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On the Cauchy problem for some higher dimensional versions of the Benjamin-Ono equation

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Author: Oscar Guillermo Riaño Castañeda
Advisor: Felipe Linares

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To the memory of my grandmother, Lucy Rey.

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Contents

Abstract

This thesis is intended to study the initial value problem associated to some higher dimensional versions of the Benjamin-Ono equation. Firstly, we consider a mathematical extension to $\mathbb{R}^d$ of a two-dimensional model implemented to describe internal waves in stratified fluids. For the initial value problem associated to this equation, we will determine some well-posedness and ill-posedness results in classical Sobolev spaces $H^s(\mathbb{R}^d)$, and we will discuss some properties of the generalized equation derived by varying the nonlinear term. We also study some unique continuation properties of solutions to a large class of nonlinear dispersive equations. Additionally, by establishing sharp well-posedness and unique continuation principles in weighted spaces, we will characterize the spatial behavior of solutions of this model. A key ingredient in our arguments is the deduction of a new commutator estimate for the Riesz transform that could be applied for different problems. We continue our analysis studying the initial value problem associated to a model arising in the study of capillary-gravity wave flows. Initially, we will prove local well-posedness in $H^s(\mathbb{R}^2)$ and in some spaces adapted to time-invariant energy of the equation. The essential part to achieve these well-posedness conclusions is the deduction of a commutator estimate concerning the Hilbert transform operator and fractional derivatives. Next, by employing the short-time Fourier restriction norm method, we shall establish local well-posedness in bi-periodic Sobolev spaces $H^s(\mathbb{T}^2)$. We follow by deducing local well-posedness in anisotropic weighted spaces and some unique continuation principles that characterize the polynomial type decay on the first variable of this model. Finally, by applying the preceding techniques, we will derive new well-posedness results for the Shrira equation that appears in the context of waves in shear flows.

Resumo

Essa tese pretende estudar o problema de valor inicial associado a algumas versões de maior dimensão da equação de Benjamin-Ono. Primeiramente, consideramos uma extensão matemática para $\mathbb{R}^d$ de um modelo bidimensional implementado para descrever ondas internas em fluidos estratificados. Para o problema de valor inicial associado a essa equação, determinaremos alguns resultados de boa e má colocação em espaços clássicos de Sobolev $H^s(\mathbb{R}^d)$ e discutiremos algumas propriedades dos modelos gerados pela variação do termo não linear da equação descrita acima. Também estudamos alguns princípios de continuação única de soluções para uma classe de equações dispersivas não lineares. Além disso, ao estabelecer princípios de boa colocação local e de continuação única em espaços com peso, caracterizaremos o comportamento espacial das soluções desse modelo. Um ingrediente chave em nossos argumentos é a dedução de uma nova estimativa de comutador para a transformada de Riesz, que produz uma ferramenta que pode ser utilizada em outros problemas. Continuaremos nossa análise estudando o problema de valor inicial determinado por um modelo que surge no estudo dos fluxos de ondas de gravidade capilar. Primeiro, provaremos a boa colocação local em $H^s(\mathbb{R}^d)$ e em alguns espaços adaptados à energia invariante no tempo da equação. A parte essencial para se chegar a essas conclusões é a deducção de uma estimativa do comutador do operador da transformada de Hilbert e derivadas fracionárias. Em seguida, empregando o método "short-time Fourier restriction norm", estabeleceremos boa colocação local nos espaços bi-periodicos de Sobolev $H^s(\mathbb{T}^2)$. Seguidamente deduziremos a boa colocação local em espaços com pesos anisotrópicos e alguns princípios de continuação única que caracterizam o decaimento do tipo polinomial na primeira variável deste modelo. Finalmente, aplicando os métodos precedentes poderemos obter novos resultados de boa colocação para a equação de Shrira que aparece no contexto de ondas em fluxos de cisalhamento.

Introduction

This work is aimed to establish several well-posedness conclusions for different models that can be regarded, at least from a mathematical point of view, as a generalization to a several variables setting of the well-known Benjamin-Ono equation (see[**1**, **29**, **43**, **63**, **64**, **74**, **85**] and the references therein):

$$\partial_t u - \mathcal{H}_x \partial_x^2 u + u\partial_x u = 0, \tag{0.1}$$

where $\mathcal{H}_x$ denotes the Hilbert transform defined by

$$\mathcal{H}_x\phi(x) = \frac{1}{\pi}\, p.v. \int \frac{\phi(z)}{x-z}\, dz = \mathcal{F}^{-1}\big(-i\,\mathrm{sign}(\xi)\widehat{\phi}(\xi)\big)(x),$$

for $\phi \in \mathcal{S}(\mathbb{R})$ and *p.v.* denotes the Cauchy principal value.

We begin our analysis studying the initial value problem (IVP) for a higher dimensional version of the Benjamin-Ono equation (HBO):

$$\begin{cases} \partial_t u - \mathcal{R}_1 \Delta u + u\partial_{x_1} u = 0, & x \in \mathbb{R}^d,\, t \in \mathbb{R}, \\ u(x,0) = u_0, \end{cases} \tag{0.2}$$

where $d \geqslant 2$, Δ stands for the Laplace operator in the spatial variables $x \in \mathbb{R}^d$ and $\mathcal{R}_1$ denotes the Riesz transform with respect to the first coordinate defined by

$$\mathcal{R}_1\phi(x) = c_d\, p.v. \int \frac{(x_1 - z_1)\phi(z)}{|x-z|^{d+1}}\, dz = \mathcal{F}^{-1}\big(\frac{-i\xi_1}{|\xi|}\widehat{\phi}(\xi)\big)(x),$$

$\phi \in S(\mathbb{R}^d)$ and $c_d = 1/(\pi V_{d-1})$, where V_{d-1} is the volume of the unit $(d-1)$-ball.

When $d = 1$, the Riesz transform coincides with the Hilbert transform, and so we recover the Benjamin-Ono equation (0.1). When $d = 2$, equation (0.2) preserves its physical relevance, it describes the dynamics of three-dimensional slightly nonlinear disturbances in boundary-layer shear flows, without the assumption that the scale of the disturbance being smaller along than across the flow, see for instance [**2**, **71**, **87**]. We emphasize that the existence and decay rate of solitary-wave solutions in this case were studied in [**62**].

Some recent works have been devoted to establish that the IVP associated to (0.2) is locally well-posed (LWP) in the space $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$ and $d \geqslant 2$. Here we adopt Kato's notion of *well-posedness*, which consists of existence, uniqueness, persistence property (i.e., if the data $u_0 \in X$

a function space, then the corresponding solution $u(\cdot)$ describes a continuous curve in X, $u \in C([0,T];X), T>0)$, and continuous dependence of the map data-solution. Regarding the IVP (0.2), in [39] LWP in $H^s(\mathbb{R}^d)$ was deduced for $s>5/3$ when $d=2$ and for $s>(d+1)/2$ when $d \geqslant 3$. In [80], LWP was improved to the range $s>3/2$ in the case $d=2$. To the best of our knowledge there are no results concerning global well-posedness (GWP) in the current literature. It is worthwhile to mention that local well-posedness issues have been addressed by compactness methods, since one cannot solve the IVP related to (0.2) by a Picard iterative method implemented on its integral formulation for any initial data in the Sobolev space $H^s(\mathbb{R}^d)$, $d \geqslant 2$ and $s \in \mathbb{R}$. This is a consequence of the results deduced in [39] (see Theorem 2.2 below), where it was established that the flow map data-solution $u_0 \mapsto u$ for (0.2) is not of class C^2 at the origin from $H^s(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ $d \geqslant 2$.

Regarding some invariants of the equation, we notice that if u solves (0.2), then so does the scaled version u_λ defined by

$$u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t),$$

for any positive λ. Thus, one can calculate that

$$\|u_\lambda(\cdot,t)\|_{\dot{H}^s} = \lambda^{1-d/2+s} \left\|u(\cdot,\lambda^2 t)\right\|_{\dot{H}^s}.$$

As a consequence, the scale-invariant regularity for (0.2) is $s = d/2-1$. In particular, the $d=2$ problem is L^2-critical.

Real solutions of (0.2) formally satisfy at least three conservation laws (time invariant quantities)

$$\begin{aligned} I(u) &= \int u(x,t)\,dx, \\ M(u) &= \int u^2(x,t)\,dx, \qquad (0.3)\\ H(u) &= \int \left|(-\Delta)^{1/4}u(x,t)\right|^2 - \frac{1}{3}u^3(x,t)\,dx. \end{aligned}$$

It should be mentioned that we do not know of any other conservation law available for (0.2), what is more, it still remains an open question to determinate if this model is completely integrable. By way of comparison, it is known that the BO equation (0.1) is a completely integrable Hamiltonian system. For further information on this regard, we refer to [13, 49] and references therein.

Additionally, this manuscript concerns the initial value problem (IVP)

$$\begin{cases} \partial_t u + \mathcal{H}_x u - \mathcal{H}_x \partial_x^2 u \pm \mathcal{H}_x \partial_y^2 u + u\partial_x u = 0, & (x,y) \in \mathbb{R}^2 \text{ (or } (x,y) \in \mathbb{T}^2), \ t \in \mathbb{R}, \\ u(x,y,0) = u_0, \end{cases} \qquad (0.4)$$

where $\mathcal{H}_x$ denotes the Hilbert transform in the x-directions defined by $\mathcal{F}(H_x\phi(x,y))(\xi,\eta) = -i\,\mathrm{sign}(\xi)\phi(\xi,\eta)$ for $\phi \in S(\mathbb{R}^2)$, and its periodic equivalent

$$\mathcal{H}_x\phi(x,y) = \frac{1}{2\pi}\, p.v. \int_{-\pi}^{\pi} \cot\big(\frac{x-z}{2}\big)\phi(z,y)\,dz = \mathcal{F}^{-1}\big(-i\,\mathrm{sign}(m)\widehat{\phi}(m,n)\big)(x,y),$$

for all $\phi \in C^{\infty}(\mathbb{T}^2)$. This model was derived in [3] as an approximation to the equations for deep water gravity-capillary waves. Numerical results determining existence of line solitary waves (solutions of the form $u(x,y,t) = \varphi(x-ct,y)$, $c>0$ and φ real valuable with suitable decay at infinity) as well as wavepacket lump solitary waves were also presented in [3].

Alternatively, the equation (0.4) can be considered as a two-dimensional extension of the so called Burgers-Hilbert equation (see, [5, 40]):

$$\partial_t u + \mathcal{H}_x u + u\partial_x u = 0. \tag{0.5}$$

We are also interested in studying the IVP associated to the Shrira equation:

$$\begin{cases} \partial_t u - \mathcal{H}_x \partial_x^2 u - \mathcal{H}_x \partial_y^2 u + u\partial_x u = 0, & (x,y)\in \mathbb{R}^2 \text{ (or } (x,y)\in\mathbb{T}^2\text{)},\ t\in\mathbb{R}, \\ u(x,y,0) = u_0. \end{cases} \tag{0.6}$$

This equation was deduced as a simplified model to describe a two-dimensional weakly nonlinear long-wave perturbation on the background of a boundary-layer type plane-parallel shear flow (see [72]). Existence and asymptotic behavior of solitary-wave solutions were studied in [24].

Concerning well-posedness for the IVP (0.4), LWP in $H^s(\mathbb{R}^2)$ and $Y^s(\mathbb{R}^2) = \{f \in H^s : \|f\|_{Y^s} = \|f\|_{H^s} + \|\partial_x^{-1} f\|_{H^s} < \infty\}$ $s>2$, were inferred in [21]. These results were provided by implementing a parabolic regularization argument in the spirit of [45]. It was also showed in the same reference that (0.4) is LWP in weighted Sobolev spaces $Y^s(\mathbb{R}^2)\cap L^2(|x|^{2r}+|y|^{2r}\,dxdy)$, $0\leqslant r\leqslant 1$ and $s>2$.

With respect to (0.6), by adapting the short-time linear Strichartz estimate approach employed in [50, 59], LWP in $H^s(\mathbb{R}^2)$ $s>3/2$ was deduced in [11]. In [10], inspired by the work of [41, 57], LWP was established in $H^s(\mathbb{T}^2)$ $s>7/4$ assuming that the initial data satisfies, $\int_0^{2\pi} u_0(x,y)\,dx = 0$ for almost every y. Recently, in [81], by employing short-time bilinear Strichartz estimates the conclusion on the periodic setting was improved to regularity $s>3/2$ without any assumption on the initial data. Furthermore, in [61], LWP was deduced in the spaces $H^{s_1,s_2}(\mathbb{R}^2)\cap L^2(|x|^{2\theta}\,dxdy)$ $s_1 \geqslant 2$, where $0\leqslant\theta<1/2$ for arbitrary initial data, and $1/2<\theta<1$ assuming that $\widehat{u}(0,\eta)=0$ for almost every η. Besides, it was also determined LWP in the spaces $H^{s_1,s_2}(\mathbb{R}^2)\cap L^2(|y|^{2r}\,dxdy)$, $s_2\geqslant r$.

It is worth pointing out that (0.4) does not enjoy of scale invariance. In contrast, if u solves (0.6), $u_\lambda(x,y,t) = \lambda u(\lambda x, \lambda y, \lambda^2 t)$ solves (0.6) whenever $\lambda>0$, and so

$$\|u_\lambda(\cdot,\cdot,t)\|_{\dot{H}^s} = \lambda^s \left\|u(\cdot,\cdot,\lambda^2 t)\right\|_{\dot{H}^s}.$$

Thus (0.6) is L^2-critical. On the other hand, real solutions of (0.4) formally satisfy the following conserved quantities (time invariant):

$$M(u) = \int u^2(x,y,t)\,dxdy, \tag{0.7}$$

$$E(u) = \frac{1}{2}\int |D_x^{1/2}u(x,y,t)|^2 + |D_x^{-1/2}u(x,y,t)|^2 \mp |D_x^{-1/2}\partial_y u(x,y,t)|^2 - \frac{1}{3}u^3(x,y,t)\,dxdy, \tag{0.8}$$

and real solutions of (0.6) preserve the quantity $M(u)$ and

$$\widetilde{E}(u) = \frac{1}{2}\int |D_x^{1/2}u(x,y,t)|^2 + |D_x^{-1/2}\partial_y u(x,y,t)|^2 - \frac{1}{3}u^3(x,y,t)\,dxdy, \tag{0.9}$$

where $D_x^{\pm 1/2}$ is the fractional derivative operator in the x variable defined by its Fourier transform as $\mathcal{F}(D_x^{\pm 1/2}u)(\xi,\eta) = |\xi|^{\pm 1/2}\widehat{u}(\xi,\eta)$. As far as we know, it has not been determined whether (0.4) and (0.6) are completely integrable.

