Complex Dirac structures with constant real index

Dan Agüero Cerna

A thesis presented for the degree of
Doctor of Philosophy

Instituto de Matematica Pura e Aplicada
Rio de Janeiro
2020

Abstract

This thesis studies complex Dirac structures (i.e., Dirac structures in the complexification $(TM \oplus T^*M)_{\mathbb{C}}$ of the generalized tangent bundle of a manifold M) with constant real index. These objects extend generalized complex structures, which arise when the real index is zero, and encode geometric structures such as presymplectic, transverse holomorphic and CR structures. We introduce a new invariant that we call *order*, which is a nonnegative integer that allows us to obtain a classification of complex Dirac structures at the linear-algebraic level. We prove that complex Dirac structures with constant real index and order carry a presymplectic foliation which comes from an underlying (real) Dirac structure (generalizing the Poisson structures associated with generalized complex structures). We prove a local splitting theorem for complex Dirac structures with constant real index and order which extends the Abouzaid–Boyarchenko's splitting theorem for generalized complex structures. Finally we focus on complex Dirac structures with real index one; we study a pairing $(\cdot,\cdot)_1$, analogous to the Chevalley-Mukai pairing, which gives information about the dimension of the intersection of the annihilators of two pure spinors. We use it to give a spinorial description of complex Dirac structures with real index one.

Acknowledgements

During my years as a PhD student I have met many people whom I would like to thank. I begin by thanking my advisors Henrique Bursztyn and Roberto Rubio; without them this project would have not been completed. Henrique introduced me to the topic and gave me some freedom to work, but at the same time he was always concerned about the state of the project, giving ideas and resolving my doubts. Roberto generously shared his knowledge and guided me patiently through the writing process. I know both of them spent a lot of time revising my mistakes.

I am thankful to the members of the thesis committee, Alejandro Cabrera, Cristian Ortiz and Raimundo Heluani, for valuable comments and suggestions. I thank the members of the "Poisson team" Clarice Netto, Daniel Alvarez, Hudson Lima, Mateus Moreira and Miquel Cueca for our discussions and all the fun we had doing mathematics. I would also like to express my gratitude to IMPA for having given me the opportunity to grow up as a mathematician in such a great environment and for the friends I made there. Those days as a PhD student will be memorable.

I thank my family for all their support, especially my mother and sister.

I am indebted to CNPq for the PhD scholarship, and CNE Faperj for travel support; this project has also received funding from the European Union's Horizon 2020 research and innovation program under the Marie Sklodowska-Curie grant agreement No 750885 GENERALIZED, and also under the ERC StG grant 637912. Finally, I thank the Weizmann Institute of Science and the Universitat Autonoma de Barcelona for their hospitality during my visits there.

Contents

Chapter 1

Introduction

The study of Dirac structures [13] grew out of Poisson geometry and is by now a well established field of research with many applications (see e.g. [2, 3, 40]). From a modern perspective, Dirac structures are viewed as part of what is now known as "generalized geometry" [25], a term that refers to a broader viewpoint to the study of geometrical structures on manifolds based on the idea of replacing the tangent bundle of a smooth manifold M by the "generalized tangent bundle" $TM \oplus T^*M$. A key observation is that $TM \oplus T^*M$ carries a natural symmetric pairing (the usual pairing of vectors and covectors) and an extension of the Lie bracket of vector fields known as the *Courant-Dorfman bracket* [13, 16]. Dirac structures are defined as subbundles $L \subseteq TM \oplus T^*M$ which are lagrangian with respect to the symmetric pairing and satisfy an integrability condition with respect to the Courant-Dorfman bracket. Basic examples include foliations, presymplectic and Poisson structures, illustrating how the Courant bracket codifies the integrability conditions of different geometrical structures. The pairing and Courant bracket found on the bundle $TM \oplus T^*M$ naturally extend to its complexification $(TM \oplus T^*M)_{\mathbb{C}} = (TM \oplus T^*M) \otimes \mathbb{C}$; the main subject of study in this thesis are the much less explored Dirac structures in $(TM \oplus T^*M)_{\mathbb{C}}$, which we refer to as *complex Dirac structures*[1].

The most studied area within generalized geometry is that of *generalized complex geometry*, as initiated by N. Hitchin in [24] in the context of low-dimensional geometry and further developed by Gualtieri in [23]. The subject has become an active field of research in the last 15 years, especially due to its strong connections with physics (see e.g. [3, 11, 28]). Generalized complex structures have a very rich geometry, encompassing complex and symplectic structures as extreme examples. An important fact is that generalized complex structures are very special types of complex Dirac structures: like any subbundle of a complex vector bundle, complex Dirac structures have a pointwise *real index*, and generalized complex structures correspond to complex Dirac structures $L \subseteq (TM \oplus T^*M)_{\mathbb{C}}$ whose real indices vanish at all points (i.e., $L \cap \overline{L} = 0$). One of our main goals is to identify the geometrical structures encoded by complex Dirac structures that do not necessarily satisfy the vanishing condition on the real index. In contrast with generalized complex manifolds, which must be even dimensional, many of the more general structures that we will encounter may exist in odd dimensions.

One of the motivations to pass from generalized complex structures to more general complex Dirac structures is entirely analogous to the original motivation for considering Dirac structures in Poisson geometry. Just as Poisson structures provide the geometrical description of phase spaces in classical mechanics, Dirac structures were introduced to provide a geometric framework for constrained mechanics. Constraints of mechanical systems are represented by submanifolds

[1]Not to be confused with holomorphic Dirac structures on complex manifolds

of their phase spaces, and the difficulty is that, in general, submanifolds of Poisson manifolds do not inherit a Poisson structure. In turn, modulo mild regularity conditions, such submanifolds are naturally equipped with Dirac structures. In a similar fashion, submanifolds of generalized complex manifolds do not generally inherit a generalized complex structure, but always carry (modulo the same regularity conditions) complex Dirac structures, usually with non-trivial real indices (see e.g. [5, 36] for a treatment of the special submanifolds which are again generalized complex). In short, just as Dirac structures arise on submanifolds of Poisson manifolds, complex Dirac structures appear on submanifolds of generalized complex manifolds.

In this thesis we begin a systematic study of complex Dirac structures; we now outline our main contributions.

We start by studying invariants of complex Dirac structures on vector spaces. A fundamental pointwise invariant of a generalized complex structure is an integer called its *type*; in fact, at the linear-algebraic level, the type completely determines the generalized complex structure (see [22, Theor. 4.13]). In this work we give a definition of type for any complex Dirac structure which, in the case of generalized complex structures, is equivalent to the original notion in [22]. A complex Dirac structure of real index r is defined on spaces of dimension $2n + r$, and its type always varies from 0 to n. But in contrast with generalized complex structures, in order to specify a complex Dirac structure at the linear-algebraic level we notice that the real index and type are not enough; so we introduce a third invariant, which we call *order* (Definition 3.12), that provides the missing information, see Proposition 3.18. The order is an integer varying from 0 to r (the real index), so for generalized complex structures, not only the real index vanishes but also the order.

In generalized complex geometry, the examples of extreme types are symplectic (type 0) and complex (type n). For complex Dirac structures, we have a richer situation: at the linear level, we obtain different examples of extreme types for each order s, see Table 3.1.4. In other words, we have one set of structures for each extreme type, 0 or n, parametrized by the order s. Passing to manifolds, we notice that the subclass of complex Dirac structures with *constant real index and order* (but not necessarily type) are the most tractable, so we focus on them. We identify their key examples of extreme types, extending the description of extreme types of generalized complex structures: in type 0 we have regular foliations with leafwise presymplectic forms with regular kernel (i.e., regular Dirac structures with regular null distribution), while in type n we have structures interpolating CR structures and transverse holomorphic structures (Proposition 4.7), which we call *transverse CR structures* (Definition 2.74).

Complex Dirac structures of constant real index and order include generalized complex structures and various aspects of their theory can be extended to this more general setting. For example, an important feature of generalized complex structures is that they have an underlying Poisson structure [14, 23, 28], which in turn determines a symplectic foliation on any generalized complex manifold. More generally, we prove (Theorem 4.21)

Theorem 1.1. *A complex Dirac structure with constant real index and order has an underlying Dirac structure, which agrees with the Poisson structure of a generalized complex structure when the real index is zero.*

Since any Dirac structure gives rise to a presymplectic foliation, we obtain presymplectic foliations associated with complex Dirac structures of constant real index and order. As we will see below, these presymplectic leaves are a key ingredient in the local description of these complex Dirac structures, playing a role similar to that of symplectic leaves in the local study of generalized complex manifolds ([1, 22]).

We single out a class of complex Dirac structures having an associated split isotropic subbundle (Definition 4.30). An interesting subset of this class is given by those having constant real index equal to their order; in this case, their underlying Dirac structures are Poisson (Corollary 4.32); they are also special instances of the generalized CR structures of [27](Remark 4.36). Particular examples include regular foliations with leafwise generalized complex structures (Proposition 4.35). Inside the complex Dirac structures having constant real index equal to their order, those of maximal type are equivalent (via a B-transformation, possibly complex) to CR structures (Proposition 4.7), so we refer to them as being of *CR-type*.

A central result in generalized complex geometry concerns the local description of generalized complex manifolds. It is proven in [1] that, around any point, a generalized complex structure is equivalent (via a diffeomorphism and B-transformation) to the direct product of a symplectic structure and another generalized complex structure that, at the given point, has "complex type" (i.e., its type at the point is maximal, or equivalently, its associated Poisson structure vanishes at the point). Here the symplectic factor is a neighbourhood of the point in the symplectic leaf through it, while the second factor is given by a transversal to this leaf. So this result should be regarded as a version of Weinstein's splitting theorem for Poisson structures [39] in generalized complex geometry. When the type is constant around the point, this local splitting gives rise to the generalized Darboux theorem [23, Theorem 4.35] of generalized complex structures. As we mentioned above, in the more general context of complex Dirac structures with constant real index and order, the analogue of points of "complex type" are the points of "CR-type" (i.e., the real index and order coincide, and the type is maximal – or the underlying Poisson structure vanishes). We prove in Theorem 5.10 the following local structure result:

Theorem 1.2. *Let L be a complex Dirac structure with constant real index r and order s, and let $p \in M$ be a point of type k. Then, locally around p, L is equivalent (via a diffeomorphism and B-transformation) to the product of a presymplectic manifold (with $(r-s)$-dimensional kernel) and a complex Dirac structure of constant real index and order equal to s and which is of CR-type at the point p.*

Analogously to the generalized complex situation, the presymplectic factor comes from the leaf through the point, while the other factor is realized by small transversals. If the type is constant around the point p, the transverse factor is a CR-manifold (Corollary 5.20). When $r = s = 0$, we recover the known local results for generalized complex structures. Our main tool to prove the result is the technique developed in [10] to obtain splitting theorems in various contexts. We remark that, for generalized complex manifolds, their local description has a further refinement proven in [4], asserting that a generalized complex structure of complex type at a point is locally equivalent to a holomorphic Poisson structure, for some complex structure near the point. It would be interesting to find a more general formulation of this result in our context.

Another issue that we consider using complex Dirac structures concerns the odd dimensional analogue of generalized complex geometry. One can view generalized complex structures as complex Dirac structures on even dimensional manifolds with the smallest possible real index (which is zero). Similarly, the smallest possible real index of a complex Dirac structure on an odd-dimensional manifold is one. This leads us to give particular attention to complex Dirac structures with real index one. We obtain in this case a complete description of these objects via pure spinors, in a way that is parallel to the spinorial viewpoint to generalized complex structures as in [22, Section 4.1]. We show that the pure spinors of complex Dirac structures of real index one satisfy an additional equation that is similar to the equation for real index zero, but now taking the order into account (Proposition 6.17). For the spinorial description of real-index one lagrangian subbundles of $(TM \oplus T^*M)_{\mathbb{C}}$, we introduce an analogue of the Chevalley-Mukai pairing used to describe zero real index [22, Section 4.1]. The construction of this pairing, that we denote by $(\cdot,\cdot)_1$, is an adaptation of one of the pairings described in [8]; similarly to the Chevalley-Mukai pairing, this pairing has the property that for a pure spinor ρ with annihilator $L \subset (TM \oplus T^*M)_{\mathbb{C}}$, the condition that $\dim(L \cap \overline{L}) = 1$ (i.e., L has real index one) is equivalent to $(\rho, \overline{\rho})_1 \neq 0$ (Proposition 6.6). It is mentioned in [8] that the family of pairings introduced there has potential applications to general relativity, twistor theory and optical geometry; it would be interesting to explore similar applications for the spinorial equation for the pairing $(\cdot,\cdot)_1$.

Looking into the future, many aspects of the theory of complex Dirac structures remain to be explored. We mention some directions. Regarding metrics, at the end of Chapter 4 we propose a metric theory corresponding to complex Dirac structures with constant real index; as an example we show how strictly pseudoconvex structures fit well into this theory. It is also natural to investigate deformations of complex Dirac structures, having the interesting deformation theory of generalized complex structures as a motivation. In another direction, one should extend the spinorial viewpoint presented for the case of real index one to complex Dirac structures with arbitrary real index; a possible way is to adapt the whole family of pairings in [8] in order to obtain the equations that the spinors associated to complex Dirac structures with constant real index should satisfy.

The thesis is structured as follows:

Chapter 2 contains some preliminaries, including a brief review of Lie and Courant algebroids, Dirac and generalized complex structures. We also recall CR structures and introduce the more general concept of *transverse CR structure*, which plays an important role in the study of complex Dirac structures.

We start Chapter 3 with a motivation for complex Dirac structures with non-zero real index coming from submanifolds of generalized complex manifolds. We then discuss foundational aspects of complex Dirac structures on vector spaces, including a definition of type (extending the notion for generalized complex structures) and the new invariant, order. The main result (Proposition 3.19) in this chapter is the full classification of complex Dirac structures on vector spaces in terms of these invariants. We also discuss some basic properties of complex Dirac structures on manifolds, some natural distributions associated with them and present examples illustrating how real index, type and order can change.

In Chapter 4, we focus our attention on complex Dirac structures with constant real index. We mention some topological obstructions for the existence of these objects. Under the additional assumption of constant order, we give a full description of examples of extreme types (Proposition 4.7). In this context we also describe the natural Dirac structure associated to a complex Dirac structure (Theorem 4.21), and discuss situations where this Dirac structure is Poisson.

In Chapter 5, after recalling some results from [10], we present the local splitting theorem for complex Dirac structures with constant real index and order (Theorem 5.10 and Corollary 5.20).

In the last Chapter 6, we study the special case of complex Dirac structures with real index one, with focus on the spinorial viewpoint.

In Appendix A we introduce a new class of structures inspired by the maximally nonintegrability of contact structures: the nondegenerate structures.

Chapter 2

Preliminaries

In this chapter we review some results of Lie and Courant algebroids, and Dirac and generalized complex structures, which are fundamental for the reading of the thesis. We also show that in order to study submanifolds of generalized complex structures we need to deal with complex Dirac structures with nontrivial real index. At the end of the chapter we recall some classical structures as CR structures and introduce the *transverse CR structures*, which will play an important role in subsequent chapters.

2.1 Lie and Courant algebroids

We begin by recalling some definitions. In the whole thesis we deal with smooth manifolds. Let E be a vector bundle over a manifold M, a **distribution on** E is an assignment to each point $p \in M$ to a subspace $D|_p \subseteq E|_p$. We say that a distribution D is of **constant rank**, if the dimension of $D|_p$ is constant for all $p \in M$. We say that the distribution is **smooth** if for every $p \in M$ and $e_p \in D|_p$, there exist an open neighbourhood U of p and a smooth section $\hat{e} \in \Gamma(E|_U)$ such that $\hat{e}|_q \in D|_q$, for all $q \in U$ and $\hat{e}|_p = e_p$. A smooth distribution of constant rank is a vector subbundle. A distribution D is called **regular** if it is of constant rank.

Every smooth distribution D on a vector bundle E defines a subsheaf $\Gamma : U \mapsto \Gamma(D|_U)$ of the sheaf of smooth section of E, where $\Gamma(D|_U) = \{e \in \Gamma(E|_U) \mid e_p \in D|_p, \ \forall p \in U\}$. The function which assigns to each point the dimension of each space $D|_p$ has a special property.

Lemma 2.1. *Let D be a smooth distribution of a vector bundle E over M. Given a point $p \in M$, there exists an open neighbourhood U of p such that* $\dim D|_q \geq \dim D|_p$*, for all $q \in U$.*

This lemma implies that if D is a smooth distribution of a vector bundle, then the function $d(p) = \dim D_p$ is lower semi-continuous.

Along this thesis a foliation is a partition $\mathcal{F} = \{l_\alpha\}$ of an $m-$dimensional manifold M in a disjoint union of immersed connected submanifolds l_α called leaves, which satisfies the **local foliation property** at each point $p \in M$: let l_p be the leaf of $\mathcal{F}$ passing through p and d the dimension of l_p. Then there exists a chart $(y_1, \dots, y_m)$ on a neighbourhood $U(\lambda)$ of p, $U(\lambda) = \{-\lambda < y_1 < \lambda, \dots, -\lambda < y_m < \lambda\}$ such that $\{y_{d+1} = \dots = y_m = 0\} = U \cap l_p$ and each submanifold $\{y_d + 1 = c_{d+1}, \dots, y_m = c_m\}$ is contained in some leaf of $\mathcal{F}$, where $c_{d+1}, \dots, c_m \in \mathbb{R}$ are small enough. Foliations are called **regular** if their leaves have the same dimension. A **simple** foliation is a regular foliation admitting a smooth manifold B and a submersion $q : M \to B$ such that the fibres of q are the leaves of the foliation. The space B is called the **leaf space** since the map q makes a one-to-one correspondence between the points of B and the leaves of the foliation. Every foliation $\mathcal{F}$ has associated a smooth distribution $T\mathcal{F}$ defined as $p \in M \mapsto T_p l_\alpha$, where l_α is the leaf passing through p.

In what follows we focus on distributions of TM, assumed to be smooth from now on. An **integral manifold** of a distribution D is an immersed connected submanifold $N \subseteq M$ such that $T_pN = D_p$ for every $p \in N$. We say that an integral manifold of a distribution D through p is **maximal** if it contains every integral manifold passing through p. A distribution is **integrable** if for every $p \in M$, there exists an integral manifold of D passing through p. Every integrable distribution defines a partition of M given by its maximal integral manifolds, this partition satisfy the local foliation property and it so is a foliation, cf. [33,34]. An **involutive** distribution on TM is a distribution D such that $\forall X, Y \in \Gamma(D)$, we have that $[X, Y] \in \Gamma(D)$, where $[\cdot,\cdot]$ denotes the Lie bracket. The classical Frobenius theorem asserts that a regular distribution is integrable if and only if it is involutive.

Involutive regular distributions are a special case of a more general kind of structures, Lie algebroids.

Definition 2.2. A **Lie algebroid** over a manifold M is a vector bundle L over M together with a Lie bracket

$$[\cdot,\cdot]_L : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$$

and a bundle map $\rho : L \to TM$ called the anchor map satisfying the Leibniz property

$$[\alpha, f\beta] = f[\alpha, \beta] + \rho(\alpha)(f)\beta,$$

for all $\alpha, \beta \in \Gamma(L)$ and $f \in C^\infty(M)$.

Example 2.3. *The tangent bundle TM is a Lie algebroid with a bracket given by the Lie bracket and the identity map as anchor map.*

Example 2.4. Poisson structures induce Lie algebroid structures on T^*M in the following way. Consider π a Poisson bivector on M, then $(T^*M, \pi, [\cdot,\cdot]_\pi)$ is a Lie algebroid. Here the anchor map is the Poisson bivector itself seen as a map $\pi : T^*M \to TM$; π defines naturally the bracket $[\cdot,\cdot]_\pi$ in the following way

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi(\alpha)}\beta - \mathcal{L}_{\pi(\beta)}\alpha - d(\pi(\alpha, \beta)), \tag{2.1}$$

where $\alpha, \beta \in \Gamma(T^*M)$.

As a consequence of the local splitting theorem for Lie algebroids [21] we have the following:

Proposition 2.5. *The distribution defined by the image of the anchor map of a Lie algebroid is integrable.*

There is a replacement for TM more appropriate in our context, the **generalized tangent bundle** $TM \oplus T^*M$. This bundle has a natural nondegenerate symmetric pairing

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y))$$

where $X + \xi, Y + \eta \in TM \oplus T^*M$ and inherits a bracket on $\Gamma(TM \oplus T^*M)$

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X\eta - \iota_Y d\xi,$$

where $X + \xi, Y + \eta \in \Gamma(TM \oplus T^*M)$, called the **Courant-Dorfman bracket** [13,16]. The bundle $TM \oplus T^*M$ with the pairing and bracket described above is a special case of a more general structure.

Definition 2.6 ([29], [31])**.** Let E be a vector bundle over the manifold M equipped with a bundle map $\rho : E \to TM$, a nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle$ and a bilinear bracket $[\cdot, \cdot]$. We say that $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$ is a **Courant algebroid** if it satisfies the following conditions:

a) $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]], \forall e_1, e_2, e_3 \in \Gamma(E)$

b) $\rho[e_1, e_2] = [\rho(e_1), \rho(e_2)], \forall e_1, e_2 \in \Gamma(E)$

c) $[e_1, fe_2] = f[e_1, e_2] + \rho(e_1)(f)e_2, \forall e_1, e_2, e_3 \in \Gamma(E)$

d) $\rho(e)\langle e_1, e_2 \rangle = \langle [e, e_1], e_2 \rangle + \langle e_1, [e, e_2] \rangle, \forall e, e_1, e_2 \in \Gamma(E)$

e) $[e, e] = D\langle e, e \rangle, \forall e \in \Gamma(E)$

here we denote D the following map

$$D : C^\infty(M) \to \Gamma(E)$$

$$Df = \frac{1}{2}\rho^* df,$$

where we see $\rho^* df$ as an element of E via the isomorphism given by twice the pairing.

From axiom e), we note that the bracket $[\cdot, \cdot]$ is not skew-symmetric. However, the following holds

$$[e_1, e_2] = -[e_2, e_1] + 2D\langle e_1, e_2 \rangle. \tag{2.2}$$

We say that a distribution of a Courant algebroid is involutive if it is closed under the bracket.

Lemma 2.7. *Let A be an involutive isotropic subbundle ($A \subseteq A^\perp$) of a Courant algebroid E. Then A inherits a Lie algebroid structure coming from the restriction of the anchor map and bracket of E.*

Proof. The bracket restricted to the isotropic A is skew-symmetric by equation (2.2), by item a) is a Lie bracket and by item c) it satisfies the Leibniz property. □

Examples 2.8. We give some examples of Courant algebroids:

a) The bundle $TM \oplus T^*M$ itself with anchor the projection onto TM, the usual pairing and the bracket

$$[X + \xi, Y + \eta]_H = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_Y \iota_X H,$$

for some $H \in \Omega^3_{cl}(M)$.

b) Let $\mathfrak{g}$ be a quadratic Lie algebra, i.e, a Lie algebra $\mathfrak{g}$ together with a nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ such that

$$\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0.$$

Then, $\mathfrak{g}$ is a Courant algebroid over a point.

c) Suppose A is Lie algebroid, with anchor map a, such that its dual A^* is a Lie algebroid with anchor map a^*. We call the pair (A, A^*) a Lie bialgebroid if the differential d_* associated to A^* satisfies

$$d_*[a_1, a_2]_A = [d_*a_1, a_2]_A + [a_1, d_*a_2]_A,$$

where $[\cdot,\cdot]_A$ is the Schouten bracket associated to A. The vector bundle $A \oplus A^*$ is a Courant algebroid with the obvious pairing and the following bracket

$$[X_1 + \alpha_1, X_2 + \alpha_2] = ([X_1, X_2] + L_{\alpha_1}X_2 - \iota_{\alpha_2}d_*X_1) + ([\alpha_1, \alpha_2] + L_{X_1}\alpha_2 - \iota_{X_2}d\alpha_1), \quad (2.3)$$

where $X_1 + \alpha_1, X_2 + \alpha_2 \in \Gamma(A \oplus A^*)$.

Definition 2.9. Let $(E_k, \langle\cdot,\cdot\rangle_k, [\cdot,\cdot]_k, \rho_k)$ be two Courant algebroids over the manifold M, where $k = 1, 2$. An **isomorphism** of the Courant algebroids E_1 and E_2 is a pair (F, f), where $F : E_1 \to E_2$ is a bundle isomorphism covering a diffeomorphism $f : M \to M$ satisfying the following:

a) $f^*\langle F(e_1), F(e_2)\rangle_2 = \langle e_1, e_2\rangle_1$, $\forall e_1, e_2 \in E_1$,

b) $F[e_1, e_2]_1 = [F(e_1), F(e_2)]_2$, $\forall e_1, e_2 \in E_1$,

c) $\rho_2 \circ F = f_* \circ \rho_1$.

When considering the same Courant algebroid $(E, \langle,\rangle, [,], \rho)$ we obtain the definition of an **automorphism** of a Courant algebroid. In what follows we focus on Courant automorphisms rather than on isomorphisms. The set of all automorphisms of a Courant algebroid is a group denoted by $\mathrm{Aut}_{CA}(E)$.

We recall that the automorphisms of a Lie algebroid are vector bundle automorphism preserving the Lie bracket. It is known that the automorphism group of the Lie algebroid TM with the Lie bracket is $\mathrm{Diff}(M)$, cf. [22]. We will see that the automorphism group of the Courant algebroid $TM \oplus T^*M$ contains the automorphism group of the Lie algebroid TM with the Lie bracket. Consider $\varphi \in \mathrm{Diff}(M)$; then the bundle map

$$\mathbb{T}\varphi : TM \oplus T^*M \to TM \oplus T^*M$$

$$\mathbb{T}\varphi(X + \xi) = \varphi_*X + (\varphi^{-1})^*\xi$$

is an automorphism of $TM \oplus T^*M$. The operator $\mathbb{T}$ is called **generalized differential**.

Given a two-form B, there is an automorphism of the vector bundle $TM \oplus T^*M$ denoted by e^B, defined in the following way

$$e^B(X + \xi) = X + \xi + \iota_X B.$$

These maps usually do not preserve the Courant-Dorfman bracket, actually we have the following

$$[e^B(X + \xi), e^B(Y + \eta)] = e^B[X + \xi, Y + \eta] + \iota_X\iota_Y dB,$$

where $X, Y \in TM$ and $\xi, \eta \in T^*M$. So when B is closed, e^B is an automorphism of $(TM \oplus T^*M, \langle\cdot,\cdot\rangle\, [\cdot,\cdot], pr_{TM})$, called B**-transformation** or **B-fields**.

The automorphisms of $(TM \oplus T^*M, \langle\cdot,\cdot\rangle, [\cdot,\cdot], pr_{TM})$ are generated by diffeomorphisms of M and B-transformations.

Proposition 2.10 ([23]). *The automorphism group of* $(TM \oplus T^*M, \langle\cdot,\cdot\rangle, [\cdot,\cdot], pr_{TM})$*, is given by the pairs* $(F, B) \in \mathrm{Diff}(M) \times \Omega^2_{cl}(M)$ *via the map*

$$\mathrm{Diff}(M) \times \Omega^2_{cl}(M) \to \mathrm{Aut}_{CA}(TM \oplus T^*M)$$

$$(F, B) \mapsto \mathbb{T}F \circ e^B.$$

From the Proposition we obtain the following exact sequence:

$$0 \longrightarrow \Omega^2_{cl}(M) \longrightarrow \mathrm{Aut}_{CA}(TM \oplus T^*M, \langle\cdot,\cdot\rangle, [\cdot,\cdot], pr_{TM}) \longrightarrow \mathrm{Diff}(M) \longrightarrow 0, . \tag{2.4}$$

Definition 2.11. The **infinitesimal automorphisms** of a Courant algebroid $\mathrm{Der}(E)$ are given by first order differential operators $D : \Gamma(E) \to \Gamma(E)$ such that there exists $X \in \mathfrak{X}(M)$ satisfying:

a) $X\langle e_1, e_2\rangle = \langle De_1, e_2\rangle + \langle e_1, De_2\rangle$,

b) $D[e_1, e_2] = [De_1, e_2] + [e_1, De_2]$,

where $e_1, e_2 \in \Gamma(E)$. The vector field X is called the symbol of D.

The set of all infinitesimal automorphisms of a Courant algebroid is a Lie algebra with bracket given by the commutator and denoted by $\mathfrak{aut}(E)$ as it is the Lie algebra of $\mathrm{Aut}_{CA}(E)$.

Corollary 2.12 ([23]). *We have that the map*

$$\mathfrak{X}(M) \times \Omega^2_{cl}(M) \to \mathfrak{aut}(TM \oplus T^*M)$$

defined by the natural action of $\mathfrak{X}(M) \times \Omega^2_{cl}(M)$ *over* $\Gamma(TM \oplus T^*M)$ *given by*

$$(X, b) \cdot (Y + \eta) = \mathcal{L}_X(Y + \eta) + \iota_Y b$$

is an isomorphism.

Examples 2.13. Consider $X + \xi \in \Gamma(TM \oplus T^*M)$. The map

$$ad_{X+\xi} : \Gamma(TM \oplus T^*M) \to \Gamma(TM \oplus T^*M)$$

$$ad_{X+\xi}(Y + \eta) = [X + \xi, Y + \eta]$$

is an infinitesimal automorphism. Moreover, we have that it corresponds to the pair $(X, -d\xi) \in \mathfrak{X}(M) \times \Omega^2(M)$.

From now on we also denote the Courant and the infinitesimal automorphisms as pairs $(F, \omega) \in \mathrm{Diff}(M) \times \Omega^2_{cl}$ and $(X, \omega) \in \mathfrak{X}(M) \times \Omega^2_{cl}$, respectively.

Definition 2.14. An **exact Courant algebroid** E is a Courant algebroid fitting into the following exact sequence:

$$0 \longrightarrow T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \longrightarrow 0\,. \tag{2.5}$$

Remark 2.15. There is a classification of exact Courant algebroids given by Severa [32]. Given an exact Courant algebroid $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \rho)$, there exist a closed three-form H such that E is isomorphic to $(TM \oplus T^*M, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H, pr_{TM})$.

Until now we have only treated Courant and Lie algebroids over real vector bundles. However, we will deal with these structures in the complex setting. So we make some remarks about $(TM \oplus T^*M)_{\mathbb{C}}$ as a "complex Courant algebroid"; later in Chapter 3 we will speak about complex Lie algebroids. The bundle $(TM \oplus T^*M)_{\mathbb{C}}$ inherits a pairing and a bracket which are the complexification of both the canonical pairing and the Courant-Dorfman bracket, and has anchor map $pr_{TM_{\mathbb{C}}}$. Its automorphisms are defined in the same way as for Courant algebroids. On one hand we have the symmetries defined by $\mathrm{Diff}(M)$, given $\varphi \in \mathrm{Diff}(M)$, we have that the complexification of $\mathbb{T}\varphi$ is a symmetry. On the other hand we have the symmetries given by complex two-forms: consider a complex closed two-form B, the bundle map e^B (defined as in the real case) is also a symmetry of $(TM \oplus T^*M)_{\mathbb{C}}$, we call these transformations **complex B-transformations**.

2.2 Dirac structures on vector spaces

In this section we begin by recalling the definition and some results related to Dirac structures on vector spaces and the spinors associated to them.

2.2.1 Dirac structures

Let V be a finite dimensional vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The vector space $V \oplus V^*$ has a canonical pairing defined as:

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X)),$$

where $X + \xi, Y + \eta \in V \oplus V^*$. A **lagrangian** subspace of $V \oplus V^*$ is an isotropic subspace of $V \oplus V^*$ of maximal dimension.

Definition 2.16 ([13]). A **Dirac structure** on a vector space V is a lagrangian subspace of $V \oplus V^*$ with respect to the canonical pairing.

There are some subspaces of V related to these structures. The **range** of a Dirac structure L is defined as $pr_V L$ and the **kernel** of L is defined as $\ker L = L \cap V$.

Note that associated to any Dirac structure, there is a map

$$\varepsilon_L : pr_V L \times pr_V L \to \mathbb{K}$$

$$\varepsilon_L(X, Y) = \eta(X),$$

where $X + \xi, Y + \eta \in L$ for some $\xi, \eta \in V^*$. The map ε_L is well defined and skew-symmetric.

The subspace $pr_V L$ and the two-form ε_L determine completely the Dirac structure L. Let $E \subseteq V$ and $\varepsilon \in \wedge^2 E^*$. We define

$$L(E, \varepsilon) = \{X + \xi \mid \xi|_E = \iota_X \varepsilon\}.$$

We recover a Dirac structure from its range and two-form as $L = L(pr_V L, \varepsilon_L)$.

Proposition 2.17. *Every Dirac structure of $V \oplus V^*$ is of the form $L(E, \varepsilon)$, for some $E \subseteq V$ and $\varepsilon \in \wedge^2 E^*$.*

Definition 2.18. Let φ be a linear map from V to W and L_V, L_W be Dirac structures on the vector spaces V and W, respectively. The **backward image** of L_W, denoted by $\mathcal{B}_\varphi(L_W)$ is the subspace

$$\{X + \varphi^*\xi \mid \varphi_* X + \xi \in L_W\} \subseteq V \oplus V^*.$$

The **forward image** of L_V, denoted by $\mathcal{F}_\varphi(L_V)$ is the subspace

$$\{\varphi_* X + \xi \mid X + \varphi^*\xi \in L_V\} \subseteq W \oplus W^*.$$

Proposition 2.19 ([9])**.** *Let φ be a linear map from V to W and L_V, L_W be Dirac structures on V and W, respectively. Then, $\mathcal{B}_\varphi(L_W)$ and $\mathcal{F}_\varphi(L_V)$ are Dirac structures on V and W, respectively.*

We will see that the backward image of a Dirac structure fits into a exact sequence, a fact that will be used in Chapter 4. Consider the linear map $\varphi : V \to W$ and a Dirac structure L_W on W. Consider the subspace

$$\Gamma_\varphi = \{(Y + \eta, X + \xi) \mid Y = \varphi(X), \xi = \varphi^*\eta\} \subseteq (W \oplus W^*) \times \overline{(V \oplus V^*)},$$

where $\overline{(V \oplus V^*)}$ denotes the vector space $V \oplus V^*$ equipped with the pairing $-\langle \cdot, \cdot \rangle$. Note that Γ_φ is a lagrangian subspace of $(W \oplus W^*) \times \overline{(V \oplus V^*)}$. Then the backward image of L_W fits into the following exact sequence

$$0 \longrightarrow \ker(\varphi^*) \cap L_W \longrightarrow (L_W \oplus (V \oplus V^*)) \cap \Gamma_\varphi \longrightarrow \mathcal{B}_\varphi(L_W) \longrightarrow 0 \tag{2.6}$$

where the first map is the inclusion $\eta \mapsto (\eta, 0)$ and the second is the projection $pr_{V \oplus V^*}$.

