4.6 is affine.*

Proof. This follows directly from the construction and its functorial property. □

Lemma 4.6. *Let $U \subset X$ be an affine open subscheme of the affine super schemem X. Then for any $p \in \mathbb{N}$, the super scheme $J_pU \subset J_pX$ is an open subscheme. More generally, the jet-scheme $JU \subset JX$ is an open subscheme.*

Proposition 4.6. *From the previous lemma, for any super scheme X we can construct the ind-super scheme JX following the rules:*

1. *Let $\{U_i\}_{i\in I}$ be a covering of X by affine open super subschemes such that U_{ij} is also affine.*

2. *For any $i \in I$ and $p \in \mathbb{N}$, construct J_pX by the gluing of the affine super schemes J_pU_i, with the marked intersection J_pU_{ij}.*

3. *The closed immersions $J_pU_i \to J_{p+1}U_i$ are well behaved, and induce the closed immersions $J_pX \to J_{p+1}X$.*

4. *The family $\{J_pX\}_{p\in\mathbb{N}}$ is an ind-scheme, called* Jet-scheme of X.

Proof. The proof follows from the Lemmas 4.5 and 4.6. □

Proposition 4.7. *Let X be a smooth super scheme, then JX is a smooth ind-scheme.*

Proof. This comes directly from Proposition 4.4. □

Definition 4.9. Let R be a super algebra, for any $p \in \mathbb{N}$, we already see that J_pR represents the functor $S \mapsto \mathrm{Hom}_{\mathrm{SAlg}}(R, S \otimes k[t/(t^{p+1})])$. Now, we will consider the elements with

$$f = f_0 + f_1t + \cdots + f_pt^p,$$

where $f_i \in \mathrm{Hom}(R, S)$, for $i = 0, \cdots, p$, and $f_1 \neq 0$. This functor is represented by the *super algebra of coordinates* $\mathrm{Aut}_p R$. The family $\{\mathrm{Aut}_p R\}_{p \in \mathbb{N}}$ is the ind-scheme that represents the functor

$$\begin{aligned} &\mathrm{SAlg}_k \to \mathrm{Sets} \\ &\quad S \mapsto \{\ f \in \mathrm{Hom}_{\mathrm{SAlg}}(R, S[[t]]) \text{ with a non vanishing differential.}\} \end{aligned}$$

As we already done, for any affine super scheme $X = \mathrm{Spec}R$ we define the *ind-scheme of coordinates* Aut_X. Finally, following the rules given in Observation 4.6 for a super scheme X we define the *ind-scheme of coordinates* Aut_X.

Proposition 4.8. *Let X be a super scheme, then* Aut_X *is an open subscheme in JX. Also, if X is smooth, then JX is smooth.*

Observation 4.7. Let X be a smooth super scheme. From the projection $\mathrm{Aut}_X \to X$, for any closed point $x \in X$ the fiber $\mathrm{Aut}_{X,x}$ represents the space of coordinates around $x \in X$, that is $\mathrm{Aut}_{X,x} = \{t_x : \mathcal{O}_{X,x} \xrightarrow{\sim} \mathcal{O}\}$. In particular, the ind-group $\mathrm{Aut}_{\mathcal{O}}$ acts on the fiber $\mathrm{Aut}_{X,x}$ by composition on the left, that is $t_x \cdot \Phi = \Phi \circ t_x$. More generally, Aut_X is an $\mathrm{Aut}\mathcal{O}$-principal bundle.

Observe that a system of coordinates around $x \in X$ define a local section of the bundle $\mathrm{Aut}_X \to X$. Also, for a group G acting on the bundle $\mathrm{Aut}_X \to X$ a global section of the quotient $\mathrm{Aut}_X/G \to X$ is equivalent to have a system of coordinates $\{\phi_i\}_{i \in I}$ defined over open sets $U_i \subset X$ such that $\phi_i \circ \phi_j^{-1} : U_i \cap U_j \to G$.

Chapter 5

Applications

5.1 The induced curve

Proposition 5.1. *Any* $1|n$*-super curve has an* $SUSY_{2n}$*-super curve associated.*

Proof. Given a super curve $C \to S$, consider the $\mathrm{Aut}[[1|n]]$-principal bundle Aut_C, then we can construct an $\mathrm{Aut}^{\omega}[[1|2n]]$-principal bundle given by

$$\mathrm{Aut}^{\omega}[[1|2n]] \times_{\mathrm{Aut}[[1|n]]} \mathrm{Aut}_C$$

for the inclusion $\mathrm{Aut}[[1|n]] \subset \mathrm{Aut}^{\omega}[[1|2n]]$ given in (2.8).

The structure of $1|2$-super curve give us family of local sections of the bundle $\mathrm{Aut}_C \to C$, $\{\phi_i\}_{i\in I}$; then the family of local sections $\{\Phi_i = (1, \phi_i)\}_{i\in I}$ gives us a family of $K(1|2n)$-super curves. ☐

The curve obtained in proposition (5.1) is going to be denoted by $\widetilde{C}$. If we have an atlas $\{\phi_i = (z|\theta^1, \theta^2)\}_{i\in I}$ over $C \to S$ with cocicles $\Phi_{ij} = \Phi_i \circ \Phi_j^{-1}$, we construct the atlas over $\widetilde{C}$ given by $\{\widetilde{\phi}_i := (z|\theta^1, \theta^2, \rho^1, \rho^2)\}$, with cocicles $\widetilde{\phi}_i\widetilde{\phi}_j^{-1} = j(\phi_{ij})$, with j given in (2.8).

Observation 5.1. In the previous construction, we get a projection locally given by

$$(z|\theta^1, \dots, \theta^n, \rho^1, \dots, \rho^n) \mapsto (z|\theta^1, \dots, \theta^n).$$

By construction, the projection $\widetilde{C} \xrightarrow{\pi} C$ well defined. For any point $p \in C$ the fiber has dimension $0|n$.

Observation 5.2. There exists a geometric description of this fact given by [8]. The space $\widetilde{C}$ is described by the space of $0|n$-subspaces of T_pC for any $p \in C$. The *SUSY*-super structure for a point (p, E) is given by the distribution $\widetilde{E} \subset \mathcal{T}_{(p,E)}\widetilde{C}$ defined by the local form $dz + \rho^1 d\theta^1 + \cdots + \rho^n d\theta^n$ for $p = (z|\theta^1, \ldots, \theta^n)$. Locally, $\widetilde{E}$ is generated by $\{\partial_{\rho^1}, \ldots, \partial_{\rho^n}, \rho^1\partial_z + \partial_{\theta^1}, \ldots, \rho^n\partial_z + \partial_{\theta^n}\}$.

In this context, the projection $\widetilde{C} \xrightarrow{\pi} C$ is given by $(p, E) \mapsto p$ and the distribution $\widetilde{E}$ is given by $d\pi^{-1}(E)$, for $T_{(p,E)}\widetilde{C} \xrightarrow{d\pi} T_pC$.

For $SUSY_n$-super curves $C \to S$ the operators $D^i = \theta^i\partial_z + \partial_i$ define a $0|n$-distribution over $T_{C/S}$ with $D^i_\alpha = (D^i_\alpha\phi^1_{\alpha,\beta})D^i_\beta + \cdots + (D^i_\alpha\phi^n_{\alpha,\beta})D^i_\beta$, $i = 1, \ldots, n$. For the change of coordinates $(F|\phi^1, \ldots, \phi^n)$ the operators D^i, for $i = 1, \ldots, n$, verify

$$D^iF = \phi^1D^i\phi^1 + \cdots + \phi^nD^i\phi^n,\ i = 1, \ldots, n. \tag{5.1}$$

Observation 5.3. The $SUSY_{2n}$-super curve $\widetilde{C}$ associated to the $1|n$-super curve $C \to S$ has a rank $0|n$ bundle locally generated by the local fields $D^i = \partial_{\rho^i}$, $i = 1, \ldots, n$. Reciprocally, suppose that $C \to S$ is a $SUSY_{2n}$-super curve with the local coordinates $(w|\theta^1, \cdots, \theta^{2n})$ with a change of coordinates $(F|\phi^1, \cdots, \phi^n)$, introducing the new variables

$$\begin{aligned} w &= z + i\left(\theta^1\theta^2 + \cdots + \theta^{2n-1}\theta^{2n}\right) \\ \zeta^j &= -\,i(-\theta^{2j-1} + i\theta^{2j}),\ j = 1, \ldots, n. \\ \rho^j &= -\,i(\theta^{2j-1} + i\theta^{2j}),\ j = 1, \ldots, n. \end{aligned}$$

For the change of coordinates

$$\begin{aligned} G &= F + i\left(\phi^1\phi^2 + \cdots + \phi^{2n-1}\phi^{2n}\right) \\ \psi^j &= -\,i(-\phi^{2j-1} + i\phi^{2j}),\ j = 1, \ldots, n, \\ \eta^j &= -\,i(\phi^{2j-1} + i\phi^{2j}),\ j = 1, \ldots, n, \end{aligned}$$

and considering the operators $D^j_\pm = \frac{1}{2}\left\{D^{2j} \pm iD^{2j-1}\right\}$, the equation (5.1) reads:

$$D^j_\pm G = \eta^1D^j_\pm\psi^1 + \cdots + \eta^nD^j_\pm\psi^n. \tag{5.2}$$

This induces the rank $0|2n$-distribution

$$D^j_{\alpha,\pm} = \sum_{k=1}^{n}\left((D^j_{\alpha,\pm}\psi^k)D^j_{\beta,-} + (D^j_{\alpha,\pm}\eta^k)D^j_{\beta,+}\right) \tag{5.3}$$

and if the following equations hold

$$D^j_+\psi^k = 0,\ j,k = 1, \ldots, n; \tag{5.4}$$

then we can define the $1|n$-super curve by the coordinates $(w|\zeta^1, \dots, \zeta^n)$. The conditions described in equation (5.4) are equivalent to have the $0|n$-subdistribution locally defined by D^j_+, $j = 1, \dots, n$.

Also, let $C \to S$ be a $SUSY_4$-super curve taking the $\mathrm{Aut}^\omega[[1|4]]$-principal bundle $\mathrm{Aut}_C \to C$, we obtain that it comes from a $1|2$-super curve if and only if the bundle $\mathrm{Aut}[[1|2]]\backslash\mathrm{Aut}_C$ has a global section.

Example 5.1. The $1|1$-super curves induce what is called oriented $SUSY_2$-super curves. For a general $SUSY_2$-super curve and change of coordinates $(F|\psi, \eta)$ equations (5.3) are:

$$\begin{aligned} D_{\alpha,+} =& D_{\alpha,+}\psi D_{\beta,-} + D_{\alpha,+}\eta D_{\beta,+}, \\ D_{\alpha,-} =& D_{\alpha,-}\psi D_{\beta,-} + D_{\alpha,-}\eta D_{\beta,+}, \end{aligned}$$

and since the matrix

$$\begin{pmatrix} D_{\alpha,+}\psi & D_{\alpha,+}\eta \\ D_{\alpha,-}\psi & D_{\alpha,-}\eta \end{pmatrix}$$

is invertible, then $D_{\alpha,-}\psi D_{\alpha,+}\eta \neq 0$. If (5.4) holds, differentiating equation (5.2), we get

$$\begin{aligned} D_{\alpha,-}D_{\alpha,-}F &= D_{\alpha,-}\left(\eta D_{\alpha,-}\psi\right), \\ &= D_{\alpha,-}\eta D_{\alpha,-}\psi - \eta D_{\alpha,-}D_{\alpha,-}\psi; \end{aligned}$$

since $D_{\alpha,-}D_{\alpha,-} = 0$, we obtain that $D_{\alpha,-}\eta D_{\alpha,-}\psi = 0$, then $D_{\alpha,-}\eta$ vanishes. Finally, there exists another bundle defined by D_- that for a change of coordinates $(F|\psi, \eta)$ we get

$$D_{\alpha,-} = D_{\alpha,-}\psi D_{\beta,-}.$$

This line bundle induce another curve $\widehat{C}$ defined by the coordinates:

$$\begin{aligned} \widehat{z} =& z - \theta\rho, \\ \widehat{\rho} =& \rho, \end{aligned}$$

this curve is called "dual" curve associated to $C \to S$. The situation was described in [2].

Example 5.2. In this example we will see a $SUSY_4$-super curve that does not come from a $1|2$-super curve.

Over the affine plane $\mathbb{C}^{1|4}$ consider the following relations

1. $S(z|\theta^1, \theta^2, \theta^3, \theta^4) = (z + 1|\theta^1, \theta^2, \theta^3, \theta^4)$.

2. $T(z|\theta^1, \theta^2, \theta^3, \theta^4) = (z + \tau - 2\theta^1\theta^2\theta^3\theta^4|\theta^1 + \theta^2\theta^3\theta^4, \theta^2 - \theta^1\theta^3\theta^4, \theta^3 + \theta^1\theta^2\theta^4, \theta^4 - \theta^1\theta^2\theta^3)$.

where τ is even. The quotient $\mathbb{C}^{1|4}/\langle S,T\rangle$ is an elliptic curve $\widetilde{\mathbb{T}}_\tau$ with a $SUSY_4$-super structure. Its tangent bundle $\mathcal{T}_{\widetilde{\mathbb{T}}_\tau}$ does not have a subbundle of dimension $0|2$ then this curve does not come from a $1|2$-super curve.

Observation 5.4. For $n=2$ the change of coordinates for our new variables (ρ^1,ρ^2) are given by the equation:

$$\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix} = A^{-1}\begin{pmatrix}\partial_1 F\\ \partial_2 F\end{pmatrix} + \mathrm{Ber}(\Phi)\left\{\begin{pmatrix}\partial_2\phi^2 & -\partial_1\phi^2\\ -\partial_2\phi^1 & \partial_1\phi^1\end{pmatrix}\begin{pmatrix}\rho^1\\ \rho^2\end{pmatrix} + \begin{pmatrix}\partial_z\phi^2\\ -\partial_z\phi^1\end{pmatrix}\rho^1\rho^2\right\}, \tag{5.5}$$

where $A=(\partial_i\phi^j)_{i,j}$. More specifically, observe that the local coordinates $\rho^1\rho^2,\rho^1,\rho^2$ define a $1|2$ bundle over $C\to S$ and since

$$\begin{pmatrix}\eta^1\eta^2\\ \eta^2\\ -\eta^1\end{pmatrix} = \det A^{-1}\left(\begin{matrix}\partial_1 F\partial_2 F\\ \begin{pmatrix}\partial_2\phi^1 & \partial_1\phi^1\\ \partial_2\phi^2 & \partial_1\phi^2\end{pmatrix}\begin{pmatrix}-\partial_1 F\\ \partial_2 F\end{pmatrix}\end{matrix}\right) + \mathrm{Ber}(\Phi)J\Phi\begin{pmatrix}\rho^1\rho^2\\ \rho^2\\ -\rho^1\end{pmatrix}.$$

Then the sheaf $\mathcal{W}_{\widetilde{C}}$ that defines $\widetilde{C}$ is given by the extension of $\mathcal{O}_C$ by $\mathrm{Ber}_{C/S}\otimes\Omega_C$:

$$0\to\mathcal{O}_C\to\mathcal{W}_{\widetilde{C}}\to\mathrm{Ber}_{C/S}\otimes\Omega_C\to 0.$$

When our curve is split, and associated to the bundle E, then $\widetilde{C}$ is a split curve associated the reduced curve C_{rd} and

$$W=E\oplus(E^*\otimes\Omega_{C_{\mathrm{rd}}}).$$

In this special case, we can consider the *dual curve* as the split curve associated to the curve C_{rd} and odd part $E^*\otimes\Omega_{C_{\mathrm{rd}}}$.