This thesis is intended to obtain well-posedness conclusions for the model (0.2) in the spaces $H^s(\mathbb{R}^d)$ and in weighted spaces. Concerning (0.4) and (0.6), we deduce well-posedness result in the spaces $H^s(\mathbb{K}^2)$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{T}\}$ and in some spaces adapted to (0.8) and (0.9). Additionally, we obtain some well-posedness conclusion for the models (0.4) and (0.6) in anisotropic weighted Sobolev spaces. In consequence, we will study the spatial behavior of solutions of the previous equations, determining that in general arbitrary polynomial type decay in the x-spatial variable is not preserved by the flow of these equations. To achieve this conclusion, we shall establish some unique continuation principles, as well as some commutator estimates for the Riesz and Hilbert transforms (see Propositions 3.8 and 4.2 respectively) that may be of independent interest and are of interest on their own in harmonic analysis.

This document is organized as follows: In Chapter 1 we set up some general notation and preliminaries that will be implemented to analyze all the previous models. Next, in Chapter 2, we proceed to study well-posedness and ill-posedness issues in $H^s(\mathbb{R}^d)$ for equation (0.2). In this part, we also compile some remarks for a generalized version of this equation, and we determine some local unique continuation principles for a large class of dispersive equations. Chapter 3 concerns the study of (0.2) in weighted Sobolev spaces. More precisely, we determine LWP and unique continuation principles in weighted spaces that characterize the spatial behavior of solutions of (0.2). A key ingredient in our arguments is the deduction of Proposition 3.8, where we find a new commutator estimate involving Riesz transform operators. Subsequently, Chapter 4 is devoted to provide different well-posedness conclusions for the equations (0.4) and (0.6). We first prove local well-posedness in $H^s(\mathbb{R}^2)$ and in some spaces adapted to the energy (0.8). Then we determine well-posedness in the periodic Sobolev spaces $H^s(\mathbb{T}^2)$. We follow by establishing well-posedness in anisotropic weighted spaces and some unique continuation principles. We conclude with an appendix where we prove the fractional commutator estimate for the Hilbert transform stated in Proposition 4.2.

Chapter 1

Preliminaries and notation

We will employ the standard multi-index notation, $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$, $|\alpha| = \sum_{j=1}^{d} \alpha_j$, $\alpha! = \alpha_1! \cdots \alpha_d!$ and $\alpha \leqslant \beta$ if $\alpha_j \leqslant \beta_j$ for all $j = 1, \dots, d$. As usual $e_k \in \mathbb{R}^d$ will denote the standard canonical vector in the k direction.

For any two positive quantities a and b, $a \lesssim b$ means that there exists $C > 0$ independent of a and b (and in our computations of any parameter involving approximations) such that $a \leqslant Cb$. Similarly, we define $a \gtrsim b$, and $a \sim b$ states that $a \lesssim b$ and $b \gtrsim a$. $[A, B]$ denotes the commutator between the operators A and B, that is

$$[A, B] = AB - BA.$$

Given $p \in [1, \infty]$, the Lebesgue spaces $L^p(\mathbb{K})$ are defined in the usual manner, the norm will be denoted by $\|f\|_{L^p} = \|f\|_{L^p(\mathbb{K})}$ (the set $\mathbb{K}$ will be easily identified according to the context). In the two dimensional case, to emphasize the dependence on the variables, we will denote by $\|f\|_{L^p} = \|f\|_{L^p_{xy}}$. We denote by $C_c^\infty(\mathbb{R}^d)$ the spaces of smooth functions of compact support and $\mathcal{S}(\mathbb{R}^d)$ the space of Schwartz functions. The Fourier transform is defined as

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx.$$

As usual, the operator $J^s = (1 - \Delta)^{s/2}$ is defined by the Fourier multiplier with symbol $\langle \xi \rangle^s = (1 + |\xi|^2)^{s/2}$, $s \in \mathbb{R}$. The norm in the Sobolev space $H^s(\mathbb{R}^d)$ is given by

$$\|f\|_{H^s} = \|J^s f\|_{L^2} = \|\langle \xi \rangle^s \widehat{f}(\xi)\|_{L^2},$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$. Similarly, the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ is determined by its norm, $\|f\|_{\dot{H}^s} = \||\xi|^s \widehat{f}(\xi)\|_{L^2}$.

The Sobolev space $W^{1,\infty}(\mathbb{R}^2)$ is defined as usual with norm $\|f\|_{W^{1,\infty}} := \|f\|_{L^\infty} + \|\nabla f\|_{L^\infty}$, and $W_x^{1,\infty}(\mathbb{R}^d)$ is defined according to $\|f\|_{W_x^{1,\infty}} := \|f\|_{L^\infty} + \|\partial_x f\|_{L_x^\infty}$. We are also interested in studying well-posedness issues in weighted spaces

$$Z_{s,r}(\mathbb{R}^d) = H^s(\mathbb{R}^d) \cap L^2(|x|^{2r}\, dx), \quad s, r \in \mathbb{R} \tag{1.1}$$

and

$$\dot{Z}_{s,r}(\mathbb{R}^d) = \left\{ f \in H^s(\mathbb{R}^d) \cap L^2(|x|^{2r}\, dx) : \widehat{f}(0) = 0 \right\}, \quad s, r \in \mathbb{R}. \tag{1.2}$$

To analyze the spatial asymptotics of (0.4) and (0.6), we consider the following anisotropic weighted Sobolev spaces:

$$Z_{s,r_1,r_2}(\mathbb{R}^2) = H^s(\mathbb{R}^2) \cap L^2(|x|^{2r_1} + |y|^{2r_2}\,dxdy), \quad s, r_1, r_2 \in \mathbb{R} \tag{1.3}$$

and

$$\dot{Z}_{s,r_1,r_2}(\mathbb{R}^2) = \left\{ f \in H^s(\mathbb{R}^2) \cap L^2(|x|^{2r_1} + |y|^{2r_2}\,dxdy) : \widehat{f}(0,\eta) = 0 \right\}, \quad s, r_1, r_2 \in \mathbb{R}. \tag{1.4}$$

Now, if A denotes a functional space (for instance any of the spaces introduced above), we define the spaces $L^p_T A$ and $L^p_t A$ according to the norms

$$\|f\|_{L^p_T A} = \|\|f(\cdot,t)\|_A\|_{L^p([0,T])} \text{ and } \|f\|_{L^p_t A} = \|\|f(\cdot,t)\|_A\|_{L^p(\mathbb{R})}, \tag{1.5}$$

respectively, for all $1 \leqslant p \leqslant \infty$.

The variable N is presumed to be dyadic, i.e., $N \in \{2^l : l \in \mathbb{Z}\}$. To study the IVP (0.4), we will mostly use the dyadic numbers $N \geqslant 1$, then we set $\mathbb{D} = \{2^l : l \in \mathbb{Z}^+ \cup \{0\}\}$. Let $\psi_0 \in C_c^\infty(\mathbb{R}^d)$ radial such that

$$0 \leqslant \psi_0 \leqslant 1, \ \ \psi_0(\xi) = 1 \text{ for } |\xi| \leqslant 1, \ \ \psi_0(\xi) = 0 \text{ for } |\xi| \geqslant 2, \tag{1.6}$$

and set $\psi(\xi) = \psi_0(\xi) - \psi_0(2\xi)$ which is supported on $1/2 \leqslant |\xi| \leqslant 2$. For any $f \in \mathcal{S}(\mathbb{R}^d)$ and N dyadic, we define the Littlewood-Paley projection operators

$$\begin{aligned} \widehat{P_N f}(\xi) &= \psi(\xi/N)\widehat{f}(\xi), \\ \widehat{P_{\leqslant N} f}(\xi) &= \psi_0(\xi/N)\widehat{f}(\xi), \ \ \xi \in \mathbb{R}^d \end{aligned} \tag{1.7}$$

and $\widetilde{P}_N = \sum_{M \sim N} P_M$ (for our considerations, $\sum_{M\sim N} P_M = \sum_{|j|\leqslant 2} P_{2^j N}$). Then by support considerations, $P_{N_1}P_{N_2} = 0$ when $N_1 > 2N_2$. Next, we recall Bony's paraproduct decomposition (see for instance [68]) for a pair of functions f, g given by

$$fg = \sum_{N>0} P_N f \widetilde{P}_N g + \sum_{N>0} P_N f P_{<N/2} g + \sum_{j} P_{<N/2} f P_N g \tag{1.8}$$

1.1. Commutators, interpolation and some additional estimates

To obtain estimates for the nonlinear terms, the following Leibniz rules for fractional derivatives will be implemented in our arguments.

Lemma 1.1. *If $s > 0$ and $1 < p < \infty$, then*

$$\|[J^s, f]g\|_{L^p(\mathbb{R}^d)} \lesssim \|\nabla f\|_{L^\infty(\mathbb{R}^d)} \|J^{s-1} g\|_{L^p(\mathbb{R}^d)} + \|J^s f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^\infty(\mathbb{R}^d)}, \tag{1.9}$$

where

$$[J^s, f]g = J^s(fg) - fJ^s g.$$

Lemma 1.1 was proved by Kato and Ponce in [48]. We also need the following lemma whose proof can be find in [34].

Lemma 1.2. *Given $d \in \mathbb{Z}^+$ and $s > 0$, it holds that*

$$\|D^s(fg)\|_{L^2(\mathbb{R}^d)} \lesssim \|D^s f\|_{L^{p_1}(\mathbb{R}^d)} \|g\|_{L^{q_1}(\mathbb{R}^d)} + \|f\|_{L^{p_2}(\mathbb{R}^d)} \|D^s g\|_{L^{q_2}(\mathbb{R}^d)}, \tag{1.10}$$

$$\|J^s(fg)\|_{L^2(\mathbb{R}^d)} \lesssim \|J^s f\|_{L^{p_1}(\mathbb{R}^d)} \|g\|_{L^{q_1}(\mathbb{R}^d)} + \|f\|_{L^{p_2}(\mathbb{R}^d)} \|J^s g\|_{L^{q_2}(\mathbb{R}^d)}, \tag{1.11}$$

with $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{2}$, $1 < p_1, p_2, q_1, q_2 \leqslant \infty$.

Lemma 1.3. *Let $\sigma, \beta \in (0,1)$, then*

$$\begin{aligned}\|D_x^\sigma D_y^\beta(fg)\|_{L^2(\mathbb{R}^2)} \lesssim & \|f\|_{L^{p_1}(\mathbb{R}^2)}\|D_x^\sigma D_y^\beta g\|_{L^{q_1}(\mathbb{R}^2)} + \|D_x^\sigma D_y^\beta f\|_{L^{p_2}(\mathbb{R}^2)}\|g\|_{L^{q_2}(\mathbb{R}^2)} \\ & + \|D_y^\beta f\|_{L^{p_3}(\mathbb{R}^2)}\|D_x^\sigma g\|_{L^{q_3}(\mathbb{R}^2)} + \|D_x^\sigma f\|_{L^{p_4}(\mathbb{R}^2)}\|D_y^\beta g\|_{L^{q_4}(\mathbb{R}^2)},\end{aligned} \tag{1.12}$$

where $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{2}$, $1 < p_j, q_j \leqslant \infty$, $j = 1,2,3,4$.

Lemma 1.3 was deduced by Muscalu, Pipher, Tao and Thiele in [67].

In addition, we require the following set of inequalities, which were deduced in the proof of [50, Lemma 2.1] (see equations (2.5), (2.6) and (2.7) in this reference). See also [59, Lemma 4.6].

Lemma 1.4. (i) *Let $0 < \delta < 1/2$, then*

$$\|D_x^{1/2+\delta} u\|_{L^\infty_{xy}} \lesssim \|u\|_{L^\infty_{xy}} + \|\partial_x u\|_{L^\infty_{xy}}. \tag{1.13}$$

(ii) *If δ_0 is a positive constant chosen small enough, then the following holds true. There exist*

$$\begin{cases} 2 < p_1, q_1 < \infty \\ 1 < r_1, s_1 < \infty \end{cases} \quad \textit{with} \quad \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2}, \ \frac{1}{r_1} + \frac{1}{s_1} = 1,$$

$0 < \theta < 1$ and $0 < \delta_1 = \delta_1(\delta_0, \theta) \ll 1$ such that

$$\|\partial_x D_x^{1/2+\delta} u\|_{L^{s_1}_T L^{q_1}_{xy}} \lesssim \|\partial_x u\|^{\theta}_{L^1_T L^\infty_{xy}} \|J_x^{3/2+\delta_0} u\|^{1-\theta}_{L^\infty_T L^2_{xy}}, \tag{1.14}$$

and

$$\|D_y^\delta u\|_{L^{r_1}_T L^{p_1}_{xy}} \lesssim \left(\|u\|_{L^1_T L^\infty_{xy}}\right)^{1-\theta} \left(\|D_y^{1/2} u\|_{L^\infty_T L^2_{xy}} + \|u\|_{L^\infty_T L^2_{xy}}\right)^{\theta}, \tag{1.15}$$

for all $0 < \delta < \delta_1$.

The following result will be useful to implement energy estimates for the equation (0.4).

Proposition 1.5. *Let $1 < p < \infty$ and $l, m \in \mathbb{Z}^+ \cup \{0\}$, $l + m \geqslant 1$ then*

$$\|\partial_x^l [\mathcal{H}_x, g] \partial_x^m f\|_{L^p(\mathbb{R})} \lesssim_{p,l,m} \|\partial_x^{l+m} g\|_{L^\infty(\mathbb{R})} \|f\|_{L^p(\mathbb{R})}. \tag{1.16}$$

The estimate (1.16) was established in [19, Lemma 3.1] and it was extended to the BMO spaces in [56, Proposition 3.8].