In order to recall the definition of a generalized complex structure, we make a discussion about real parts that will be also useful in the following chapter.

Definition 2.20. Let W be a real vector space and $A \subseteq W_{\mathbb{C}}$. If $A = \overline{A}$ or equivalently $A = V_{\mathbb{C}}$ for some subspace V of W, we say that A is **real**, and we call V the **real part** of A and denote it as $\operatorname{Re} A$.

Note that the space $A \cap \overline{A}$ is always real and actually

$$\operatorname{Re}(A \cap \overline{A}) = A \cap W = \{s \in A \mid s \text{ is real}\},$$

here we use the identification $W_{\mathbb{C}} = W \oplus iW$, where real elements are elements of W. Note that $A \cap \overline{A}$ is the maximal real subspace contained in A.

Definition 2.21. The **real index** of A is $\dim_{\mathbb{C}} A \cap \overline{A}$.

The real index measures how big the space of real elements of the vector space is. For example if A is real, i.e. $A = \overline{A}$, then its real index is $\dim A$. However, if its real index is zero, then $A \cap \overline{A} = 0$ implying that $A = \ker(J_{\mathbb{C}} - iId)$ for a complex structure J on $\operatorname{Re}(A \oplus \overline{A})$.

Definition 2.22. Let V be a real vector space. A **generalized complex structure** on V is a lagrangian subspace of $(V \oplus V^*)_{\mathbb{C}}$ which has real index zero.

As the existence of complex structures on a vector space implies that the dimension of the vector space is even, the same happens for generalized complex structures.

Proposition 2.23 ([22])**.** *If a vector space V admits a generalized complex structure, then V is even dimensional.*

We will talk more about these structures in the next sections

2.2.2 Spinors

Let V be a complex or real $m-$dimensional vector space and consider $S = \bigwedge^\bullet V^*$, the **space of spinors** associated to V. The elements of S are called **spinors**. There exists an action of $V \oplus V^*$ on S, given by

$$(X+\xi)\cdot\rho = \iota_X\rho + \xi\wedge\rho,$$

where $X+\xi \in V\oplus V^*$ and $\rho\in S$. Given a spinor ρ we associate an isotropic space

$$L_\rho = \{X+\xi \in V\oplus V^* \mid (X+\xi)\cdot\rho = 0\},$$

which is called the **annihilator** of ρ. Note that the annihilator depends on the conformal class of the spinor.

Definition 2.24. A spinor ρ is said to be **pure** if its annihilator L_ρ is a lagrangian subspace of $V\oplus V^*$.

We present some examples of pure spinors.

Example 2.25. *(Exponential of two-forms)* Let $B\in\wedge^2V^*$, the **exponential** of B is the spinor

$$e^B = \sum_j \frac{B^j}{j!},$$

where $B^j = B\wedge\cdots\wedge B$, j times. We note that the annihilator of e^B is the graph of B and thus e^B is a pure spinor.

Example 2.26. *(Annihilator)* Let $\theta_1,\dots,\theta_k\in V^*$ linearly independent; the spinor $\Omega = \theta_1\wedge\ldots\wedge\theta_k$ is pure and its annihilator is $L(E,0)$, where $E=\bigcap_j\ker\theta_j$.

We also have the following.

Proposition 2.27 ([12])**.** *Any lagrangian subspace of $V\oplus V^*$ is the annihilator of some pure spinor. This spinor is unique up to multiplication by scalars.*

A spinor associated to a lagrangian subspace is given in the following way: given a lagrangian L by Proposition 2.17, there exist $E\subseteq V$ and $\varepsilon\in\wedge^2E^*$ such that $L=L(E,\varepsilon)$. Consider a two-form B extending ε and pick out a generator Ω of $\det\operatorname{Ann}E$. The spinor $\rho = e^B\wedge\Omega$ is pure and actually its annihilator is L. Then we have the following.

Corollary 2.28 ([12])**.** *Let ρ be a pure spinor over a complex vector space V. Then there exists $c\in\mathbb{C}-\{0\}$, $B,\omega\in\wedge^2V^*$ and linearly independent $\theta_1,\dots,\theta_k\in V^*$ such that*

$$\rho = ce^{B+i\omega}\wedge\theta_1\wedge\ldots\wedge\theta_k.$$

The space of spinors decomposes as

$$S = S^{ev}\oplus S^{odd},$$

where $S^{ev} = \wedge^{ev}V^*$ and $S^{odd}=\wedge^{odd}V^*$. Let $\intercal : S\to S$ denote the anti-involution on S defined on decomposable form in the following way

$$(e_1\wedge\cdots\wedge e_k)^\intercal = e_k\wedge\cdots\wedge e_1,$$

where $e_j \in V^*$. Then we extend linearly for the rest of elements of S. Recall that if $\alpha \in \wedge^k V^*$, then

$$\alpha^{\intercal} = (-1)^{\frac{k(k-1)}{2}} \alpha. \tag{2.7}$$

One important property of the spinors is that they can detect some transversality properties of lagrangian subspaces of $V \oplus V^*$.

Definition 2.29 ([12])**.** The **Chevalley pairing** is defined as

$$(\cdot,\cdot)_0 : S \times S \to \det(V^*)$$

$$(\rho, \tau)_0 = (\rho^{\intercal} \wedge \tau)_{top},$$

where ρ_{top} denotes the homogeneous component of ρ with degree equal to the dimension of the vector space.

The Chevalley pairing satisfies the following properties:

Lemma 2.30. *If* $\dim V$ *is even then* $(\cdot,\cdot)_0$ *is zero when restricted to* $S^{ev} \times S^{odd}$ *and to* $S^{odd} \times S^{ev}$*. On the other hand, if* $\dim V$ *is odd then* $(\cdot,\cdot)_0$ *is zero when restricted to* $S^{ev} \times S^{ev}$ *and to* $S^{odd} \times S^{odd}$*.*

Lemma 2.31. *On an* $m-$*dimensional vector space* V*, the pairing* $(\cdot,\cdot)_0$ *satisfies*

$$(\rho, \tau)_0 = (-1)^{\frac{m(m-1)}{2}} (\tau, \rho)_0.$$

Lemma 2.32. *Let* ρ_1*,* ρ_2 *be two spinors. Then*

$$(u \cdot \rho_1, u \cdot \rho_2)_0 = (\rho_1, \rho_2)_0,$$

for all $u \in Spin_0(V \oplus V^*)$*, where* $Spin_0(V \oplus V^*)$ *is the identity component of* $Spin(V \oplus V^*) = \{v_1 \dots v_r \,|\, \langle v_i, v_i \rangle = \pm 1 \text{ and } r \text{ is even}\}$ *in the Clifford algebra of* $V \oplus V^*$ *with the canonical pairing.*

The main property of the Chevalley pairing is the following.

Proposition 2.33. *Let* ρ *and* τ *be pure spinors. Then* $L_\rho \cap L_\tau = \{0\}$ *if and only if* $(\rho, \tau)_0 \neq 0$*.*

Next we recall the spinorial description of a generalized complex structure on a vector space V.

Proposition 2.34. *Let* ρ *be a pure spinor on* $(V \oplus V^*)_{\mathbb{C}}$*, where* V *is a* $2n$*-dimensional real vector space and let* $B, \omega \in \wedge^2 V^*$*,* $\theta_1, \dots, \theta_k$ *such that* $\rho = c e^{B + i\omega} \wedge \theta_1 \wedge \dots \wedge \theta_k$*. Then,* L_ρ *is a generalized complex structure on* V *if and only if*

$$\omega^{n-k} \wedge \theta_1 \wedge \dots \wedge \theta_k \wedge \overline{\theta_1} \wedge \dots \wedge \overline{\theta_k} \neq 0.$$

2.3 Dirac structures on manifolds

2.3.1 Dirac manifolds

Now we review some properties of Dirac structures on manifolds.

General theory

Definition 2.35 ([13]). A **Dirac structure** is a lagrangian subbundle of $TM \oplus T^*M$, whose space of sections is closed under the Courant-Dorfman bracket.

Examples 2.36. We give some examples of Dirac structures

a) Presymplectic structures: let $\omega \in \Omega^2(M)$ be a closed two-form. Then

$$L_\omega = \{X + \iota_X \omega \mid X \in TM\}$$

is a Dirac structure.

b) Poisson structures: let π be a Poisson structure. Then

$$Graph(\pi) = \{\iota_\xi \pi + \xi \mid \xi \in T^*M\}$$

is a Dirac structure.

The **range distribution** is defined as in the linear case by

$$E = pr_{TM} L.$$

This is a smooth distribution which is not necessarily regular. By Lemma 2.7, Dirac structures are Lie algebroids. Then by Proposition 2.5, the range distribution is integrable. Given a Dirac structure L, we note that $L|_p$ is a Dirac structure on T_pM. Consequently we obtain, as in the linear case, a skew-symmetric bilinear map $\varepsilon_L : E \times E \to \mathbb{R}$. If we take a leaf S of the range distribution, the two-form $\varepsilon_L|_{TS \times TS}$ becomes a presymplectic two-form on S.

Proposition 2.37 ([13]). *The range distribution of a Dirac structure is integrable and each leaf inherits a presymplectic form.*

If we look back at the examples, we have that in the case of a Poisson structure we obtain a symplectic foliation and in the case of a presymplectic structure the foliation consists of the connected components of the manifold with the presymplectic structure itself. The leaves of the range distribution associated to a Dirac structure satisfy a parity property.

Proposition 2.38 ([20]). *Given a Dirac structure on a connected manifold, then the leaves of its presymplectic foliation are all even-dimensional or all odd-dimensional.*

Proposition 2.39 ([20]). *Let L be a Dirac structure and a point $p \in M$. If the presymplectic leaf passing through p is a single point, then on a neighbourhood of p, L is the graph of a Poisson structure.*

There is another distribution associated to a Dirac structure. The one given by

$$p \mapsto \ker L|_p = L|_p \cap T_pM$$

which is called the **null distribution**. It is not always smooth, although when the null distribution is of constant rank then it is smooth and integrable. Its associated foliation is called the **null foliation**. In the case of the graph of a Poisson structure, the null distribution is trivial. In the case of the graph of a presymplectic structure the null distribution is the kernel of the presymplectic structure. In fact, we have that if $E \subseteq TM$ and $\varepsilon \in \wedge^2 E^*$

$$L(E, \varepsilon) \cap TM = \ker \varepsilon.$$

We have the following characterization of Poisson and presymplectic structures in terms of its intersection with TM and T^*M.

Proposition 2.40. *Let L be a Dirac structure. Then*

1. *L is the graph of a Poisson structure if and only if $L \cap TM = 0$.*

2. *L is the graph of a presymplectic structure if and only if $L \cap T^*M = 0$.*

A **regular Dirac structure** is a Dirac structure whose range distribution is regular. Regular Dirac structures are of the form $L(E, \omega)$ where E is a regular distribution of TM and $\omega \in \wedge^2 E^*$. Actually we have the following.

Proposition 2.41 ([23])**.** *Let L be a lagrangian subbundle of $TM \oplus T^*M$. Assume that there exists a regular distribution E on TM and $\varepsilon \in \wedge^2 E^*$ such that $L = L(E, \varepsilon)$. Then L is a Dirac structure if and only if E is involutive and $d_E \varepsilon = 0$, where d_E is the differential along the directions of E.*

Backward and forward images

Next we study the backward and forward image of Dirac structures. Consider a map $\varphi : M \to N$ and the Dirac structures L_M and L_N over M and N respectively. The backward and forward image of these Dirac structures are defined as follows

$$\mathcal{B}_\varphi(L_N)|_x = \{X + \varphi^*\xi \mid \varphi_* X + \xi \in L_N|_{\varphi(x)}\} \subseteq T_xM \oplus T_x^*M,$$

$$\mathcal{F}_\varphi(L_M)|_x = \{\varphi_* X + \xi \mid X + \varphi^*\xi \in L_M|_x\} \subseteq T_{\varphi(x)}N \oplus T^*_{\varphi(x)}N,$$

where $x \in M$. Note that $\mathcal{B}_\varphi(L_N) \subseteq (TM \oplus T^*M)$ and $\mathcal{F}_\varphi(L_M) \subseteq \varphi^*(TN \oplus T^*N)$ are pointwise lagrangian but they are not necessarily smooth vector bundles. In order to assure smoothness we need to impose some conditions on the map φ and the Dirac structures.

Proposition 2.42 ([9])**.** *Suppose $\varphi : M \to N$ is a smooth map and L_N is a lagrangian subbundle of $TN \oplus T^*N$. If $\ker((d\varphi)^*) \cap \varphi^* L_N$ has constant rank, then $\mathcal{B}_\varphi(L_N)$ is a lagrangian subbundle. If $\mathcal{B}_\varphi(L_N)$ is smooth and L_N is integrable, then $\mathcal{B}_\varphi(L_N)$ is a Dirac structure.*

The case of the forward image $\mathcal{F}_\varphi(L_M)$ is more delicate, since it does not define a Dirac structure on the whole manifold N. For this reason we need an invariance condition on the Dirac structure L_M with respect to the map φ. We say that the Dirac structure L_M is **φ-invariant** if

$$\mathcal{F}_\varphi(L_M)|_x = \mathcal{F}_\varphi(L_M)|_{x'}$$

for every $x, x' \in M$ such that $\varphi(x) = \varphi(x')$.

Proposition 2.43 ([9])**.** *Suppose $\varphi : M \to N$ is a surjective submersion and L_M is a lagrangian subbundle of $TM \oplus T^*M$. If $\ker(d\varphi) \cap L_M$ has constant rank, then $\mathcal{F}_\varphi(L_M)$ is a lagrangian subbundle of $\varphi^*(TN \oplus T^*N)$. If $\mathcal{F}_\varphi(L_M)$ is smooth and L_M is φ-invariant, then $\mathcal{F}_\varphi(L_M)$ defines a lagrangian subbundle of $TN \oplus T^*N$, which is integrable in case L_M is integrable.*

The property of $\ker((d\varphi)^*) \cap \varphi^* L_N$ or $\ker(d\varphi) \cap L_M$ having constant rank is usually referred to as **clean intersection**.

The following well-known result tells us that the forward image and the null distribution play a role in the reduction of a Dirac structure to a Poisson structure.

Proposition 2.44 ([13])**.** *Consider a Dirac structure L with simple null foliation and leaf space B realized by the submersion $\varphi : M \to B$. Then $\mathcal{F}_\varphi(L)$ defines a Poisson structure on B.*

The proof is based on the following lemma:

Lemma 2.45 ([9]). *If $\varphi : M \to N$ is a surjective submersion whose fibres are connected and L is a lagrangian subbundle of $TM \oplus T^*M$ such that $\ker d\varphi \subseteq L \cap TM$, then L is φ-invariant.*

Proof of Proposition 2.44. We note that $\ker d\varphi = L \cap TM$. Since the fibres of φ are leaf of $L \cap TM$, they are connected. Consequently, L is φ-invariant and applying the clean intersection property we obtain that $\mathcal{F}_\varphi(L)$ is a Dirac structure over B. Note also that $\mathcal{F}_\varphi(L) \cap TB = 0$ and hence L is the graph of a Poisson structure. □

For this reason Dirac structures are regarded as pre-Poisson structures.

Preliminaries on complex Dirac structures

Until now we only saw Dirac structure in the real setting, now we present its complex counterpart: the complex Dirac structures.

Definition 2.46. A **complex Dirac structure** is an involutive lagrangian subbundle of $(TM \oplus T^*M)_\mathbb{C}$.

We present one example of a complex Dirac structure. In Sections 2.5 and 3.2 we will give more examples.

Examples 2.47. Consider $\omega \in \Omega^2_{cl}(M)$. Then $L_{i\omega} = L(TM_\mathbb{C}, i\omega)$ is a complex Dirac structure.

Complex Dirac structures do not satisfy the same properties of Dirac structures, to start with the range of a complex Dirac structure is not a real distribution and so we cannot obtain a foliation from the range distribution. But other properties are satisfied with slight modifications.

The definitions of backward and forward images remain the same. We next see the conditions for the backward image being smooth. Let $\varphi : M \to N$ be a smooth map. Its differential defines the bundle map $d\varphi : TM \to \varphi^*TN$. Let $d\varphi_\mathbb{C}$ denote the complexification of this bundle map.

Proposition 2.48. *Suppose $\varphi : M \to N$ is a smooth map and L_N is a lagrangian subbundle of $(TN \oplus T^*N)_\mathbb{C}$. If $\ker((d\varphi_\mathbb{C})^*) \cap \varphi^* L_N$ has constant rank, then $\mathcal{B}_\varphi(L_N)$ is a lagrangian subbundle of $(TM \oplus T^*M)_\mathbb{C}$. If $\mathcal{B}_\varphi(L_N)$ is smooth and L_N is integrable, then $\mathcal{B}_\varphi(L_N)$ is a complex Dirac structure.*

The proof is identical to the Dirac case. When the distribution $\ker((d\varphi_\mathbb{C})^*) \cap \varphi^* L_N$ has constant dimension, we say that the map φ and the complex Dirac structure satisfy the clean intersection property.

Example 2.49. Let $C \overset{\iota}{\hookrightarrow} M$ be a submanifold and L a complex Dirac structure over M. The clean intersection property is equivalent to $\mathrm{Ann}(TC_\mathbb{C}) \cap L|_C$ having constant rank.

Definition 2.50. Let C be a submanifold of M and L a complex Dirac structure. We say that C is **transversal** to L if

$$TC_\mathbb{C} + pr_{TM_\mathbb{C}} L|_C = (TM_\mathbb{C})|_C.$$

Remark 2.51. We note that if C is transversal to the complex Dirac structure L, then L satisfies the clean intersection property with respect to the inclusion ι and so $\mathcal{B}_\iota(L)$ is a complex Dirac structure over C.

In the next chapter we will develop the general theory of complex Dirac structures.

2.3.2 Spinors on manifolds

Let M be a manifold, consider the bundle S

$$S = \wedge^{\bullet}(T^*M)_{\mathbb{C}}.$$

The sections of S are called spinors. As in Section 2.2.2, there is an action of $(TM \oplus T^*M)_{\mathbb{C}}$ on S, given by

$$(X + \xi) \cdot \rho = \iota_X \rho + \xi \wedge \rho,$$

where $X \in TM_{\mathbb{C}}$, $\xi \in TM^*_{\mathbb{C}}$ and $\rho \in S$.

The annihilator of a spinor and pure spinors are defined as in Section 2.2.2. The annihilator of a pure spinor is, by definition a lagrangian subbundle of $(TM \oplus T^*M)_{\mathbb{C}}$. If a lagrangian L is the annihilator of a pure spinor ρ, we say that ρ is the spinor associated to L; note that if L has associated a spinor ρ then ρ multiplied by any nowhere vanishing function is also a spinor associated to L, consequently, is unique up to multiplication by scalars. It usually happens that a lagrangian subbundle does not necessarily have associated a pure spinor. However, around any point of M, there is neighbourhood U such that $L|_U$ has associated a pure spinor ρ_U and so the trivial line subbundle of S having as generator ρ_U. Consequently, there is a cover of open subsets of M and trivial lines subbundles of S on each open of this cover. Gluing these trivial lines subbundle we obtain a line subbundle of S usually called the **pure spinor line bundle**. Thus the object associated to a lagrangian subbundle is not a pure spinor but a line subbundle of S.

All the other results and examples of Section 2.2.2 apply to the spinors on M, including the results related to the Chevalley pairing. The involutivity of a lagrangian subbundle is represented in term of spinors in the following way.

Proposition 2.52 ([23]). *Let L be a lagrangian subbundle of $(TM \oplus T^*M)_{\mathbb{C}}$. Then L is involutive if and only if for any local trivialization ρ of its associated spinor line bundle, there exist a local section $X + \xi$ of $(TM \oplus T^*M)_{\mathbb{C}}$ such that*

$$d\rho = (X + \xi) \cdot \rho.$$

2.4 Generalized complex structures

In this section we recall the basic properties of generalized complex structures. We can mimic the definition of an almost complex structure for any vector bundle $p : E \to M$ by just asking for maps $J : E \to E$ covering the identity map on M such that $J^2 = -Id$. In order to define the integrability condition for these almost complex structures, we need a bracket on sections of E. Courant algebroids, in particular $TM \oplus T^*M$, have this feature. Most of the material presented here comes from [22].

Definition 2.53. A **generalized almost complex structure** is a bundle map

$$\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M,$$

$$\mathcal{J} = \begin{pmatrix} A & \pi \\ \omega & -A^* \end{pmatrix}$$

such that $\mathcal{J}^2 = -1$ and $\mathcal{J}^* = -\mathcal{J}$. A generalized almost complex structure which satisfies that $N_{\mathcal{J}} = 0$, where

$$N_{\mathcal{J}} = [\mathcal{J}X, \mathcal{J}Y] - [X, Y] - \mathcal{J}([\mathcal{J}X, Y] + [X, \mathcal{J}Y])$$

is the Nijenhuis tensor associated to the Courant-Dorfman bracket, is called a **generalized complex structure**.

In the classical setting, there are many alternative ways to define a complex structure. We can define it as a maximal rank involutive complex distribution of $TM_{\mathbb{C}}$ which is transversal to its conjugate. Also as a local holomorphic volume form. In the generalized setting we obtain a similar description.

Proposition 2.54. *The following are equivalent:*

a) A generalized complex structure.

b) A complex Dirac structure such that $L \cap \overline{L} = 0$.

c) A line subbundle K *of* $\wedge^\bullet T^*M_{\mathbb{C}}$ *satisfying the following*

  i) If $\rho_p \in K|_p - \{0\}$, *then* ρ_p *is a pure spinor.*
  ii) If $\rho_p \in K|_p - \{0\}$, *then* $(\rho_p, \overline{\rho_p})_0 \neq 0$.
  iii) For any local trivialization ρ *of* K, *there exists* $X + \xi \in \Gamma(TM \oplus T^*M)_{\mathbb{C}}$ *such that* $(X + \xi) \cdot \rho = d\rho$.

There are some obstructions for the existence of generalized almost complex structures.

Proposition 2.55. *A manifold admits an generalized almost complex structure if and only if it admits an almost complex structure.*

As an immediate consequence we get the following:

Corollary 2.56. *If a manifold admits an generalized almost complex structure, then it has even dimension.*

After this corollary we can ask what kind of structure an odd-dimensional manifold could admit. We see in the following chapter that we need to weaken the condition of real index zero on complex Dirac structures.

Examples 2.57. We present some basic examples:

a) Symplectic structures: let ω be a symplectic structure on a $2n$-dimensional manifold M. Then

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

is a generalized complex structure with associated subbundle

$$L_{i\omega} = \{X - i\iota_X\omega \mid X \in TM_{\mathbb{C}}\}$$

and spinor $\rho = e^{i\omega}$.

b) Complex structures: let J be a complex structure on M^{2n}. Then

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$$

is a generalized complex structure with associated subbundle

$$L_J = T_{0,1} \oplus T^*_{1,0}$$

and spinor $\rho = \Omega^{n,0}$ a holomorphic volume form which is defined locally.

c) Holomorphic Poisson structures ([26]): let J be a complex structure over M and π be a holomorphic Poisson structure with respect to J, i.e. $\pi \in \wedge^2 TM_{\mathbb{C}}$ such that $\pi = P + iQ$, where $P, Q \in \wedge^2 TM$, P is a Poisson structure, $P = QJ^*$, $JQ = QJ^*$ and

$$[\alpha, \beta]_{QJ^*} = [J^*\alpha, \beta]_Q + [\alpha, J^*\beta]_Q - J^*[\alpha, \beta]_Q,$$

where $\alpha, \beta \in \Gamma(T^*M)$ and $[\cdot, \cdot]_Q$ and $[\cdot, \cdot]_{QJ^*}$ are the brackets associated to Q and QJ^* as in equation (2.1). Consider the bundle map given by

$$\mathcal{J} = \begin{pmatrix} -J & P \\ 0 & J^* \end{pmatrix}.$$

Then $\mathcal{J}$ defines a generalized complex structure.

Each generalized complex structure L with associated bundle map $\mathcal{J}$ has associated the following distributions

$$E = pr_{TM_{\mathbb{C}}} L \subseteq TM_{\mathbb{C}}, \qquad \Delta = \operatorname{Re} E \cap \overline{E} \subseteq TM$$

and the pointwise defined two-form

$$\omega_\Delta = \operatorname{Im} \varepsilon|_\Delta,$$

where Im means the imaginary part of the two-form. We have the following:

Proposition 2.58. *The distribution (Δ, ω_Δ) is integrable and every leaf inherits a symplectic structure.*

In this case the symplectic distribution (Δ, ω_Δ) corresponds to the symplectic distribution associated to the Poisson structure $\pi = pr_{TM}\mathcal{J}|_{T^*M}$:

Proposition 2.59 ([23])**.** *The bivector π is a Poisson structure and its associated symplectic foliation is given by (Δ, ω_Δ).*

Now we present the principal invariant of generalized complex structures.

Definition 2.60. The **type** of a lagrangian subbundle L of $(TM \oplus T^*M)_{\mathbb{C}}$ at a point $p \in M$ is defined as $\operatorname{codim}_{\mathbb{C}}(pr_{TM_{\mathbb{C}}} L|_p)$. If L is a generalized complex structure over a $2n$-dimensional manifold, then its type varies from 0 to n.

From the examples we note that symplectic structures have type 0, complex structures type n and it can be proved that holomorphic Poisson structures has a type that could vary in-between 0 and n. Moreover we see that the extreme types are well identified.

Proposition 2.61. *Let L be a generalized complex structures over a $2n$-dimensional manifold. If L has constant type 0, then L is a $B-$transformation of a symplectic structure. If L has constant type n, then $L = e^B L_J$, where L_J is the generalized complex structure associated to a complex structure J on M and $B \in \Omega^2(M, \mathbb{C})$ is a ∂-closed $(2,0)$-form.*

Consider generalized complex structures L_1 and L_2 over the manifolds M_1 and M_2 respectively. Let $\pi_i : M_1 \times M_2 \to M_i$ denote the canonical projections of the product manifold.

Proposition 2.62. *The bundle $L = \pi_1^* L_1 \oplus \pi_2^* L_2^*$ is a generalized complex structure over $M_1 \times M_2$.*

We call L the **product of generalized complex structures**. If the spinors associated to the generalized complex structures L_1 and L_2 are ρ_1 and ρ_2 respectively, then the spinor associated to the product of generalized structures is $\pi_1^*\rho_1 \wedge \pi_2^*\rho_2$.

Abouzaid and Boyarshenko proved a Weinstein splitting-like theorem for generalized complex structures.

Theorem 2.63 ([1]). *Let L be a generalized complex structure over M and let $p \in M$. Then, there exists a neighborhood U, a closed two-form B, a symplectic structure ω and a generalized complex structure L' such that*

$$L|_U \cong e^B(L' \times L_{i\omega}).$$

Moreover, the Poisson structure π' associated to L' vanishes at p.

The previous theorem is more accurate for generalized complex structures having a regular presymplectic distribution or equivalently having constant type.

Definition 2.64. Let L be a generalized complex structure. We say that a point p in M is **regular** if there exist a neighborhood of p where the type is constant.

We next recall a more precise splitting theorem for generalized complex structures around regular points which tells us what happens in-between the extreme types and also shows the importance of the type in the local geometry of generalized complex structures.

Theorem 2.65. *(Darboux theorem for regular generalized complex structures, [22]) Around any regular point of type k of a generalized complex structure L, we can find a neighbourhood U of the point such that $L|_U$ is equivalent via diffeomorphism and B-transformation to the product of the generalized complex structure associated to the canonical complex structure of $\mathbb{C}^k$ and the generalized complex structure associated to the canonical symplectic structure of $\mathbb{R}^{2(n-k)}$.*

2.5 Submanifolds of generalized complex structures

Now we study submanifolds of generalized complex structures from the point of view of pull-backs of complex Dirac structures as suggested in [5,36]. This point of view is different from the proposed originally in [22].

Consider a manifold M, a submanifold $N \overset{\iota}{\hookrightarrow} M$ and a generalized complex structure L on M. The complex Dirac structure $\mathcal{B}_\iota(L)$ is no longer a generalized complex structure, as we can see in the following examples.

Examples 2.66. Let $N \overset{\iota}{\hookrightarrow} M$ be a submanifold.

a) Let $\omega \in \Omega^2(M)$ be a symplectic structure and $L_{i\omega}$ the generalized complex structure given by the graph of $i\omega_{\mathbb{C}}$. Note that $\mathcal{B}_\iota(L_{i\omega})$ is the graph of the presymplectic structure $i\iota^*\omega_{\mathbb{C}}$. Moreover, $\mathcal{B}_\iota(L_{i\omega}) \cap \overline{\mathcal{B}_\iota(L_{i\omega})} = (\ker \iota^*\omega)_{\mathbb{C}}$ and then $L_{i\iota^*\omega} = \mathcal{B}_\iota(L_{i\omega})$ has as real index the dimension of $\ker \iota^*\omega$. Consequently, $L_{i\omega}$ is not a generalized complex structure.

b) Let J be an almost complex structure on M, let L_J denote its associated generalized almost complex structure and assume that N has codimension-one on M. Consider $D = TN \cap J(TN)$, since $J(D) = D$, we have that $(J|_D)^2 = -Id$, i.e. (D, J) is an almost CR structure of corank one in N. Let $L = L(\ker((J|_D)_{\mathbb{C}} - iId), 0)$; then we have that $\mathcal{B}_\iota(L_J) = L$. We will see in Example 3.16 that $L \cap \overline{L} = (\operatorname{Ann} D)_{\mathbb{C}}$. Therefore, L has real index one and is not a generalized complex structure.

Actually there is a characterization for $\mathcal{B}_\iota(L)$ to be a generalized complex structure on N. But first we need the following definition.

Definition 2.67. Let M be a manifold with a Poisson structure π and consider a submanifold $N \overset{\iota}{\hookrightarrow} M$. We say that N is a **Dirac-Poisson submanifold** of (M,π) if $\mathcal{B}_\iota(L_\pi)$ is the graph of a Poisson structure on N.

Proposition 2.68 ([36])**.** *Let $N \overset{\iota}{\hookrightarrow} M$ be a submanifold of M and let L be a generalized complex structure on M with associated bundle map*

$$\mathcal{J} = \begin{pmatrix} A & \pi \\ \sigma & -A^* \end{pmatrix}.$$

Then $\mathcal{B}_\iota(L)$ is a generalized complex structure on N if and only if

i) N is a Dirac-Poisson submanifold of (M,π).

ii) $A(TN) \subseteq TN + \pi(T^*M)|_N = TN \oplus \pi(\operatorname{Ann} TN)$.

iii) $pr_{TN} \circ A$ is differentiable, where pr_{TN} comes from the projection onto TN of the direct sum of ii).

The examples above point to the arising of lagrangian subbundles with nonzero real index when studying submanifolds of generalized complex structures. We begin our study of lagrangian subbundles with nontrivial real index by giving a bound for the real index we can obtain in submanifolds of generalized complex structures.

Lemma 2.69. *Let N be a codimension-r submanifold of M with $\iota : N \to M$ the inclusion map. Let L be a lagrangian subbundle of $(TM \oplus T^*M)_{\mathbb{C}}$ with real index zero. Then*

$$\dim(\mathcal{B}_\iota(L)|_p \cap \overline{\mathcal{B}_\iota(L)|_p}) \leq r,$$

for all $p \in N$. Furthermore, if $\mathcal{B}_\iota(L)$ is smooth and L is involutive, then $\mathcal{B}_\iota(L)$ is involutive too.

Proof. Suppose that $\dim \mathcal{B}_\iota(L)|_p \cap \overline{\mathcal{B}_\iota(L)|_p} > r$. As a consequence there exist linearly independent real elements

$$X_1 + \xi_1, \ldots, X_{r+1} + \xi_{r+1} \in \mathcal{B}_\iota(L)|_p,$$

where $X_j \in T_pN$ and $\xi_j \in T_p^*N$, for $k = 1, \ldots r+1$. By the definition of $\mathcal{B}_\iota(L)_p$ there exist

$$X_1 + \tau_1 + i\eta_1, \ldots, X_{r+1} + \tau_{r+1} + i\eta_{r+1} \in L|_p$$

such that $\tau_k, \eta_k \in T_p^*M$, $\tau_k|_{T_pN} = \xi_k$ and $\eta_k \in \operatorname{Ann} T_pN$, for $k = 1, \ldots r+1$. Since $\dim \operatorname{Ann} T_pN = r$, then there exist non all vanishing constants $c_1, \ldots, c_{r+1} \in \mathbb{C}$ such that $\sum_{j=1}^{r+1} c_j\eta_j = 0$. Consequently,

$$\sum_{j=1}^{r+1} c_j(X_j + \tau_j + i\eta_j) = \sum_{j=1}^{r+1} c_j(X_j + \tau_j)$$

is real in $L|_p$, yielding that $\sum_{j=1}^{r+1} c_j(X_j + \tau_j) = 0$ and thus

$$\sum_{j=1}^{r+1} c_j(X_j + \xi_j) = 0.$$

The last part of the lemma follows from Proposition 2.48. □

In submanifolds of codimension-one we control completely the real index.