Example 5.3. Here, we will see a $S(2)$ super curve that is not an $K(2)$ super curve. Over the affine plane $\mathbb{C}^{1|2}$ consider the following relations

1. $S(z|\theta^1,\theta^2)=(z+1|\theta^1,\theta^2)$

2. $T(z|\theta^1,\theta^2)=(z+\tau+\theta^1\rho|\theta^1,\theta^2)$

where τ is even and ρ a nonzero odd constant. The quotient $\mathbb{C}^{1|4}/\langle S,T\rangle$ is an elliptic curve $\mathbb{T}_{\tau,\rho}$ with the Berezinian $[dz|d\theta^1 d\theta^2]$. Notice that its tangent bundle has a well definite $1|0$ vector field ∂_z and suppose that $E\subset\mathcal{T}_{\mathbb{T}_{\tau,\rho}}$ is a $0|2$-subbundle. Since the projection is $E\to\mathcal{T}_{\mathbb{T}_{\tau,\rho}}/\langle\partial_z\rangle$ is an isomorphism and $\mathcal{T}_{\mathbb{T}_{\tau,\rho}}/\langle\partial_z\rangle$ is trivial, with global sections $\{\partial_1,\partial_1\}$, then E is trivial. Now, the étale covering

$$\begin{array}{ccc} \mathcal{T}_{\mathbb{C}^{1|2}} & \longrightarrow & \mathcal{T}_{\mathbb{T}_{\tau,\rho}} \\ \big\downarrow & & \big\downarrow \\ \mathbb{C}^{1|2} & \xrightarrow{\pi} & \mathbb{T}_{\tau,\rho} \end{array}$$

implies that the pullback π^*E is trivial. A global vector field X has the form $a\partial_z + b\partial_1 + c\partial_2$ that should verifies the relations

1. $a(z+1,\theta^1,\theta^2) = a(z,\theta^1,\theta^2)$, $a(z+\tau+\theta^1\rho,\theta^1,\theta^2) = a(z,\theta^1,\theta^2) + \rho b(z,\theta^1,\theta^2)$.
2. $b(z+1,\theta^1,\theta^2) = b(z,\theta^1,\theta^2)$, $b(z+\tau+\theta^1\rho,\theta^1,\theta^2) = b(z,\theta^1,\theta^2)$.
3. $c(z+1,\theta^1,\theta^2) = c(z,\theta^1,\theta^2)$, $c(z+\tau+\theta^1\rho,\theta^1,\theta^2) = c(z,\theta^1,\theta^2)$.

we deduce that a, b, c are constants, and moreover, $\rho b = 0$. Then π^*E is not trivial. Then $\mathcal{T}_{\mathbb{T}_{\tau,\rho}}$ is a curve with a trivial Berezinian that is not a $SUSY_2$-super curve.

5.2 $S(2)$-super curves and $SUSY_4$-super curves

In this section we define the most important family of curves on our work.

Fix a base super scheme S, we will consider curves and bundles relative to S. Recall that for a $1|2$ super curve $C \to S$ we write $\mathrm{Ber}_{C/S} = \mathrm{Ber}(\Omega_{C/S})$ and that an $S(1|2)$-super curve is a pair (C, Δ), where $C \to S$ is a super curve and a nonvanishing section $\Delta \in H^0(C, \mathrm{Ber}_{C/S})$.

Definition 5.1. An $S(2)$-*super curve* is an $S(1|2)$-super curve (C, Δ) such that there exists an atlas $\{U_i, \Phi_i\}$ compatible with Δ and change of coordinates $\Phi_{ij} = \Phi_i \circ \Phi_j^{-1} \in \mathrm{Aut}^{\Delta}[[1|2]]$. We say that a flat family of curves $C \to S$ is a family of $S(2)$-super curves, if for any closed point $y \in S$, C_y is an $S(2)$-super curve.

Observation 5.5. For an $S(1|2)$-super curve (C, Δ) we can construct the bundle of coordinates preserving the Berezinian, Aut_C^{δ}, considered as the set of pairs (Z, Φ) for Z a S-point in $C \to S$ and Φ a local system of coordinates compatible with the section Δ. This bundle is an $\mathrm{Aut}^{\delta}(R[[1|2]])$-bundle, and observe that the quotient:

$$\mathrm{Aut}^{\Delta}[[1|2]] \backslash \mathrm{Aut}_C^{\delta} \to C \tag{5.1}$$

is an $\mathbb{G}_a$-bundle. From (4.3), we get that $\mathrm{Aut}^{\Delta}[[1|2]] \backslash \mathrm{Aut}_C^{\delta}$ is isomorphic to $\mathcal{A}$.

Now, we can reformulate the definition of $S(2)$-super curves:

Proposition 5.2. *An $S(2)$-super curve is an $S(1|2)$-super curve (C, Δ) such that the $\mathbb{G}_a$-bundle* (5.1) *is trivial.*

Proof. Observe that the bundle (5.1) is trivial if and only if it has a section. In this case, a section is an atlas $\{(U_i, \Phi_i)\}_i$ such that the change of coordinates $\Phi_{ij} := \Phi_j \circ \Phi_i^{-1} \in \mathrm{Aut}^{\Delta}(R[[1|2]])$.

Finally, the bundle is trivial if and only if there exists a covering for $C \to S$ with trivializations $\{(U_i, \Phi_i)\}_i$ compatible with Δ such that the change of coordinates $\Phi_{ij} \in \mathrm{Aut}^{\Delta}[[1|2]]$, that is, (C, Δ) is an $S(2)$-super curve. □

The adjoint action $\mathrm{Ad}_\alpha : \mathrm{Aut}^\omega[[1|4]] \to \mathrm{Aut}^\omega[[1|4]]$ given by $\alpha \in \mathrm{Aut}^\omega[[1|4]]$:

$$\alpha(z|\theta^1, \theta^2, \rho^1, \rho^2) := (z - \theta^1\rho^1 - \theta^2\rho^2, \rho^1, \rho^2, \theta^1, \theta^2).$$

Observe that α is involutive and Ad_α fix the subspace $\mathrm{Aut}^\omega[[1|2]] \subset \mathrm{Aut}^\omega[[1|4]]$.

For the automorphism Ad_α we have that $\mathrm{Aut}[[1|2]] \cap \mu\left(\mathrm{Aut}^\delta[[1|2]]\right) = \mathrm{Aut}^\Delta[[1|2]]$. We recall the commutative diagram

$$\begin{array}{ccc} \mathrm{Aut}^\Delta[[1|2]] & \xrightarrow{\mathrm{Ad}_\alpha} & \mathrm{Aut}^\Delta[[1|2]] \\ \downarrow j & & \downarrow j \\ \mathrm{Aut}^\omega[[1|4]] & \xrightarrow{\mathrm{Ad}_\alpha} & \mathrm{Aut}^\omega[[1|4]] \end{array}$$

Let $C \to S$ be an $S(1|2)$-super curve, over Aut_C^δ we construct the $\mathrm{Aut}^\omega[[1|4]]$-principal bundle, $\mathrm{Aut}^\omega[[1|4]] \times_{\mathrm{Aut}^\delta[[1|2]]} \mathrm{Aut}_C^\delta$, after this we obtain a morphism given by:

$$\begin{aligned} \mu : \mathrm{Aut}^\omega[[1|4]] \times_{\mathrm{Aut}^\delta[[1|2]]} \mathrm{Aut}_C^\delta &\to \mathrm{Aut}^\omega[[1|4]] \times_{\mathrm{Aut}^\delta[[1|2]]} \mathrm{Aut}_C^\delta \\ (g, (x, \Phi)) &\mapsto (\alpha \circ g, (x, \Phi)). \end{aligned} \tag{5.2}$$

If $C \to S$ is an $S(2)$-super curve, then the bundle $\mathcal{A} = \mathrm{Aut}^\Delta \backslash \mathrm{Aut}_C^\delta \to C$ is trivial, then we have a global section $s : C \to \mathcal{A}$, defined by local sections $\{\phi_i\}_i$ of the bundle $\mathrm{Aut}_C^\delta \to C$ such that $\phi_i \circ \phi_j^{-1} \in \mathrm{Aut}^\Delta[[1|2]]$. Using (5.2), we obtain the local sections of $\mathrm{Aut}^\omega[[1|4]] \times_{\mathrm{Aut}^\delta[[1|2]]} \mathrm{Aut}_C^\delta \to C$ given by $\{\mathrm{Ad}_\alpha((1, \phi_i)) = (\alpha, \phi_i)\}_i$, since $\phi_i \circ \phi_j^{-1} \in \mathrm{Aut}^\Delta[[1|2]]$, then $\mu(\phi_i \circ \phi_j^{-1}) \in \mathrm{Aut}^\Delta[[1|2]]$. In particular, such sections gives us a global section of $\mathrm{Aut}[[1|2]] \backslash \mathrm{Aut}^\omega[[1|4]] \times_{\mathrm{Aut}^\delta[[1|2]]} \mathrm{Aut}_C^\delta \to C$. Then, we obtain another family of $S(2)$-super curves $\widehat{C} \to S$. Such family is called dual family of curves, or simply dual curve.

Then the bundle $\mathrm{Ad}_\alpha(\mathcal{A})$ defines a family of $S(2)$-super curves if $C \to S$ is a family of $S(2)$-super curves.

Observation 5.6. Let us consider an element $\Phi \in \mathrm{Aut}^{\Delta}[[1|2]]$ such that $\mu(\Phi) = \Phi$. Then we have the condition $\Phi \circ \alpha = \alpha \circ \Phi$, expanding this we obtain:

$$\begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} \circ \alpha.$$

Using the formula given by the inclusion (2.8), we obtain

$$\begin{pmatrix} D_1\phi^1 & D_1\phi^2 \\ D_2\phi^1 & D_2\phi^2 \end{pmatrix}^{-1} \begin{pmatrix} D_1 F \\ D_2 F \end{pmatrix} = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} \circ \alpha. \tag{5.3}$$

Using the relations:

$$\begin{aligned} D_i &= D^i - (\theta^i - \rho^i)\partial_z, \\ H \circ \alpha &= H - (\theta^1 - \rho^1)D^1 H - (\theta^2 - \rho^2)D^2 H - (\theta^2 - \rho^2)(\theta^1 - \rho^1)D^1 D^2 H, \end{aligned}$$

where $H = H(z, \theta^1, \theta^2)$ is a regular function. Replacing this relations on the left hand side of (5.3) we obtain:

$$\begin{aligned} \mathrm{LHS} &= \left(\begin{pmatrix} D^1\phi^1 & D^1\phi^2 \\ D^2\phi^1 & D^2\phi^2 \end{pmatrix} - \begin{pmatrix} \theta^1 - \rho^1 \\ \theta^2 - \rho^2 \end{pmatrix} \begin{pmatrix} \partial_z\phi^1 & \partial_z\phi^2 \end{pmatrix} \right)^{-1} \left(\begin{pmatrix} D^1 F \\ D^2 F \end{pmatrix} - \partial_z F \begin{pmatrix} \theta^1 - \rho^1 \\ \theta^2 - \rho^2 \end{pmatrix} \right), \\ &= A^{-1} \begin{pmatrix} D^1 F \\ D^2 F \end{pmatrix} - (\det A) A^{-1} \begin{pmatrix} \theta^1 - \rho^1 \\ \theta^2 - \rho^2 \end{pmatrix} - (\theta^1 - \rho^1)(\theta^2 - \rho^2) \begin{pmatrix} -\partial_z\phi^2 \\ \partial_z\phi^1 \end{pmatrix}, \end{aligned}$$

where $A = \begin{pmatrix} D^1\phi^1 & D^1\phi^2 \\ D^2\phi^1 & D^2\phi^2 \end{pmatrix}$. Similarly, expanding the right hand side we obtain:

$$\mathrm{RHS} = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} - A^t \begin{pmatrix} \theta^1 - \rho^1 \\ \theta^2 - \rho^2 \end{pmatrix} - (\theta^1 - \rho^1)(\theta^2 - \rho^2) \begin{pmatrix} D^1 D^2 \phi^1 \\ D^1 D^2 \phi^2 \end{pmatrix}.$$

Comparing both sides, and replacing in (5.3), we obtain that the only parts that contain the variables z, θ^1, θ^2 are

$$\begin{pmatrix} D^1\phi^1 & D^1\phi^2 \\ D^2\phi^1 & D^2\phi^2 \end{pmatrix}^{-1} \begin{pmatrix} D^1 F \\ D^2 F \end{pmatrix} = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix},$$

so, we should have that $\Phi \in \mathrm{Aut}^{\omega}[[1|2]]$. In order to see that $\Phi \in \mathrm{Aut}^{\omega}[[1|2]]$ verifies (5.3), we simply recall the equations (3.14) and recall that $\mathrm{Aut}^{\omega}[[1|2]] \cap \mathrm{Aut}^{\Delta}[[1|2]] = \mathrm{Aut}^{\omega,+}[[1|2]]$

should preserve the orientation, that is we use the identities (3.16) and we are done.

Finally for $\Phi \in \mathrm{Aut}^{\omega,+}[[1|2]]$ we have $\mu(\Phi) = \Phi$, then for a $SUSY_2$-super curve the construction given in Proposition 5.2 the dual curve is canonically isomorphic to the original one.

Similar to 5.2 we obtain:

Proposition 5.3. *Given a family of $S(1|2)$-super curves $C \to S$, the image $\mathrm{Ad}_\alpha(\mathcal{A})$ in (5.2) defines a family of $1|2$-super curves if and only if $C \to S$ is a family of $S(2)$-super curves. In such case, we obtain the family of dual curves. Also, if our curve is an oriented $SUSY_2$-super curve, the dual curve is isomorphic to the original one.*

Proof. The bundle $\mathrm{Ad}_\alpha(\mathcal{A})$ defines a $1|2$-super curve if and only if the projection of $\mathrm{Ad}_\alpha(\mathcal{A})$ over $\mathrm{Aut}[[1|2]]\backslash(\mathrm{Aut}^\omega[[1|4]]\times_{\mathrm{Aut}^\delta[[1|2]]}\mathrm{Aut}^\delta_C)$ has a global section, since $\mathrm{Aut}[[1|2]]\cap\mu\left(\mathrm{Aut}^\delta[[1|2]]\right) = \mathrm{Aut}^\Delta[[1|2]]$ then such projection is isomorphic to $\mathcal{A}$, then $\mathrm{Ad}_\alpha(\mathcal{A})$ is trivial if and only if $\mathcal{A}$ is trivial, that is if $C \to S$ is an $S(2)$-super curve.