Our arguments require the following proposition due to Coifman-Meyer (see [14, 15] and [33]).

Proposition 1.6. *Let $\sigma(\xi,\eta) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d \backslash (0,0))$ satisfying*

$$|\partial_\xi^{\gamma_1}\partial_\eta^{\gamma_2}\sigma(\xi,\eta)| \lesssim_{\gamma_1,\gamma_2} (|\xi|+|\eta|)^{-(|\gamma_1|+|\gamma_2|)} \tag{1.17}$$

for all multi-index γ_1, γ_2 and for all $(\xi,\eta) \neq (0,0)$. Define

$$\sigma(D)(f,g)(x) = \int e^{ix\cdot(\xi+\eta)}\sigma(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)d\xi d\eta. \tag{1.18}$$

Then for any $1 < p < \infty$,

$$\|\sigma(D)(f,g)\|_{L^p} \lesssim \|f\|_{L^\infty}\|g\|_{L^p}.$$

We shall use the following Fefferman-Stein inequality.

Lemma 1.7. *([26]) Let $f = (f_j)_{j=1}^\infty$ be a sequence of locally integrable functions in $\mathbb{R}^d$. Let $1 < p < \infty$. Then*

$$\left\|(\mathcal{M}f_j)_{l_j^2}\right\|_{L^p} \lesssim \left\|(f_j)_{l_j^2}\right\|_{L^p}$$

where $\mathcal{M}f$ is the usual Hardy-Littlewood maximal function.

Denoting by $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions, we have:

Lemma 1.8. *Let $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}(\phi) \subset \{|\xi| \leqslant R\}$ for some $R > 0$. Consider the operator $P^\phi f$ determined by $\widehat{P^\phi f}(\xi) = \phi(\xi)\widehat{f}(\xi)$. Then*

$$\sup_{z\in\mathbb{R}^d}\frac{|P^\phi f(x-z)|}{(1+R|z|)^d} \lesssim \mathcal{M}(f)(x). \tag{1.19}$$

Proof. See for instance [56, Lemma 2.3]. □

1.2. Preliminaries weighted spaces

For a given $n \in \mathbb{Z}^+$, we introduce the truncated weights $\tilde{w}_n : \mathbb{R} \to \mathbb{R}$ satisfying

$$\tilde{w}_n(x) = \begin{cases} \langle x \rangle, \text{ if } |x| \leqslant n, \\ 2n, \text{ if } |x| \geqslant 3n \end{cases} \tag{1.20}$$

in such a way that $\tilde{w}_n(x)$ is smooth and non-decreasing in $|x|$ with $\tilde{w}_n'(x) \leqslant 1$ for all $x > 0$ and there exists a constant c independent of N from which $|\tilde{w}_n''(x)| \leqslant c\partial_x^2\langle x\rangle$. We then define the d-dimensional weights by the relation

$$w_n(x) = \tilde{w}_n(|x|), \text{ where } |x| = \sqrt{x_1^2 + \cdots + x_d^2}. \tag{1.21}$$

We require some point-wise bounds for the product between powers of the weight w_n and a polynomial with variables in $\mathbb{R}^d$. More specifically, for a given $\theta \in (0,2]$ and multi-indexes α and β with $1 \leqslant |\alpha| \leqslant 2$, by the definition of w_N one finds

$$|\partial^\alpha w_n^\theta(x)x^\beta| \lesssim w_n^{\theta+|\beta|-|\alpha|}(x), \tag{1.22}$$

where the implicit constant is independent of n and θ. In particular, when $\theta \leqslant |\alpha|$ and $\beta = 0$, $|\partial^\alpha w_n^\theta| \lesssim 1$.

The definition of the $A_p(\mathbb{R}^d)$ condition is essential in our analysis. For a more detailed discussion on this regard, we refer to [20, 82].

Definition 1.9. *A non-negative function $w \in L^1_{loc}(\mathbb{R}^d)$ satisfies the $A_p(\mathbb{R}^d)$ inequality with $1 < p < \infty$ if there exists a constant C independent of the cube Q, such that*

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int w(x)^{1-p'}dx\right)^{p-1} = Q_p(w) \leqslant C \tag{1.23}$$

where the supremum runs over cubes in $\mathbb{R}^d$ and $1/p + 1/p' = 1$.

For instance we have

$$|x|^\theta \in A_p(\mathbb{R}^d), \text{ whenever } -d < \theta < d(p-1).$$

Since we are concerned with weighted energy estimates, we require some continuity properties of Riesz transforms in weighted spaces (we refer to [83] for further information).

Theorem 1.10. *([73]) For $1 < p < \infty$ and $l = 1, \dots, d$ there exists a constant c depending on p and d so that for all weights $w \in A_p(\mathbb{R}^d)$ the Riesz transforms as operators in weighted space $\mathcal{R}_l : L^p(w(x)\,dx) \mapsto L^p(w(x)\,dx)$ satisfies*

$$\left(\int_{\mathbb{R}^d} |\mathcal{R}_l f(x)|^p w(x)dx\right)^{1/p} \leqslant c\, Q_p(w)^r \left(\int_{\mathbb{R}^d} |f(x)|^p w(x)dx\right)^{1/p} \tag{1.24}$$

where $Q_p(w)$ is defined by (1.23), $r = \max\{1, p'/p\}$. Moreover, this result is sharp.

One can verify that for fixed $\theta \in (-d, d)$, $w_n^\theta(x)$, $n \in \mathbb{Z}^+$, satisfies the $A_2(\mathbb{R}^d)$ inequality with a constant $Q_2(w_n^\theta)$ independent of n. From this fact and Theorem 1.10, we infer:

Proposition 1.11. *For any $\theta \in (-d, d)$ and any $n \in \mathbb{Z}^+$, $w_n^\theta(x)$ satisfies the $A_2(\mathbb{R}^d)$ inequality (1.23).*

Moreover, the Riesz transform is bounded in $L^2(w_n^\theta(x)\,dx)$ with a constant depending on θ but independent of $n \in \mathbb{Z}^+$.

Proposition 1.11 is helpful to show that our computations in the proof of Theorem 3.2 are independent of the parameter n defining the weight w_n. We also require the following commutator relation.

Proposition 1.12. *Let $\theta \in (0, 1)$ and $1 \leqslant p_1, p_2 < \infty$ such that $\frac{3}{2} = \frac{1}{p_1} + \frac{1}{p_2}$. Then*

$$\|[D^\theta, g]f\|_{L^2} \lesssim \||\cdot|^\theta \widehat{g}\|_{L^{p_1}} \|\widehat{f}\|_{L^{p_2}}. \tag{1.25}$$

The following characterization of the spaces $L^p_s(\mathbb{R}^d) = J^{-s}L^p(\mathbb{R}^d)$ is fundamental in our considerations.

Theorem 1.13. *([84]) Let $b \in (0, 1)$ and $2d/(d + 2b) < p < \infty$. Then $f \in L^p_b(\mathbb{R}^d)$ if and only if*

(i) $f \in L^p(\mathbb{R}^d)$,

(ii) $\mathcal{D}^b f(x) = \left(\int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2b}}\,dy\right)^{1/2} \in L^p(\mathbb{R}^d)$,

with

$$\|J^b f\|_{L^p} = \|(1-\Delta)^{b/2} f\|_{L^p} \sim \|f\|_{L^p} + \|\mathcal{D}^b f\|_{L^p} \sim \|f\|_{L^p} + \|D^b f\|_{L^p}.$$

Above we have introduced the notation $D^s = (-\Delta)^{s/2}$.

Next, we proceed to show several consequences of Theorem 1.13. When $p = 2$ and $b \in (0,1)$ one can deduce that

$$\|\mathcal{D}^b(fg)\|_{L^2} \lesssim \|f\mathcal{D}^b g\|_{L^2} + \|g\mathcal{D}^b f\|_{L^2}, \tag{1.26}$$

and

$$\|\mathcal{D}^b h\|_{L^\infty} \lesssim \left(\|h\|_{L^\infty} + \|\nabla h\|_{L^\infty}\right). \tag{1.27}$$

The estimates (1.26) and (1.27) yield:

PROPOSITION 1.14. *Let $h \in L^\infty(\mathbb{R}^d)$ with $\nabla h \in L^\infty(\mathbb{R}^d)$. Then*

$$\|hf\|_{H^{1/2}} \lesssim \left(\|h\|_{L^\infty} + \|\nabla h\|_{L^\infty} \right) \|f\|_{H^{1/2}}.$$

As a further consequence of Theorem 1.13 one has the following interpolation inequality.

Lemma 1.15. *Let $a, b > 0$. Assume that $J^a f = (1-\Delta)^{a/2} f \in L^2(\mathbb{R}^d)$ and $\langle x \rangle^b f = (1+|x|^2)^{b/2} f \in L^2(\mathbb{R}^d)$. Then for any $\mathfrak{r} \in (0,1)$,*

$$\left\| J^{\mathfrak{r}a}(\langle x \rangle^{(1-\mathfrak{r})b} f) \right\|_{L^2} \lesssim \|\langle x \rangle^b f\|_{L^2}^{1-\mathfrak{r}} \|J^a f\|_{L^2}^{\mathfrak{r}}. \tag{1.28}$$

Moreover, the inequality (1.28) is still valid with $w_n(x)$ instead of $\langle x \rangle$ with a constant c independent of n.

PROOF. The proof follows the ideas in [29, Lemma 1]. □

Chapter 2

Study of the HBO equation in $H^s(\mathbb{R}^d)$

In this chapter, we study local well-posedness in $H^s(\mathbb{R}^d)$ for the initial value problem (0.2). Additionally, we determine some ill-posedness conclusions for this model and we review some results for the generalized equation determined by (0.2). We conclude by showing some unique continuation principles for a family of dispersive equations that includes the equation (0.2). The main results in this chapter are contained in [39].

2.1. Statement of results

To motive our conclusions, we combine the Kato–Ponce commutator estimate [48] with Gronwall's inequality to obtain that any smooth solution of (0.2) defined on an interval $[0, T]$ satisfies

$$\sup_{t\in[0,T]} \|u(t)\|_{H^s} \leqslant \|u(0)\|_{H^s} \exp\Big(c\int_0^T \|\nabla u(t)\|_{L^\infty}\, dt\Big). \tag{2.1}$$

Thus, if we could control the norm $\|\nabla u\|_{L^1_T L^\infty_x}$ of the exponential function by the $H^s(\mathbb{R}^d)$-norm, we could argue by compactness in order to establish existence of solutions with less regularity. If this were to be done by using the Sobolev embedding $H^s(\mathbb{R}^d) \hookrightarrow W^{1,\infty}(\mathbb{R}^d)$ with order of regularity $s > d/2 + 1$, we would not take into account the dispersive effect of the equation (0.2).

Instead, we follow the short-time Strichartz linear approach introduced by Koch and Tzvetkov [54] implemented to study the local well-posedness of the one-dimensional Benjamin-Ono equation. Roughly this consists of determining a refined Strichartz estimate (see Lemma 2.7 below) that allows us to control the $L^1([0, T]; W^{1,\infty}(\mathbb{R}^d))$-norm of smooth solutions without relying on Sobolev's embeddings. Extensions of this method were given by Kenig and König [51], and in two dimensions by Kenig [50], and Linares, Pilod and Saut [59].

Let us now state our results. Our first conclusion improves the standard well-posedness results provided by a parabolic regularization argument on (0.2) (see Lemma 2.10 below).

Theorem 2.1. *Let $s > s_d$ where $s_d := d/2 + 1/2$ for $d \geqslant 3$ and $s_2 := 5/3$. Then, for any $u_0 \in H^s(\mathbb{R}^d)$, there exist a time $T = T(\|u_0\|_{H^s})$ and a unique solution u to (0.2) that belongs to*

$$C\big([0, T); H^s(\mathbb{R}^d)\big) \cap L^1\big([0, T); W^{1,\infty}(\mathbb{R}^d)\big).$$

Moreover, the flow map $u_0 \mapsto u(t)$ is continuous from $H^s(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$.

To the best of our knowledge, the previous theorem determines the first non-standard result regarding local well-posedness for equation (0.2). In [**80**], by means of the short-time Fourier restriction norm method developed by Ionescu, Kenig and Tataru [**44**], the result of Theorem 2.1 was improved to regularity $s > 3/2$ for $d = 2$. However, our well-posedness conclusions are the best known for (0.2) for dimension $d \geqslant 3$.

Remarks. (i) *Concerning the Benjamin-Ono equation* (0.1)*, Tao* [**85**] *introduced a gauge transformation which allowed him to establish local and global results in $H^1(\mathbb{R}^2)$. Moreover, it was possible to go all the way to $L^2(\mathbb{R}^2)$ by using this gauge transformation; see* [**43, 64**]*. However, we do not know if there is such a gauge transformation for the equation* (0.2)*. Additionally, we do not know of a maximal estimate function for* (0.2) *that would help us to adapt the arguments in* [**51**] *to improve our conclusion in Theorem 2.1.*

(ii) *Our well-posedness results require too much regularity to take advantage of $H(u)$ in* (0.3)*. As a matter of fact, we do not know of any result concerning global well-posedness for the equation* (0.2)*.*

Next, we will show that the flow map $u_0 \mapsto u(t)$ is not of class C^2 for any $s \in \mathbb{R}$. In particular, this implies that (0.2) cannot be solved by using the Duhamel formulation combined with the contraction mapping principle in $H^s(\mathbb{R}^d)$.

Theorem 2.2. *Let $s \in \mathbb{R}$. Then* (0.2) *does not admit a solution u such that the flow map $u_0 \mapsto u(t)$ is C^2-differentiable from $H^s(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$.*

With $d = 2$, we use the existence of solitary wave solutions [**62**] to show that the flow map cannot be uniformly continuous in $L^2(\mathbb{R}^2)$.

