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Heavy Tails and Random Polymers

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Abstract

Directed polymers have been an object of study in statistical mechanics for many decades. The usual set-up is the following: Space-time impurities (also referred to as the disorder or environment) are introduced as a collection of i.i.d. random variables, to each site on the integer d-dimensional lattice, to alter the law of a nearest-neighbor random walk that will roughly get attracted (repelled) to sites where the disorder presents higher (lower) values. The influence of this environment is tuned by some parameter called the inverse temperature. It has been established that there is a phase transition, for the inverse temperature, between two regimes respectively referred to as weak and strong disorder. In the weak disorder regime, the distribution of the re-scaled paths converges to a Brownian motion as if no impurities were added. When strong disorder occurs, the paths have non-Gaussian scaling limits, and exhibits a localized behavior. When the corresponding random environment has finite second moment, it is known that this phase transition is trivial when the dimension of the lattice is either one and two (strong disorder occurring at every temperature) and occurs at a positive and finite temperature in dimension three and larger. The first part of this thesis, is dedicated to study the effect of heavier tailed environment (exponent between one and two) on the phase transition. We showed for every tail exponent of the disorder, a necessary and sufficient condition on the dimension to observe a phase transition which strictly differs from the one obtained in the case of second moment. In the second part of the thesis, we study the two notions of strong disorder found in the literature: usual strong disorder (partition function going to zero) and very strong disorder (negative Lyapunov exponent) when the random walk is long-ranged and has very heavy tail increments - that is, with associated tail exponent equal to one. Surprisingly, in the very heavy tail setup, we show that very strong disorder does not occur at any temperature whereas the existence of a strong disorder phase at low temperature depends on finer properties of the tail decay of the increments. This results sheds a new light on a classical conjecture in the field that can be summarized as follows: "Strong disorder and very strong disorder are equivalent except possibly at the critical point". Indeed we prove that there is a directed polymer setup for which a strong disorder phase exists without having a very strong disorder one. While this does not fully invalidate the conjecture (which has been formulated for the nearest-neighbor directed polymer), it questions its range of validity.

Keywords: directed polymer, heavy tail, free energy

Resumo

Polímeros dirigidos têm sido objeto de estudo em mecânica estatística por muitas décadas. A configuração usual é a seguinte: uma coleção de variáveis aleatórias i.i.d. (chamadas genericamente de impurezas, desordem ou ambiente) é introduzida em cada vértice no retículo d-dimensional, alterando a lei de um passeio aleatorio simples. Informalmente, o passeio aleatório será atraído (repelido) por sítios onde a impureza apresenta valores mais altos (mais baixos). A influência desse ambiente é ajustada por um parâmetro (chamado temperatura inversa). Existe uma transição de fase, na temperatura inversa, entre dois regimes, respectivamente, referidos como desordem fraca e forte. No regime de desordem fraca, a distribuição dos caminhos re-escalados converge para um movimento browniano, como se nenhuma impureza fosse adicionada. No regime de desordem forte, os caminhos têm limites de escala não gaussianos e exibem um comportamento localizado. Quando o ambiente aleatório correspondente possui segundo momento finito, sabe-se que essa transição de fase é trivial quando a dimensão é um e dois (desordem forte ocorre para cada valor da temperatura) e ocorre a uma temperatura positiva e finita em dimensão maior ou igual do que três. A primeira parte desta tese é dedicada ao estudo do efeito do ambiente de cauda pesada (expoente entre um e dois) na transição de fase. Mostramos, para cada expoente da cauda da desordem, uma condição necessária e suficiente na dimensão para observar uma transição de fase que difere estritamente daquela obtida no caso do segundo momento. Na segunda parte da tese, estudamos as duas noções de desordem forte encontradas na literatura: desordem forte usual (função de partição indo para zero) e desordem muito forte (expoente de Lyapunov estritamente negativo) quando o passéio aleatório é de longo alcance e possui incrementos com cauda muito pesados - ou seja, com o expoente de cauda associado igual a um. Surpreendentemente, na configuração de cauda muito pesada, mostramos que desordem muito forte não ocorre em nenhuma temperatura, enquanto a existência de uma fase de desordem forte a baixa temperatura depende de propriedades mais refinadas da cauda dos incrementos. Esses resultados lançam uma nova luz sobre uma conjectura clássica que pode ser resumida da seguinte forma: "Desordem forte e desordem muito forte são equivalentes, exceto possivelmente no ponto crítico". De fato, provamos que existem cassos nos quais existe uma fase de desordem forte sem que exista uma fase de desordem muito forte. Embora isso não invalide completamente a conjectura (que foi formulada para o polímero dirigido no passeio aleatório simples), questiona o seu alcance de validade.

Palavras chaves: polímero dirigido, cauda pesada, energia livre

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Contents

CHAPTER 1

Introduction

1. Directed polymers and the effect of disorder

Disordered systems have been object of study in statistical mechanics for many decades. The usual set-up is the following: Disorder is introduced in a well-understood homogeneous system as a sample of a random field with ergodic properties or even a field of independent identically distributed random variables, and then one tries to compare the properties of the original model (such as phase transition, critical exponents, path behavior) to the one where disorder has been added.

The introduction of impurities may modify the behavior of the original model in a drastic manner or, on the contrary, have a negligible effect. A rigorous understanding of the phenomenon is in general a difficult task [**21, 24**]. The directed polymer model (which is the main object of study of the present thesis) offers an ideal setup to study this question of disorder effect since it is based on one of the most well-understood model of statistical mechanics, the simple random walk.

The directed polymer with random environment model was introduced originally in physics literature in 1985, to study the interface in the Ising model subjected to random impurities [**25**]. Then it was brought into the probability realm in [**26**] and [**9**]. For a recent review on the subject see [**16, 19**].

A polymer is a large molecule composed of many repeated sub-units called monomers. They play an essential role in everyday lives due to their abundance in nature and large range of properties. Depending on their physical attributes, they admit different types of classification. Various types of models are considered in the literature to study and describe these properties.

In the directed polymer model, polymers are modeled by the graph of a random path in $\mathbb{Z}^d$ of finite length N, $S = (S_0, S_1, \ldots, S_N)$. The monomers are the vertices (n, S_n) on the graph of S, and the chemical bonds connecting the monomers are the edges $[(n, S_n), (n+1, S_{n+1})]$.

The model consists in a probability measure on the set of paths. Starting with a reference measure $\mathbf{P}$ which is that of a simple random walk in $\mathbb{Z}^d$ we modify it by considering an Hamiltonian $H_N(S)$ associated with each path realization to codify its energy. The polymer measure $\mathbf{P}_N$ is defined by its Radon-Nikodym derivative with respect to the law $\mathbf{P}$ of the original random walk:

$$\frac{d\mathbf{P}_N}{d\mathbf{P}}(S) = \frac{1}{Z_N} e^{H_N(S)}. \tag{1.1}$$

This is also called the Gibbs measure, and it describes the polymer in equilibrium, at fixed temperature and fixed polymer length. The partition function Z_N is the normalizing factor that makes $\mathbf{P}_N$ a probability measure. The Hamiltonian we consider is defined later in Section 3. Since we are interested in comparing the properties of $\mathbf{P}_N$ with those of the simple random walk let us recall some classic facts about it (see **[31]** for an extensively review on the subject).

2. Random walk on $\mathbb{Z}^d$

Let us describe the reference measure $\mathbf{P}$ we are considering and some of its well-known properties. On the path space $\left(\left(\mathbb{Z}^d\right)^{\mathbb{N}}, \mathcal{P}(\mathbb{Z}^d)^{\otimes\mathbb{N}}\right)$ of sequences $S := (S_n)_{n\geq 0}$, let $\mathbf{P}^x$ be a probability measure that satisfies:

$$\begin{aligned} &\mathbf{P}^x\left[S_0 = x\right] = 1 \text{ and} \\ &\{S_n - S_{n-1}\}_{n\geq 1} \text{ is an i.i.d. sequence.} \end{aligned} \tag{2.1}$$

We say that $\mathbf{P}^x$ is a random walk on $\mathbb{Z}^d$, starting at $x \in \mathbb{Z}^d$. If the random walk starts at 0 we write $\mathbf{P}$ instead of $\mathbf{P}^0$ for simplicity. We say that $\mathbf{P}$ is the law of the simple random walk (SRW) or the nearest-neighbor random walk when

$$\mathbf{P}[S_1 = e_j] = \mathbf{P}[S_1 = -e_j] = \frac{1}{2d}, \tag{2.2}$$

where $\{e_1, ..., e_d\}$ is the canonical basis of $\mathbb{R}^d$. Many of its properties rely on the fact that it is a sum of independent identically distributed random variables . Let us recall some of their well-known properties.

THEOREM 2.1 (Law of Large Numbers). *Let $\{X_1, X_2, ...\}$ a sequence of i.i.d. random variables with expectation μ and $S_n = X_1 + X_2 + ... + X_n$. Then,*

$$\lim_{n\to\infty} \frac{S_n}{n} = \mu, \qquad \mathbf{P}\text{-}a.s. \tag{2.3}$$

THEOREM 2.2 (Central Limit Theorem). *Assuming that $\{X_1, X_2, ...\}$ is an i.i.d. sequence of random variables with expectation μ and variance $\sigma^2 \in (0, \infty)$, then*

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \to^{\mathbf{P}} N[0,1], \tag{2.4}$$

as $n \to \infty$.

THEOREM 2.3 (Donsker's Invariance Principle). *Let $\{X_1, X_2, ...\}$ be sequence of i.i.d. random variables with with mean 0 and variance 1. For each $n \in \mathbb{N}$ define the re-scaled random function*

$$\widehat{S}^{(n)}(t) := \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}, \qquad t \in [0,1]. \tag{2.5}$$

Then, the random functions $\widehat{S}^{(n)} := (\widehat{S}^{(n)}(t))_{t\in[0,1]}$ converges in distribution to a standard Brownian motion $W := (W(t))_{t\in[0,1]}$. That is, for every bounded continuous function $f : C([0,1]) \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \mathbf{E}\left[f(\widehat{S}^{(n)})\right] = \mathbf{E}\left[f(W)\right]$$

Hence under $\mathbf{P}$ the polymer trajectories typically extend to a range $\sqrt{N}$ and their behavior is diffusive.

It is also possible to consider long-ranged heavier-tailed distributions for the random walk. Specifically if

$$\mathbf{P}\left[|S_1| \geq n\right] = \frac{L(n)}{n^{\alpha}}, \tag{2.6}$$

where $\alpha \in (0,2)$ and $L(\cdot)$ is a slowly varying function at $\pm\infty$ then the distribution of S_1 belongs to the domain of attraction of an α-stable law. We remit the reader to [**8**] for a complete overview on stable laws and the analogous versions of the Central Limit Theorem and Donsker's Invariance Principle in this context. If $\alpha = 0$ in the equation (2.6) above, we say S_1 has a very heavy-tailed distribution. The growth of the random walk in this case is faster than any polynomial. For instance,

$$\forall k > 0, \lim_{n\to\infty} \mathbf{P}\left[\ |S_n| \leq n^k\right] = 0, \tag{2.7}$$

see Lemma 4.3 for a result in this spirit with its proof. The polymer we study in Chapter 3 is based on the random walk with such a distribution.

3. Directed Polymer and Random Environments

Let us now introduce the disorder to our model. To each vertex $(n,z) \in \mathbb{N}\times\mathbb{Z}^d$ is associated a real random variable $\omega_{n,z}$ from a probability space $(\Lambda, \mathcal{F}, \mathbb{P})$. The collection $\omega := \{\omega_{n,z} : n \in \mathbb{N}, z \in \mathbb{Z}^d\}$, from now on referred to as the environment, is independent identically distributed. Assume that the distribution of this random variables satisfy

$$\mathbb{E}\left[\exp(\beta\omega_{n,z})\right] < \infty, \tag{3.1}$$

for any $\beta \in \mathbb{R}$. The *polymer measure* $\mathbf{P}_N^{\beta,\omega}$ is the probability measure in $\left(\mathbb{Z}^d\right)^{\mathbb{N}}$ described by its Radon-Nikodym derivative with respect to $\mathbf{P}$: For a fixed value of β (called the inverse temperature), $N \in \mathbb{N}$ and a fixed realization of the environment ω,

$$\frac{\mathrm{d}\mathbf{P}_N^{\beta,\omega}}{\mathrm{d}\mathbf{P}}(S) = \frac{1}{Z_N^{\beta,\omega}}\exp\left(\beta\sum_{n=1}^{N}\omega_{n,S_n}\right). \tag{3.2}$$

The positive random variable $Z_N^{\beta,\omega}$, called the partition function, makes $\mathbf{P}_N^{\beta,\omega}$ a probability measure. Its value is given by

$$Z_N^{\beta,\omega} = \mathbf{E}\left[\exp\left(\beta\sum_{n=1}^{N}\omega_{n,S_n}\right)\right]. \tag{3.3}$$

Roughly speaking, this measure $\mathbf{P}_N^{\beta,\omega}$ rewards (penalizes) walks that visits sites with higher (smaller) values of the environment. The parameter β is used to increase or decrease the influence of the environment over the measure $\mathbf{P}_N^{\beta,\omega}$.

3.1. Overview of known results. *Weak, Strong and Very Strong disorder.* Most of the results below were proved in the context of the nearest-neighbor random walk. Nevertheless, they are all still valid in the general framework (2.1). It was observed by Bolthausen in [**9**] that by the i.i.d. structure of the environment, the sequence

$$W_N^{\beta,\omega} := \frac{Z_N^{\beta,\omega}}{\mathbb{E}\left[Z_N^{\beta,\omega}\right]}, \tag{3.4}$$

is a positive martingale with respect to the filtration $\{\mathcal{G}_N\}_{N\geq 1}$ defined as

$$\mathcal{G}_N := \sigma\{\omega_{n,z} : 1 \leq n \leq N, z \in \mathbb{Z}\}. \tag{3.5}$$

Notice that

$$\mathbb{E}\left[Z_N^{\beta,\omega}\right] = \mathbb{E}\left[\exp(\beta\omega_{1,0})\right]^N = e^{\lambda(\beta)N}, \tag{3.6}$$

where $\lambda(\beta) := \log \mathbb{E}\exp(\beta\omega_{1,0})$. It follows that the limit

$$W_\infty^{\beta,\omega} := \lim_{N\to\infty} W_N^{\beta,\omega}, \tag{3.7}$$

exists $\mathbb{P}$-*a.s.* and is a non-negative random variable. Moreover, the event $\{W_\infty^{\beta,\omega} = 0\}$ belongs to the tail σ-field of $\{\mathcal{G}_N, N \geq 0\}$ as for $M < N$,

$$W_N^{\beta,\omega} = \sum_{m_1,\ldots m_M \in \mathbb{Z}^d} \mathbf{P}\left[\cap_{i=1}^M \{S_i = m_i\}\right] e^{\sum_{i=1}^M \beta\omega_{i,m_i} - \lambda(\beta)} W_{N-M,m_M}^{\beta,\Theta^M\omega}, \tag{3.8}$$

where Θ is the shift operator defined as $\Theta\omega_{n,z} := \omega_{n+1,z}$ and $Z_{N,x}^{\beta,\omega}$ (and analogously $W_{N,x}^{\beta,\omega}$) is the partition function when the law of the random walk starts at x:

$$Z_{N,x}^{\beta,\omega} := \mathbf{E}^x\left[\exp\left(\beta\sum_{n=1}^N \omega_{n,S_n}\right)\right]. \tag{3.9}$$

Then,

$$W_\infty^{\beta,\omega} = 0 \iff \lim_{N\to\infty} W_{N-M,m_M}^{\beta,\theta^M\omega} = 0, \qquad \forall m_M \in \mathbb{Z}. \tag{3.10}$$

By Kolmogorov's $0-1$ Law, this implies that

$$\mathbb{P}\left\{W_\infty^\beta > 0\right\} \in \{0,1\}. \tag{3.11}$$

This dichotomy allows to define a natural manner to characterize the influence of disorder. We say that we have *weak disorder* if $W_\infty^\beta > 0$ $\mathbb{P}$-*a.s.* and *strong disorder* if $W_\infty^\beta = 0$ $\mathbb{P}$-*a.s.* It is known [**13**] that there exists a critical value $\beta_c = \beta_c(d) \in [0,\infty]$ such that there is weak disorder for $\beta \in \{0\} \cup (0,\beta_c)$ and strong disorder for $\beta > \beta_c$. A lot of information about the model is also encoded in the free energy, defined as

$$F(\beta) := \lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta,\omega} = \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_N^{\beta,\omega}. \tag{3.12}$$

It is known that this limit exists (see [**12**, Proposition 2.5] for the nearest-neighbor case and [**6**] for the general case), except on a set of measure zero. By Jensen's Inequality,

$$F(\beta) \leq \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z_N^{\beta,\omega} = \lambda(\beta). \tag{3.13}$$

Furthermore, since $F(\beta) - \lambda(\beta) = \lim_{N\to\infty} \dfrac{1}{N} \log W_N^{\beta,\omega}$ $\mathbb{P}$-a.s. we have that

$$F(\beta) - \lambda(\beta) < 0 \implies \lim_{N\to\infty} W_N^{\beta,\omega} = 0. \tag{3.14}$$

Thus, the case $F(\beta) - \lambda(\beta) < 0$ is called the *very strong disorder*. As a function, $p(\beta) := F(\beta) - \lambda(\beta)$ is continuous and non-increasing [**12**, Theorem 3.2 (b)]. Thus, there is a critical value $\bar{\beta}_c$ such that $p(\beta) = 0$ if $\beta \in [0, \bar{\beta}_c]$ and $p(\beta) < 0$ if $\beta > \bar{\beta}_c$. As noted before, $\beta_c \leq \bar{\beta}_c$.

The weak disorder regime. Let us first show a simple sufficient condition for the system to be in the weak disorder regime. To this purpose we reproduce a second moment computation due to Bolthausen [**9**]. Notice that, $\sup_N \mathbb{E}\left[\left(W_N^{\beta,\omega}\right)^2\right] < \infty$ implies that the sequence $\left\{W_N^{\beta,\omega}\right\}_N$ is uniformly integrable which implies convergence in $\mathcal{L}^1$. As $\mathbb{E}\left[\left(W_N^{\beta,\omega}\right)\right] = 1$ for all N this would imply that $W_\infty^{\beta,\omega} > 0$ $\mathbb{P}$-a.s. On the other hand, by considering an independent replica of the random walk,

$$\begin{aligned}\mathbb{E}\left[\left(W_N^{\beta,\omega}\right)^2\right] &= \mathbb{E}\left[\mathbf{E}\otimes\mathbf{E}'\left[\exp\left(\sum_{n=1}^{N} \beta\omega_{n,S_n} - \lambda(\beta) + \beta\omega_{n,S'_n} - \lambda(\beta)\right)\right]\right] \\ &= \mathbf{E}\otimes\mathbf{E}'\left[e^{(\lambda(2\beta)-2\lambda(\beta))I_N}\right],\end{aligned} \tag{3.15}$$

where $I_N := \sum_{n=1}^N \mathbf{1}_{\{S_n = S'_n\}}$ is the number of times the paths S and S' coincide up to time N. I_N is upper bounded by I_∞ (the total number of times both paths intersect) which is a geometrically distributed random variable with success probability $1-\pi_{\mathbf{P}}$ with

$$\pi_{\mathbf{P}} := \mathbf{P}\otimes\mathbf{P}'\left[\exists n \geq 1 : S_n = S'_n\right]. \tag{3.16}$$

As $\lambda(2\beta) - 2\lambda(\beta) \to 0$ as $\beta \to 0$, we have that $\beta > 0$ small is in the weak disorder regime whenever the random walk is transient. Specifically,

THEOREM 3.1. [**9**] *For all β such that*

$$\lambda(2\beta) - 2\lambda(\beta) < \log\frac{1}{\pi_{\mathbf{P}}} \tag{3.17}$$

weak disorder holds.

EXAMPLE 3.2. *For the SRW, $\pi_{\mathbf{P}} < 1$ when $d \geq 3$. This implies that $\beta_c > 0$ when $d \geq 3$ and it does not give any information for the cases $d = 1, 2$.*

EXAMPLE 3.3. *If ω is Bernoulli-distributed with success probability p then*

$$\lim_{\beta\to\infty} \lambda(2\beta) - 2\lambda(\beta) = \log\frac{1}{p}.$$

This implies that if $\mathbf{P}$ is transient and p is sufficiently close to one, the condition (3.17) *is satisfied for arbitrarily high values of β. In this case there is no strong disorder regime and $\beta_c = \infty$.*

Let us mention a property of the paths in the weak disorder regime. Roughly speaking, weak disorder implies that the polymers paths show the same behavior as the ones under the unaltered law $\mathbf{P}$. Specifically,

THEOREM 3.4. [**13**, Theorem 1.2] *Assume* $\mathbf{P}$ *is the SRW,* $d \geq 3$ *and weak disorder. Then, for all bounded continuous functions* f *on the path space, we have the following convergence in probability*

$$\mathbf{E}_N^{\beta,\omega}\left[f(\widehat{S}^{(N)})\right] \xrightarrow{\mathbb{P}} \mathbf{E}\left[f(W)\right],$$

as $N \to \infty$ *where* $\widehat{S}^{(N)}$ *is the re-scaled path defined in* (2.5) *and* W *is the Brownian motion with diffusion matrix* $d^{-1}I_d$.

Strong and very strong disorder regime. We start with a simple computation that appeared originally in [**12**] for the SRW yielding a sufficient condition for very strong disorder. Fix $\theta \in (0,1]$. By Jensen's Inequality,

$$F(\beta) = \lim_{N\to\infty} \frac{1}{\theta N}\mathbb{E}\log\left(Z_N^{\beta,\omega}\right)^\theta \leq \liminf_N \frac{1}{\theta N}\log\mathbb{E}\left(Z_N^{\beta,\omega}\right)^\theta. \tag{3.18}$$

Recall that

$$\left(\sum_i a_i\right)^\theta \leq \sum a_i^\theta, \tag{3.19}$$

for any $a_i \geq 0$ and $\theta \leq 1$. Then, replacing $Z_N^{\beta,\omega} = \sum_{x\in\mathbb{Z}^d}\mathbf{P}\left[S_1 = x\right]e^{\beta\omega(1,x)}Z_{N-1,x}^{\beta,\Theta\omega}$ it follows,

$$\mathbb{E}\left(Z_N^{\beta,\omega}\right)^\theta \leq \sum_{x\in\mathbb{Z}^d}\mathbf{P}\left[S_1 = x\right]^\theta\mathbb{E}e^{\theta\beta\omega(1,x)}\mathbb{E}\left(Z_{N-1,x}^{\beta,\Theta\omega}\right)^\theta. \tag{3.20}$$

By the i.i.d. structure of the environment, $\mathbb{E}\left(Z_{N-1,x}^{\beta,\Theta\omega}\right)^\theta = \mathbb{E}\left(Z_{N-1}^{\beta,\omega}\right)^\theta$. Applying the argument above recursively N times,

$$\mathbb{E}\left(Z_N^{\beta,\omega}\right)^\theta \leq e^{N\lambda(\theta\beta)}\left(\sum_{x\in\mathbb{Z}^d}\mathbf{P}\left[S_1 = x\right]^\theta\right)^N. \tag{3.21}$$

This allows to upper bound the free energy as $F(\beta) \leq \frac{v(\theta)}{\theta}$, where

$$v(\theta) = \lambda(\theta\beta) + \log\sum_{x\in\mathbb{Z}^d}\mathbf{P}\left[S_1 = x\right]^\theta, \tag{3.22}$$

and $\theta \in (0,1]$. For $\theta = 1$, $\frac{v(\theta)}{\theta} = \lambda(\beta)$. This implies that to show that $F(\beta) < \lambda(\beta)$ it is sufficient that $\frac{d}{d\theta}\frac{v(\theta)}{\theta}|_{\theta=1} > 0$, which is equivalent to

$$\beta\lambda'(\beta) - \lambda(\beta) + \sum_{x\in\mathbb{Z}^d}\mathbf{P}\left[S_1 = x\right]\log\mathbf{P}\left[S_1 = x\right] > 0. \tag{3.23}$$

We just proved the following,

THEOREM 3.5. [**17**, Theorem 5.1] *For any* $d \geq 1$*, we have* $p(\beta) < 0$ *whenever*

$$\beta\lambda'(\beta) - \lambda(\beta) > -\sum_{x\in\mathbb{Z}^d} \mathbf{P}\left[S_1 = x\right] \log \mathbf{P}\left[S_1 = x\right]. \tag{3.24}$$

In particular, for the SRW this implies that $p(\beta) < 0$ if $\beta\lambda'(\beta) - \lambda(\beta) > \log(2d)$. Notice that this implies that $\bar{\beta}_c(d) < \infty$ (there is a very strong disorder regime) whenever

$$\lim_{\beta\to\infty} \beta\lambda'(\beta) - \lambda(\beta) = \infty,$$

which always occur when the distribution of the environment is not bounded from above (see Proposition 3.A.1)

Strong disorder implies that the polymer is largely influenced by the environment and is attracted to sites with the higher values. In this direction we may cite:

THEOREM 3.6. [**12**, Theorem 2.1] *For* $\beta > 0$*,*

$$\left\{W_\infty^{\beta,\omega} = 0\right\} = \left\{\sum_{n\geq 1} \left(\mathbf{P}_{n-1}^{\beta,\omega}\right)^{\otimes 2} \left[S_n = S_n'\right] = \infty\right\} \ \mathbb{P}\text{-}a.s., \tag{3.25}$$

where S *and* S' *are two independent polymers with distribution* $\mathbf{P}_{n-1}^{\beta,\omega}$*. Moreover, if* $\mathbb{P}[W_\infty^{\beta,\omega} = 0] = 1$*, then there exists some constants* c_1*,* $c_2 \in (0,\infty)$ *such that,*

$$-c_1 \log W_N^{\beta,\omega} \leq \sum_{n\geq 1}^{N} \left(\mathbf{P}_{n-1}^{\beta,\omega}\right)^{\otimes 2} \left[S_n = S_n'\right] \leq -c_2 \log W_N^{\beta,\omega}, \tag{3.26}$$

for N *large enough,* $\mathbb{P}$*-a.s.*

In particular when $p(\beta) < 0$, (3.26) implies that the Cesaro mean of

$$\left(\mathbf{P}_{n-1}^{\beta,\eta}\right)^{\otimes 2} \left[S_n = S_n'\right]$$

is bounded away from zero, showing a strong instance of path localization (though only for the end point of the trajectory). Note that further research efforts have been made to obtain stronger characterization of localization, let us mention non-exhaustively [**6**] and references therein.