Corollary 2.70. *Consider a codimension-one submanifold N of M with inclusion map ι. Then*

$$\operatorname{rank}(\mathcal{B}_\iota(L) \cap \overline{\mathcal{B}_\iota(L)}) = 1.$$

Proof. By the lemma above $\operatorname{rank}(\mathcal{B}_\iota(L) \cap \overline{\mathcal{B}_\iota(L)}) \leq 1$. Since $\dim N$ is odd, then N does not admit generalized almost complex structures and the corollary holds. □

2.6 Other geometrical structures

The purpose of this section is to recall some classical structures that will appear in the following chapters and to introduce a structure that will play an important role in the theory of complex Dirac structures, the transverse CR structures. A **cosymplectic structure** on a $2n+1$-dimensional manifold M is a pair $(\theta, \omega) \in \Omega^1(M) \times \Omega^2(M)$ such that $\omega^n \wedge \theta \neq 0$, $d\theta = 0$ and $d\omega = 0$.

Definition 2.71 ([18])**.** An almost **CR structure** (Cauchy-Riemann/Complex-Real) is a pair (D, J), where D is a regular distribution on M and $J : D \to D$ is a bundle map such that $J^2 = -Id$. A **CR structure** is an almost CR structure satisfying

$$[\Gamma(T_{1,0}), \Gamma(T_{1,0})] \subseteq \Gamma(T_{1,0}),$$

where $T_{1,0} = \ker(J_{\mathbb{C}} - iId) \subseteq TM_{\mathbb{C}}$. A CR structure is equivalently an involutive regular distribution $T_{1,0}$ of $TM_{\mathbb{C}}$ such that $T_{1,0} \cap \overline{T_{1,0}} = 0$.

CR structures appear naturally when studying real submanifolds of complex manifolds as we have seen in the previous section. The distribution D is not necessarily involutive and, actually, we can detect how far it is from being involutive with the following symmetric tensor.

Definition 2.72. Let (D, J) be CR structure and assume that there exists a $\eta \in \Omega^1(M)$ such that $D = \ker \eta$. The **Levi form** of the CR structure is

$$L(X, Y) = d\eta(X, JY).$$

Note that when $L = 0$, then the distribution D is involutive and so we obtain a foliation where each leaf carries a holomorphic structure.

The following definition plays an important role in the metric theory of CR structures.

Definition 2.73 ([18])**.** Let (D, J) be a CR structure and assume that there exists a contact form $\eta \in \Omega^1(M)$ such that $D = \ker \eta$. We say that it is **strictly pseudoconvex** if the Levi form is positive or negative definite.

Until now we just have treated the basic notions of CR geometry, next we introduce a definition that will play a key role in the following chapters.

Definition 2.74. A **transverse CR structure** is a triple (R, S, J) consisting of two regular distributions $R \subseteq S \subseteq TM$, where R is integrable and a bundle map $J : S/R \to S/R$ such that $J^2 = -Id$ and $q^{-1}(\ker(J_{\mathbb{C}} - iId))$ is involutive on $TM_{\mathbb{C}}$, where $q : S_{\mathbb{C}} \to (S/R)_{\mathbb{C}}$ is the quotient map.

Examples 2.75. We give some examples of transverse CR structures.

a) Let (D, J) be a CR structure. Then $(0, D, J)$ is a transverse CR structure.

b) If $S = TM$, then (R, TM, J) recovers the transverse holomorphic structures.

Lemma 2.76. *Consider the transverse CR structure* (R, S, J)*. Then, for any point* $p \in M$*, there exists a neighbourhood* U *of* p *such that* $U/\mathcal{F}_U$ *carries a CR structure, where* $\mathcal{F}_U$ *is the foliation associated to* R *restricted to* U*.*

Proof. Let U be a neighbourhood of p where $\mathcal{F}_U$ is simple. Let P be the leaf space associated to $\mathcal{F}_U$ and let $t : M \to P$ be a submersion such that its fibres are the leaves of $\mathcal{F}_U$; note that $\ker t_* = R$. Consider the distribution $H_{1,0} = q^{-1}(\ker(J_{\mathbb{C}} - iId))$. Note that the integrability of $H_{1,0}$ implies that $[\Gamma(H_{1,0}), \Gamma(R_{\mathbb{C}})] \subseteq \Gamma(H_{1,0})$. The last fact implies that $H_{1,0}$ descends to an involutive regular distribution $T_{1,0}$ on $TP_{\mathbb{C}}$. Note that $T_{1,0} \cap \overline{T_{1,0}} = 0$. Therefore, $T_{1,0}$ defines a CR structure on P. □

Remark 2.77. *There is in the literature an alternative definition of transverse CR structures, cf. [17] which appears naturally in the context of CR geometry. However, our definition arises naturally in the generalized context representing a whole family of structures (up to certain transformations) as we will see in the next chapter. In particular, Definition 2.74 generalizes both transverse holomorphic and CR structures.*

Chapter 3

Complex Dirac structures

In this chapter we focus on complex Dirac structures. We begin by studying its linear algebra and then we pass to study them on manifolds.

3.1 Complex Dirac structures on vector spaces

In Section 2.2 we studied Dirac structures on real and complex vector spaces. Now we focus on Dirac structures on $V_{\mathbb{C}}$ or equivalently on lagrangian subspaces of $V_{\mathbb{C}} \oplus V_{\mathbb{C}}^* \cong (V \oplus V^*)_{\mathbb{C}}$.

Definition 3.1. A complex Dirac structure on V is a lagrangian subspace of $(V \oplus V^*)_{\mathbb{C}}$.

Given that complex Dirac structures are special cases of Dirac structures, they satisfy all the properties mentioned on section 2.2. What we will see along this section is that actually they carry far more information than its real counterpart (Dirac structures on V). They have more associated distributions, more invariants and they carry a complex map.

3.1.1 Associated subspaces

We know that any complex Dirac structure on a vector space V has associated two natural complex subspaces of $V_{\mathbb{C}}$ called the range and the kernel. However we can also associate some real subspaces that differentiate them from the usual Dirac structures on V and will be useful in the future when working on real manifolds. So consider the complex Dirac structure $L \subseteq (V \oplus V^*)_{\mathbb{C}}$, we obtain naturally the following subspaces:

$$\begin{aligned} K &= \operatorname{Re}(L \cap \overline{L}) \subseteq V \oplus V^*, & E &= pr_{V_{\mathbb{C}}} L \subseteq V_{\mathbb{C}}, \\ \Delta &= \operatorname{Re}(E \cap \overline{E}) \subseteq V, & D &= \operatorname{Re}(E + \overline{E}) \subseteq V, \end{aligned} \tag{3.1}$$

note that E is the range and the other three subspaces are real and exclusively defined from a complex Dirac structure. Note that K is the real part of L and so

$$K = \{X + \xi \in L \mid X + \xi \text{ is real}\}.$$

We observe that K is isotropic since

$$K^{\perp} = \operatorname{Re}(L \cap \overline{L})^{\perp} = \operatorname{Re}(L + \overline{L}), \tag{3.2}$$

and we also have $D = pr_V K^{\perp}$. There exists a complex two-form $\varepsilon \in \wedge^2 E^*$ such that $L = L(E, \varepsilon)$. Now consider the real two-form

$$\omega_{\Delta} := \operatorname{Im}(\varepsilon)|_{\Delta} \in \wedge^2 \Delta^*,$$

where $\text{Im}(\varepsilon)$ means the imaginary part of ε. So we have a natural Dirac structure defined from L,

$$L_\Delta = L(\Delta, \omega_\Delta).$$

Remark 3.2. *Let $E \subseteq V_{\mathbb{C}}$ and $\varepsilon \in \wedge^2 E^*$, denote by $E_{\mathbb{R}}$ the space E considered as a real vector space. Then $\varepsilon = \varepsilon_1 + i\varepsilon_2$, where $\varepsilon_1, \varepsilon_2 \wedge^2 E_{\mathbb{R}}^*$. We denote ε_2 by $\text{Im}(\varepsilon)$. It is easily seen that given any extension $B + i\omega \in \wedge^2 V_{\mathbb{C}}^*$ of ε, we have that $\omega|_\Delta = \text{Im}(\varepsilon)|_\Delta$.*

We have the following relationship between K and ω_Δ.

Lemma 3.3. *For any linear complex Dirac structure L we have that*

$$pr_V K = \ker \omega_\Delta.$$

Proof. Let $E \subseteq V_{\mathbb{C}}$ and $B + i\omega \in \wedge^2 V^*$ such that $L = L(E, \iota^*(B + i\omega))$, where ι is the inclusion map of E into $V_{\mathbb{C}}$. First, we note that

$$\overline{L} = L(\overline{E}, \iota^*(B - i\omega)). \tag{3.3}$$

Given any $X \in \ker \omega_\Delta \subseteq \Delta$, we will construct a $\xi \in V_{\mathbb{C}}^*$ such that $X + \xi \in L \cap \overline{L}$. Let U be a complement to $(E + \overline{E})$ in $V_{\mathbb{C}}$, note that U could be the trivial subspace. Consider $\tau \in V_{\mathbb{C}}^*$ such that $\tau(Y) = \iota_X \omega_{\mathbb{C}}$, whenever $Y \in E$, and $\tau(Y) = -\iota_X \omega_{\mathbb{C}}$, whenever $Y \in \overline{E} \oplus U$. Note that τ is well defined because $X \in \ker \omega_\Delta$ and τ in general is not real. Now consider $\xi = \iota_X B + i\tau$; we see that $X + \xi \in L$ since

$$\xi|_E = \iota_X \iota^* B + i\iota^* \tau = \iota_X(\iota^*(B + i\omega)).$$

Using equation (3.3), we obtain that $X + \xi \in L \cap \overline{L}$ in a similar way as above. Let $\tau_1, \tau_2 \in V^*$ such that $\tau = \tau_1 + i\tau_2$. Then

$$\frac{1}{2}((X + \xi) + (X + \overline{\xi})) = X + \iota_X B - \tau_2 \in K,$$

and one inclusion follows.

Conversely, consider $X + \xi \in K$, i.e. $X + \xi$ is a real element of L. Then $\iota_X(\iota^*(B + i\omega)) = \xi|_E$, comparing the real and imaginary parts we obtain that $X \in \ker \omega_\Delta$. □

Denote by Δ_0 the subspace $\ker \omega_\Delta$. Note that we have the following inclusions

$$\Delta_0 \subseteq \Delta \subseteq D.$$

Given a complex Dirac structure, we note that D just depends on K and by Lemma 3.3, Δ_0 too. So from now on we associate to any isotropic subspace of $V \oplus V^*$ the real subspaces $D = pr_V K^\perp$ and $\Delta_0 = pr_V K$, independently of any lagrangian subspace K.

3.1.2 Generalized complex viewpoint

In this section we will see that there exists a correspondence between complex Dirac structures on a vector space and generalized complex structures on a possibly different vector space. For that purpose we need to describe the geometry associated to the isotropic subspaces of $V \oplus V^*$.

So consider an isotropic subspace K of $V \oplus V^*$. The vector space $K^\perp / K$ naturally inherits a pairing coming from $V \oplus V^*$

$$\langle e_1 + K, e_2 + K \rangle_K = \langle e_1, e_2 \rangle,$$

where $e_1, e_2 \in K^\perp$, it is easy to see that the pairing $\langle \cdot, \cdot \rangle_K$ is nondegenerate. The following lemma will be useful later.

Lemma 3.4. *Let L be a lagrangian subspace of $V \oplus V^*$, where V is a complex or real vector space. If K is an isotropic subspace of $V \oplus V^*$, then $S_K = L \cap K^\perp + K$ is a lagrangian subspace of $V \oplus V^*$ and $L_K = (L \cap K^\perp + K)/K$ is a lagrangian subspace of $K^\perp/K$.*

Proof. First we note that $S_K \subseteq L + K$ and since K is isotropic $S_K \subseteq K^\perp$. Then $S_K \subseteq (L + K) \cap K^\perp = S_K^\perp$. Computing the dimensions

$$\begin{aligned}\dim S_K &= \dim(L \cap K^\perp) + \dim K - \dim(L \cap K)\\ &= \dim L + \dim K^\perp - \dim(L + K^\perp) + \dim K - (\dim L + \dim K - \dim(L + K))\\ &= \dim K^\perp - \dim(L + K^\perp) + \dim(L + K)\\ &= \dim S_K^\perp,\end{aligned}$$

thus we obtain that $S_K = S_K^\perp$. The result for L_K follows from the previous one. □

The vector space $K^\perp/K$ has many similarities with $V \oplus V^*$, actually we have the following proposition which also appears in [19].

Proposition 3.5. *Let K be an isotropic subspace of $V \oplus V^*$, where $\dim V = m$ and $\dim K = r$. Then there exist a lagrangian subspace W of $K^\perp/K$ such that*

$$(K^\perp/K, \langle\cdot,\cdot\rangle_K) \cong (W \oplus W^*, \langle\cdot,\cdot\rangle_{can}),$$

where $\langle\cdot,\cdot\rangle_{can}$ is the canonical pairing of $W \oplus W^*$. Consequently, the pairing $\langle\cdot,\cdot\rangle_K$ has signature $(m - r, m - r)$.

Proof. First we need a lagrangian subspace S of $V \oplus V^*$ such that $K \subseteq S \subseteq K^\perp$. For that it is enough to take any lagrangian subspace L and set $S = L \cap K^\perp + K$; note that S satisfies the desired properties. If $S = L(E, \varepsilon)$, for some $E \subseteq V$ and $\varepsilon \in \wedge^2 E^*$, then the lagrangian subspace $T = L(E', 0)$, where $E \cap E' = 0$, is a complement of S.

So we have that $S \oplus T = V \oplus V^*$, we next see that there exists a subspace $T_0 \subseteq T$ such that $S \oplus T_0 = K^\perp$ and $T_0 \cong (S/K)^*$. Since the pairing is nondegenerate, we have that $T \cong S^*$ via the isomorphism given by the pairing itself; let denote this isomorphism by $\Phi : S^* \to T$. Since $K \subseteq S$, we consider $\operatorname{Ann} K \subseteq S^*$, the annihilator of K in S. Consider $T_0 = \Phi(\operatorname{Ann} K)$; note that $T_0 \cong (S/K)^*$. We can see that by construction $T_0 \subseteq K^\perp$, implying that $S \oplus T_0 \subseteq K^\perp$ and by counting dimension we obtain the equality. As a result $K^\perp/K \cong S/K \oplus T_0 \cong S/K \oplus (S/K)^*$.

Let denote by $q : K^\perp \to K^\perp/K$ the quotient map; then $q(S)$ and $q(T_0)$ are complementary isotropic subspaces of $K^\perp/K$ with dimension $m - r$, i.e. lagrangian subspaces. Therefore, the pairing in $K^\perp/K$ has signature $(m - r, m - r)$. □

As we saw in the proof of Proposition 3.5, the isomorphism of $K^\perp/K$ with $W \oplus W^*$ is not canonical, it depends on the choice of a lagrangian subspace $K \subseteq S \subseteq K^\perp$. Instead we obtain a canonical short exact sequence for $K^\perp/K$, similar to the usual exact sequence associated to $W \oplus W^*$. Consider the subspaces

$$D = pr_{TM} K^\perp, \qquad \Delta_0 = pr_{TM} K,$$

and the map

$$\begin{aligned}p_K : K^\perp/K &\to D/\Delta_0\\ X + \xi + K &\mapsto X + \Delta_0.\end{aligned}$$

Proposition 3.6 ([19]). *The vector space $K^\perp/K$ fits into the following short exact sequence:*

$$0 \longrightarrow (D/\Delta_0)^* \xrightarrow{p_K^*} K^\perp/K \xrightarrow{p_K} D/\Delta_0 \longrightarrow 0\,.$$

Any generalized complex structure on V is a complex Dirac structure on V. We have a kind of converse for this statement.

Proposition 3.7. *Let $K \subseteq V \oplus V^*$ be an isotropic subspace. There exist a one-to-one correspondence between complex Dirac structures of $(V \oplus V^*)_{\mathbb{C}}$ with associated isotropic $K \subseteq V \oplus V^*$ and linear maps $\mathcal{J} : K^\perp/K \to K^\perp/K$ such that $\mathcal{J}^2 = -1$ and $\mathcal{J}^* + \mathcal{J} = 0$.*

Proof. Let L be a complex Dirac structure with associated isotropic K. We proceed to construct the map $\mathcal{J}$. We can see that $L + \overline{L} = K^\perp \otimes \mathbb{C}$. Consider $L_0 = q(L)$, where $q : K^\perp_{\mathbb{C}} \to (K^\perp/K)_{\mathbb{C}}$ denote the quotient map. We see that $L_0 \cap \overline{L_0} = \{0\}$ and then $L_0 \oplus \overline{L_0} = (K^\perp/K)_{\mathbb{C}}$. Let $\mathcal{J}'$ be the linear map defined as $\mathcal{J}'|_{L_0} = iId_{L_0}$ and $\mathcal{J}'|_{\overline{L_0}} = -iId_{\overline{L_0}}$. Constructed in this way, $\mathcal{J}'$ is a real map and its real component $\mathcal{J}$ is the desired map.

Conversely, we construct L from $\mathcal{J}$. Assume that K and $\mathcal{J}$ are as in the statement of the theorem and $t : (V \oplus V^*)_{\mathbb{C}} \to (V \oplus V^*/K)_{\mathbb{C}}$ is the quotient map. Then $L = t^{-1}(\ker(\mathcal{J}_{\mathbb{C}} - iId))$ is a complex Dirac structure such that $L \cap \overline{L} = K_{\mathbb{C}}$. □

As a consequence of this proposition we get a first obstruction to the existence of a complex Dirac structure:

Corollary 3.8. *Let V be an $m-$dimensional vector space. If V admits a complex Dirac structure with real index r, then there exist $n \in \mathbb{N}$ such that $m = 2n + r$.*

Proof. By Proposition 3.5,

$$K^\perp/K = S/K \oplus T_0/K \cong S/K \oplus (S/K)^*,$$

with S a maximal isotropic containing K. According to Proposition 3.7 we have a map $\mathcal{J}$: $S/K \oplus (S/K)^* \to S/K \oplus (S/K)^*$ such that $\mathcal{J}^2 = -1$ and $\mathcal{J}^* + \mathcal{J} = 0$. Consequently, by Corollary 2.56, S/K is even-dimensional and thus $\dim V \equiv r \ (mod\ 2)$. □

3.1.3 Order and type

In this section we introduce a new invariant associated to complex Dirac structures, the order. We next see how this invariant identifies when a lagrangian has a certain real index. We also give a new definition of the type that together with the real index and the order characterize complex Dirac structures up to B-transformations as we will see in the next section. We recall the following proposition.

Proposition 3.9 ([22]). *The lagrangian subspace $L(E, \varepsilon)$ of $(V \oplus V^*)_{\mathbb{C}}$ has real index zero if and only if $E + \overline{E} = V_{\mathbb{C}}$ and ω_Δ is nondegenerate.*

We generalize the previous proposition to arbitrary real index but first we need a lemma.

Lemma 3.10. *Consider a subspace S of $V \oplus V^*$, then the following holds*

$$\mathrm{Ann}(pr_V S^\perp) = S \cap V^* \ \ and \ \ \mathrm{Ann}(pr_{V^*} S^\perp) = S \cap V.$$

Proof. Note that $\mathrm{Ann}(pr_V S^\perp) = S \cap V^*$, since $\xi \in \mathrm{Ann}(pr_V S^\perp)$ if and only if $\langle \xi, e\rangle = 0, \forall e \in S^\perp$, i.e., $\xi \in S \cap V^*$. $\square$

Proposition 3.11. *Let $L(E, \varepsilon)$ be a complex Dirac structure of V. If L has real index r then,*

$$\mathrm{codim}(E + \overline{E}) + \dim \ker \omega_\Delta = r.$$

Proof. Let K be the isotropic subspace associated to L. Note that K fits in the following short exact sequence

$$0 \longrightarrow K \cap V^* \xrightarrow{\iota} K \xrightarrow{pr_V} pr_V K \longrightarrow 0. \tag{3.4}$$

As a consequence we have that

$$\dim K = \dim pr_V K + \dim K \cap V^*.$$

By Lemma 3.10, we have that $K \cap V^* = \mathrm{Ann}\, D$, by Lemma 3.3, $pr_V K = \ker \omega_\Delta$ and then we have

$$r = \dim K = \dim \ker \omega_\Delta + \dim \mathrm{Ann}\, D = \dim \ker \omega_\Delta + \mathrm{codim}(E + \overline{E}).$$

$\square$

In particular this proposition retrieves Proposition 3.9. It also motivates the following definition.

Definition 3.12. Let L be a linear complex Dirac structure. The **order** of L is defined as

$$\mathrm{order}(L) = \mathrm{codim}\, D.$$

By Proposition 3.11, the order is always less than or equal to the real index of the linear complex Dirac structure. In particular, in a generalized complex structure the order is always zero. Moreover, note that the order depends exclusively on K as $\mathrm{order}(L) = \mathrm{codim}\, pr_V K^\perp$.

The other invariant we introduce is a redefinition of the type, presented in Definition 2.60, more appropriate to the study complex Dirac structures:

Definition 3.13. The **type** of a complex Dirac structure L is

$$\mathrm{type}(L) = \dim(E + \overline{E}) - \dim E.$$

Lemma 3.14. *Let L be a complex Dirac structure on V with real index r and assume that $\dim V = 2n + r$ as in Proposition 3.8; then the order, the real index and the type satisfy the following:*

a) $\mathrm{order}(L) + \mathrm{type}(L) = \mathrm{codim}\, E$.

b) The type is always between 0 *and* n.

c) If L has order s and type k, then $\dim \Delta = 2(n - k) + r - s$.

Proof. a)It is straightforward.

b)Counting dimensions, we have that:

$$2(\dim(E + \overline{E}) - \dim E) = \dim(E + \overline{E}) - \dim(E \cap \overline{E}).$$

Since $\Delta_{\mathbb{C}} = E \cap \overline{E}$ and $\dim(E + \overline{E}) = 2n + r - s$, we have

$$\dim(E + \overline{E}) - \dim E = n - \frac{\dim \Delta + s - r}{2}. \tag{3.5}$$

By Proposition 3.11 and the inclusion $\Delta_0 \subseteq \Delta$, we have that $\dim \Delta + s \geq \dim \Delta_0 + s = r$ and thus

$$0 \leq \dim(E + \overline{E}) - \dim E \leq n.$$

c)It follows after equation (3.5). □

As a consequence of item a) we have that our definition of type coincides with Definition 2.60 in the case of real index zero.

We present some examples of complex Dirac structures on V.

Example 3.15. *(Presymplectic subspaces)* Let S be a subspace of a $(2n+r)$-dimensional vector space V and $\omega \in \wedge^2 S^*$ such that $\operatorname{codim} S = s$ and $\dim \ker \omega = r - s$. Consider $L = L(S_{\mathbb{C}}, i\omega_{\mathbb{C}})$, note that

$$L = \{X_1 + iX_2 + \zeta_1 + i\zeta_2 \mid X_1, X_2 \in S, \zeta_1|_S = -\iota_{X_2}\omega, \zeta_2|_S = \iota_{X_1}\omega\}$$

and consequently

$$L \cap \overline{L} = (\ker \omega \oplus \operatorname{Ann} S)_{\mathbb{C}}.$$

In this case its associated spaces are:

$$E = S_{\mathbb{C}}, \qquad D = S,$$
$$\Delta = S, \qquad \Delta_0 = \ker \omega.$$

Therefore L has real index r, order s and type 0.

Now we compute the associated map $\mathcal{J} : K^\perp/K \to K^\perp/K$. First, note that

$$K^\perp/K \cong D/\ker \omega \oplus (D/\ker \omega)^*.$$

Then for $X \in D$ and $\xi \in V^*$

$$\mathcal{J}_{\mathbb{C}}(X + K_{\mathbb{C}}) = \mathcal{J}_{\mathbb{C}}(\frac{1}{2}(X + i\omega(X)) + \frac{1}{2}(X - i\omega(X)) + K_{\mathbb{C}}) = -\omega(X) + K_{\mathbb{C}},$$

$$\mathcal{J}_{\mathbb{C}}(\xi + K_{\mathbb{C}}) = \mathcal{J}_{\mathbb{C}}(-\frac{i}{2}(\omega^{-1}(\xi) + i\xi) - \frac{i}{2}(-\omega^{-1}(\xi) + i\xi) + K_{\mathbb{C}}) = \omega^{-1}(\xi) + K_{\mathbb{C}}.$$

Thus the map $\mathcal{J}$ has the form

$$\mathcal{J} = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}.$$

Finally we compute the spinor associated to L. Let U be a complement of D in V. Consider a basis of D, $\{d_1, \ldots, d_{2n+r-s}\}$ and complete it by elements of U to a basis $\{d_1, \ldots, d_{2n+r-s}, u_1, \ldots, u_s\}$ with dual basis $\{d_1^*, \ldots, d_{2n+r-s}^*, u_1^*, \ldots, u_s^*\}$. Then taking $\omega_0 \in \wedge^2 V^*$ any extension of ω, we have that the spinor associated to L is $\rho = e^{i\omega} \wedge u_1^* \wedge \ldots \wedge u_s^*$.

Example 3.16. *(Transverse CR structures on linear spaces)* Consider subspaces S, R of a $(2n+r)$-dimensional vector space V such that $R \subseteq S$. Assume that there exists a map $J : S/R \to S/R$ such that $J^2 = -Id$. Let $q : S_{\mathbb{C}} \to (S/R)_{\mathbb{C}}$ denote the quotient map. Let E denote $q^{-1}(\ker(J_{\mathbb{C}} - iId))$; consider the lagrangian subspace $L_{(R,S,J)} = L(E, 0)$. Now we pass to describe its associated subspaces.
Claim A: We have the following

$$E = \{X_1 + iX_2 \in S_{\mathbb{C}} \mid J(X_1 + R) = -X_2 + R\}$$

and $pr_V E = S$.

Proof. An element $X_1 + iX_2$ is in $E = q^{-1}(\ker(J_{\mathbb{C}} - iId))$ if and only if

$$J(X_1 + R) + iJ(X_2 + R) = -X_2 + iX_1 + R_{\mathbb{C}},$$

comparing the real and imaginary parts and using that $J^2 = -Id$, this is equivalent to

$$J(X_1 + R) = -X_2 + R.$$

For the second assertion of the claim, given $X \in S$, then $X + iY \in E$ for some $Y \in S$ such that $J(X + R) = -Y + R$. □

Claim B: We have that $L_{(R,S,J)}$ has as associated isotropic subspace

$$K = R \oplus \operatorname{Ann} S.$$

Proof. Let $X + \xi \in K$, i.e. a real element of $L_{(R,S,J)}$. Then X is a real element of E, i.e. $J(X + R) = 0 + R$ and since J is an isomorphism, $X \in R$. We have that $\xi \in \operatorname{Ann} E$, so $\xi(Y_1) + i\xi(Y_2) = 0$ for all $Y_1 + iY_2 \in E$ and then $\xi \in \operatorname{Ann} S$. For the other inclusion assume that $\xi \in \operatorname{Ann} S$, take $Y_1 + iY_2 \in E$, note that $Y_2 - iY_1 \in E$ and then $\xi(Y_1 + iY_2) = 0$. □

With the previous claims we compute the subspaces associated to this complex Dirac structure:

$$E = q^{-1}(\ker(J_{\mathbb{C}} - iId)), \qquad \Delta = \Delta_0 = R,$$
$$D = S.$$

Consequently we can see that $L_{(R,S,J)}$ has real index $r = \dim R + \operatorname{codim} S$ and order $\operatorname{codim} S$. Let n be the positive integer such that $\dim V = 2n + r$; then L has type n. Indeed,

$$K^{\perp}/K = S/R \oplus (S/R)^*$$

and its associated map

$$\mathcal{J} : S/R \oplus (S/R)^* \to S/R \oplus (S/R)^*$$

is of the form

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}.$$

Now we compute the associated spinor of $L_{(R,S,J)}$. Let $E_0 = \ker(J_{\mathbb{C}} - iId) \subseteq (V/R)_C$ and let $\Omega_0 \in \wedge^{n+s}(V/R)^*_{\mathbb{C}}$ be a generator of $\det \operatorname{Ann} E_0$. Note that $t^*\Omega_0 \in \wedge^{n+s}V^*_{\mathbb{C}}$ is a generator of $\det \operatorname{Ann} E$, where $t : V_{\mathbb{C}} \to (V/R)_{\mathbb{C}}$ is the quotient map. Consequently, the spinor associated to $L_{(R,S,J)}$ is $\rho = t^*\Omega_0$.

Remark 3.17. A transverse CR structure (R, S, J) on a vector space V is equivalent to a CR structure $(S/R, J)$ on V/R. Consider $E_0 = \ker(J_{\mathbb{C}} - iId)$ and $t^{-1}E_0$. The complex Dirac structure associated to (R, S, J) seen as a transverse CR structure is $L(E, 0)$ and seen as CR structure is $L(E_0, 0)$. Their associated spinors are $t^*\Omega_0$ and Ω_0, where Ω_0 is a generator of $\det \operatorname{Ann} E_0$, as it has been showed at the previous example. And thus we see that at the spinorial level these complex Dirac structures are similar. Moreover, $\mathcal{B}_t(L(E_0, 0)) = L(E, 0)$.

3.1.4 Classification

In this section we present a classification of complex Dirac structures in terms of its real index, order and type. This classification has as key ingredients the associated presymplectic subspace (Δ, ω_Δ) and a certain complex structure on D/Δ, and states that a complex Dirac structure is (up to B-transformations) the product of (Δ, ω_Δ) with the complex structure on D/Δ. The complex structure above mentioned is the one obtained from $E/\Delta_{\mathbb{C}}$, since $E/\Delta_{\mathbb{C}} \cap \overline{E/\Delta_{\mathbb{C}}} = 0$. We next see how these two structures characterize the complex Dirac structure up to B-transformation.

Proposition 3.18. *Let L be a complex Dirac structure with real index r and order s. Then L is isomorphic to a B-transformation of the product of a complex Dirac structure defined by a presymplectic structure with $(r-s)$-dimensional kernel with a complex Dirac structure defined by a codimension-s CR structure.*

Proof. Let $N \subseteq V$ such that $\Delta \oplus N = V$. By Proposition 3.14, $\dim N = 2k+s$. Since $E + N_{\mathbb{C}} = V_{\mathbb{C}}$, we have that $\dim(E \cap N) = k$. Consider a basis $\{\gamma_1, \ldots, \gamma_{r_0}\}$ of $\Delta_{\mathbb{C}}$ consisting of real elements, where $r_0 = 2(n-k) + r - s$. Then complete to a basis of E, $\{\gamma_1, \ldots, \gamma_{r_0}, \alpha_1, \ldots \alpha_k\}$. Note that $\{\gamma_1, \ldots, \gamma_{r_0}, \overline{\alpha}_1, \ldots \overline{\alpha}_k\}$ is a basis of $\overline{E}$.

Let $U \subseteq V$ such that $(E + \overline{E}) \oplus U_{\mathbb{C}} = V_{\mathbb{C}}$. Note that $\dim U = s$. Taking $\{u_1, \ldots, u_s\}$ as a basis of U, we can see that

$$\{\overline{\alpha}_1, \ldots \overline{\alpha}_k, \alpha_1, \ldots \alpha_k, u_1, \ldots, u_s\}$$

is a basis of $N_{\mathbb{C}}$ and

$$\{\gamma_1, \ldots, \gamma_{r_0}, \overline{\alpha}_1, \ldots \overline{\alpha}_k, \alpha_1, \ldots \alpha_k, u_1, \ldots, u_s\}$$

is a basis of $V_{\mathbb{C}}$ with dual basis

$$\{\gamma_1^*, \ldots, \gamma_{r_0}^*, \overline{\alpha}_1^*, \ldots \overline{\alpha}_k^*, \alpha_1^*, \ldots \alpha_k^*, u_1^*, \ldots, u_s^*\}.$$

Then

$$\Omega = \overline{\alpha}_1^* \wedge \ldots \wedge \overline{\alpha}_k^* \wedge u_1^* \wedge \ldots \wedge u_s^* \in \wedge^{k+s} V_{\mathbb{C}}^*$$

is a generator of $\det \operatorname{Ann} E$. Note that

$$\Omega|_N \wedge \iota_{u_1} \ldots \iota_{u_s} \overline{\Omega}|_N \neq 0$$

is a generator of $\det N_{\mathbb{C}}^*$. Consider $H_{1,0} = span_{\mathbb{C}}\{\overline{\alpha_1}, \ldots, \overline{\alpha_k}\}$, we note that $H_{1,0} \cap \overline{H_{1,0}} = 0$ and $H = \operatorname{Re}(H_{1,0} \oplus \overline{H_{1,0}}) \subseteq N$, so we have that there exists a complex structure J on H such that $H_{1,0} = \ker(J_{\mathbb{C}} - iId)$; also note that $U \oplus H = N$.

Note that first, we can see that when considering

$$\Omega_{1,0} = \overline{\alpha}_1^* \wedge \ldots \wedge \overline{\alpha}_k^* \in \wedge^k V_{\mathbb{C}}^*,$$

we have that $\Omega_{1,0}|_{H_{1,0}}$ is a generator of $\det H_{1,0}^*$. Second, $\Omega|_N$ is the spinor associated to the complex Dirac structure defined from the CR structure (H, J) on N.