Proposition 2.3. *Let $d = 2$. Then* (0.2) *does not admit a solution u such that the flow map $u_0 \mapsto u(t)$ is uniformly continuous from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$.*

Additionally, we are interested in study ill-posedness issues for the following generalized equation associated to (0.2),

$$u_t - \mathcal{R}_1 \Delta u + u^k u_{x_1} = 0 \quad (x,t) \in \mathbb{R}^{d+1} \tag{2.2}$$

with $k \geqslant 2$ integer. To motivate our result, we notice that if u solves (2.2) with initial data u_0, then $u_\lambda(x,t) = \lambda^{1/k} u(\lambda x, \lambda^2 t)$ also solves (2.2) with initial condition $u_\lambda(x,0) = \lambda^{1/k} u_0(\lambda x)$ for all $\lambda > 0$. Consequently, since

$$\|u_\lambda(\cdot,0)\|_{\dot{H}^s} = \lambda^{1/k-d/2+s} \|u_0\|_{\dot{H}^s}, \tag{2.3}$$

we deduce that the scale-invariant Sobolev space to study (2.2) is $\dot{H}^{s_{crit(k)}}(\mathbb{R}^d)$ where $s_{crit}(k) = d/2 - 1/k$. Thus, the natural spaces $H^s(\mathbb{R}^d)$ to address well-posedness issues are those with regularity $s \geqslant s_{crit(k)}$.

The next result establishes that below the critical index $s_{crit(k)}$ the flow-map data solution associated to (2.2) fails to be of class C^{k+1}. In particular, this implies that (2.2) cannot be solved in $H^s(\mathbb{R})$, $s < s_{crit(k)}$ employing a contraction argument. This type of ill-posedness result can be view as an extension of those deduced in [**65**] for the generalized Benjamin-Ono equation.

PROPOSITION 2.4. *Let $k \geqslant 2$ integer and $s < d/2 - 1/k$. Then for any $T > 0$, the flow-map $u_0 \mapsto u$ (if it exists) is not of class C^{k+1} from $H^s(\mathbb{R}^d)$ to $C([-T,T]; H^s(\mathbb{R}^d))$ at the origin.*

We follow by presenting some known facts concerning the stability of solitary wave solutions for the equation (0.2). Finally, we conclude this chapter establishing some unique continuation properties of solutions to a large class of nonlinear dispersive equations.

This chapter is organized as follows: we begin by showing some linear and energy estimates. Theorem 2.1 is deduced in the following subsection. Next we prove the ill-posedness conclusions of Theorem 2.2 and Proposition 2.3 in Section 2.4. In Section 2.5, we deduce Proposition 2.4 and we discuss some aspects regarding stability and instability of solitary wave solutions. Finally, Section 2.6 is aimed to deduce some local unique continuation principles.

2.2. Preliminary estimates

2.2.1. Linear estimates. This subsection is devoted to deduce some estimates for the linear equation determined by (0.2):

$$\begin{cases} u_t - \mathcal{R}_1 \Delta u = 0, \; x \in \mathbb{R}^d, \quad t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{2.4}$$

whose solution are defined trough the unitary group

$$U(t)u_0(x) = \int e^{i\xi_1|\xi|t + ix\xi} \widehat{u_0}(\xi)\, d\xi, \tag{2.5}$$

$t \in \mathbb{R}$. We begin by recalling the following Strichartz estimates for the group $\{U(t)\}_{t\in\mathbf{R}}$ established in [39].

PROPOSITION 2.5. *The following estimates hold*

$$\|U(t)f\|_{L^r_t L^q_x} \lesssim \|f\|_{\dot{H}^{s_d}} \tag{2.6}$$

and

$$\left\| D^{-2\widetilde{s}_d} \int_0^t U(t-t')G(t')\, dt' \right\|_{L^r_t L^q_x} \leqslant c \, \|G\|_{L^{r'}_t L^{q'}_x} \tag{2.7}$$

for $q < \infty$ with

$$\begin{cases} \dfrac{10}{q} + \dfrac{12}{r} \leqslant 5 \quad \text{and} \quad \widetilde{s}_2 = 1 - \dfrac{2}{q} - \dfrac{2}{r}, & \text{if } d = 2 \\ \dfrac{1}{q} + \dfrac{1}{r} \leqslant \dfrac{1}{2} \quad \text{and} \quad \widetilde{s}_d = d(\dfrac{1}{2} - \dfrac{1}{q}) - \dfrac{2}{r}, & \text{if } d \geqslant 3. \end{cases}$$

Remarks. (i) *Actually, the conclusions in [39] determined that (2.6) is sharp with respect to the regularity and the Lebesgue exponents.*

(ii) *We will only work on the Sharp lines of indexes determined by Proposition 2.5. More precisely,*

$$\begin{cases} 2 \leqslant q < \infty \\ 12/5 \leqslant r \leqslant \infty \end{cases} \text{satisfying } \frac{10}{q} + \frac{12}{r} = 5 \text{ and } \widetilde{s}_2 = \frac{2}{5r}, \text{ if } d = 2$$

and

$$\begin{cases} 2 \leqslant q < \infty \\ 2 \leqslant r \leqslant \infty \end{cases} \text{satisfying } \frac{1}{q} + \frac{1}{r} = \frac{1}{2} \text{ and } \widetilde{s}_d = \frac{d-2}{r}, \text{ if } d \geqslant 3.$$

Since the endpoint Strichartz estimate corresponding to $(r, q) = (2, \infty)$ is not known, we need to lose a little bit of regularity to control this norm.

Corollary 2.6. *Let $s > s_d - 3/2$, where $s_d = d/2 + 1/2$ for $d \geqslant 3$ and $s_2 = 5/3$. Then for each $T > 0$ and $0 < \delta < s - s_d + 3/2$, there exist $\kappa_\delta \in (0, 1/2)$ such that*

$$\|U(t)f\|_{L^2_T L^\infty_x} \leqslant c_\delta T^{\kappa_\delta} \|f\|_{H^s} \tag{2.8}$$

PROOF. We take r sufficiently large to assure that $\delta > \frac{d}{r}$ and the conditions in Proposition 2.5 are satisfied. Then, Sobolev's embedding and (2.6) yield

$$\|U(t)f\|_{L^2_T L^\infty_x} \leqslant c_\delta T^{\frac{r-2}{2r}} \|U(t) J^\delta f\|_{L^r_T L^q_x} \leqslant c_\delta T^{\frac{r-2}{2r}} \|J^{s_d - 3/2 + \delta} f\|_{L^2}. \tag{2.9}$$

Therefore, setting $k_\delta := \frac{r-2}{2r}$, $k_\delta \in (1/12, 1/2)$ for $d = 2$, and $k_\delta \in (0, 1/2)$ for $d \geqslant 3$, the proof is completed. □

A key ingredient for our arguments is the following refined Strichartz estimate. This estimate has been deduced in different context for other dispersive models, see for instance [50, 59].

Lemma 2.7. *Let $s > s_d - 1$ where $s_d := d/2 + 1/2$ for $d \geqslant 3$ and $s_2 := 5/3$. Then there exists $\kappa_\delta \in (1/2, 1)$ and $\delta > 0$ such that*

$$\|w(\cdot, t)\|_{L^1_T L^\infty_x} \lesssim T^{k_\delta} \Big(\sup_{t \in [0,T]} \|w(\cdot, t)\|_{H^s} + \int_0^T \|F(\cdot, t')\|_{H^{s-1}} \, dt' \Big) \tag{2.10}$$

whenever $T \leqslant 1$ and w is a solution to

$$\partial_t w - \mathcal{R}_1 \Delta w = F. \tag{2.11}$$

PROOF. We will follow the arguments in [51]. We write $[0, T] = \bigcup_{m=1}^n I_m$ such that $I_m = [a_m, b_m]$ and $b_m - a_m = T/N$. Recalling the projectors (1.7), we apply the triangle inequality to obtain

$$\begin{aligned} \|w\|_{L^1_T L^\infty_x} &\leqslant \|P_{\leqslant 1} w\|_{L^1_T L^\infty_x} + \sum_{N > 1} \|P_N w\|_{L^1_T L^\infty_x} \leqslant \|P_{\leqslant 1} w\|_{L^1_T L^\infty_x} + \sum_{N > 1} \sum_{m=1}^N \|P_N w\|_{L^1_{I_m} L^\infty_x} \\ &\lesssim \|P_{\leqslant 1} w\|_{L^1_T L^\infty_x} + \sum_{N > 1} \sum_{m=1}^N (T/N)^{1/2} \|P_N w\|_{L^2_{I_m} L^\infty_x}, \end{aligned} \tag{2.12}$$

where the last line is obtained by Hölder's inequality. Now, we proceed to estimate each term on the rand-hand side of the above inequality. Let us deal first with the low frequency term. Let $0 < \delta < (s - s_d - 1)/2$, then since $P_{\leqslant 1} w$ solves the integral equation

$$P_{\leqslant 1} w(t) = U(t) P_{\leqslant 1} w(0) + \int_0^t U(t - t') P_{\leqslant 1} F(\cdot, t') \, dt', \tag{2.13}$$

we deduce from Hölder's inequality in time and Corollary 2.6 that

$$\begin{aligned}\|P_{\leqslant 1}w\|_{L^1_TL^\infty_x} &\lesssim T^{1/2}\Big(\|U(t)P_{\leqslant 1}w(0)\|_{L^2_TL^\infty_x}+\int_0^T\|U(t-t')P_{\leqslant 1}F(\cdot,t')\|_{L^2_TL^\infty_x}\,dt'\Big)\\ &\lesssim T^{1/2+\widetilde{\kappa}_\delta}\Big(\|J^{s_d-1+\delta}w(0)\|_{L^2_x}+\int_0^T\|J^{s_d-2+\delta}F(\cdot,t')\|_{L^2}\,dt'\Big),\end{aligned}\tag{2.14}$$

for some $\widetilde{\kappa}_\delta\in(0,1/2)$ and where we have employed that

$$\|J^\delta P_{\leqslant 1}F(t)\|_{L^2}=\|J^{s_d-2+\delta}J^{2-s_d}P_{\leqslant 1}F(t)\|_{L^2}\lesssim_s\|J^{s_d-2+\delta}F(t)\|_{L^2}.$$

This estimate completes the analysis of the first term on the r.h.s of (2.12). On the other hand, by employing Duhamel's formula on each I_m, it is seen

$$P_Nw(t)=U(t-a_m)P_Nw(\cdot,a_m)+\int_{a_m}^t U(t-t')P_NF(t')\,dt'\tag{2.15}$$

for all $t\in I_m$ and each $N>1$, thus Corollary 2.6 yields

$$\begin{aligned}&\sum_{N>1}\sum_{m=1}^N(T/N)^{1/2}\|P_Nw\|_{L^2_TL^\infty_x}\\ &\quad\lesssim\sum_{N>1}T^{1/2+\widetilde{\kappa}_\delta}N^{-1/2}\big(\sum_{m=1}^N\|J^{s_d-3/2+\delta}P_Nw(a_m)\|_{L^2}+\int_{I_m}\|J^{s_d-3/2+\delta}P_NF(t')\|_{L^2}\,dt'\big)\\ &\quad\lesssim T^{1/2+\widetilde{\kappa}_\delta}\big(\sum_{N>1}N^{-\delta}\big)\Big(\sup_{t\in[0,T]}\|J^{s_d-1+2\delta}w(t)\|_{L^2}+\int_0^T\|J^{s_d-2+2\delta}F(t')\|_{L^2}\,dt'\Big).\end{aligned}\tag{2.16}$$

Plugging the previous estimates in (2.12) completes the proof. □

2.2.2. Energy estimates. By means of the Kato-Ponce commutator estimate Lemma 1.1, we deduce the following *a priori* estimate.

Lemma 2.8. *Let $T>0$ and $u\in C([0,T];H^\infty(\mathbb{R}^d))$ be the solution of the IVP (0.2). Then there exists a positive constant c_0 such that*

$$\|u\|^2_{L^\infty_TH^s}\leqslant\|u_0\|^2_{H^s}+c_0\|\nabla u\|_{L^1_TL^\infty_x}\|u\|^2_{L^\infty_TH^s}\tag{2.17}$$

for any $s>0$.

PROOF. Applying J^s to the equation in (0.2), multiplying by J^su and integrating in space yields to

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}(J^su)^2\,dx=-\int_{\mathbb{R}^d}[J^s,u]\partial_{x_1}uJ^su\,dx-\int_{\mathbb{R}^d}uJ^s\partial_{x_1}uJ^su\,dx,$$

where it is not difficult to see that the factor concerning the dispersive term in (0.2) is zero, since $\mathcal{R}_1$ defines a skew-symmetric operator. The first term on the right-hand side (r.h.s) of the equality above is bounded by using Hölder's inequality and Lemma 1.1, while the second is controlled by integrating by parts and using Hölder inequality again. Summarizing

$$\frac{d}{dt}\|J^su\|^2_{L^2}\lesssim\|\nabla u\|_{L^\infty}\|J^su\|^2_{L^2}.\tag{2.18}$$

Integrating on time the inequality above yields the proof of the lemma. □

Now we derive some energy estimates for the Strichartz norm. More precisely, we establish *a priori* estimates for the norms $\|u\|_{L^1_T L^\infty_x}$ and $\|\nabla u\|_{L^1_T L^\infty_x}$. Our arguments relay on the refined Strichartz estimate deduced in Lemma 2.7.

Lemma 2.9. *Let $s \in (s_d, d+1]$ where $s_d = d/2 + 1/2$ for $d \geqslant 3$ and $s_2 = 5/3$ for $d = 2$. For $T \leqslant 1$, let*

$$f(T) := \|\nabla u\|_{L^1_T L^\infty_x} + \|u\|_{L^1_T L^\infty_x}\,. \tag{2.19}$$

Then there exists a constant and $c_s > 0$ such that

$$f(T) \leqslant c_s T^{\kappa_\delta}(1 + f(T))\, \|u\|_{L^\infty_T H^s}\,, \tag{2.20}$$

whenever $u \in C([0,T]; H^\infty(\mathbb{R}^d))$ solves the IVP (0.2).