The critical values $\beta_c, \bar{\beta}_c$ are known for dimensions $d = 1$ and $d = 2$. It was first proved in [**10**] that $\beta_c = 0$ for Gaussian environment, in $d = 1$ and $d = 2$. This was extended for the general environment in [**12**]. In respect of the free energy, it was shown $\bar{\beta}_c = 0$ for $d = 1$ in [**15**] and for $d = 2$ in [**27**].

It is conjectured that there is no intermediate phase between weak disorder and very strong disorder (i.e., $\beta_c = \bar{\beta}_c$) but so far this has only been proved in the case when $\beta_c = \bar{\beta}_c = 0$, this includes the simple symmetric directed polymer on dimensions $d = 1$ and $d = 2$ and for the long-range directed polymer where the underlying random walk is in the domain of attraction of an α-stable law for some $\alpha \in (1, 2]$ in [**44**] for $d = 1$.

Free energy asymptotics at high temperature. In the cases where $\beta_c = 0$, some precise statements have been obtained concerning the behavior of the free energy when β is close to this critical value (high temperature region). In dimension $d = 1$, it is known that $p(\beta)$ is of order $-\beta^4$ as $\beta \to 0$ **[27, 43, 2]**. In **[34]**, it is shown that, under some conditions on the environment,

$$\lim_{\beta\to 0} \frac{p(\beta)}{\beta^4} = -\frac{1}{6}. \tag{3.27}$$

In dimension $d = 2$, it has been proved in **[27]** that $p(\beta)$ is smaller than any power of β at the neighborhood of 0 and in **[5]** that,

$$\lim_{\beta\to 0} \beta^2 \log |p(\beta)| = -\pi. \tag{3.28}$$

4. Our contribution in this thesis

This thesis is divided into two parts. In Chapter 2, we study the influence of a heavier-tailed random environment on the phase transition observed for the polymer measure. In Chapter 3, we assume the random walk associated with the polymer has a very heavy tailed distribution. Let us give a brief introduction of these results below.

4.1. Power-tail environment. In Chapter 2 we present our work **[41]** which has been submitted to the journal Annals of Institute Henri Poincare, in which we slightly modify the formalism for the polymer measure with respect to that presented in the previous section by setting

$$\frac{\mathrm{d}\mathbf{P}_N^{\beta,\omega}}{\mathrm{d}\mathbf{P}}(S) = \frac{1}{Z_N^{\beta,\omega}} \left(\prod_{n=1}^{N} (1 + \beta\omega_{n,S_n}) \right). \tag{4.1}$$

The collection $\{\omega_{n,z} : n \in \mathbb{N}, z \in \mathbb{Z}^d\}$ of i.i.d. random variables belong to a probability space $(\Lambda, \mathcal{A}, \mathbb{P})$ and it satisfies: $\omega_{n,z} \geq -1$ (we also consider $\beta \in [0, 1)$ to ensure the expression above is positive) and $\mathbb{E}\omega_{n,z} = 0$ (for the disorder to be centered). As before, β (the inverse temperature) tunes the influence of the environment ω on the polymer measure.

This alternative setup is not new and it was the one adopted in the earliest mathematical works on directed polymers **[26, 9]**. For most purposes it is equivalent to the exponential form considered in Section 3 (one can be obtained from the other by a change of variable). Our motivation to adopt this formalism is that we want to study the case where the multiplicative Boltzman weight of the monomers have the following power law distribution

$$\mathbb{P}[\omega_{n,z} > x] \overset{x\to\infty}{\sim} C_{\mathbb{P}} x^{-\gamma}, \text{ for } \gamma \in (1, 2). \tag{4.2}$$

An exponential form for these weight would make the tail exponent depend on β and would be inconvenient for our analysis.

Some of the standard results for directed polymer such as the existence of the free energy, its continuity and monotonicity of $\beta \mapsto F(\beta)$ (and thus in particular the existence of $\bar{\beta}_c$) have been proven only in the exponential setup for the disorder. We provide in an appendix (Appendix 2.A) a proof of these results in our context.

Our interest for the case of power-tail distribution without second moment comes from the idea that this kind of disorder may yield a different criterion for the existence of a weak disorder regime. As $\mathbb{E}[\omega_{n,z}^2] = \infty$, our assumption (4.2) makes the second moment of the partition function infinite and thus makes the computation (3.15) invalid (the case $\gamma > 2$ does not interest us since when $\omega_{n,z}$ is square-integrable, (3.15) yields that the partition function is uniformly bounded in L_2 for small values of β which implies that the phase diagram should not change too much). Since the second moment method plays such a crucial role in the original analysis of the weak disorder regime [**9, 26**], it is reasonable to expect that the informal criterion "a weak disorder phase exists if and only if the corresponding random walk is transient" loses its validity in our setup.

Indeed, the phenomenology observed here is different. We found a critical value $\gamma_c := 1 + 2/d$ for the exponent γ such that in dimensions $d \geq 3$,

$$\begin{aligned} \gamma \leq \gamma_c \quad &\Longrightarrow \quad \beta_c = \bar{\beta}_c = 0, \\ \gamma > \gamma_c \quad &\Longrightarrow \quad \beta_c > 0. \end{aligned} \tag{4.3}$$

Some of the results we prove in this part are also valid for dimensions $d = 1$ and $d = 2$ (See the details in Chapter 2). However we decided to focus on dimensions $d \geq 3$ as we want to present a picture as complete as possible of the phase diagram and to contrast our results with what occurs in the original model. Here we mention again Theorem 3.1 in which it is proven that there is always weak disorder for small β in dimensions $d \geq 3$.

Beyond establishing that $\bar{\beta}_c = 0$ when $\gamma \leq \gamma_c$, we prove some asymptotics on the free energy at high temperature (notice that $F(\beta) = p(\beta)$ in this context). For $\gamma < \gamma_c$ we show that $p(\beta)$ behaves like a power of β, as β approaches zero and we identify the critical exponent. For the marginal case $\gamma = \gamma_c$ we prove that $|p(\beta)|$ is smaller than any power of β. The following three theorems summarize all the main results of this part. Assuming $d \geq 3$, there exists a critical value $\gamma_c := 1 + 2/d$ such that,

THEOREM 4.1. *If $\gamma < \gamma_c$ then for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that*

$$-c_\varepsilon \beta^{\alpha - \varepsilon} \leq p(\beta) \leq -C_1 \beta^\alpha \tag{4.4}$$

where $\alpha = \alpha(d, \gamma) := \frac{\gamma(\gamma_c - 1)}{\gamma_c - \gamma}$ and $C_1 > 0$. In particular $\beta_c = \bar{\beta}_c = 0$.

THEOREM 4.2. *If $\gamma = \gamma_c$ then*

$$p(\beta) \leq -C_2 e^{c_3/\beta^{2\gamma}} \tag{4.5}$$

where $C_2 > 0$. In particular $\beta_c = \bar{\beta}_c = 0$

THEOREM 4.3. *If $\gamma > \gamma_c$ then $\beta_c > 0$.*

REMARK 4.4. *In the body of the proof of Theorem 4.1 we show that $p(\beta) \leq -C\beta^\alpha$ for $d \geq 1$. Notice that, in dimension $d = 1$ when γ approaches 2 (the finite second moment case) it yields $p(\beta) \leq -C\beta^4$. This is the same exponent that has been proved before for the free energy in the original setup* [**27, 43, 2, 34**].

4.2. Very heavy tailed random walks. In Chapter 3, we present our work [**42**], in which we study the polymer measure (3.2) when the underlying random walk distribution is given by

$$\mathbf{P}\left[X_1 = n\right] =: K(n) = \frac{L(n)}{n}, \tag{4.6}$$

where $L(\cdot)$ is a *slowly varying function* at ∞. That is, for all $k > 0$

$$\lim_{n\to\infty} \frac{L(kn)}{L(n)} = 1. \tag{4.7}$$

We focus on dimension $d = 1$ although our results can be adapted for higher dimensions. Heavier-tailed random walks model super-diffusivity and have a wide range of applications from chemical reactions and physics to mathematical finance and statistics [**11**].

Our motivation to study heavier tail walks comes from the following observation: Since the heavy-tail of the random walk distribution allows for wild fluctuations, there are more possibility to witness a self-averaging of the environment along the trajectories. Hence a strong disorder regime seems to be less likely in this context.

In this direction, notice that the proof of Theorem 3.5 (which gives a sufficient condition for $p(\beta) < 0$) only yields a trivial upper bound when $\sum_{n\in\mathbb{Z}} K(n)^\theta = \infty$ (which is the case here for every $\theta \in (0, 1)$).

Another indication that disorder might have a weaker influence in this context is a work on pinning model [**3**] (see [**22**] for an introduction to the subject). Under an assumption analogous to (4.6) (for the underlying renewal process), the authors of [**3**] proved a result establishing that disorder has only as small influence on the system in the context of pinning (namely coincidence of the quenched critical point and the annealed one for any given value of $\beta \geq 0$).

Our first result is the analogous version of the one in [**3**] for directed polymers. We show that the quenched and annealed free energy coincide at any temperature.

THEOREM 4.5. *Consider the polymer measure* (3.2)*. Assuming that the distribution of the increments satisfies* (4.6) *and that* $K(n) > 0$ *for all* $n \in \mathbb{Z}$ *then,*

$$p(\beta) = 0, \tag{4.8}$$

for all $\beta \in \mathbb{R}$*, which implies that there is no very strong disorder regime.*

Our second result gives a sufficient condition for having a non-trivial strong disorder regime ($\beta_c < \infty$).

THEOREM 4.6. *If the distributions of the increments and the environment satisfy*

$$\beta\lambda'(\beta) - \lambda(\beta) > -\sum_{n\geq 1} K(n) \log K(n), \tag{4.9}$$

for some $\beta > 0$ *then* $\beta_c < \beta$*. In particular, there is a strong disorder phase whenever*

$$\lim_{\beta\to\infty} \beta\lambda'(\beta) - \lambda(\beta) = \infty \quad \textit{and} \quad \sum_{n\geq 1} K(n)\log(1/K(n)) < \infty.$$

Note that the condition $\lim_{\beta\to\infty}\beta\lambda'(\beta)-\lambda(\beta)=\infty$ is satisfied in particular whenever the environment is unbounded (see Proposition 3.A.1). Our third result of this part provides a sufficient condition to have $\beta_c=\infty$.

THEOREM 4.7. *Under the following conditions on the law of the increments:*

(a) *$K(\cdot)$ is unimodal and symmetric around 0 (that is, $K(n)=K(-n)$ and $K(n)\geq K(n+1)$ for all $n\geq 0$)*

(b)
$$K(n)\geq\frac{(\log\log n)^{\alpha}}{n(\log n)^2},\tag{4.10}$$
for all n sufficiently large, and some $\alpha>1$,

(c) *and*
$$\frac{\mathbf{P}\left[X_1\in(s_n,2ns_n)\right]}{\mathbf{P}\left[X_1\geq s_n\right]}\leq\frac{1}{n^{\gamma}},\tag{4.11}$$
where $\gamma>\frac{1}{2}$ and
$$s_n:=\min\left\{s\in\mathbb{N}:\mathbf{P}\left[X_1\geq s\right]\leq\frac{(\log n)^2}{n}\right\},\tag{4.12}$$
for all n sufficiently large,

then, $\beta_c=\infty$.

We finish this introduction with some final remarks. By Theorem 4.6, the polymer presents a strong disorder phase if the asymptotic of the increment distribution has the following form
$$K(n)=\frac{(\log\log|n|)^{\alpha}}{|n|(\log|n|)^2}(1+o(1))\tag{4.13}$$
where $\alpha<-1$. By Theorem 4.5, this provides an example for directed polymer model for which the two critical points do not coincide ($\beta_c<\bar\beta_c$). We had mentioned before that the equality of the critical values β_c and $\bar\beta_c$ has been conjectured to be true and while it is not invalidating the conjecture concerning the nearest neighbor model, it sheds a new light on it.

If K satisfies (4.13) with $\alpha>1$ then Theorem 4.7 applies and we can conclude that $\beta_c=\infty$.

In the case where $\alpha\in[-1,1]$ we are not able to say whether strong disorder holds or not although we believe that the sufficient condition Theorem 4.6 might also be necessary to have $\beta_c<\infty$. More precisely,

CONJECTURE 4.8. *Assuming that the environment is unbounded form above, we have the following equivalence*
$$\beta_c<\infty\quad\Leftrightarrow\quad\sum_{n\geq 1}K(n)\log\frac{1}{K(n)}<\infty.\tag{4.14}$$

CHAPTER 2

Directed Polymer with γ-stable random environments

1. Introduction

1.1. The model. A directed polymer system consists in a random distribution of walks or paths in $\mathbb{Z}^d$ parametrized by time. The graph of the walk in $\mathbb{Z}^{d+1}$ is the *polymer* which stretches in the time direction and so is called *directed*. We consider walks interacting with a random space-time environment, with power-law distribution and we show bounds on the free energy at high temperature. Directed polymers in a random environment have appeared originally in the physics literature as an effective model for the interface in two-dimensional Ising model with random exchange interactions **[25]** and has become an interesting subject of study for many authors ever since (see **[14, 16]** for a review on the matter).

Consider the following version of the directed polymer model (we opted to introduce the model first in the more conventional setup with the environment randomness appearing in exponential form as it is the more convenient option when referring to the existing literature. We introduce and justify our modified setup in Section 1.4): Let $\mathbf{P}_x$ be the probability measure on the space $(\Omega, \mathcal{F}) := \left(\left(\mathbb{Z}^d\right)^{\mathbb{N}}, \mathcal{P}(\mathbb{Z}^d)^{\otimes \mathbb{N}} \right)$ of sequences $S := (S_n)_{n \geq 0}$ such that:

$$\begin{aligned} &S_0 = x, \\ &\{S_n - S_{n-1}\}_{n \geq 1} \text{ is an i.i.d. sequence, and} \\ &\mathbf{P}_x[S_1 = x + e_j] = \mathbf{P}_x[S_1 = x - e_j] = \frac{1}{2d}, \end{aligned} \tag{1.1}$$

for all $j \leq d$ where $\{e_1, ..., e_d\}$ is the canonical basis of $\mathbb{R}^d$. The set of points $\{(n, S_n) : n \geq 0\} \subset \mathbb{Z}^{1+d}$ represents the graph of a simple random walk on $\mathbb{Z}^d$.

Independently, also consider a sequence of i.i.d. random variables $\eta := \{\eta_{n,z} : n \in \mathbb{N}, z \in \mathbb{Z}^d\}$, called *the environment*, defined on a probability space $(\Lambda, \mathcal{F}, \mathbb{P})$, that satisfies,

$$\begin{aligned} &\mathbb{E}[\eta_{0,0}] = 0 \text{ and} \\ &\mathbb{E}[\exp(\beta \eta_{0,0})] < \infty, \text{ for all } \beta \in \mathbb{R}. \end{aligned} \tag{1.2}$$

For a given $\beta > 0$, $N \in \mathbb{N}$ and a fixed realization of the environment η, we define the measure $\mathbf{P}_N^{\beta,\eta}$ on the space Ω, called the *polymer measure*, by its Radon-Nikodym derivative with respect to $\mathbf{P}_0$:

$$\frac{\mathrm{d}\mathbf{P}_N^{\beta,\eta}}{\mathrm{d}\mathbf{P}_0}(S) = \frac{1}{Z_N^{\beta,\eta}} \exp\left(\beta \sum_{n=1}^{N} \eta_{n,S_n} \right), \tag{1.3}$$

where $Z_N^{\beta,\eta}$ is the positive normalization factor that makes $\mathbf{P}_N^{\beta,\eta}$ a probability measure. We call $Z_N^{\beta,\eta}$ the *partition function* of the system and its value is given by

$$Z_N^{\beta,\eta} = \mathbf{E}_0\left[\exp\left(\beta\sum_{n=1}^{N}\eta_{n,S_n}\right)\right] = (2d)^{-N}\sum_{S\in\Omega_N}\exp\left(\beta\sum_{n=1}^{N}\eta_{n,S_n}\right), \tag{1.4}$$

where

$$\Omega_N := \left\{S\in\mathbb{Z}^N : S_0=0, |S_n-S_{n-1}|=1, \forall n\in[1,N]\cap\mathbb{Z}\right\}. \tag{1.5}$$

The goal for this model is to study how the presence of the environment affects the distribution of the random walk. Intuitively, this new measure $\mathbf{P}_N^{\beta,\eta}$ rewards (penalizes) walks that visit sites with higher (smaller) values of the environment. The parameter β (the inverse temperature) is used to increase or decrease the possible influence of the environment over the measure $\mathbf{P}_N^{\beta,\eta}$. Notice that when $\beta=0$, $\mathbf{P}_N^{\beta,\eta}$ becomes $\mathbf{P}_0$.

1.2. Known facts. In [**9**], Bolthausen observed that the renormalized partition function

$$W_N^{\beta,\eta} := \frac{Z_N^{\beta,\eta}}{\mathbb{E}\left[Z_N^{\beta,\eta}\right]}, \tag{1.6}$$

is a positive martingale with respect to the sequence of σ-fields $\{\mathcal{G}_N\}_{N\geq 0}$ where $\mathcal{G}_N := \sigma\{\eta_{n,z} : 0\leq n\leq N, z\in\mathbb{Z}^d\}$. By the Martingale Convergence Theorem, it follows that the limit

$$W_\infty^{\beta,\eta} := \lim_{N\to\infty} W_N^{\beta,\eta}, \tag{1.7}$$

exists $\mathbb{P}$-*a.s.* and is a non-negative random variable. The event $\{W_\infty^{\beta,\eta}=0\}$ belongs to the tail σ-field of $\{\mathcal{G}_N, N\geq 0\}$. Hence, by Kolmogorov's $0-1$ Law,

$$\mathbb{P}\left\{W_\infty^{\beta}>0\right\}\in\{0,1\}. \tag{1.8}$$

This dichotomy allows to define a natural manner to characterize the influence of disorder. Following standard terminology we say that we have *weak disorder* if $W_\infty^\beta>0$ $\mathbb{P}$-*a.s.* and *strong disorder* if $W_\infty^\beta=0$ $\mathbb{P}$-*a.s.*.

Roughly speaking, weak disorder implies that the polymer paths have the same behavior as the simple random walk (delocalized phase). A series of papers [**26, 9, 1, 37, 13**] lead to the following: Assuming $d\geq 3$ and weak disorder, the measures $\mathbf{P}_N^{\beta,\eta}$, after rescaling, converge in law to the Brownian motion, for almost all realizations of the environment.

On the other hand, strong disorder implies that the polymer is largely influenced by the disorder and is attracted to sites with favorable environment (localized phase). We mention [**12**, Theorem 2.1], where it is shown that for $\beta>0$,

$$\left\{W_\infty^{\beta,\eta}=0\right\} = \left\{\sum_{n\geq 1}\left(\mathbf{P}_{n-1}^{\beta,\eta}\right)^{\otimes 2}[S_n=S_n']=\infty\right\}\ \mathbb{P}\text{-a.s.}, \tag{1.9}$$

where S and S' are two independent polymers with distribution $\mathbf{P}_{n-1}^{\beta,\eta}$. Moreover, if $\mathbb{P}[W_\infty^{\beta,\eta}=0]=1$, then there exists some constants c_1, $c_2 \in (0,\infty)$ such that,

$$-c_1 \log W_N^{\beta,\eta} \leq \sum_{n\geq 1}^{N} \left(\mathbf{P}_{n-1}^{\beta,\eta}\right)^{\otimes 2} [S_n = S_n'] \leq -c_2 \log W_N^{\beta,\eta}, \tag{1.10}$$

for N large enough, $\mathbb{P}$-a.s. This result suggests that when we have strong disorder, the polymer is more attracted to sites with favorable environment and the probability of two of them to occupy the same last site increases (recall that for the simple random walk, $\mathbf{P}_0^{\otimes 2}[S_n = S_n'] \sim \frac{C_d}{n^{d/2}}$). Also the decay property of W_N is reflected in some specific localization property of the path.

In **[13]**, it was also shown that there exists a critical value $\beta_c = \beta_c(d) \in [0,\infty]$ with

$$\beta_c = 0 \text{ for } d = 1,2 \text{ and} \tag{1.11}$$

$$\beta_c > 0 \text{ for } d \geq 3, \tag{1.12}$$

such that there is weak disorder for $\beta \in [0,\beta_c)$ and strong disorder for $\beta > \beta_c$.

1.3. Free energy. A lot of information about the model is encoded in the following quantity

$$F(\beta) := \lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta,\eta}, \tag{1.13}$$

called the *free energy* of the model. This limit exists and is non-random [**12**, Proposition 2.5]. Moreover, by Jensen's inequality, $F(\beta) \leq \lambda(\beta) := \log \mathbb{E} e^{\beta\eta}$ and the function $\beta \mapsto p(\beta) := F(\beta) - \lambda(\beta)$ is continuous and non-increasing. In particular, there exists $\bar\beta_c = \bar\beta_c(d)$ with

$$0 \leq \beta_c \leq \bar\beta_c \leq \infty, \tag{1.14}$$

such that

$$p(\beta) = \begin{cases} = 0 & \text{if } \beta \leq \bar\beta_c \\ < 0 & \text{if } \beta > \bar\beta_c \end{cases} \tag{1.15}$$

Notice that if $W_\infty^{\beta,\eta} > 0$ then $p(\beta) = 0$. In view of this, we say that *very strong disorder* holds when $p(\beta) < 0$.

Some estimates have been proved for the free energy. In dimension $d = 1$, it is known that $p(\beta)$ is of order $-\beta^4$ as $\beta \to 0$ **[27, 43, 2]**. In **[34]** is been shown that, under some conditions on the environment,

$$\lim_{\beta\to 0} \frac{p(\beta)}{\beta^4} = -\frac{1}{6}. \tag{1.16}$$

In dimension $d = 2$, it has been proved **[5]** that,

$$\lim_{\beta\to 0} \beta^2 \log |p(\beta)| = -\pi. \tag{1.17}$$

In particular, we have that $\beta_c = \bar\beta_c = 0$, for $d = 1,2$.