The second part comes from the observation above. □

Finally, we obtain:

Theorem 5.1. *There exists an involution μ of the moduli space $\mathcal{M}_{S(2)}$ of $S(2)$-super curves. The fixed point set of μ contains the moduli space $\mathcal{M}_{K(1|2)}$ of orientable $SUSY_2$-super curves.*

Observation 5.7. This duality was observed in [15] as an involution over the super algebra $S(2)$.

5.3 Splitting curves

The criterium given in Proposition 5.2 is very complicated to use in practice, since the Aut_C bundle is infinite dimensional, there exists a simplest criterium given by the operator defined over $\mathrm{Aut}(1|2) \ni \Phi = (F|\phi^1, \phi^2)$:

$$\gamma(\Phi) = \mathrm{Ber}\begin{pmatrix} D^1D^2F & D^1D^2\phi^1 & D^1D^2\phi^2 \\ D^1F & D^1\phi^1 & D^1\phi^2 \\ D^2F & D^2\phi^1 & D^2\phi^2 \end{pmatrix},$$

where $D^i = \theta^i\partial_z + \partial_{\theta^i}$. That give us the useful lemma that is going to be proved later:

Lemma 5.1. *Suppose that $\Phi \in \mathrm{Aut}^\delta(1|2)$, then $\Phi \in \mathrm{Aut}^\Delta(1|2)$ if and only if $\gamma(\Phi)$ vanishes.*

Observation 5.8. We already see in Proposition 3.6 that any oriented $SUSY_2$-super curve is naturally an $S(1|2)$-super curve. Now suppose that we fix coordinates compatible with the $SUSY$-structure, and consider the change of coordinates $\Phi = (F|\phi^1, \phi^2)$. From the relations given in (2.2) we calculate:

$$\begin{aligned}\gamma(\Phi) &= \operatorname{Ber}\begin{pmatrix} D^1D^2F & D^1D^2\phi^1 & D^1D^2\phi^2 \\ D^1F & D^1\phi^1 & D^1\phi^2 \\ D^2F & D^2\phi^1 & D^2\phi^2 \end{pmatrix} \\ &= \left(D^1D^2F - \begin{pmatrix} D^1D^2\phi^1 & D^1D^2\phi^2 \end{pmatrix} A^{-1} \begin{pmatrix} D^1F \\ D^2F \end{pmatrix} \right) \det A^{-1},\end{aligned} \tag{5.1}$$

where $A = (D^i\phi^j)$. Derivating by D^1 the condition $D^2F = \phi^1D^2\phi^1 + \phi^1D^2\phi^1$ given in (2.2) an using (3.14) we obtain that

$$D^1D^2F = -\phi^1D^1D^2\phi^1 - \phi^2D^1D^2\phi^2.$$

Finally, replacing this in (5.1) we get that

$$\begin{aligned}\gamma(\Phi) &= \left(D^1D^2F - \begin{pmatrix} D^1D^2\phi^1 & D^1D^2\phi^2 \end{pmatrix} A^{-1} \begin{pmatrix} D^1F \\ D^2F \end{pmatrix} \right) \det A^{-1} \\ &= \left(D^1D^2F - \begin{pmatrix} D^1D^2\phi^1 & D^1D^2\phi^2 \end{pmatrix} \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} \right) \det A^{-1} \\ &= \left(D^1D^2F + \phi^1D^1D^2\phi^1 + \phi^2D^1D^2\phi^2 \right) \det A^{-1} \\ &= 0.\end{aligned}$$

So we obtain the following proposition:

Proposition 5.4. *Let C be an oriented $SUSY_2$-super curve, then C is an $S(2)$-super curve.*

In particular, if C be an $S(2)$-super curve over a point, that comes from an oriented $SUSY_2$-super curve from Proposition 3.7 we obtain that C is split.

Observation 5.9. Let $\{\Phi_i = (z_i|\theta_i^1, \theta_i^2)\}$ local coordinates over $C \to S$. For the change of coordinates ϕ_{ij} we get

$$\begin{aligned} z_j &= F_{ij}(z_i) + G_{ij}(z_i)\theta_i^1\theta_i^2 \\ \theta_j^1 &= \theta_i^1 a_{11}(z_i) + \theta_i^2 a_{12}(z_i) \\ \theta_j^2 &= \theta_i^1 a_{21}(z_i) + \theta_i^2 a_{22}(z_i). \end{aligned}$$

Since $C \to S$ has a trivial Berezinian, then $G_{ij} = \lambda_{ij}\partial_{z_i}F_{ij}$, for λ_{ij} a constant, then we have

$$\begin{aligned} z_j &= F_{ij}(z_i) + \lambda_{ij}\partial_{z_i}F_{ij}(z_i)\theta_i^1\theta_i^2 = F_{ij}(z_i + \lambda_{ij}\theta_i^1\theta_i^2) \\ \theta_j^1 &= \theta_i^1 a_{11}(z_i) + \theta_i^2 a_{12}(z_i) = g_{ij}^1(z_i|\theta_i^1, \theta_i^2) \\ \theta_j^2 &= \theta_i^1 a_{21}(z_i) + \theta_i^2 a_{22}(z_i) = g_{ij}^2(z_i|\theta_i^1, \theta_i^2). \end{aligned}$$

From the generators given in Observation (2.15), we get that $\widetilde{\Phi}_{ij} = (F_{ij}|g_{ij}^1, g_{ij}^2) \in \mathrm{Aut}^{\Delta}[[1|2]]$ and $z_i + \lambda_{ij}\theta^1\theta_i^2 = \exp(\lambda_{ij}\theta_i^1\theta_i^2\partial_{z_i})(z_i)$, then $\Phi_{ij} = \widetilde{\Phi}_{ij} \circ \exp(\lambda_{ij}\theta_i^1\theta_i^2\partial_{z_i})$, that is $\exp(\lambda_{ij}\theta_i^1\theta_i^2\partial_{z_i})$ gives the cocicle in (5.1).

The relation of (3.8) with the class (5.1) is the following:

$$\omega_{ij}(f) = \gamma_{ij}\theta_i^1\theta_i^2\partial_{z_i}f$$

for local coordinates $(z_i|\theta_i^1, \theta_i^2)$ over U_i.

In [10] it was proved that $C \to S$ is projected if and only if $\{\omega_{ij}\} \in H^1(C, \mathcal{O}_{C/S})$ vanishes. Then we obtain

Theorem 5.2. *Every $S(2)$-super curve over a purely even base S is split.*

In order to get a geometric interpretation of this,consider a $1|2$-super curve $C \to S$, the inclusion $C_{\mathrm{rd}} \xrightarrow{j} C_0$ and the space of differentials over Ω_{C_0}, we obtain that $j^*\Omega_{C_0}$ is a rank 2 bundle over C_{rd} with a projection $j^*\Omega_{C_0} \to \Omega_{C_{\mathrm{rd}}} \to 0$. Actually, we get the sequence of $\mathcal{O}_{C_0}$-modules:

$$0 \to \det \mathcal{F} \to j^*\Omega_{C_0} \to \Omega_{C_{\mathrm{rd}}} \to 0. \tag{5.2}$$

As an extension of $\mathcal{O}_{C_{\mathrm{rd}}}$-modules, (5.2), is defined by an element of

$$\mathrm{Ext}^1(\Omega_{C_{\mathrm{rd}}}, \det \mathcal{F}) = H^1(\Omega^*_{C_{\mathrm{rd}}} \otimes \det \mathcal{F}) = H^1(\mathcal{T}_{C_{\mathrm{rd}}} \otimes \det \mathcal{F}),$$

and such element is the class $\{\omega_{ij}\}$ defined above.

Now, if $\det \mathcal{F} = \Omega_{C_{\mathrm{rd}}}$, then we have the sequence

$$0 \to \Omega_{C_{\mathrm{rd}}} \to j^*\Omega_{C_0} \to \Omega_{C_{\mathrm{rd}}} \to 0. \tag{5.3}$$

When $C \to S$ is a $S(1|2)$ curve, to distinguish an element in $H^1(C_{\mathrm{rd}}, \pi^*\mathcal{O}_S) \subseteq H^1(C_{\mathrm{rd}}, \mathcal{O}_{C_{\mathrm{rd}}})$, where $\pi : C \to S$, we have to notice that the sequence (5.3) fits in the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Omega_{C_{\mathrm{rd}}} & \longrightarrow & L \otimes \Omega_{C_{\mathrm{rd}}} & \longrightarrow & \Omega_{C_{\mathrm{rd}}} & \longrightarrow & 0 \\
 & & d \uparrow & & 1\otimes d \uparrow & & d \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_{C_{\mathrm{rd}}} & \xrightarrow{\alpha} & L \otimes \mathcal{O}_{C_{\mathrm{rd}}} & \longrightarrow & \mathcal{O}_{C_{\mathrm{rd}}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \pi^*\mathcal{O}_S & \longrightarrow & L & \longrightarrow & \pi^*\mathcal{O}_S & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (5.4)$$

where $L \otimes \Omega_{C_{\mathrm{rd}}} = j^*\Omega_{C_0}$ and L as an extension of $\pi^*\mathcal{O}_S$-modules represents the class $\Gamma_C \in H^1(C_{\mathrm{rd}}, \pi^*\mathcal{O}_S)$.

Finally, in [16] and [7] it is proved that each $1|2$ super curve, over a point, is defined by the data of $(C_{\mathrm{rd}}, \mathcal{F}, \{\omega_{ij}\})$, where $\{\omega_{ij}\}$ represents an extension of $\mathcal{O}_{C_{\mathrm{rd}}}$-modules:

$$0 \to \det \mathcal{F} \to \mathcal{L} \to \Omega_{C_{\mathrm{rd}}} \to 0.$$

For an $S(1|2)$ we need $(C_{\mathrm{rd}}, \mathcal{F}, \Gamma_C)$, where Γ_c represents an extension of $\pi^*\mathcal{O}_S$-modules

$$0 \to \pi^*\mathcal{O}_S \to L \to \pi^*\mathcal{O}_S \to 0$$

which gives rise to the diagram (5.4).

Chapter 6

Moduli Spaces

6.1 Families of super curves

6.1.1 A family of $S(2)$-super curves

Observe that for an $S(2)$-super curve, over its $SUSY_4$-super curve $\widetilde{C}$ we have the local fields $D^i_+ = \partial_{\rho^i}$ and $D^j_- = \rho^j\partial_z + \partial_{\theta^j}$ on the coordinates $(z|\theta^1, \theta^2, \rho^1, \rho^2)$ given by 5.1 satisfying the relation $\{D^i_+, D^j_-\} = \delta_{ij}\partial_z$. The change of coordinates satisfies the equations (5.2):

$$D^j_- F = \eta^1 D^j_-\phi^1 + \eta^2 D^j_-\phi^2,$$

thus we have

$$\{D^i_+, D^j_-\}F = D^i_+\eta^1 D^j_-\phi^1 - \eta^1\{D^i_+, D^j_-\}\phi^1 + D^i_+\eta^2 D^j_-\phi^2 - \eta^2\{D^i_+, D^j_-\}\phi^2,$$

and that implies:

$$\delta_{ij}(\partial_z F + \eta^1\partial_z\phi^1 + \eta^2\partial_z\phi^2) = D^i_+\eta^1 D^j_-\phi^1 + D^i_+\eta^2 D^j_-\phi^2.$$

Follows from (5.5) that

$$\begin{pmatrix} D^1_+\eta^1 & D^1_+\eta^2 \\ D^2_+\eta^1 & D^2_+\eta^2 \end{pmatrix} = \begin{pmatrix} D^2_-\phi^2 & -D^2_-\phi^1 \\ -D^1_-\phi^2 & D^1_-\phi^1 \end{pmatrix}.$$

Finally, we get

$$\partial_z F + \eta^1\partial_z\phi^1 + \eta^2\partial_z\phi^2 = D^1_-\phi^1 D^2_-\phi^2 - D^1_-\phi^2 D^2_-\phi^1, \tag{6.1}$$

then we have that the projection $\det E \to \mathcal{T}_{\widetilde{C}}/(E + \widehat{E})$ is an isomorphism.

Conversely, suppose that we start with a $1|0$-family over a purely even base S, $\pi_0 : C_0 \to S$, and a rank $0|2$ bundle E with an isomorphism $\beta : \det E \to \Omega_{C_0/S}$. Given such data, we construct $\pi : C \to S$ given by

$$\begin{aligned} C_{\mathrm{rd}} &= C_0, \\ (\mathcal{O}_C)_0 &= \mathcal{O}_{C_0} \oplus \det E, \\ (\mathcal{O}_C)_1 &= \Pi E. \end{aligned}$$

The $S(2)$ structure is given (locally) by the coordinates z and local sections θ^1, θ^2 of E such that $\beta(\theta^1 \otimes \theta^2) = dz$. This data defines the Berezinian $[dz|d\theta^1 d\theta^2]$. Since β is an isomorphism, then such class is well defined. Also, we already see in Proposition 3.4 its associated bundle $\mathcal{A}_C$ is trivial, then $C \to S$ is an $S(2)$-super curve.

Similar to [2] we obtain:

Proposition 6.1. *For any family $\pi_0 : C_0 \to S$ of relative dimension $1|0$ over a purely even base, the following data are equivalent:*

1. *An $S(2)$-family of curves $\pi : C \to S$ over S with $C_{\mathrm{rd}} = C_0$ and $\mathcal{O}_{C,0} = \mathcal{O}_{C_0}(1)$, where $C_0(1)$ is the first neighborhood of the diagonal in $C_0 \times_S C_0$.*

2. *A rank 2 bundle E joint with an isomorphism $\det E \overset{\beta}{\xrightarrow{\sim}} \Omega_{C_0/S}$ up to equivalence: a pair (E, β) is equivalent to (E', β') if there exists an isomorphism, such that the following diagram commutes:*

$$\begin{array}{ccc} E & \longrightarrow & E' \\ & \underset{\beta}{\searrow} & \downarrow{\scriptstyle \beta'} \\ & & \Omega_{C_0/S} \end{array}$$

Proof. The previous comment shows (2) $\to$ (1).