Proof. We first estimate $\|\nabla u\|_{L^1_T L^\infty_x}$. Let

$$(F_1, \dots, F_d) = \big((-\partial_{x_1}(u\partial_{x_1}u), \dots, -\partial_{x_d}(u\partial_{x_1}u)\big) = -\frac{1}{2}\partial_{x_1}\nabla(u^2),$$

by considering the corresponding equations determined after setting $F = F_j$ in (2.11) for each $j = 1, \dots, d$, Lemma 2.7 reveals that

$$\begin{aligned}\|\nabla u\|_{L^1_T L^\infty_x} &\sim \sum_{j=1}^{d} \|\partial_{x_j} u\|_{L^1_T L^\infty_x} \\ &\lesssim T^{1/2}\Big(\sup_{[0,T]} \|J^{s-1}\nabla u(t)\|_{L^2} + \int_0^T \|J^{s-2}\nabla(u\partial_{x_1}u)(t')\|_{L^2}\, dt'\Big) \\ &\lesssim T^{1/2}\Big(\sup_{[0,T]} \|J^{s}u(t)\|_{L^2} + \int_0^T \|J^{s}(u^2)(t')\|_{L^2}\, dt'\Big).\end{aligned} \tag{2.21}$$

To estimate the r.h.s of the above inequality, we apply Lemma 1.2 to find

$$\|J^s(u^2)(t)\|_{L^2} \lesssim \|u(t)\|_{L^\infty} \|J^s u(t)\|_{L^2}. \tag{2.22}$$

Gathering together (2.21) and (2.22), we find

$$\|\nabla u\|_{L^1_T L^\infty_x} \lesssim T^{1/2}(1 + \|u\|_{L^1_T L^\infty_x})\|u\|_{L^\infty_T H^s}. \tag{2.23}$$

On the other hand, Lemmas 2.7 and 1.2 yield

$$\begin{aligned}\|u\|_{L^1_T L^\infty_x} &\lesssim T^{1/2}\Big(\|J^{s-1}u_0\|_{L^2} + \int_0^T \|J^{s-2}(u\partial_{x_1}u)(t')\|_{L^2}\, dt'\Big) \\ &\lesssim T^{1/2}\Big(\|J^{s-1}u_0\|_{L^2} + \int_0^T \|u(t')\|_{L^\infty}\|J^{s-1}u(t')\|_{L^2}\, dt'\Big) \\ &\lesssim T^{1/2}(1 + \|u\|_{L^1_T L^\infty_x})\|u\|_{H^{s-1}}.\end{aligned} \tag{2.24}$$

This estimate completes the proof of the lemma. ∎

2.3. LWP in $H^s(\mathbb{R}^d)$, $s > s_d$, where $s_d = d/2+1/2$ for $d \geqslant 3$ and $s_2 = 5/3$

This section is devoted to proving Theorem 2.1. Considering that this result relays on a compactness method, we will obtain solutions in low-regularity spaces as a certain limit of smooth solutions.

Accordingly, we require to assure existence of smooth solutions for the initial value problem (0.2). But first, we notice that for $s > d/2+1$, Theorem 2.1 is obtained by implementing a parabolic regularization argument in the spirit of [**45**, **47**, **60**]. Roughly speaking, an additional term $-\mu\Delta u$ is added to the equation, after which the limit $\mu \to 0$ is taken. The precise consequence of this technique is stated in the following lemma:

Lemma 2.10. *Let $d \geqslant 2$ integer and $s > d/2+1$. Then for any $u_0 \in H^s(\mathbb{R}^d)$, there exist $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution $u \in C([0,T]; H^s(\mathbb{R}^d))$ of the IVP (0.2). Furthermore, the flow-map $u_0 \mapsto u(t)$ is continuous in the H^s-norm and there exists a function $\rho \in C([0,T];[0,\infty))$ such that*

$$\|u(t)\|_{H^s} \leqslant \rho(t), \quad t \in [0,T].$$

Moreover, the existence time does not depend on s, in the sense that u can be extended, if necessary, to the interval $[0, T(\|u_0\|_{H^{s_0}})]$, if $s \geqslant s_0 > d/2+1$, with u_0 viewed as an element of $H^{s_0}(\mathbb{R}^d)$.

Lemma 2.10 allow us to deduce existence of smooth solutions and a blow-up criterion.

Proposition 2.11. *Let $d \geqslant 2$ and $u_0 \in H^\infty(\mathbb{R}^d)$. Then there exists $u \in C([0,T^*); H^\infty(\mathbb{R}^d))$ solution of (0.2) with initial data u_0, where T^* is the maximal time of existence of u such that $T^* \geqslant T(\|u_0\|_{H^{d+1}})$. Moreover, it follows*

$$\lim_{t \uparrow T^*} \|u(t)\|_{H^{d+1}} = +\infty \quad \text{if} \quad T^* < \infty. \tag{2.25}$$

Consequently, in virtue of Lemma 2.10, we will restrict our considerations to prove the case $s_d < s \leqslant d+1$, where $s_d = d/2+1/2$ for $d \geqslant 3$ and $s_2 = 5/3$.

2.3.1. A priori estimates. We first prove that the smooth solutions determined by Proposition 2.11 exist long enough for our purposes. We then provide some additional *a priori* estimates whose prove follow closely the arguments in [**59**].

Lemma 2.12. *Let $s \in (s_d, d+1]$ where $s_d = d/2+1/2$ for $d \geqslant 3$ and $s_2 = 5/3$. Then there exists a constant $A_s > 0$ such that for all $u_0 \in H^{d+1}(\mathbf{R}^d)$ there is a solution $u \in C([0, T^*(H^{d+1}(\mathbf{R}^d))))$ of (0.2) with $T^* \geqslant (1 + A_s\|u_0\|_{H^s})^{-2}$. Moreover there exists a constant K_0 such that*

$$\|u\|_{L^\infty_T H^s} \leqslant 2\,\|u_0\|_{H^s}\,, \text{ and } f(T) \leqslant K_0, \tag{2.26}$$

whenever $T \leqslant (1 + A_s\|u_0\|_{H^s})^{-2}$.

Proof. We define

$$T_0 = \sup\left\{T \in (0, T^*) \,:\, \|u\|^2_{L^\infty_T H^s} \leqslant 4\|u_0\|^2_{H^s}\right\}.$$

Since $u \in C([0,T^*); H^\infty))$, we have that T_0 is well-defined. Arguing by contradiction, let us suppose that $0 < T_0 < (1 + A_s\|u_0\|_{H^s})^{-2}$. Then continuity yields $\|u\|^2_{L^\infty_{T_0} H^s} \leqslant 4\|u_0\|^2_{H^s}$, and Lemma 2.9 determines

$$f(T_0) \leqslant 2c_s T_0^{1/2}(1 + f(T_0))\,\|u_0\|_{H^s}\,.$$

Thus, if we choose $A_s = 8(1+c_0)c_s$, where c_0 and c_s are defined as in Lemmas 2.8 and 2.9 respectively, we find

$$f(T_0) \leqslant \frac{1}{3c_0}.$$

This result and Lemma 2.8 show

$$\|u\|^2_{L^\infty_{T_0} H^{d+1}} \leqslant \frac{3}{2}\|u_0\|^2_{H^{d+1}}, \text{ and } \|u\|^2_{L^\infty_{T_0} H^s} \leqslant \frac{3}{2}\|u_0\|^2_{H^s}.$$

In view of the blow-up alternative in Proposition 2.11, the former estimate implies that $T_0 < T^*$. On the other hand, the latter estimate and continuity establish that for some $T_0 < T < T^*$, $\|u\|^2_{L^\infty_T H^s} \leqslant 3\|u_0\|^2_{H^s}$, which in turn contradicts the definition of T_0. The proof is completed. □

2.3.2. Uniqueness. Let u_1 and u_2 be two solutions of equation (0.2) in the class

$$C([0,T], H^s(\mathbb{R}^d)) \cap L^1([0,T], W^{1,\infty}(\mathbb{R}^d))$$

with respective initial data $u_1(\cdot,0) = \varphi_1$ and $u_2(\cdot,0) = \varphi_2$. By setting $v := u - v$, we find that

$$\partial_t v - \mathcal{R}_1 \Delta v + \frac{1}{2}\partial_{x_1}((u_1+u_2)v) = 0.$$

Then, multiplying the previous equation by v and integrating in space, it follows that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} v^2\,dx = -\frac{1}{4}\int_{\mathbb{R}^d} \partial_{x_1}(u_1+u_2)v^2\,dx,$$

where the factor concerning the dispersion is zero since $\mathcal{R}_1$ is a skew-symmetric operator, and the right-hand side of the above expression is obtained by two integration by parts. Thus Hölder's inequality determines

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2} \lesssim (\|\nabla u_1(t)\|_{L^\infty} + \|\nabla u_2(t)\|_{L^\infty})\|v(t)\|^2_{L^2}.$$

An application of Gronwall's inequality (see for example [86, Theorem 1.12]) gives

$$\sup_{t\in[0,T]} \|u_1(t) - u_2(t)\|_{L^2} \leqslant e^{cK}\|\varphi_1 - \varphi_2\|_{L^2} \tag{2.27}$$

where $K = \|\nabla u_1\|_{L^1_T L^\infty_x} + \|\nabla u_2\|_{L^1_T L^\infty_x}$. Uniqueness is now a consequence of (2.27).

2.3.3. Existence. We shall implement the Bona-Smith argument [7]. We state some properties of the projectors defined in (1.7).

Lemma 2.13. *Let $\sigma \geqslant 0$ and $N \in \mathbb{D} = \{2^k : k \in \mathbb{Z}^+\} \cup \{1\}$. Then,*

$$\|P_{\leqslant N} J^{s+\sigma} u_0\|_{L^2} \lesssim N^\sigma \|J^s u_0\|_{L^2}. \tag{2.28}$$

Moreover, let $M, N \in \mathbb{D}$ with $M \geqslant N$ and $0 \leqslant \sigma \leqslant s$, then

$$\|J^{s-\sigma}(P_{\leqslant N}u_0 - P_{\leqslant M}u_0)\|_{L^2} \underset{N\to\infty}{=} o(N^{-\sigma}). \tag{2.29}$$

PROOF. The estimate (2.28) follows directly from the properties of the projectors $P_{\leqslant N}$. On the other hand, by support considerations

$$|\langle\xi\rangle^{s-\sigma}(\psi_{\leqslant N}(\xi) - \psi_{\leqslant M}(\xi))\widehat{u_0}(\xi)|^2 \lesssim N^{-2\sigma}|\psi_{\leqslant N}(\xi) - \psi_{\leqslant M}(\xi)|^2|\langle\xi\rangle^s \widehat{u_0}(\xi)|^2 \tag{2.30}$$

and so if $\sigma < s$ the result follows by Plancherel's identity. For $\sigma = s$, an application of Lebesgue dominated convergence theorem yields the result. □

Let $s \in (s_d, d+1]$ where $s_d = d/2 + 1/2$ for $d \geqslant 3$ and $d_2 = 5/3$. For each dyadic $N \in \mathbb{D}$, we consider the solutions $u_N \in C([0,T]; H^\infty(\mathbb{R}^d))$ of (0.2) emanating from $P_{\leqslant N} u_0$:

$$\begin{cases} \partial_t u_N - \mathcal{R}_1 \Delta u_N + u_N \partial_{x_1} u_N = 0, & x \in \mathbb{R}^d, \quad t \in (0,T], \\ u_N(x,0) = P_{\leqslant N} u_0, \end{cases} \tag{2.31}$$

where in virtue of Lemma 2.12 we can find a time

$$0 < T \leqslant (1 + A_s \|u_0\|_{H^s})^{-2} \tag{2.32}$$

(for some constant $A_s > 0$) independent of N, such that

$$\|u_N\|_{L^\infty_T H^s} \leqslant 2\|u_0\|_{H^s} \tag{2.33}$$

and

$$K := \sup_{N \geqslant 1} \left\{ \|u_N\|_{L^1_T L^\infty_x} + \|\nabla u_N\|_{L^1_T L^\infty_x} \right\} < \infty. \tag{2.34}$$

Now, given $M \geqslant N \geqslant 1$ dyadic numbers, we set $v_{N,M} := u_N - u_M$ so that $v_{N,M}$ satisfies

$$\partial_t v_{N,M} - \mathcal{R}_1 \Delta v_{N,M} + u_N \partial_{x_1} u_N - u_M \partial_{x_1} u_M = 0, \tag{2.35}$$

with initial datum $v_{N,M}(\cdot, 0) = P_{\leqslant N} u_0 - P_{\leqslant M} u_0$.

Arguing as in the deduction of (2.27) and applying (2.29) we find

$$\|v_{N,M}\|_{L^\infty_T L^2_x} \leqslant e^{cK} \|P_{\leqslant N} u_0 - P_{\leqslant M} u_0\|_{L^2} \underset{n\to\infty}{=} o(N^{-s}) \tag{2.36}$$

and so interpolating with (2.33), we get

$$\|J^\sigma v_{N,M}\|_{L^\infty_T L^2_x} \leqslant \|J^s v_{N,M}\|_{L^\infty_T L^2_x}^{\frac{\sigma}{s}} \|v_{N,M}\|_{L^\infty_T L^2_x}^{1-\frac{\sigma}{s}} \underset{N\to\infty}{=} o(N^{-(s-\sigma)}) \tag{2.37}$$

for all $0 \leqslant \sigma < s$.

Below we will show that $\{v_{N,M}\}$ determines a Cauchy sequence in $C([0,T]; H^s(\mathbb{R}^d))$, but first we prove that the sequence is a Cauchy sequence in $L^1([0,T]; W^{1,\infty}(\mathbb{R}))$. In fact, we prove a stronger result that will be helpful later

Lemma 2.14. *Let $M, N \in \mathbb{D} = \{2^k : k \in \mathbb{Z}^+ \cup \{0\}\}$, $M \geqslant N \geqslant 1$. Then,*

$$\|v_{N,M}\|_{L^1_T L^\infty_x} = o(N^{-1}) \tag{2.38}$$

and

$$\|\nabla v_{N,M}\|_{L^1_T L^\infty_x} = o(1) \tag{2.39}$$

provided that T as in (2.32) is chosen sufficiently small (that is, A_s large enough).