1.4. Our work. The techniques used to prove weak disorder in dimension $d \geq 3$ rely, in a crucial way, on the boundedness of the second moment of the partition function [**13**]. In the present paper, we study the model in the case where the environment is i.i.d. but with a distribution belonging to the domain of attraction of a stable law with parameter $\gamma \in (1,2)$; In this case the partition function has an infinite second moment. Specifically we consider the sequence of i.i.d. random variables $\omega = \{\omega_{n,z} : n \in \mathbb{N}, z \in \mathbb{Z}^d\}$, that satisfies,

$$\begin{aligned} &\omega_{0,0} \geq -1 \quad \mathbb{P}\text{-a.s.}, \\ &\mathbb{E}[\omega_{0,0}] = 0 \text{ and} \\ &\mathbb{P}[\omega_{0,0} > x] \overset{x\to\infty}{\sim} C_{\mathbb{P}} x^{-\gamma}, \text{ for } \gamma \in (1,2), \end{aligned} \tag{1.18}$$

as the environment, where $f(x) \overset{x\to\infty}{\sim} g(x)$ means $\lim_{x\to\infty} f(x)/g(x) = 1$. For $\beta \in [0,1)$, $N \in \mathbb{N}$ and a fixed realization of the environment ω, we write the polymer measure $\mathbf{P}_N^{\beta,\omega}$ as

$$\frac{\mathrm{d}\mathbf{P}_N^{\beta,\omega}}{\mathrm{d}\mathbf{P}_0}(S) = \frac{1}{Z_N^{\beta,\omega}} \left(\prod_{n=1}^{N} (1 + \beta\omega_{n,S_n}) \right), \tag{1.19}$$

where as before, the partition function $Z_N^{\beta,\omega} := \mathbf{E}_0 \left[\prod_{n=1}^{N} (1 + \beta\omega_{n,S_n}) \right]$. Notice that this measure is well defined since the environment is bounded below by our assumption. Our purpose in this model is to understand how the parameters β and γ, affect the measure $\mathbf{P}_0$ and the existence of the localized phase. The expression (1.19) differs substantially from (1.3), however by setting $\widehat{\omega}_{n,z}^{\beta} := \log(1 + \beta\omega_{n,z})$ we can rewrite it in the Gibbsian framework, with

$$Z_N^{\beta,\omega} = \mathbf{E}_0 \left[\exp\left(\sum_{n=1}^{N} \widehat{\omega}_{n,S_n}^{\beta} \right) \right]. \tag{1.20}$$

We choose to work with the expression (1.19) for our polymer measure, because as we said before, we want to study the phenomenology when the partition function is no longer square integrable and we do not want changes in disorder intensity β to affect the power-tail exponent of the environment's distribution γ which is the parameter whose influence on the phase transition we wish to examine.

Our assumption $\gamma \in (1,2)$ makes the second moment of the partition function infinite. Since the second moment method plays such a crucial role in the analysis in [**9, 26**], it is reasonable to expect that the picture differs in this case, possibly due to the influence of extreme values of the field ω as often observed in heavy tailed setups. We prove that for some values of γ, we have strong disorder for all $\beta > 0$, in all dimensions: specifically, for $d \geq 3$, there is a critical value $\gamma_c = \gamma_c(d) := 1 + \frac{2}{d}$, such that $\gamma \in (1, \gamma_c]$ implies strong disorder, for all $\beta \in (0,1)$ and $\gamma \in (\gamma_c, 2]$ implies weak disorder, for all $\beta > 0$ sufficiently small. We summarize our results bellow. Notice that we can now write

$$p(\beta) = \lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta,\omega}, \tag{1.21}$$

since $\mathbb{E}\left[Z_N^{\beta,\omega}\right] = 1$. We show this convergence and the continuity and monotonicity of $\beta \to p(\beta)$ in Theorem 2.A.1. From now on, we assume the environment always satisfies (1.18) and unless otherwise specified, the polymer measure/partition function is the one defined in (1.19). Assuming $d \geq 3$ and given $\gamma_c := 1 + 2/d$,

THEOREM 1.1. *If $\gamma < \gamma_c$ then for all $\varepsilon > 0$ there exists $\beta_0(\varepsilon) > 0$ such that for all $\beta \in [0, \beta_0(\varepsilon))$,*

$$-C_1\beta^{\alpha-\varepsilon} \leq p(\beta) \leq -C_2\beta^{\alpha} \tag{1.22}$$

where $\alpha = \alpha(d,\gamma) := \frac{\gamma(\gamma_c-1)}{\gamma_c-\gamma}$ and $C_1, C_2 > 0$. In particular $\beta_c = \bar{\beta}_c = 0$.

THEOREM 1.2. *If $\gamma = \gamma_c$ then*

$$p(\beta) \leq -C_4 e^{C_3/\beta^{2\gamma}} \tag{1.23}$$

for $C_3, C_4 > 0$. In particular $\beta_c = \bar{\beta}_c = 0$

THEOREM 1.3. *If $\gamma > \gamma_c$ then $\beta_c > 0$.*

1.5. Related works. Among other works that deal with heavy tail environments we mention **[7]** and **[4]**, where in the setup (1.3) the environment η is allowed to belong to the domain of attraction of a α-stable law and it is studied properties of paths trajectories drawn from the polymer measure. In those contexts there is no free energy so the work is fundamentally different from ours.

In **[17]**, it is studied the influence of the jump distribution on the delocalization-localization transition and the interplay between jump tails, spatial dimension and existence of the delocalized phase, when nearest neighbor walks are replaced by long range jumps. Our results most likely extend to that setup, the criterion for having a weak disorder phase in dimension $d = 1$ becoming $\gamma > \gamma_c = 1 + \alpha$ where $\alpha \in (0, 1)$ is the exponent of the random walk. We also mention **[29]** as another case where a change in the environment setup (that is: moving from the i.i.d. setup to a strong spatial correlation in the environment) modifies the criterion for having no phase transition.

1.6. Organization of the chapter. We show upper bounds for the free energy in Sections 2 and 3, for the cases $\gamma < \gamma_c(d)$ and $\gamma = \gamma_c(d)$ respectively. In Section 4 we bound some fractional moments of the partition function, when $\gamma > \gamma_c(d)$. This leads to Theorem 1.2 through a uniform integrability argument. In Section 5 we show a lower bound for the free energy when $\gamma < \gamma_c(d)$. This completes the proof of Theorem 1.1.

1.7. Notation. For simplicity, we write $\mathbf{E}[\cdot]$ and $\mathbf{P}[\cdot]$ instead of $\mathbf{E}_0[\cdot]$ and $\mathbf{P}_0[\cdot]$ for the law of the simple random walk, starting form the origin. We also sometimes omit brackets from expectations when it is clear from the context with respect to which random variable it is integrating. For example, we may write $\mathbf{E}[(\cdot)^q]$ as $\mathbf{E}(\cdot)^q$. Also, to avoid ambiguities, $\mathbf{E}[\cdot]^q$ always means $(\mathbf{E}[\cdot])^q$.

2. The disorder relevance case $\gamma < \gamma_c(d)$.

In this section we show an upper bound for the free energy that is required for the first limit of Theorem 1.1. We base upon the proof of the upper bound of [**30**, Theorem 1.4], where an analogous bound is proved, in the setup (1.3). The proof combines coarse graining, a fractional moment method and a different idea for the change of measure: we penalize sites whose values are above a certain threshold. These ideas have appeared originally in [**20**] for the pinning model and in [**39**] for the copolymer model.

PROPOSITION 2.1. *Assuming the environment's distribution satisfies condition* (1.18) *and* $\gamma < \gamma_c(d)$ *we have,*

$$p(\beta) \leq -C\beta^{\alpha}, \tag{2.1}$$

where $\alpha = \frac{\gamma(\gamma_c-1)}{\gamma_c-\gamma}$, *for some positive constant* C *and all* β *sufficiently small.*

PROOF. Fix $n \in \mathbb{N}$ and $\theta \in (0,1)$. By Jensen's Inequality,

$$p(\beta) \leq \lim_{m\to\infty} \frac{1}{nm\theta} \log \mathbb{E}\left(Z_{mn}^{\beta,\omega}\right)^{\theta}. \tag{2.2}$$

Notice that we replace the expectation of a logarithm by the estimation of a fractional moment, which in principle should be easier to handle. The goal is to prove that, for all $m \in \mathbb{N}$,

$$\mathbb{E}\left(Z_{mn}^{\beta,\omega}\right)^{\theta} \leq \exp(-m), \tag{2.3}$$

for some convenient value of n. In fact, assuming (2.3) and letting n be a square integer such that $\frac{C_1}{2n} \leq \beta^{\alpha} < \frac{C_1}{n}$, where $C_1 > 0$ is a constant to be defined later, then (2.3) implies

$$p(\beta) \leq -\frac{\beta^{\alpha}}{C_1\theta}. \tag{2.4}$$

Let us now prove (2.3). We first decompose the partition function $Z_{mn}^{\beta,\omega}$ according to the position of the walk at times $n, 2n, 3n, ..., mn$:

$$Z_{mn}^{\beta,\omega} = \sum_{y_1,\dots,y_m\in\mathbb{Z}^d} \mathbf{E}\left[\prod_{i=1}^{nm}(1+\beta\omega_{i,S_i})\prod_{k=1}^{m}\mathbf{1}_{\left\{S_{kn}\in I_{y_k}\right\}}\right], \tag{2.5}$$

where the region

$$I_z := \{x = (x_1, ..., x_d) \in \mathbb{Z}^d : z_j\sqrt{n} \leq x_j < (z_j+1)\sqrt{n} \quad \forall j \leq d\}, \tag{2.6}$$

is defined for any $z = (z_1, ..., z_d) \in Z^d$. Using the inequality

$$(a_1 + \cdots + a_t)^{\theta} \leq a_1^{\theta} + \cdots + a_t^{\theta}, \tag{2.7}$$

which holds for any $\theta \in [0,1]$ and $a_i \geq 0$, we deduce

$$\mathbb{E}\left(Z_{mn}^{\beta,\omega}\right)^{\theta} \leq \sum_{y_1,\dots,y_m\in\mathbb{Z}^d} \mathbb{E}\left(\widetilde{Z}_{y_1,\dots,y_m}\right)^{\theta}, \tag{2.8}$$

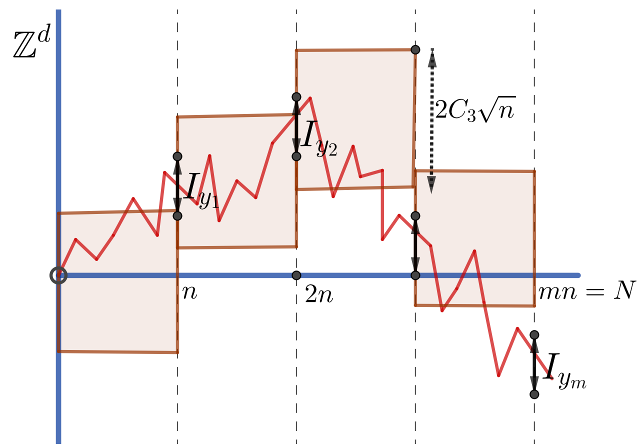

FIGURE 1. The change of measure used to upper bound $\mathbb{E}\left(\widetilde{Z}_{y_1,\dots,y_m}\right)$ penalizes higher values of the environment in regions where the polymer likely visits.

where

$$\widetilde{Z}_{y_1,\dots,y_m} := \mathbf{E}\left[\prod_{i=1}^{nm}(1+\beta\omega_{i,S_i})\prod_{k=1}^{m}\mathbf{1}_{\{S_{kn}\in I_{y_k}\}}\right].$$

In order to bound the expectation of $\widetilde{Z}_{y_1,\dots,y_m}$, we introduce a change of measure that penalizes higher values of the environment on regions that paths are likely to visit, increasing the value of the partition function. Consider the function

$$\begin{aligned} g &: \mathbb{R} \to \mathbb{R} \\ &x \mapsto 1 - \tfrac{1}{2}\mathbf{1}_{\{x \geq C_2 n^q\}}, \end{aligned} \tag{2.9}$$

for C_2 and q constants whose values are chosen later. For $\mathbf{Y} := (y_1, \dots, y_m) \in \left(\mathbb{Z}^d\right)^{\otimes m}$, define the region

$$J_{\mathbf{Y}} := \{(kn+i, \sqrt{n}y_k+z) \in \mathbb{Z}^{1+d} : k = 0,\dots,m-1, i = 1,\dots,n, |z_j| \leq C_3\sqrt{n} \quad \forall j \leq d\}, \tag{2.10}$$

where C_3 is a positive constant. This region is defined to take advantage of the concentration properties of the simple random walk. By Holder's Inequality,

$$\begin{aligned} \mathbb{E}\left(\widetilde{Z}_{y_1,\dots,y_m}\right)^{\theta} &= \mathbb{E}\left[\prod_{(i,z)\in J_{\mathbf{Y}}} g(\omega_{i,z})^{-(1-\theta)} \prod_{(i,z)\in J_{\mathbf{Y}}} g(\omega_{i,z})^{1-\theta}\left(\widetilde{Z}_{y_1,\dots,y_m}\right)^{\theta}\right] \\ &\leq \mathbb{E}\left[\prod_{(i,z)\in J_{\mathbf{Y}}} g(\omega_{i,z})^{-1}\right]^{1-\theta} \mathbb{E}\left[\prod_{(i,z)\in J_{\mathbf{Y}}} g(\omega_{i,z})^{\frac{1-\theta}{\theta}}\left(\widetilde{Z}_{y_1,\dots,y_m}\right)\right]^{\theta}, \end{aligned} \tag{2.11}$$

Notice that $|J_{\mathbf{Y}}| = nm(2C_3\sqrt{n})^d$. Hence, for the first factor, we have

$$\begin{aligned}
\mathbb{E}\left[\prod_{(i,z)\in J_{\mathbf{Y}}} g(\omega_{i,z})^{-1}\right] &= \mathbb{E}\left[g(\omega_{0,0})^{-1}\right]^{|J_{\mathbf{Y}}|} \\
&\le \left(1+2\mathbb{P}\left[\omega_{1,0} > C_2 n^q\right]\right)^{2^d C_3^d m n^{1+d/2}} \\
&\le \exp\left(2^{d+1}C'' m (C_2 n^q)^{-\gamma} C_3^d n^{1+d/2}\right) \\
&= \exp\left(2^{d+1}C'' m\right),
\end{aligned} \tag{2.12}$$

by choosing $C_2 = C_3^{d/\gamma}$ and $q = \frac{2+d}{2\gamma}$. Notice that for the second inequality we used

$$C' x^{-\gamma} \le \mathbb{P}[\omega_{1,0} > x] \le C'' x^{-\gamma}, \tag{2.13}$$

for some constants C', C'' with $C' < C_{\mathbb{P}} < C''$, for all x sufficiently large, by (1.18). By Fubini's Theorem, we have for the second factor,

$$\begin{aligned}
\mathbb{E}\left[\prod_{(i,z)\in J_Y} g(\omega_{i,z})^{\frac{1-\theta}{\theta}}\left(\widetilde{Z}_{y_1,\dots,y_m}\right)\right] &= \mathbb{E}\left[\prod_{(i,z)\in J_Y} g(\omega_{i,z})^{\frac{1-\theta}{\theta}}\mathbf{E}\left[\prod_{i=1}^{nm}(1+\beta\omega_{i,S_i})\prod_{k=1}^{m}\mathbf{1}_{\left\{S_{kn}\in I_{y_k}\right\}}\right]\right] \\
&= \mathbf{E}\left[\mathbb{E}\left[\prod_{(i,z)\in J_Y} g(\omega_{i,z})^{\frac{1-\theta}{\theta}}\prod_{i=1}^{nm}(1+\beta\omega_{i,S_i})\prod_{k=1}^{m}\mathbf{1}_{\left\{S_{kn}\in I_{y_k}\right\}}\right]\right] \\
&\le \mathbf{E}\left[\prod_{i\in\mathcal{I}(S,J_Y)}\mathbb{E}\left[(1+\beta\omega_{1,0})g(\omega_{1,0})^{\frac{1-\theta}{\theta}}\right]\prod_{k=1}^{m}\mathbf{1}_{\left\{S_{kn}\in I_{y_k}\right\}}\right],
\end{aligned} \tag{2.14}$$

where for a given walk S and a finite subset $J \subset \mathbb{Z}^{1+d}$ we define

$$\mathcal{I}(S,J) := \{i \in \mathbb{N} : (i,S_i) \in J\}. \tag{2.15}$$

In the last inequality above, we neglect sites on J_Y, which the paths do not visit, since $\mathbb{E}[g(\omega_{(1,0)})] \le 1$ and sites for which paths do visit but are outside J_Y, since $\mathbb{E}[1+\beta\omega_{1,0}] = 1$. A simple computation shows that, for $\theta = 1/2$,

$$\begin{aligned}
\mathbb{E}\left[(1+\beta\omega_{1,0})g(\omega_{1,0})^{\frac{1-\theta}{\theta}}\right] &= \mathbb{E}\left[(1+\beta\omega_{1,0})\left(1-\tfrac{1}{2}\mathbf{1}_{\{\omega_{1,0}\ge C_2 n^q\}}\right)\right] \\
&\le \exp(-\tfrac{1}{2}C'\beta(C_2 n^q)^{-(\gamma-1)}).
\end{aligned} \tag{2.16}$$

Using the Markov Property, we obtain

$$\begin{aligned}
\mathbb{E}\left[\prod_{(i,z)\in J_Y} g(\omega_{i,z})^{\frac{1-\theta}{\theta}}\left(\widetilde{Z}_{y_1,\dots,y_m}\right)\right] &\le \mathbf{E}\left[\exp(-\tfrac{1}{2}C'\beta(C_2 n^q)^{-(\gamma-1)}\left|\mathcal{I}(S,J_Y)\right|)\prod_{k=1}^{m}\mathbf{1}_{\left\{S_{kn}\in I_{y_k}\right\}}\right] \\
\le \prod_{k=1}^{m}\max_{x\in I_0}\mathbf{E}_x\left[\exp(-\tfrac{1}{2}C'\beta(C_2 n^q)^{-(\gamma-1)}|\mathcal{I}(S,\widetilde{J})|)\mathbf{1}_{\left\{S_n\in I_{y_k-y_{k-1}}\right\}}\right]&,
\end{aligned} \tag{2.17}$$

where we define the set $\widetilde{J}$ as

$$\widetilde{J} := \{(i,z) \in \mathbb{Z}^{1+d} : i = 1,...,n, |z_j| \le C_3\sqrt{n} \quad \forall j \le d\}. \tag{2.18}$$

Combining (2.8) (2.11), (2.12) and (2.17) we obtain

$$\log \mathbb{E}\left[\left(Z_{mn}^{\beta,\omega}\right)^{\theta}\right] \le m2^dC'' + m\log\sum_{y\in\mathbb{Z}^d}\max_{x\in I_0}\mathbf{E}_x\left[e^{-\frac{1}{2}C'\beta(C_2n^q)^{-(\gamma-1)}|\mathcal{I}(S,\widetilde{J})|}\mathbf{1}_{\{S_n\in I_y\}}\right]^{1/2}. \tag{2.19}$$

Then it is enough to show that the expression

$$\sum_{y\in\mathbb{Z}^d}\max_{x\in I_0}\mathbf{E}_x\left[e^{-\frac{1}{2}C'\beta(C_2n^q)^{-(\gamma-1)}|\mathcal{I}(S,\widetilde{J})|}\mathbf{1}_{\{S_n\in I_y\}}\right]^{1/2}, \tag{2.20}$$

is sufficiently small. For values of y far form the origin, we neglect the contribution of the change of measure:

$$\sum_{|y|_\infty>C_4}\max_{x\in I_0}\mathbf{E}_x\left[e^{-\frac{1}{2}C'\beta(C_2n^q)^{-(\gamma-1)}|\mathcal{I}(S,\widetilde{J})|}\mathbf{1}_{\{S_n\in I_y\}}\right]^{1/2} \le \sum_{|y|_\infty>C_4}\max_{x\in I_0}\mathbf{P}_x\left[S_n\in I_y\right]^{1/2}. \tag{2.21}$$

Applying standard results on sums of i.i.d. random variables, we can bound

$$\max_{x\in I_0}\mathbf{P}_x\left[S_n\in I_y\right] \le \mathbf{P}\left[|S_n|_\infty \ge (|y|_\infty-1)\sqrt{n}\right] \le e^{-c|y|_\infty^2}, \tag{2.22}$$

for a fixed constant $c>0$. In this manner, we make the sum (2.21) arbitrarily small, by choosing C_4 large enough. For values of y near from the origin, we neglect the condition over S_n:

$$\begin{aligned}\sum_{|y|_\infty\le R}&\max_{x\in I_0}\mathbf{E}_x\left[e^{-\frac{1}{2}C'\beta(C_2n^q)^{-(\gamma-1)}|\mathcal{I}(S,\widetilde{J})|}\mathbf{1}_{\{S_n\in I_y\}}\right]^{1/2}\\ &\le (2R)^d\max_{x\in I_0}\mathbf{E}_x\left[e^{-\frac{1}{2}C'\beta(C_2n^q)^{-(\gamma-1)}|\mathcal{I}(S,\widetilde{J})|}\right]^{1/2}\\ &\le (2R)^d\mathbf{E}\left[e^{-\frac{1}{2}C'\beta(C_2n^q)^{-(\gamma-1)}|\mathcal{I}(S,\bar{J})|}\right]^{1/2},\end{aligned} \tag{2.23}$$

where the set $\bar{J}$ is defined as

$$\bar{J} := \{(i,z) \in \mathbb{Z}^{1+d} : i = 1,...,n, |z_j| \le (C_3-1)\sqrt{n} \quad \forall j \le d\}. \tag{2.24}$$

In the last line, we use the fact that for any walk S, starting at zero and $x \in I_0$,

$$\{i : (i,S_i) \in \bar{J}\} \subset \{i : (i,x+S_i) \in \widetilde{J}\}. \tag{2.25}$$

The last expression can be bounded as

$$\mathbf{E}\left[e^{-\frac{1}{2}C'\beta(C_2n^q)^{-(\gamma-1)}|\mathcal{I}(S,\widetilde{J})|}\right] \le \mathbf{P}\left[\exists i : (i,S_i) \notin \bar{J}\right] + e^{-\frac{1}{2}C'\beta(C_2n^q)^{-(\gamma-1)}n}. \tag{2.26}$$

The first term in the last sum can be made arbitrarily small, by choosing C_3 sufficiently big. For the second term, using our initial assumption: $n > \frac{C_1}{2\beta^\alpha}$ and the values of $\alpha = \frac{\gamma(\gamma_c - 1)}{\gamma_c - \gamma}$, $q = \frac{2+d}{2\gamma}$ and $\gamma_c = 1 + \frac{2}{d}$, we obtain,

$$e^{-\frac{1}{2}C'\beta(C_2 n^q)^{-(\gamma-1)}n} \leq e^{-\frac{1}{2}C'\beta C_2^{-(\gamma-1)}n^{1/\alpha}} \leq e^{-\frac{1}{2}C'C_2^{-(\gamma-1)}\left(\frac{C_1}{2}\right)^{1/\alpha}}, \tag{2.27}$$

which can also be made arbitrarily small, by choosing C_1 sufficiently big. This completes the proof of the theorem. □

REMARK 2.2. *The value of* $\gamma_c = 1 + \frac{2}{d}$ *appears naturally in the computations. Notice that in* (2.27)*, for the value of* α *to be positive we need* $1 - q(\gamma - 1) > 0$ *which implies* $\gamma < 1 + 2/d$.