We have that

$$V_{\mathbb{C}} = \Delta_{\mathbb{C}} \oplus H_{1,0} \oplus H_{0,1} \oplus U_{\mathbb{C}},$$

where $H_{0,1} = \overline{H_{1,0}}$. Thus,

$$\wedge^2 V_{\mathbb{C}}^* = \bigoplus_{p+q+t+w=2} \wedge^p \Delta_{\mathbb{C}}^* \otimes \wedge^q H_{1,0}^* \otimes \wedge^t H_{0,1}^* \otimes \wedge^w U_{\mathbb{C}}^*.$$

and so we obtain a four-graduation (p,q,t,w) on two-forms of $V_{\mathbb{C}}$. Let ρ be the associated spinor to $L = L(E,\varepsilon)$, then $\rho = e^{B+i\omega} \wedge \Omega$, where $B + i\omega \in \wedge^2 V_{\mathbb{C}}^*$ is an extension of ε. Since U is complementary to $D = \mathrm{Re}(E + \overline{E})$ and ε only depends on E, we can take the components on U of $B + i\omega$ to be zero; since Ω has as a factor $\Omega_{1,0}$ that restricts to a volume form of $H_{1,0}^*$, we have that just the elements $(1,0,1,0), (2,0,0,0)$ and $(0,0,2,0)$ have an effect on Ω when doing a wedge product. Let $(B+i\omega)^{(p,q,t,w)}$ denote the (p,q,t,w) component of $B+i\omega$ in the four-graduation. We note that

$$\omega_\Delta = -\frac{i}{2}((B+i\omega)^{(2,0,0,0)} - \overline{(B+i\omega)^{(2,0,0,0)}}).$$

Consider the real two-form

$$\begin{aligned} B' =& \frac{1}{2}((B+i\omega)^{(2,0,0,0)} + \overline{(B+i\omega)^{(2,0,0,0)}}) + (B+i\omega)^{(1,0,1,0)} + (B+i\omega)^{(0,0,2,0)} \\ &+ \overline{(B+i\omega)^{(1,0,1,0)}} + \overline{(B+i\omega)^{(0,0,2,0)}}. \end{aligned}$$

Note that the two-forms $\overline{(B+i\omega)^{(1,0,1,0)}}$ and $\overline{(B+i\omega)^{(0,0,2,0)}}$ have no effect on Ω. Consequently we have that

$$e^{B+i\omega} \wedge \Omega = e^{B'+i\omega_\Delta} \wedge \Omega = e^{B'}(e^{i\omega_\Delta} \wedge \Omega|_N).$$

So the result holds. □

From the proof we can see that the CR structure (H,J) on N is equivalent to the complex structure on D/Δ described at the beginning of the section.

As a consequence of the previous proposition we have the description of the extreme-type complex Dirac structures. Example 3.15 and Example 3.16 are a clear evidence of this description.

Corollary 3.19. *Let L be a complex Dirac structure with real index r on a vector space V of dimension $2n+r$. Then if L has order s and type 0, then L is a B-transformation of a complex Dirac structure $L(D_{\mathbb{C}}, i\omega_{\mathbb{C}})$ as in Example 3.15 such that* $\dim D = s$ *and* $\dim\ker\omega = r - s$. *If L has order s and type n, then it is a B-transformation of a complex Dirac structure $L(E,0)$ as in Example 3.16 such that* $\operatorname{codim} S = s$ *and* $\dim R = r - s$.

In the following table we organize the information of the previous proposition up to B-transformation. Recall that we are considering complex Dirac structures with order r over a $(2n+r)$-dimensional vector space V.

order r	(Δ, ω) presymplectic subspace $\operatorname{codim} \Delta = r$ $\omega \in \wedge^2 \Delta^*$ nondegenerate	$\cdots$	$(\Delta^{2(n-k)}, \omega) \times (N^{2k+r}, D, J)$ (Δ, ω) symplectic space $\omega \in \wedge^2 \Delta^*$ nondegenerate (D, J) codimension-r CR structure on N	$\cdots$	(D, J) codimension-r CR structure
$\vdots$	$\vdots$		$\vdots$		$\vdots$
order s	(Δ, ω) presymplectic subspace $\operatorname{codim} \Delta = s$ $\omega \in \wedge^2 \Delta^*$, $\dim \ker \omega = r - s$	$\cdots$	$(\Delta^{2(n-k)+r-s}, \omega) \times (N^{2k+s}, D, J)$ (Δ, ω) presymplectic space $\dim \ker \omega = r - s$ (D, J) codimension-s CR structure on N	$\cdots$	(Δ_0, D, J) transverse CR structure $\operatorname{codim} D = s$, $\dim \Delta_0 = r - s$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
order 0	(V, ω) presymplectic space $\dim \ker \omega = r$	$\cdots$	$(\Delta^{2(n-k)+r}, \omega) \times (N^{2k}, J)$ (Δ, ω) presymplectic space $\dim \ker \omega = r$ J is a complex structure on N	$\cdots$	(Δ_0, V, J) transverse CR structure $\dim \Delta_0 = r$
	type 0	$\cdots$	type k	$\cdots$	type n

In extreme order with extreme type we have a more accurate result than the one given in Proposition 3.19:

Corollary 3.20. *If L is a linear complex Dirac structure with real index r, order r and type n over a vector space V of dimension $2n+r$, then L is a B-transformation of a CR structure of codimension r.*

We end this section with an alternative description of the type. Consider a complex Dirac structure L over a vector space V with associated isotropic $K = \mathrm{Re}(L \cap \overline{L})$ and associated map $\mathcal{J}$. By Lemma 3.4, since V^* is a lagrangian subspace, then $V_K^* = (V^* \cap K^\perp + K)/K$ is a lagrangian subspace of $K^\perp/K$. We have the following characterization of the type which depends on $\mathcal{J}$.

Proposition 3.21. *If L is a complex Dirac structure with associated isotropic K and associated map $\mathcal{J}$, then*

$$\frac{1}{2}\dim(\mathcal{J}(V_K^*) \cap V_K^*) = \mathrm{type}(L).$$

Proof. The subspace $\mathcal{J}(V_K^*) \cap V_K^*$ is invariant under $\mathcal{J}$ and then $\mathcal{J}|_{\mathcal{J}(V_K^*) \cap V_K^*}$ is a complex map. Let $E' = \ker((\mathcal{J}|_{\mathcal{J}(V_K^*) \cap V_K^*})_{\mathbb{C}} - iId)$; then, we have the decomposition

$$(\mathcal{J}(V_K^*) \cap V_K^*)_{\mathbb{C}} = E' \oplus \overline{E'}.$$

We have that

$$E' = \{\xi + K - i\mathcal{J}(\xi + K) \mid \xi \in V^* \text{ and } \xi + K = \mathcal{J}(\eta + K) \ \ for \ \ some \ \ \eta + K \in V_K^*\},$$

giving that $E' = L_0 \cap (V_K^*)_{\mathbb{C}}$, where $L_0 = \ker(\mathcal{J}_{\mathbb{C}} - iId)$. Consider the quotient map $q : K_{\mathbb{C}}^\perp \to (K^\perp/K)_{\mathbb{C}}$. Note that $q(L \cap V_{\mathbb{C}}^*) = L_0 \cap (V_K^*)_{\mathbb{C}}$ and consequently

$$\begin{aligned}
\frac{1}{2}\dim \mathcal{J}(V_K^*) \cap V_K^* &= \dim L_0 \cap (V_K^*)_{\mathbb{C}} \\
&= \dim L \cap V_{\mathbb{C}}^* - \dim K_{\mathbb{C}} \cap V_{\mathbb{C}}^* \\
&= \mathrm{codim}\, E - \mathrm{codim}(E + \overline{E}) \\
&= \mathrm{type}(L).
\end{aligned}$$

□

This gives another justification of our definition of type as [23, Prop 3.6.] did in the generalized complex setting. As a consequence we have another characterization of extreme-type complex Dirac structures.

Corollary 3.22. *Let L be a complex Dirac structure with real index r on a $(2n+r)$-dimensional vector space V and $\mathcal{J}$ be its associated map. Then L has type 0 if and only if $\mathcal{J}(V_K^*) \cap V_K^* = 0$, and L has type n if and only if $\mathcal{J}(V_K^*) = V_K^*$.*

3.2 Complex Dirac structures on manifolds

In this section we continue our study of complex Dirac structures on manifolds. First we concentrate on complex Lie algebroids, defining some of its invariants which are also extended to complex Dirac structures. After that we study some examples of complex Dirac structures.

Definition 3.23. A **complex Lie algebroid** over a manifold M is a complex vector bundle L over M together with a Lie bracket on $\Gamma(L)$

$$[\cdot,\cdot]_L : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$$

and a bundle map $\rho : L \to TM_{\mathbb{C}}$ called the anchor map satisfying the Leibniz property

$$[e_1, fe_2]_L = f[e_1, e_2]_L + \rho(e_1)(f)e_2,$$

for all $e_1, e_2 \in \Gamma(L)$ and for any function $f \in C^\infty(M, \mathbb{C})$.

Examples 3.24. We give some examples of complex Lie algebroids:

1. Involutive structures: an involutive structure E is an involutive vector subbundle of $TM_{\mathbb{C}}$, cf. [35]. Then the pair $(E, pr_{TM_{\mathbb{C}}})$ defines a complex Lie algebroid. So complex and CR structures define complex Lie algebroids.

2. Complexification of real Lie algebroids: if (E, ρ) is a Lie algebroid, then the complexification of the bundle E and the anchor map ρ, $(E_{\mathbb{C}}, \rho_{\mathbb{C}})$ is a complex Lie algebroid with the complexification of the bracket of (E, ρ).

3. Generalized complex structures: if L is a generalized complex structure (Definition 2.53), then the Courant-Dorfman bracket restricted to $\Gamma(L)$ becomes a Lie bracket. Since L is a lagrangian subbundle, the pair $(L, pr_{TM_{\mathbb{C}}})$ is a complex Lie algebroid with respect to the restriction of the Courant-Dorfman bracket to L.

Note that the real index is not necessarily defined over arbitrary complex Lie algebroids; we need an embedding into the complexification of some real vector bundle in order to define it as we have for complex Dirac structures, see Section 2.3. Actually, the real index depends on the embedding. However, we can define the order and the type in a natural way.

Definition 3.25. Consider a complex Lie algebroid L over M with anchor map ρ. The order of L at $p \in M$ is

$$\mathrm{order}_p(L) = \mathrm{codim}(\rho(L|_p) + \overline{\rho(L|_p)})$$

and the type at p is

$$\mathrm{type}_p(L) = \mathrm{codim}\,\rho(L|_p) - \mathrm{codim}(\rho(L|_p) + \overline{\rho(L|_p)}).$$

We can naturally associate to any complex Lie algebroid the following distributions:

$$E := \rho(L) \subseteq TM_{\mathbb{C}}, \;\; \Delta := \mathrm{Re}(E \cap \overline{E}) \subseteq TM \text{ and } D := \mathrm{Re}(E + \overline{E}) \subseteq TM. \qquad (3.6)$$

These distributions are not necessarily vector bundles since their fibres are not necessarily of constant dimension. The distribution Δ is not necessarily smooth as we will see in the Example 3.37 in the next section. It is a different case for E and D.

Lemma 3.26. *The distributions E and D are smooth.*

Proof. Since E is the image of L under the anchor map, we have that E is smooth. Since E is smooth, $\overline{E}$ is smooth and then $E + \overline{E}$ is smooth as well. Then

$$E + \overline{E} = D \oplus iD \subseteq TM \oplus iTM = TM_{\mathbb{C}}$$

and thus $pr_{TM}(E + \overline{E}) = D$ is smooth. □

Given a complex Lie algebroid, its order is defined pointwise, so we obtain a function $\text{order}(L) : M \to \mathbb{Z}$, which satisfies the following:

Corollary 3.27. *The order of a complex Lie algebroid is an upper semi-continuous function.*

Proof. We see this from the fact that $order_p(L) = \operatorname{codim} D|_p$ and that the dimension of D is a lower semi-continuous function by Lemma 2.1. □

In the next section we will prove the same property for the real index of a lagrangian subbundle. We recall the definition of a complex Dirac structure.

Definition 3.28. A complex Dirac structure over a manifold M is a lagrangian subbundle of $(TM \oplus T^*M)_{\mathbb{C}}$ which is involutive with respect to the Courant-Dorfman bracket.

Clearly a complex Dirac structure is a complex Lie algebroid and then it has associated the distributions E, Δ and D. Since complex Dirac structures are embedded in $(TM \oplus T^*M)_{\mathbb{C}}$ we have two additional distributions.

$$K = \operatorname{Re}(L \cap \overline{L}) \qquad \text{and} \qquad \Delta_0 = pr_{TM}K. \tag{3.7}$$

We see in Example 3.31 that K and Δ_0 are not necessarily smooth. Note that

$$\Delta_0 \subseteq \Delta \subseteq D.$$

We give some examples of complex Dirac structures and show its associated distributions.

Example 3.29. *(Complexification of a Dirac structure)* Let L be the complexification of a Dirac structure L' on M. Note that L is a complex Dirac structure and has associated the following distributions:

$$\begin{aligned} &K = L' \\ &E = \Delta_{\mathbb{C}} = D_{\mathbb{C}} = (\Delta_0)_{\mathbb{C}} = (pr_{TM}L')_{\mathbb{C}}. \end{aligned} \tag{3.8}$$

In this case all distributions are smooth. Moreover, the real index of L is constant and equal to $\dim M$ and

$$\text{order}_p(L) = \text{type}_p(L) = \operatorname{codim}(pr_{TM}L'|_p),$$

for all $p \in M$.

Example 3.30. *(Regular Dirac structures)* Let $S \subseteq TM$ be a regular distribution and $\omega \in \wedge^2 S^*$. Consider $L = L(S_{\mathbb{C}}, i\omega_{\mathbb{C}})$. By Proposition 2.41, L is a complex Dirac structure if and only if S is involutive and $d_S\omega = 0$. By Example 3.15, we have that

$$K = \ker\omega \oplus \operatorname{Ann} S$$

and the associated distributions are:

$$\begin{aligned} E &= S_{\mathbb{C}}, & D &= S, \\ \Delta &= S, & \Delta_0 &= \ker\omega. \end{aligned}$$

When $S = TM$, we will use the notation $L_{i\omega}$ for $L(S_{\mathbb{C}}, i\omega_{\mathbb{C}})$. Note that L has real index $\dim\ker\omega|_p + \operatorname{corank} S$ at $p \in M$,

$$\text{order}_p(L) = \operatorname{corank} S \qquad \text{and} \qquad \text{type}_p(L) = 0.$$

We note that in this example the real index could change depending on $\dim\ker\omega|_p$. We use this fact in the next example.

Example 3.31. *(K and Δ_0 not necessarily smooth)* Consider $M = \mathbb{R}^2$, $\omega = xdx \wedge dy \in \Omega^2_{cl}(M)$ and $L = L_{i\omega}$, then L is a complex Dirac structure. The distribution $K = \Delta_0 = \ker \omega$ is not smooth. Along the line $Z = \{x = 0\}$, we have that $L = TM_{\mathbb{C}}$ so the real index is 2 whereas outside Z we have that L has real index zero.

Example 3.32. *(Transverse CR structure)* Consider the transverse CR structure (R, S, J); let $q : TM_{\mathbb{C}} \to (TM/R)_{\mathbb{C}}$ denote the quotient map and $E = q^{-1}(\ker(J_{\mathbb{C}} - iId))$. Since E is involutive in $TM_{\mathbb{C}}$, the lagrangian subbundle $L_{(R,S,J)} = L(E, 0)$ is a complex Dirac structure. By Example 3.16 we have that

$$K = R \oplus \operatorname{Ann} S$$

and the distributions associated to this complex Dirac structure are:

$$E = q^{-1}(\ker(J_{\mathbb{C}} - iId)), \qquad \Delta = \Delta_0 = R,$$
$$D = S. \qquad .$$

Consequently, the real index of $L_{(R,S,J)}$ is constant and equal to $r = \operatorname{corank} S + \operatorname{rank} R$. Let n be the nonnegative integer such that $\dim M = 2n + r$. We have that

$$\operatorname{order}(L_{(R,S,J)}) = \operatorname{corank} S \qquad \text{and} \qquad \operatorname{type}(L_{(R,S,J)}) = n.$$

We have two important cases of transverse CR structures.

Example 3.33. *(CR structures)* Let (S, J) be an almost CR structure. Consider $E = \ker(J_{\mathbb{C}} - iId)$; the lagrangian subbundle $L_{(S,J)} = L(E, 0)$ is involutive if and only if E is involutive, which is the integrability condition for the almost CR structure (S, J). Indeed, a CR structure is a special case of Example 3.32, where $R = 0$. As a result,

$$K = \operatorname{Ann} S, \qquad D = S \qquad \text{and} \qquad \Delta = \Delta_0 = 0.$$

Therefore, if $r = \operatorname{codim} S$, then $L_{(S,J)}$ has constant real index r, constant order r and type n, where $\dim M = 2n + r$. The involutivity of E does not imply necessarily the involutivity of S. If we assume that S is additionally involutive, then by the Frobenius Theorem we get a complex foliation.

Example 3.34. *(Transverse holomorphic structures)* Let (TM, R, J) be a transverse holomorphic structure, i.e., $R \subseteq TM$ is an r-dimensional distribution and $J : TM/R \to TM/R$ is an almost complex structure on TM/R such that $q^{-1}(\ker(J_{\mathbb{C}} - iId))$ is involutive in $TM_{\mathbb{C}}$, where $q : TM_{\mathbb{C}} \to (TM/R)_{\mathbb{C}}$ denotes the quotient map. This structure is a special case of Example 3.32, by taking $S = TM$. As in Example 3.32, consider $L_{(TM,R,J)} = L(q^{-1}(\ker(J_{\mathbb{C}} - iId), 0)$ and note that

$$K = R, \qquad D = TM \qquad \text{and} \qquad \Delta = \Delta_0 = R.$$

Thus $L_{(TM,R,J)}$ has constant real index r, order 0 and type n, where $\dim M = 2n + r$.

3.3 Jumping phenomena

In Corollary 3.27 we observed how the order of a complex Lie algebroid changes. We will see that the real index of a complex Dirac structure varies in the same way; we also provide examples of these changes.

Corollary 3.35. *Let L be a lagrangian subbundle of $(TM \oplus T^*M)_{\mathbb{C}}$, then the function $r(p) = \dim(L|_p \cap \overline{L}|_p)$ is upper semi-continuous. Furthermore, the function $p \mapsto r(p)(mod\, 2)$ is constant.*

Proof. Since $r(p) = \dim(L|_p \cap \overline{L}|_p) = 2\dim M - \dim(L|_p + \overline{L}|_p)$ and $L + \overline{L}$ is a smooth distribution, then by Lemma 2.1, r is upper semi-continuous. The second part of the statement follows from Corollary 3.8. □

In Examples 3.30 and 3.31 we have observed that the real index of a complex Dirac structure can change, remaining the order and the type constant. We next see examples of complex Dirac structures with some invariants changing while other remain constant. We also provide an example of a complex Dirac structure with distribution Δ not smooth.

Example 3.36. *(Type change with constant order)* Let $M = \mathbb{C}^2 \times \mathbb{R}$, with coordinates (z_1, z_2, t). Consider the spinor $\rho = z_1 + dz_1 \wedge dz_2$ and the submanifold $Z = \{z_1 = 0\}$. Then,

$$\rho_p = \begin{cases} dz_1 \wedge dz_2, & \text{if } p \in Z \\ z_1 e^{\frac{dz_1 \wedge dz_2}{z_1}}, & \text{if } p \notin Z. \end{cases}$$

Let L be the annihilator of ρ. We can see that on the points of Z, we have that $L = L(E, 0)$, where $E = \mathrm{Ann}\{dz_1, dz_2\}$, then L is a lagrangian subbundle over Z and thus ρ is pure on Z. On the points of $M - Z$, the spinor comes from the complex two-form $\frac{1}{z_1} dz_1 \wedge dz_2$, then ρ is pure on $M - Z$ and so on the whole manifold. Since

$$d\rho = dz_1 = \iota_{\partial_{z_2}} \cdot \rho,$$

by Proposition 2.52, L is a complex Dirac structure.

Now we study in more detail the distributions associated to L as well as its order, type and real index. We consider real coordinates (x_1, y_1, x_2, y_2, t), where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. On the points of Z, as we have observed above, $L = L(E, 0)$ with $E = \mathrm{Ann}\{dz_1, dz_2\}$ or more explicitly

$$E = span_{\mathbb{C}}\{\partial_{x_1} - i\partial_{y_1}, \partial_{x_2} - i\partial_{y_2}, \partial_t\}.$$

Implying that $E = q^{-1}(\ker(J_{\mathbb{C}} - iId))$, where $q : \mathbb{C}^2 \times \mathbb{R} \to \mathbb{C}^2$ denotes the projection onto $\mathbb{C}^2$ and J the canonical complex structure on $\mathbb{C}^2$. Note that $\mathbb{C}^2 = TM/\mathbb{R} \cdot \partial_t$ and the map q is also the quotient map, yielding us that L is defined by a transverse holomorphic structure. By Example 3.34,

$$D|_Z = TM|_Z \qquad \text{and} \qquad K|_Z = \Delta|_Z = \Delta_0|_Z = \mathbb{R} \cdot \partial_t.$$

We observe that on the points of Z the order of L is zero, the rank of $\ker \omega_\Delta$ is one and the type of L is 2 and so by Proposition 3.11 the real index is one.

On the points of $M - Z$, we have that $L = L(TM_C, B + i\omega) = e^B L(TM_{\mathbb{C}}, i\omega)$, where $\frac{1}{z_1} dz_1 \wedge dz_2 = B + i\omega$,

$$B = \frac{x_1}{x_1^2 + y_1^2}(dx_1 \wedge dx_2 - dy_1 \wedge dy_2) - \frac{y_1}{x_1^2 + y_1^2}(dx_1 \wedge dy_2 - dy_1 \wedge dx_2)$$

and

$$\omega = \frac{x_1}{x_1^2 + y_1^2}(dx_1 \wedge dy_2 - dy_1 \wedge dx_2) - \frac{y_1}{x_1^2 + y_1^2}(dx_1 \wedge dx_2 - dy_1 \wedge dx_2).$$

We can see that the two-form $\omega \in \Omega^2(M - Z)$ is presymplectic and $\ker\omega = \mathbb{R}\cdot\partial_t$. Since L is a B-transformation of a presymplectic structure, we have that the associated distributions remain the same as in the presymplectic case, then

$$E = TM_{\mathbb{C}}, \qquad D = \Delta = TM \qquad \text{and} \qquad \Delta_0 = \ker\omega = \mathbb{R}\cdot\partial_t \ \text{ on } \ M - Z.$$

However, since the B-transformation is real then

$$L \cap \overline{L} = e^B(L(TM_{\mathbb{C}}, i\omega) \cap \overline{L(TM_{\mathbb{C}}, i\omega)})$$

and thus

$$K = e^B(\mathbb{R}\cdot\partial_t) = \mathbb{R}\cdot\partial_t \ \text{ on } \ M - Z.$$

Hence ρ defines a complex Dirac structure with constant real index one, constant order 0 which is type 0 along $M - Z$ and type 2 along Z.

Example 3.37. *(Order and type change with constant real index)*
Consider $M = \mathbb{R}^5$, with coordinates (x_1, x_2, y_1, y_2, z), the canonical two-form

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 \in \Omega^2(\mathbb{R}^5)$$

and the regular distribution

$$E_p = span_{\mathbb{C}}\{\partial_{x_1}, \partial_{y_1}, \partial_{x_2}, (a(p)\partial_{y_2} + ib(p)\partial_z)\},$$

where a and b are real functions and $a(p) \neq 0$ for all p.

Consider the lagrangian subbundle

$$L = L(E, i\iota_E^*\omega)$$

and the set $Z = \{p \mid b(p) = 0\}$. Since E is a regular distribution, L is smooth.

Since

$$\operatorname{Ann} E = \mathbb{C}\cdot(adz - ibdy_2),$$

we have that the spinor associated to L is

$$\rho = e^{i(dx_1\wedge dy_1 + dx_2\wedge dy_2)} \wedge (adz - ibdy_2)$$

Taking $a(x_1, y_1, x_2, y_2, z) = e^{y_2}$ and $b(x_1, y_1, x_2, y_2, z) = f(y_2)$, where $f \in C^\infty(\mathbb{R})$ has non-empty zero set, we have that

$$d\rho = e^{y_2} \wedge dy_2 \wedge dz + ie^{y_2} \wedge dx_1 \wedge dy_1 \wedge dy_2 \wedge dz = dy_2 \cdot \rho$$

and thus L is a complex Dirac structure.

We study the real index, type and order of L. First, we note that $\operatorname{rank} E = 4$. In case $p \in Z$, the bundle E is real and thus

$$\Delta|_p = D|_p = span_{\mathbb{R}}\{\partial_{x_1}, \partial_{x_2}, \partial_{y_1}, \partial_{y_2}\},$$

moreover, $\operatorname{order}_p(L) = \operatorname{codim} D|_p = 1$ and $\operatorname{type}_p(L) = 0$. In case $p \in M - Z$, we have that

$$\Delta|_p = span_{\mathbb{R}}\{\partial_{x_1}, \partial_{x_2}, \partial_{y_1}\}, \qquad D|_p = T_pM,$$

and thus $\text{order}_p(L) = \text{codim}\, D|_p = 0$ and $\text{type}_p(L) = 1$.

Note that $\omega_\Delta = \iota_\Delta^* \omega$ and then

$$\omega_\Delta|_p = \begin{cases} dx_1 \wedge dy_1 + dx_2 \wedge dy_2, & \text{whenever } p \in Z \\ dx_1 \wedge dy_1, & \text{whenever } p \notin Z \end{cases}$$

Thus ω_Δ is nondegenerate on Z and has one-dimensional kernel outside Z. Using Proposition 3.11, we obtain that the real index of L is one.

We have seen that L has real index one, and type and order changing along Z.

It remains to compute K; we will prove that

$$K = \mathbb{R} \cdot (b\partial_{x_2} + adz).$$

By Lemma 3.3, we have that $pr_{TM}K = \ker \omega_\Delta = \mathbb{R} \cdot b\partial_{x_2}$ and so

$$K = \mathbb{R} \cdot (b\partial_{x_2} + \zeta),$$

for some $\zeta \in T^*M$. Since $b\partial_{x_2} + \zeta$ is a real element of L, and $L = L(E, i\iota_E^*\omega)$, we have that

$$\zeta|_E = \iota_{b\partial_{x_2}} i\iota_E^* \omega_{\mathbb{C}} = ibdy_2.$$

Consequently, $\zeta - ibdy_2 \in \text{Ann}\, E = \mathbb{C} \cdot (adz - ibdy_2)$ and then

$$\begin{aligned} \zeta &= (\beta_1 + i\beta_2)(adz - ibdy_2) + ibdy_2 \\ &= \beta_1 adz + \beta_2 bdy_2 + i(\beta_2 adz + (1 - \beta_1) bdy_2) \end{aligned}$$

for some $\beta_1 + i\beta_2 \in \mathbb{C}$. Since ζ is real, then

$$\beta_2 adz + (1 - \beta_1) bdy_2 = 0$$

and thus $\beta_2 = 0$ and $b(1 - \beta_1) = 0$. Then $\zeta = \beta_1 adz$. Whenever $p \in M - Z$, i.e. $b(p) \neq 0$, we have that $\beta_1 = 1$ and then $\zeta = adz$, otherwise $pr_{T_pM}K|_p = 0$.

Another fact about L is that Δ is not smooth, since it has dimension 4 on Z and dimension 3 elsewhere.

Chapter 4

Complex Dirac structures with constant real index

In this chapter we study the phenomena associated to complex Dirac structures with constant real index. We begin by describing the conditions to assure the smoothness of the distributions associated to a complex Dirac structure. We introduce the (real) Dirac structure associated to a complex Dirac structure with constant real index and order as a generalization of the Poisson structure associated to generalized complex structure. We also introduce the class of split isotropic subbundles and study the complex Dirac structures related to them. At the end of this chapter we discuss briefly a new notion of generalized metric.

4.1 More on complex Dirac structures on manifolds

In this section we study several properties related to complex Dirac structures with constant real index. We also provide some topological obstructions for their existence.

4.1.1 Smoothness for the associated distributions

We now study the conditions for guaranteeing the smoothness of the distributions associated to complex Dirac structures. Given a complex Dirac structure L, we saw that it has associated real distributions $K \subseteq TM \oplus T^*M$, $D, \Delta, \Delta_0 \subseteq TM$ and the complex distribution E; we saw in Proposition 3.26 that E and D are always smooth. However in Example 3.31 and Example 3.37 we observed that K, Δ and Δ_0 are not necessarily smooth. We will show that under certain conditions we can assure the smoothness of these distributions and even the integrability of Δ and Δ_0. First we recall an elementary lemma.

Lemma 4.1. *Let S be a smooth distribution of a vector bundle A over M. If S has constant rank, then S is a vector subbundle of A.*

Proof. Since S is a smooth distribution and has constant rank r, we have that around any point $p \in M$ we can construct a local frame of r sections of S and with them we obtain a local trivialization for S, showing that S is a vector bundle of rank r. □

The previous lemma together with Lemma 2.7 give us the following.

Corollary 4.2. *Let L be a lagrangian subbundle of $(TM \oplus T^*M)_{\mathbb{C}}$. If the real index of L is constant and equal to r, then K and $K^{\perp}$ are vector bundles of rank r and $2\dim M - r$ respectively. Moreover, if L is involutive, then K is a Lie algebroid.*

Corollary 4.3. *Let L be a lagrangian subbundle of $(TM \oplus T^*M)_{\mathbb{C}}$. If the order of L is constant and equal to s, then D is a vector bundle of rank $\dim M - s$.*

Remark 4.4. The involutivity of D does not follow from the involutivity of L. An example of this is the complex Dirac structure defined by the CR structure of S^3 inherited by the complex structure of $\mathbb{C}^2$, cf. [6].

For Δ_0 we have the following.

Corollary 4.5. *If L is a lagrangian subbundle of $(TM \oplus T^*M)_{\mathbb{C}}$ with constant real index, then Δ_0 is smooth. Moreover, if L is involutive, then Δ_0 is integrable.*

Proof. Since $\Delta_0 = pr_{TM}K$, the first part follows after Lemma 4.2. If L is involutive then K is a Lie algebroid and by Proposition 2.5, Δ_0 is integrable. □

For the smoothness of Δ we have the following.

Lemma 4.6. *If L has constant order, then the distribution Δ is smooth.*

Proof. Consider the bundle map

$$\Phi : L \to D$$
$$\Phi(l) = i(pr_{TM_{\mathbb{C}}}(l) - \overline{pr_{TM_{\mathbb{C}}}(l)}).$$

First we see that Φ is surjective. Note that $E+\overline{E} = D \oplus iD$. Given $d \in D$, since $pr_{TM_{\mathbb{C}}}(L+\overline{L}) = D_{\mathbb{C}}$ there exists $\tau \in T^*M_{\mathbb{C}}$ such that $d + \tau \in L + \overline{L}$. Take $l_1 = X_1 + iX_2 + \xi \in L$ and $l_2 = Y_1 + iY_2 + \eta \in \overline{L}$, where $X_1, X_2, Y_1, Y_2 \in TM$ and $\eta, \xi \in TM_{\mathbb{C}}$ such that $d + \tau = l_1 + l_2$. Consequently, $\overline{l_2} = d - X_1 + iX_2 + \overline{\eta} \in L$ and $\Phi(-\frac{i}{2}(l_1 + \overline{l_2})) = d$.

Since L has constant order, D is a vector bundle. As we have proved that Φ is surjective, $\ker \Phi$ is a vector bundle and so $\Delta = pr_{TM}(\ker \Phi)$ is smooth. □

Constant real index does not imply the smoothness of Δ as we observed in Example 3.37, where Δ is not smooth. We will prove in Section 4.2 that if the real index and the order of a complex Dirac structure are constant, then Δ is the presymplectic distribution associated to a Dirac structure and so it is integrable.

4.1.2 Extreme-type complex Dirac structures

Now we give the smooth version of Proposition 3.19 and more examples of complex Dirac structures with constant real index.

Proposition 4.7. *Let L be a complex Dirac structure with constant real index r over a $(2n+r)$-dimensional manifold. We have the following:*

a) If L has order s and type 0, then $L = e^B L(D_{\mathbb{C}}, \omega_{\mathbb{C}})$, where $L(D_{\mathbb{C}}, \omega_{\mathbb{C}})$ is the complex Dirac structure associated to a regular Dirac structure, $L(D, \omega)$ with D a corank-r involutive distribution and $\omega \in \wedge^2 D^*$ is a presymplectic structure with $(r-s)-$dimensional kernel, and B is a not necessarily closed real two-form. Moreover, the differential of B vanishes in the direction of the distribution D.

b) If L has order s and type n, then $L = e^B L(E,0)$, where $L(E,0)$ is the complex Dirac structure associated to a transverse CR structure and $B \in \Omega^2(M,\mathbb{C})$ is not necessarily closed.