PROOF. We first deduce (2.38). Since $v_{n,m}$ solves equation (2.35), we apply Lemma 2.7 with

$$F = -\frac{1}{2} \partial_{x_1} (u_N + u_M(v_{N,M}))$$

to deduce

$$\begin{aligned} \|v_{N,M}\|_{L^1_T L^\infty_x} &\lesssim T^{1/2} \Big(\|J^{s_d - 1 + \delta} v_{N,M}\|_{L^\infty_T L^2_x} + \int_0^T \|J^{s_d - 2 + \delta} \partial_{x_1} ((u_N + u_M) v_{N,M})(t')\|_{L^2} \, dt' \Big) \\ &=: T^{1/2} (\mathcal{A}_1 + \mathcal{A}_2), \end{aligned} \tag{2.40}$$

where $0<\delta<(s-s_d)$, $s_d = d/2+1/2$ and $s_2 = 5/3$. Our choice of δ and (2.37) imply

$$\mathcal{A}_1 \underset{N\to\infty}{=} o(N^{-1}). \tag{2.41}$$

To estimate $\mathcal{A}_2$, we combine (1.11) and (2.37) to find

(2.42)
$$\begin{aligned}\mathcal{A}_2 &\lesssim \int_0^T \|J^{s_d-1+\delta}(u_N+u_M)(t')\|_{L^2}\|v_{N,M}(t')\|_{L^\infty} + \|(u_N+u_M)(t')\|_{L^\infty}\|J^{s_d-1+\delta}v_{N,M}(t')\|_{L^2}\,dt' \\ &\lesssim (\|u_N\|_{L^\infty_T H^s}+\|u_M\|_{L^\infty_T H^s})\|v_{N,M}\|_{L^1_T L^\infty_x} + (\|u_N\|_{L^1_T L^\infty_x}+\|u_M\|_{L^1_T L^\infty_x})\|J^{s_d-1+\delta}v_{N,M}\|_{L^\infty_T L^2_x} \\ &= O(\|u_0\|_{H^s}\|v_{N,M}\|_{L^1_T L^\infty_x}) + o(N^{-1}).\end{aligned}$$

Therefore, gathering (2.40)-(2.42) we have

$$\|v_{N,M}\|_{L^1_T L^\infty_x} \underset{N\to\infty}{=} o(N^{-1}) + O(T^{\frac{1}{2}}\|u_0\|_{H^s}\|v_{N,M}\|_{L^1_T L^\infty_x}). \tag{2.43}$$

Then taking T sufficiently small in (2.32) with respect to the implicit constant above (which does not depend on N), we conclude (2.38).

On the other hand, (2.39) is deduced by a similar reasoning as above. Indeed, we apply Lemma 2.7 with $F = -\frac{1}{2}\nabla\partial_{x_1}(u_N+u_M(v_{N,M}))$ and (1.11) to deduce

(2.44)
$$\begin{aligned}\|\nabla v_{N,M}\|_{L^1_T L^\infty_x} &\lesssim T^{1/2}\Big(\|J^{s_d+\delta}v_{N,M}\|_{L^\infty_T L^2} + \int_0^T \|J^{s_d-1+\delta}\partial_{x_1}((u_N+u_M)v_{N,M})(t')\|_{L^2}\,dt'\Big) \\ &\lesssim T^{1/2}\Big(\|J^{s_d+\delta}v_{N,M}\|_{L^\infty_T L^2} + (\|u_N\|_{L^\infty_T H^s}+\|u_M\|_{L^\infty_T H^s})\|v_{N,M}\|_{L^1_T L^\infty_x} \\ &\quad + (\|u_N\|_{L^1_T L^\infty_x}+\|u_M\|_{L^1_T L^\infty_x})\|J^{s_d+\delta}v_{N,M}\|_{L^\infty_T L^2_x}\Big).\end{aligned}$$

Hence, our choice of δ, (2.37) and the preceding estimate for $\|v_{N,M}\|_{L^1_T L^\infty_x}$ determine

$$\|\nabla v_{N,M}\|_{L^1_T L^\infty_x} \underset{N\to\infty}{=} O(\|J^{s_d+\delta}v_{N,M}\|_{L^\infty_T L^2_x}) + O(\|v_{N,M}\|_{L^1_T L^\infty_x}) \underset{N\to\infty}{=} o(1). \tag{2.45}$$

The proof is completed. □

PROPOSITION 2.15. *Let $N, M \in \mathbb{D}$ with $M \geqslant N$. Then,*

$$\|v_{N,M}\|_{H^s} \underset{n\to\infty}{\to} 0. \tag{2.46}$$

PROOF. Applying the operator J^s to (2.35), multiply then by $J^s v_{N,M}$, integrating in space we deduce

(2.47)
$$\begin{aligned}\frac{d}{dt}\|J^s v_{N,M}(t)\|_{L^2}^2 &= -2\int_{\mathbb{R}^d} J^s(u_N\partial_{x_1}u_N - u_M\partial_{x_1}u_M)J^s v_{N,M} \\ &= -2\int_{\mathbb{R}^d} J^s(u_M\partial_{x_1}(u_N-u_M))J^s v_{N,M} - 2\int_{\mathbb{R}^d} J^s((u_N-u_M)\partial_{x_1}u_N)J^s v_{N,M} \\ &=: -2(A_1+A_2).\end{aligned}$$

Integrating by parts we obtain

$$A_1 = \int_{\mathbb{R}^d}[J^s, u_M]\partial_{x_1}v_{N,M}J^s v_{N,M} - \frac{1}{2}\int_{\mathbb{R}^d}\partial_{x_1}u_M(J^s v_{N,M})^2. \tag{2.48}$$

Then from Lemma 1.1, Hölder's inequality and (2.33) it follows

$$\begin{aligned} |A_1| &\lesssim \|\nabla u_M(t)\|_{L^\infty}\|J^s v_{N,M}(t)\|_{L^2}^2 + \|\partial_{x_1} v_{N,M}(t)\|_{L^\infty}\|J^s u_M(t)\|_{L^2}\|J^s v_{N,M}(t)\|_{L^2} \\ &\lesssim \|\nabla u_M(t)\|_{L^\infty}\|J^s v_{N,M}(t)\|_{L^2}^2 + \|\nabla v_{N,M}(t)\|_{L^\infty}\|u_0\|_{H^s}^2. \end{aligned} \tag{2.49}$$

On the other hand,

$$A_2 = \int_{\mathbb{R}^d} [J^s, v_{N,M}]\partial_{x_1} u_N J^s v_{N,M} + \int_{\mathbb{R}^d} v_{N,M}(J^s \partial_{x_1} u_N) J^s v_{N,M}. \tag{2.50}$$

Hence Lemma 1.1, Hölder's inequality and (2.33) yield

$$|A_2| \lesssim \|\nabla v_{N,M}(t)\|_{L^\infty}\|J^s u_N(t)\|_{L^2}\|J^s v_{N,M}(t)\|_{L^2} + \|v_{N,M}(t)\|_{L^\infty}\|J^{s+1}u_N(t)\|_{L^2}\|J^s v_{N,M}(t)\|_{L^2}. \tag{2.51}$$

To control the norm $\|J^{s+1}u_N(t)\|_{L^2}$, we employ energy estimates and Lemma 1.1 to observe

$$\frac{d}{dt}\|J^{s+1}u_N(t)\|_{L^2}^2 \lesssim \|\nabla u_N(t)\|_{L^\infty}\|J^{s+1}u_N(t)\|_{L^2}^2, \tag{2.52}$$

so that Gronwall's inequality and (2.28) yield

$$\|J^{s+1}u_N(t)\|_{L^2} \leqslant e^{cK}\|J^{s+1}P_{\leqslant N}u_0\|_{L^2} \lesssim N\|u_0\|_{H^s}, \tag{2.53}$$

where K is defined as in (2.34). Thus in view of (2.33)

$$|A_2| \lesssim (\|\nabla v_{N,M}(t)\|_{L^\infty} + N\|v_{N,M}(t)\|_{L^\infty})\,\|u_0\|_{H^s}^2. \tag{2.54}$$

Summing up our estimates for A_1 and A_2, we find that

$$\frac{d}{dt}\|J^s v_{N,M}(t)\|_{L^2}^2 \leqslant a(t)\|J^s v_{N,M}(t)\|_{L^2}^2 + b(t) \tag{2.55}$$

where

$$\begin{aligned} a(t) &:= C_0\Big(\|\nabla u_N(t)\|_{L^\infty} + \|\nabla u_M(t)\|_{L^\infty}\Big), \\ b(t) &:= C_1\Big(N\|v_{N,M}(t)\|_{L^\infty} + \|\nabla v_{N,M}(t)\|_{L^\infty}\Big)\|u_0\|_{H^s}^2. \end{aligned}$$

Now, if $g(t)$ solves

$$\begin{cases} \dfrac{d}{dt}g(t) = a(t)g(t) + b(t), \\ g(0) = \|P_{\leqslant N}u_0 - P_{\leqslant M}u_0\|_{H^s}^2, \end{cases}$$

then

$$\frac{d}{dt}\left(\|J^s v_{N,M}(t)\|_{L^2}^2 - g(t)\right) \leqslant a(t)\left(\|J^s v_{N,M}(t)\|_{L^2}^2 - g(t)\right)$$

with initial condition, $\|J^s v_{N,M}(0)\|_{L^2}^2 - g(0) = 0$. Then by an application of Gronwall's inequality, we find that $\|J^s v_{N,M}(t)\|_{L^2}^2 \leqslant g(t)$ for all $t \geqslant 0$. Now, since $g(t)$ has the explicit form

$$g(t) = g(0)e^{\int_0^t a(t')\,dt'} + \int_0^t b(\tau)e^{\int_\tau^t a(t')\,dt'}\,d\tau,$$

if follows

$$\|J^s v_{N,M}(t)\|_{L^\infty_T L^2}^2 \lesssim e^{cK}\Big(\|P_{\leqslant N}u_0 - P_{\leqslant M}u_0\|_{H^s}^2 + \|u_0\|_{H^s}^2\left(N\|v_{N,M}\|_{L^1_T L^\infty_x} + \|\nabla v_{N,M}\|_{L^1_T L^\infty_x}\right)\Big) \underset{n\to\infty}{=} 0,$$

where we have employed Lemmas 2.13 and 2.14. The proof is completed. □

We deduce from Proposition 2.15 and Lemma 2.14 that u_N has a limit u in

$$C([0,T];H^s(\mathbb{R}^d)) \cap L^1([0,T];W^{1,\infty}(\mathbb{R}^d)).$$

Now, recalling that

$$u_N(t) = U(t)P_{\leqslant N}u_0 - \frac{1}{2}\int_0^t U(t-t')\partial_{x_1}(u_N(t'))^2\,dt', \tag{2.56}$$

and the estimate

$$\begin{aligned}\left\|\int_0^t U(t-t')\partial_{x_1}(u_N(t')^2 - u(t')^2)\,dt'\right\|_{H^{s-1}} &\leqslant \int_0^t \|u_N(t')^2 - u(t')^2\|_{H^s}\,dt' \\ &\lesssim \int_0^t \|u_N(t') + u(t')\|_{H^s}\|u_N(t') - u(t')\|_{H^s}\,dt',\end{aligned}$$

we see that u also solves the integral formulation of (0.2) in the $C([0,T];H^{s-1}(\mathbb{R}^d))$ sense. This completes the existence part of Theorem 2.1.

2.3.4. Continuity of the flow map data-solution. Let $s \in (s_d, d+1]$ where $s_d = d/2+1/2$ for $d \geqslant 3$ and $s_2 = 5/3$. Let $u_0 \in H^s(\mathbb{R}^d)$ fixed. By the existence and uniqueness parts above, we know that there exist a positive time $T = T(\|u_0\|_{H^s})$ and a unique solution $u \in C([0,T];H^s(\mathbb{R}^d)) \cap L^1([0,T];W^{1,\infty}(\mathbb{R}^d))$ to (0.2). Now, since T is a nonincreasing function of its argument, for any $0 < T' < T$ there exists $\tilde{\delta} > 0$ such that for all

$$v_0 \in B_{\tilde{\delta}}(u_0) := \left\{v_0 \in H^s(\mathbb{R}^d) \,:\, \|u_0 - v_0\|_{H^s} < \tilde{\delta}\right\}$$

the corresponding solution v of (0.2) is defined at least on the time interval $[0,T']$.

We require to prove that for all $\epsilon > 0$, there exists $\delta > 0$ with $0 < \delta < \tilde{\delta}$ such that for any initial data $v_0 \in B_\delta(u_0)$, the solution $v \in C([0,T'];H^s(\mathbb{R}^d))$ emanating from v_0 satisfies

$$\|u - v\|_{L^\infty_{T'}H^s} < \epsilon. \tag{2.57}$$

Therefore, for any $N \in \mathbb{D}$, let $u_N, v_N \in C([0,T'];H^\infty(\mathbb{R}^d))$ be the smooth solutions of (0.2) with regularized initial data $P_{\leqslant N}u_0$ and $P_{\leqslant N}v_0$ respectively. Then we have

$$\|u - v\|_{L^\infty_{T'}H^s} \leqslant \|u - u_N\|_{L^\infty_{T'}H^s} + \|u_N - v_N\|_{L^\infty_{T'}H^s} + \|v - v_N\|_{L^\infty_{T'}H^s}. \tag{2.58}$$

The proof of existence assures that for some dyadic number $N_0 \geqslant 1$ large,

$$\|u - u_N\|_{L^\infty_{T'}H^s} + \|v - v_N\|_{L^\infty_{T'}H^s} < 2\epsilon/3, \tag{2.59}$$

for all dyadic $N \geqslant N_0$. On the other hand,

$$\|P_{\leqslant N_0}u_0 - P_{\leqslant N_0}v_0\|_{H^{d+1}} \lesssim N_0^{d+1-s}\delta\|u_0 - v_0\|_{H^s} \lesssim N_0^{d+1-s}\delta. \tag{2.60}$$

Then, by using the continuity of the flow map for smooth solutions, we can choose $\delta > 0$ small enough (according to (2.60)) such that

$$\|u_{N_0} - v_{N_0}\|_{L^\infty_{T'}H^s} \leqslant \|u_{N_0} - v_{N_0}\|_{L^\infty_{T'}H^{d+1}} \leqslant \epsilon/3. \tag{2.61}$$

Consequently, (2.57) follows by combining (2.59) and (2.61).