To see (1) $\to$ (2) consider the $SUSY_4$-super curve $\widetilde{C}$ associated to $C \to S$. Since $C \to S$ is $S(2)$, we have the well defined $(0|2)$ bundle

$$\widehat{E} = \langle \rho^1 \partial_z + \partial_1, \rho^2 \partial_z + \partial_2 \rangle$$

over $\widetilde{C}$. The pullback $E := i^*(\Pi \widehat{E})$ over the inclusion $i : C_0 \hookrightarrow \widetilde{C}$ is a rank 2 bundle. We get from equation (6.1) that $\det E \simeq \Omega_{C_0/S}$, where $C_0 = C_{\mathrm{rd}}$. Finally, since $C \to S$ is $S(2)$, then $\mathcal{O}_{C,0} = \mathcal{O}_{C_0} \oplus \det E$ and $\mathcal{O}_{C,1} = \Pi E$. $\square$

6.1.2 A family of $S(1|2)$-super curves

Example 6.1. Here we will see an example of a $S(1|2)$-super curve over a point that is not an $S(2)$-super curve. Over the affine plane $\mathbb{C}^{1|2}$ consider the following relations

1. $T(z|\theta^1,\theta^2) = (z+1|\theta^1,\theta^2)$,
2. $S(z|\theta^1,\theta^2) = (z+\tau+\theta^1\theta^2|\theta^1,\theta^2)$,

where τ is even. The quotient $\mathbb{C}^{1|2}/\langle T,S\rangle$ is an elliptic curve $\mathbb{T}_\tau$ with Berezinian $[dz|d\theta^1 d\theta^2]$. Consider the $SUSY_4$-super curve associated, $\widetilde{\mathbb{T}}_\tau$, since $\mathcal{T}_{\widetilde{\mathbb{T}}_{\tau,\rho}}$ does not have a split then $\mathbb{T}_\tau$ does not have an $S(2)$-structure.

A general family of $S(1|2)$-super curve $C \to S$ over the even base S, with a nonvanishing section $\Delta \in H^0(C, \mathrm{Ber}_{C/S})$, defines a class

$$\Gamma_C \in H^1(C, \pi^*\mathcal{O}_S), \tag{6.2}$$

given by the bundle $\mathcal{A}$ in (5.1) and Proposition 5.9. Suppose that this class is defined by an atlas $\{(U_i,\Phi_i)\}_{i\in I}$ compatible to Δ and $\Gamma_C = \{\gamma_{ij}\}$, then the change of coordinates is given by

$$\begin{aligned} z_j &= F_{ij}(z_i) + \gamma_{ij}\partial_{z_i}F_{ij}(z_i)\theta_i^1\theta_i^2 \\ \theta_j^1 &= \theta_i^1 a_{11}(z_i) + \theta_i^2 a_{12}(z_i) \\ \theta_j^2 &= \theta_i^1 a_{21}(z_i) + \theta_i^2 a_{22}(z_i). \end{aligned}$$

Here the covering $\{U_i\}$ and the change of coordinates $\{F_{ij}\}$ defines a curve C_0 and

$$A_{ij} = \begin{pmatrix} a_{11}(z_i) & a_{12}(z_i) \\ a_{21}(z_i) & a_{22}(z_i) \end{pmatrix}$$

defines a rank 2 bundle E over C_0 with $\det E \simeq \Omega_{C_0}$.

We can define the new curve $C' = C_0(E)$ that is actually an $S(2)$-super curve.

The sheaf $(\mathcal{O}_C)_0$ defined over C_0 is a sheaf of algebras that is isomorphic to $C_0(1)$, the first neighbourhood of the diagonal on $C_0 \times_S C_0$, if and only if $\{\gamma_{ij}\} \in H^1(C_0, \pi^*\mathcal{O}_S)$ vanishes. In general, each class $\Gamma \in H^1(C_0,\pi^*\mathcal{O}_S)$ defines a curve $C_\Gamma = (C_0,(\mathcal{O}_C)_0)$ by the diagram from (5.4). First take $\mathcal{L} = L\otimes\mathcal{O}_{C_0}$ and $\mathcal{B} = \alpha(\mathcal{O}_{C_0})$, with these we construct the sheaf

$$(\mathcal{O}_C)_0 := \frac{T^{\bullet\geq 1}\mathcal{L}}{\langle x\otimes y - y\otimes x, a\otimes a : x,y\in\mathcal{L}, a\in\mathcal{B}\rangle}.$$

Observe that this sheaf fits on the diagram of algebras:

$$0 \to \Omega_{C_0} \to (\mathcal{O}_C)_0 \to \mathcal{O}_{C_0} \to 0,$$

where the image of Ω_{C_0} have the zero multiplication. Since we consider terms of degree greater then zero, then $(\mathcal{O}_C)_0$ does not necessarily has the structure of O_{C_0} algebra.

Finally, we consider

$$(\mathcal{O}_C)_1 := \Pi E,$$

with multiplication given by the isomorphism

$$\det E \to \Omega_{C_0}.$$

So, we get the $S(2)$-super curve $(C, \mathcal{O}_C)$. Finally:

Proposition 6.2. *For any family over a pure even base $\pi_0 : C_0 \to S$ of relative dimension $1|0$, the following data are equivalent:*

1. *An $S(1|2)$-family of curves $\pi : C \to S$ over S with $C_{\mathrm{rd}} = C_{0,\mathrm{rd}}$ and $(\mathcal{O}_C)_0 = \mathcal{O}_{C_\Gamma}$.*
2. *A class $\Gamma \in H^1(C_0, \pi^*\mathcal{O}_S)$ and a rank $(2|0)$ bundle E with an isomorphism $\det E \stackrel{\beta}{\to} \Omega_{C_0/S}$ up to equivalence: a pair (E, β) is equivalent to (E', β') is there exists an isomorphism, such that the diagram*

$$\begin{array}{ccc} E & \longrightarrow & E' \\ & {\scriptstyle\beta}\searrow & \downarrow{\scriptstyle\beta'} \\ & & \Omega_{C_0/S} \end{array}$$

commutes.

Proof. The previous comment shows (2) $\to$ (1).

To see (1) $\to$ (2) we get the class $\Gamma \in H^1(C_0, \pi^*\mathcal{O}_S)$ by taking $\gamma_{ij} = \gamma(\Phi_{ij})$ as in (6.2), and considering the $S(2)$-super curve $C' = C(E)$, from the comment, the associated C' super curve verifies what we want. □

6.1.3 Example: The genus 1 curve

For the special case of genus 1 curves, we have that the even part is given by an ordinary curve E_0 and an element of $H^1(E_0, SL(2, \mathbb{C}))$.

Is known that the space of elliptic curves is given by a quotient of the upper half space $\mathbb{H}$ by the group $SL(2,\mathbb{Z})$, where any element $\tau \in \mathbb{H}$ defines a quotient of $\mathbb{C}$ by the action of the group $\mathbb{Z}_\tau = \{a + b\tau : a, b \in \mathbb{Z}\}$. Similarly, any element $(\tau, a) \in \mathbb{H} \times \mathfrak{sl}(2,\mathbb{C})$, the quotient of $\mathbb{C}^{1|2}$ by the action of $S(z|\theta^1,\theta^2) = (z+1|\theta^1,\theta^2)$ and $T(z|\theta^1,\theta^2) = (z+\tau|(\theta^1,\theta^2)\exp(2\pi i a))$.

A family of even deformation is given by $\mathbb{H} \times \mathfrak{sl}(2,\mathbb{C})$ considering the action of the following three groups: $SL(2,\mathbb{Z})$, $\mathbb{Z}^2$, $SL(2,\mathbb{C})$.

1. The group $SL(2,\mathbb{Z})$: Consider the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau, a) = \left(\frac{a\tau + b}{c\tau + d}, \frac{a}{c\tau + d} \right)$$

   and for the quotient, we have the isomorphism induced by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z|\theta^1,\theta^2) = \left(\frac{z}{c\tau + d} \middle| (\theta^1,\theta^2) \exp\left(-2\pi i z \frac{ca}{c\tau + d} \right) \right)$$

   and observe that such isomorphism preserves the Berezinian if and only if $c = 0$ and $d = 1$.

2. The group $\mathbb{Z}^2$: Consider the action

$$(m,n) \cdot (\tau, a) = (\tau, a + m\tau + n)$$

   and for the quotient, we have the isomorphism induced by

$$(m,n) \cdot (z|\theta^1,\theta^2) = (z|(\theta^1,\theta^2)\exp(2m\pi i z))$$

   and observe that such isomorphism preserves the Berezinian if and only if $m = 0$.

3. The group $SL(2,\mathbb{C})$: Consider the action

$$C \cdot (\tau, a) = (\tau, CaC^{-1})$$

   and for the quotient, we have the isomorphism induced by

$$C \cdot (z|\theta^1,\theta^2) = (z|(\theta^1,\theta^2)C)$$

   and observe that such isomorphism preserves the Berezinian.

Finally, if we consider the group $SL(2,\mathbb{Z}) \times \mathbb{Z}^2 \times SL(2,\mathbb{C})$ with the product

$$(A,\gamma,C)\cdot(A',\gamma',C') = (AA',\gamma'+\gamma A', CC')$$

we get the fine moduli space $\mathcal{M}_0 = \mathbb{H} \times \mathfrak{sl}(2,\mathbb{C}) \mathbin{/\!\!/} SL(2,\mathbb{Z}) \times \mathbb{Z}^2 \times SL(2,\mathbb{C})$ of the even families of $S(2)$-super curves.

Observation 6.1. The moduli space of oriented $SUSY_2$-super curves is given by (τ, a), where $a \in \mathfrak{sl}(2,\mathbb{C})$ is diagonal. In this case and coordinates given by $(z|\theta^1,\theta^2)$, the $SUSY$-structure is obtained by $\omega = dz + \theta^2 d\theta^1$.

6.2 The moduli space of curves with a trivial Berezinian

It was studied in 5 the moduli space of super curves with a fixed Berezinian. Also, a deformation is viewed as a deformation of a curve together with a deformation of a section $\Delta \in H^0(C, \mathrm{Ber}_{C/S})$. Observe that the same happens when we study the moduli space of $S(2)$-super curves.

The routine to study this is by the following recipe:

1. Suppose that the moduli is a super scheme SM_g.

2. The reduced space $(SM_g)_{\mathrm{rd}}$ represent the functor $SM_g|_{\mathrm{Sch}}$.

3. The odd part is given (locally) by the sheaf $(j^*\mathcal{T}_{SM_g})_1$. The deformations over the odd part are represented by families over the super scheme $Spec(k[\epsilon_0,\epsilon_1])$, where $\epsilon_0^2 = \epsilon_1^2 = \epsilon_0\epsilon_1 = 0$ and ϵ_j has parity j.

4. Locally, the scheme SM_g is given by $(SM_g)_{\mathrm{rd}}(E)$, with E a sheaf over $(SM_g)_{\mathrm{rd}}$.

Observation 6.2. The process described above is justified by the following construction: Suppose that we have a family $C \to S$ of $S(2)$ curves, then the diagram

$$\begin{array}{ccc} C_0 := X \times_{S_{\mathrm{rd}}} S & \longrightarrow & X \\ \downarrow & & \downarrow \\ S_{\mathrm{rd}} & \longrightarrow & S \end{array} \tag{6.1}$$

Since S_{rd} is a scheme and $C_0 \to S_{\mathrm{rd}}$ is family of $S(2)$ super curves, then this family is given by a morphism $\phi_0 : S_{\mathrm{rd}} \to (SM_g)_{\mathrm{rd}}$ and the pullback $\phi_0^*(\mathcal{SM}_g)_{\mathrm{rd}} \to S_{\mathrm{rd}}$. Since locally, S is a split scheme, then in an open neighbourhood $U \subset S_{\mathrm{rd}}$ there exists a fiber bundle E

such that, locally, $S|_U = U(E)$. Assuming that U is affine we will proof uniqueness on the extension of such family $C_0(U) \to U$.

Finally the family $\pi : X \to S$ is going to be described as the gluing of local open pieces $\{\pi^{-1}(U_i) \to U_i\}_i$ through morphisms $U_i \to SM_g$. Finally, the inner automorphism is going to give us an orbifold description of such object.

We first work on the moduli space of super curves with a fixed Berezinian, that is for a super scheme S and the functor $M_{S(1|2),g}$ the set $M_{S(1|2),g}(S)$ denotes the collection of $1|2$ dimensional families $C \to S$ of genus g super curves with a global non-vanishing section $\Delta \in H^0(C, \mathrm{Ber}_{C/S})$. This moduli was described by 5 as an orbifold. Here we give a local description.

6.2.1 The reduced space

It follows from proposition (6.1) that an even deformation of an $S(2)$-super curve is given by deformation of curves and a rank 2 bundle with determinant the canonical divisor. In this case, we have that such moduli space is parametrized by the space of curves, and a rank 2 fiber bundles E joint with an isomorphism $\det E \simeq \Omega_{C_0}$.

In order to do this, first consider the moduli space of curves joint with a rank 2 bundles $M_{g,2}$, over it consider the natural transformation

$$\begin{aligned} \det : M_{g,2} \to & Jac \\ (C_0, E) \to & (C, \det E). \end{aligned}$$

The preimage of pairs (C_0, Ω_{C_0}) representing genus g curves joint with the canonical bundle, represents the pairs (C_0, E), where C_0 is a curve and E is a rank 2 bundle joint with a isomorphism $\det E \xrightarrow{\sim} \Omega_{C_0}$.

Observation 6.3. For simplicity (and smoothness) we are going to consider only stable bundles. It was seen in [17] that this space is not smooth and is not endowed with a universal curve. This space has an stratification given by the subspaces

$$M_{g,\Omega,k} := \{E \in M_{g,\Omega} : \dim H^0(C_{\mathrm{rd}}, E) \geq k\}.$$

the singular locus is given by $M_{g,\Omega,k+1}$ and, in general, the dimension of $M_{g,\Omega,k}$ is $6g - 6 - \binom{k+1}{2}$, for $g \geq 2$. However, the dimension of an open part of this space is $6g-6$, when $g \geq 2$.

Observation 6.4. Suppose that we consider families $C \to S$ of genus g super curves with

$g \geq 2$ and S a scheme. If we have an automorphism of the family given by

$$\begin{array}{ccc} C & \xrightarrow{\Phi} & C \\ \downarrow & & \downarrow \\ S & \xrightarrow{\phi} & S \end{array}$$

such that they coincide over a point. Considering the reduced scheme $C_{\mathrm{rd}} \to S$, this is a family of genus g curves over S. Since this family does not have automorphism different from the identity, then $\phi = \mathrm{id}_S$. Recall that Φ is induced by an automorphism of a rank 2 bundle $\psi : E \to E$ that preserves the isomorphism $\det E \simeq \Omega_{C/S}$, that is $\psi \in SL(E)$.

That is, the set of automorphism preserving the family $C \to S$ of genus g super curves with $g \geq 2$ over a scheme S are given by isomorphisms of rank 2 bundles $\psi \in SL(E)$. In particular the functor representing families of $S(2)$ super curves over schemes is an orbifold.

6.2.2 Odd part

Let $\pi : C \to S$ be a family of $S(2)$-super curves with a fixed genus g, for a split supermanifold S, with reduced space S_{rd} and $\mathcal{O}_{S_{\mathrm{rd}}}$-free module W. Considering the closed point $s_0 \in S$ and the fiber $C_{s_0} \to \{s_0\}$, there exists a covering by affine open sets $\{U_i\}$ of C_{s_0} such that the family $C \to S$ restricts to $U_i \times S \to S$. This observation follows from [18, Theorem 1.2.4]. Then the global family is determined by the change of coordinates in $\Phi_{ij} \in \mathrm{Aut}((U_{ij}) \times S)$.