Proof. a) The first assertion follows from Proposition 3.19. Suppose that $L = e^B L(D_{\mathbb{C}}, i\omega)$ where $D \overset{\iota}{\hookrightarrow} TM$ is a corank-r smooth distribution, $\omega \in \wedge^2 D^*$ is the presymplectic structure with $(r-s)-$dimensional kernel and $B \in \Omega^2(M)$. Since $L = L(D_{\mathbb{C}}, \iota^* B + i\omega)$ is involutive, then D is a regular integrable distribution, $d_D(\iota^* B + i\omega) = 0$, implying that $d_D \iota^* B = 0$, where d_D represents the differential on the directions of D.

b) Let $E = pr_{TM_{\mathbb{C}}} L$, $\Delta = \mathrm{Re}(E \cap \overline{E})$ and $D = \mathrm{Re}(E + \overline{E})$. We see that E defines a bundle map $J : D/\Delta \to D/\Delta$ satisfying $J^2 = -Id$ such that $E = q^{-1}(\ker(J_{\mathbb{C}} - iId))$, where $q : TM_{\mathbb{C}} \to (TM/R)_{\mathbb{C}}$ denotes the quotient map. By Lemma 4.6 and the fact that L has constant real index, order and type, Δ and D are regular distributions. The involutivity of L implies that E and Δ are involutive and so (Δ, D, J) defines a transverse CR structure. It follows that $L = e^B L_{(\Delta,D,J)}$, where B is an extension of ε. □

Remark 4.8. Recalling that B-transformations are real and closed two-forms. In the previous proposition, the two-form B does not define in general a B-transformation. For the case of type 0, around any point we obtain a neighbourhood where B can be taken to be an honest B-transformation. This is done using a foliated chart and extending ε remaining the same in the directions of the distribution and vanishing on the other directions. It is not always possible to obtain a global B-transformation, actually there is an obstruction that relies on the foliated cohomology of the manifold with respect to the foliation associated to D, cf. [15]. On the other hand, in type n, we find some difficulties to find a globally or locally defined $B-$transformation taking L to the complex Dirac structure associated to a transverse CR structure. This differs from what happens to generalized complex structures with type n, where the two-form is not necessarily real and closed but locally we obtain an honest B-transformation, [22, Prop. 4.22].

Proposition 4.9. *Let L be a complex Dirac structure with constant real index r and order r over a $(2n+r)$-dimensional manifold. If L has type n, then there exist a CR structure (D, J) and a real two-form B such that L is a transformation by B of the complex Dirac structure associated to (D, J).*

Proof. Since L has type n, we have that its associated Poisson structure is zero. So, if $\mathcal{J}_N : D \oplus D^* \to D \oplus D^*$ is its associated bundle map, then

$$\mathcal{J} = \begin{pmatrix} A & 0 \\ \Omega & -A^* \end{pmatrix}.$$

Since $\mathcal{J}$ is a generalized complex structure, $\Omega A = A^* \Omega$ and $A^2 = -Id$. Then we have that (D, A) is an almost CR structure. We next prove that it is actually a CR structure. Since

$$\ker(\mathcal{J}_{\mathbb{C}} - iId) = \{X - iA(X) + \xi - i(\Omega(X) - A^*\xi) \in (D \oplus D^*)_{\mathbb{C}} \mid X + \xi \in D \oplus D^*\},$$

we have that

$$L = \{X - iA(X) + \hat{\xi} - i\hat{\eta} \in (D \oplus T^*M)_{\mathbb{C}} \mid X \in D, \hat{\xi}|_D = \xi \text{ and } \hat{\eta}|_D = \Omega(X) - A^*\xi\}.$$

Note that $pr_{TM} L = \ker(A_{\mathbb{C}} - iId)$, so the involutivity of L implies the involutivity of $\ker(A_{\mathbb{C}} - iId)$. Thus, we have obtained the condition of integrability of (D, A).

Consider the lagrangian subbundle $L_0 = L/(\operatorname{Ann} D)_{\mathbb{C}} \subseteq (D \oplus D^*)_{\mathbb{C}}$ and the quotient map $q : (D \oplus T^*M)_{\mathbb{C}} \to (D \oplus D^*)_{\mathbb{C}}$. Now consider $B_0 = -\Omega A/2 \in \Gamma(\wedge^2 D^*)$; note that

$$e^{-B_0}\mathcal{J}e^{B_0} = \begin{pmatrix} A & 0 \\ \Omega - A^*B_0 - B_0A & -A^* \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} = \mathcal{J}_A,$$

since $A^*B_0 + B_0A = (-A^*\Omega A - \Omega A^2)/2 = \Omega$. Let $L_0' = \ker((\mathcal{J}_A)_{\mathbb{C}} - iId)$, we observe that $L_0' = e^{B_0}L_0$. Let B_1 be an extension of B_0 to $\Omega^2(M)$. We note that $L' = q^{-1}L_0'$ is the complex Dirac structure associated to the CR structure (D, A) and $L' = e^{B_1}L$. This implies that $e^{B_1}(L)$ is the complex Dirac structure associated to a CR structure. □

Examples 4.10. We give additional examples of complex Dirac structures with constant real index.

a) Let L be a generalized complex structure. Note that $L \cap \overline{L} = 0$ and consequently we have that $K = 0$, $D = TM$ and $\Delta_0 = 0$.

b) Let (θ, ω) be a cosymplectic structure on a $(2n+1)$-dimensional manifold, i.e. $\theta \in \Omega^1_{cl}(M)$, $\omega \in \Omega^2_{cl}(M)$ such that $\omega^n \wedge \theta$ is a volume form. The last condition implies that $\ker \iota^*\omega = 0$, where $\iota : \ker\theta \to TM$ is the inclusion map. We can see that a cosymplectic structure defines a regular Dirac structure $L(\ker\theta, \omega)$ and so a complex Dirac structure $L = L((\ker\theta)_{\mathbb{C}}, i\iota^*\omega)$ with real index one and type 0.

c) Let $p : M \times \mathbb{R} \to M$ be the projection of $M \times \mathbb{R}$ onto M. Let L be a generalized complex structure on M. Then $\mathcal{B}_p(L)$ has real index one and order zero, since

$$\mathcal{B}_p(L) \cap \overline{\mathcal{B}_p(L)} = \ker(p_*)_{\mathbb{C}} = \mathbb{C} \cdot \partial_t.$$

4.1.3 Obstructions

In this section we study some topological obstructions to the existence of a complex Dirac structure with real index r. In Corollary 3.8 we obtained a first obstruction that relies on the dimension of the manifold: if M admits a lagrangian subbundle L of $(TM \oplus T^*M)_{\mathbb{C}}$ with constant real index r, then $\dim M \cong r \pmod 2$.

Let K be an isotropic subbundle of $TM \oplus T^*M$ with rank r and assume that $\dim M = 2n + r$; then $\operatorname{rank} K^\perp/K = 4n$. By Proposition 3.5, $K^\perp/K$ admits a pairing of signature $(2n, 2n)$, thus the structural group of the frame bundle of $K^\perp/K$ admits a reduction to $O(2n, 2n)$. In the same manner as generalized almost complex structures over a $2n$-dimensional manifold M are obtained from reduction of the structure group $O(2n, 2n)$ on the frame bundle of $TM \oplus T^*M$ to $U(n, n)$, we will see that complex Dirac structures L with associated isotropic subbundle K(this implies that L would have constant real index) are obtained from reduction of the structure group $O(2n, 2n)$ on the frame bundle $K^\perp/K$. Since the existence of a complex Dirac structure with associated isotropic subbundle K is equivalent to the existence of a pairing-preserving-map $\mathcal{J} : K^\perp/K \to K^\perp/K$ such that $\mathcal{J}^2 = -Id$, we have a reduction to $U(n, n) = GL(2n, \mathbb{C}) \cap O(2n, 2n)$. Thus we obtain the following.

Lemma 4.11. *Let K be an isotropic subbundle of $TM \oplus T^*M$ with rank r. A lagrangian subbundle L of $TM \oplus T^*M$ with isotropic subbundle K is equivalent to a reduction of structure from $O(2n, 2n)$ to $U(n, n)$ on the frame bundle of $K^\perp/K$.*

Consider a lagrangian subbundle L of $TM \oplus T^*M$ with isotropic subbundle K, by Lemma 4.11, there is a reduction from $O(2n, 2n)$ to $U(n, n)$ on $K^\perp/K$. Since $U(n) \times U(n)$ is a maximal compact subgroup of $U(n, n)$, it is homotopic to $U(n, n)$, so we can find a complex subbundle C_+ inside $K^\perp/K$ with orthogonal bundle $C_- = C_+^\perp$ such that $K^\perp/K = C_+ \oplus C_-$. As a result the bundle map

$$p_K : C_+ \to D/\Delta_0$$
$$X + \xi + K \mapsto X + \Delta_0$$

is well-defined since $pr_{TM}K^\perp = D$ and $pr_{TM}K = \Delta_0$ and actually is an isomorphism. Therefore, the vector bundle D/Δ_0 admits an almost complex structure by transporting the almost complex structure of C_+ via p_K to D/Δ_0. Thus we have proved the following.

Proposition 4.12. *Let L be a real index r Dirac structure. Then there exists an almost complex structure on the bundle D/Δ_0.*

In particular, when the order is equal to the real index, i.e. $K^\perp/K = D \oplus D^*$, the distribution D admits an almost CR structure.

Corollary 4.13. *If a manifold M admits a lagrangian subbundle with constant real index r and order r, then it admits an almost CR structure of codimension r as well.*

Proposition 4.12 gives a constraint on the existence of complex Dirac structure with underlying associated isotropic subbundle K such that $pr_{TM}K^\perp = D$ and $pr_{TM}K = \Delta_0$. We recall the following proposition about the existence of almost complex structures on vector bundles is proved exactly as in [22, Prop. 4.16].

Proposition 4.14. *Let E be a vector bundle of rank $2n$ over M admitting an almost complex map J. Then:*

1. *The odd Stiefel-Whitney classes $w_{2i+1}(E) \in H^{2i+1}(M, \mathbb{Z}/2\mathbb{Z})$ vanishes.*
2. *The Chern classes $c_i(E) \in H^{2i}(M, \mathbb{Z})$ and the Pontrjagin classes $p_i(E) \in H^{4i}(M, \mathbb{Z})$ satisfy*

$$\sum_{i=0}^{[n/2]} (-1)^i p_i = \sum_{j=0}^{n} c_j \cup \sum_{k=0}^{n} (-1)^k c_k.$$

So if the proposition above fails for D/Δ_0, we obtain that there is no lagrangian subbundle with associated isotropic K such that $pr_{TM}K = \Delta_0$ and $pr_{TM}K^\perp = D$. Note that the closer rank of K is to $\dim M$, the less information the proposition gives. In the case of real index zero, the obstruction is the same as the existence of almost complex structure. On the other hand, in the case where $\operatorname{rank} K = \dim M$, i.e. K is a lagrangian subbundle, Proposition 4.12 gives no information, since $D/\Delta_0 = 0$.

4.2 Associated Dirac structures

In this section we show another feature of generalized complex structures that is extended to complex Dirac structures with constant real index and order, that is that complex Dirac structures with constant real index and order carry a presymplectic foliation which comes from a Dirac structure.

Let L be a complex Dirac structure with constant real index r and order s. The presymplectic distribution (Δ, ω_Δ) has as kernel the distribution Δ_0 which, by Proposition 3.11, is a regular distribution of rank $r - s$. We see in this section that Δ is integrable and actually

$$L_\Delta := L(\Delta, \omega_\Delta)$$

is a Dirac structure. Firstly, we study the relationship between L_Δ and L, as well as its associated bundle map $\mathcal{J}$.

We recall that

$$K^\perp \cap T^*M = \operatorname{Ann} \Delta_0 \cong (TM/\Delta_0)^*.$$

The following three lemmas concern a complex Dirac structure L with constant real index r.

Lemma 4.15. *If $\xi \in K^\perp \cap T^*M$, then there exist $X \in TM$ and $\eta \in T^*M$ such that $X + i\xi + \eta \in L$. If $X' \in TM$ and ξ', η', we have that $X' + i\xi + \eta' \in L$ if and only if $X + \eta - (X' + \eta') \in K$.*

Proof. Since $\xi \in K^\perp \cap T^*M \subseteq (L + \overline{L})$, then there exists $l_1 \in L$, $l_2 \in \overline{L}$ with $l_k = X_k + iY_k + \eta_k + i\tau_k$, such that $\xi = l_1 + l_2$. We can see that $l_2 = -X_1 - iY_1 - (\xi - \eta_1) - i\tau_1$. Then

$$i(l_1 + \overline{l_2}) = -2Y_1 + i\xi - 2\tau_1 \in L.$$

□

The previous lemma allows us to see ξ as an element of $L + \overline{L}$, by taking $e = -\frac{i}{2}(X + i\xi + \eta)$ and seeing that $\xi = e + \overline{e}$.

Lemma 4.16. *Let $X \in TM$ and $\xi, \eta \in T^*M$. If $X + i\xi + \eta \in L$ then $X + \xi \in L_\Delta$. Conversely, if $X + \xi \in L_\Delta$, then there exists $\eta \in T^*M$ such that $X + i\xi + \eta \in L$.*

Proof. $(\to)$ Let $E \subseteq T_pM_\mathbb{C}$ and $\varepsilon \in \wedge^2 E|_p^*$ such that $L|_p = L(E|_p, \varepsilon)$. Let $B + i\omega \in \wedge^2 T_p^*M$ be an extension of ε. Since $X + i\xi + \eta \in L|_P$, we have that

$$(i\xi + \eta)|_E = \iota_X \varepsilon = \iota_X(B + i\omega) = \iota_X B + i\iota_X \omega.$$

Restricting to $\Delta_\mathbb{C}$ and comparing the imaginary component we get that $\xi|_\Delta = \iota_X \omega|_\Delta = \iota_X \omega_\Delta$. Consequently, $X + \xi \in L_\Delta$.

$(\leftarrow)$ Consider $X + \xi \in L_\Delta$; then, $\xi \in pr_{T^*M}(L_\Delta) = \operatorname{Ann} \Delta_0$. Thus by Lemma 4.15, there exist $Y \in TM$, $\eta \in T^*M$ such that $Y + i\xi + \eta \in L$. Applying the right-side direction of the proof to $Y + i\xi + \eta \in L$, we obtain that $Y + \xi \in L_\Delta$. Then $X - Y \in L_\Delta \cap TM = \Delta_0 = pr_{TM} K$ and thus there exists $\zeta \in T^*M$ such that $X - Y + \zeta \in K$. So by Lemma 4.15, $X + i\xi + \eta + \zeta \in L$. □

Lemma 4.17. *If $\xi \in K^\perp \cap T^*M$ and $Y \in TM$, $\tau \in T^*M$ satisfy $\mathcal{J}(\xi + K) = Y + \tau + K$, then $Y + \xi \in L_\Delta$. Conversely, if $X \in TM$, $\xi \in T^*M$ satisfy that $X + \xi \in L_\Delta$, then there exists $\tau \in T^*M$ such that $\mathcal{J}(\xi + K) = X + \tau + K$.*

Proof. $(\to)$ By Lemma 4.15, there exist $X \in TM$ and $\eta \in T^*M$ such that $X + i\xi + \eta \in L = L(E, \varepsilon)$ and by Lemma 4.16, $X + \xi \in L_\Delta$.

Applying $\mathcal{J}_\mathbb{C}$ to $\xi + K_\mathbb{C}$ and taking $e = -\frac{i}{2}(X + i\xi + \eta)$, we get that

$$\mathcal{J}_\mathbb{C}(\xi + K_\mathbb{C}) = \mathcal{J}_\mathbb{C}(e + \overline{e} + K_\mathbb{C}) = i(e - \overline{e}) + K_\mathbb{C} = X + \eta + K_\mathbb{C}.$$

If $Y + \tau + K \in K^\perp / K$ is such that $\mathcal{J}(\xi + K) = Y + \tau + K$. Then

$$Y + \tau + K_\mathbb{C} = X + \eta + K_\mathbb{C}$$

and thus $Y - X \in pr_{TM_{\mathbb{C}}} K_{\mathbb{C}}$. Finally, $Y - X \in \Delta_0 = L_\Delta \cap TM$ and then $Y + \xi \in L_\Delta$.

($\leftarrow$) Consider $X + \xi \in L_\Delta$. By Lemma 4.16, there exist $\tau \in T^*M$ such that $l = X + i\xi + \tau \in L$. Taking $e = -\frac{i}{2} l$, we have that $\xi = e + \bar{e}$ and

$$\mathcal{J}_{\mathbb{C}}(\xi + K_{\mathbb{C}}) = X + \tau + K_{\mathbb{C}}.$$

□

Remark 4.18. If $\mathcal{J}$ is a generalized complex structure and $\xi \in T^*M$, then $\mathcal{J}(\xi) = \pi(\xi) + \eta$, where π is the Poisson structure associated to $\mathcal{J}$ and η some element in T^*M. We can see that in this case $L_\Delta = Graph(\pi)$.

From now on we consider a complex Dirac structure with constant real index r and order s.

Consider $F = pr_{T^*M}(L_\Delta)$. We know that $F = \operatorname{Ann}(\ker \omega_\Delta) = \operatorname{Ann} \Delta_0 \cong (TM/\Delta_0)^*$. Since the order is constant, F is smooth and of constant rank.

Now consider

$$\gamma : F \cong (TM/\Delta_0)^* \to TM/\Delta_0$$
$$\xi \mapsto p_K \mathcal{J}(\xi + K)$$

where

$$p_K : K^\perp / K \to TM/\Delta_0$$
$$X + \xi + K \mapsto X + \Delta_0.$$

Note that $\gamma \in \Gamma(\wedge^2(TM/\Delta_0))$ is well defined globally. Let

$$r : TM \to TM/\Delta_0$$

be the quotient map.

Lemma 4.19. *Let L_γ denote the graph of γ in $TM/\Delta_0 \oplus (TM/\Delta_0)^*$. Then*

$$\mathcal{B}_{r_p}(L_\gamma|_p) = L_\Delta|_p,$$

for all $p \in M$. Here we are taking the backward image pointwise.

Proof. Let $X + \xi \in \mathcal{B}_{r_p}(L_\gamma)|_p$. Then there exists $\hat{\xi} \in (TM/\Delta_0)^*$ such that $r(X) + \hat{\xi} \in L_\gamma|_p$ and $r^*\hat{\xi} = \xi$, and then $\xi \in \operatorname{Ann} \Delta_0$. Thus

$$X + \Delta_0 = \gamma(\hat{\xi}) = p_K \mathcal{J}(\xi + K).$$

Then by Lemma 4.17, $X + \xi \in L_\Delta|_p$. Consequently, $\mathcal{B}_{r_p}(L_\gamma)|_p \subseteq L_\Delta|_p$ and the equality holds since both are lagrangian subspaces of $(TM \oplus T^*M)|_p$. □

Before proving the main theorem, we prove a technical lemma.

Lemma 4.20. *Let A and B be two vector subbundles of a vector bundle V. If the rank of $A \cap B$ is constant, then $A \cap B$ is a vector subbundle of V*

Proof. Consider the map

$$\Psi : V \times V \to V$$
$$(v_1, v_2) \mapsto v_1 - v_2.$$

Note that $\Psi|_{A \times B}$ is smooth. By hypothesis $\ker \Psi|_{A\times B} = (A \cap B) \times (A \cap B)$ has constant rank, then $\Psi(A \times B)$ is a vector bundle. Consequently, $\ker \Psi|_{A\times B}$ is a vector bundle and so $A \cap B$. □

Theorem 4.21. *If L is a complex Dirac structure with constant real index and order, then the space L_Δ is a Dirac structure.*

Proof. We divide the proof in two claims. The smoothness of γ allows us to prove the first claim.

Claim 4.22. *The space L_Δ is smooth.*

By equation (2.6) and Lemma 4.19, we have the following exact sequence of bundles

$$0 \longrightarrow \ker r^* \cap L_\gamma \longrightarrow (L_\gamma \oplus (TM/\Delta_0 \oplus (TM/\Delta_0)^*)) \cap \Gamma_r \longrightarrow L_\Delta \longrightarrow 0. \tag{4.1}$$

Recall that

$$\Gamma_r = \{(Y+\eta, X+\xi) \mid Y = rX, \xi = r^*\eta\} \subseteq (TM/\Delta_0 \oplus (TM/\Delta_0)^*) \times \overline{(TM \oplus T^*M)}$$

is a lagrangian subbundle. Since $\ker r^* \cap L_\gamma = 0$, we obtain that

$$(L_\gamma \oplus (TM/\Delta_0 \oplus (TM/\Delta_0)^*)) \cap \Gamma_r \cong L_\Delta.$$

Finally, by Lemma 4.20, L_Δ is a vector bundle.

Claim 4.23. *The lagrangian subbundle L_Δ is involutive.*

Let $X+\xi, Y+\eta \in \Gamma(L_\Delta)$. Then, there exists $\xi_0, \eta_0 \in \Gamma(T^*M)$ such that

$$X + i\xi + \xi_0,\ Y + i\eta + \eta_0 \in \Gamma(L).$$

Since L is involutive

$$\begin{aligned}[X+i\xi+\xi_0, Y+i\eta+\eta_0] &= [X,Y] + \mathcal{L}_X(i\eta+\eta_0) - \iota_Y d(i\xi+\xi_0)\\ &= [X,Y] + i(\mathcal{L}_X\eta - \iota_Y d\eta) + \mathcal{L}_X\eta_0 - \iota_Y d\xi_0 \in \Gamma(L).\end{aligned}$$

Then by Lemma 4.16,

$$[X,Y] + \mathcal{L}_X\eta - \iota_Y d\eta \in \Gamma(L_\Delta).$$

□

Corollary 4.24. *If L is a complex Dirac structure with constant real index and order, then the distribution Δ is integrable.*

Given F a subbundle of T^*M and $\gamma \in \wedge^2 F^*$ we consider

$$L(F,\gamma) = \{X + \xi \mid \iota_\xi\gamma = X|_F\},$$

which is a lagrangian subbundle.

Note that in our case we have that

$$L_\Delta = L(\operatorname{Ann}\Delta_0, \gamma).$$

In conclusion we get that to each complex Dirac structure with constant real index r and order s we assign a Dirac structure with $(r-s)-$dimensional kernel.

Examples 4.25.

1. Let (D,ω) be a presymplectic distribution, where D is involutive and regular. Consider the complex Dirac structure $L(D_{\mathbb{C}}, i\omega_{\mathbb{C}})$, then the associated Dirac structure is given by $L_\Delta = L(D,\omega)$, since D is regular L_Δ is a Dirac structure.

2. If we consider the transverse CR structure (R, S, J) and its associated complex Dirac structure, we obtain that in this case $L_\Delta = R \oplus \operatorname{Ann} R$.

We do not have a direct relationship between K and L_Δ, but we have that

$$pr_{TM} K = \Delta_0 \subseteq \Delta = pr_{TM} L_\Delta \text{ and}$$

$$pr_{T^*M} K^\perp = \operatorname{Ann} D \subseteq \operatorname{Ann} \Delta_0 = pr_{T^*M} L_\Delta$$

The distribution L_Δ is always well defined. In Proposition 4.21 we ask for the complex Dirac to have constant real index and order in order to assure the smoothness of L_Δ. When there are variations of the real index or the order we can no longer assure the smoothness of L_Δ.

It is known that the leaves of the foliation associated to a real Dirac structure have the same parity. Given a complex Dirac structure L and a point $p \in M$, denote by $r(p)$ the real index of L at p, $s(p)$ to the order of L at p and $k(p)$ to the type of L at p. By Remark 3.14.5, we know that $\dim \Delta|_p = 2(n - k(p)) + r(p) - s(p)$. Hence, if we have a jumping of real index and order then we have that depending on $r - s$ the leaves could have different parities. Consider

$$\delta_L : M \to \mathbb{Z}_2$$

$$p \to r(p) - s(p)\ (mod\ 2) = s(p)\ (mod\ 2),$$

where the last equality follows from Corollary 3.35. Then we have the following.

Proposition 4.26. *Let L be a complex Dirac structure. If δ_L is not constant, then L_Δ is not smooth.*

Now we present some examples:

Example 4.27. *(Real index changing and constant order)* Let D be a regular involutive distribution and $\omega \in \wedge^2 D^*$ a presymplectic form with changes in the rank of its null-distribution. Consider $L = L(D_\mathbb{C}, i\omega_\mathbb{C})$; then $\operatorname{codim} D + \operatorname{rank} \ker \omega|_p = r(p)$. Since the changes in the rank of the null-distribution of a two-form is always modulo 2, we have that δ_L is constant. Indeed, we have that $L_\Delta = L(D, \omega)$ is a Dirac structure.

Example 4.28. *(Order changing with associated smooth Dirac structure)* Let L' be a Dirac structure and take $L = L'_\mathbb{C}$. Then, its associated Dirac structure is L' itself. Since $pr_{TM} L'$ could be non-regular, then the order of L changes but the Dirac structure remains smooth.

Example 4.29. *(Constant real index does not imply smoothness of the distribution L_Δ)* Let L be the complex Dirac structure of example b) of Section 3.5, we have that the real index is one but the order changes along the submanifold Z. We had observed that $\text{order}(L) = 1$ on Z, whereas $\text{order}(L) = 0$ on $M - Z$. By Proposition 4.26, we get that the bundle L_Δ is not a Dirac structure. Consequently, constant real index does not necessarily assure that L_Δ is a Dirac structure.

4.3 Split isotropic subbundles

In this section we focus on complex Dirac structure having associated an isotropic subbundle K that splits; we focus in the case of splitting K such that $pr_{TM} K = 0$ and observe how in this case we obtain a foliation where each leaf carries a generalized complex structure.

We observed in Examples 3.30 and 3.32 that some complex Dirac structures have an associated isotropic subbundle K which is the direct sum of a subspace of TM with a subspace of T^*M. In this section we study this kind of isotropic subbundles and the complex Dirac structures having them as associated isotropic subbundles. We give special attention to isotropic subbundles of the form $K = \operatorname{Ann} D$.

Definition 4.30. Consider a rank-r isotropic subbundle K of $TM \oplus T^*M$. We say that K **splits** if the exact sequence at (3.4)(extended to vector bundles) splits and $\Delta_0 = pr_{TM}K$ has constant rank.

Note that the condition on the exact sequence is equivalent to $pr_{TM}K = K \cap TM$. Consider $D = pr_{TM}K^\perp$, by the additivity of the ranks of the exact sequence (3.4) and the fact that $K \cap T^*M = \operatorname{Ann} D$, if K splits then D has constant rank. As a result,

$$K = \Delta_0 \oplus \operatorname{Ann} D \ \text{ and } \ K^\perp = D \oplus \operatorname{Ann} \Delta_0, \tag{4.2}$$

yielding that

$$K^\perp/K \cong D/\Delta_0 \oplus (D/\Delta_0)^*,$$

via the isomorphism

$$X + \xi + K \mapsto X + \Delta_0 + \widehat{\xi|_D},$$

where $\widehat{\xi|_D}$ is the unique linear map satisfying $\widehat{\xi|_D} \circ r = \xi|_D$, where $r : D \to D/\Delta_0$ is the quotient map.

Now, consider a complex Dirac structure L with real part K as above. By Proposition 3.7 this is equivalent to a map

$$\mathcal{J} : D/\Delta_0 \oplus (D/\Delta_0)^* \to D/\Delta_0 \oplus (D/\Delta_0)^*$$

$$\mathcal{J} = \begin{pmatrix} A & \Phi \\ \Omega & -A^* \end{pmatrix}$$

such that $\mathcal{J}^2 = -I$ and $\mathcal{J}^* + \mathcal{J} = 0$. By Lemma 4.17, we have that

$$L_\Delta = \{X + \xi \in D \oplus \operatorname{Ann} \Delta_0 \mid X + \Delta_0 = \Phi(\widehat{\xi|_D})\}.$$

If we take regular distributions $\Delta_0 \subseteq D \subseteq TM$ and $K = \Delta_0 \oplus \operatorname{Ann} D$, and consider the class of all complex Dirac structures with real part K, we have that inside this class the complex Dirac structures of extreme type are represented by presymplectic structures defined on D with kernel Δ_0 (type 0) and transverse CR structures (Δ_0, D, J) (type n).

There is a case when the splitting of K occurs naturally.

Lemma 4.31. *Let K be an isotropic subbundle of $TM \oplus T^*M$. We have $\operatorname{rank} K = \operatorname{corank}(pr_{TM}K^\perp)$ if and only if $\Delta_0 = 0$ and K splits (or equivalently $K = \operatorname{Ann} D$ for some subbundle of TM).*

Proof. By Lemma 3.10, $\dim K \cap T^*M = \dim \operatorname{Ann} pr_{TM}K^\perp = \dim \operatorname{Ann} D = r$. Therefore $K \cap T^*M = K$ and K splits. □

We easily see that $K = \operatorname{Ann} D$ and $K^\perp = D \oplus T^*M$. From the perspective of a complex Dirac structure, this is the case when the order and the real index of the complex Dirac structure are constant and equal. From now on we focus on complex Dirac structures with associated isotropic subbundle of the form $\operatorname{Ann} D$.

As a consequence of the previous lemma and the discussion at the beginning of the section, we obtain the following.

Corollary 4.32. *Let L be a complex Dirac structure with constant real index with associated isotropic subbundle* $\operatorname{Ann} D$*. Then, we have that the map associated to L is a map $\mathcal{J} : D \oplus D^* \to D \oplus D^*$ and its associated Dirac structure comes from the Poisson structure $\pi(\xi) = pr_D \circ \mathcal{J}(\xi|_D)$*

The map of the corollary above resembles a generalized complex structure. Although in general $D \oplus D^*$ does not inherit the structure of a Courant algebroid, we next see that the involutivity of D will assure that condition.

Lemma 4.33. *If D is involutive then $D \oplus D^*$ is a Courant algebroid.*

Proof. First we see how the bracket of $TM \oplus T^*M$ descends to $D \oplus D^*$. Since $D \oplus D^* = D \oplus T^*M / \operatorname{Ann} D$ and the involutivity of D implies that $\Gamma(D \oplus T^*M)$ is closed under the Courant-Dorfman bracket, it is enough to show that $\Gamma(\operatorname{Ann} D)$ is an ideal of $\Gamma(D \oplus T^*M)$. Consider $\xi \in \Gamma(\operatorname{Ann} D)$ and $Y + \eta \in \Gamma(D \oplus T^*M)$, then

$$[\xi, Y + \eta] = -\iota_Y d\xi.$$

Take $Z \in \Gamma(D)$,

$$\iota_Y d\xi(Z) = Y(\xi(Z)) - Z(\xi(Y)) - \xi([Y, Z]) = 0.$$

Consequently $[\xi, Y + \eta] \in \Gamma(\operatorname{Ann} D)$ and thus $\Gamma(\operatorname{Ann} D)$ is an ideal.

We can see that $(D \oplus D^*, \langle \cdot, \cdot \rangle_D, [\cdot, \cdot]_D, pr_D)$ is a Courant algebroid, where $\langle \cdot, \cdot \rangle_D$ is the canonical pairing of $D \oplus D^*$ and $[\cdot, \cdot]_D$ is the bracket inherited by $TM \oplus T^*M$. □

Under the hypothesis of the lemma above, the explicit form of the bracket of $D \oplus D^*$ is

$$[X + \xi, Y + \eta]_D = [X, Y] + \mathcal{L}_X^D \eta - \iota_Y d_D \xi,$$

where $X + \xi, Y + \eta \in D \oplus D^*$, d_D denotes the differential along D and

$$\mathcal{L}_X^D = \iota_X d_D + d_D \iota_X.$$

Also, the involutivity of L does not imply the integrability of D, as we saw in Remark 4.4.

Let $q : D \oplus T^*M \to \frac{D \oplus T^*M}{\operatorname{Ann} D} = D \oplus D^*$ denote the quotient map $q(X + \xi) = X + \xi|_D$. There exist a one-to-one correspondence between lagrangian subbundles of $(TM \oplus T^*M)_{\mathbb{C}}$ such that $L \cap \overline{L} = (\operatorname{Ann} D)_{\mathbb{C}}$ and lagrangian subbundles L_0 of $(D \oplus D^*)_{\mathbb{C}}$ such that $L_0 \cap \overline{L_0} = 0$, via the identification $L \mapsto q_{\mathbb{C}}(L)$, where $q_{\mathbb{C}}$ is the complexification of the map q. Since $L \cap \overline{L} = (\operatorname{Ann} D)_{\mathbb{C}}$, we have that $L \subseteq (D \oplus T^*M)_{\mathbb{C}}$. Then it is natural to ask whether this identification restricts to complex Dirac structures. So we have the following.

Proposition 4.34. *Let L be a lagrangian subbundle of $(TM \oplus T^*M)_{\mathbb{C}}$ such that $L \cap \overline{L} = (\operatorname{Ann} D)_{\mathbb{C}}$. If D is involutive, then L is involutive if and only if $q_{\mathbb{C}}(L)$ is involutive.*

Proof. First we note that

$$q[X + \xi, Y + \eta] = [q(X + \xi), q(Y + \eta)]_D,$$

for every $X + \xi, Y + \eta \in D \oplus T^*M$ as the following holds

$$(\mathcal{L}_X \eta - \iota_Y d\xi)|_D = \mathcal{L}_X^D \eta - \iota_Y d_D \xi.$$

Since $L \cap \overline{L} = (\operatorname{Ann} D)_{\mathbb{C}}$, we have that $L \subseteq (D \oplus T^*M)_{\mathbb{C}}$. So by the first part of the proof and the fact that $(q_{\mathbb{C}})^{-1}(q_{\mathbb{C}}(L)) = L$, the proposition holds. □

As an application of the proposition above we obtain a family of generalized complex structures on each leaf of the foliation associated to D.

Proposition 4.35. *Let L be a complex Dirac structure with constant real index r, order r and bundle map $\mathcal{J}$, such that $L \cap \overline{L} = (\operatorname{Ann} D)_{\mathbb{C}}$. If the distribution D is involutive, then $\mathcal{J}$ defines a family of generalized complex structures given by the restriction of $\mathcal{J}$ to the leaves of D.*

Proof. Let $N_{\mathcal{J}}$ denote the Nijenhuis tensor associated to $[\cdot,\cdot]_D$. In a similar way as in complex structures, we can see that L is involutive if and only if $N_{\mathcal{J}} = 0$. Let S be a leaf of D, when we restrict $\mathcal{J}$ to $TS \oplus T^*S$ we obtain a map

$$\mathcal{J}_S : TS \oplus T^*S \to TS \oplus T^*S$$

such that $\mathcal{J}_S^2 = -Id$ and $\mathcal{J}_S + \mathcal{J}_S^* = 0$. Finally note that $N_{\mathcal{J}_S} = N_{\mathcal{J}}|_S = 0$ and thus $\mathcal{J}_S$ is a generalized complex structure. □

We deduce from the previous proposition that one way to glue generalized complex structures on a regular foliation is via a complex Dirac structure of nonzero real index.