3. The marginal case $\gamma = \gamma_c(d)$.

In this section we show an upper bound for the free energy for the case $\gamma = \gamma_c$. This completes the proof that very strong disorder holds for all values of $\beta > 0$, when $\gamma \leq \gamma_c$. The first part of the proof shares steps from the case when $\gamma < \gamma_c$ up to the point of choosing the change of measure function, since that is no longer suitable in this case (cf. Remark 2.2). The approach we take here is to penalize regions of the environment that contain a pair of sites whose values are above a certain threshold that depends on the distance between each other. The idea behind this is to account the fact that pairs of sites that are closer to each other produces a more noticeable effect on the partition function. The construction of this change of measure is inspired by the one used in **[28]** to prove disorder relevance for the pinning model.

PROPOSITION 3.1. *When* $\gamma = \gamma_c$*, and assuming the usual hypothesis on the environment's distribution, we have*

$$p(\beta) \leq -C \exp\left(-\frac{c}{\beta^{2\gamma}}\right), \tag{3.1}$$

for all $\beta > 0$ *sufficiently small, and some fixed constants* $C, c > 0$.

PROOF. Let $n \in \mathbb{N}$ be a squared natural number satisfying

$$\exp\left(\frac{C_5}{\beta^{2\gamma}}\right) \leq n \leq \exp\left(\frac{2C_5}{\beta^{2\gamma}}\right). \tag{3.2}$$

where C_5 is a constant to be chosen later. As before, our goal is to show,

$$\mathbb{E}\left[\left(Z_{mn}^{\beta,\omega}\right)^\theta\right] \leq \exp\left(-m\right), \tag{3.3}$$

for all $m \in \mathbb{N}$ and some $\theta \in (0,1)$. Using the notation from the previous section,

$$\mathbb{E}\left[\left(Z_{mn}^{\beta,\omega}\right)^\theta\right] \leq \sum_{y_1,\dots,y_m \in \mathbb{Z}^d} \mathbb{E}\left[\left(\widetilde{Z}_{y_1,\dots,y_m}\right)^\theta\right], \tag{3.4}$$

where $\widetilde{Z}_{y_1,\dots,y_m} = \mathbf{E}\left[\prod_{i=1}^{nm}(1+\beta\omega_{(i,S_i)})\prod_{k=1}^{m}\mathbf{1}_{\left\{S_{kn}\in I_{y_k}\right\}}\right]$ for $y_1,\dots,y_m \in \mathbb{Z}^d$. Fixing the collection of vertex $\mathbf{Y} := (y_1,\dots,y_m)$ we define the blocks $(B_k)_{1\le k\le m}$ as

$$B_k := \left\{(i,z)\in\mathbb{N}\times\mathbb{Z}^d : \lceil i/n\rceil = k, \left|z-\sqrt{n}y_{k-1}\right|_\infty \le C_6\sqrt{n}\right\}, \tag{3.5}$$

where $C_6 > 0$ is chosen later. We also define the set of functions $\{g_k(\omega)\}_{1\le k\le m}$ as

$$g_k(\omega) := \exp\left(-M 1_{A_k}\right), \tag{3.6}$$

where

$$A_k := \left\{\omega : \exists (i,z),(j,z')\in B_k \text{ with } |z-z'|\le C_7\sqrt{|i-j|}, \omega_{i,z}\wedge\omega_{j,z'}\ge V(|i-j|)\right\}, \tag{3.7}$$

and

$$V(t) := \exp(M^2)\left(C_6^d C_7^d n^{1+\frac{d}{2}} t^{1+\frac{d}{2}}\log n\right)^{\frac{1}{2\gamma}}, \tag{3.8}$$

for $t > 0$ and $V(0) := \infty$. We choose the expression above for $V(t)$ so that the probability of the event A_k is well controlled (see the computation (3.12) below). Now we can define our change of measure function $G_{\mathbf{Y}}(\omega)$ as

$$G_{\mathbf{Y}}(\omega) := \prod_{k=1}^{m} g_k(\omega). \tag{3.9}$$

Apply Hölder's Inequality to obtain

$$\mathbb{E}\left[\left(\widetilde{Z}_{y_1,\dots,y_m}\right)^\theta\right] \le \mathbb{E}\left[G_I^{-\frac{\theta}{1-\theta}}\right]^{1-\theta}\mathbb{E}\left[G_I\widetilde{Z}_{y_1,\dots,y_m}\right]^\theta. \tag{3.10}$$

Notice first that

$$\mathbb{E}\left[g_k(\omega)^{-\frac{\theta}{1-\theta}}\right] = \mathbb{E}\left[g_k(\omega)^{-\frac{\theta}{1-\theta}}1_{A_k}\right] + \mathbb{E}\left[g_k(\omega)^{-\frac{\theta}{1-\theta}}1_{A_k^c}\right] \le \exp\left(\frac{M\theta}{1-\theta}\right)\mathbb{P}\left[A_k\right]+1. \tag{3.11}$$

The probabilities of the events A_k are sufficiently small so that the product above is well controlled,

$$\begin{aligned}
\mathbb{P}\left[A_k\right] &\le 2\sum_{(i,z)\in B_k}\sum_{t=1}^{n}\sum_{\substack{z'\in\mathbb{Z}^d\\ |z-z'|\le C_7\sqrt{t}}}\mathbb{P}\left[\omega_{i,z}\wedge\omega_{i+t,z'}\ge V(t)\right]\\
&\le 2\left(2C_6\sqrt{n}\right)^d n\sum_{t=1}^{n}\left(2C_7\sqrt{t}\right)^d C_{\mathbb{P}}^2\left(\exp(M^2)\left(C_6^d C_7^d n^{1+\frac{d}{2}}t^{1+\frac{d}{2}}\log n\right)^{\frac{1}{2\gamma}}\right)^{-2\gamma}\\
&= 2^{2d+1}C_{\mathbb{P}}^2\exp\left(-2\gamma M^2\right)\frac{1}{\log n}\sum_{t=1}^{n}\frac{1}{t},
\end{aligned} \tag{3.12}$$

and we can choose M sufficiently large, such that $\mathbb{E}\left[g_k(\omega)^{-\frac{\theta}{1-\theta}}\right] \le 2$. Then,

$$\mathbb{E}\left[G_I^{-\frac{\theta}{1-\theta}}\right] \le 2^m. \tag{3.13}$$

Now we are left with the estimation of the second term. As before, we have

$$\mathbb{E}\left[G_I \widetilde{Z}_{y_1,\dots,y_m}\right] \le \prod_{k=1}^{m} \max_{x\in I_0} \mathbf{E}_x\left[\mathbb{E}\left[g_1(\omega)\prod_{i=1}^{n}(1+\beta\omega_{i,S_i})\right]\mathbf{1}_{\left\{S_n\in I_{y_k-y_{k-1}}\right\}}\right]. \tag{3.14}$$

Plugging (3.13) and (3.14) in Equation (3.4), we obtain that

$$\mathbb{E}\left[\left(Z_{mn}^{\beta,\omega}\right)^{\theta}\right] \le 2^{m(1-\theta)}\left(\sum_{y\in\mathbb{Z}^d}\max_{x\in I_0}\mathbf{E}_x\left[\mathbb{E}\left[g_1(\omega)\prod_{i=1}^{n}(1+\beta\omega_{i,S_i})\right]\mathbf{1}_{\{S_n\in I_y\}}\right]^{\theta}\right)^{m}, \tag{3.15}$$

so it will be sufficient to show that

$$\sum_{y\in\mathbb{Z}^d}\max_{x\in I_0}\mathbf{E}_x\left[\mathbb{E}\left[g_1(\omega)\prod_{i=1}^{n}(1+\beta\omega_{(i,S_i)})\right]\mathbf{1}_{\{S_n\in I_y\}}\right]^{\theta} \tag{3.16}$$

is small. The contribution of y far from the origin can be controlled as in Equation (2.21). Thus it is sufficient to check that

$$\max_{x\in I_0}\mathbf{E}_x\left[\mathbb{E}\left[g_1(\omega)\prod_{i=1}^{n}(1+\beta\omega_{i,S_i})\right]\mathbf{1}_{\{S_n\in I_y\}}\right]^{\theta} \le \varepsilon, \tag{3.17}$$

for some arbitrarily small $\varepsilon > 0$. We choose C_6 sufficiently big, such that

$$\mathbf{P}\left[|S_i| > (C_6-1)\sqrt{n},\ \text{for some } i\le n\right] \le \frac{\varepsilon}{2}. \tag{3.18}$$

Then it is enough to prove,

$$\mathbf{E}\left[\mathbb{E}\left[g_1(\omega)\prod_{i=1}^{n}(1+\beta\omega_{i,S_i})\right]\mathbf{1}_{\{S_i\in B_1,\ \text{for all } i\le n\}}\right] \le \frac{\varepsilon}{2}. \tag{3.19}$$

Notice that when S is fixed, the change of measure induced by the function

$$\omega \mapsto \prod_{i=1}^{n}(1+\beta\omega_{i,S_i}) \tag{3.20}$$

retains the independence of the elements of the environment but tilts the distribution of the ones that belong to the graph of S by a factor of $(1+\beta\omega_{i,S_i})$. This allows us to consider an i.i.d. set of random variables $\widetilde{\omega} = \{\widetilde{\omega}_{n,z} : n\in\mathbb{N}, z\in\mathbb{Z}^d\}$ from a probability space $(\widetilde{\Lambda}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ of distribution given by:

$$\widetilde{\mathbb{P}}\left(\widetilde{\omega}_{1,0}\in\cdot\right) = \mathbb{E}\left[(1+\beta\omega_{1,0})\,\mathbf{1}_{\{\omega_{1,0}\in\cdot\}}\right], \tag{3.21}$$

and express the inequality above as

$$\mathbf{E}\left[\mathbb{E}\otimes\widetilde{\mathbb{E}}\left[g_1(\widehat{\omega}^S)\right]\mathbf{1}_{\{S_i\in B_1,\ \text{for all } i\le n\}}\right] \le \frac{\varepsilon}{2}. \tag{3.22}$$

where for all $i \in \mathbb{N}$ and $z \in \mathbb{Z}^d$ we define $\widehat{\omega}^S_{(i,z)}$ as:

$$\widehat{\omega}^S_{(i,z)} := \omega_{(i,z)} \mathbf{1}_{\{z \neq S_i\}} + \widetilde{\omega}_{(i,z)} \mathbf{1}_{\{z = S_i\}}. \tag{3.23}$$

The idea for the rest of the proof is to show that, under the measure $\mathbb{P} \otimes \widetilde{\mathbb{P}}$, the event $\{\widehat{\omega}^S \in A_1\}$ is very likely, so $g_1(\widehat{\omega}^S)$ is equal to $\exp(-M)$ with high probability. Then, taking M large will be sufficient. The following estimates hold for the distributions of the tilted environment, where $C > 0$ is some constant and $x \geq \beta^{-1}$:

$$\frac{1}{C} \beta x^{-\gamma+1} \leq \widetilde{\mathbb{P}} \left[\widetilde{\omega}_{1,0} \geq x \right] \leq C \beta x^{-\gamma+1}. \tag{3.24}$$

Define the random variable

$$X(\widehat{\omega}^S) := \sum_{0 \leq i,j < n} \mathbf{1}_{\left\{ |S_i - S_j| \leq C_7 \sqrt{|i-j|}, \widehat{\omega}^S_{i,S_i} \wedge \widehat{\omega}^S_{j,S_j} \geq V(|i-j|) \right\}}. \tag{3.25}$$

Notice that $X(\widehat{\omega}^S) \geq 1$ implies that $\widehat{\omega}^S \in A_1$ and that we can lower bound the expectation of $X(\widehat{\omega}^S)$ under $\mathbb{P} \otimes \widetilde{\mathbb{P}}$ by,

$$\mathbb{E} \otimes \widetilde{\mathbb{E}} \left[X(\widehat{\omega}^S) \right] \geq \sum_{i=0}^{n/2} \sum_{t=1}^{n/2} \widetilde{\mathbb{P}} \left[\widetilde{\omega}_{1,0} \geq V(t) \right]^2 \mathbf{1}_{\left\{ |S_i - S_{i+t}| \leq C_7 \sqrt{t} \right\}}. \tag{3.26}$$

Since

$$\sum_{i=0}^{n/2} \sum_{t=1}^{n/2} \widetilde{\mathbb{P}} \left[\widetilde{\omega}_{1,0} \geq V(t) \right]^2 \mathbf{1}_{\left\{ |S_i - S_{i+t}| \leq C_7 \sqrt{t} \right\}} \leq \sum_{i=0}^{n/2} \sum_{t=1}^{n/2} \widetilde{\mathbb{P}} \left[\widetilde{\omega}_{1,0} \geq V(t) \right]^2, \tag{3.27}$$

we can choose C_7 sufficiently big, such that, by Markov Inequality,

$$\mathbf{P} \left[\sum_{i=0}^{n/2} \sum_{t=1}^{n/2} \widetilde{\mathbb{P}} \left[\widetilde{\omega}_{1,0} \geq V(t) \right]^2 \mathbf{1}_{\left\{ |S_i - S_{i+t}| \leq C_7 \sqrt{t} \right\}} \leq \frac{1}{2} \sum_{i=0}^{n/2} \sum_{t=1}^{n/2} \widetilde{\mathbb{P}} \left[\widetilde{\omega}_{1,0} \geq V(t) \right]^2 \right] \leq \varepsilon/4. \tag{3.28}$$

Then,

$$\mathbf{P} \left[\mathbb{E} \otimes \widetilde{\mathbb{E}} \left[X(\widehat{\omega}^S) \right] \geq \frac{1}{2} \sum_{i=0}^{n/2} \sum_{t=1}^{n/2} \widetilde{\mathbb{P}} \left[\widetilde{\omega}_{(1,0)} \geq V(t) \right]^2 \right] \geq 1 - \varepsilon/4. \tag{3.29}$$

In the same event, we have that

$$\mathbb{E} \otimes \widetilde{\mathbb{E}} \left[X(\widehat{\omega}^S) \right] \geq \frac{1}{2} \sum_{i=0}^{n/2} \sum_{t=1}^{n/2} \widetilde{\mathbb{P}} \left[\widetilde{\omega}_{1,0} \geq V(t) \right]^2 \geq \frac{n}{4} \sum_{t=1}^{n/2} \left(C^{-1} \beta V(t)^{-\gamma+1} \right)^2 \tag{3.30}$$

$$= \frac{n}{4} \sum_{t=1}^{n/2} \left(C^{-1} \beta \left(\exp(M^2) \left(C_6^d C_7^d t^{1+\frac{d}{2}} n^{1+\frac{d}{2}} \log n \right)^{\frac{1}{2\gamma}} \right)^{-\gamma+1} \right)^2 \tag{3.31}$$

$$\geq C' \beta^2 (\log n)^{\frac{1}{\gamma}}. \tag{3.32}$$

for a constant C' that might depend on M, C_6 and C_7, whose values have already been chosen. On the other hand, we cancel all terms with covariance zero and bound the variance of X as

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}\otimes\widetilde{\mathbb{P}}}\left[X(\widehat{\omega}_S)\right] \leq &\sum_{0\leq i,j<n} \widetilde{\mathbb{P}}\left[\widetilde{\omega}_{i,S_i}\wedge\widetilde{w}_{j,S_j}\geq V(|i-j|)\right]\\ &+4\sum_{0\leq i,j,k<n}\widetilde{\mathbb{P}}\left[\widetilde{w}_{i,S_i}\wedge\widetilde{w}_{j,S_j}\geq V(|i-j|),\widetilde{w}_{i,S_i}\wedge\widetilde{w}_{k,S_k}\geq V(|i-k|)\right].\end{aligned} \tag{3.33}$$

The first term in the sum is similar to the expectation of X, and by an analogous computation, we have that,

$$\mathbb{E}\otimes\widetilde{\mathbb{E}}[X(\widehat{\omega}_S)]\leq C''\beta^2(\log n)^{1/\gamma}. \tag{3.34}$$

Let us called Y the second term in the sum. Rearranging the terms of Y, we have that,

$$\begin{aligned}Y &\leq 32\sum_{i=0}^{n-1}\sum_{t=1}^{n}\sum_{t'=1}^{t}\widetilde{\mathbb{P}}\left[\widetilde{w}_{i,S_i}\wedge\widetilde{w}_{(i+t,S_{i+t})}\geq V(t),\widetilde{w}_{i,S_i}\wedge\widetilde{w}_{i+t',S_{i+t'}}\geq V(t')\right]\\ &\leq 32\sum_{i=0}^{n-1}\sum_{t=1}^{n}\sum_{t'=1}^{t}\widetilde{\mathbb{P}}\left[\widetilde{w}_{i,S_i}\geq V(t)\right]\widetilde{\mathbb{P}}\left[\widetilde{w}_{i+t',S_{i+t'}}\geq V(t')\right]\widetilde{\mathbb{P}}\left[\widetilde{w}_{i+t,S_{i+t}}\geq V(t)\right]\\ &\leq 32\sum_{i=0}^{n-1}\sum_{t=1}^{n}\sum_{t'=1}^{n}\mathbb{P}\left[w_{1,0}\geq V(t)\right]^2\mathbb{P}\left[w_{1,0}\geq V(t')\right]\\ &\leq C'''n\sum_{t=1}^{n}\beta\left(\left(n^{1+\frac{d}{2}}t^{1+\frac{d}{2}}\log n\right)^{\frac{1}{2\gamma}}\right)^{2(-\gamma+1)}\sum_{t'=1}^{n}\beta\left(\left(n^{1+\frac{d}{2}}t'^{1+\frac{d}{2}}\log n\right)^{\frac{1}{2\gamma}}\right)^{-\gamma+1}\\ &\leq C'''\beta^3(\log n)^{\frac{3}{2}\frac{1}{\gamma}}.\end{aligned} \tag{3.35}$$

By Chebychev's Inequality, in the event $\{\mathbb{E}\otimes\widetilde{\mathbb{E}}\left[X(\widehat{\omega}_S)\right]\geq\frac{1}{2}\sum_{i=0}^{n/2}\sum_{t=1}^{n/2}\widetilde{\mathbb{P}}\left[\widetilde{w}_{1,0}\geq V(t)\right]^2\}$, we have that

$$\mathbb{P}\otimes\widetilde{\mathbb{P}}\left[X(\widehat{\omega}_S)=0\right]\leq\frac{\mathrm{Var}_{\mathbb{P}\otimes\widetilde{\mathbb{P}}}\left[X(\widehat{\omega}_S)\right]}{\mathbb{E}\otimes\widetilde{\mathbb{E}}\left[X(\widehat{\omega}_S)\right]^2}\leq\frac{C''\beta^2(\log n)^{1/\gamma}+C'''\beta^3(\log n)^{\frac{3}{2}\frac{1}{\gamma}}}{\left(C'\beta^2(\log n)^{\frac{1}{\gamma}}\right)^2}. \tag{3.36}$$

Recall Equation (3.2) and choose C_5 large enough such that the last term is smaller than $\varepsilon/8$ and M such that $\exp(-M)\leq\varepsilon/8$. Using this and (3.29) we obtain (3.22). □

4. The disorder irrelevance case $\gamma > \gamma_c(d)$.

In this section we are going to prove Theorem 1.2. The idea is to show that when $\gamma>\gamma_c$, with $\gamma_c = 1+2/d$, the sequence of partition functions is uniformly integrable so that $\mathbb{E}\left[Z_\infty\right]=1$. This is performed by bounding the $(1+q)$-th moment of the partition function for some positive q. In order to do this, we rewrite the problem as the estimation of the q-th moment of the partition function of the system where the environment's distribution has been tilted along a quenched path. This technique has appeared originally in [**30**] for the pinning model case.

PROPOSITION 4.1. *When $\gamma > \gamma_c$, $d \geq 3$ and assuming the usual hypothesis on the environment's distribution, we have*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[\left(Z_N^{\beta,\omega}\right)^{1+q}\right] < \infty, \tag{4.1}$$

for all $\beta > 0$ sufficiently small, and some $q \in (0, \gamma - 1)$.

PROOF. Rewrite partition function above as

$$\begin{aligned}\mathbb{E}\left[\left(Z_N^{\beta,\omega}\right)^{1+q}\right] &= \mathbb{E}\left[\mathbf{E}\left[\prod_{i=1}^{N}(1+\beta\omega_{i,S_i})\right]\left(Z_N^{\beta,\omega}\right)^{q}\right] \\ &= \mathbf{E}\left[\mathbb{E}\left[\left(Z_N^{\beta,\omega}\right)^{q}\prod_{i=1}^{N}(1+\beta\omega_{i,S_i})\right]\right].\end{aligned} \tag{4.2}$$

Using the notation introduced in Equation (3.6) we write the expectation above as

$$\mathbb{E}\left[\left(Z_N^{\beta,\omega}\right)^{q}\prod_{i=1}^{N}(1+\beta\omega_{i,S_i})\right] = \mathbb{E}\otimes\widetilde{\mathbb{E}}\left[\left(Z_N^{\beta,\widehat{\omega}^S}\right)^{q}\right], \tag{4.3}$$

where

$$\widehat{\omega}_{i,z}^{S} := \omega_{i,z}\mathbf{1}_{\{z\neq S_i\}} + \widetilde{\omega}_{i,z}\mathbf{1}_{\{z=S_i\}}, \tag{4.4}$$

for all $i \in \mathbb{N}, z \in \mathbb{Z}^d$. The reason for which this consideration might be useful is that more techniques are available to control p-moments for p in the interval $(0, 1)$ than for $p > 1$. We can also express $Z_N^{\beta,\widehat{\omega}^S}$ as

$$Z_N^{\beta,\widehat{\omega}^S} = \mathbf{E}'\prod_{i=1}^{N}\left(1+\beta\widehat{\omega}_{i,S_i'}^{S}\right), \tag{4.5}$$

where $(\Omega', \mathbf{P}', S')$ is an independent copy of $(\Omega, \mathbf{P}, S)$. By Fubini's Theorem and Jensen's Inequality

$$\begin{aligned}\mathbb{E}\left[\left(Z_N^{\beta,\omega}\right)^{1+q}\right] &= \mathbf{E}\otimes\widetilde{\mathbb{E}}\otimes\mathbb{E}\left[\left(\mathbf{E}'\prod_{i=1}^{N}\left(1+\beta\widehat{\omega}_{i,S_i'}^{S}\right)\right)^{q}\right] \\ &\leq \mathbf{E}\otimes\widetilde{\mathbb{E}}\left[\left(\mathbf{E}'\prod_{i=1}^{N}\left(1+\beta\widetilde{\omega}_{i,S_i'}\mathbf{1}_{\{S_i=S_i'\}}\right)\right)^{q}\right].\end{aligned} \tag{4.6}$$

We used Jensen Inequality here to obtain a more tractable expression to estimate. Notice that we cannot simply apply it for $\widetilde{\mathbb{E}}$ as $\widetilde{\omega}$ has infinite mean. Also notice that, we can rewrite the expectation inside as

$$\widetilde{\mathbb{E}}\left[\mathbf{E}'\left[\prod_{i=1}^{N}\left(1+\beta\widetilde{\omega}_{i,0}\mathbf{1}_{\{S_i=S_i'\}}\right)\right]^{q}\right], \tag{4.7}$$

since for a fixed path S, the joint distributions of $\left\{\widetilde{\omega}_{(i,S_i)}\right\}_{i=1}^N$ and $\left\{\widetilde{\omega}_{(i,0)}\right\}_{i=1}^N$ are identical. By simplicity we write $\widetilde{\omega}_{i,0}$ as $\widetilde{\omega}_i$. Using Jensen's Inequality and Fubini's Theorem one more time, we have

$$\mathbb{E}\left[\left(Z_N^{\beta,\omega}\right)^{1+q}\right] \leq \widetilde{\mathbb{E}}\left[\left(\mathbf{E}'\otimes\mathbf{E}\prod_{i=1}^N\left(1+\beta\widetilde{\omega}_i\mathbf{1}_{\left\{S_i=S'_i\right\}}\right)\right)^q\right]. \tag{4.8}$$