Considering the reduced family $C_0 \to S_{\mathrm{rd}}$ given by (6.1), fixing the reduced family, then the reduced part of the change of coordinates Φ_{ij} is fixed, denote it by $\phi_{ij} \in \mathrm{Aut}((U_{ij}) \times S)_{\mathrm{rd}}$. Then, the $\phi_{ij}^{-1}\Phi_{ij}$ is an automorphism being the identity over the reduced space, then this morphism is the exponential of a nilpotent vector field relative to S. To get a family of $S(2)$ curves we need the distribution:

$$\mathcal{D} := S(2) \cap \mathcal{T}_{C/S}$$

where $S(2)$ was described in (3.11), and the sheaf $\mathcal{N}$ of nilpotent elements of $\bigwedge^{\bullet} \pi^* W$. Then, $\phi_{ij}^{-1}\Phi_{ij} = \exp(X_{ij})$ for an even vector field $X_{ij} \in \mathcal{N} \otimes \mathcal{D}$. The group bundle

$$G = \exp((\mathcal{N} \otimes \mathcal{D})_0), \tag{6.2}$$

and the deformations are parametrized by $H^1(C_0, G)$.

Finally, we get

Proposition 6.3. *Let S be a split super scheme and a family $C \to S$ of $S(2)$-super curves, then this family correspond to a extension of $C_0 \to S_{\mathrm{rd}}$ and a class in $H^1(C_0, G)$, for G in (6.2).*

Observation 6.5. We are going to calculate the dimension of the odd part. Suppose that we take an $S(2)$-super curve over a point, given by a curve $C_{\rm rd}$ and a rank 2 bundle over $C_{\rm rd}$. Take the *fat point* $\mathrm{Spec}(k[\epsilon_1])$, where ϵ_1 is odd. Any extension is given by an element in $H^1(C_{\rm rd}, G)$, with G in (6.2). Any element in (6.2) is given by a class $\{\exp(X_{ij})\}$ with $X_{ij} \in (\mathcal{N} \otimes \mathcal{D})_0$. In this case the nilpotent part is described by:

$$\epsilon_1 A_{ij}, A_{ij} \in \mathcal{D}(U_{ij})$$

In this case, $\mathcal{D}$ is described as a sum of the distributions $E_1 = \langle \partial_{\theta^1}, \partial_{\theta^2} \rangle, E_2 = \langle \theta^1 \partial_z, \theta^2 \partial_z \rangle$ and observe that $E_1 \simeq E$ as bundles over $C_{\rm rd}$. Since E is stable, then the odd dimension is $\dim H^0(C_{\rm rd}, E)$ when $g \geq 2$. Doing the same calculation for $E_2 \simeq E^* \otimes \Omega_C$, we obtain that the dimension of the odd part is $2 \dim H^0(C_{\rm rd}, E)$. It was proved in Theorem 1.1 [19] that in general the dimension of this space is zero.

6.2.3 Inner Automorphism

Now, we will check which automorphisms preserves the family of curves. We are looking for maps $\psi : S \to S$ such that the families of $S(2)$-super curves $C \to S$, $\psi^* C \to S$ are equal, in this case we are going to say that $\psi : S \to S$ *preserves the family.* First consider the following lemma:

Lemma 6.1. *Let $\pi : C \to S$ be a family of curves, suppose that $\psi : S \to S$ preserves the family and $\psi_{\rm rd} : S_{\rm rd} \to S_{\rm rd}$ is the identity, then ψ is the identity.*

Proof. Since, we can cover S by open split super schemes $\{S_i\}_i$, then $\psi|_{C_i} : C_i \to S_i$, with $C_i = \pi^{-1}(S_i)$, is the identity over $S_{i,\rm rd}$. The family $C_i \to S_i$ is defined by $C_{i,0} \to S_{i,\rm rd}$ and a class $H^1(C_{i,0}, G_i)$. Since ψ define the same family, then both classes should coincide, then ψ should be also the identity. □

From the previous lemma, it follows that any automorphism of the family $X \to S$ is given by an automorphism over the reduced space $S_{\rm rd}$.

Let $\pi : C \to S$ be a family of $S(2)$-super curves, S is a scheme and E be the rank 2 bundle defining π. Considering the reduction $C_{\rm rd} \to S$, we obtain a family of genus g curves over S. From now on we consider only the case $g \geq 2$. There does not exists any automorphism different from the identity. Then, in order to study such automorphism we have to check what happens in the odd part generated by the bundle E, that is we have to study the automorphisms of the rank 2 bundle $E \to C_{\rm rd}$, with the chosen isomorphism $\det E \to C_{\rm rd}$. Considering the stable bundles, we have $\mathrm{Aut}_{C_{\rm rd}}(E) = (\pi^* \mathcal{O}_S)^*$, since we need that such automorphism preserve the isomorphism $\det E \to C_{\text{rd}}$, this group reduces to ± 1. The corresponding automorphism given by 1 is the identity, while the automorphism given by -1 is denoted by Ψ. Then we get the lemma:

Lemma 6.2. *Let $\pi : C \to S$ be a family of genus g $S(2)$-super curves, S is a scheme and E be the rank 2 stable bundle defining π. Then any automorphism $\psi : S \to S$ preserving the family corresponds to a class $\sigma_\psi \in H^1(S, \mathbb{Z}/2\mathbb{Z})$.*

Finally, we get

Proposition 6.4. *Let $\pi : C \to S$ be a family of genus g $S(2)$-super curves. Then any automorphism $\psi : S \to S$ preserving the family corresponds to a class $\sigma_\psi \in H^1(S_{\text{rd}}, \mathbb{Z}/2\mathbb{Z})$.*

From now on, let $M_{g,2,\omega}$ be the moduli space of genus g curves C joint with a rank 2 bundle $E \to C$ with a fixed isomorphism $\det E \simeq \Omega_C$. With this notation, the previous proposition could be rephrased as follows:

Theorem 6.1. *Let $C \to S$ be a family of $S(2)$ super curves, then from Proposition 6.1 define a map $h : S_{\text{rd}} \to M$ and from Proposition 6.3 we define a class $\tau \in H^1(C_0, G)$. The pair (h, τ) are unique up to the equivalences $(h, \tau) \sim (\widehat{h}, \widehat{\tau})$ if and only if $\widehat{h} = h \circ \psi$, $tau = \widehat{t}au$, for ψ as Proposition* (6.4).

Observation 6.6. From Observations 6.5 and 6.3 over the point (C_{rd}, E) the orbifold (in general) has dimension $6g - 6|0$, when $g \geq 2$.

6.3 Automorphisms over $S(2)$

6.3.1 Automorphisms on super manifolds

We start asking what kind of automorphisms are allowed between super manifolds. Let M be a super manifold with reduced space M_{rd}, then observe that any $\Phi \in \text{Aut}(M)$ induce by reduction a $\Phi_{\text{rd}} \in \text{Aut}(M_{\text{rd}})$, so we get a natural map extending (2.1):

$$\text{Aut}(M) \to \text{Aut}(M_{\text{rd}}) \tag{6.1}$$

Observation 6.7. More generally, let M, N be super manifolds. We saw a natural map $\text{Hom}_{\text{SSch}}(M, N) \to \text{Hom}_{\text{Sch}}(M_{\text{rd}}, N_{\text{rd}})$, there is no chance that this map is surjective. For example, take $N = M_{\text{rd}}$, then if there exists a morphism $\phi : M \to M_{\text{rd}}$ such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M_{\mathrm{rd}} \\ {\scriptstyle j}\Big\uparrow & & \Big\uparrow{\scriptstyle id} \\ M_{\mathrm{rd}} & \xrightarrow{id} & M_{\mathrm{rd}} \end{array}$$

That is equivalent to M being projected. Since this is not true for super manifolds, then, in general, the correspondence $\mathrm{Hom}_{\mathrm{SSch}}(M, N) \to \mathrm{Hom}_{\mathrm{Sch}}(M_{\mathrm{rd}}, N_{\mathrm{rd}})$ is not surjective.

When we restrict to automorphisms, we firstly ask if (6.1) is surjective. Let M be a split super manifold, associated to a manifold M_{rd} joint to a vector bundle E. Suppose that $\phi : M_{\mathrm{rd}} \to M_{\mathrm{rd}}$ is an automorphism, any extension $\Phi : M \to M$ is an isomorphism of vector bundles

$$E \to \phi^* E.$$

This is true when ϕ is homotopic to the identity. In general, we cannot assure that $E, \phi^* E$ are isomorphic.

Example 6.2. Let C be a torus and $\phi : C \to C$ the degree -1 map, that is, consider $\phi(P) = -P$ with respect to the abelian structure. Any point $P \in C$ defines a divisor $[P]$ over C, and the pullback is given by $\phi^*[P] = [-P]$. Then $D \sim \phi^* D$ if and only if $2D \sim 0$. Then, if a line bundle L is isomorphic to $\phi^* L$, then $L^2 \simeq \mathcal{O}_C$, that is, L is a spin structure over C.

On the other side, for any translation $t_P : C \to C, Q \mapsto Q + P$, we get that for any divisor D the corresponding pullback $t_P D = D + \deg(D)P$, then such divisors are equivalent if and only if $\deg(D) = 0$ or $P = 0$. In particular, for a line bundle L the line bundles $L, t_P^* L$ are isomorphic if and only if L has degree zero or $P = 0$.

Finally, we get a complete description of the image in (6.1) for a super manifold M with $M_{\mathrm{rd}} = C$ and associated to the line bundle L. If L is a spin structure, then any composition $\phi \circ t_P$, for ϕ a morphism of degree ± 1 and t_P is a translation, is in the image of (6.1). When L has degree zero but is not a spin structure, then just the translations t_P are in the image of (6.1). Finally, if $\deg L \neq 0$ then just the identity is in the image of (6.1).

We just see that (6.1) is not surjective, the next question is if this map is injective.

Example 6.3. (The canonical automorphism) Let M be a super scheme. From the decomposition $\mathcal{O}_M = (\mathcal{O}_M)_{\bar{0}} + (\mathcal{O}_M)_{\bar{1}}$ we can define the automorphism

$$\begin{aligned} \tau : (\mathcal{O}_M)_{\bar{0}} + (\mathcal{O}_M)_{\bar{1}} \to& (\mathcal{O}_M)_{\bar{0}} + (\mathcal{O}_M)_{\bar{1}} \\ a_0 + a_1 \mapsto& a_0 - a_1. \end{aligned}$$

This automorphism is the identity when we restrict to M_{rd}. Such automorphism τ is called *canonical automorphism*, and observe that $\tau^2 = \mathrm{id}_M$.

More generally, let $\Phi \in \mathrm{Aut}(M)$ with reduction $\phi = \mathrm{id}_{M_{\mathrm{rd}}}$, then for any $f \in \mathcal{O}_M(U)$ we get that $\Phi(f) = f \mod J$. In particular, Φ preserves the filtration

$$\mathcal{O}_M \supset J \supset J^2 \supset \dots .$$

From this, we obtain the isomorphism $\overline{\Phi} : J/J^2 \to J/J^2$ of vector bundles over M_{rd}. Also, the induced map for topological spaces is the identity and since for a covering of M_{rd} given by $\{U_i\}$ we get $\mathcal{O}_M(U_i) \simeq \mathcal{O}_{M_{\mathrm{rd}}}(J/J^2)(U_i)$ and we already know how this homomorphism behaves on J/J^2. Finally, to fully determine Φ is still miss a family of automorphism $\Phi_{ij} \in \mathcal{O}_M(U_{ij})$ that represents a class in $H^1(M_{\mathrm{rd}}, \mathrm{Aut}^{(2)}(\mathcal{O}_M))$, where $\mathrm{Aut}^{(2)}(\mathcal{O}_M)$ is the sheaf of automorphism ϕ with $\phi(f) = f \mod J^2$.

Observation 6.8. Let $U \subset M_{\mathrm{rd}}$ be an open set and $\phi \in \mathrm{Aut}^{(2)}(\mathcal{O}_M)(U)$, then such automorphism is given by the exponential of a nilpotent vector field X. We could suppose that there exists a distribution $\mathcal{D} \subset \mathcal{T}_M$ that vector fields corresponds with elements in $\mathrm{Aut}^{(2)}(\mathcal{O}_M)$, but this is false, since the exponential map $\exp : \mathcal{D} \to \mathrm{Aut}^{(2)}(\mathcal{O}_M)$ is not linear.

To finish the section we mention how to understand the automorphism over a super manifold. Firstly, we have to obtain the image of (6.1), that is all the automorphism over M_{rd} that extends to M. Secondly, we check when $\Phi \in \mathrm{Aut}(M)$ with $\Phi_{\mathrm{rd}} = \mathrm{id}$. In order to do this, we just classify this by a subset in $\mathrm{Aut}_{M_{\mathrm{rd}}}(J/J^2) \ltimes H^1(M_{\mathrm{rd}}, \mathrm{Aut}^{(2)}(\mathcal{O}_M))$.

Example 6.4. Let M be an $n|1$-dimensional super manifold, then $\mathcal{O}_M \simeq \mathcal{O}_{M_{\mathrm{rd}}} \oplus L$ for some line bundle L over M_{rd}. Also, observe that $\mathrm{Aut}^{(2)}(\mathcal{O}_M)$ is a trivial group, because $L^2 = 0$. Finally, we get that any map $\Phi \in \mathrm{Aut}(M)$ is represented by a pair (ϕ, ψ), where $\phi \in \mathrm{Aut}(M_{\mathrm{rd}})$ and $\psi : L \to \phi^* L$ is an isomorphism of vector bundles over M_{rd}.

In particular, following Example 6.2, we get a complete description of the automorphisms over M a $1|1$ super manifold with reduced space a torus.

6.3.2 Automorphisms on the reduced space

We are interested on natural automorphism $\mathcal{F}$ of the functor $S(2)_g : \mathrm{SSch} \to \mathrm{Sets}$. Observe that such natural automorphism induce two reductions: $\mathcal{F}|_{\mathrm{Sch}}$ is the natural transformations defined in $S(2)_g|_{\mathrm{Sch}}$ corresponding to

$$\{C \to S\} \mapsto \{\mathcal{F}(C) \times_{\mathcal{F}(S)} \mathcal{F}(S)_{\mathrm{rd}} \to \mathcal{F}(S)_{\mathrm{rd}}\}.$$

The second one is defined as a natural transformation $\mathcal{F}_{\text{rd}}$ given by

$$\{C_{\text{rd}} \to S\} \mapsto \{\mathcal{F}(C)_{\text{rd}} \to \mathcal{F}(S)_{\text{rd}}\}.$$

Such projection corresponds to the process of having a family of $S(2)$ super curves $C \to S$ and construct the family of curves $C_{\text{rd}} \to S_{\text{rd}}$. Then, any automorphism on $S(2)_g$ will induce an automorphism over M_g. Since $g \geq 2$, following [20] Proposition 3.5, such projection is just the identity.