Remark 4.36. The complex Dirac structure having constant real index equal to its order are referred to as generalized CR structures in [27].

4.4 Generalized metrics and strictly pseudoconvex structures

In this section we present some ideas towards a generalized metric theory in the more general context of complex Dirac structures. We begin with a motivation for that construction. Let M be an oriented manifold and assume it admits a strictly pseudoconvex structure, i.e. it admits a codimension-one CR structure (D, J) such that D admits a contact form $\theta \in \Omega^1(M)$ and its associated Levi form $G_\theta(X, Y) = d\theta(X, JY)$ is positive or negative definite. For simplicity assume that G_θ is positive definite.

The contact form θ defines a presymplectic structure on the distribution D and thus a complex Dirac structure of real index one. This structure is $L_{-id\theta} = L(D_{\mathbb{C}}, -id\theta)$, it has associated the bundle map $\mathcal{J}_1 : D \oplus D^* \to D \oplus D^*$

$$\mathcal{J}_1 = \begin{pmatrix} 0 & -(d\theta)^{-1} \\ d\theta & 0 \end{pmatrix}.$$

On the other hand the CR structure (D, J) also has associated a complex Dirac structure of real index one, $L_{(D,J)} = L(\ker(J_{\mathbb{C}} - iId), 0)$ with associated bundle map $\mathcal{J}_2 : D \oplus D^* \to D \oplus D^*$

$$\mathcal{J}_2 = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}.$$

By the symmetry of G_θ, we have that $\mathcal{J}_1\mathcal{J}_2 = \mathcal{J}_2\mathcal{J}_1$. Consider $G = -\mathcal{J}_1\mathcal{J}_2$; we can see that

$$G = \begin{pmatrix} 0 & G_\theta^{-1} \\ G_\theta & 0 \end{pmatrix}.$$

This justifies the following definition.

Definition 4.37. Let K be an isotropic subbundle of $TM \oplus T^*M$. A K**-generalized metric** is a bundle map $G : K^\perp/K \to K^\perp/K$ such that $G^2 = Id$ and it is positive definite, i.e. $\langle G(e), e\rangle > 0$ for all nontrivial $e \in K^\perp/K$.

When $K = 0$ we retrieve the definition of generalized metric on $TM \oplus T^*M$ as defined in [23].

Our construction is mainly based on generalized Kahler structures. We recall that a generalized Kahler structure is a pair (L_1, L_2) of generalized complex structures on the same manifold with associated bundle maps $\mathcal{J}_1$ and $\mathcal{J}_2$, respectively, such that $\mathcal{J}_1\mathcal{J}_2 = \mathcal{J}_1\mathcal{J}_2$ and $G = -\mathcal{J}_1\mathcal{J}_2$ is a generalized metric. If (M, ω, J) is a Kahler structure, then $(L_{i\omega}, L_J)$ is a generalized Kahler structure. Note that an important ingredient for a generalized Kahler structure is the integrability of its underlying generalized complex structures. However, the example we presented at the beginning of the section fails in that, $L_{-id\theta}$ is not involutive, actually it is a nondegenerate structure, see Appendix A. This suggests the study of pairs (L_1, L_2) of lagrangian subbundles of $(TM \oplus T^*M)_{\mathbb{C}}$ such that $L_1 \cap \overline{L_1} = L_2 \cap \overline{L_2} = K_{\mathbb{C}}$, L_1 is nondegenerate (see Definition A.7), L_2 is a complex Dirac structure, $\mathcal{J}_1\mathcal{J}_2 = \mathcal{J}_1\mathcal{J}_2$ and $G = -\mathcal{J}_1\mathcal{J}_2$ is a K-generalized metric, where $\mathcal{J}_1$ and $\mathcal{J}_2$ are the bundle maps associated to L_1 and L_2, respectively. Something that we leave for future work.

Chapter 5

Splitting theorems

In this chapter we obtain a splitting theorem for complex Dirac structures with constant real index and order. In the first half of this chapter we give a general review of some the results from [10]. In the second half we adapt the previously presented ideas to proceed to the proof of the splitting theorem. The second part is completely independent of the first one, although we are inspired by it. Along this chapter we use the notation $\varphi^! L$ instead of $\mathcal{B}_\varphi(L)$ for the backward image of a Dirac structure L by the map φ.

5.1 Splitting theorems in Poisson and related geometries

In this section we make an overview of the techniques introduced in [10] where all the material presented here is from. One consequence of the results is the normal form for Dirac structures. No results of this section are needed in the proof of our main theorem, though they motivate it.

The flow of a complete vector field $X \in \mathfrak{X}(M)$ is the one-parameter group $(\varphi_s)_{s\in\mathbb{R}}$ defined by the following

$$X(f) = \frac{d}{dt}\bigg|_{t=0} \varphi^*_{-s} f.$$

Let E be a vector bundle over M. For any $t \in \mathbb{R}$ consider the bundle map $\kappa_t : E \to E$, $\kappa_t e = te$; note that $\kappa_t \in \mathrm{Aut}(E)$ for every $t \in \mathbb{R} - \{0\}$, where $\mathrm{Aut}(E)$ denotes the automorphism group of the vector bundle E. Moreover, $s \mapsto \kappa_{e^{-s}}$ is a one-parameter group and since $\mathrm{Aut}(E) \subseteq \mathrm{Diff}(E)$ we have a one-parameter group in $\mathrm{Diff}(E)$ and thus a vector field on E.

Definition 5.1. Given a vector bundle over a manifold, the **Euler vector field** is the vector field defined by the one-parameter group $s \mapsto \kappa_{e^{-s}}$ usually denoted by $\mathcal{E}$.

The Euler vector field has the following local description, let (x_i, y_i) be a fibred local coordinates for E, where x_i are the fibre directions and y_i the base directions, we obtain that locally

$$\mathcal{E} = \sum_i x_i \frac{\partial}{\partial x_i}.$$

Given a submanifold N of M, we denote the normal bundle of N by

$$\nu(M, N) = TM|_N / TN,$$

we usually denote its projection map by $p : \nu(M, N) \to N$. Let $\varphi : (M', N') \to (M, N)$ be a map of pairs, i.e. a map $\varphi : M \to M$ satisfying that $\varphi(N') \subseteq N$. Then we can associate a map $\nu(\varphi) : \nu(M', N') \to \nu(M, N)$.

Given $X \in \mathfrak{X}(M)$ tangent to N, then it defines a map of pairs $X : (M, N) \to (TM, TN)$. Using the fact that $T\nu(M, N) = \nu(TM, TN)$, we have that $\nu(X)$ is a vector field on $\nu(M, N)$.

Definition 5.2. Let N be a submanifold of M. A **tubular neighbourhood embedding** is an embedding $\psi : \nu(M, N) \to M$ such that it takes N_0, the zero section of $\nu(M, N)$, to N and $\nu(\psi) = Id$, where $\nu(\psi)$ is induced by $\psi : (\nu(M, N), N_0) \to (M, N)$ and we are using the identification $\nu(\nu(M, N), N) = \nu(M, N)$.

Let N be a submanifold and X be a vector field on M that is tangent to N. We say that X is **linearizable** if there exist a tubular neighborhood embedding ψ such that $\nu(X)$ agrees with $\psi^* X$ on a neighborhood of N.

Lemma 5.3. *Consider a submanifold N of M and X a vector field of M such that $X|_N = 0$ and $\nu(X)$ is the Euler vector field associated to $\nu(M, N)$. Then X is linearizable.*

Euler vector field are just defined on vector bundles, now we present an extension of this definition to manifolds.

Definition 5.4. Let N be a submanifold of M and let $X \in \mathfrak{X}(M)$. The vector field X is called **Euler-like** (along N) if it is complete, $X|_N = 0$ and $\nu(X)$ is the Euler vector field of $\nu(M, N)$.

Given a submanifold N of M, there is a one-to-one correspondence between tubular neighbourhood embeddings and Euler-like vector fields. Given a tubular neighbourhood embedding ψ, we retrieve a Euler-like vector field just by pushing forward the Euler vector field associated to the normal bundle $\nu(M, N)$ to $\psi(\nu(M, N))$ via ψ. For the other side we have the following.

Proposition 5.5. *Let $X \in \mathfrak{X}(M)$ be a Euler-like vector field along $N \subseteq M$. Then, there exists a unique tubular neighborhood embedding $\psi : \nu(M, N) \to M$ such that*

$$\mathcal{E} \sim_\psi X,$$

where $\mathcal{E}$ denotes the Euler vector field of $\nu(M, N)$.

The main feature of the technique developed in [10] is the production of normal forms for geometrical structures, such as Lie algebroids or Dirac structures L with anchor map ρ, from a special section $\epsilon \in \Gamma(L)$ such that $\epsilon|_N = 0$ and $\rho(\epsilon)$ is an Euler-like vector field for certain submanifold N. In the following lemma we observe a condition for the existence of such sections.

Lemma 5.6. *Let L be a Lie algebroid over M with anchor map ρ and let N be a transversal submanifold, i.e. $\rho(L)|_N + TN = TM|_N$. Then there exists $\epsilon \in \Gamma(L)$ such that $\epsilon|_N = 0$ and $\rho(\epsilon)$ is an Euler-like vector field.*

The existence of sections as in the corollary carry many consequences. We recall that given a diffeomorphism $\psi : M \to N$, its generalized differential is defined as

$$\mathbb{T}\psi : TM \oplus T^*M \to TN \oplus T^*N$$

$$\mathbb{T}(X + \xi) = \psi_* X + (\psi^{-1})^* \xi.$$

Proposition 5.7. *(Normal form for Dirac structures) Let $L \subseteq TM \oplus T^*M$ be a Dirac structure and $N \overset{\iota}{\hookrightarrow} M$ a submanifold transversal to L. Choose $\varepsilon = X + \alpha \in \Gamma(L)$ with $\varepsilon|_N = 0$, X an Euler-like vector field along N and let $\psi : \nu(M,N) \to M$ be the associated tubular neighborhood embedding. Then, there exist a neighbourhood U of N and a two-form $B \in \Omega^2(\nu(M,N))$ such that $\mathbb{T}\psi : T\nu(M,N) \oplus T^*\nu(M,N) \to TM \oplus T^*M$ restricts to an isomorphism of Dirac structures*

$$L|_U \cong e^B(p^!\iota^!L),$$

where $B = \int_0^1 \frac{1}{\tau}\kappa_\tau^*\psi^* d\alpha d\tau$.

Proof. We give a sketch of the proof. Let $\varepsilon = X+\alpha \in \Gamma(L)$ as specified in the hypothesis. Denote by φ_s the flow associated to X and by ψ its tubular neighbourhood embedding. By Proposition 5.5, X and the Euler vector field $\mathcal{E}$ of $\nu(M,N)$ are related by ψ and so $\varphi_s \circ \psi = \psi \circ \kappa_{e^{-s}}$, recall that the flow of $\mathcal{E}$ is given by $s \mapsto \kappa_{e^{-s}}$. Consequently, we have $\lambda_s \circ \psi = \psi \circ \kappa_s$, for $s > 0$, where $\lambda_s = \varphi_{-log(s)}$. Consider

$$B_t = \int_t^1 \frac{1}{\tau}\kappa_\tau^*\psi^* d\alpha d\tau.$$

Let $L_t = e^{B_t}(\kappa_t^!\psi^!L)$. It is proved that L_t is independent of t (see proof of Theorem 5.10), when $t \geq 0$. So, we have that $L_1 = L_0$, i.e.

$$\psi^!L = L_1 = L_0 = e^{B_0}(\kappa_0^!\psi^!L) = e^{B_0}(p^!\iota^!L).$$

Taking $B = B_0$, the result holds. □

The following lemma is straightforward.

Lemma 5.8. *Let M and N two manifolds and L be a Dirac structure on a manifold M. Then, we have*

$$pr_M^!L = L \times TN,$$

where $pr_M : M \times N \to M$ is the usual projection map.

As a consequence of the previous proposition we present the splitting theorem for Dirac structures.

Corollary 5.9. *(Splitting theorem for Dirac structures, [7]) Let L be a Dirac structure on M and $p \in M$. Let $N \overset{\iota}{\hookrightarrow} M$ be a submanifold containing p, such that T_pN is complement to $P = pr_{TM}L|_p$. Then, there exist a neighbourhood U of p, a two-form $B \in \Omega^2_{cl}(P \times N)$ such that*

$$L|_U \cong e^B(\iota^!L \times L_\omega) \subseteq T(P \times N) \oplus T^*(N \times P),$$

where L_ω is the Dirac structure associated to the presymplectic leaf passing through p. Moreover, there exist a Poisson structure π over N such that $\iota^!L = Graph(\pi)$ and π vanishes at p.

Proof. Since $T_pN \oplus P = T_pM$, we obtain, using an adapted chart of N, that exists a neighbourhood U' of p such that $N \times P = U'$, here we shrink N if necessary. Note that $\nu(M,N) = TM|_N/TN = N \times P$ and the projection map $p : \nu(M,N) \to N$ became the projection $p : N \times P \to N$ and by Lemma 5.8, $p^!\iota^!L = \iota^!L \times TP$. By Lemma 5.6, there exists a section $\varepsilon \in \Gamma(L)$ such that $\varepsilon|_N = 0$ and $pr_{TM}\varepsilon$ is an Euler-like vector field. Then we apply Proposition 5.7 choosing ε, so there exist a neighbourhood U of N and a closed two-form B' such that

$$L|_U \cong e^{B'}(p^!\iota^!L) = e^{B'}(\iota^!L \times TP).$$

Let $\omega_0 \in \Omega^2(P)$ be he presymplectic structure such that locally $(P, \omega_0) \cong (S, \omega)$, where (S, ω) is the presymplectic leaf passing through p. Consider $B = B' - pr_N^* \omega$, where $pr_N : N \times P \to N$ is the projection map. Then

$$L|_U \cong e^{B'}(\iota^! L \times TP) = e^B(\iota^! L \times L(TP, \omega_0)) \cong e^B(\iota^! L \times L_\omega).$$

The presymplectic foliation associated to $\iota^! L$ is given by the intersection of the leaves of the presymplectic foliation of L with N. Consequently, the presymplectic leaf of $\iota^! L$ passing through p is $\{p\}$ and by Proposition 2.39, there exist a neighbourhood $V \subseteq U \cap U'$ of p and a Poisson bivector π such that $\iota^! L = Graph(\pi)$ and vanishing at p. □

In the next section, we will prove a complex Dirac version of the previous result, more specifically a splitting theorem for complex Dirac structures L with constant real index and order. As we have seen the splitting theorem for Dirac structures has two well identified factors, the presymplectic leaf passing through p carrying the graph of its presymplectic two-form and a local transversal submanifold $N \overset{\iota}{\hookrightarrow} M$ carrying the Dirac structure $\iota^! L$ which is the graph of a Poisson bivector. On the other hand the splitting theorem for complex Dirac structure with constant real index and order will have one factor which consists of the presymplectic leaf associated to L passing through p carrying the complex Dirac structure associated to its presymplectic two-form (note that in the previous case, the presymplectic leaf is locally isomorphic with the image of the anchor map with the canonical presymplectic structure differing to this case when the presymplectic leaf is locally isomorphic to the real part of the image of the anchor map) and a second factor that will be a local transversal submanifold $N \overset{\iota}{\hookrightarrow} M$ carrying the complex Dirac structure $\iota^! L$ with same constant real index and order having its associated Poisson bivector vanishing at p.

5.2 Splitting theorem for complex Dirac structures with constant real index and order

In Chapter 3, we proved that a complex Dirac structure L on a vector space is equivalent (up to B-transformations) to the product of its associated presymplectic structure (Δ, ω_Δ) and a CR structure. This result is no longer true for complex Dirac structures on manifolds as we observed in Section 4.1.2 that a complex Dirac structure with constant real index and order and type 0 is not necessarily a globally defined B-transformation of a presymplectic structure. Instead, we will prove that around any point $p \in M$, a complex Dirac structure with constant real index and order is a B-transformation of a presymplectic structure and a complex Dirac structure with same constant real index and order with associated Poisson bivector vanishing at p. In other words we will prove a local splitting theorem.

Theorem 5.10. *Let L be a complex Dirac structure of $(TM \oplus T^*M)_{\mathbb{C}}$ with constant real index r and order s, a point $p \in M$ be a point of type k and n be a nonnegative integer such that $\dim M = 2n + r$. Consider a $(2k + s)$-dimensional submanifold $N \subseteq U \overset{\iota}{\hookrightarrow} M$ transversal to L_Δ at p, i.e. $T_pN \oplus \Delta|_p = T_pM$. Then there exist a neighbourhood U of p and a closed real two-form B defined on U, such that*

$$L|_U \cong e^B(\iota^! L \times L_{i\omega}),$$

where $\iota^! L$ is a complex Dirac structure with constant real index s and order s and having associated Poisson bivector vanishing at p, $L_{i\omega}$ is the complex Dirac structure associated to the presymplectic leaf S passing through p.

The proof of the previous propositions is divided on two parts:

Step 1: The study of the properties of backward images of transversal submanifolds to L, Lemma 5.11 and Lemma 5.13.

Step 2: To prove that given a submanifold $N \overset{\iota}{\hookrightarrow} M$ with dimension complementary to the dimension of the presymplectic leaf S passing through p and transversal to L_Δ at p, we can find a section $\varepsilon \in \Gamma(L)$ such that $\varepsilon|_N = 0$ and $pr_{TM}\varepsilon$ is Euler-like along N.

Also, we will see in the next section that the splitting theorem for complex Dirac structures with constant real index and order induces a splitting theorem on their underlying Dirac structures which coincides with the usual splitting theorem for Dirac structures as stated in Corollary 5.9 in the following way: the splitting for a complex Dirac structure L with constant real index and order around a point p is

$$L|_U \cong e^B(\iota^! L \times L_{i\omega}),$$

whereas the splitting for its associated Dirac structure L_Δ is

$$L_\Delta|_U \cong e^{B'}(\iota^! L_\Delta \times L_\omega),$$

where $B - B' = pr_N^*\omega$. Note that $\iota^! L_\Delta$ and L_ω are the real Dirac structures associated to $\iota^! L$ and $L_{S,\omega}$, respectively. Moreover, $\iota^! L_\Delta = Graph(\pi_N)$, where π_N is the Poisson bivector associated to $\iota^! L$ which vanishes at p.

5.2.1 Step 1

As we mentioned above, the step 1 is about the properties of $\iota^! L$. We see along this section that the Dirac structure associated to $\iota^! L$ is the backward image of the Dirac structure associated to L. Furthermore, the complete transversality of N with respect to L_Δ at a point p implies that the Dirac structure associated to $\iota^! L$ is near p the graph of a Poisson bivector vanishing at p. At the end of this section, we also provide a result about the additivity of the real index and order with respect to the product.

Let $N \overset{\iota}{\hookrightarrow} M$ be a submanifold and L a complex Dirac structure over M with $E = pr_{TM_\mathbb{C}} L$, $\Delta = \mathrm{Re}(E \cap \overline{E})$ and ε a skew-symmetric bilinear map such that $L = L(E, \varepsilon)$. Assume that $\iota^! L$ is a complex Dirac structure. Let $E_N = pr_{TN_\mathbb{C}} \iota^! L$ and $\Delta_N = \mathrm{Re}(E_N \cap \overline{E_N})$; then

$$E_N = E|_N \cap TN_\mathbb{C} \qquad \text{and} \qquad \Delta_N = \Delta|_N \cap TN.$$

Since $\iota^! L$ is a lagrangian subbundle, then there exists a skew-symmetric bilinear map $\varepsilon_N \in \wedge^2 E_N{}^*$ such that $\iota^! L = L(E_N, \varepsilon_N)$. Note that $(\varepsilon_N)_n = (\varepsilon|_{E \cap TN_\mathbb{C}})_n$ for all $n \in N$. The Dirac structure associated to $\iota^! L$ is, by definition, $L_{\Delta_N} = L(\Delta_N, \omega_{\Delta_N})$, where $\omega_{\Delta_N} = \mathrm{Im}\, \varepsilon_N|_{\Delta_N}$. So we have obtained the following:

Lemma 5.11. *Let L be a complex Dirac structure and $N \overset{\iota}{\hookrightarrow} M$ be a submanifold. If L_Δ and L_{Δ_N} are Dirac structures, then $\iota^! L_\Delta = L_{\Delta_N}$.*

A point of a generalized complex structure L is called of **complex type** if L has maximum type at p. This is equivalent to its associated Poisson bivector vanishes at p. In our context we have the following definition.

Definition 5.12. Let L be a complex Dirac structure. We say that a point $p \in M$ is of CR-type if $L_\Delta|_p = T_p^*M$. We say that L is of CR-type if it has real index constant and equal to its order and every point of M is of CR-type.

We easily see that if a complex Dirac structure L with constant real index and order admits a point of CR-type, then L has real index and order coinciding and its associated Poisson bivector vanishes at p; additionally L has maximum type at p. So, by Proposition 4.9, a complex Dirac structure of CR-type is a transformation by a real two-form of a CR structure. When L has real index constant and equal to its order, its associated Dirac structure is the graph of a Poisson structure(Corollary 4.32). So the set of points of CR-type is the zero set of the Poisson structure.

Proposition 5.13. *Let L be a complex Dirac structure with constant real index and order, and $N \overset{\iota}{\hookrightarrow} M$ a submanifold. If $T_pN \oplus \Delta|_p = T_pM$, then near p, $\iota^! L$ is a complex Dirac structure and L_{Δ_N} is a Dirac structure. Moreover, near p, L_{Δ_N} is the graph of a Poisson bivector vanishing at p.*

Proof. Let l_p be the leaf of Δ passing through p. Since Δ is integrable, it satisfies the local foliation property, so there exists a chart $(y_1, \dots y_m)$ on a neighbourhood $U = U(\lambda)$ of p such that $\{y_{d+1} = \dots = y_m = 0\} = U \cap l_p$ and each dimensional disk $\{y_{d+1} = c_{d+1} \dots = y_m = c_m\}$ is contained in some leaf. As a consequence each leaf close to l_p contains a disk. Since $T_pN \oplus T_pl_p = T_pN \oplus \Delta|_p = T_pM$, we have that each disk close enough to l_p intersects with N in a single point. Let B such disk and n be its intersection with N; since the disks form a regular foliation, $T_nN \oplus T_nB = T_nM$. Let $n \in N$ be a point near p and B a disk containing it; then $T_nN + T_nl = T_nN + \Delta|_n = T_nM$, where l is the leaf containing B. So we have that there exist a neighbourhood U of p such that, N is transversal to L_Δ, i.e. $TN|_{U\cap N} + \Delta|_{U\cap N} = TM|_{U\cap N}$, after taking complexification and observing that $\Delta_\mathbb{C} \subseteq E$, we obtain $(TN_\mathbb{C})|_{U\cap N} + E|_{U\cap N} = (TM_\mathbb{C})|_{U\cap N}$, i.e. N is transversal to L. By Remark 2.51, $\iota^! L|_U$ is a complex Dirac structure and by Proposition 2.42, $L_{\Delta_N}|_U$ is a Dirac structure. Lemma 5.11 tells us that $L_{\Delta_N}|_U = \iota^! L_\Delta$, so its presymplectic leaves are the intersection of $N \cap U$ with the presymplectic leaves of L_Δ. Consequently, we have that the leaf of $L_{\Delta_N}|_U$ passing through p is a single point (we shrink U if necessary) and by Proposition 2.39 the last part of the lemma follows. ☐

The following result about the additivity of the real index and order of the product of complex Dirac structures will be useful in the proof of Theorem 5.10.

Lemma 5.14. *Let L_1 and L_2 be two complex Dirac structures over the manifolds M_1 and M_2 respectively and let $\pi_i : M_1 \times M_2 \to M_i$ for $i = 1, 2$, be the projection maps. Denote by K_1 and K_2 to the real part of L_1 and L_2 respectively (which are not necessarily smooth). Then the complex Dirac structure $L_1 \times L_2$ defined over $M_1 \times M_2$ has real part $\pi_1^* K_1 \oplus \pi_2^* K_2$ (which is not necessarily smooth).*

Proof. Let $X + \xi$ be a real element of $L_1 \times L_2 = \pi_1^* L_1 \oplus \pi_2^* L_2$. Since $X + \xi$ is real it decomposes as $X + \xi = X_1 + \xi_1 + X_2 + \xi_2$, where $X_1 + \xi_1 \in \pi_1^* L_1$ and $X_2 + \xi_2 \in \pi_2^* L_2$ are real, so $X_1 + \xi_1 \in \pi_1^* K_1$ and $X_2 + \xi_2 \in \pi_2^* K_2$. Consequently, we have that $\mathrm{Re}(L_1 \times L_2) \subseteq \pi_1^* K_1 \oplus \pi_2^* K_2$. The other inclusion is straightforward. ☐

Corollary 5.15. *Let L_1 and L_2 be two complex Dirac structures over the manifolds M_1 and M_2. The real index of $L_1 \times L_2$ at $(p_1, p_2) \in M_1 \times M_2$ is the sum of the real index of L_1 at p_1 with the real index of L_2 at p_2. The same happens for the order.*

Proof. The additivity of the real index is a consequence of the lemma above. The additivity of the order follows from the fact that $(\pi_1^* K_1 \oplus \pi_2^* K_2)^\perp = \pi_1^*(K_1^\perp) \oplus \pi_2^*(K_2^\perp)$. ☐

5.2.2 Step 2

In this section we focus on finding a section of $\Gamma(L)$ with the properties specified in step 2 above.

Lemma 5.16. *Let L be a complex Dirac structure with constant real index r and order s, a point $p \in M$ of type k and let $N \overset{\iota}{\hookrightarrow} M$ be a submanifold such that $T_pN \oplus \Delta|_p = T_pM$. Then there exist a neighbourhood U of p, a diffeomorphism $\psi : N \times \mathbb{R}^{2(n-k)+r-s} \to U$ sending $N \times 0$ to N and $\{p\} \times \mathbb{R}^{2(n-k)+r-s}$ to the presymplectic leaf passing through p, and a section $\varepsilon = X + i\alpha + \beta \in \Gamma(\psi^! L)$ such that $\varepsilon|_{N\times 0} = 0$, X is Euler-like,*

$$\alpha = \sum_{i=1}^{n-k} q_i dp_i - p_i dq_i$$

and $\beta \in \Gamma(T^*M)$ where $(q_1, \ldots, q_{n-k}, p_1, \ldots p_{n-k}, z_1, \ldots z_{r-s})$ are the coordinates of $\mathbb{R}^{2(n-k)+r-s}$.

Proof. Since L has constant real index r and order s, we have that $\operatorname{rank} \ker \omega_\Delta = r - s$ and the null foliation of L_Δ is regular. Since the type of L at p is k, then by Lemma 3.14, the dimension of the presymplectic leaf passing through p is $2(n-k) + r - s$.

Consider a small enough neighbourhood U of p such that the null foliation is simple. Let P denote the leaf space associated to the null foliation, with submersion map $u : U \to P$. By Proposition 2.44, P inherits a Poisson structure π from L_Δ and the presymplectic leaf passing through p descends via u to a $2(n-k)$-dimensional symplectic leaf of the Poisson structure passing through $u(p)$. Since Δ_0 is regular and $T_pN \cap \Delta_0|_p = 0$, we have that $TN|_{U\cap N} \cap \Delta_0|_{U\cap N} = 0$ and that $u|_{U\cap N}$ is a diffeomorphism (again we shrink U if necessary). So $N_0 = u(U \cap N)$ is transversal to π at p. By applying the rank theorem to u and then applying the Weinstein splitting theorem around $u(p)$, [39, Theorem 1.4.5], with transversal $N \cong N_0$, we can assume that $M = N \times \mathbb{R}^{2(n-k)+r-s}$ with coordinates (y_k) for the submanifold N, $(q_1, \ldots q_{n-k}, p_1 \ldots p_{n-k}, z_1, \ldots z_{r-s})$ for $\mathbb{R}^{2(n-k)+r-s}$; leaf space $P = N \times \mathbb{R}^{2(n-k)}$ with coordinates (y_k, q_i, p_i) and with $u = pr_{N\times\mathbb{R}^{2(n-k)}}$ (the projection from M to its first coordinates, deleting the last $r-s$ coordinates) and the Poisson structure splits as $\pi = \pi_0 + \pi_N$, where $\pi_0 = \sum \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} \in \wedge^2(\mathbb{R}^{2(n-k)+r-s})$ and $\pi_N \in \wedge^2(TN)$ a Poisson bivector vanishing at p. Furthermore, the Poisson structure π on $N \times \mathbb{R}^{2(n-k)}$ is such that $u^! L_\pi = L_\Delta$.

Since L has real index r, order s and we have that $\ker \omega_\Delta = \Delta_0 = \mathbb{R} \cdot (\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{r-s}})$, by Lemma 3.3, we obtain a frame

$$\{\frac{\partial}{\partial z_1} + \zeta_1, \ldots, \frac{\partial}{\partial z_{r-s}} + \zeta_{r-s}, \zeta_{r-s+1}, \ldots, \zeta_r\} \tag{5.1}$$

for K, where $\zeta_j \in \Gamma(T^*M)$ and ζ_j never vanishes whenever $j \geq r - s + 1$ and could vanish whenever $j \leq r - s$.

Consider

$$X_0 = \sum_i (p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i}) \in \mathfrak{X}(N \times \mathbb{R}^{2n-2k}) \text{ and}$$

$$\alpha_0 = \sum_i (q_i dp_i - p_i dq_i) \in \Omega^1(N \times \mathbb{R}^{2n-2k}).$$

We observed that, $\pi(\alpha_0) = \pi_0(\alpha_0) = X_0$ and thus $X_0 + \alpha_0 \in L_\pi$.

Now consider $\alpha = q^*\alpha_0 \in \Omega^1(M)$, where $q : TM \to TM/\Delta_0$ is the quotient map; note that q coincides with u_* since P is the leaf space of the foliation associated to Δ_0. Since $u^! L_\pi = L_\Delta$, we have that taking

$$Y = \sum_i (p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i}) \in \mathfrak{X}(N \times \mathbb{R}^{2n-2k+r-s}),$$

we obtain that $Y + \alpha \in L_\Delta$. Denote by $\mathcal{J}$ the associated bundle map to L. By Lemma 4.17, there exists $\beta_0 \in \Gamma(T^*M)$ such that $\mathcal{J}(\alpha + K) = Y + \beta_0 + K$. If we evaluate the previous expression on points of N, we observe that $\beta_0|_N \in (K \cap T^*M)|_N$. By equation (5.1),

$$\{\zeta_{r-s+1}, \dots, \zeta_r\}$$

is a frame for $K \cap T^*M$. So, there exist functions $c_j \in C^\infty(N)$, for $j = r - s + 1, \dots, r$ such that $-\beta_0|_N = \sum_{j=r-s+1}^r c_j \zeta_j|_N$. Extending the functions c_j to M, we obtain a section $\beta_1 \in K \cap T^*M$ such that $\beta_1|_N = -\beta_0|_N$. As a result, considering $\beta_2 = \beta_0 + \beta_1$, we obtain that $\mathcal{J}(\alpha + K) = Y + \beta_0 + K = Y + \beta_0 + \beta_1 + K = Y + \beta_2 + K$, where $\beta_2|_N = 0$.

Note that $\mathcal{J}(\alpha + K) + i(\alpha + K)$ is always in $\Gamma(L/L \cap \overline{L})$. Thus,

$$\mathcal{J}(\alpha + K) + i(\alpha + K) = Y + i\alpha + \beta_2 + K_{\mathbb{C}} = Y + i\alpha + \beta_2 + \sum z_j (\frac{\partial}{\partial z_j} + \zeta_j) + K_{\mathbb{C}}.$$

Let $X = Y + \sum z_j \frac{\partial}{\partial z_j}$ and $\beta = \beta_2 + \sum z_j \zeta_j$, we have that

$$X + i\alpha + \beta + K_{\mathbb{C}} \in \Gamma(L/L \cap \overline{L}).$$

Consequently, we obtain that

$$\varepsilon = X + i\alpha + \beta \in \Gamma(L). \tag{5.2}$$

We see that X is the Euler vector field of the trivial bundle $N \times \mathbb{R}^{2n-2k+r-s}$ over N and $\varepsilon|_N = 0$, since $N = \{p_i = q_j = z_k = 0\}$. □

5.2.3 Proof of Theorem 5.10

Before we prove the splitting theorem, we make a review of the flow associated to derivations.

Proposition 5.17. *[23] Let $(X, B) \in \mathfrak{X}(M) \times \Omega^2_{cl}(M)$ be an infinitesimal automorphism of the Courant algebroid $TM \oplus T^*M$(see Section 2.1), with X complete and let φ_s be its flow. Then (φ_s, γ_s) is the one-parameter group associated to (X, B), where*

$$\gamma_s = -\int_0^s (\varphi_u)^* B du.$$

Remark 5.18. Consider $X + \xi \in \Gamma(TM \oplus T^*M)$, then $ad_{X+\xi}$ is an infinitesimal automorphism and so is a derivation of the Courant algebroid $TM \oplus T^*M$ as a consequence it has associated a flow by linear automorphism of $TM \oplus T^*M$, which in this case is the one-parameter group $(\varphi_s, \gamma_s) = \mathbb{T}\varphi_s \circ e^{\gamma_s}$; note that the main property of the flow (φ_s, γ_s) is that

$$\frac{d}{dt}(\varphi_t, \gamma_t)(Y + \eta) = \frac{d}{dt}\mathbb{T}\varphi_t \circ e^{\gamma_t}(Y + \eta) = [X + \xi, Y + \eta].$$

If L is a real Dirac structure and $X + \xi \in \Gamma(L)$, then $[X + \xi, \Gamma(L)] \subseteq \Gamma(L)$. Implying that its flow preserves L, $\mathbb{T}\varphi_s \circ e^{\gamma_s}(L) = L$ and so $\mathbb{T}\varphi_{-s}(L) = e^{\gamma_s}(L)$. Exactly the same applies to $(TM \oplus T^*M)_{\mathbb{C}}$ and complex Dirac structures.