Observe that the expression on the right corresponds to the q-th moment for the partition function of a one dimensional pinning model, with free boundary condition, associated with a transient renewal process $\bar{\tau}$ whose inter-arrival distribution $\overline{\mathbf{P}}$ satisfies:

$$\overline{\mathbf{P}}\left[\bar{\tau}=n\right] =: K(n) = \mathbf{P}\otimes\mathbf{P}'\left[S_1\neq S'_1,\ldots,S_{n-1}\neq S'_{n-1},S_n=S'_n\right], \tag{4.9}$$

and an environment given by a realization of $\{\widetilde{\omega}_1,\ldots,\widetilde{\omega}_N\}$ (See **[23]** for a review on pinning model). With this notation we can write

$$\mathbf{E}'\otimes\mathbf{E}\prod_{i=1}^N\left(1+\beta\widetilde{\omega}_i\mathbf{1}_{\left\{S_i=S'_i\right\}}\right) = \overline{\mathbf{E}}\prod_{i=1}^N\left(1+\beta\widetilde{\omega}_i\mathbf{1}_{\{i\in\bar{\tau}\}}\right). \tag{4.10}$$

Let us consider the constrained version of the pinning model:

$$\bar{Z}_N^{\beta,\widetilde{\omega}} := \overline{\mathbf{E}}\left[\prod_{i=1}^N\left(1+\beta\widetilde{\omega}_i\mathbf{1}_{\{i\in\bar{\tau}\}}\right)\mathbf{1}_{\{N\in\bar{\tau}\}}\right]. \tag{4.11}$$

Notice that it is sufficient to show that

$$\sum_{N=1}^{\infty}\widetilde{\mathbb{E}}\left[\left(\bar{Z}_N^{\beta,\widetilde{\omega}}\right)^q\right] < \infty, \tag{4.12}$$

since we can write, by the Markov property,

$$\begin{aligned}\overline{\mathbf{E}}\left[\prod_{i=1}^N\left(1+\beta\widetilde{\omega}_i\mathbf{1}_{\{i\in\bar{\tau}\}}\right)\right] &= \sum_{j=0}^N\overline{\mathbf{E}}\left[\prod_{i=1}^N\left(1+\beta\widetilde{\omega}_i\mathbf{1}_{\{i\in\bar{\tau}\}}\right)\mathbf{1}_{\{j\in\bar{\tau},j+1\notin\bar{\tau},\ldots,N\notin\bar{\tau}\}}\right] \\ &= \sum_{j=1}^N\bar{Z}_j^{\beta,\widetilde{\omega}}\overline{\mathbf{P}}\left[\bar{\tau}>N-j\right] \leq \sum_{j=1}^N\bar{Z}_j^{\beta,\widetilde{\omega}},\end{aligned} \tag{4.13}$$

and

$$\widetilde{\mathbb{E}}\left[\left(\sum_{j=1}^N\bar{Z}_j^{\beta,\widetilde{\omega}}\right)^q\right] \leq \sum_{N=1}^{\infty}\widetilde{\mathbb{E}}\left[\left(\bar{Z}_N^{\beta,\widetilde{\omega}}\right)^q\right]. \tag{4.14}$$

Similarly as we did in the previous sections, we apply Hölder's Inequality and a change of measure to get rid of the exponent q. By Hölder's Inequality, we have

$$\widetilde{\mathbb{E}}\left[\left(\bar{Z}_N^{\beta,\widetilde{\omega}}\right)^q\right] \leq \widetilde{\mathbb{E}}\left[\left(\prod_{i=1}^N h(\widetilde{\omega}_i)^{-1}\right)^{\frac{1}{1-q}}\right]^{1-q}\widetilde{\mathbb{E}}\left[\left(\prod_{i=1}^N h(\widetilde{\omega}_i)\right)^{1/q}\bar{Z}_N^{\beta,\widetilde{\omega}}\right]^q, \tag{4.15}$$

for some positive function $h : \mathbb{R} \to \mathbb{R}$. We choose to use

$$\begin{aligned} h : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto (1+\beta x)^{-q(1-q)}, \end{aligned} \tag{4.16}$$

for the change of measure as in [**30**] since it gives us the same weight $\widetilde{\mathbb{E}}\left[(1+\beta\widetilde{\omega}_i)^q\right]$ for both factors after applying Hölder's Inequality, as it is seen below in Equations (4.17) and (4.18). We compute the first expectation using the i.i.d. structure of the environment:

$$\widetilde{\mathbb{E}}\left[\left(\prod_{i=1}^{N} h(\widetilde{\omega}_i)^{-1}\right)^{\frac{1}{1-q}}\right] = \widetilde{\mathbb{E}}\left[(1+\beta\widetilde{\omega}_i)^q\right]^N = \mathbb{E}\left[(1+\beta\omega_i)^{1+q}\right]^N. \tag{4.17}$$

For the second expectation,

$$\begin{aligned} \widetilde{\mathbb{E}}\left[\left(\prod_{i=1}^{N} h(\widetilde{\omega}_i)\right)^{1/q} \bar{Z}_N^{\beta,\widetilde{\omega}}\right] &= \widetilde{\mathbb{E}}\left[\left(\prod_{i=1}^{N}(1+\beta\widetilde{\omega}_i)\right)^{-(1-q)} \overline{\mathbf{E}}\left[\prod_{i=1}^{N}\left(1+\beta\widetilde{\omega}_i\mathbf{1}_{\{i\in\overline{\tau}\}}\right)\mathbf{1}_{\{N\in\overline{\tau}\}}\right]\right] \\ &\leq \overline{\mathbf{E}}\left[\prod_{i\in\overline{\tau}\cap[1,N]} \mathbb{E}\left[(1+\beta\omega_1)^{1+q}\right] \prod_{i\in[1,N]\setminus\overline{\tau}} \mathbb{E}\left[(1+\beta\omega_1)^q\right]\mathbf{1}_{\{N\in\overline{\tau}\}}\right] \\ &\leq \overline{\mathbf{E}}\left[\prod_{i=1}^{N} \mathbb{E}\left[(1+\beta\omega_i)^{1+q}\right]\mathbf{1}_{\{i\in\overline{\tau}\}}\mathbf{1}_{\{N\in\overline{\tau}\}}\right]. \end{aligned} \tag{4.18}$$

In the last inequality, we neglect that contribution of sites which the renewal process does not visit, since $\mathbb{E}\left[(1+\beta\omega_i)^q\right] \leq 1$. By Dominated Convergence we have

$$\lim_{\beta\to 0} \mathbb{E}\left[(1+\beta\omega_{(1,0)})^{1+q}\right] = 1, \tag{4.19}$$

since $q < \gamma - 1$. For a given $\delta > 0$, let $\beta_0 = \beta_0(\delta)$ be such that, $\mathbb{E}\left[(1+\beta\omega_{(1,0)})^{1+q}\right] \leq 1+\delta$ for all $\beta \leq \beta_0$. Then, (4.17), (4.18) and standard asymptotic results on the returning time of the simple random walk yield,

$$\begin{aligned} \widetilde{\mathbb{E}}\left[\left(\bar{Z}_N^{\beta,\widetilde{\omega}}\right)^q\right] &\leq (1+\delta)^N\, \overline{\mathbf{P}}\left[N\in\overline{\tau}\right]^q \\ &\leq (1+\delta)^N \left(\frac{C_d}{N^{d/2}}\right)^q. \end{aligned} \tag{4.20}$$

With the help of the following criterion, whose proof can be found in [**20**, Proposition 2.5], we see that this last bound suffices for our purpose. Let $A_n := \widetilde{\mathbb{E}}\left[\left(\bar{Z}_n^{\beta,\widetilde{\omega}}\right)^q\right]$.

LEMMA 4.2. *If $k \in \mathbb{N}$ is such that*

$$\rho := \widetilde{\mathbb{E}}\left[(1+\beta\widetilde{w}_{(1,0)})^q\right] \sum_{n=k}^{\infty}\sum_{j=1}^{k-1} K(n-j)^q A_j < 1 \tag{4.21}$$

Then there exists $C = C(\rho, q, k, K(\cdot)) > 0$ *such that*

$$A_N \leq C(K(N))^q, \tag{4.22}$$

for every $N \in \mathbb{N}$.

Assuming we can get (4.21), the proof is complete since we can use (2.14), $K(N) \overset{N\to\infty}{\sim} \frac{C_d'}{N^{d/2}}$ since $d \geq 3$ and $q > 2/d$ to obtain (4.12). By (4.23) we see that

$$\begin{aligned}\rho &\leq (1+\delta)^{k+1} \sum_{n=k}^{\infty} \sum_{j=1}^{k-1} K(n-j)^q \frac{(C_d)^q}{j^{dq/2}} \\ &\leq C'(1+\delta)^k \sum_{j=1}^{k-1} \frac{1}{j^{dq/2}} \frac{1}{(k-j)^{dq/2-1}} \\ &\leq C'(1+\delta)^k \left(\sum_{j=1}^{k/2} \frac{1}{j^{dq/2}} \frac{1}{(k-j)^{dq/2-1}} + \sum_{j=k/2}^{k-1} \frac{1}{j^{dq/2}} \frac{1}{(k-j)^{dq/2-1}} \right) \\ &\leq C'(1+\delta)^k \left(\left(\sum_{j=1}^{\infty} \frac{1}{j^{dq/2}} \right) \frac{1}{(k/2)^{dq/2-1}} + \frac{1}{(k/2)^{dq/2-1}} \right),\end{aligned} \tag{4.23}$$

for some constant C'. Then, for a given $\varepsilon > 0$ we can choose $k \in \mathbb{N}$ such that

$$\left(\sum_{j=1}^{\infty} \frac{1}{j^{dq/2}} \right) \frac{1}{(k/2)^{dq/2-1}} + \frac{1}{(k/2)^{dq/2-1}} < \varepsilon/(2C') \tag{4.24}$$

and then choose $\delta > 0$ such that $(1+\delta)^k < 2$. This proves that we can make the value of ρ arbitrary small. □

5. Lower bound.

In this section, we show a lower bound for the free energy, assuming $\gamma < \gamma_c$. This completes the proof of Theorem 1.1.

PROPOSITION 5.1. *Given* $\varepsilon > 0$, *under usual hypothesis on the environment's distribution,* $\gamma < \gamma_c(d)$ *and* $d \geq 3$ *we have*

$$p(\beta) \geq -C\beta^{\alpha-\varepsilon}, \tag{5.1}$$

for all $\beta \in [0, \beta_0(\varepsilon))$ *and a constant* C *depending on the environment distribution.*

PROOF. Consider the following partition function of a truncated version of the environment:

$$\breve{Z}_N^{\beta,\omega} := \mathbf{E}\left[\prod_{i=1}^{N} \frac{1 + \beta\left(\omega_{i,S_i} \wedge \beta^{-\kappa}\right)}{c_\beta} \right], \tag{5.2}$$

where $a \wedge b := \min\{a, b\}$, $c_\beta := \mathbb{E}\left(1 + \beta\left(\omega_{1,0} \wedge \beta^{-\kappa}\right)\right)$ and $\kappa > 0$ is a constant to be fixed soon. Let us show first that to prove the proposition, it suffices that

$$\lim_{N\to\infty} \frac{1}{N} \log \breve{Z}_N^{\beta,\omega} = 0. \tag{5.3}$$

In fact, observe that

$$\lim_{N\to\infty}\frac{1}{N}\log Z_N^{\beta,\omega} \geq \lim_{N\to\infty}\frac{1}{N}\log Z_N^{\beta,\omega\wedge\beta^{-\kappa}} = \log c_\beta + \lim_{N\to\infty}\frac{1}{N}\log \breve{Z}_N^{\beta,\omega}. \tag{5.4}$$

As $\log(1-x)+2x \geq 0$ for $x \in [0,1/2]$ we have that $\log c_\beta \geq -2\beta\mathbb{E}\left[\omega_{1,0}\mathbf{1}_{\{\omega_{1,0}\geq\beta^{-\kappa}\}}\right]$ for sufficiently small $\beta > 0$. Then, using that $\mathbb{E}\left[X\mathbf{1}_{\{X\leq a\}}\right] = a\mathbb{P}\left[X \geq a\right]+\int_a^\infty \mathbb{P}\left[X \geq z\right]dz$ and taking $\kappa \geq \frac{\gamma_c}{\gamma_c-\gamma} - \frac{\varepsilon}{\gamma_c-\gamma}$ we obtain

$$\log c_\beta \geq -C\beta^{\kappa(\gamma-1)+1} \geq -C\beta^{\frac{\gamma(\gamma_c-1)}{\gamma_c-\gamma}-\varepsilon}, \tag{5.5}$$

for a constant $C > 0$ that depends on the environment's distribution.

To prove (5.3), we adapt the same strategy as Section 4: showing that

$$\sup_{N\geq 1}\mathbb{E}\left[\left(\breve{Z}_N^{\beta,\omega}\right)^{1+q}\right] < \infty, \tag{5.6}$$

for some $q \in (\gamma_c - 1, 1)$. As we did before, we can write this expectation as

$$\begin{aligned}\mathbb{E}\left[\left(\breve{Z}_N^{\beta,\omega}\right)^{1+q}\right] &= \mathbb{E}\left[\mathbf{E}\left[\prod_{i=1}^N \frac{1+\beta\omega_{i,S_i}\wedge\beta^{-y}}{c_\beta}\right]\left(\breve{Z}_N^{\beta,\omega}\right)^q\right]\\ &= \mathbf{E}\left[\mathbb{E}\left[\left(\breve{Z}_N^{\beta,\omega}\right)^q\prod_{i=1}^N \frac{1+\beta\omega_{i,S_i}\wedge\beta^{-y}}{c_\beta}\right]\right].\end{aligned} \tag{5.7}$$

Then, considering the i.i.d. random variables $\widetilde{\omega} = \{\widetilde{\omega}_{n,z} : n \in \mathbb{N}, z \in \mathbb{Z}^d\}$ from a probability space $(\widetilde{\Lambda}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ of distribution given by:

$$\widetilde{\mathbb{P}}\left(\widetilde{\omega}_{1,0} \in A\right) = \mathbb{E}\left[\frac{1+\beta\omega_{1,0}\wedge\beta^{-\kappa}}{c_\beta}\mathbf{1}_{\{\omega_{1,0}\in A\}}\right], \tag{5.8}$$

we could express the expectation above as

$$\mathbb{E}\left[\left(\breve{Z}_N^{\beta,\omega}\right)^q\prod_{i=1}^N \frac{1+\beta\omega_{i,S_i}\wedge\beta^{-y}}{c_\beta}\right] = \mathbb{E}\otimes\widetilde{\mathbb{E}}\left[\left(Z_N^{\beta,\widehat{\omega}^S}\right)^q\right], \tag{5.9}$$

where for all i, z we use the notation introduced in Equation (3.6):

$$\widehat{\omega}_{i,z}^S := \omega_{i,z}\mathbf{1}_{\{z\neq S_i\}} + \widetilde{\omega}_{i,z}\mathbf{1}_{\{z=S_i\}}. \tag{5.10}$$

Let us denote $\omega'_{i,S'_i} := \omega_{i,S'_i}\wedge\beta^{-y}$ and $\widetilde{\omega}'_{i,S'_i} := \widetilde{\omega}_{i,S'_i}\wedge\beta^{-y}$. Fubini's Theorem and Jensen's Inequality yields

$$\begin{aligned}\mathbb{E}\left[\left(\breve{Z}_N^{\beta,\omega}\right)^{1+q}\right] &\le \mathbf{E}\otimes\widetilde{\mathbb{E}}\left[\left(\mathbb{E}\left[\breve{Z}_N^{\beta,\widehat{\omega}^S}\right]\right)^q\right] \\ &= \mathbf{E}\otimes\widetilde{\mathbb{E}}\left[\left(\mathbb{E}\otimes\mathbf{E}'\prod_{i=1}^N \frac{1+\beta\omega'_{i,S'_i}\mathbf{1}_{\{S'_i\neq S_i\}}+\beta\widetilde{\omega}'_{i,S'_i}\mathbf{1}_{\{S'_i=S_i\}}}{c_\beta}\right)^q\right] \\ &\le \mathbf{E}\otimes\widetilde{\mathbb{E}}\left[\left(\mathbf{E}'\prod_{\substack{1\le i\le N\\ S_i=S'_i}}\frac{1+\beta\widetilde{\omega}'_{i,S_i}}{c_\beta}\right)^q\right],\end{aligned} \tag{5.11}$$

where $(\Omega',\mathbf{P}',S')$ is an independent copy of $(\Omega,\mathbf{P},S)$. As before, we replace $\widetilde{\omega}'_{i,S_i}$ by $\widetilde{\omega}'_{i,0}$ (which we now simply denote by $\widetilde{\omega}'_i$) and apply Jensen's Inequality one more time to get,

$$\mathbb{E}\left[\left(\breve{Z}_N^{\beta,\omega}\right)^{1+q}\right] \le \widetilde{\mathbb{E}}\left[\left(\mathbf{E}'\otimes\mathbf{E}\prod_{\substack{1\le i\le N\\ S_i=S'_i}}\frac{1+\beta\widetilde{\omega}'_i}{c_\beta}\right)^q\right]. \tag{5.12}$$

Using the same notation defined in (4.9) and by the argument in (4.13), we are left with showing that

$$\sum_{N=1}^{\infty}\widetilde{\mathbb{E}}\left[\left(\bar{\mathbf{E}}\prod_{\substack{1\le i\le N\\ i\in\bar{\tau}}}\frac{1+\beta\widetilde{\omega}'_i}{c_\beta}\mathbf{1}_{\{N\in\bar{\tau}\}}\right)^q\right]<\infty. \tag{5.13}$$

Let us call

$$\breve{Z}_{N,b}^{\beta,\widetilde{\omega}'} := \bar{\mathbf{E}}\left[\prod_{\substack{1\le i\le N\\ i\in\bar{\tau}}}\frac{1+\beta\widetilde{\omega}'_i}{c_\beta}\mathbf{1}_{\{N\in\bar{\tau}\}}\right]. \tag{5.14}$$

Define the function

$$\begin{aligned} h:\mathbb{R}&\to\mathbb{R} \\ x&\mapsto\left(\frac{1+\beta x}{c_\beta}\right)^{-q(1-q)}. \end{aligned} \tag{5.15}$$

By Hölder's Inequality,

$$\widetilde{\mathbb{E}}\left[\left(\breve{Z}_{N,b}^{\beta,\widetilde{\omega}'}\right)^q\right] \le \widetilde{\mathbb{E}}\left[\left(\prod_{i=1}^N h(\widetilde{\omega}'_i)^{-1}\right)^{\frac{1}{1-q}}\right]^{1-q}\widetilde{\mathbb{E}}\left[\left(\prod_{i=1}^N h(\widetilde{\omega}'_i)\right)^{1/q}\breve{Z}_{N,b}^{\beta,\widetilde{\omega}'}\right]^q. \tag{5.16}$$

For the first expectation we have,

$$\widetilde{\mathbb{E}}\left[\left(\prod_{i=1}^{N} h(\widetilde{\omega}_i')^{-1}\right)^{\frac{1}{1-q}}\right]^{1-q} = \widetilde{\mathbb{E}}\left[\left(\frac{1+\beta\widetilde{\omega}_1\wedge\beta^{-\kappa}}{c_\beta}\right)^{q}\right]^{N(1-q)} = \mathbb{E}\left[\left(\frac{1+\beta\omega_1\wedge\beta^{-\kappa}}{c_\beta}\right)^{1+q}\right]^{N(1-q)}. \tag{5.17}$$

For the second expectation,

$$\widetilde{\mathbb{E}}\left[\left(\prod_{i=1}^{N} h(\widetilde{\omega}_i')\right)^{1/q} \breve{Z}_{N,b}^{\beta,\widetilde{\omega}'}\right]^{q} = \widetilde{\mathbb{E}}\left[\prod_{i=1}^{N}\left(\frac{1+\beta\widetilde{\omega}_i\wedge\beta^{-\kappa}}{c_\beta}\right)^{-(1-q)} \bar{\mathbf{E}}\left[\prod_{\substack{1\le i\le N\\ i\in\bar\tau}} \frac{1+\beta\widetilde{\omega}_i\wedge\beta^{-\kappa}}{c_\beta}\mathbf{1}_{\{N\in\bar\tau\}}\right]\right]^{q}$$
$$\le \bar{\mathbf{E}}\left[\prod_{\substack{1\le i\le N\\ i\in\bar\tau}} \mathbb{E}\left[\left(\frac{1+\beta\omega_1\wedge\beta^{-\kappa}}{c_\beta}\right)^{1+q}\right]\mathbf{1}_{\{N\in\bar\tau\}}\right]^{q}. \tag{5.18}$$

As we did in the previous section, we now need to show that

$$\mathbb{E}\left[\left(\frac{1+\beta\omega_1\wedge\beta^{-\kappa}}{c_\beta}\right)^{1+q}\right] \tag{5.19}$$

is arbitrarily close to one, for all β sufficiently small. This is done in the next Lemma. After this, the rest of the proof follows the exact same lines as the proof from Section 4. □

LEMMA 5.2. *For some $q > \gamma_c - 1$, we have*

$$\lim_{\beta\to 0}\mathbb{E}\left[\left(\frac{1+\beta\omega_1\wedge\beta^{-\kappa}}{c_\beta}\right)^{1+q}\right] = 1. \tag{5.20}$$

PROOF. Let us called $\omega = \omega_1$. Notice that we only need to focus on the numerator of the fraction since $c_\beta = \mathbb{E}\left[1+\beta\omega\wedge\beta^{-\kappa}\right] \to 1$, as $\beta \to 0$. For the numerator we have that, for a fixed $\delta > 0$,

$$\mathbb{E}\left[\left(1+\beta\omega\wedge\beta^{-y}\right)^{1+q}\right] = \mathbb{E}\left[(1+\beta\omega)^{1+q}\mathbf{1}_{\{\omega\le\delta\beta^{-1}\}}\right] + \mathbb{E}\left[(1+\beta\omega)^{1+q}\mathbf{1}_{\{\delta\beta^{-1}\le\omega\le\beta^{-y}\}}\right] + \mathbb{E}\left[\left(1+\beta^{-y+1}\right)^{1+q}\mathbf{1}_{\{\beta^{-y}\le\omega\}}\right]. \tag{5.21}$$

For the first summand, we have that

$$\mathbb{E}\left[(1+\beta\omega)^{1+q}\mathbf{1}_{\{\omega\le\delta\beta^{-1}\}}\right] \le (1+\delta)^{1+q}. \tag{5.22}$$

For the third summand, we have that

$$\mathbb{E}\left[\left(1+\beta^{-\kappa+1}\right)^{1+q}\mathbf{1}_{\{\beta^{-\kappa}\le\omega\}}\right] \le C\beta^{(-\kappa+1)(1+q)+\kappa\gamma}. \tag{5.23}$$

Since $\kappa < \frac{\gamma_c}{\gamma_c - \gamma}$ is fixed, we can choose the value of q sufficiently close to $\gamma_c - 1$ so that the exponent $(-\kappa+1)(1+q)+\kappa\gamma > 0$, making the third summand arbitrarily close to zero. Finally, using the identity

$$\mathbb{E}\left[X^p \mathbf{1}_{\{a \le X \le b\}}\right] = a^p \mathbb{P}\left[X \ge a\right] - b^p \mathbb{P}\left[X \ge b\right] + p\int_a^b z^{p-1}\mathbb{P}\left[X \ge z\right] dz, \tag{5.24}$$

valid for any random variable $X \ge 0$, we obtain for the second summand,

$$\begin{aligned}\mathbb{E}\left[(1+\beta\omega)^{1+q}\,\mathbf{1}_{\{\delta\beta^{-1}\le\omega\le\beta^{-\kappa}\}}\right] &\le (\delta+1)^{1+q}\mathbb{P}\left[\beta\omega\ge\delta\right] + \int_\delta^{\beta^{-\kappa+1}+1}(1+q)z^q\mathbb{P}\left[1+\beta\omega\ge z\right]dz \\ &\le C(\delta+1)^{1+q}\delta^{-\gamma}\beta^\gamma + C\beta^\gamma\int_\delta^{\beta^{-\kappa+1}}(z^{q-\gamma}+z^{-\gamma})dz \\ &\le C(\delta+1)^{1+q}\delta^{-\gamma}\beta^\gamma + C\beta^{\gamma+(-\kappa+1)(q-\gamma+1)} + C\beta^\gamma\delta^{-\gamma+1}.\end{aligned} \tag{5.25}$$

Using one more time the fact that $(-\kappa+1)(1+q)+\kappa\gamma > 0$ we can make the last term arbitrarily small, concluding the proof of the Lemma. □

Appendix

2.A. Properties of the free energy.