For the first one, once we define the automorphism over M_g, we note that the automorphism $\mathcal{F}|_{\text{Sch}}$ is going to define an automorphism over each fiber of $S(2)_g|_{\text{Sch}} \to M_g$. Since for each curve C the fiber is given by $M_C(2, \Omega)$. Now, in [21] Theorem 5.3 is proven that if $g \geq 4$ then any automorphism of such space is given by a line bundle L with $L^{\otimes 2} = \mathcal{O}_C$ or $L^{\otimes 2} = \omega_C^{\otimes 2}$. In order to define $\mathcal{F}|_{\text{Sch}}$ we need to choose a line bundle L_C for any $C \in M_g$. It was proven in [22] Theorem 2 that if $g \geq 3$ then any section of the Picard bundle $P_{g,d} \to M_g$ has the form $C \mapsto \Omega_C^{\otimes(d/2g-2)}$. Then, we can only take $C \mapsto \mathcal{O}_C$ or $C \mapsto \Omega_C$. These sections correspond to the identity and the automorphism given by to $\{C(E) \to S\} \mapsto \{C(E^* \otimes \Omega_C) \to S\}$, respectively.

Finally we get that for $g \geq 4$ the only automorphisms in $S(2)_g|_{\text{Sch}}$ are the identity and $\{C(E) \to S\} \mapsto \{C(E^* \otimes \Omega_C) \to S\}$. The second automorphism is going to be denoted by σ and we get

Proposition 6.5. *Let $g \geq 4$, then the unique automorphism on $S(2)_g|_{\text{Sch}}$ are the identity and $\sigma : C(E) \mapsto C(E^* \otimes \Omega_C)$.*

Now, let C be an split $S(2)$-super curve over a point, defined by its reduction and the rank two bundle E. Since, for any rank two bundle over a curve C_{rd} we have the natural isomorphism of bundles:

$$\begin{aligned} E^* \otimes \det E &\to E \\ \alpha \otimes u \wedge v &\mapsto i_\alpha(u \wedge v) = \alpha(u)v - \alpha(v)u \end{aligned}$$

we obtain that the automorphism σ is the identity on Sch, since $\det E = \Omega_{C_{\text{rd}}}$. Nevertheless, we will see later an example (Example 6.5) where the automorphism is not the identity.

Observation 6.9. If we repeat the same argument to the functor $S(1|2)$ we obtain that any automorphism is given by id or σ and an automorphism of the bundle $\mathcal{L}$.

6.3.3 The Automorphism μ

Recall the involution given in Theorem 5.1. Let C be an $S(2)$ super curve and consider local coordinates $(z|\theta^1, \theta^2)$, the automorphism μ induces the coordinates on $\widehat{C}$ by the rule:

$$\begin{aligned}\widehat{z} &= z - \theta^1\rho^1 - \theta^2\rho^2\\ \widehat{\rho}^1 &= \rho^1\\ \widehat{\rho}^2 &= \rho^2.\end{aligned}$$

Since, C is split and described by the reduced curve C_{rd} with a rank two bundle E over C_{rd}, then for a change of coordinates $\Phi(z|\theta^1, \theta^2) = (f(z)|\phi^1(z|\theta^1, \theta^2), \phi^2(z|\theta^1, \theta^2))$ we obtain that the change of coordinates over $\widehat{C}$ given by $\mathrm{Ad}_\alpha(\Phi)$. To obtain an explicit expression, first we will see what happens in the inclusion $\mathrm{Aut}[[1|2]] \hookrightarrow \mathrm{Aut}^\omega[[1|4]]$. Considering the coordinates $\Phi = (f|\phi^1, \phi^2)$, with $f(z|\theta^1, \theta^2) = f(z)$, the construction given in (2.7) is given by

$$\begin{aligned}\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix} &= \begin{pmatrix}D_1\phi^1 & D_1\phi^2\\ D_2\phi^1 & D_2\phi^2\end{pmatrix}^{-1}\begin{pmatrix}D_1 f\\ D_2 f\end{pmatrix}\\ &= \begin{pmatrix}D_1\phi^1 & D_1\phi^2\\ D_2\phi^1 & D_2\phi^2\end{pmatrix}^{-1}\begin{pmatrix}\rho^1\partial_z f\\ \rho^2\partial_z f\end{pmatrix}.\end{aligned}$$

Here $\phi^i(z|\theta^1, \theta^2) = \theta^1 a_{i1} + \theta^2 a_{i1}$, for $i = 1, 2$, that means

$$[\phi^1 \quad \phi^2] = [\theta^1 \quad \theta^2]A,$$

where $A = (a_{ij})$. Observe that $\det A = \partial_z f$.

Now, to calculate $\mu(\Phi)$, first we have to get the expression:

$$
\begin{aligned}
f - [\phi^1 \quad \phi^2]\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix} &= f - [\theta^1 \quad \theta^2] A \begin{pmatrix} D_1\phi^1 & D_1\phi^2 \\ D_2\phi^1 & D_2\phi^2 \end{pmatrix}^{-1} \begin{pmatrix}\rho^1\partial_z f\\ \rho^2\partial_z f\end{pmatrix} \\
&= f - [\theta^1 \quad \theta^2] \left(\begin{pmatrix} D_1\phi^1 & D_1\phi^2 \\ D_2\phi^1 & D_2\phi^2 \end{pmatrix} A^{-1}\right)^{-1} \begin{pmatrix}\rho^1\partial_z f\\ \rho^2\partial_z f\end{pmatrix} \\
&= f - [\theta^1 \quad \theta^2] \left(\mathrm{id} + \begin{pmatrix}\rho^1\\ \rho^2\end{pmatrix}\begin{pmatrix}\partial_z\phi^1 & \partial_z\phi^2\end{pmatrix} A^{-1}\right)^{-1} \begin{pmatrix}\rho^1\partial_z f\\ \rho^2\partial_z f\end{pmatrix} \\
&= f - [\theta^1 \quad \theta^2]\begin{pmatrix}\rho^1\partial_z f\\ \rho^2\partial_z f\end{pmatrix} + [\theta^1 \quad \theta^2]\begin{pmatrix}\rho^1\\ \rho^2\end{pmatrix}\begin{pmatrix}\partial_z\phi^1 & \partial_z\phi^2\end{pmatrix} A^{-1}\begin{pmatrix}\rho^1\partial_z f\\ \rho^2\partial_z f\end{pmatrix} \\
&= f - (\theta^1\rho^1 + \theta^2\rho^2)\partial_z f - \rho^1\rho^2\partial_z(\phi^1\phi^2) \\
&= f - (\theta^1\rho^1 + \theta^2\rho^2)\partial_z f - \theta^1\theta^2\rho^1\rho^2\partial_z^2 f \\
&= f(z - \theta^1\rho^1 - \theta^2\rho^2).
\end{aligned}
$$

Similarly, we can calculate $\eta^i(\widehat{z}|\widehat{\rho}^1, \widehat{\rho}^2)$ and we obtain that

$$
\begin{aligned}
\widetilde{f} =& f \\
[\widetilde{\eta}^1 \quad \widetilde{\eta}^2] =& A^t[\widehat{\rho}^1 \quad \widehat{\rho}^2],
\end{aligned}
$$

in $(\widehat{z}|\widehat{\rho}^1, \widehat{\rho}^2)$ coordinates.

Then, we obtain that such coordinates corresponds to a the same reduced curve C_{rd} and bundle $E^* \otimes \Omega_{C_{\mathrm{rd}}}$. Finally, we obtain:

Proposition 6.6. *For a split $S(2)$ curve $C \to S$ associated to C_{rd} and the vector bundle E, the dual curve $\widehat{C}$ is also split and is associated to C_{rd} and the vector bundle $E^* \otimes \Omega_{C_{\mathrm{rd}}}$.*

In particular, the involution $C \mapsto \widehat{C}$ is explictly the automorphism σ over Sch. Finally, we get:

Proposition 6.7. *Let $g \geq 4$, then the unique automorphisms over $S(2)_g$ are the identity and μ, up to an affine transformation and a unipotent morphism.*

Example 6.5. We are going to see that μ is not the identity.

Let C be the $S(2)$-super curve described on Example 5.3, observe that the dual curve $\widehat{C}$ is given by the quotient of the plane by the relations:

1. $\widehat{S}(w|\rho^1, \rho^2) = (w + 1|\rho^1, \rho^2)$.
2. $\widehat{T}(w|\rho^1, \rho^2) = (w + \tau|\rho^1 - \rho, \rho^2)$.

Observe that the tangent bundle $\mathcal{T}_C$ has a rank $0|2$ distribution given by the global sections $\langle \partial_{\theta^1}, \partial_{\theta^2} \rangle$, but does not have a $1|0$ global section that over the reduction C_{rd} generates the tangent space. On the other side, over $\widehat{C}$, the global section ∂_w generates the tangent space $\mathcal{T}_{C_{\mathrm{rd}}}$. That is, C and $\widehat{C}$ are not isomorphic.

Also, observe that $SUSY_2$-super curves are fixed points of this involution. Finally, we resume this as follows:

Theorem 6.2. *The fixed points of the involution μ consists on the $SUSY_2$-super curves and the reduced space $S(2)_{g,\mathrm{rd}}$.*

Appendix A

An explicit calculation

Let us recall that $R[[1|2]] = R[[z|\theta^1, \theta^2]]$ and fix the differentials $D^i = \theta^i \partial_z + \partial_{\theta^i}$.

Definition A.1. Define the operator γ as follows:

$$\begin{aligned} \gamma : \mathrm{Aut}(R[[1|2]]) &\rightarrow R[[1|2]] \\ \Phi = (F|\phi^1, \phi^2) &\mapsto \gamma(\Phi) := \mathrm{Ber} \begin{pmatrix} D^1 D^2 F & D^1 D^2 \phi^1 & D^1 D^2 \phi^2 \\ D^1 F & D^1 \phi^1 & D^1 \phi^2 \\ D^2 F & D^2 \phi^1 & D^2 \phi^2 \end{pmatrix} (z|\theta^1, \theta^2). \end{aligned}$$

Since $\Phi = (F|\phi^1, \phi^2) \in \mathrm{Aut}(R[[1|2]])$ and the matrix $(D^i \phi^j)$ is invertible, then $\gamma(\Phi)$ is well defined.

Proposition A.1. *For the restriction:*

$$\gamma : \mathrm{Aut}^\delta(R[[1|2]]) \rightarrow R[[1|2]]$$

we have that $\ker(\gamma|_{\mathrm{Aut}^\delta(R[[1|2]])}) = \mathrm{Aut}^\Delta(R[[1|2]])$.

Proof. We divide the proof into three steps:

1. For any $\Phi = (F|\phi^1, \phi^2) \in \mathrm{Aut}^\delta(R[[1|2]])$

$$\gamma(\Phi) = \mathrm{Ber} \begin{pmatrix} \partial_1 \partial_2 F & \partial_1 \partial_2 \phi^1 & \partial_1 \partial_2 \phi^2 \\ \partial_1 F & \partial_1 \phi^1 & \partial_1 \phi^2 \\ \partial_2 F & \partial_2 \phi^1 & \partial_2 \phi^2 \end{pmatrix}, \tag{A.1}$$

in particular, $\gamma(T) = 0$ for any $T \in \mathrm{Aut}_0^\delta(R[[1|2]])$.

2. For any $\Phi, \Psi \in \mathrm{Aut}^{\delta}(R[[1|2]])$

$$\gamma(\Phi * \Psi) = \gamma(\Phi) + \gamma(\Psi)\left(\Phi(z|\theta^1, \theta^2)\right), \tag{A.2}$$

where $\Phi(z|\theta^1, \theta^2) = (F(z|\theta^1, \theta^2)|\phi^1(z|\theta^1, \theta^2), \phi^2(z|\theta^1, \theta^2))$.

3. Let $X \in S(1|2)$ and $\tau \in \mathbb{C}$ put $\Phi_\tau = \exp(\tau X)$, then

$$\left.\frac{d\gamma(\Phi_\tau)}{d\tau}\right|_{\tau=0} = \partial_1\partial_2(X \cdot z), \tag{A.3}$$

in particular, $\left.\frac{d\gamma(\Phi_\tau)}{d\tau}\right|_{\tau=0} = 0$ if and only if $X \in S(2)$.

Let us show that the Proposition follows from these three statements.

For a change of coordinates $\Phi = (F|\phi^1, \phi^2) = \exp(X) \circ T$, where $X \in S(1|2)_+$ and $T \in \mathrm{Aut}_0^{\delta}(R[[1|2]])$. From (A.2) we get that $\gamma(\Phi) = \gamma(\exp(X))(T)$.

Let $\Phi_\tau = \exp(\tau X)$, for $X \in S(1|2)_+$ and $\tau \in \mathbb{C}$, we have

$$\Phi_{\tau+\sigma} = \Phi_\tau \circ \Phi_\sigma = \Phi_\sigma \circ \Phi_\tau.$$

By (A.2) $\gamma(\Phi_{\tau+\sigma}) = \gamma(\Phi_\sigma) + \gamma(\Phi_\tau)(\Phi_\sigma)$. Taking $\left.\frac{d\gamma}{d\tau}\right|_{\tau=0}$ we get:

$$\left.\frac{d\gamma(\Phi_\tau)}{d\tau}\right|_{\tau=\sigma}(z|\theta^1, \theta^2) = \left.\frac{d\gamma(\Phi_\tau)}{d\tau}\right|_{\tau=0}(\Phi_\sigma(z|\theta^1, \theta^2)). \tag{A.4}$$

Observe that $\partial_1\partial_2(X \cdot z)$ is constant. Indeed $\partial_i\partial_1\partial_2(X \cdot z) = 0$ for $i = 1, 2$, and for ∂_z we write $X = A_0\partial_z + A_1\partial_1 + A_2\partial_2$

$$\begin{aligned}
\partial_z\partial_1\partial_2(X \cdot z) =& \partial_1\partial_2\partial_z(X \cdot z) \\
=& \partial_1\partial_2(\partial_z A_0) \\
=& \partial_1\partial_2\left(-(-1)^{A_1}\partial_1 A_1 - (-1)^{A_2}\partial_2 A_2\right) \\
=& 0.
\end{aligned} \tag{A.5}$$

Therefore, from (A.4) and (A.5), then we have $\gamma(\Phi_\tau) = \tau\partial_1\partial_2(X \cdot z)$, so $X \in S(2)$ if and only if $\gamma(\Phi_1) = 0$. So the assertion follows.