Now we are prepared for the proof of Theorem 5.10.

Proof of Theorem 5.10. By Lemma 5.16, we can assume that $M = N \times \mathbb{R}^{2(n-k)+r-s}$ and that we have found a section $\varepsilon = X + i\alpha + \beta \in \Gamma(L)$, where $X \in \Gamma(TM)$, $\alpha, \beta \in \Gamma(T^*M)$ such that X is the Euler vector field associated to the trivial bundle $N \times \mathbb{R}^{2(n-k)+r-s}$ and $\varepsilon|_N = 0$. Note that the flow of X is $\varphi_t = \kappa_{e^{-t}}$, where κ_t denotes the multiplication by t on the fibres of $N \times \mathbb{R}^{2n-2k+r-s}$. Consider the following two-forms

$$\omega_t = \int_t^1 \frac{1}{\tau}\kappa_\tau^*(d\alpha)d\tau, \tag{5.3}$$

$$B_t = \int_t^1 \frac{1}{\tau}\kappa_\tau^*(d\beta)d\tau \tag{5.4}$$

and also consider

$$L_t = e^{B_t + i\omega_t}(\kappa_t^! L).$$

Let $\omega = \sum dq_i \wedge dp_i \in \Omega^2(N \times \mathbb{R}^{2n-2k+r-s})$; we have that

$$\omega_0 = \int_0^1 \frac{1}{\tau}\kappa_\tau^*(d\alpha)d\tau = \int_0^1 \frac{1}{\tau}\kappa_\tau^*(d(\sum_i (q_i dp_i - p_i dq_i)))d\tau = 2\int_0^1 \frac{1}{\tau}\kappa_\tau^*\omega d\tau = \omega.$$

Let $p : M = N \times \mathbb{R}^{2(n-k)+r-s} \to N$ denote the projection and ι denote the inclusion $N \xhookrightarrow{\iota} M$. Since $\kappa_0 = \iota \circ p$, we have that

$$L_0 = e^{B_0 + i\omega_0}(\kappa_0^! L) = e^{B_0 + i\omega_0}(p^! \iota^! L). \tag{5.5}$$

We will prove that L_t is independent of t. If that happens, then $L_0 = L_1$ and since $L_1 = L$, we would obtain that

$$e^{-B_0}L = e^{i\omega_0}(p^! \iota^! L).$$

By Proposition 5.17 and Remark 5.18, since $X + i\alpha + \beta \in \Gamma(L)$ and L is involutive, $X + i\alpha + \beta$ induces an infinitesimal automorphism $(X, -d\beta - id\alpha)$ with associated one-parameter group of automorphisms $(\kappa_{e^{-s}}, \sigma_s)$, where

$$\sigma_s = -\int_0^s \kappa_{e^{-u}}^*(d\beta + id\alpha)du \in \Omega^2(M).$$

Implying that

$$(\kappa_{e^{-s}})^! L = \mathbb{T}\kappa_{e^s}(L) = e^{\sigma_s}L.$$

After applying the substitution rule using the function $u = -log(\tau)$ we get that

$$\sigma_s = -\int_{e^{-s}}^1 \frac{1}{\tau}\kappa_\tau^*(d\beta + id\alpha)d\tau$$

and taking $s = -log(t)$ we get that

$$\sigma_{-log(t)} = -\int_t^1 \frac{1}{\tau}\kappa_\tau^*(d\beta + id\alpha)d\tau = -(B_t + i\omega_t).$$

Then,

$$\kappa_t^! L = e^{\sigma_{-log(t)}}L = e^{-(B_t + i\omega_t)}L \tag{5.6}$$

and thus $L_t = L$, for all $t > 0$. By continuity, it follows that $L_0 = L$.

Finally, it follows from equations (5.5) and (5.6) that

$$e^{-B}L = e^{i\omega}(p^! i^! L) = e^{i\omega}(i^! L \times T\mathbb{R}^{(2n-k)+r-s}) = \iota^! L \times L_{i\omega}.$$

Now we focus on the properties of $\iota^! L$. By Corollary 5.15, $\iota^! L$ has constant real index s and order s. By Proposition 5.13, $\iota^! L$ is a complex Dirac structure with associated Poisson structure vanishing at p. Taking $B = B_0$, the theorem holds. □

Remark 5.19. As we mentioned at the beginning of Section 5.2, Theorem 5.10 induces a splitting on the associated Dirac structures. Let L be a complex Dirac structure with constant real index r and order s, $p \in M$ a point of type k and chose a $N \overset{\iota}{\hookrightarrow} M$ submanifold completely transversal to L_Δ at p, i.e. $T_pN \oplus \Delta|_p = T_pM$; then by Theorem 5.10, there exist a closed two-form B and a neighbourhood U of p such that $L|_U \cong e^B(\iota^! L \times L_{i\omega})$, where $L_{i\omega}$ denotes the complex Dirac structure associated to the presymplectic leaf of L passing through p. Now we prove that

$$L_\Delta|_U \cong e^{B'}(Graph(\pi_N) \times L_\omega),$$

where π_N is the Poisson structure associated to $\iota^! L$, L_ω is the real Dirac structure associated to the presymplectic leaf (S, ω) of L(and so of L_Δ) passing through p and $B - B' = pr_N^*\omega$. By Lemma 5.16, there exists a local section $\varepsilon = X + \alpha + i\beta \in \Gamma(L)$ such that $\varepsilon|_N = 0$ and X is an Euler-like vector field along N. By Lemma 4.16, $\hat{\varepsilon} = X + \beta \in \Gamma(L_\Delta)$ and satisfies the same properties as ε. By Corollary 5.9, applied to L_Δ, using explicitly the section $\hat{\varepsilon}$, we have that

$$L_\Delta|_U \cong e^{B'}(Graph(\pi) \times L_\omega),$$

where L_ω is the real Dirac structure associated to the presymplectic leaf (S, ω) passing through p and π is a Poisson structure over N such that $\iota^! L_\Delta = Graph(\pi)$, so $\pi = \pi_N$. Along the proof of Corollary 5.9 we can see that $B - B' = pr_N^*\omega$.

Corollary 5.20. *Let L be a complex Dirac structure with constant real index r and order s and let p be a regular point of type k. Then there exist a neighbourhood U of p, a $(2k+s)$-dimensional submanifold N such that*

$$L|_U \cong e^B(L_{(D,J)} \times L_{i\omega_{can}}),$$

where $L_{i\omega_{can}}$ is the graph of the canonical presymplectic structure on $\mathbb{R}^{2(n-k)+r-s}$ with kernel of dimension $r-s$, $L_{(D,J)}$ is the complex Dirac structure associated to a CR structure of codimension s over N and B is a real two-form on M which is closed on the directions of $\mathbb{R}^{2(n-k)+r-s}$.

Proof. By Theorem 5.10, there exist a neighbourhood U' of p, a local transversal submanifold $N \overset{\iota}{\hookrightarrow} M$ such that $L|_U \cong e^{B'}(\iota^! L \times L_\omega)$, where B' is a closed two-form on U. We take U' such that the type of $L|_{U'}$ is constant and equal to k. So the foliation associated to Δ is regular on U'. Since $T_pN \oplus \Delta = T_pM$ at p, we have that $TN|_{U\cap N} \oplus \Delta|_{U\cap N} = TM|_{U\cap N}$, here we shrink U' if necessary. Consequently, $\iota^! L$ is complex Dirac structure of constant real index s, order s and maximum type, by Proposition 4.9 we have that there exists a real two-form $B_1 \in \Omega^2(U)$ not necessarily closed defined on that neighbourhood such that $e^{B_1}(\iota^! L)$ is a CR structure (D, J) on N.

Consequently, we have that $L|_U \cong e^{B' - pr_N^* B_1}(L_\omega \times L_{(D,J)})$, where pr_N is the projection onto N given by the splitting of M. Finally we note that $B = B' - pr_N^* B_1$ is only closed on the directions of $\mathbb{R}^{2(n-k)+r-s}$. □

Another consequence of Proposition 5.10 is the local form of generalized complex structures given by Abouzaid and Boyarshenko.

Corollary 5.21 ([1])**.** *Consider a generalized complex structure L and a point $p \in M$. Then there exist a neighbourhood U of p, such that $L|_U$ is isomorphic to a B-transformation of the product of a generalized complex structure carrying a Poison bivector vanishing at p with a a generalized complex structure defined by a symplectic structure.*

Chapter 6

Complex Dirac structures with real index one

In this chapter we focus on the case of complex Dirac structures with real index one. First we study a pairing on spinors that could detect when the intersection of the annihilator of two pure spinors has dimension one. At the end of the chapter we present the description of the spinors associated to complex Dirac structures as an application of this pairing.

6.1 Spinors revisited

6.1.1 The pairing $(\cdot,\cdot)_1$

Consider a vector space V over the over the fields $\mathbb{R}$ or $\mathbb{C}$. We observed in Chapter 2 that the Chevalley pairing characterizes when the annihilators of two pure spinors are transversal. In the same spirit we characterize when the intersection of the annihilators of two pure spinors has dimension one. For that reason we adapt the pairing B_1 from [8] to the space of spinors $S = \wedge^\bullet V^*$, putting it in a more geometrical context.

Definition 6.1. Let ρ and τ be two spinors, consider

$$(\rho,\tau)_1 : V \oplus V^* \to \det(V^*)$$
$$(\rho,\tau)_1(X+\xi) = (\rho^\intercal \wedge (X+\xi)\cdot\tau)_{top}.$$

This defines a pairing $(\cdot,\cdot)_1$ on the space of spinors S, with values on $Hom(V\oplus V^*, \det(V^*))$.

Examples 6.2. We compute the pairing of some pure spinors.

a) Let ω be a two-form with one-dimensional kernel on a $(2n+1)$-dimensional vector space V and let $\theta \in V^*$ such that $\omega^n \wedge \theta \neq 0$. Consider the spinors $\rho = e^{i\omega}$ and $\overline{\rho} = e^{-i\omega}$ in $\wedge^\bullet V^*_{\mathbb{C}}$. Then

$$(\rho,\overline{\rho})_1\theta = (e^{i\omega}\wedge\theta\wedge e^{i\omega})_{top} = (e^{2i\omega}\wedge\theta)_{top} = \frac{(2i)^n}{n!}\omega^n\wedge\theta \neq 0,$$

whereas for the Chevalley pairing, we have

$$(\rho,\overline{\rho})_0 = (e^{i\omega}\wedge e^{i\omega})_{top} = (e^{2i\omega})_{top} = 0.$$

Note that the lagrangian subspace $L(V_{\mathbb{C}}, i\omega)$ is the annihilator of ρ, so ρ and $\overline{\rho}$ are pure spinors. By Example 3.15, $L(V_{\mathbb{C}}, i\omega)$, has real index one, checking the previous computation.

b) Let (θ, ω) be a cosymplectic structure, i.e., $\theta \in V^*$ nontrivial, $\omega \in \wedge^2 V^*$ over a $(2n+1)$-dimensional vector space such that $\omega^n \wedge \theta \neq 0$. Let $R \in V$ such that $\theta(R) = 1$. Consider the spinor $\rho = e^{i\omega} \wedge \theta$ and $\overline{\rho} = e^{-i\omega} \wedge \theta$ in $\wedge^\bullet V^*_{\mathbb{C}}$. Then

$$\begin{aligned}(\rho, \overline{\rho})_1 R &= (\theta \wedge e^{i\omega} \wedge \iota_R(e^{i\omega} \wedge \theta))_{top} \\ &= (\theta \wedge e^{2i\omega})_{top} \\ &= \frac{(2i)^n}{n!} \omega^n \wedge \theta \neq 0,\end{aligned}$$

yielding that $(\rho, \overline{\rho})_1 \neq 0$. We also have that

$$(\rho, \overline{\rho})_0 = (\theta \wedge e^{i\omega} \wedge e^{i\omega} \wedge \theta)_{top} = 0.$$

We saw in Example 3.30 that the lagrangian subspace $L_\rho = L((\ker \theta)_{\mathbb{C}}, i\iota^*\omega)$ has real index one.

From the previous examples we note that there is a relationship between the pairing $(\cdot, \cdot)_1$ and the dimension of $L \cap \overline{L}$, we will confirm this relationship later.

Next we study the main properties of the pairing $(\cdot, \cdot)_1$.

Lemma 6.3. *Let S^{ev} and S^{odd} denote the space of even and odd spinors. Then the two scenarios follow: first if* $\dim V$ *is odd then $(\cdot, \cdot)_1$ is zero when restricted to $S^{ev} \times S^{odd}$ and to $S^{odd} \times S^{ev}$. Second, if* $\dim V$ *is even then $(\cdot, \cdot)_1$ is zero when restricted to $S^{ev} \times S^{ev}$ and to $S^{odd} \times S^{odd}$.*

This is completely opposite to what happened in Section 2.2.1 with the Chevalley pairing. However, the symmetry or skew-symmetry of the pairing $(\cdot, \cdot)_1$ depends on the dimension of V in the same fashion as with the Chevalley pairing.

Lemma 6.4. *On an $m-$dimensional vector space V, the pairing $(\cdot, \cdot)_1$ satisfies*

$$(\rho, \tau)_1 = (-1)^{\frac{m(m-1)}{2}} (\tau, \rho)_1.$$

Proof. It is enough to prove the identity for ρ, $\tau \in S$ homogeneous of degree m_1 and m_2, respectively, since we can extend the identity by linearity. If $m_1 + m_2 \neq m+1$ and $m-1$, then $(\rho, \tau)_1 = 0 = (\tau, \rho)_1$.

First, we suppose $m_1 + m_2 = m + 1$, thus $(\tau, \rho)_1 \xi = 0$ for any $\xi \in V^*$ and we just need to evaluate on vectors of V

$$(\rho, \tau)_1 X = (\rho^\intercal \wedge \iota_X \tau)_{top} = \rho^\intercal \wedge \iota_X \tau.$$

Since $m_1 + m_2 = m + 1$, we have that $\rho^\intercal \wedge \tau = 0$, implying that

$$0 = \iota_X(\rho^\intercal \wedge \tau) = \iota_X \rho^\intercal \wedge \tau + (-1)^{m_2} \rho^\intercal \wedge \iota_X \tau.$$

Therefore,

$$\begin{aligned}(\rho, \tau)_1 X &= \rho^\intercal \wedge \iota_X \tau \\ &= (-1)^{m_1+1} \iota_X \rho^\intercal \wedge \tau \\ &= (-1)^{1+m_1+(m_1-1)m_2+\frac{m_2(m_2-1)}{2}+\frac{m_1(m_1-1)}{2}} \tau^\intercal \wedge \iota_X \rho \\ &= (-1)^{\frac{m(m-1)}{2}} (\tau, \rho)_1 X.\end{aligned}$$

The next case is when $m_1 + m_2 = m - 1$. Note that $(\tau, \rho)_1 X = 0$ for any $X \in V$, so it remains to evaluate on one-forms of V:

$$\begin{aligned}
(\rho, \tau)_1 \xi &= (\rho^\intercal \wedge \xi \wedge \tau)_{top} \\
&= \rho^\intercal \wedge \xi \wedge \tau \\
&= (-1)^{m_1 m_2 + \frac{m_2(m_2-1)}{2} + \frac{m_1(m_1-1)}{2} + m_1 + m_2} \tau^\intercal \wedge \xi \wedge \rho \\
&= (-1)^{\frac{m(m-1)}{2}} \tau^\intercal \wedge \xi \wedge \rho \\
&= (-1)^{\frac{m(m-1)}{2}} (\rho, \tau)_1 \xi.
\end{aligned}$$

Combining both parts of the proof we obtain the identity. □

The Chevalley pairing is invariant under the action of B-transformations. That is no longer true for the pairing $(\cdot, \cdot)_1$. However, we have the following.

Lemma 6.5. *For $B \in \wedge^2 V^*$,*

$$(e^B \cdot \rho, e^B \cdot \tau)_1 (X + \xi) = (\rho, \tau)_1 e^{-B}(X + \xi).$$

Proof. We check the identity separately on vectors and one-forms. Consider $X \in V$. Then

$$\begin{aligned}
(e^B \cdot \rho, e^B \cdot \tau)_1 X &= ((e^{-B} \wedge \rho)^\intercal \wedge \iota_X (e^{-B} \wedge \tau))_{top} \\
&= (\rho^\intercal \wedge e^B (-\iota_X B \wedge e^{-B} \wedge \tau + e^{-B} \wedge \iota_X \tau))_{top} \\
&= (-\rho^\intercal \wedge e^B \wedge \iota_X B \wedge e^{-B} \tau + \rho^\intercal \wedge e^B \wedge e^{-B} \iota_X \tau)_{top} \\
&= (-\rho \intercal \wedge \iota_X B \wedge \tau + \rho^\intercal \wedge \iota_X \tau))_{top} \\
&= (\rho, \tau)_1 e^{-B} X.
\end{aligned}$$

On the other hand, for $\xi \in V^*$, we have the following

$$\begin{aligned}
(e^B \cdot \rho, e^B \cdot \tau)_1 \xi &= ((e^{-B} \wedge \rho)^\intercal \wedge \xi \wedge (e^{-B} \wedge \tau))_{top} \\
&= (\rho^\intercal \wedge e^B \wedge \xi \wedge e^{-B} \wedge \tau)_{top} \\
&= (\rho^\intercal \wedge \xi \wedge \tau)_{top} \\
&= (\rho, \tau)_1 \xi.
\end{aligned}$$

□

The main property of the Chevalley pairing is Proposition 2.33. Next we prove a similar statement for the pairing $(\cdot, \cdot)_1$.

Proposition 6.6. *Let ρ and τ be two pure spinors. Then $\dim(L_\rho \cap L_\tau) = 1$ if and only if $(\rho, \tau)_1 \neq 0$.*

We broke the proof in several step.

Lemma 6.7. *Let ρ be a pure spinor and $m = \dim V$. Then dim$(L_\rho \cap V) = r$ if and only if $\rho_{m-r} \neq 0$ and $\rho_j = 0$, for all $j > m - r$.*

Proof. We can deduce the case $r = 0$ from Proposition 2.33.

($\Rightarrow$) Suppose that r is odd. Let $E \subseteq V$ and $\varepsilon \in \wedge^2 E$ such that $L_\rho = L(E, \varepsilon)$. Since $L(E, \varepsilon) \cap V = \ker \varepsilon$, we have $\dim \ker \varepsilon = r$, thus there exist q such that $\dim(E) = 2q + r$ and also a basis $\{e_1, \cdots, e_{2q}\}$ of E with dual basis $\{e^1, \cdots, e^{2q}\}$ such that

$$\varepsilon = e^1 \wedge e^{q+1} + \cdots + e^q \wedge e^{2q}.$$

Completing the previous basis, we get another basis $\{e_j, f_j\}$ of V with dual basis $\{\hat{e}^j, f^j\}$ and consider

$$B = \hat{e}^1 \wedge \hat{e}^{q+1} + \cdots + \hat{e}^q \wedge \hat{e}^{2q}.$$

Note that B extends ε.

Since ρ is a pure spinor, then by Proposition 2.28 we have that $\rho = e^B \wedge \Omega$, where Ω is a generator of the determinant of $\mathrm{Ann}(E)$. We take $\Omega = f^1 \wedge \ldots \wedge f^k$. More explicitly,

$$\rho = \Omega + B \wedge \Omega \cdots + \frac{1}{(q + (r-1)/2)!} B^{q + \frac{r-1}{2}} \wedge \Omega.$$

We note that $B^q \neq 0$ and $B^{q+1} = 0$. Let ρ_j denote the homogeneous components of ρ in the usual grading of $\wedge^\bullet V^*$. Then all the terms from ρ_{m-r+2} vanish since they have a factor B^{q+1}, the term ρ_{m-r+1} also vanishes by the parity of ρ. Also, we note that

$$\rho_{m-r} = \frac{1}{q!} B^q \wedge \Omega = \frac{1}{q!} \hat{e}^1 \wedge \cdots \wedge \hat{e}^{2q} \wedge f^1 \wedge \ldots \wedge f^k \neq 0,$$

since $\{\hat{e}^j, f^j\}$ is a basis.

The case for r even is similar.

($\Leftarrow$) Let ρ be a pure spinor and a nonnegative integer s such that $\rho_{m-s} \neq 0$ and $\rho_j = 0$, for all $j > m - s$. Suppose that $\dim(L_\rho \cap V) = r$. By the first implication, $\rho_{m-r} \neq 0$ and $\rho_j = 0$, for all $j > m - r$, so if we assume $r \neq s$ we reach to a contradiction. □

Lemma 6.8. *Let ρ be a pure spinor and $\tau = vol_{\det \mathrm{Ann}\, E_2}$, where $E_2 \subseteq V$ such that* $\dim(L_\rho \cap L_\tau) =$ 1. *Consider $E_1 \subseteq V \oplus V^*$ and $\varepsilon \in \wedge^2 E^*$ such that $L_\rho = L(E_1, \varepsilon)$. Then,*

a) $\dim(\ker \iota^*_{E_1 \cap E_2} \varepsilon) \leq 1$,

b) $pr_V(L_\rho \cap L_\tau) \subseteq \ker \iota^*_{E_1 \cap E_2} \varepsilon_1$,

c) If $pr_V(L_\rho \cap L_\tau) = 0$, then there exists $X \in V$ such that $(\rho, \tau)_1 X \neq 0$,

d) If $pr_V(L_\rho \cap L_\tau) \neq 0$, then there exists $\xi \in V$ such that $(\rho, \tau)_1 \xi \neq 0$.

Proof. Since V is a real or complex vector space, we denote $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.
a) Let $E_1 \subseteq V$ and $\varepsilon_1 \in \wedge^2 V^*$ such that $L_\rho = L(E_1, \varepsilon_1)$; note that $L_\tau = L(E_2, 0)$. Let k_1 and k_2 be the respective codimensions of E_1 and E_2.

Since $L_\rho + L_\tau = (L_\rho \cap L_\tau)^\perp$, we have that $\mathrm{codim}(E_1 + E_2) \leq 1$. Let $B \in \wedge^2 V^*$ be a two-form extending ε_1 such that if $Y \in \ker \iota^*_{E_1 \cap E_2} \varepsilon_1$, then $B(Y) \in \mathrm{Ann}(E_2)$ (we can extend ε_1 by keeping the extension vanishing on the remaining directions of E_2). Then,

$$\{Z + \iota_Z B | Z \in \ker(\iota^*_{E_1 \cap E_2} \varepsilon_1)\} \subseteq L(E_1, \varepsilon_1) \cap L(E_2, 0), \tag{6.1}$$

where the left-hand side is the graph of the two-form $\iota^*_{\ker(\iota^*_{E_1 \cap E_2} \varepsilon_1)} B$, the right-hand side has dimension one by hypothesis. Consequently, $\dim(\ker \iota^*_{E_1 \cap E_2} \varepsilon) \leq 1$ and depends precisely on the parity of $\dim(E_1 \cap E_2)$.

b) Let $X + \xi \in L_\rho \cap L_\tau$. Then $X \in E_1 \cap E_2$, $\xi \in \operatorname{Ann} E_2$ and $\xi|_{E_1} = \iota_X \varepsilon_1$. Restricting the last equation to $E_1 \cap E_2$, as $\xi|_{E_2} = 0$ we obtain that $X \in \ker \iota^*_{E_1 \cap E_2} \varepsilon_1$.

c) Let $\zeta \in V^*$ such that $L_\rho \cap L_\tau = \mathbb{K} \cdot \zeta$. In this case

$$L_\rho + L_\tau = (L_\rho \cap L_\tau)^\perp = (\mathbb{K} \cdot \zeta)^\perp = \ker \zeta \oplus V^*.$$

Then $E_1 + E_2 = pr_V(L_\rho + L_\tau)$ equals $\ker \zeta$ and so $\operatorname{codim}(E_1 + E_2) = 1$. Now we prove that $\dim(E_1 \cap E_2)$ is even and $\ker \iota^*_{E_1 \cap E_2} \varepsilon_1 = 0$. We proceed by contradiction. Assume that $\dim(E_1 \cap E_2)$ is odd. Then $\dim \ker \iota^*_{E_1 \cap E_2} \varepsilon_1 = 1$ and by projecting to V both terms of equation (6.1), we get that $\ker \iota^*_{E_1 \cap E_2} \varepsilon_1 \subseteq pr_V(L_\rho \cap L_\tau)$, by the hypothesis we get a contradiction. Therefore, $\dim(E_1 \cap E_2)$ is even and $\ker \iota^*_{E_1 \cap E_2} \varepsilon_1 = 0$.

Since $\iota^*_{E_1 \cap E_2} \varepsilon_1$ is nondegenerate there exists a nonnegative integer q and a basis $\{e_i\}$ of E such that $\dim(E_1 \cap E_2) = 2q$ and

$$\iota^*_{E_1 \cap E_2} \varepsilon_1 = e^1 \wedge e^{q+1} + \ldots + e^q \wedge e^{2q}.$$

Consider $X \in V$ such that

$$(L_\rho + L_\tau) \oplus \mathbb{K} \cdot X = V \oplus V^*.$$

Completing the basis $\{e_i\}$ to a larger basis of E_1, $\{e_i, f_j\}$, and to a basis of E_2, $\{e_i, g_j\}$, we construct the basis $\{e_i, f_j, g_k, X\}$ of V with dual basis $\{e^i, f^j, g^k, \theta\}$. Note that

$$vol_{\det \operatorname{Ann} E_1} = \theta \wedge g^1 \wedge \ldots \wedge g^{k_1 - 1} \text{ and}$$

$$vol_{\det \operatorname{Ann} E_2} = \theta \wedge f^1 \wedge \ldots \wedge f^{k_2 - 1}.$$

Since $\dim E_1 \cap E_2 = 2q$, we have $2q + k_1 + k_2 - 1 = \dim V$. Let $B \in \wedge^2 V^*$ such that $\rho_1 = e^B \wedge vol_{\det \operatorname{Ann} E_1}$. Then

$$\begin{aligned}
(\rho, \tau)_1 X &= (-1)^s B^q \wedge vol_{\det \operatorname{Ann} E_1} \wedge \iota_X vol_{\det \operatorname{Ann} E_2} \\
&= (-1)^s e^1 \wedge e^{q+1} \wedge \ldots \wedge e^q \wedge e^{2q} \wedge vol_{\det \operatorname{Ann} E_1} \wedge \iota_X vol_{\det \operatorname{Ann} E_2} \\
&= (-1)^s e^1 \wedge e^{q+1} \wedge \ldots \wedge e^q \wedge e^{2q} \wedge \theta \wedge g^1 \ldots g^{k_1 - 1} \wedge \iota_X \theta \wedge f^1 \wedge \ldots \wedge f^{k_2 - 1} \\
&= (-1)^s e^1 \wedge e^{q+1} \wedge \ldots \wedge e^q \wedge e^{2q} \wedge \theta \wedge g^1 \ldots g^{k_1 - 1} \wedge f^1 \wedge \ldots \wedge f^{k_2 - 1} \neq 0
\end{aligned}$$

where $s = \frac{2q + k_1(2q + k_1 - 1)}{2}$.

d) We have that $L_\rho \cap L_\tau = \mathbb{K} \cdot (Z + \zeta)$, for some $Z + \zeta \in V \oplus V^*$, where $Z \neq 0$. In this case, we decompose

$$L_\rho + L_\tau = \ker \zeta \oplus \operatorname{Ann} Z \oplus \mathbb{K} \cdot (W + \beta).$$

Since $Z \neq 0$, there exists $\xi \in V^*$ such that $\xi \oplus \operatorname{Ann}(Z) = V^*$ and then

$$(L_\rho + L_\tau) \oplus \mathbb{K} \cdot \xi = V \oplus V^*.$$

By item b), we have that $pr_V(L_\rho \cap L_\tau) \subseteq \ker \iota^*_{E_1 \cap E_2} \varepsilon_1$, implying that that $\ker \iota^*_{E_1 \cap E_2} \varepsilon_1 = \mathbb{K} \cdot Z$. Consequently, $\dim(E_1 \cap E_2)$ is odd and $\dim(\ker \iota^*_{E_1 \cap E_2} \varepsilon_1) = 1$.

Let q be a nonnegative integer such that $\dim(E_1 \cap E_2) = 2q+1$. As in the previous case $\dim V - k_1 - k_2 = 2q+1$.

Then there exists a basis $\{e_i\}$ of $E_1 \cap E_2$ such that $e_{2q+1} = Z$ and

$$\iota^*_{E_1 \cap E_2} \varepsilon_1 = e^1 \wedge e^{q+1} + \ldots + e^q \wedge e^{2q}.$$

Since $E_1 + E_2 = V$, we complete $\{e_i\}$ to a basis $\{e_i, f_j\}$ of E_1, to a basis $\{e_i, g_j\}$ of E_2 and to a basis $\{e_i, f_j, g_k\}$ of V with dual basis $\{e^i, f^j, g^k\}$. Since $\{e^1, \ldots e^{2q-1}, e^{2q}, f^j, g^k\}$ is a basis for $\operatorname{Ann} Z$ and $\operatorname{Ann} Z \oplus \mathbb{K} \cdot \xi = V^*$, we have that $\{e^1, \ldots e^{2q-1}, e^{2q}, \xi, f^j, g^k\}$ is a basis for V^*. Then,

$$\begin{aligned}(\rho, \tau)_1 \xi &= B^q \wedge \Omega_1 \wedge \xi \wedge \Omega_2 \\ &= e^1 \wedge e^{q+1} \ldots \wedge e^{2q} \wedge g^1 \wedge \ldots \wedge g^{k_1} \wedge e^{2q+1} \wedge f^1 \wedge \ldots \wedge f^{k_2} \neq 0\end{aligned}$$

□

Lemma 6.9. *Let ρ be a pure spinor and $\tau = vol_{\det \operatorname{Ann} E_2}$, where E_2 is a subspace of V. Then, $\dim(L_\rho \cap L_\tau) = 1$ if and only if $(\rho, \tau)_1 \neq 0$.*

Proof. ($\Rightarrow$) It follows from the previous lemma.

($\Leftarrow$) We first prove the following technical identity: let $X + \xi \in L_\rho \cap L_\tau$ such that $X \neq 0$ and $Z + \zeta \in V \oplus V^*$. Then, the following identity holds

$$\iota_X[(\rho, \tau)_1(Z + \zeta)] = \frac{(-1)^{\frac{k_1(k_1+1)}{2} + c_0}}{c_0!} \langle X + \xi, Z + \zeta \rangle B^{c_0} \wedge \Omega_1 \wedge \Omega_2, \tag{6.2}$$

where $c_0 = \frac{\dim V - k_1 - k_2}{2}$, $B \in \wedge^2 V^*$ is an extension of ε_1, and Ω_1 and Ω_2 are generators of $\det \operatorname{Ann}_1$ and $\det \operatorname{Ann}_2$ respectively.

We note that $\rho = e^B \wedge \Omega_1$ and $\tau = \Omega_2$. Let $Z + \zeta \in TM \oplus T^*M$; we make the following computation

$$\begin{aligned}\iota_X[(\rho, \tau)_1(Z + \zeta)] &= (-1)^{\frac{k_1(k_1-1)}{2}} \iota_X[(e^{-B} \wedge \Omega_1 \wedge (Z + \zeta) \cdot \Omega_2)_{top}] \\ &= (-1)^{\frac{k_1(k_1-1)}{2}} \iota_X((-1)^{c_0+1} \frac{B^{c_0+1}}{(c_0+1)!} \wedge \Omega_1 \wedge \iota_Z \Omega_2 + (-1)^{c_0} \frac{B^{c_0}}{c_0!} \wedge \Omega_1 \wedge \zeta \wedge \Omega_2) \\ &= \frac{(-1)^{\frac{k_1(k_1-1)}{2} + c_0}}{(c_0+1)!} \iota_X((c_0+1) B^{c_0} \wedge \Omega_1 \wedge \zeta \wedge \Omega_2 - B^{c_0+1} \wedge \Omega_1 \wedge \iota_Z \Omega_2). \quad (6.3)\end{aligned}$$

We need to develop each term of the right side of the previous. Since $X + \xi \in L_\rho$, we have $\iota_X \rho + \xi \wedge \rho = 0$. Looking at the components of lower degree in the previous equation, we have

$$\xi \wedge \Omega_1 = -\iota_X B \wedge \Omega_1. \tag{6.4}$$

On the one hand, using equation (6.4) and the fact that $X \in E_1 \cap E_2$ and $\xi \in \operatorname{Ann} E_2$, we have

$$\iota_X(B^{c_0} \wedge \Omega_1 \wedge \zeta \wedge \Omega_2) = (-1)^{k_1} \zeta(X) B^{c_0} \wedge \Omega_1 \wedge \Omega_2.$$

On the other hand, as $X \in E_1 \cap E_2$, we have

$$\iota_X(B^{c_0+1} \wedge \Omega_1 \wedge \iota_Z \Omega_2) = (c_0+1) \iota_X B \wedge B^{c_0} \wedge \Omega_1 \wedge \iota_Z \Omega_2.$$

Since $\xi \in \operatorname{Ann} E_2$, we obtain that

$$0 = \iota_Z(\xi \wedge \Omega_2) = \xi(Z)\Omega_2 - \xi \wedge \iota_Z \Omega_2.$$

Then, using (6.4) and that $\xi \in \operatorname{Ann} E_2$, we have that

$$\iota_X B \wedge B^{c_0} \wedge \Omega_1 \wedge \iota_Z \Omega_2 = -\xi \wedge B^{c_0} \wedge \Omega_1 \wedge \iota_Z \Omega_2 = (-1)^{k_1+1} \xi(Z) B^{c_0} \wedge \Omega_1 \wedge \Omega_2.$$

Gathering the parts above we obtain

$$\begin{aligned}\iota_X[(\rho_1, \rho_2)_1(Z+\zeta)] &= \frac{(-1)^{\frac{k_1(k_1-1)}{2}+c_0}}{c_0!}((-1)^{k_1}\zeta(X)B^{c_0} \wedge \Omega_1 \wedge \Omega_2 - (-1)^{k_1+1}\xi(Z)B^{c_0} \wedge \Omega_1 \wedge \Omega_2) \\ &= c_1(\zeta(X)B^{c_0} \wedge \Omega_1 \wedge \Omega_2 + \xi(Z)B^{c_0} \wedge \Omega_1 \wedge \Omega_2) \\ &= c_1\langle X+\xi, Z+\zeta\rangle B^{c_0} \wedge \Omega_1 \wedge \Omega_2\end{aligned}$$

where $c_1 = \frac{(-1)^{\frac{k_1(k_1+1)}{2}+c_0}}{c_0!}$. So the identity holds.