THEOREM 2.A.1. *As $N \to \infty$, the limit*

$$p(\beta) = \lim \frac{1}{N}\ln Z_N^{\beta,\omega}, \tag{2.A.1}$$

exists $\mathbb{P}$-a.s. We also have that

$$p(\beta) = \lim \frac{1}{N}\mathbb{E}\left[\ln Z_N^{\beta,\omega}\right]. \tag{2.A.2}$$

Moreover, the function $p : [0,1) \to \mathbb{R}$ is continuous and non-increasing.

PROOF. Following the lines of **[12]**, we first show that the limit in (2.A.2) converges. Notice that we can write the partition function as

$$\begin{aligned}Z_{N+M}^{\beta,\omega} &= \mathbf{E}\left[\prod_{i=1}^{N+M}\left(1+\beta\omega_{(i,S_i)}\right)\right] = \mathbf{E}\left[\prod_{i=1}^{N}\left(1+\beta\omega_{(i,S_i)}\right)\prod_{j=1}^{M}\left(1+\beta\omega_{(N+j,S_{N+j})}\right)\right] \\ &= \sum_{z\in\mathbb{Z}^d}\mathbf{E}\left[\prod_{i=1}^{N}\left(1+\beta\omega_{(i,S_i)}\right)\prod_{j=1}^{M}\left(1+\beta\omega_{(N+j,S_{N+j})}\right)\mathbf{1}_{\{S_N=z\}}\right] \\ &= \sum_{z\in\mathbb{Z}^d}\mathbf{E}\left[\prod_{i=1}^{N}\left(1+\beta\omega_{(i,S_i)}\right)\mathbf{1}_{\{S_N=z\}}\right]\mathbf{E}\left[\prod_{j=1}^{M}\left(1+\beta\omega_{(N+j,S_{N+j})}\right)\,\middle|\, S_N=z\right] \\ &= Z_N^{\beta,\omega}\sum_{z\in\mathbb{Z}^d}\mathbf{P}_N^{\beta,\omega}\left[S_N=z\right]\mathbf{E}\left[\prod_{j=1}^{M}\left(1+\beta\omega_{(N+j,S_{N+j})}\right)\,\middle|\, S_N=z\right].\end{aligned} \tag{2.A.3}$$

Using the concavity of the logarithm we have that

$$\log Z_{N+M}^{\beta,\omega} \geq \log Z_N^{\beta,\omega} + \sum_{z\in\mathbb{Z}^d} \mathbf{P}_N^{\beta,\omega}\left[S_N = z\right] \log \mathbf{E}\left[\prod_{j=1}^{M}\left(1+\beta\omega_{(N+j,S_{N+j})}\right) \,\middle|\, S_N = z\right], \tag{2.A.4}$$

and by the i.i.d. structure of the environment

$$\begin{aligned}\mathbb{E}\left[\log Z_{N+M}^{\beta,\omega}\right] &\geq \mathbb{E}\left[\log Z_N^{\beta,\omega}\right] + \sum_{z\in\mathbb{Z}^d}\mathbb{E}\left[\mathbf{P}_N^{\beta,\omega}\left[S_N=z\right]\right]\mathbb{E}\left[\log \mathbf{E}\left[\prod_{j=1}^{M}\left(1+\beta\omega_{(N+j,S_{N+j})}\right) \,\middle|\, S_N = z\right]\right]\\ &= \mathbb{E}\left[\log Z_N^{\beta,\omega}\right] + \sum_{z\in\mathbb{Z}^d}\mathbb{E}\left[\mathbf{P}_N^{\beta,\omega}\left[S_N=z\right]\right]\mathbb{E}\left[\log \mathbf{E}\left[\prod_{j=1}^{M}\left(1+\beta\omega_{(j,S_j)}\right)\right]\right]\\ &= \mathbb{E}\left[\log Z_N^{\beta,\omega}\right] + \mathbb{E}\left[\log Z_M^{\beta,\omega}\right].\end{aligned} \tag{2.A.5}$$

Then, by super-additivity,

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[\log Z_N^{\beta,\omega}\right] = \sup_N \frac{1}{N}\mathbb{E}\left[\log Z_N^{\beta,\omega}\right], \tag{2.A.6}$$

which by Jensen's Inequality is always a finite, non-positive number. In particular, we have that

$$\log(1-\beta) \leq \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[\log Z_N^{\beta,\omega}\right] \leq 0. \tag{2.A.7}$$

For the limit (2.A.1), let's assume first that $|\omega_{(1,0)}| \leq K$, for some $K \in \mathbb{R}$. Consider the martingale sequence $\left\{\mathbb{E}\left[\log Z_N^{\beta,\omega}|\mathcal{G}_j\right] : 0 \leq j \leq N\right\}$. Define $\widehat{Z}_{N,j}^{\beta,\omega}$ as

$$\widehat{Z}_{N,j}^{\beta,\omega} := \mathbf{E}\left[\prod_{n\in[\![1,N]\!]\setminus\{j\}}\left(1+\beta\omega_{(n,S_n)}\right)\right], \tag{2.A.8}$$

and notice that

$$\mathbb{E}\left[\log \widehat{Z}_{N,j+1}^{\beta,\omega}|\mathcal{G}_j\right] = \mathbb{E}\left[\log \widehat{Z}_{N,j+1}^{\beta,\omega}|\mathcal{G}_{j+1}\right]. \tag{2.A.9}$$

Then, we have that

$$\left|\mathbb{E}\left[\log Z_N^{\beta,\omega}|\mathcal{G}_{j+1}\right] - \mathbb{E}\left[\log Z_N^{\beta,\omega}|\mathcal{G}_j\right]\right| = \left|\mathbb{E}\left[\log\frac{Z_N^{\beta,\omega}}{\widehat{Z}_{N,j+1}^{\beta,\omega}} \,\middle|\, \mathcal{G}_{j+1}\right] - \mathbb{E}\left[\log\frac{Z_N^{\beta,\omega}}{\widehat{Z}_{N,j+1}^{\beta,\omega}} \,\middle|\, \mathcal{G}_j\right]\right|, \tag{2.A.10}$$

with

$$\left|\log\frac{Z_N^{\beta,\omega}}{\widehat{Z}_{N,j+1}^{\beta,\omega}}\right| \leq K' := \max\{\log(1+\beta K), |\log(1-\beta)|\}. \tag{2.A.11}$$

since

$$\log \frac{Z_N^{\beta,\omega}}{Z_{N,j+1}^{\beta,\omega}} = \log \sum_{x\in\mathbb{Z}^d} (1+\beta\omega_{(j+1,x)}) \mathbf{E}\left[\frac{\prod_{n\in[\![1,N]\!]\setminus\{j+1\}}(1+\beta\omega_{(n,S_n)})}{\widehat{Z}_{N,j+1}^{\beta,\omega}} \mathbf{1}_{\{S_{j+1}=x\}} \right], \tag{2.A.12}$$

and

$$\sum_{x\in\mathbb{Z}^d} \mathbf{E}\left[\frac{\prod_{\substack{n=1\\n\neq j+1}}^N (1+\beta\omega_{(n,S_n)})}{\widehat{Z}_{N,j+1}^{\beta,\omega}} \mathbf{1}_{\{S_{j+1}=x\}} \right] = 1. \tag{2.A.13}$$

Then,

$$\left| \mathbb{E}\left[\log Z_N^{\beta,\omega} | \mathcal{G}_{j+1}\right] - \mathbb{E}\left[\log Z_N^{\beta,\omega} | \mathcal{G}_j\right] \right| \leq 2K', \tag{2.A.14}$$

and by Azuma's Inequality, we have that

$$\mathbb{P}\left[\left| \log Z_N^{\beta,\omega} - \mathbb{E}\log Z_N^{\beta,\omega} \right| \geq \varepsilon N \right] \leq 2\exp\left(\frac{N\varepsilon^2}{8K'^2} \right), \tag{2.A.15}$$

which by Borel-Cantelli's Lemma, implies the convergence we wanted. For the unbounded case, we define the truncated version of the environment $\{\widetilde{\omega}_{(n,z)} : (n,z) \in \mathbb{Z}^{1+d}\}$ as

$$\widetilde{\omega}_{(n,z)} := \omega_{(n,z)} \mathbf{1}_{\left\{\omega_{(n,z)} \leq N^q\right\}}, \tag{2.A.16}$$

with $q > d+2$. We define the event A_N as

$$A_N := \left\{ \exists (n,z) : 0 \leq n \leq N, |z| \leq N, \omega_{(n,z)} > N^q \right\}. \tag{2.A.17}$$

In this case, we can bound the probability in (2.A.15) as

$$\mathbb{P}\left[\left| \log Z_N^{\beta,\omega} - \mathbb{E}\log Z_N^{\beta,\omega} \right| \geq \varepsilon N \right] \leq \mathbb{P}\left[A_N\right] + \mathbb{P}\left[\left| \log Z_N^{\beta,\widetilde{\omega}} - \mathbb{E}\log Z_N^{\beta,\omega} \right| \geq \varepsilon N \right]. \tag{2.A.18}$$

We bound the two terms separately. In the fist case we have that

$$\mathbb{P}\left[A_N\right] \leq N(2N)^d \mathbb{E}[|w_{(1,0)}|] N^{-q} = 2^d \mathbb{E}[|w_{(1,0)}|] N^{-(q-d-1)}. \tag{2.A.19}$$

For the second term, using Azuma's Inequality as before, we have that

$$\mathbb{P}\left[\left| \log Z_N^{\beta,\widetilde{\omega}} - \mathbb{E}\log Z_N^{\beta,\widetilde{\omega}} \right| \geq \varepsilon N \right] \leq 2\exp\left(\frac{N\varepsilon^2}{8\log^2(1+\beta N^q)} \right), \tag{2.A.20}$$

for all N sufficiently large. Then, it will be sufficient to show that

$$\frac{1}{N}\left(\mathbb{E}\log Z_N^{\beta,\omega} - \mathbb{E}\log Z_N^{\beta,\widetilde{\omega}} \right) \to 0 \quad \text{as} \quad N\to\infty, \tag{2.A.21}$$

since we can apply Borel-Cantelli's Lemma as we did above. In this case, we have that

$$\frac{1}{N}\left(\mathbb{E}\log Z_N^{\beta,\omega} - \mathbb{E}\log Z_N^{\beta,\widetilde{\omega}} \right) = \frac{1}{N}\mathbb{E}\left[\left(\log Z_N^{\beta,\omega} - \log Z_N^{\beta,\widetilde{\omega}} \right) \mathbf{1}_{\{A_N\}} \right], \tag{2.A.22}$$

and applying Hölder's Inequality,

$$\left| \frac{1}{N}\mathbb{E}\left[\left(\log Z_N^{\beta,\omega} \right) \mathbf{1}_{\{A_N\}} \right] \right| \leq \frac{1}{N}\sqrt{\mathbb{E}\left[\left(\log Z_N^{\beta,\omega} \right)^2 \right]} \sqrt{\mathbb{P}\left[A_N\right]}. \tag{2.A.23}$$

For the first factor, we have

$$\begin{aligned}\mathbb{E}\left[\left(\log Z_N^{\beta,\omega}\right)^2\right] &\leq \mathbb{E}\left[\left(\log Z_N^{\beta,\omega}\right)^2 \mathbf{1}_{\left\{Z_N^{\beta,\omega}\leq 1\right\}}\right] + \mathbb{E}\left[\left(\log Z_N^{\beta,\omega}\right)^2 \mathbf{1}_{\left\{Z_N^{\beta,\omega}>1\right\}}\right] \\ &\leq N^2\log^2(1-\beta) + \mathbb{E}\left[1+\beta\omega_{(0,1)}\right]^N \\ &= N^2\log^2(1-\beta)+1,\end{aligned} \tag{2.A.24}$$

where in the second inequality we used that $N\log(1-\beta) \leq \log Z_N^{\beta,\omega} \leq 0$ if $Z_N^{\beta,\omega} \leq 1$, for the first summand and that $x-(\log x)^2 > 0$ if $x \geq 1$, for the second summand. This bound also holds replacing ω by $\widetilde{\omega}$. Then, joining (2.A.22),(2.A.23) and (2.A.24) we get (2.A.21).

Notice that the function $\beta \mapsto p_N(\beta) := \frac{1}{N}\mathbb{E}\left[\log Z_N^{\beta,\omega}\right]$ is differentiable and

$$\begin{aligned}\frac{\partial}{\partial\beta}p_N(\beta) &= \frac{1}{N}\mathbb{E}\left[\frac{1}{Z_N^{\beta,\omega}}\mathbf{E}\left[\sum_{i=1}^N \omega_{(i,S_i)} \prod_{j\in[\![1,N]\!]\setminus\{i\}} (1+\beta\omega_{(j,S_j)})\right]\right] \\ &= \frac{1}{N}\mathbb{E}\left[\mathbf{E}_N^{\beta,\omega}\left[\sum_{i=1}^N \frac{\omega_{(i,S_i)}}{1+\beta\omega_{(i,S_i)}}\right]\right].\end{aligned} \tag{2.A.25}$$

Then $\left|\frac{\partial}{\partial\beta}p_N(\beta)\right| \leq K_\beta := \max\left\{\frac{1}{\beta},\frac{1}{1-\beta}\right\}$ which implies that the limit $p(\beta)$ is a continuous function in $(0,1)$. Also, since the functions

$$\omega \mapsto \frac{1}{Z_N^{\beta,\omega}}, \tag{2.A.26}$$

and

$$\omega \mapsto \mathbf{E}\left[\sum_{i=1}^N \omega_{(i,S_i)} \prod_{j\in[\![1,N]\!]\setminus\{i\}} (1+\beta\omega_{(j,S_j)})\right] \tag{2.A.27}$$

are decreasing and increasing respectively, applying FKG Inequality, we have that

$$\frac{\partial}{\partial\beta}p_N(\beta) \leq \frac{1}{N}\mathbb{E}\left[\frac{1}{Z_N^{\beta,\omega}}\right]\mathbb{E}\left[\mathbf{E}\left[\sum_{i=1}^N \omega_{(i,S_i)} \prod_{j\in[\![1,N]\!]\setminus\{i\}} (1+\beta\omega_{(j,S_j)})\right]\right] = 0, \tag{2.A.28}$$

which implies that $p(\beta)$ is non-increasing. □

CHAPTER 3

Directed Polymer for very heavy tailed random walks

1. Introduction

Directed polymer in random environment is a model for elastic molecules interacting with random impurities. It appeared originally in the physics literature in the study of the interface for the Ising model [**25**] and has become an interesting subject of study for many authors ever since (see [**14, 16**] for a review on the matter).

Loosely speaking, the model consists on a random walk (of law denoted by $\mathbf{P}$) on the integer lattice $\mathbb{Z}^{1+d}$, which stretches in the time direction, and interacts with a random space-time environment (of law denoted by $\mathbb{P}$) whose intensity is parameterized by some constant $\beta \geq 0$ (inverse temperature). Given a fixed realization of the environment (sometimes also referred as the disorder), new weights are assigned to the walks. The $\mathbf{P}$-expectation of this weight is the partition function of the system and the Liapunov exponent of this expectation is the quenched free energy (see the formal definitions later).

Most of the literature concerning the study of directed polymers associates it with a simple symmetric random walk [**13, 9, 12, 27**] or when the distribution of the increments belongs to the domain of attraction of an α-stable law for some $\alpha \in (0, 2]$ [**17, 33, 44**].

It is known that there is a phase transition both in the limit of the partition functions and also in the free energy. In particular, there is a critical value β_c below which the sequence of normalized partition functions has a strictly positive limit $\mathbb{P}$-a.s. (weak disorder), while above β_c the limit is zero $\mathbb{P}$-a.s. (strong disorder). Moreover, there is a second critical value $\bar{\beta}_c$ below which the quenched free energy is equal to its annealed counterpart, while above it is strictly smaller than it (very strong disorder). It is not hard to see that $\beta_c \leq \bar{\beta}_c$ and a question of interest is whether these critical points are different.

Informally, in the weak disorder regime, the polymer paths are globally not affected by the environment, for instance, showing diffusivity when $\mathbf{P}$ is the SRW while displaying localization phenomena and superdiffusivity in the strong disorder regime.

So far, it has been shown that $\beta_c = \bar{\beta}_c = 0$ for the nearest-neighbor directed polymer on $\mathbb{Z}^{d+1}$ for $d = 1$ in [11] and $d = 2$ in [15], and for the long-range directed polymer with underlying random walks in the domain of attraction of an α-stable law for some $\alpha \in (1, 2]$ in [16] for $d = 1$. A second moment computation of the partition function shows that $\beta_c > 0$ whenever the random walk is transient [**9**] (see [**41**] for a study of the phase diagram when the environment displays a heavier tails), but the question of whether these two critical points coincide remains open whenever $\beta_c > 0$. It has been conjectured that $\beta_c = \bar{\beta}_c$.

Our aim in this paper is to examine the case when the exponent of the distribution of the increments is equal to one. Specifically, for $d = 1$, assuming that the random walk is defined as $S_n = X_1 + ... + X_n$, where $\{X_i : i \in \mathbb{N}\}$ is a sequence of i.i.d. random variables (also known as the increments) taking values in $\mathbb{Z}$. We assume that the increments have symmetric distribution and that for $n \in \mathbb{Z} \setminus \{0\}$ we have

$$\mathbf{P}\left[X_1 = n\right] =: K(n) = \frac{L(n)}{n}, \tag{1.1}$$

where $L(\cdot)$ is a slowly varying function at $\pm\infty$.

Interestingly, the phenomenology in this case is different than what has been seen before. We show that the quenched free energy is equal to the annealed free energy at every temperature ($\bar{\beta}_c = \infty$) and that under some additional hypothesis, the strong regime is non-trivial ($\beta_c < \infty$), proving that the conjecture cannot hold in complete generality.

The first result is inspired by the work in [**3**] in which an analogous result is proven, for the pinning model: the quenched critical point and the annealed one coincide for any given value of $\beta \geq 0$ when the law of the renewal process τ has loop exponent one, i.e.

$$\mathbb{P}\left[\tau = n\right] = \frac{L(n)}{n}. \tag{1.2}$$

for some slowly varying function L. We also mention the work in [**36**] where low disorder relevance is proven, in the hierarchical pinning model at every temperature, in the $b = s$ case. These are analogous notions of very strong disorder for the pinning model and the hierarchical pinning model respectively.

In our second and third results we prove a sufficient and a necessary condition on $\mathbf{P}$ and $\mathbb{P}$ for $\beta_c < \infty$. This has no analogous version for the pinning model, as there is no notion of weak disorder developed in that context so far.

The organization of the rest of the introduction goes as follows: In the next section we give the formal definition of the model and state already known facts. Then we present our results and give some comments on the extra hypothesis needed and methods used in the proofs.

1.1. Polymer measure. On the space $\left(\left(\mathbb{Z}^d\right)^{\mathbb{N}}, \mathcal{P}(\mathbb{Z}^d)^{\otimes\mathbb{N}}\right)$ of sequences $S := (S_n)_{n\geq 0}$, let $\mathbf{P}$ be a probability measure that satisfies:

$$\begin{aligned} &S_0 = 0, \\ &\{S_n - S_{n-1}\}_{n\geq 1} \text{ is an i.i.d. sequence.} \end{aligned} \tag{1.3}$$

We say that $\mathbf{P}$ is a random walk on $\mathbb{Z}^d$. Most of the results in the literature assumes that $\mathbf{P}$ is the law of the nearest-neighbor symmetric random walk:

$$\mathbf{P}[S_1 = e_j] = \mathbf{P}[S_1 = -e_j] = \frac{1}{2d}, \tag{1.4}$$

where $\{e_1, ..., e_d\}$ is the canonical basis of $\mathbb{R}^d$, but the results stated in this subsection are true in the general setting (1.3).

Independently, also consider a set of i.i.d. random variables $\omega := \{\omega_{n,z} : n \in \mathbb{N}, z \in \mathbb{Z}^d\}$, called *the environment*, defined on a probability space $(\Lambda, \mathcal{F}, \mathbb{P})$, that satisfies,

$$\mathbb{E}\left[\exp(\beta\omega_{n,z})\right] < \infty, \tag{1.5}$$

for any $\beta \in \mathbb{R}$. The *polymer measure* $\mathbf{P}_N^{\beta,\eta}$ is the probability measure in $\left(\left(\mathbb{Z}^d\right)^{\mathbb{N}}, \mathcal{P}(\mathbb{Z}^d)^{\otimes\mathbb{N}}\right)$ describe by its Radon-Nikodym derivative with respect to $\mathbf{P}$: For a fixed value of β (called the inverse temperature) and $N \in \mathbb{N}$ we let

$$\frac{\mathrm{d}\mathbf{P}_N^{\beta,\omega}}{\mathrm{d}\mathbf{P}}(S) = \frac{1}{Z_N^{\beta,\omega}} \exp\left(\beta \sum_{n=1}^{N} \omega_{n,S_n}\right). \tag{1.6}$$

The positive normalization factor $Z_N^{\beta,\omega}$ (called the *partition function*) makes $\mathbf{P}_N^{\beta,\omega}$ a probability measure. Consider the re-normalized partition function

$$W_N^{\beta,\eta} := \frac{Z_N^{\beta,\eta}}{\mathbb{E}\left[Z_N^{\beta,\eta}\right]}. \tag{1.7}$$

In [**9**], Bolthausen observed that the sequence $\{W_N, \mathcal{G}_N\}_{N\in\mathbb{N}}$ is a positive martingale, where $\{\mathcal{G}_N\}_{N\geq 0}$ is the filtration defined by $\mathcal{G}_N := \sigma\{\omega_{n,z} : 1 \leq n \leq N, z \in \mathbb{Z}\}$. By the classical martingale theory, it follows that the limit

$$W_\infty^{\beta,\omega} := \lim_{N\to\infty} W_N^{\beta,\omega}, \tag{1.8}$$

exists $\mathbb{P}$-*a.s.* and is a non-negative random variable. Moreover, the event $\{W_\infty^{\beta,\eta} = 0\}$ belongs to the tail σ-field of $\{\mathcal{G}_N, N \geq 0\}$. Hence, by Kolmogorov's $0-1$ Law,

$$\mathbb{P}\left\{W_\infty^{\beta,\omega} > 0\right\} \in \{0, 1\}. \tag{1.9}$$

Following standard terminology we say that we have *weak disorder* if $W_\infty^\beta > 0$ $\mathbb{P}$-*a.s.* and *strong disorder* if $W_\infty^\beta = 0$ $\mathbb{P}$-*a.s.* In [**13**], it is shown that there exists a critical value $\beta_c \in [0, \infty]$, depending possibly on the environment distribution, such that there is weak disorder for $\beta \in [0, \beta_c)$ and strong disorder for $\beta > \beta_c$. The *quenched free energy* is defined as

$$F(\beta) := \lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta,\omega} = \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_N^{\beta,\omega}. \tag{1.10}$$

It is known that this limit exists and does not depend on ω (see [**12**, Proposition 2.5] for the nearest-neighbor case and [**6**] for the general case), except on a set of measure zero. By Jensen's Inequality we have that $F(\beta) \leq \lambda(\beta)$, where $\lambda(\beta) := \log \mathbb{E} \exp(\beta\omega)$ (the *annealed free energy*). Also, it is not hard to see that

$$F(\beta) < \lambda(\beta) \implies \lim_{N\to\infty} W_N^{\beta,\omega} = 0 \quad \mathbb{P} - a.s. \tag{1.11}$$

Thus, the case $p(\beta) := F(\beta) - \lambda(\beta) < 0$ is called the *very strong disorder*. As a function, $p(\cdot)$ is continuous and non-increasing. There is a critical value $\bar{\beta}_c$ such that $p(\beta) = 0$ if $\beta \in [0, \bar{\beta}_c]$ and $p(\beta) < 0$ if $\beta > \bar{\beta}_c$. As noted before, $\beta_c \leq \bar{\beta}_c$.

It is conjectured that there is no intermediate phase between weak disorder and very strong disorder (i.e., $\beta_c = \bar{\beta}_c$) but so far this has only been proved for the simple symmetric directed polymer on dimensions $d = 1$ and $d = 2$ in which $\beta_c = \bar{\beta}_c = 0$ [**27**] and for the long-range directed polymer where the underlying random walk in the domain of attraction of an α-stable law for some $\alpha \in (1, 2]$ in [**44**] for $d = 1$.