2.4. Lack of C^2-regularity and uniformly continuity for the flow-map data solution

Here we prove that (0.2) cannot be solved in $H^s(\mathbb{R}^d)$ by a Picard iterative scheme based on the Duhamel formula. This result can be viewed as an extension of [66], where the C^2 ill-posedness in $H^s(\mathbb{R})$ is established for the Benjamin-Ono equation.

PROOF OF THEOREM 2.2. Suppose that there exists $T > 0$ such that (0.2) is locally well-posed in $H^s(\mathbb{R}^d)$ on the time interval $[0, T)$ and such that the flow map

$$\Phi(t) : H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d), \;\; u_0 \mapsto u(t)$$

is C^2 differentiable at the origin. When $\phi \in H^s(\mathbb{R}^d)$, we have that $\Phi(\cdot)\phi$ is a solution of (0.2) with initial data ϕ, so by Duhamel's principle $\Phi(t)\phi$ must satisfy the integral equation

$$\Phi(t)\phi = U(t)\phi - \frac{1}{2}\int_0^t U(t-t')\partial_{x_1}\left(\Phi(t')\phi\right)^2 dt'.$$

We compute the Fréchet derivative of $\Phi(t)$ at ψ with direction ϕ_1,

$$d_\psi\Phi(t)(\phi_1) = U(t)\phi_1 - \int_0^t U(t-t')\partial_{x_1}\left(\Phi(t')\psi\, d_\psi\Phi(t')(\phi_1)\right) dt'. \tag{2.62}$$

Supposing that (0.2) is well-posed, uniqueness implies that $\Phi(t)(0) = 0$, so that $d_0\Phi(t)(\phi_1) = U(t)\phi_1$. Differentiating again we find that

$$\begin{aligned} d_0^2\Phi(t)(\phi_1,\phi_2) &= \frac{\partial}{\partial\gamma}\Big(\gamma \mapsto d_{\gamma\phi_2}\Phi(t)(\phi_1)\Big)\Big|_{\gamma=0} \\ &= -\int_0^t U(t-t')\partial_{x_1}\left(d_{\gamma\phi_2}\Phi(t)(\phi_2)d_{\gamma\phi_2}\Phi(t)(\phi_1)\right) dt'\Big|_{\gamma=0} \\ &\quad -\int_0^t U(t-t')\partial_{x_1}\left(\Phi(t)(\gamma\phi_2)d^2_{\gamma\phi_1}\Phi(t)(\phi_1,\phi_2)\right) dt'\Big|_{\gamma=0}, \end{aligned}$$

which implies

$$d_0^2\Phi(t)(\phi_1,\phi_2) = -\int_0^t U(t-t')\partial_{x_1}\left((U(t')\phi_1)(U(t')\phi_2)\right) dt'.$$

Now, if the flow map were C^2 then $d_0^2\Phi(t)$ would be bounded from $H^s \times H^s$ to H^s, i.e.,

$$\left\|\int_0^t U(t-t')\partial_{x_1}\left((U(t')\phi_1)(U(t')\phi_2)\right) dt'\right\|_{H^s} \lesssim \|\phi_1\|_{H^s}\|\phi_2\|_{H^s}.$$

We will prove that this does not hold in general, following the arguments in [66].

Indeed, we will construct two sequences of functions, $\phi_{1,N}$ and $\phi_{2,N}$, such that

$$\|\phi_{1,N}\|_{H^s}, \|\phi_{2,N}\|_{H^s} \leqslant C \tag{2.63}$$

and

$$\lim_{N\to\infty}\left\|\int_0^t U(t-t')\partial_{x_1}\left((U(t')\phi_{1,N})(U(t')\phi_{2,N})\right] dt'\right\|_{H^s} = \infty. \tag{2.64}$$

We define $\phi_{1,N}$ and $\phi_{2,N}$ via their Fourier transforms as

$$\begin{cases} \widehat{\phi_{1,N}}(\xi) = \lambda^{\frac{1-2d}{2d}} N^{-s}\chi_{\mathcal{A}_1}(\xi), & \text{with } \mathcal{A}_1 = [N, N+\lambda] \times [\lambda^{1/d}/2, \lambda^{1/d}]^{d-1}, \\ \widehat{\phi_{2,N}}(\xi) = \lambda^{\frac{1-2d}{2d}} \chi_{\mathcal{A}_2}(\xi), & \text{with } \mathcal{A}_2 = [3\lambda, 4\lambda] \times [\lambda^{1/d}/2, \lambda^{1/d}]^{d-1} \end{cases}$$

where $N \gg 1$, $\lambda = N^{-(1+\epsilon)}$ and $0 < \epsilon < 1/(2d-1)$. First, we observe that $\phi_{1,N}$ and $\phi_{2,N}$ satisfy (2.63).

On the other hand, taking the Fourier transform with respect to the space variable,

$$\begin{aligned} \widehat{I_N}(\xi, t) &:= \left\{ \int_0^t U(t-t')\partial_{x_1}\left((U(t')\phi_{1,N})(U(t')\phi_{2,N})\right) dt' \right\}^{\wedge}(\xi) \\ &= \int_{K_\xi} \xi_1 e^{it\xi_1|\xi|} \frac{e^{iZ(\xi,\eta)t}-1}{Z(\xi,\eta)} \widehat{\phi_{1,N}}(\eta)\widehat{\phi_{2,N}}(\xi-\eta)\, d\eta \end{aligned} \tag{2.65}$$

where the resonant function is given by

$$Z(\xi, \eta) := -\xi_1|\xi| + (\xi_1 - \eta_1)|\xi - \eta| + \eta_1|\eta|$$

and

$$K_\xi := \left\{ \eta \in \mathbb{R}^d \,:\, \eta \in \mathcal{A}_1,\ \xi - \eta \in \mathcal{A}_2 \right\}.$$

When $\eta \in \mathcal{A}_1$ and $\xi - \eta \in \mathcal{A}_2$, we claim that

$$|Z(\xi, \eta)| \sim \lambda N. \tag{2.66}$$

Indeed, using that $\widehat{I}_N(\xi)$ is supported on

$$\mathcal{A}_3 = [N+3\lambda, N+5\lambda] \times [\lambda^{1/d}, 2\lambda^{1/d}]^{d-1}$$

we easily obtain

$$(\xi_1 - \eta_1)|\xi - \eta| \sim \lambda^{(d+1)/d}. \tag{2.67}$$

Moreover, from the inequality

$$|\xi| \leqslant \left((N+5\lambda)^2 + 4(d-1)\lambda^{2/d}\right)^{1/2} \leqslant N + 6\lambda$$

which holds for N large, $\lambda = N^{-(1+\epsilon)}$ with $0 < \epsilon < 1/(2d-1)$, we have

$$(N+3\lambda)^2 \leqslant \xi_1|\xi| \leqslant (N+6\lambda)^2. \tag{2.68}$$

Analogously, we get

$$N^2 \leqslant \eta_1|\eta| \leqslant (N+2\lambda)^2. \tag{2.69}$$

Then, (2.66) follows from (2.67), (2.68) and (2.69).

Now, since $\lambda N = N^{-\epsilon}$ and $|Z(\xi,\eta)| \sim \lambda N$ it follows

$$\left| \frac{e^{iZ(\xi,\eta)t}-1}{Z(\xi,\eta)} \right| = |t| + O\left(\frac{1}{N^\epsilon}\right). \tag{2.70}$$

From (2.70) and $|K_\xi| \sim \lambda^{(2d-1)/d}$, we infer that

$$|\widehat{I_N}(\xi,t)|\chi_{\mathcal{A}_3}(\xi) \gtrsim \frac{N\lambda^{(2d-1)/d}}{N^s\lambda^{(2d-1)/d}}|t|\,\chi_{\mathcal{A}_3}(\xi).$$

Therefore we arrive at

$$\|I_N(t)\|_{H^s} \gtrsim N\lambda^{(2d-1)/2d}|t| = N^{1/2d-\epsilon((2d-1)/2d)}|t|.$$

Since $0 < \epsilon < 1/(2d-1)$, we deduce (2.64), which completes the proof. □

The following corollary (of the proof) shows that it is not possible to solve (0.2) in $H^s(\mathbb{R}^d)$ via the usual contraction argument.

Corollary 2.16. *Let $s \in \mathbb{R}$ and $T > 0$. Then there does not exist a space X_T continuously embedded in $C([0,T]; H^s(\mathbb{R}^d))$ such that*

$$\|U(t)\phi\|_{X_T} \leqslant C\|\phi\|_{H^s} \tag{2.71}$$

and

$$\left\| \int_0^t U(t-t')\big(F(\cdot,t')\partial_{x_1}F(\cdot,t')\big)\,dt' \right\|_{X_T} \leqslant C\,\|F(\cdot,t)\|^2_{X_T}. \tag{2.72}$$

PROOF. We write $F = F_1 + F_2$ and note that

$$\left\| \int_0^t U(t-t')\big[F\partial_{x_1}F(\cdot,t')\big]\,dt' \right\|_{X_T} \geqslant \left\| \int_0^t U(t-t')\partial_{x_1}\big(F_1F_2(\cdot,t')\big)\,dt' \right\|_{X_T}$$
$$- \left\| \int_0^t U(t-t')\big(F_1\partial_{x_1}F_1(\cdot,t')\big)\,dt' \right\|_{X_T} - \left\| \int_0^t U(t-t')\big(F_2\partial_{x_1}F_2(\cdot,t')\big)\,dt' \right\|_{X_T}.$$

Now taking $F_1(\cdot,t') := U(t')\phi_{1,N}$ and $F_2(\cdot,t') := U(t')\phi_{2,N}$, by (2.71) and (2.63), we have

$$\|F\|_{X_T}, \quad \|F_1\|_{X_T}, \quad \|F_2\|_{X_T} \leqslant C.$$

Thus, if (2.72) holds, we would find that

$$\left\| \int_0^t U(t-t')\partial_{x_1}\big(U(t')\phi_{1,N})U(t')\phi_{2,N}\big)\,dt' \right\|_{X_T}$$

is uniformly bounded in N, contradicting (2.64). □

Next we prove that the flow map could not be uniformly continuous in $L^2(\mathbb{R}^2)$. We recall that Mariş [62] proved that there exist solitary wave solutions of the form $u_c(x_1,x_2,t) = \varphi(x_1 - ct, x_2)$ with $c > 0$. That is, φ_c is a solution of the equation

$$-c\varphi - (-\Delta)^{1/2}\varphi + \frac{1}{2}\varphi^2 = 0 \tag{2.73}$$

where $\varphi_c \in H^s(\mathbb{R}^2)$ for all $s \geqslant 0$, and where $(-\Delta)^{1/2} = D$ is defined by the Fourier symbol $\mathcal{F}((-\Delta)^{1/2}\phi)(\xi) = \mathcal{F}(D\phi)(\xi) = |\xi|\widehat{\phi}(\xi) = \sqrt{\xi_1^2 + \cdots + \xi_d^2}\,\widehat{\phi}(\xi)$.

PROOF OF PROPOSITION 2.3. Let $\varphi_c(x_1,x_2) := c\varphi_1(cx_1,cx_2)$ where φ_1 solves (2.73) with $c = 1$. Then φ_c solves (2.73) with $c > 0$ and we consider solutions

$$u_c(x_1,x_2,t) := c\varphi_1(cx_1 - c^2t, cx_2)$$

to (0.2). In particular we will consider solutions u_{c_1} and u_{c_2} with $c_1 \neq c_2$.

By a change of variables it is easy to see that, for all $t > 0$,

$$\|u_{c_1}(\cdot,t)\|_{L^2} = \|\varphi_1\|_{L^2} = \|u_{c_2}(\cdot,t)\|_{L^2},$$

so that

$$\|u_{c_1}(\cdot,t) - u_{c_2}(\cdot,t)\|_{L^2}^2 = 2\,\|\varphi_1\|_{L^2}^2 - 2\langle u_{c_1}(\cdot,t), u_{c_2}(\cdot,t)\rangle_{L^2}. \tag{2.74}$$

Changing variables by $c_2x_1 - c_2^2 t \to x_1$ and $c_2x_2 \to x_2$, we see that

$$\big\langle u_{c_1}(\cdot,t), u_{c_2}(\cdot,t)\big\rangle_{L^2} = \frac{c_1}{c_2}\int \varphi_1\big(\tfrac{c_1}{c_2}(x_1 - c_2(c_1-c_2)t), \tfrac{c_1}{c_2}x_2\big)\overline{\varphi_1(x)}\,dx.$$

Therefore, taking $c_1 = n+1$, $c_2 = n$, from the Lebesgue dominated convergence theorem, it follows that, for all $t>0$,

$$\big\langle u_{c_1}(\cdot,t), u_{c_2}(\cdot,t)\big\rangle_{L^2} = \frac{c_1}{c_2}\int \varphi_1\big(\tfrac{c_1}{c_2}(x_1 - nt, x_2)\big)\overline{\varphi_1(x)}\,dx \to 0 \text{ as } n\to\infty,$$

while

$$\big\langle u_{c_1}(\cdot,0), u_{c_2}(\cdot,0)\big\rangle_{L^2} \to \|\varphi_1\|_{L^2}^2 \text{ as } n\to\infty.$$

Thus, in view of (2.74), we deduce

$$\|u_{c_1}(\cdot,0) - u_{c_2}(\cdot,0)\|_{L^2} \to 0 \text{ as } n\to\infty,$$

while on the other hand, for all $t>0$,

$$\|u_{c_1}(\cdot,t) - u_{c_2}(\cdot,t)\|_{L^2} \to 2^{1/2}\,\|\varphi_1\|_{L^2} \text{ as } n\to\infty,$$

completing the proof. □

2.5. Some remarks on the generalized equation

2.5.1. Ill-posedness conclusions. This part is aimed to prove Proposition 2.4.

PROOF OF PROPOSITION 2.4. We consider the flow map $\phi \mapsto u(x,t;\phi)$ and define u_{k+1} by

$$u_{k+1} := \frac{\partial^{k+1}u}{\partial^{k+1}\varphi}\bigg|_{\varphi=0}(h_N,\ldots,h_N) \tag{2.75}$$

where the sequence h_N will be constructed below. Uniqueness yields $u(\cdot,\cdot;0)=0$, and so by some simple calculations

$$u_{k+1} = (k+1)!\int_0^t U(t-t')\partial_{x_1}\big((U(t')h_N)^{k+1}\big)\,dt'. \tag{2.76}$$