Now we are prepared to retake the proof. Suppose that $\dim(L_\rho \cap L_\tau) \neq 1$. If we assume that $\dim(L_\rho \cap L_\tau) = 0$, then $(\rho, \tau)_0 \neq 0$. Consequently, by Lemma 2.30 and by Lemma 6.3, we reach to a contradiction.

Then we have that, $\dim(L_\rho \cap L_\tau) \geq 2$, yielding that there exist at least two linearly independent elements $X + \xi, Y + \eta \in L_\rho \cap L_\tau$. We have three situations:

a) $X \neq 0$ and $Y \neq 0$: Since $X+\xi, Y+\eta \in L_\rho \cap L_\tau$ we have that $X, Y \in E_1 \cap E_2$, $\xi, \eta \in \operatorname{Ann} E_2$. By (6.2), as $X \neq 0$ and $Y \neq 0$

$$(X+\xi)^\perp \subseteq \ker(\rho,\tau)_1, \ (Y+\eta)^\perp \subseteq \ker(\rho,\tau)_1.$$

Since $\langle \cdot, \cdot \rangle$ is nondegenerate, the different subspaces $(X+\xi)^\perp$ and $(Y+\eta)^\perp$ both have codimension one and thus they generate $V \oplus V^*$. Therefore, $(\rho,\tau)_1 = 0$.

b) $X = 0$ and $Y \neq 0$: In this case $\xi \in \operatorname{Ann} E_1 \cap \operatorname{Ann} E_2$, implying that $\Omega_1 \wedge \Omega_2 = 0$. Then we have

$$(\rho,\tau)_1\zeta = (e^{-B} \wedge \Omega_1 \wedge \zeta \wedge \Omega_2)_{top} = 0, \ \forall \zeta \in V^*.$$

Consequently, $V^* \subseteq \ker(\rho,\tau)_1$ and by equation (6.2), $(Y+\eta)^\perp \subseteq \ker(\rho,\tau)_1$. Since, the subspace $(Y+\eta)^\perp$ has codimension one and does not contain V^*, these two subspace generate $V \oplus V^*$, yielding that $(\rho,\tau)_1 = 0$.

c) $X = 0$ and $Y = 0$: Since $\xi, \eta \in \operatorname{Ann} E_1 \cap \operatorname{Ann} E_2$ are linearly independent, then $\Omega_1 \wedge \Omega_2 = 0$ and $\Omega_1 \wedge \iota_Z \Omega_2 = 0$ for all $Z \in V$. Then

$$(\rho,\tau)_1 Z = (e^{-B} \wedge \Omega_1 \wedge \iota_Z \Omega_2)_{top} = 0, \ \forall Z \in V.$$

Thus, $V \subseteq \ker(\rho,\tau)_1$. Since $\Omega_1 \wedge \Omega_2 = 0$, we have that $V^* \subseteq \ker(\rho,\tau)_1$ and finally $(\rho,\tau)_1 = 0$.

□

Proof of Proposition 6.6. Let ρ and $\tau = e^B \wedge \Omega$ be two spinors such that $\dim(L_\rho \cap L_\tau) = 1$. Then, $\dim(L_{e^{-B}\rho} \cap L_\Omega) = 1$. By Lemma 6.9, $(e^{-B}\rho, \Omega)_1 \neq 0$ and by Lemma 6.5, the conclusion follows. □

Corollary 6.10. *If* $\dim(L_\rho \cap L_\tau) = 1$*, then* $\ker(\rho, \tau)_1 = L_\rho + L_\tau$ *and thus*

$$\frac{V \oplus V^*}{L_\rho + L_\tau} = \det(V^*).$$

Proof. By definition of the pairing, $L_\rho \subseteq \ker(\rho, \tau)_1$; by Lemma 6.4, $L_\tau \subseteq \ker(\rho, \tau)_1$. Since $(\rho, \tau)_1 \neq 0$, the subspaces $L_\rho + L_\tau$ and $\ker(\rho, \tau)_1$ have the same dimension and thus are equal. □

This corollary has a stronger meaning in a geometrical context. Let M be a m-dimensional manifold. Then Corollary 6.10 tells us that any line bundle complementary to the subbundle $L_\rho + L_\tau$ is isomorphic to $\det(T^*M)$. In particular for orientable manifolds this line bundle is trivial.

6.1.2 r-dimensional intersection of lagrangian subspaces

Until now we have focused on the study of spinors ρ and τ satisfying that $\dim L_\rho \cap L_\tau = 1$. Now we present an elementary method to see when $\dim L_\rho \cap L_\tau = r$.

The idea is the following: the parity of a lagrangian vector space L is the parity of $\dim pr_V L$. The group $SO(V \oplus V^*)$ acts on the space of lagrangian subspaces of $V \oplus V^*$ preserving the parity, actually the action is transitive when restricted to the set of lagrangian having the same parity, cf. [30, Chap. 1]. Let ρ, τ be two pure spinors with annihilators L_ρ and L_τ, respectively. Then, if L_τ has the same parity as V then there exists a linear map $A \in SO(V \oplus V^*)$ such that $A(V) = L_\tau$. Consider the lagrangian subspace $\hat{L} = A^{-1}(L_\rho)$, then $L_\rho \cap L_\tau = A(\hat{L} \cap V)$. Let σ be a pure spinor such that $L_\rho \cap L_\tau = A(L_\sigma \cap V)$. Then, $\dim L_\rho \cap L_\tau = r$ if and only $\dim L_\sigma \cap V = r$ and by Lemma 6.7, the last happens if and only if $\sigma_{m-r} \neq 0$ and $\sigma_j = 0$ for all $j > m - r$. In what follows we give a method for constructing the map A and so the spinor σ under certain hypotheses.

Given a lagrangian L there exist $E \subseteq V$ and $\varepsilon \in \wedge^2 E^*$ such that $L = L(E, \varepsilon)$. Also we know that

$$L = L(F, \gamma) = \{X + \xi \in V \oplus F \mid \iota_\xi \gamma = X|_F\},$$

for some $F \subseteq V^*$ and $\gamma \in \wedge^2 F^*$; note that $F = pr_{V^*}(L)$. We will show that we can retrieve F and γ from E and ε, and vice versa. Note that

$$\ker \varepsilon = V \cap L = \operatorname{Ann}(F) \text{ and}$$

$$\ker \gamma = V^* \cap L = \operatorname{Ann}(E),$$

obtaining that $E = \operatorname{Ann}(\ker \gamma)$ and $F = \operatorname{Ann}(\ker \varepsilon)$. For the relationship between ε and γ we recall the definition of both,

$$\varepsilon : E \to E^*$$
$$X \in E \mapsto \xi + \operatorname{Ann} E \in V^* / \operatorname{Ann} E \cong E^*,$$

where ξ is such that $X + \xi \in L$, the definition of γ is just the same but replacing E by F. Then we note that $\varepsilon(E) = F/\operatorname{Ann} E$ and $\ker \varepsilon = \operatorname{Ann} F$, defining the isomorphism

$$\hat{\varepsilon} : E/\operatorname{Ann} F \to F/\operatorname{Ann} E$$

such that $\varepsilon = \hat{\varepsilon} \circ q$, where $q : E \to E/\operatorname{Ann} F$ is the quotient map. Analogously for γ we obtain the isomorphism

$$\hat{\gamma} : F/\operatorname{Ann} E \to E/\operatorname{Ann} F$$

such that $\gamma = \hat{\gamma} \circ p$, where $p : F \to F/Ann(E)$ is the quotient map. From the definitions of ε and γ, we observe that $\hat{\gamma} = \hat{\varepsilon}^{-1}$. In summary we have obtained.

Lemma 6.11. *Given the data $E \subseteq V$, $\varepsilon \in \wedge^2 E^*$, we have that $\gamma = \hat{\varepsilon} \circ p$ and $F = \mathrm{Ann}(\ker(\varepsilon))$ are such that $L(E,\varepsilon) = L(F,\gamma)$. Conversely, given the data $F \subseteq V^*$, $\gamma \in \wedge^2 F^*$, we have that $\varepsilon = \hat{\gamma} \circ q$ and $E = \mathrm{Ann}(\ker(\gamma))$ are such that $L(F,\gamma) = L(E,\varepsilon)$.*

Finally we are prepared for the main result of this section.

Proposition 6.12. *Let L be a lagrangian subspace of $V \oplus V^*$ and let $F \subseteq V^*$, $\gamma \in \wedge^2 F^*$ such that $L = L(F,\gamma)$. If F is even dimensional, then there exists $B \in \wedge^2 V^*$ and $\beta \in \wedge^2 V$, such that $L = e^{-\beta} e^B(V)$.*

Proof. Since $\dim F$ is even, there exists $\beta \in \wedge^2 V$ such that $\gamma + \iota^*\beta$ is nondegenerate on F. By Lemma 6.11

$$e^{\beta} L(E,\varepsilon) = L(F, \gamma + \iota^*\beta) = L(V, \widehat{(\gamma + \iota^*\beta)}^{-1} \circ q),$$

where $q : \mathrm{Ann}(\ker(\gamma + \iota^*\beta)) \to \mathrm{Ann}(\ker(\gamma + \iota^*\beta)) / \mathrm{Ann}\, F$ is the quotient map and the last equality is given because $\mathrm{Ann}(\ker(\gamma + \iota^*\beta)) = V$. Finally we get that

$$e^{\beta} L(E,\varepsilon) = L(V,B) = e^B V,$$

where $B = \widehat{(\gamma + \iota^*\beta)}^{-1} \circ q \in \wedge^2 V^*$ and therefore $L = e^{-\beta} e^B V$. □

Corollary 6.13. *If ρ is a pure spinor such that $pr_{V^*}(L_\rho)$ has even dimension, then there exist $B \in \wedge^2 V^*$ and $\beta \in \wedge^2 V$ such that $\rho = e^{-\beta} e^B \cdot 1$.*

Corollary 6.14. *Let ρ and τ be two spinors, with annihilators L_ρ and L_τ, respectively. Assume that $pr_{V^*}(L_\tau)$ has even dimension and that $B \in \wedge^2 V^*$ and $\beta \in \wedge^2 V$ are as in the corollary above. Then $\dim(L_\rho \cap L_\tau) = r$ if and only if $(e^{-B} e^{\beta} \cdot \tau)_{m-r} \neq 0$ and $(e^{-B} e^{\beta} \cdot \tau)_j = 0$, whenever $j > m - r$.*

6.2 Spinors associated to complex Dirac structures with real index one

By Corollary 3.8 the lowest possible real index on an odd-dimensional manifold is one and on an even-dimensional manifold is zero. Thus, we study lagrangian subspaces of $(V \oplus V^*)_{\mathbb{C}}$ with real index one and show its resemblance to generalized complex structures.

The following proposition is a straightforward consequence of Proposition 3.11.

Proposition 6.15. *Let V be a $(2n+1)$-dimensional vector space, $E \subseteq V_{\mathbb{C}}$, $\varepsilon \in \wedge^2 E^*$ and $\omega_\Delta = \mathrm{Im}(\varepsilon|_{E \cap \overline{E}})$. Then $L = L(E,\varepsilon)$ has real index one if and only if one of the following conditions are satisfied:*

1. *the form ω_Δ is degenerate with $\dim(\ker \omega_\Delta) = 1$ and $E + \overline{E} = V_{\mathbb{C}}$, or*
2. *the form ω_Δ is nondegenerate and $\mathrm{codim}(E + \overline{E}) = 1$.*

Remark 6.16. From the previous proposition we deduce that $\omega_\Delta^{n-k} \neq 0$, where $k = \mathrm{type}(L)$.

The fact that L has real index one imposes new constraints on its spinor line. These constraints depend on the type and the order as we see next.

Proposition 6.17. *Let L be a lagrangian subspace of $(V \oplus V^*)_{\mathbb{C}}$ with real index one, where $\dim V = 2n+1$, and let $\rho = e^{B+i\omega} \wedge \Omega$ be its associated spinor. Then, we have:*

a) For L with order one and type k, there exist $X \in V_{\mathbb{C}}$, such that

$$\omega^{n-k} \wedge \Omega \wedge \iota_X \overline{\Omega} \neq 0.$$

b) For L with order zero and type k,

$$\omega^{n-k} \wedge \Omega \wedge \overline{\Omega} \neq 0$$

Proof. Let E and $\varepsilon \in \wedge^2 E^*$ such that $L = L(E, \varepsilon)$. Then Ω is a generator of the space $\det \operatorname{Ann} E$.

a) If L has order one and type k, then by Proposition 3.11, $pr_V K = 0$ and so we have that $e^{-(B-i\omega)}L$ and $e^{-(B-i\omega)}\overline{L}$ satisfy the conditions of Lemma 6.8.c. Then there exists $X \in V$ such that

$$\begin{aligned} 0 \neq (e^{-(B-i\omega)}\rho, e^{-(B-i\omega)}\overline{\rho})_1 X &= (e^{2i\omega} \wedge \Omega, \overline{\Omega})_1 X = ((e^{2i\omega} \wedge \Omega)^{\intercal} \wedge \iota_X \overline{\Omega})_{top} \\ &= (-1)^{\frac{k(k-1)}{2}} \frac{(-2i)^{n-k}}{(n-k)!} \omega^{n-k} \wedge \Omega \wedge \iota_X \overline{\Omega}. \end{aligned}$$

b) If L has order zero and type k, then by Proposition 3.11, $pr_V K \neq 0$. Consequently, $e^{-(B-i\omega)}L$ and $e^{-(B-i\omega)}\overline{L}$ satisfy the conditions of Lemma 6.8.d and so there exist $\xi \in V^*$ such that

$$\begin{aligned} 0 \neq (e^{-(B-i\omega)}\rho, e^{-(B-i\omega)}\overline{\rho})_1 \xi &= (e^{2i\omega} \wedge \Omega, \overline{\Omega})_1 \xi = ((e^{2i\omega} \wedge \Omega)^{\intercal} \wedge \xi \wedge \overline{\Omega})_{top} \\ &= (-1)^{\frac{k(k-1)}{2}} \frac{(-2i)^{n-k}}{(n-k)!} \omega^{n-k} \wedge \Omega \wedge \xi \wedge \overline{\Omega}. \end{aligned}$$

□

Appendix A

Coorientable contact structures in generalized geometry

In this appendix we will present some ideas of how contact forms could be seen inside the lagrangian subbundles of $(TM \oplus T^*M)_{\mathbb{C}}$ with real index one.

Let $\theta \in \Gamma(T^*M)$ be a precontact form on M, i.e. θ is nowhere vanishing. Consider $D = \ker\theta$ the two-form $\varepsilon_\theta \in \wedge^2 D^*$ given by $\varepsilon_\theta(X,Y) = -\theta([X,Y])$. Note that $\varepsilon_\theta = \iota^* d\theta$, where ι is the inclusion map of D into TM. A precontact form is **contact** if and only if $\theta \wedge d\theta^n \neq 0$. The last condition is usually called the **maximally nonintegrability condition**, which is equivalent to the nondegeneracy of the two-form $d\theta$ on D. In other words, ε_θ is nondegenerate, which is equivalent to the following condition: given $X \in \Gamma(D)$, there exist $Y \in \Gamma(D)$ such that $[X,Y]$ is not in $\Gamma(D)$. The last condition is much stronger that simply being non integrable and could be adapted to other structures, as we do next.

Definition A.1. Let E be a smooth distribution on TM or $TM \oplus T^*M$ or $(TM \oplus T^*M)_{\mathbb{C}}$, we say that E is **maximally nonintegrable** if for every e_1 in $\Gamma(E)$, there exists e_2 in $\Gamma(E)$ such that $[e_1, e_2]$ is not in $\Gamma(E)$.

Let $\theta \in \Omega^1(M)$ be a contact form, then $D = \ker\theta$ and $D_{\mathbb{C}}$ are maximally nonintegrable, so $D_{\mathbb{C}}$ is non integrable and we cannot define the operator $d_{D_{\mathbb{C}}}$, as this operator is defined only on regular involutive distributions. Thus $L(D_{\mathbb{C}}, i(\varepsilon_\theta)_{\mathbb{C}})$ is not involutive, by Proposition 2.41. Actually we have the following.

Proposition A.2. *If E is a maximally nonintegrable regular distribution on TM or TM_C, then for any $\varepsilon \in \wedge^2 E^*$, we have that $L(E, \varepsilon)$ is maximally nonintegrable.*

Proof. Consider $X + \xi \in \Gamma(L)$, we have that $X \in \Gamma(E)$ and then by the maximally nonintegrability of E, there exists $Y \in \Gamma(E)$ such that $[X,Y] \notin \Gamma(E)$. Since $Y \in \Gamma(E)$, there exists $\eta \in \Omega^1(M)$ such that $Y + \eta \in \Gamma(L)$. Consequently, $[X+\xi, Y+\eta] \notin \Gamma(L)$, since $pr_{TM}[X+\xi, Y+\eta] \notin \Gamma(E)$. □

Now we present some properties of maximally nonintegrable distributions.

Proposition A.3. *Let E be a regular distribution on TM or $TM_{\mathbb{C}}$. If $L(E,0)$ is maximally nonintegrable, then E is maximally nonintegrable*

Proof. Given $X \in \Gamma(E) \subseteq \Gamma(L(E,0))$, there exists $Y + \eta \in \Gamma(L(E,0))$, such that

$$[X, Y+\eta] = [X,Y] + \mathcal{L}_X \eta \notin \Gamma(L(E,0)).$$

Then we have just two options: $[X,Y] \notin \Gamma(E)$ or $[X,Y] \in \Gamma(E)$. If the first happens we are done. So supose that $[X,Y] \in \Gamma(E)$, then $\mathcal{L}_X\eta \notin \mathrm{Ann}(E)$, implying that there exists $Z \in \Gamma(E)$ such that $\mathcal{L}_X\eta(Z) = -\eta([X,Z]) \neq 0$. Consequently, $[X,Z] \notin \Gamma(E)$ and so E is maximally nonintegrable. □

Proposition A.4. *The maps e^B, where $B \in \Omega^2_{cl}(M)$ preserve the condition of being maximally nonintegrable on lagrangian subbundles of $TM \oplus T^*M$ or $(TM \oplus T^*M)_{\mathbb{C}}$.*

Proposition A.5. *Let L be a Dirac or complex Dirac on M. Consider $E = pr_{TM}L$ in case L is a Dirac structure or $E = pr_{TM_{\mathbb{C}}}L$ in case L is a complex Dirac structure with inclusion map ι. Let B be a two-form on M, such that ι^*dB is nondegenerate, i.e. for each $X \in \Gamma(E)$, there exist $Y, Z \in \Gamma(E)$ such that $dB(X,Y,Z) \neq 0$. Then $e^B L$ is maximally nonintegrable.*

Proof. Let $X + \xi \in \Gamma(L)$; then there exists $Y \in \Gamma(E)$ such that $\iota_Y\iota_X dB \neq 0$. We also note that there exists $\eta \in \Omega^1(M)$ such that $Y + \eta \in \Gamma(L)$. We proceed by contradiction. Suppose that $[e^B(X+\xi), e^B(Y+\eta)] \in \Gamma(e^B L)$. Then

$$[e^B(X+\xi), e^B(Y+\eta)] = e^B([X+\xi, Y+\eta] + \iota_Y\iota_X dB) \in \Gamma(e^B L).$$

Since L is involutive, $[X+\xi, Y+\eta] \in \Gamma(L)$. Consequently, $\iota_Y\iota_X dB) \in \Gamma(L)$ and so $\iota_Y\iota_X dB \in \mathrm{Ann}\, E$, reaching to a contradiction. Finally $[e^B(X+\xi), e^B(Y+\eta)] \notin \Gamma(e^B L)$. □

The proposition above says that we can take a complex Dirac structure into a maximally nonintegrable lagrangian subbundle.

Proposition A.6. *Let L be a maximally nonintegrable lagrangian subbundle of $TM \oplus T^*M$ or $(TM \oplus T^*M)_{\mathbb{C}}$ such that $E = pr_{TM}L$ or $E = pr_{TM_{\mathbb{C}}}L$ is involutive and ι be the inclusion map of E into TM or $TM_{\mathbb{C}}$. Then $d_E\varepsilon$ is nondegenerate.*

Proof. Let $\varepsilon \in \wedge^2 E^*$ such that $L = L(E,\varepsilon)$. Let $X \in \Gamma(E)$, then there exists $\xi \in \Omega^1(M)$ such that $X + \xi \in \Gamma(L)$. Since L is maximally nonintegrable, there exists $Y + \eta \in \Gamma(L)$ such that $[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_x\eta - \iota_Y d\xi \notin \Gamma(L)$. Since E is involutive, $[X,Y] \in \Gamma(E)$. Thus, $(\mathcal{L}_x\eta - \iota_Y d\xi)|_E - \iota_{[X,Y]}\varepsilon \neq 0$, but the left-hand side is equal to $\iota_Y\iota_X d_E\varepsilon$ (see the proof of [23]). Consequently, there exists $Y \in \Gamma(E)$ such that $\iota_Y\iota_X d_E\varepsilon \neq 0$ and thus $d_E\varepsilon$ is non degenerate. □

As a consequence, any maximally nonintegrable lagrangian subbundle of the form $L(E,\varepsilon)$ with E a regular involutive distribution, is of the form $e^B L(E,0)$, where $B \in \Omega^2(M)$ is such that ι^*dB is nondegenerate for ι denotes the inclusion map of E into TM or TM_C.

Definition A.7. A **nondegenerate** structure of $TM \oplus T^*M$ or $(TM \oplus T^*M)_{\mathbb{C}}$ is a lagrangian subbundle which is maximally nonintegrable.

Examples A.8.

a) Let E be a coorientable codimension-one distribution on TM, let $\theta \in \Omega^1(M)$ such that $\ker\theta = E$. Let $\iota : E \to TM$ the inclusion map and let $\varepsilon_\theta \in \Gamma(\wedge^2 E^*)$, given by $\varepsilon_\theta(X,Y) = -\theta([X,Y])$. We have seen that θ is a contact form if and only if ε is nondegenerate which is equivalent to the maximally nonintegrability of E. Thus we get that $L(E_{\mathbb{C}}, i(\varepsilon_\theta)_{\mathbb{C}})$ is a non degenerate structure with real index one.

b) Let (D,J) be an almost CR structure with codimension r and let $L = L(\ker(J_{\mathbb{C}} - iId), 0)$. Let $T_{1,0} = \ker(J_{\mathbb{C}} - iId) \subseteq TM_{\mathbb{C}}$ and let

$$N_J : \Gamma(T_{1,0}) \times \Gamma(T_{1,0}) \to \Gamma(T_{1,0})$$

$$N_J(X,Y) = J([JX,Y] + [X,JY]) + [X,Y] - [JX,JY]$$

be the Nijenhuis tensor. We know that $[\Gamma(T_{1,0}), \Gamma(T_{1,0})] \subseteq \Gamma(T_{1,0})$ if and only if $N_J = 0$ and that is equivalent to L being a complex Dirac structure. On the other hand we prove that L is nondegenerate if and only if $T_{1,0}$ is maximally nonintegrable: Let $X, Y \in \Gamma(D)$;

$$[X - iJX, Y - iJY] = [X,Y] - [JX,JY] - i([JX,Y] + [X,JY]),$$

thus $[X - iJX, Y - iJY] \notin \Gamma(E)$ if and only if $J([X,Y] - [JX,JY]) \neq [JX,Y] + [X,JY]$ which is equivalent to $N_J(X,Y) \neq 0$. Thus $\Gamma(T_{1,0})$ is maximally nonintegrable if and only if N_J is nondegenerate.

c) Let $\omega \in \Omega^2(M)$, such that $\dim\ker\omega = 1$. Consider $L = L(TM_{\mathbb{C}}, i\omega_{\mathbb{C}})$. Note that L has real index one. And note that L is a nondegenerate structure if and only if $d\omega$ is non degenerate.

d) Any almost lagrangian subbundle L has an associated trilinear skew-symmetric map $T_L \in \wedge^3 E^*$ defined in the following way $T_L(X_1, X_2, X_3) = \langle [e_1, e_2], e_3\rangle$, where $e_j = X_j + \xi_j \in \Gamma(L)$; the previous expression defines an element of $\wedge^3 E^*$. We have that L is a Dirac structure if and only if $T_L = 0$. If L is a locally conformal Dirac, cf. [38], we have that $T_L = \omega_L \wedge \varepsilon$ for some $\omega_L \in E^*$ called the Lee form and ε denotes the skew-symmetric bilinear map associated to L. Finally, we have that L is a non degenerate structure if and only if T_L is nondegenerate, in the sense that given e_1, there exist e_2 and e_3 such that $T_L(e_1, e_2, e_3) \neq 0$.

Let L be a lagrangian subbundle of $TM \oplus T^*M$. Let ρ be a local trivialization of its associated spinor line bundle. Consider the $\Gamma(\wedge^\bullet T^*M|_U)$-valued two-form,

$$\Omega_L|_U : \Gamma(L|_U) \times \Gamma(L|_U) \to \Gamma(\wedge^\bullet T^*M|_U)$$

$$\Omega_L|_U(X + \xi, Y + \eta) = [X + \xi, Y + \eta] \cdot \rho.$$

We note that Ω_L is nondegenerate if and only if L is a nondegenerate structure. The same applies for lagrangian subbundle of $(TM \oplus T^*M)_{\mathbb{C}}$. In the same way we can see the generalized almost complex structures from the point of view of the maximally nonintegrability condition, obtaining the following.

Definition A.9. A **nondegenerate generalized almost complex structure** is a nondegenerate structure of $(TM \oplus T^*M)_{\mathbb{C}}$ with real index zero. This is equivalent to a map $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$ such that $\mathcal{J}^2 = -Id$, $\mathcal{J} + \mathcal{J}^* = 0$ and with nondegenerate associated Nijenhuis tensor N_J.

Examples A.10. *a) Let ω be a nondegenerate two-form, such that $d\omega$ is non degenerate. Then $L(TM_{\mathbb{C}}, i\omega)$ is a nondegenerate generalized almost complex structure, since*

$$[X + i\iota_X\omega, Y + i\iota_Y\omega] = [X,Y] + i\iota_{[X,Y]}\omega + i\iota_Y\iota_X d\omega.$$

b) Let J be an almost complex structure with nondegenerate Nijenhuis tensor. Then $L(\ker(J_{\mathbb{C}} - iId), 0)$ is a nondegenerate generalized almost complex structure.

Remark A.11. Manifolds with nondegenerate Nijenhuis tensor are mentioned in the context of nearly Kahler manifolds in [37].

Bibliography

[1] Mohammed Abouzaid and Mitya Boyarchenko. Local structure of generalized complex manifolds. *Journal of Symplectic Geometry*, 4(1):43–62, 2006.

[2] Anton Alekseev, Henrique Bursztyn, and Eckhard Meinrenken. Pure spinors on Lie groups. *Asterisque*, (327):129–197, 2010.

[3] Anton Alekseev and Thomas Strobl. Current algebras and differential geometry. *Journal of High Energy Physics*, 2005(03):035, 2005.

[4] Michael Bailey. Local classification of generalized complex structures. *Journal of Differential Geometry*, 95(1):1–37, 2013.

[5] Oren Ben-Bassat and Mitya Boyarchenko. Submanifolds of generalized complex manifolds. *Journal of Symplectic Geometry*, 2(3):309–355, 2004.

[6] John S Bland et al. Contact geometry and CR structures on S^3. *Acta Mathematica*, 172(1):1–49, 1994.

[7] Christian Blohmann. Removable presymplectic singularities and the local splitting of Dirac structures. *International Mathematics Research Notices*, 2017(23):7344–7374, 2017.

[8] Paolo Budinich and Andrzej Trautman. Fock space description of simple spinors. *Journal of Mathematical Physics*, 30(9):2125–2131, 1989.

[9] Henrique Bursztyn. A brief introduction to Dirac manifolds. *Geometric and topological methods for quantum field theory*, pages 4–38, 2013.

[10] Henrique Bursztyn, Hudson Lima, and Eckhard Meinrenken. Splitting theorems for Poisson and related structures. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2019(754):281–312, 2019.

[11] Gil R Cavalcanti and Marco Gualtieri. Generalized complex geometry and T-duality. In P. Robert Kotiuga, editor, *A celebration of the mathematical legacy of Raoul Bott*, page 341–365. American Mathematical Soc., Providence, RI, 2010.

[12] Claude Chevalley. *The Algebraic Theory of Spinors and Clifford Algebras: Collected Works*, volume 2. Springer Science & Business Media, 1996.

[13] Theodore James Courant. Dirac manifolds. *Transactions of the American Mathematical Society*, 319(2):631–661, 1990.

[14] Marius Crainic. Generalized complex structures and Lie brackets. *Bulletin of the Brazilian Mathematical Society*, 42(4):559–578, 2011.

[15] Pierre Dazord, Thomas Delzant, et al. Le probleme général des variables actions-angles. *Journal of Differential Geometry*, 26(2):223–251, 1987.

[16] Irene Ya Dorfman. Dirac structures of integrable evolution equations. *Physics Letters A*, 125(5):240–246, 1987.

[17] E Barletta-S Dragomir. Transversally CR foliations. *Rendiconti di Matematica*, 17:51–85, 1997.

[18] Sorin Dragomir and Giuseppe Tomassini. *Differential geometry and analysis on CR manifolds*, volume 246. Springer Science & Business Media, 2007.

[19] Thiago Drummond. *Generalized reduction and pure spinors*. PhD thesis, IMPA, 2014.

[20] Jean-Paul Dufour and Aissa Wade. On the local structure of Dirac manifolds. *Compositio Mathematica*, 144(3):774–786, 2008.

[21] Rui Loja Fernandes. Lie algebroids, holonomy and characteristic classes. *Advances in Mathematics*, 170(1):119–179, 2002.

[22] Marco Gualtieri. *Generalized complex geometry*. PhD thesis, Oxford University, 2004.

[23] Marco Gualtieri. Generalized complex geometry. *Annals of mathematics*, 174(1):75–123, 2011.

[24] Nigel Hitchin. Generalized Calabi–Yau manifolds. *Quarterly Journal of Mathematics*, 54(3):281–308, 2003.

[25] Nigel Hitchin. Lectures on generalized geometry. *arXiv preprint arXiv:1008.0973*, 2010.

[26] Camille Laurent-Gengoux, Mathieu Stiénon, and Ping Xu. Holomorphic Poisson manifolds and holomorphic Lie algebroids. *International Mathematics Research Notices*, 2008, 2008.

[27] David Li-Bland. AV-Courant algebroids and generalized CR structures. *Canadian Journal of Mathematics*, 63(4):938–960, 2011.

[28] Ulf Lindström, Ruben Minasian, Alessandro Tomasiello, and Maxim Zabzine. Generalized complex manifolds and supersymmetry. *Communications in mathematical physics*, 257(1):235–256, 2005.

[29] Zhang-Ju Liu, Alan Weinstein, and Ping Xu. Manin triples for Lie bialgebroids. *J. Differential Geometry*, (45):547–574, 1997.

[30] Eckhard Meinrenken. *Clifford algebras and Lie theory*, volume 58. Springer, 2013.

[31] Dmitry Roytenberg. *Courant algebroids, derived brackets and even symplectic supermanifolds*. PhD thesis, University of California, Berkeley, 1999.

[32] Pavol Ševera. Letters to Alan Weinstein about Courant algebroids. *arXiv preprint arXiv:1707.00265*, 2017.

[33] Peter Stefan. Accessible sets, orbits, and foliations with singularities. *Proceedings of the London Mathematical Society*, 3(4):699–713, 1974.

[34] Héctor J Sussmann. Orbits of families of vector fields and integrability of distributions. *Transactions of the American Mathematical Society*, 180:171–188, 1973.

[35] François Trèves. *Hypo-Analytic Structures (PMS-40), Volume 40: Local Theory (PMS-40)*. Princeton University Press, 2014.

[36] Izu Vaisman. Reduction and submanifolds of generalized complex manifolds. *Differential Geometry and its Applications*, 25(2):147–166, 2007.

[37] Misha Verbitsky. An intrinsic volume functional on almost complex 6-manifolds and nearly Kähler geometry. *Pacific Journal of Mathematics*, 235(2):323–344, 2008.

[38] Aïssa Wade. Locally conformal Dirac structures and infinitesimal automorphisms. *Communications in mathematical physics*, 246(2):295–310, 2004.

[39] Alan Weinstein. The local structure of Poisson manifolds. *Journal of Differential Geometry*, 18(3):523–557, 1983.

[40] Hiroaki Yoshimura and Jerrold E Marsden. Dirac structures in Lagrangian mechanics part I: Implicit Lagrangian systems. *Journal of Geometry and Physics*, 57(1):133–156, 2006.