1.2. The results. For the rest of the paper, we assume that the law $\mathbf{P}$ of the random walk satisfies (1.1).

THEOREM 1.1. *Consider the polymer measure* (1.6) *and assume that the distribution of the increments satisfies* (1.1) *and that* $K(n) > 0$ *for all* $n \in \mathbb{Z}$ *then,*

$$p(\beta) = 0, \tag{1.12}$$

for all $\beta \in \mathbb{R}$*, which implies that there is no very strong disorder regime.*

The extra assumption $K(n) > 0$ appears only in Lemma 2.1 and is not really necessary. It is used to avoid technical details that are not part of the main ideas of the proof.

The result of the first theorem contrasts with the cases that have been studied before, in particular, in the α-stable case, $p(\beta) < 0$ for sufficiently large β [**17**, Proposition 5.1]. The next result gives a sufficient condition for which $\beta_c < \infty$ which means there is a strong disorder phase. Important quantities here are the entropy $-\sum_{n\in\mathbb{Z}} K(n) \log K(n)$ of the walk and the mass on the essential supremum of the marginal distribution of $\omega_{n,z}$.

THEOREM 1.2. *If the distributions of the increments and the environment satisfy*

$$\beta\lambda'(\beta) - \lambda(\beta) > \sum_{n\in\mathbb{Z}} K(n) \log \frac{1}{K(n)}, \tag{1.13}$$

then

$$W_\infty^{\beta,\omega} = 0 \quad \mathbb{P} - a.s.$$

In particular if $\lim_{\beta\to\infty} \beta\lambda'(\beta) - \lambda(\beta) = \infty$ *then*

$$\sum_{n\in\mathbb{Z}} K(n) \log \frac{1}{K(n)} < \infty \quad \Rightarrow \quad \beta_c < \infty. \tag{1.14}$$

Note that the condition (1.13) appears in [**17**, Proposition 5.1] (which studies the case of polymer based on α stable walks) as a sufficient condition to have very strong disorder ($p(\beta) < \infty$). However here very strong disorder cannot hold in our case (since it would contradict Theorem (1.1)) and the criterion (1.13) emerges from a proof which is of a different nature than the (fractional moment based) one in [**17**, Proposition 5.1].

Setting $s = \text{ess sup}\{\omega\}$, we have that $\log \frac{1}{\mathbb{P}[\eta=s]} > \sum_{n\in\mathbb{Z}} K(n) \log \frac{1}{K(n)}$ implies that $\beta_c < \infty$ as

$$\lim_{\beta\to\infty} \beta\lambda'(\beta) - \lambda(\beta) = \log \frac{1}{\mathbb{P}[\eta = s]}. \tag{1.15}$$

This known property of the exponential moments is proven in the Appendix for completeness. (Lemma 3.A.1). The assumption $\lim_{\beta\to\infty} \beta\lambda'(\beta) - \lambda(\beta) = \infty$ is equivalent to say that ω is either unbounded or almost surely does not attain its essential supremum.

We note that the condition $\lim_{\beta\to\infty} \beta\lambda'(\beta) - \lambda(\beta) = \infty$ is necessary to have (1.14). To illustrate our point let us consider the case of the Bernoulli environment with parameter p. Then there is weak disorder for all β, if p is sufficiently close to one. More specifically, as shown in **[17]**, a sufficient condition for which the sequence of polymer measures $W_N^{\beta,\omega}$ is uniformly bounded in $\mathcal{L}^2$ for all β (which implies weak disorder) is that

$$p > \mathbf{P} \otimes \mathbf{P}' \left[\exists n \geq 1 : S_n = S'_n\right],$$

where S, S' are two independent walks.

Assuming that the environment is unbounded, Theorem 1.2 permits to conclude that if for some $\alpha < -1$,

$$K(n) \leq \frac{(\log\log n)^\alpha}{n(\log n)^2}, \tag{1.16}$$

for all n sufficiently large, the polymer presents a strong disorder phase. More importantly it provides an example of a directed polymer model for which the two critical points do not coincide ($\beta_c < \bar{\beta}_c$). To our knowledge, the existence of such a setup was not predicted in the literature, and while it is not invalidating the conjecture concerning the nearest neighbor model, it sheds a new light on it.

In opposition, in the next theorem, we show that under some extra assumptions, if $\alpha > 1$ in (1.16), then there is no strong disorder phase.

THEOREM 1.3. *Under the following conditions on the law of the increments:*

(a) $K(\cdot)$ *is unimodal and symmetric around 0,*
(b) *For some* $\alpha < -1$,

$$K(n) \geq \frac{(\log\log n)^\alpha}{n(\log n)^2}, \tag{1.17}$$

for all n sufficiently large,
(c) *and*

$$\frac{\mathbf{P}\left[X_1 \in (s_n, 2ns_n)\right]}{\mathbf{P}\left[X_1 \geq s_n\right]} \leq \frac{1}{n^\gamma}, \tag{1.18}$$

where $\gamma > \frac{1}{2}$ *and*

$$s_n := \min\left\{ s \in \mathbb{N} : \mathbf{P}\left[X_1 \geq s\right] \leq \frac{(\log n)^2}{n} \right\}, \tag{1.19}$$

for all n sufficiently large,

then, $\beta_c = \infty$.

Condition (c) might seem artificial at first sight but it is satisfied by most distribution with sufficiently regular tails, as $\mathbf{P}\left[X_1 \geq n\right] = \frac{L(\log n)}{(\log n)^\alpha}$ where $\alpha \leq 1$ or $\mathbf{P}\left[X_1 \geq n\right] = \frac{L(\log\log n)}{(\log\log n)^\beta}$ where $\beta > 0$ and L a slowly varying function.

1.3. Conjecture and future research directions. At the present moment we are not able to answer whether a strong disorder phase exists if $\alpha \in [-1,1]$ and $K(n) \asymp \frac{c(\log\log n)^\alpha}{n(\log n)^2}$ although we believe that the condition (1.13) on the entropy might be necessary to the existence of the strong disorder phase. Let us make this point more precise.

CONJECTURE 1.4. *Assuming that the environment is unbounded from above, we have the following equivalence*

$$\beta_c < \infty \quad \Leftrightarrow \quad \sum_{n\geq 1} K(n)\log\frac{1}{K(n)} < \infty. \tag{1.20}$$

2. Lower bound for the free energy

Idea of the proof.

As we said before, our proof shares some ideas with [**3**]. Specifically, since $\mathbf{P}[X_1 \geq n]$ is a slowly varying function of n, the longest of the first m excursions typically has length greater than any power of m. This enables the polymer to travel further distances, avoiding some regions of insufficiently unfavorable values at low cost. With this in mind, we partition the environment into rectangles of size $N \times 2N^2$, where N is a scaling factor and restrict attention to the ones whose higher values contributes more to the partition function. Roughly speaking, the partition function, when restricted to a *good* rectangle, has a value higher than some appropriate threshold. Further we will lower bound the partition function by considering paths that only travel through these good rectangles. In Lemma 2.1 we will lower bound the probability of a path to stay inside a rectangle and in Lemma 2.2, we control the cost of jumping to a good rectangle. In the proof we make an energy-entropy balancing of the paths that travel only through good rectangles.

PROOF OF THEOREM 1.1. Fix $\varepsilon > 0$ arbitrarily small and let $N = N(\beta,\varepsilon) \in \mathbb{N}$ be a scaling factor whose value is defined later. Consider the following collection of disjoint rectangles $\cup_{(i,j)\in\mathbb{Z}^2} R_{i,j} = \mathbb{Z}^2$, each one of size $N \times 2N^2$, defined as

$$R_{i,j} := \left\{(x,y) \in \mathbb{Z}^2 : iN+1 \leq x \leq (i+1)N, (2j-1)N^2 \leq y < (2j+1)N^2\right\}. \tag{2.1}$$

In order to lower bound the free energy, we consider only paths that visit rectangles which contribute the most to the partition function. Consider the following restricted version of the normalized partition function to the rectangle $R_{i,j}$:

$$\widetilde{W}_N(i,j) := \mathbf{E}^{2jN^2}\left[\exp\left(\sum_{k=0}^{N-1} \beta\omega_{iN+k+1,2jN^2+S_k} - \lambda(\beta)\right)\middle| S \in \mathcal{A}_N\right], \tag{2.2}$$

where we define $\mathcal{A}_N$ as the event,

$$\mathcal{A}_N := \left\{(S_k)_{k=0}^N : S_{N-1} = S_0, |S_k - S_0| < N^2 \text{ for } 0 \leq k < N\right\}. \tag{2.3}$$

Paths considered in the expectation above start at $(iN+1, 2jN^2)$ and remain inside the rectangle until ending up at the vertex $((i+1)N, 2jN^2)$. Notice that by the i.i.d. structure of the environment, $\left\{\widetilde{W}_N(i,j) : (i,j) \in \mathbb{Z}^2\right\}$ is an i.i.d. collection of random variables with

$$\mathbb{E}\left[\widetilde{W}_N(i,j)\right] = 1. \tag{2.4}$$

Depending on the environment's realization, we say that a rectangle $R_{i,j}$ is $\eta-$*good* when

$$\widetilde{W}_N(i,j) \geq \eta, \tag{2.5}$$

for some constant $\eta > 0$. Let p_η be the probability of a rectangle to be $\eta-$good. Given a realization of the environment, let us define the random sequence $\{J_{-1}, J_0, J_1, ...\}$ inductively: Let $J_{-1} = 0$ and for $i \geq 0$,

$$J_i = \min\{j > J_{i-1} : R_{i,j} \text{ is } \eta\text{-good}\}. \tag{2.6}$$

We lower bound the partition function $W_{Nm}^{\beta,\omega}$ by considering trajectories that only visit good rectangles. Specifically, let us consider the trajectories $(S_k)_{k=0}^{Nm}$ that belong to Ξ_{Nm} where

$$\Xi_{Nm} := \left\{(S_k)_{k=0}^{Nm} : S_{iN+1} = S_{(i+1)N} = 2J_iN^2, |S_{iN+k} - S_{iN}| < N^2, \text{ for } 0 \leq i < m, 1 \leq k < N\right\}. \tag{2.7}$$

In other words, when considering the graph of these paths, in $\mathbb{Z}^2$, they do the following

- Starting from (0,0), they jump to the site $(1, 2J_0N^2)$ and remain inside R_{0,J_0} until the ending up at the site $(N, 2J_0N^2)$.
- Inductively for $1 \leq i < m$, after visiting the last site of $R_{i-1,J_{i-1}}$, they jump to the site $(iN+1, 2J_iN^2)$ and remain inside R_{i,J_i} until ending up at the site $((i+1)N, 2J_iN^2)$.

Let $W_{Nm}^{\beta,\omega}(\Xi_{Nm})$ be the partition function restricted to the trajectories that belong to Ξ_{Nm}. By the Markov Property

$$W_{Nm}^{\beta,\omega}(\Xi) = \prod_{i=0}^{m-1} \mathbf{P}\left[X_1 = 2(J_i - J_{i-1})N^2\right] \widetilde{W}(i, J_i) \mathbf{P}\left[\mathcal{A}_N\right]. \tag{2.8}$$

We then have that

$$\frac{1}{Nm} \log W_{Nm}^{\beta,\omega} \geq \frac{\log \eta}{N} + \frac{1}{Nm} \sum_{i=0}^{m-1} \log K(2(J_i - J_{i-1})N^2) + \frac{\log \mathbf{P}\left[\mathcal{A}_N\right]}{N}. \tag{2.9}$$

Letting $m \to \infty$, the left hand side of (2.9) converges to the free energy. Notice that since the events

$$\left\{R_{i,j} \text{ is } \eta - \text{good} : (i,j) \in \mathbb{Z}^2\right\} \tag{2.10}$$

are independent, $\{J_i - J_{i-1} - 1\}_{i \geq 0}$ is an i.i.d. collection of random variables. Therefore, by the Law of Large Numbers,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \log K(2(J_i - J_{i-1})N^2) = \mathbb{E}\left[\log K(2J_0N^2)\right] = \mathbb{E}\left[\log \frac{L(2J_0N^2)}{2J_0N^2}\right]. \tag{2.11}$$

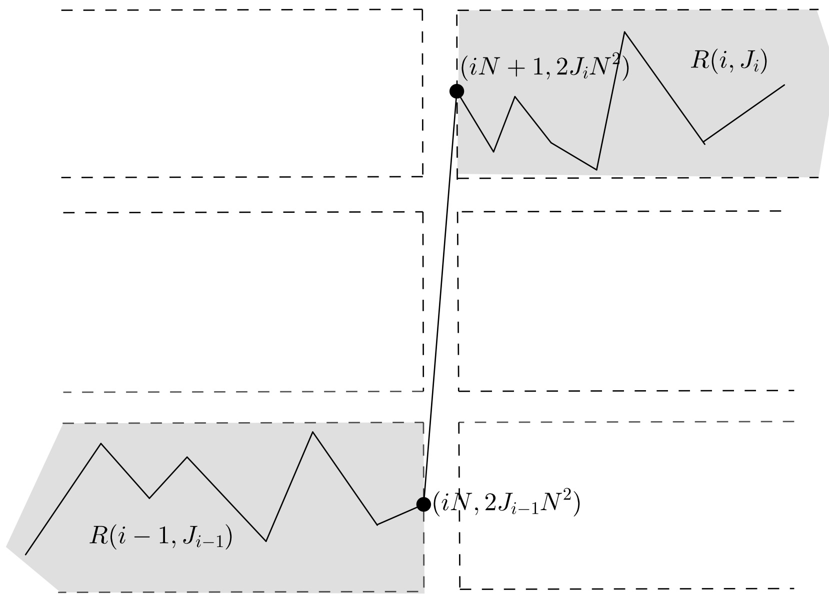

FIGURE 2.1. A path that belongs to Ξ_{Nm}, after visiting the last site of the good rectangle $R(i-1, J_{i-1})$, jumps to the site $(iN, 2J_iN^2)$ of the first good rectangle $R(i, J_i)$ from the next column.

Let

$$C_{L,\varepsilon} := \inf \{x^\varepsilon L(x) : x \geq K\} > 0, \tag{2.12}$$

for some $K = K(\varepsilon) > 0$ sufficiently large. Then, assuming $2N^2 \geq K$ we have, by Jensen's Inequality,

$$\mathbb{E}\left[\log \frac{L(2J_0N^2)}{2J_0N^2}\right] \geq \mathbb{E}\left[\log \frac{C_{L,\varepsilon}}{(2J_0N^2)^{1+\varepsilon}}\right] \geq \log \frac{C_{L,\varepsilon}}{(2N^2)^{1+\varepsilon}} - (1+\varepsilon)\log \mathbb{E}[J_0]. \tag{2.13}$$

As $J_0 - 1$ is a geometric random variable with parameter p_η, we have that $\mathbb{E}[J_0] = \frac{1}{p_\eta} + 1$. Then,

$$p(\beta) \geq \frac{\log \eta}{N} + \frac{1}{N}\log \frac{C_{L,\varepsilon}}{(2N^2)^{1+\varepsilon}} - \frac{(1+\varepsilon)}{N}\log\left(\frac{1}{p_\eta}+1\right) + \frac{\log \mathbf{P}[\mathcal{A}_N]}{N}. \tag{2.14}$$

Let us state the following two lemmas, whose proofs are presented at the end of the section. The first one is a straightforward lower bound for $\mathbf{P}[\mathcal{A}_N]$. The second one is more subtle and shows that we can choose a suitable value for η such that it compensates the cost p_η of the jump. We use these to bound $\frac{\log \mathbf{P}[\mathcal{A}_N]}{N}$ and $\frac{\log \eta}{N} - \frac{(1+\varepsilon)}{N}\log\left(\frac{1}{p_\eta}+1\right)$ respectively.

LEMMA 2.1. *With $\mathcal{A}_N$ defined in (2.3) we have*

$$\lim_{N\to\infty} \frac{\log \mathbf{P}[\mathcal{A}_N]}{N} = 0. \tag{2.15}$$

Lemma 2.2. *There exists* $\eta \in [1/2, e^{C_\beta N}]$ *such that* $p_\eta \geq \frac{c}{\eta(2+\log\eta)^2}$, *where* c *and* C_β *are constants, the last one depending only on* β.

Let us finish the proof of the theorem using the lemmas above. As $p_\eta \geq \frac{c}{\eta(2+\log\eta)^2}$,

$$p(\beta) \geq \frac{\log\eta}{N} + \frac{1}{N}\log\frac{C_{L,\varepsilon}}{(2N^2)^{1+\varepsilon}} + \frac{1+\varepsilon}{N}\left(\log c - \log\eta - 2\log(2+\log\eta)\right) \quad (2.16)$$

$$- \frac{1+\varepsilon}{N}\log 2 + \frac{\log \mathbf{P}\left[\mathcal{A}_N\right]}{N}. \quad (2.17)$$

Since $\eta \in [1/2, e^{C_\beta N}]$ we obtain,

$$p(\beta) \geq -\varepsilon C_\beta + \frac{1}{N}\log\frac{C_{L,\varepsilon}}{(2N^2)^{1+\varepsilon}} + \frac{1+\varepsilon}{N}\left(-\log c - 2\log(2+C_\beta N)\right) \quad (2.18)$$

$$- \frac{1+\varepsilon}{N}\log 2 + \frac{\log \mathbf{P}\left[\mathcal{A}_N\right]}{N}. \quad (2.19)$$

which can be made arbitrarily small by choosing ε sufficiently small and N sufficiently large. □

Proof of Lemma 2.1. Notice that

$$\begin{aligned}\mathbf{P}\left[\mathcal{A}_N\right] &\geq \mathbf{P}\left[|X_1| < N, ..., |X_{N-2}| < N, S_{N-1} = 0\right] \\ &= \mathbf{E}\left[\mathbf{1}_{\{|X_1|<N\}}...\mathbf{1}_{\{X_{N-2}<N\}}\mathbf{E}^{S_{N-2}}\left[X_1 = 0\right]\right] \\ &\geq \frac{C_{L,\varepsilon}}{N^{2+\varepsilon}}\mathbf{E}\left[\mathbf{1}_{\{|X_1|<N\}}..., \mathbf{1}_{\{|X_{N-2}|<N\}}\right].\end{aligned} \quad (2.20)$$

In the last inequality we use (2.12) (if the last jump X_{N-1} is smaller than K its probability can be lower bounded by a positive constant). Finally,

$$\mathbf{P}\left[\mathcal{A}_N\right] \geq \frac{C_{L,\varepsilon}}{N^{2+\varepsilon}}\left(1 - \mathbf{P}\left[X_1 \geq N\right]\right)^{N-2}, \quad (2.21)$$

which implies

$$\frac{\log \mathbf{P}\left[\mathcal{A}_N\right]}{N} \geq \frac{1}{N}\log\left(\frac{C_{L,\varepsilon}}{N^{2+\varepsilon}}\right) + \frac{N-2}{N}\log\left(1 - \mathbf{P}\left[X_1 \geq N\right]\right), \quad (2.22)$$

which converges to 0 as $N \to \infty$. □

Proof of Lemma 2.2. Let us denote by $\widetilde{W}$ a random variable that has the same distribution as $\widetilde{W}_N(i,j)$. Notice that, as $\mathbb{E}\widetilde{W} = 1$,

$$\frac{1}{2}\mathbb{E}\widetilde{W} \leq \mathbb{E}\left[\widetilde{W}\mathbf{1}_{\left\{\widetilde{W} > \frac{1}{2}\mathbb{E}\widetilde{W}\right\}}\right] \leq \sum_{n=0}^{\infty}\mathbb{E}\left[2^n\mathbf{1}_{\left\{2^{n-1}\leq\widetilde{W}\right\}}\right]. \quad (2.23)$$

Using the fact that $\sum_{n=0}^{\infty}\frac{1}{(n+1)^2} = \frac{\pi^2}{6}$, we obtain

$$\frac{3}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(n+1)^2} \leq \sum_{n=0}^{\infty}\mathbb{E}\left[2^n\mathbf{1}_{\left\{2^{n-1}\leq\widetilde{W}\right\}}\right]. \quad (2.24)$$

We now may define $n_0 \geq 0$, as the smallest integer such that

$$\frac{3}{\pi^2}\frac{1}{(n_0+1)^2} \leq 2^{n_0}\mathbb{P}\left[2^{n_0-1} \leq \widetilde{W}\right]. \tag{2.25}$$

Letting $\eta = 2^{n_0-1}$ this implies that,

$$p_\eta \geq \frac{3}{\pi^2}\frac{2^{-n_0}}{(n_0+1)^2}. \tag{2.26}$$

On the other hand, by computing the second moment of $\widetilde{W}$ and considering S' as and independent copy of S we obtain

$$\begin{aligned}\mathbb{E}\left[\widetilde{W}^2\right] &= \mathbb{E}\left[\mathbf{E}^{\otimes 2}\left[\exp\left(\sum_{k=0}^{N-1}\beta\omega_{i,S_i}+\beta\omega_{i,S'_i}-2\lambda(\beta)\right)\middle| S,S'\in\mathcal{A}_N\right]\right]\\ &= \mathbf{E}^{\otimes 2}\left[\exp\left((\lambda(2\beta)-2\lambda(\beta))\sum_{k=0}^{N-1}\mathbf{1}_{\{S_i=S'{}_i\}}\right)\mathbf{1}_{\{S,S'\in\mathcal{A}_N\}}\right]\mathbf{P}\left[\mathcal{A}_N\right]^2\\ &\leq \exp\left((\lambda(2\beta)-2\lambda(\beta))\,N\right),\end{aligned} \tag{2.27}$$

We lower bound the expectation above as

$$\mathbb{E}\left[\widetilde{W}^2\right] \geq 2^{2(n_0-1)}\mathbb{P}\left[2^{n_0-1}\leq\widetilde{W}\right]. \tag{2.28}$$

By Equations (2.25) and (2.28) we get

$$\frac{3}{\pi^2}\frac{1}{(n_0+1)^2} \leq 2^{n_0}\exp((\lambda(2\beta)-2\lambda(\beta))\,N)2^{-2(n_0-1)}, \tag{2.29}$$

which implies

$$\frac{2^{n_0}}{(n_0+1)^2} \leq \frac{4\pi^2}{3}\exp((\lambda(2\beta)-2\lambda(\beta))\,N). \tag{2.30}$$

Let $N_0 \in \mathbb{N}$ be such that if $n > N_0$ then $(3/2)^n \leq \frac{2^n}{(n+1)^2}$. Then either $n_0 \leq N_0 \leq N$ by taking N sufficiently large, or

$$\frac{n_0}{N}\log(3/2) \leq \lambda(2\beta)-2\lambda(\beta)+\frac{\log(4\pi^2/3)}{N}, \tag{2.31}$$

which finishes the proof of the lemma. □

3. Strong Disorder for small temperature

In this Section we show that under the assumptions of Theorem 1.2, the polymer measure has a strong disorder phase, for large enough β. Along with Theorem 1.1, this allows us to construct a family of polymer measures in which there is a strong disorder phase with $p(\beta) = 0$.

3.1. Size Biasing. Notice that since

$$\mathbb{E}\left[W_N^{\beta,\omega}\right] = 1, \tag{3.1}$$

there exists a well defined probability measure $\widetilde{\mathbb{P}}_N^\beta$, called the **size biasing** measure, absolutely continuous with respect to $\mathbb{P}$ such that

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_N^\beta}{\mathrm{d}\mathbb{P}} = W_N^{\beta,\omega}. \tag{3.2}$$

The following result states that a sequence of positive, mean one random variables, converges to 0, if and only if it converges to infinity, in probability, under the size biazed distribution. This gives us a condition for which strong disorder holds, in terms of the size biasing measure.