In order to prove 1, we fix $A = (\partial_i\phi^j)_{1\le i,j\le 2}$ and $B = (D^i\phi^j)_{1\le i,j\le 2}$. Then $B = A +$

$(\theta^i\partial_z\phi^j)$. From Lemma 2.2 B is invertible and expanding in the geometric series we obtain:

$$\begin{aligned} B^{-1} =& A^{-1} - A^{-1}\begin{pmatrix}\theta^1\partial_z\phi^1 & \theta^1\partial_z\phi^2\\ \theta^2\partial_z\phi^1 & \theta^2\partial_z\phi^2\end{pmatrix}A^{-1}\\ &+ A^{-1}\begin{pmatrix}\theta^1\partial_z\phi^1 & \theta^1\partial_z\phi^2\\ \theta^2\partial_z\phi^1 & \theta^2\partial_z\phi^2\end{pmatrix}A^{-1}\begin{pmatrix}\theta^1\partial_z\phi^1 & \theta^1\partial_z\phi^2\\ \theta^2\partial_z\phi^1 & \theta^2\partial_z\phi^2\end{pmatrix}A^{-1}\end{aligned}$$

and

$$\begin{aligned} \det B =& \det A\Bigg\{1 + \mathrm{tr}\left(A^{-1}\begin{pmatrix}\theta^1\partial_z\phi^1 & \theta^1\partial_z\phi^2\\ \theta^2\partial_z\phi^1 & \theta^2\partial_z\phi^2\end{pmatrix}\right)\\ &+ \det\left(A^{-1}\begin{pmatrix}\theta^1\partial_z\phi^1 & \theta^1\partial_z\phi^2\\ \theta^2\partial_z\phi^1 & \theta^2\partial_z\phi^2\end{pmatrix}\right)\Bigg\}\\ =& \det A\left\{1 - \begin{pmatrix}\partial_z\phi^1 & \partial_z\phi^2\end{pmatrix}A^{-1}\begin{pmatrix}\theta^1\\ \theta^2\end{pmatrix} - 2\theta^1\theta^2\partial_z\phi^1\partial_z\phi^2\det A^{-1}\right\}\end{aligned}$$

Let $\widetilde{\gamma}(\Phi)$ the right hand side of (A.1). Let $a := \gamma(\Phi)\det B$, we have

$$a = \theta^1\theta^2 A_{12} + \theta^2 A_2 + \theta^1 A_1 + A_0,$$

where

$$\begin{aligned} A_{12} &= \partial_z^2 F - \begin{pmatrix}\partial_z^2\phi^1 & \partial_z^2\phi^2\end{pmatrix}A^{-1}\begin{pmatrix}\partial_1 F\\ \partial_2 F\end{pmatrix}\\ A_2 &= -\partial_1\partial_z F + \begin{pmatrix}\partial_1\partial_z\phi^1 & \partial_1\partial_z\phi^2\end{pmatrix}\left\{A^{-1} - A^{-1}\begin{pmatrix}\theta^1\partial_z\phi^1 & \theta^1\partial_z\phi^2\\ 0 & 0\end{pmatrix}A^{-1}\right\}\begin{pmatrix}\partial_1 F + \theta^1\partial_z F\\ \partial_2 F\end{pmatrix}\\ A_1 &= \partial_2\partial_z F - \begin{pmatrix}\partial_2\partial_z\phi^1 & \partial_2\partial_z\phi^2\end{pmatrix}\left\{A^{-1} - A^{-1}\begin{pmatrix}0 & 0\\ \theta^2\partial_z\phi^1 & \theta^2\partial_z\phi^2\end{pmatrix}A^{-1}\right\}\begin{pmatrix}\partial_1 F\\ \partial_2 F + \theta^2\partial_z F\end{pmatrix}\\ A_0 &= \partial_1\partial_2 F - \begin{pmatrix}\partial_1\partial_2\phi^1 & \partial_1\partial_2\phi^2\end{pmatrix}\Bigg\{A^{-1} - A^{-1}\begin{pmatrix}\theta^1\partial_z\phi^1 & \theta^1\partial_z\phi^1\\ \theta^1\partial_z\phi^1 & \theta^1\partial_z\phi^1\end{pmatrix}A^{-1}\\ &\quad + A^{-1}\begin{pmatrix}\theta^1\partial_z\phi^1 & \theta^1\partial_z\phi^1\\ \theta^1\partial_z\phi^1 & \theta^1\partial_z\phi^1\end{pmatrix}A^{-1}\begin{pmatrix}\theta^1\partial_z\phi^1 & \theta^1\partial_z\phi^1\\ \theta^1\partial_z\phi^1 & \theta^1\partial_z\phi^1\end{pmatrix}A^{-1}\Bigg\}\begin{pmatrix}D^1 F\\ D^2 F\end{pmatrix}\end{aligned}$$

Since $\mathrm{Ber}\Phi = 1$, expanding we obtain

$$\partial_z F = \begin{pmatrix}\partial_z\phi^1 & \partial_z\phi^2\end{pmatrix}A^{-1}\begin{pmatrix}\partial_1 F\\ \partial_2 F\end{pmatrix} + \det A \tag{A.6}$$

differentiating (A.6) we get:

$$\partial_1\partial_z F = \begin{pmatrix} \partial_1\partial_z\phi^1 & \partial_1\partial_z\phi^2 \end{pmatrix} A^{-1} \begin{pmatrix} \partial_1 F \\ \partial_2 F \end{pmatrix} - \begin{pmatrix} \partial_z\phi^1 & \partial_z\phi^2 \end{pmatrix} A^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \widetilde{\gamma}(\Phi) \det A + \partial_1(\det A)$$

$$\partial_2\partial_z F = \begin{pmatrix} \partial_2\partial_z\phi^1 & \partial_2\partial_z\phi^2 \end{pmatrix} A^{-1} \begin{pmatrix} \partial_1 F \\ \partial_2 F \end{pmatrix} + \begin{pmatrix} \partial_z\phi^1 & \partial_z\phi^2 \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \widetilde{\gamma}(\Phi) \det A + \partial_2(\det A)$$

$$\partial_z^2 F = \begin{pmatrix} \partial_z^2\phi^1 & \partial_z^2\phi^2 \end{pmatrix} A^{-1} \begin{pmatrix} \partial_1 F \\ \partial_2 F \end{pmatrix} - 2\partial_z\phi^1\partial_z\phi^2\widetilde{\gamma}(\Phi) + \begin{pmatrix} \partial_z\phi^1 & \partial_z\phi^2 \end{pmatrix} \begin{pmatrix} \partial_1\partial_2\phi^2 \\ -\partial_1\partial_2\phi^1 \end{pmatrix} + \partial_z(\det A)$$

Replacing we get

$$\begin{aligned} a =& \theta^1\theta^2 \Big\{ -2\partial_z\phi^1\partial_z\phi^2\widetilde{\gamma}(\Phi) + \begin{pmatrix} \partial_z\phi^1 & \partial_z\phi^2 \end{pmatrix} \begin{pmatrix} \partial_1\partial_2\phi^2 \\ -\partial_1\partial_2\phi^1 \end{pmatrix} + \partial_z(\det A) \Big\} \\ & - \theta^2 \Big\{ - \begin{pmatrix} \partial_z\phi^1 & \partial_z\phi^2 \end{pmatrix} A^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \widetilde{\gamma}(\Phi) \det A + \partial_1(\det A) \\ & \qquad - \begin{pmatrix} \partial_1\partial_z\phi^1 & \partial_1\partial_z\phi^2 \end{pmatrix} A^{-1} \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} \det A \Big\} \\ & + \theta^1 \Big\{ \begin{pmatrix} \partial_z\phi^1 & \partial_z\phi^2 \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \widetilde{\gamma}(\Phi) \det A + \partial_2(\det A) \\ & \qquad - \begin{pmatrix} \partial_2\partial_z\phi^1 & \partial_2\partial_z\phi^2 \end{pmatrix} A^{-1} \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} \det A \Big\} \\ & + \widetilde{\gamma}(\Phi) \det A + \begin{pmatrix} \partial_1\partial_2\phi^1 & \partial_1\partial_2\phi^2 \end{pmatrix} \begin{pmatrix} -\partial_z\phi^2 \\ \partial_z\phi^1 \end{pmatrix} \theta^1\theta^2 \\ =& \widetilde{\gamma}(\Phi) \det A \left\{ 1 - \begin{pmatrix} \partial_z\phi^1 & \partial_z\phi^2 \end{pmatrix} A^{-1} \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} - 2\partial_z\phi^1\partial_z\phi^2\theta^1\theta^2 \det A^{-1} \right\} \end{aligned}$$

so we obtain, $\gamma(\Phi) \det B = \widetilde{\gamma}(\Phi) \det B$ and (A.1) follows.

For the second statement, and a general derivation $D = A_0\partial_z + A_1\partial_{\theta^1} + A_2\partial_{\theta^2}$, a function $F(z|\theta^1, \theta^2)$ and a change of coordinates $\Psi = (G|\psi^1, \psi^2)$ we get

$$\begin{aligned} D(F(\Psi))(z|\theta^1, \theta^2) =& DG(z|\theta^1, \theta^2)\partial_z F(\Psi(z|\theta^1, \theta^2)) \\ & + D\psi^1(z|\theta^1, \theta^2)\partial_1 F(\Psi(z|\theta^1, \theta^2)) \\ & + D\psi^2(z|\theta^1, \theta^2)\partial_2 F(\Psi(z|\theta^1, \theta^2)) \end{aligned}$$

so we obtain,

$$\begin{aligned}\partial_1\partial_2(F(\Psi))(z|\theta^1,\theta^2) = \partial_1\partial_2 G(z|\theta^1,\theta^2)\partial_z F(\Psi(z|\theta^1,\theta^2))\\ + \partial_1\partial_2\psi^1(z|\theta^1,\theta^2)\partial_1 F(\Psi(z|\theta^1,\theta^2))\\ + \partial_1\partial_2\psi^2(z|\theta^1,\theta^2)\partial_2 F(\Psi(z|\theta^1,\theta^2))\\ + \partial_1 G\partial_2 G(z|\theta^1,\theta^2)\partial_z^2 F(\Psi(z|\theta^1,\theta^2))\\ + (\partial_1 G\partial_2\psi^1 - \partial_2 G\partial_1\psi^1)(z|\theta^1,\theta^2)\partial_1\partial_z F(\Psi(z|\theta^1,\theta^2))\\ + (\partial_1 G\partial_2\psi^2 - \partial_2 G\partial_1\psi^2)(z|\theta^1,\theta^2)\partial_2\partial_z F(\Psi(z|\theta^1,\theta^2))\\ + (\partial_1\psi^1\partial_2\psi^2 - \partial_2\psi^1\partial_1\psi^2)(z|\theta^1,\theta^2)\partial_1\partial_2 F(\Psi(z|\theta^1,\theta^2))\end{aligned}$$

joining terms, we get

$$\gamma(\Phi\circ\Psi) = \gamma(\Psi)\mathrm{Ber}(\Phi)(\Psi) + a_0 + a_1 + a_2 + a_3, \tag{A.7}$$

where

$$\begin{aligned}
a_0 &= \partial_1 G\partial_2 G\,\mathrm{Ber}\begin{pmatrix}\partial_z^2 F(\Psi) & \partial_z^2\phi^1(\Psi) & \partial_z^2\phi^2(\Psi)\\ \partial_1(F(\Psi)) & \partial_1(\phi^1(\Psi)) & \partial_1(\phi^2(\Psi))\\ \partial_2(F(\Psi)) & \partial_2(\phi^1(\Psi)) & \partial_2(\phi^2(\Psi))\end{pmatrix},\\
a_1 &= (\partial_1 G\partial_2\psi^1 - \partial_2 G\partial_1\psi^1)\mathrm{Ber}\begin{pmatrix}\partial_1\partial_z F(\Psi) & \partial_1\partial_z\phi^1(\Psi) & \partial_1\partial_z\phi^2(\Psi)\\ \partial_1(F(\Psi)) & \partial_1(\phi^1(\Psi)) & \partial_1(\phi^2(\Psi))\\ \partial_2(F(\Psi)) & \partial_2(\phi^1(\Psi)) & \partial_2(\phi^2(\Psi))\end{pmatrix},\\
a_2 &= (\partial_1 G\partial_2\psi^2 - \partial_2 G\partial_1\psi^2)\mathrm{Ber}\begin{pmatrix}\partial_2\partial_z F(\Psi) & \partial_2\partial_z\phi^1(\Psi) & \partial_2\partial_z\phi^2(\Psi)\\ \partial_1(F(\Psi)) & \partial_1(\phi^1(\Psi)) & \partial_1(\phi^2(\Psi))\\ \partial_2(F(\Psi)) & \partial_2(\phi^1(\Psi)) & \partial_2(\phi^2(\Psi))\end{pmatrix},\\
a_3 &= (\partial_1\psi^1\partial_2\psi^2 - \partial_2\psi^1\partial_1\psi^2)\mathrm{Ber}\begin{pmatrix}\partial_1\partial_2 F(\Psi) & \partial_1\partial_2\phi^1(\Psi) & \partial_1\partial_2\phi^2(\Psi)\\ \partial_1(F(\Psi)) & \partial_1(\phi^1(\Psi)) & \partial_1(\phi^2(\Psi))\\ \partial_2(F(\Psi)) & \partial_2(\phi^1(\Psi)) & \partial_2(\phi^2(\Psi))\end{pmatrix}.
\end{aligned}$$

We have

$$\begin{pmatrix}\partial_1(F(\Psi))\\ \partial_2(F(\Psi))\end{pmatrix} = \partial_z F(\Psi)\begin{pmatrix}\partial_1 G\\ \partial_2 G\end{pmatrix} + A\begin{pmatrix}\partial_1 F\\ \partial_2 F\end{pmatrix}(\Psi)$$

where $A = (\partial_i \psi^j)$. Writing $B = (\partial_i \phi^j)$, then

$$\begin{pmatrix} \partial_1(\phi^1(\Psi)) & \partial_1(\phi^2(\Psi)) \\ \partial_2(\phi^1(\Psi)) & \partial_2(\phi^2(\Psi)) \end{pmatrix} = AB(\Psi) + \begin{pmatrix} \partial_1 G \\ \partial_2 G \end{pmatrix} \begin{pmatrix} \partial_z \phi^1 & \partial_z \phi^2 \end{pmatrix} (\Psi).$$

After, we have

$$\begin{aligned} \begin{pmatrix} \partial_1(\phi^1(\Psi)) & \partial_1(\phi^2(\Psi)) \\ \partial_2(\phi^1(\Psi)) & \partial_2(\phi^2(\Psi)) \end{pmatrix}^{-1} \begin{pmatrix} \partial_1(F(\Psi)) \\ \partial_2(F(\Psi)) \end{pmatrix} =& B^{-1} \begin{pmatrix} \partial_1 F \\ \partial_2 F \end{pmatrix} (\Psi) \\ &+ \operatorname{Ber}(\Phi) \det B \{AB\}^{-1} \begin{pmatrix} \partial_1 G \\ \partial_2 G \end{pmatrix} \\ &- \operatorname{Ber}(\Phi) \det A^{-1} \partial_1 G \partial_2 G \begin{pmatrix} -\partial_z \phi^2 \\ \partial_z \phi^1 \end{pmatrix} (\Psi) \end{aligned}$$

and

$$\begin{aligned} m =& \det \begin{pmatrix} \partial_1(\phi^1(\Psi)) & \partial_1(\phi^2(\Psi)) \\ \partial_2(\phi^1(\Psi)) & \partial_2(\phi^2(\Psi)) \end{pmatrix} \\ =& \det AB \Big\{ 1 - \begin{pmatrix} \partial_z \phi^1 & \partial_z \phi^2 \end{pmatrix} (\Psi) \{AB\}^{-1} \begin{pmatrix} \partial_1 G \\ \partial_2 G \end{pmatrix} \\ &- 2 \left(\det AB\right)^{-1} \partial_1 G \partial_2 G \partial_z \phi^1 \partial_z \phi^2 (\Psi) \Big\} \end{aligned}$$