LEMMA 3.1. *Let $\{W_1, W_2, ...\}$ be a sequence of positive random variables with $\mathbb{E}[X_N] = 1$ for all N. The following are equivalent:*

- $$\lim_{N\to\infty} W_N = 0, \tag{3.3}$$
  $\mathbb{P}-a.s.$
- *For all $L > 0$,*
  $$\lim_{N\to\infty} \widetilde{\mathbb{P}}_N\left[W_N \geq L\right] = 1. \tag{3.4}$$
  where the size biased measure $\widetilde{\mathbb{P}}$ is defined by its Radon-Nikodym derivative with respect to $\mathbb{P}$,
  $$\frac{d\widetilde{\mathbb{P}}_N}{d\mathbb{P}} = W_N. \tag{3.5}$$

PROOF. See [**32**, Proposition 4.2]. □

As in [**30**], we give the following description of the size-biasing measure. Consider an i.i.d. set of random variables $\widetilde{\omega} = \{\widetilde{\omega}_{n,z} : n \in \mathbb{N}, z \in \mathbb{Z}\}$ from a probability space $(\widetilde{\Lambda}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ of distribution given by:

$$\widetilde{\mathbb{P}}\left(\widetilde{\omega}_{1,0} \in \cdot\right) = \mathbb{E}\left[e^{\beta\omega_{1,0}-\lambda(\beta)}\mathbf{1}_{\{\omega_{1,0}\in\cdot\}}\right]. \tag{3.6}$$

For a fixed path S, and a given realization of the environments $\{\omega_{n,z} : n \in \mathbb{N}, z \in \mathbb{Z}\}$ and $\{\widetilde{\omega}_{n,z} : n \in \mathbb{N}, z \in \mathbb{Z}\}$, we define $\{\widehat{\omega}_{n,z}^S : n \in \mathbb{N}, z \in \mathbb{Z}\}$ as:

$$\widehat{\omega}_{i,z}^S := \omega_{i,z}\mathbf{1}_{\{z\neq S_i\}} + \widetilde{\omega}_{i,z}\mathbf{1}_{\{z=S_i\}}. \tag{3.7}$$

One can see that for any bounded continuous function $F : \mathbb{R} \to \mathbb{R}$,

$$\widetilde{\mathbb{E}}_N^\beta\left[F(\omega)\right] = \mathbf{E} \otimes \mathbb{E} \otimes \widetilde{\mathbb{E}}\left[F(\widehat{\omega}^S)\right], \tag{3.8}$$

as the change of measure induced by the density,

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_N^{\beta,S}}{\mathrm{d}\mathbb{P}} = \exp\left(\sum_{k=1}^N \beta\omega_{i,S_i} - \lambda(\beta)\right), \tag{3.9}$$

retains the independence of the elements of the environment but tilts the distribution of the ones that belong to the graph of S by a factor of $\exp(\beta\omega - \lambda(\beta))$. This implies that, given Lemma 3.1, the following is sufficient to prove Theorem 2.

PROPOSITION 3.2. *If*

$$-\sum_{n\geq 1} K(n)\log K(n) < \beta\lambda'(\beta) - \lambda(\beta) \tag{3.10}$$

for some $\beta > 0$ *(in particular,* $-\sum_{n\geq 1} K(n)\log K(n) < \infty$*), then sequence* $\left\{W_N^{\beta,\widehat{\omega}^S}\right\}_{N\geq 1}$ *converges to infinity* $\mathbf{P}\otimes\mathbb{P}\otimes\widetilde{\mathbb{P}}$*-a.s.*

PROOF. Let us write

$$W_N^{\beta,\widehat{\omega}^S} = \mathbf{E}'\left[\exp\left(\sum_{k=1}^N \beta\widehat{\omega}^S_{k,S'_k} - \lambda(\beta)\right)\right], \tag{3.11}$$

where $(\mathbf{P}', S')$ is an independent copy of $(\mathbf{P}, S)$. Then

$$W_N^{\beta,\widehat{\omega}^S} \geq \mathbf{P}'\left[S'_1 = S_1, ..., S'_N = S_N\right]\exp\left(\sum_{k=1}^N \beta\widehat{\omega}^S_{k,S_k} - \lambda(\beta)\right) \tag{3.12}$$

$$= \prod_{k=1}^N K(X_k)\exp\left(\beta\widetilde{\omega}_{k,S_k} - \lambda(\beta)\right), \tag{3.13}$$

Since it suffices to show that $\lim_N \log W_N^{\beta,\widehat{\omega}^S} \to \infty$, we are left with proving that

$$\lim_{N\to\infty}\sum_{k=1}^N \left(\log K(X_k) + \beta\widetilde{\omega}_{k,S_k} - \lambda(\beta)\right) \to \infty, \tag{3.14}$$

$\mathbf{P}\otimes\widetilde{\mathbb{P}}$-a.s. Notice that, if $h_\beta := \mathbf{E}\otimes\widetilde{\mathbb{E}}\left[\log K(X_k) + \beta\widetilde{\omega}_{k,S_k} - \lambda(\beta)\right] > 0$, then (3.14) is a consequence of the Law of Large Numbers, applied to the i.i.d. sequence $\{\log K(X_k) + \beta\widetilde{\omega}_{k,S_k} - \lambda(\beta)\}_{k\geq 1}$ as

$$\lim_{N\to\infty}\frac{\sum_{k=1}^N \left(\log K(X_k) + \beta\widetilde{\omega}_{k,S_k} - \lambda(\beta)\right)}{N} = h_\beta, \tag{3.15}$$

$\mathbf{P}\otimes\widetilde{\mathbb{P}}$-a.s. This is a direct consequence of the assumption of the proposition as

$$\begin{aligned}\mathbf{E}\otimes\widetilde{\mathbb{E}}\left[\log K(X_k) + \beta\widetilde{\omega}_{k,S_k} - \lambda(\beta)\right] &= \mathbf{E}\left[\log K(X_k)\right] + \widetilde{\mathbb{E}}\left[\beta\widetilde{\omega} - \lambda(\beta)\right]\\ &= \sum_{n\in\mathbb{Z}}\left(K(n)\log K(n)\right) + \beta\lambda'(\beta) - \lambda(\beta).\end{aligned} \tag{3.16}$$

□

4. No strong disorder case

In this section, we prove Theorem 1.3. In the proposition below we use the size biased measure description from the previous section to show that the sequence $\{W_N\}$, under the sized biased measure, is tight (which is equivalent to proving that $\{W_N\}$ is uniformly integrable). This proves that weak disorder holds at every temperature, under the conditions on the increments distribution.

PROPOSITION 4.1. *Under the conditions of Theorem 1.3,*

$$\lim_{N\to\infty} \mathbf{P}\mathbb{P}\otimes\widetilde{\mathbb{P}}\left[W_N^{\beta,\widehat{\omega}} \geq L\right] \neq 1, \tag{4.1}$$

for some L sufficiently large.

PROOF. The idea for this proof is to fix a path S and average with respect to the other variables, then show that the resulting sequence is uniformly bounded. By Markov's Inequality and Fubini's Theorem we have,

$$\begin{aligned}\mathbb{P}\otimes\widetilde{\mathbb{P}}\left[W_N^{\beta,\widehat{\omega}} \geq L\right] &\leq \frac{1}{L}\mathbb{E}\otimes\widetilde{\mathbb{E}}\left[\mathbf{E}'\left[\exp\left(\sum_{n=1}^{N}\beta\left(\omega_{i,S'_i}\mathbf{1}_{\{S_i\neq S'_i\}}+\widetilde{\omega}_{i,S'_i}\mathbf{1}_{\{S_i=S'_i\}}\right)-\lambda(\beta)\right)\right]\right]\\ &= \frac{1}{L}\mathbf{E}'\left[F(\beta)^{|S_1^N\cap S'^N_1|}\right],\end{aligned} \tag{4.2}$$

where we write

$$W_N^{\beta,\widehat{\omega}} = \mathbf{E}'\left[\exp\left(\sum_{n=1}^{N}\left(\beta\omega_{i,S'_i}\mathbf{1}_{\{S_i\neq S'_i\}}+\widetilde{\omega}_{i,S'_i}\mathbf{1}_{\{S_i=S'_i\}}\right)-\lambda(\beta)\right)\right] \tag{4.3}$$

with $(\mathbf{P}', S')$, an independent copy of $(\mathbf{P}, S)$, $F(\beta) := \widetilde{\mathbb{E}}\left[\exp\left(\beta\widetilde{\omega}-\lambda(\beta)\right)\right]$ and

$$S_m^n := \{(i, S_i) : m \leq i \leq n\}. \tag{4.4}$$

Notice that it suffices to show that there exists some constant $K_\infty > 0$ such that

$$\mathbf{P}\left[\mathbf{E}'\left[F(\beta)^{|S_1^N\cap S'^N_1|}\right] \leq K_\infty\right] \geq 1/2, \tag{4.5}$$

since we might have

$$\mathbf{P}\mathbb{P}\otimes\widetilde{\mathbb{P}}\left[Z_N^{\beta,\widehat{\omega}} \geq L\right] \leq \frac{K_\infty}{L} + 1/2, \tag{4.6}$$

which proves (4.1) by taking L large enough. On the other hand, by considering the last time the paths S and S' intersect, we have

$$\begin{aligned}\mathbf{E}'\left[F(\beta)^{|S_1^N\cap S'^N_1|}\right] &= \sum_{n=0}^{N}\mathbf{E}'\left[F(\beta)^{|S_1^n\cap S'^n_1|}\mathbf{1}_{\{S'_n=S_n\}}\mathbf{1}_{\{S_{n+1}^N\cap S'^N_{n+1}=\emptyset\}}\right]\\ &\leq \sum_{n=0}^{N}F(\beta)^n\mathbf{P}'\left[S_n=S'_n\right]\end{aligned} \tag{4.7}$$

In the second line, we use that $F(\beta) \geq 1$. In fact, as we mentioned before, if the distribution of ω is unbounded, $F(\beta) \to \infty$ as $\beta \to \infty$ (see Lemma 3.A.1). To finish the proof we use two lemmas stated below. The first one is Theorem 2.1 from **[35]** and states that the independent sum of two symmetric unimodal distributions is again unimodal. This implies that the distribution of S_n is also symmetric and unimodal and that

$$\mathbf{P}'\left[S_n = S_n'\right] \leq \frac{1}{|S_n|}. \tag{4.8}$$

In the second lemma below, we show that S_n grows faster that any exponential, eventually almost surely. Here we use the crucial fact that $\alpha > 1$, the lemma being false otherwise. This implies that there exists $K_S > 0$, that might also depend on β, such that

$$\sum_{n=0}^{\infty} F(\beta)^n \frac{1}{|S_n|} < K_S. \tag{4.9}$$

This is sufficient to obtain (4.5) and conclude the proof of the theorem. □

Lemma 4.2. *Given $\{X_1, X_2, ...\}$ i.i.d. integer valued random variables, and the distribution of X_1 being unimodal and symmetric, then the distribution of $S_n = X_1 + ... + X_n$ is also unimodal and symetric and*

$$\mathbf{P}\left[S_n = x\right] \leq \frac{1}{|x|}, \tag{4.10}$$

for any $x \in \mathbb{Z}$.

Proof. In **[35]**, they show that the sum of two independent unimodal and symmetric random variables is also unimodal and symmetric. For (4.10), notice that

$$1 \geq \sum_{0 \leq y \leq x} \mathbf{P}\left[S_n = y\right] \geq x \mathbf{P}\left[S_n = x\right]. \tag{4.11}$$

□

Lemma 4.3. *Given $\{X_1, X_2, ...\}$ i.i.d. integer valued random variables and assuming that the distribution of X_1 satisfies*

$$\mathbf{P}\left[X_1 \geq n\right] \geq \frac{C(\log\log n)^\alpha}{\log n}, \tag{4.12}$$

for $\alpha > 1$, $C > 0$ and

$$\frac{\mathbf{P}\left[X_1 \in (s_n, 2ns_n)\right]}{\mathbf{P}\left[X_1 \geq s_n\right]} \leq \frac{1}{n^\gamma}, \tag{4.13}$$

where $\gamma > \frac{1}{2}$ and

$$s_n := \min\left\{ s \in \mathbb{N} : \mathbf{P}\left[X_1 \geq s\right] \leq \frac{(\log n)^2}{n} \right\}, \tag{4.14}$$

for all n sufficiently large, then for all constant $K > 0$,

$$|S_n| > K^n, \tag{4.15}$$

for all paths S, eventually for all n large enough, $\mathbf{P}$-a.s.

Proof. Given the first increments $\{X_1, ..., X_n\}$, let $X_n^{(n)}$ and $X_n^{(n-1)}$ be the highest and second highest values among $\{|X_1|, ..., |X_n|\}$. The proof of the lemma relies on two facts: the maximum $X_n^{(n)}$ satisfies (4.15), i.e.,

$$X_n^{(n)} > K^n, \tag{4.16}$$

eventually $\mathbf{P}-a.s.$, and that $X_n^{(n)}$ and S_n have roughly the same order, since

$$X_n^{(n-1)} \leq \frac{1}{2n} X_n^{(n)}, \tag{4.17}$$

for some constant $\delta > 0$, eventually $\mathbf{P}-a.s.$ In fact, inequalities (4.16) and (4.17) imply,

$$|S_n| \geq X_n^{(n)} - (n-1)X_n^{(n-1)} \geq X_n^{(n)} - \frac{(n-1)}{2n} X_n^{(n)} \geq \frac{1}{2} K^n, \tag{4.18}$$

eventually $\mathbf{P}-a.s.$ To show (4.16), observe that

$$\mathbf{P}\left[X_n^{(n)} \leq K^n\right] = (1 - \mathbf{P}\left[X_1 > K^n\right])^n \leq \left(1 - \frac{C_K (\log n)^\alpha}{n}\right)^n \leq e^{-C_K (\log n)^\alpha}, \tag{4.19}$$

for some constant $C_K > 0$, which by Borel-Cantelli's Lemma, implies (4.16). For (4.17) we have that, for $s \leq t \in \mathbb{N}$

$$\mathbf{P}\left[X_n^{(n-1)} = s, X_n^{(n)} = t\right] \leq \binom{n}{2} \mathbf{P}\left[X_1 \leq s\right]^{n-2} \mathbf{P}\left[X_1 = s\right] \mathbf{P}\left[X_1 = t\right]. \tag{4.20}$$

Then,

$$\mathbf{P}\left[X_n^{(n-1)} > \frac{1}{2n} X_n^{(n)}\right] \leq \sum_{s=0}^{\infty} \binom{n}{2} \mathbf{P}\left[X_1 \leq s\right]^{n-2} \mathbf{P}\left[X_1 = s\right] \mathbf{P}\left[X_1 \in (s, 2ns)\right]. \tag{4.21}$$

We split the last sum into two parts. The sum up to $s = s_n - 1$ can be bounded by

$$\sum_{s=0}^{s_n - 1} \binom{n}{2} \left(1 - \frac{(\log n)^2}{n}\right)^{n-2} \mathbf{P}\left[X_1 = s\right] \leq C' n^2 e^{-(\log n)^2}. \tag{4.22}$$

The second part of the sum can be bounded by

$$\sum_{s=s_n}^{\infty} \binom{n}{2} \mathbf{P}\left[X_1 = s\right] \mathbf{P}\left[X_1 \in (s_n, 2ns_n)\right] \leq \frac{n^2 \mathbf{P}\left[X_1 \geq s_n\right]^2}{n^\gamma} \leq \frac{(\log n)^4}{n^\gamma}. \tag{4.23}$$

Unfortunately, the last inequality is not enough to directly conclude (4.16) by Borel-Cantelli's Lemma, as γ might be smaller or equal that 1. In order to overcome this, let us consider $\{U_i : i \in \mathbb{N}\}$ a sequence of independent, Uniform-$[0,1]$ random variables, and let us couple the i.i.d. sequence $\{|X_1|, |X_2|, ...\}$ with the sequence $\{F^{-1}(U_1), F^{-1}(U_2), ...\}$, where F is the cumulative distribution function $F(x) := \mathbf{P}\left[|X_1| \leq x\right]$ and F^{-1}, the generalized inverse distribution function, defined as

$$F^{-1}(p) = \inf\{x \in \mathbb{R} : F(x) \geq p\}, \tag{4.24}$$

for $p \in [0,1]$. As before, let us denote by $U_n^{(n)}$ and $U_n^{(n-1)}$, the highest and second highest values among $\{U_1, ..., U_n\}$. In particular, this implies that $X_n^{(n)} = F^{-1}(U_n^{(n)})$

and $X_n^{(n-1)} = F^{-1}(U_n^{(n-1)})$. Consider the random variable $\tau_n^{(1)}$, as the first time after n, such us the second maximum $U_k^{(k-1)}$ needs to be updated, i.e.,

$$\tau_n^{(1)} := \min\left\{k > n : U_k > U_n^{(n-1)}\right\}. \tag{4.25}$$

and analogously, let $\tau_n^{(2)}$ be the second time after n, such us the second maximum is updated:

$$\tau_n^{(2)} := \min\left\{k > \tau_n^{(1)} : U_k > U_{\tau_n^{(1)}}^{(\tau_n^{(1)}-1)}\right\}. \tag{4.26}$$

Define the events

$$\mathcal{B} := \left\{X_n^{(n)} > K^n, \text{ for all } n \text{ sufficiently large}\right\}, \tag{4.27}$$

$$\mathcal{C} := \left\{X_{n^2}^{(n^2-1)} \le \frac{1}{2n^2} X_{n^2}^{(n^2)}, \text{ for all } n \text{ sufficiently large}\right\}, \tag{4.28}$$

and

$$\mathcal{D} := \left\{\tau_{n^2}^{(2)} \ge (n+1)^2, \text{ for all } n \text{ sufficiently large}\right\}. \tag{4.29}$$

We show that on the intersection of the three events, Inequality (4.15) holds. In fact, on $\mathcal{B}\cap\mathcal{C}$ we have that $|S_{n^2}| \ge \frac{1}{2}K^{n^2}$ for all n large enough. By intersecting with the event $\mathcal{D}$ we have that between n^2 and $(n+1)^2$, the second maximum is updated at most one time. This implies that for all $t \in (n^2, (n+1)^2)$, the pair $\left(X_t^{(t-1)}, X_t^{(t)}\right)$ is either $\left(X_{n^2}^{(n^2-1)}, X_{n^2}^{(n^2)}\right)$ or $\left(X_{(n+1)^2}^{((n+1)^2-1)}, X_{(n+1)^2}^{((n+1)^2)}\right)$. If it is equal to $\left(X_{n^2}^{(n^2-1)}, X_{n^2}^{(n^2)}\right)$ we have that

$$|S_t| \ge X_{n^2}^{(n^2)} - (t-1)X_{n^2}^{(n^2-1)} \ge X_{n^2}^{(n^2)} - \frac{(n+1)^2-1}{2n^2} X_{n^2}^{(n^2)} \ge \frac{1}{3}K^t. \tag{4.30}$$

If it is equal to $\left(X_{(n+1)^2}^{((n+1)^2-1)}, X_{(n+1)^2}^{((n+1)^2)}\right)$ we obtain

$$|S_t| \ge X_{(n+1)^2}^{((n+1)^2)} - (t-1)X_{(n+1)^2}^{((n+1)^2-1)} \ge X_{(n+1)^2}^{((n+1)^2)} - \frac{(n+1)^2-1}{2(n+1)^2} X_{(n+1)^2}^{((n+1)^2)} \ge \frac{1}{2}K^t. \tag{4.31}$$

To finish the proof of the lemma we verify that the three events have probability one. $\mathbf{P}[\mathcal{B}] = 1$ was already shown in (4.18). As $\gamma > 1/2$, the upper bound obtained in (4.23) suffices to obtain that $\mathbf{P}[\mathcal{C}] = 1$. As for $\mathcal{D}$ we have

$$\begin{aligned} \mathbf{P}\left[\tau_{n^2}^{(2)} < (n+1)^2\right] &\le \mathbf{P}\left[\exists i,j \in (n^2,(n+1)^2) : U_i > U_{n^2}^{(n^2-1)}, U_j > U_{n^2}^{(n^2-1)}\right] \\ &\le (2n)^2 \mathbf{P}\left[U_{n^2+1} > U_{n^2}^{(n^2-1)}\right]^2 \end{aligned} \tag{4.32}$$

Since U_{n^2+1} and $U_{n^2}^{(n^2-1)}$ are independent, their joint distribution can be computed explicitly [**18**]. This yields

$$\mathbf{P}\left[U_{n^2+1} > U_{n^2}^{(n^2-1)}\right] = \int_0^1\int_0^1 n^2(n^2-1)u^{n^2-2}(1-u)\mathbf{1}_{\{v>u\}}dvdu = \frac{2}{n^2+1}, \tag{4.33}$$

which proves the lemma.

□

Appendix

3.A. Properties of the exponential moment

LEMMA 3.A.1. *Let* ω *be a random variable with* $\mathbb{E}\left[e^{\beta\omega}\right] < \infty$ *for all* $\beta > 0$, $s =$ *ess sup* ω *and* $\lambda(\beta) := \log \mathbb{E}\left[e^{\beta\omega}\right]$.

a) *Let* $K < s$, *then*

$$\lim_{\beta\to\infty} \widetilde{\mathbb{P}}^{\beta}\left[\widetilde{\omega} < K\right] \to 0, \tag{3.A.1}$$

b) $\lim_{\beta\to\infty} \lambda'(\beta) = s$,

c) $\lim_{\beta\to\infty} \beta\lambda'(\beta) - \lambda(\beta) = -\log \mathbb{P}\left[\eta = s\right]$.

PROOF. The idea for this proof is that the sequence of measures $\mathbb{P}^{\beta}$, induced by the density

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}^{\beta}}{\mathrm{d}\mathbb{P}} = e^{\beta\omega-\lambda(\beta)}, \tag{3.A.2}$$

tend to put all the mass on the essential supremum s of ω, as $\beta \to \infty$.

a) Let $K' > 0$ such that $K < K' < s$. Then $\mathbb{P}\left[\omega > K'\right] = \delta > 0$ and

$$\widetilde{\mathbb{P}}^{\beta}\left[\widetilde{\omega} < K\right] = \frac{\mathbb{E}\left[e^{\beta\omega}\mathbf{1}_{\{\omega<K\}}\right]}{\mathbb{E}\left[e^{\beta\omega}\right]} \leq \frac{e^{\beta K}}{\mathbb{E}\left[e^{\beta\omega}\mathbf{1}_{\{\omega>K'\}}\right]} \leq \frac{e^{-\beta(K'-K)}}{\delta} \to 0, \tag{3.A.3}$$

as $\beta \to \infty$.

b) Assume $s < \infty$ for the rest of the proof. The case $s = \infty$ is analogous. Fix $\varepsilon > 0$. Let $K > 0$ such that $0 < s - K < \varepsilon$. Notice that $\lambda'(\beta) = \widetilde{\mathbb{E}}^{\beta}\left[\widetilde{\omega}\right] = \mathbb{E}\left[\omega e^{\beta\omega-\lambda(\beta)}\right] \leq s$ and

$$\widetilde{\mathbb{E}}^{\beta}\left[\widetilde{\omega}\right] \geq \widetilde{\mathbb{E}}^{\beta}\left[\widetilde{\omega}\mathbf{1}_{\{\widetilde{\omega}\geq K\}}\right] \geq (s-\varepsilon)\mathbb{P}\left[\widetilde{\omega} \geq K\right] \geq (s-\varepsilon)(1-\varepsilon), \tag{3.A.4}$$

for some large enough β, by the previous item. This proves b).

c) Given $\varepsilon > 0$, let $K > 0$ such that $0 < s - K < \varepsilon$ and $\mathbb{P}\left[\omega \geq K\right] \leq \mathbb{P}\left[\omega = s\right] + \varepsilon$. Then,

$$\begin{aligned} e^{\beta s-\lambda(\beta)}\left(\mathbb{P}\left[\omega = s\right] + \varepsilon\right) &\geq \mathbb{E}\left[e^{\beta\omega-\lambda(\beta)}\mathbf{1}_{\{\omega\geq K\}}\right] = \widetilde{\mathbb{P}}^{\beta}\left[\widetilde{\omega} \geq K\right] \\ &\geq e^{\beta K-\lambda(\beta)}\mathbb{P}\left[\omega \geq K\right] \geq e^{\beta(s-\varepsilon)-\lambda(\beta)}\mathbb{P}\left[\omega = s\right]. \end{aligned} \tag{3.A.5}$$

Applying logarithms and taking $\beta \to \infty$ we obtain

$$\lim_{\beta\to\infty} \beta s - \lambda(\beta) = -\log \mathbb{P}\left[\omega = s\right], \tag{3.A.6}$$

which proves c).

∎

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