Expanding the second term on the right hand side of equation (A.7), we get:

$$a_0 = \partial_1 G \partial_2 G \Big\{ \partial_z^2 F(\Psi) - \begin{pmatrix} \partial_z^2 \phi^1 & \partial_z^2 \phi^2 \end{pmatrix} (\Psi) B^{-1} \begin{pmatrix} \partial_1 F \\ \partial_2 F \end{pmatrix} (\Psi) \Big\} m^{-1},$$

now, if we differentiate (A.6):

$$\begin{aligned} a_0 =& \partial_1 G \partial_2 G \Big\{ -2 \partial_z \phi^1 \partial_z \phi^2 (\Psi) \gamma(\Phi) \\ &+ \begin{pmatrix} \partial_z \phi^1 & \partial_z \phi^2 \end{pmatrix} (\Psi) \begin{pmatrix} \partial_1 \partial_2 \phi^2 \\ -\partial_1 \partial_2 \phi^1 \end{pmatrix} (\Psi) + \partial_z (\det B(\Psi^{-1})) \Big\} m^{-1}. \end{aligned}$$

Repeating the process, we obtain:

$$
\begin{aligned}
a_1 =&(\partial_1 G\partial_2\psi^1 - \partial_2 G\partial_1\psi^1)\Big\{-\begin{pmatrix}\partial_z\phi^1 & \partial_z\phi^2\end{pmatrix}(\Psi)\begin{pmatrix}-\partial_1\phi^2\\ \partial_1\phi^1\end{pmatrix}(\Psi)\gamma(\Phi)\\
&+\partial_1(\det B(\Psi^{-1})) - \det B\begin{pmatrix}\partial_1\partial_z\phi^1 & \partial_1\partial_z\phi^2\end{pmatrix}(\Psi)\{AB\}^{-1}\begin{pmatrix}\partial_1 G\\ \partial_2 G\end{pmatrix}\Big\}m^{-1}\\
a_2 =&(\partial_1 G\partial_2\psi^2 - \partial_2 G\partial_1\psi^2)\Big\{\begin{pmatrix}\partial_z\phi^1 & \partial_z\phi^2\end{pmatrix}(\Psi)\begin{pmatrix}\partial_2\phi^2\\ -\partial_2\phi^1\end{pmatrix}(\Psi)\gamma(\Phi)\\
&+\partial_2(\det B(\Psi^{-1})) - \det B\begin{pmatrix}\partial_2\partial_z\phi^1 & \partial_2\partial_z\phi^2\end{pmatrix}(\Psi)\{AB\}^{-1}\begin{pmatrix}\partial_1 G\\ \partial_2 G\end{pmatrix}\Big\}m^{-1}\\
a_3 =&\det A\Big\{\gamma(\Phi)\det B - \det B\begin{pmatrix}\partial_1\partial_2\phi^1 & \partial_1\partial_2\phi^2\end{pmatrix}(\Psi)\{AB\}^{-1}\begin{pmatrix}\partial_1 G\\ \partial_2 G\end{pmatrix}\\
&+\det A^{-1}\partial_1 G\partial_2 G\begin{pmatrix}\partial_1\partial_2\phi^1 & \partial_1\partial_2\phi^2\end{pmatrix}(\Psi)\begin{pmatrix}-\partial_z\phi^2\\ \partial_z\phi^1\end{pmatrix}(\Psi)\Big\}m^{-1}
\end{aligned}
$$

adding all in (A.7) we obtain:

$$
\gamma(\Phi(\Psi)) = \gamma(\Psi) + (\partial_1 G\partial_2 G b_{12} + \partial_1 G b_1 + \partial_2 G b_2 + b_0)m^{-1},
$$

where

$$
\begin{aligned}
b_{12} =& -2\partial_z\phi^1\partial_z\phi^2(\Psi)\gamma(\Phi) + \begin{pmatrix}\partial_z\phi^1 & \partial_z\phi^2\end{pmatrix}\begin{pmatrix}\partial_1\partial_2\phi^2\\ -\partial_1\partial_2\phi^1\end{pmatrix}(\Psi) + \partial_z\partial_1\phi^1\partial_2\phi^2(\Psi)\\
&+\partial_1\phi^1\partial_z\partial_2\phi^2(\Psi) - \partial_z\partial_2\phi^1\partial_1\phi^2(\Psi) - \partial_2\phi^1\partial_z\partial_1\phi^2(\Psi)\\
&-\det B\begin{pmatrix}\partial_1\partial_z\phi^1 & \partial_1\partial_z\phi^2\end{pmatrix}(\Psi)B^{-1}\begin{pmatrix}1\\0\end{pmatrix} - \det B\begin{pmatrix}\partial_2\partial_z\phi^1 & \partial_2\partial_z\phi^2\end{pmatrix}(\Psi)B^{-1}\begin{pmatrix}0\\1\end{pmatrix}\\
&+\begin{pmatrix}\partial_1\partial_2\phi^1 & \partial_1\partial_2\phi^2\end{pmatrix}\begin{pmatrix}-\partial_z\phi^2\\ \partial_z\phi^1\end{pmatrix}(\Psi)\\
b_1 =& \gamma(\Phi)\{-\partial_2\psi^1\begin{pmatrix}\partial_z\phi^1 & \partial_z\phi^2\end{pmatrix}\begin{pmatrix}-\partial_1\phi^2\\ \partial_1\phi^1\end{pmatrix}(\Psi) + \partial_2\psi^2\begin{pmatrix}\partial_z\phi^1 & \partial_z\phi^2\end{pmatrix}\begin{pmatrix}\partial_2\phi^2\\ -\partial_2\phi^1\end{pmatrix}(\Psi)\}\\
&+\partial_2\psi^1(\partial_1\phi^1\partial_1\partial_2\phi^2 - \partial_1\phi^2\partial_1\partial_2\phi^1)(\Psi) + \partial_2\psi^2(\partial_2\phi^1\partial_1\partial_2\phi^2 - \partial_2\phi^2\partial_1\partial_2\phi^1)(\Psi)\\
&+\det A\det B\begin{pmatrix}\partial_1\partial_2\phi^1 & \partial_1\partial_2\phi^2\end{pmatrix}(\Psi)\{AB\}^{-1}\begin{pmatrix}1\\0\end{pmatrix}
\end{aligned}
$$

and

$$b_2 = \gamma(\Phi)\{\partial_1\psi^1 \begin{pmatrix} \partial_z\phi^1 & \partial_z\phi^2 \end{pmatrix} \begin{pmatrix} -\partial_1\phi^2 \\ \partial_1\phi^1 \end{pmatrix} (\Psi) - \partial_1\psi^2 \begin{pmatrix} \partial_z\phi^1 & \partial_z\phi^2 \end{pmatrix} \begin{pmatrix} \partial_2\phi^2 \\ -\partial_2\phi^1 \end{pmatrix} (\Psi)\}$$
$$+ \partial_2\psi^1(\partial_1\phi^1\partial_1\partial_2\phi^2 - \partial_1\phi^2\partial_1\partial_2\phi^1)(\Psi) + \partial_2\psi^2(\partial_2\phi^1\partial_1\partial_2\phi^2 - \partial_2\phi^2\partial_1\partial_2\phi^1)(\Psi)$$
$$+ \det A \det B \begin{pmatrix} \partial_1\partial_2\phi^1 & \partial_1\partial_2\phi^2 \end{pmatrix} (\Psi)\{AB\}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Replacing such terms, we get

$$\gamma(\Phi(\Psi)) = \gamma(\Psi) + \gamma(\Phi)(\Psi) \det AB \Big\{ 1 - \begin{pmatrix} \partial_z\phi^1 & \partial_z\phi^2 \end{pmatrix} (\Psi)\{AB\}^{-1} \begin{pmatrix} \partial_1 G \\ \partial_2 G \end{pmatrix}$$
$$- 2\partial_1 G\partial_2 G\partial_z\phi^1\partial_z\phi^2(\Psi)\{\det AB\}^{-1} \Big\} m^{-1}$$

Hence, cleaning terms we obtain:

$$\gamma(\Phi \circ \Psi) = \gamma(\Psi) + \gamma(\Phi)(\Psi),$$

so we get (A.2).

Finally, consider the function

$$\gamma(\Phi_\tau) = \left(\partial_1\partial_2 F_\tau - \begin{pmatrix} \partial_1\partial_2\phi^1_\tau & \partial_1\partial_2\phi^2_\tau \end{pmatrix} A_\tau^{-1} \begin{pmatrix} \partial_1 F_\tau \\ \partial_2 F_\tau \end{pmatrix} \right) \det A_\tau^{-1}$$

Since Φ_0 is the identity then $\partial_i F_0 = 0$, $\partial_1\partial_2\phi_0^j = 0$ and $(\partial_i\phi_0^j)$ is the identity. Then the derivative is

$$\frac{d\gamma(\Phi_\tau)}{d\tau}\Big|_{\tau=0} = \Bigg(\frac{d(\partial_1\partial_2 F_\tau)}{d\tau}\Big|_{\tau=0} - \begin{pmatrix} \frac{d(\partial_1\partial_2\phi^1_\tau)}{d\tau} & \frac{d(\partial_1\partial_2\phi^1_\tau)}{d\tau} \end{pmatrix}\Big|_{\tau=0} A_0^{-1} \begin{pmatrix} \partial_1 F_0 \\ \partial_2 F_0 \end{pmatrix}$$
$$+ \begin{pmatrix} \partial_1\partial_2\phi^1_0 & \partial_1\partial_2\phi^1_0 \end{pmatrix} A_0^{-1} \frac{d(A_\tau)}{d\tau}\Big|_{\tau=0} A_0^{-1} \begin{pmatrix} \partial_1 F_0 \\ \partial_2 F_0 \end{pmatrix}$$
$$- \begin{pmatrix} \partial_1\partial_2\phi^1_0 & \partial_1\partial_2\phi^1_0 \end{pmatrix} A_0^{-1} \begin{pmatrix} \frac{d(\partial_1 F_\tau)}{d\tau} \\ \frac{d(\partial_2 F_\tau)}{d\tau} \end{pmatrix} \Big|_{\tau=0} \Bigg) \det A_\tau^{-1} -$$
$$- \gamma(\Phi_\tau) \det A_\tau^{-1}\Big|_{\tau=0} \frac{d(\det A_\tau)}{d\tau}\Big|_{\tau=0}$$
$$= \frac{d(\partial_1\partial_2 F_\tau)}{d\tau}\Big|_{\tau=0}.$$

Finally, since ∂_τ and ∂_i commutes and $F_\tau = \exp(\tau X)z$, then we obtain

$$\begin{aligned}\frac{d\gamma(\Phi_\tau)}{d\tau}\Big|_{\tau=0} =& \partial_1\partial_2\left(\frac{d(F_\tau)}{d\tau}\Big|_{\tau=0}\right)\\ =&\partial_1\partial_2(X\cdot z).\end{aligned}$$

So we get (A.3). □

Observation A.1. The Proposition A.1 give us that $\gamma : \mathrm{Aut}^\delta_R(R[[1|2]]) \to R$ is a character. For such character, the kernel is given by $\mathrm{Aut}^\Delta_R(R[[1|2]])$.

In particular, we obtain:

Proposition A.2. *The group* $\mathrm{Aut}^\Delta_{\mathbb{C}}(\mathbb{C}[[1|2]])$ *is simple.*

Observation A.2. The operator γ is still additive if we consider the group defined by the preimage of R with respect to the map $\mathrm{Ber} : \mathrm{Aut}_R(R[[1|2]]) \to R[[1|2]]$. Such group, is denoted by $\mathrm{Aut}^d(R[[1|2]])$ and denote the kernel by $\mathrm{Aut}^D(R[[1|2]])$. In this case $\mathrm{Aut}^\delta(R[[1|2]]) \subset \mathrm{Aut}^d(R[[1|2]])$ and $\mathrm{Aut}^\delta(R[[1|2]]) \cap \mathrm{Aut}^D(R[[1|2]]) = \mathrm{Aut}^\Delta(R[[1|2]])$.

In this case we still can define the dual curve, that is the construction given in Teorema 5.1 still holds for curves with change of coordinates inside $\mathrm{Aut}^D(R[[1|2]])$.

There is another way to characterize $S(2)$ super curves $C \xrightarrow{\pi} S$. From the condition

$$\gamma(\Psi * \Phi) = \gamma(\Psi) + \Psi(\gamma(\Phi))$$

we obtain a class $\Gamma_C = \{\gamma(\Phi_{ij})\}_{ij} \in H^1(C, \pi^*\mathcal{O}_S)$, where $\pi^*\mathcal{O}_S$ is the sheaf given by the pullback $C \xrightarrow{\pi} S$, and observe that this class is zero if our curve is an $S(2)$ super curve. For the converse, suppose that the class Γ_C is null, then there exists a 1-cycle $\{\gamma_i\}_i$ with

$$\gamma_i - \gamma_j = \gamma(\Phi_{ij}),$$

so we can modify the local coordinates Φ_i by

$$\widehat{\Phi}_i(z_i|\theta_i^1, \theta_i^2) = \exp(-\gamma_i\theta_i^1\theta_i^2\partial_{z_i})(\Phi_i(z_i|\theta_i^1, \theta_i^2)),$$

That generates the cocicles

$$\widehat{\Phi}_{ij} = \exp(-\gamma_j\theta_i^1\theta_i^2\partial_{z_i})(\Phi_{ij}(z_i|\theta_i^1, \theta_i^2))\exp(\gamma_i\theta_i^1\theta_i^2\partial_{z_i})$$

and observe that is well defined.

Finally, from (A.2) we get that

$$\begin{aligned}\gamma(\widehat{\Phi}_{ij}) &= \gamma(\exp(-\gamma_i\theta_i^1\theta_i^2\partial_{z_i}) \circ \Phi_{ij} \circ \exp(\gamma_j\theta_j^1\theta_j^2\partial_{z_j}))\\ &= -\gamma_i + \gamma(\Phi_{ij}) + \gamma_j\\ &= 0\end{aligned}$$

so we get that the curve is an $S(2)$ super curve.

Observe that the class $\{\gamma(\phi_{ij})\}$ only depends on (C, Δ), we denote this class by $\Gamma_{C,\Delta} \in H^1(C, \pi^*\mathcal{O}_S)$.

Theorem A.1. *A family of $S(2)$ super curves $C \xrightarrow{\pi} S$ is a family of $S(1|2)$ super curves (C, Δ) such that the class $\Gamma_{C,\Delta} \in H^1(C, \pi^*\mathcal{O}_S)$ vanishes.*

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