Then, the assumption that $\phi\mapsto u(x,t;\phi)$ is of class C^{k+1} at the origin assures that there exists a positive constant $c>0$ such that

$$\sup_{t\in[-T,T]}\|u_{k+1}(t)\|_{H^s} \leqslant c\,\|h_N\|_{H^s}^{k+1}. \tag{2.77}$$

In the sequel, we will show that (2.77) fails for a suitable sequence of functions (h_N). Let A and B be positive real numbers (which will be chosen later) such that $A<B$. Consider the real-valued function h_N defined via its Fourier transform by

$$\widehat{h_N}(\xi) = N^{-(2s+d)/2}\left(\psi_+(\xi/N) + \psi_-(\xi/N)\right), \tag{2.78}$$

where $\xi\in\mathbb{R}^d$, $N\gg 1$ and ψ_+ is a smooth nonnegative function supported in the d–cube $[A,B]^d$ and such that

$$\psi_+(\xi) = 1, \quad \forall\,\xi\in[A+(B-A)/4, B-(B-A)/4]^d, \tag{2.79}$$

and $\psi_-(\xi) = \psi_+(-\xi)$. Note that for all $s \in \mathbb{R}$ and $N \geqslant 1$,

$$\|h_N\|_{H^s} \sim 1.$$

On the other hand, by Fubini's theorem we compute

$$\begin{aligned}\widehat{u}_{k+1}(\xi,t) &= \int_0^t \xi_1 e^{i(t-t')\xi_1|\xi|}\mathcal{F}([U(t')h_N]^{k+1})(\xi)\,dt' \\ &= N^{-(k+1)(2s+d)/2} i\xi_1 e^{it\xi_1|\xi|} \\ &\qquad \times \sum_{p=0}^{k+1}\binom{k+1}{p}\int_0^t e^{it'\xi_1|\xi|}\mathcal{F}_x(U(t')\mathcal{F}^{-1}\psi_+)^{k+1-p}(U(t')\mathcal{F}^{-1}\psi_-)^p).\end{aligned} \tag{2.80}$$

Since the Fourier transform of $(U(t')\mathcal{F}^{-1}\psi_+)^{k+1-p}(U(t')\mathcal{F}^{-1}\psi_-)^p$ is supported in the d-cube

$$[(k+1)AN - pN(A+B), (k+1)BN - pN(A+B)]^d,$$

for all $t \in [-T, T]$, we choose A and B such that $A > kB/(k+2)$ to obtain

$$\begin{aligned}&\widehat{u}_{k+1}(\xi,t)\chi_{[(k+1)AN,(k+1)BN]^d}(\xi) \\ &\quad = N^{-(k+1)(2s+d)/2} i\xi_1 e^{it\xi_1|\xi|}\int_0^t e^{-it'\xi_1|\xi|}\mathcal{F}_x((U(t')\mathcal{F}^{-1}\psi_+)^{k+1}) \\ &\quad = N^{-(k+1)(2s+d)/2}\xi_1 e^{it\xi_1|\xi|}\int_{\mathbb{R}^{dk}}\frac{e^{itG(\xi,\eta^1,\dots,\eta^k)}-1}{G(\xi,\eta^1,\dots,\eta^k)} \\ &\qquad \times \psi_+\left(\frac{\xi-\eta^1}{N}\right)\dots\psi_+\left(\frac{\eta^{k-1}-\eta^k}{N}\right)\psi_+\left(\frac{\eta^k}{N}\right)\,d\eta^1\dots d\eta^k,\end{aligned} \tag{2.81}$$

where

$$G(\xi,\eta^1,\dots,\eta^k) = -(\xi_1|\xi| - (\xi_1-\eta_1^1)|\xi-\eta| - \sum_{j=1}^{k-1}(\eta_1^j-\eta_1^{j+1})|\eta^j-\eta^{j+1}| - \eta_1^k|\eta^k|).$$

Notice that on the support of the integral on the right hand side (2.81), we have

$$d^{1/2}A^2N^2 \leqslant (\xi_1-\eta_1^1)|\xi-\eta^1|, (\eta_1^j-\eta_1^{j+1})|\eta^j-\eta^{j+1}|, \eta_1^k|\eta^k| \leqslant d^{1/2}B^2N^2,$$

for all $j = 1, \dots k-1$. Moreover, $\xi \in [(k+1)AN, (k+1)BN]^d$ determines

$$d^{1/2}(k+1)^2A^2N^2 \leqslant \xi_1|\xi| \leqslant d^{1/2}(k+1)^2B^2N^2.$$

Then, combining the above estimates we arrive at

$$\begin{aligned}d^{1/2}(k+1)N^2A^2((k+1)-(B/A)^2) &\leqslant |G(\xi,\eta^1,\dots,\eta^k)| \\ &\leqslant d^{1/2}(k+1)B^2N^2((k+1)-(A/B)^2).\end{aligned} \tag{2.82}$$

By choosing A close enough to B (which is compatible with $A > kB$) and

$$t_N = \frac{N^{-2}}{d^{1/2}(k+1)A^2((k+1)-(B/A)^2)},$$

it follows that

$$|\widehat{u}_{k+1}(\xi,t)|\chi_{[(k+1)AN,(k+1)BN]^d} \gtrsim N^{-(k+1)(2s+d)/2}N^{-2}|\xi_1| \times \int_{\mathbb{R}^{dk}} \psi_+\left(\frac{\xi-\eta^1}{N}\right)\dots\psi_+\left(\frac{\eta^{k-1}-\eta^k}{N}\right)\psi_+\left(\frac{\eta^k}{N}\right)\,d\eta^1\dots d\eta^k.$$

In view of (2.79),

$$|\widehat{u}_{k+1}(\xi,t)|\chi_{[(k+1)AN,(k+1)BN]^d} \gtrsim N^{-(k+1)(2s+d)/2}N^{-1}(\chi_{[\mathfrak{a}N,\mathfrak{b}N]^d}*\dots*\chi_{[\mathfrak{a}N,\mathfrak{b}N]^d})(\xi),$$

where

$$\mathfrak{a} = A+(B-A)/4, \qquad \mathfrak{b} = B-(B-A)/4.$$

Now, recalling that

$$\widehat{\chi}_{[\mathfrak{a}N,\mathfrak{b}N]^d}(\xi) = 2^d\prod_{j=1}^{d} e^{-i(\mathfrak{a}+\mathfrak{b})N\xi_j/2}\frac{\sin((\mathfrak{a}-\mathfrak{b})N\xi_j/2)}{\xi_j},$$

we find

$$\chi_{[\mathfrak{a}N,\mathfrak{b}N]^d}*\dots*\chi_{[\mathfrak{a}N,\mathfrak{b}N]^d}(x) = 2^{d(k+1)}\prod_{j=1}^{d}\int_{\mathbb{R}} e^{ix_j\cdot i\xi_j - i(k+1)(\mathfrak{a}+\mathfrak{b})N\xi_j/2}\left(\frac{\sin((\mathfrak{a}-\mathfrak{b})N\xi_j/2)}{\xi_j}\right)^{k+1} d\xi_j. \tag{2.83}$$

Hence, changing variables

$$\chi_{[\mathfrak{a}N,\mathfrak{b}N]^d}*\dots*\chi_{[\mathfrak{a}N,\mathfrak{b}N]^d}((k+1)(A+B)N/2+rN,\dots,(k+1)(A+B)N/2+rN) = 2^{d(1-k)}(B-A)^{kd}N^{kd}\Big(\int_{\mathbb{R}}\cos(4r/(B-A))\left(\frac{\sin w}{w}\right)^{k+1}dw\Big)^d. \tag{2.84}$$

This proves by continuity that for $r>0$ small enough, there exists $c>0$ which does not depend on N, such that

$$|\widehat{u}_{k+1}(\xi,t)|\chi_{[(k+1)(A+B)N/2-rN,(k+1)(A+B)N/2+rN]^d} \geqslant cN^{-(k+1)(2s+d)/2}N^{-1}N^{kd}\chi_{[(k+1)(A+B)N/2-rN,(k+1)(A+B)N/2+rN]^d}$$

so that

$$\|u_{k+1}(t)\|_{H^s} \gtrsim N^{-ks+kd/2-1}$$

from which (for fixed $T>0$)

$$\lim_{N\to\infty}\sup_{t\in[-T,T]}\|u_{k+1}(t)\|_{H^s} = +\infty, \tag{2.85}$$

as soon as $s<d/2-1/k$. This yield a contradiction to (2.77). The proof is completed. □

2.5.2. Solitary wave solutions. This subsection is devoted to present a survey of known result regarding the existence of solitary-wave solutions for the equations,

$$u_t + \mathfrak{v}\mathcal{R}_1 \Delta u + u^p u_{x_1} = 0, \quad (x,t) \in \mathbb{R}^{d+1}, \tag{2.86}$$

with $\mathfrak{v} \neq 0$. Motivated by the two-dimensional model (see [**2**, **71**, **87**]), we are interested in study solitary–wave solutions of the form $u(x,t) = \varphi(x_1 - ct, \bar{x})$, where $\bar{x} \in \mathbb{R}^{d-1}$, and c denotes the speed of propagation. Substituting $\varphi(x_1 - ct, \bar{x})$ into (2.86), integrating once with respect to the variable $z = x_1 - ct$, and assuming that φ has an appropriated decay for suitably large values of $|z|$, we observe that φ satisfies

$$-c\varphi + \mathfrak{v}(-\Delta)^{1/2}\varphi + \frac{1}{p+1}\varphi^{p+1} = 0. \tag{2.87}$$

We will assume that the power $p = k/m$, where k and m are relatively prime and m is odd. Consequently, we can define a branch of the map $r \mapsto r^{1/m}$ real on the real axis.

Next, we establish a non-existence result for solutions of (2.87).

Proposition 2.17. *Equation (2.87) cannot have a smooth non-trivial solitary-wave solution unless either*

(i) $\mathfrak{v} < 0$, $c > 0$, $p < \frac{2}{d-1}$,
(ii) $\mathfrak{v} > 0$, $c < 0$, $p < \frac{2}{d-1}$,
(iii) $\mathfrak{v} > 0$, $c > 0$, $p > \frac{2}{d-1}$, *or*
(iv) $\mathfrak{v} < 0$, $c < 0$, $p > \frac{2}{d-1}$.

By "smooth", we mean that the functions have sufficient regularity to justify the following computations. We emphasize that only the case (i) with $p = 1$ is of physical relevance. Additionally, cases (ii) and (iii) are the same as (i) and (iv) respectively except that the sign of the nonlinearity is reversed.

Proof. We will deduce some Pohozaev–type identities to derive the desired conclusion. Multiplying (2.86) by φ and integrating on $\mathbb{R}^d$ it is seen

$$\int -c\varphi^2 + \mathfrak{v}(-\Delta)^{1/2}\varphi\varphi + \frac{1}{p+1}\varphi^{p+2}\, dx = 0. \tag{2.88}$$

On the other hand, we claim

$$\int ((-\Delta)^{1/2}\varphi) x_j \varphi_{x_j}\, dx = -\frac{1}{2}\int (-\Delta)^{1/2}\varphi\varphi + (-\Delta)^{-1/2}\partial_{x_j}^2\varphi\varphi\, dx \tag{2.89}$$

for all $j = 1, \dots, d$. Indeed, by Plancherel's identity and integration by parts it follows that

$$\begin{aligned}
\int ((-\Delta)^{1/2}\varphi) x_j \varphi_{x_j}\, dx &= -\int |\xi|\widehat{\varphi}(\xi)\frac{\partial}{\partial_{\xi_j}}\overline{(\xi_j\widehat{\varphi}(\xi))}\, d\xi \\
&= -\int |\xi||\widehat{\varphi}(\xi)|^2\, d\xi - \int |\xi|\xi_j\widehat{\varphi}(\xi)\frac{\partial}{\partial_{\xi_j}}\overline{\widehat{\varphi}(\xi)}\, d\xi \\
&= -\int |\xi||\widehat{\varphi}(\xi)|^2\, d\xi + \int |\xi|^{-1}\xi_j^2|\widehat{\varphi}(\xi)|^2\, d\xi + \int |\xi|\frac{\partial}{\partial\xi_j}(\xi_j\widehat{\varphi}(\xi))\overline{\widehat{\varphi}(\xi)}\, d\xi \\
&= -\int (-\Delta)^{1/2}\varphi\varphi\, dx - \int (-\Delta)^{-1/2}\partial_{x_j}^2\varphi\varphi\, dx + \int ((-\Delta)^{1/2}\varphi)x_j\varphi_{x_j}\, dx.
\end{aligned} \tag{2.90}$$

This establishes (2.89). Now, multiplying (2.87) by $x_j\varphi_{x_j}$ and using (2.89), we get

$$\int c\varphi^2 - \mathfrak{v}(-\Delta)^{1/2}\varphi\varphi - \mathfrak{v}(-\Delta)^{-1/2}\partial^2_{x_j}\varphi\varphi - \frac{2}{(p+1)(p+2)}\varphi^{p+2}\,dx = 0$$

which leads after summing over $j = 1, \ldots, d$ to

$$\int c\varphi^2 - \mathfrak{v}\left(\frac{d-1}{d}\right)(-\Delta)^{1/2}\varphi\varphi - \frac{2}{(p+1)(p+2)}\varphi^{p+2}\,dx = 0. \tag{2.91}$$

Finally, substituting (2.88) into (2.91) we have

$$\int c\varphi^2 - \mathfrak{v}\left(\frac{(d-1)p-2}{dp}\right)|(-\Delta)^{1/4}\varphi|^2\,dx = 0.$$

The above identity yields the proof of Proposition 2.17. □

Concerning the cases (i) and (ii) of Proposition 2.17, existence of solitary waves was established in [62] for the two-dimensional problem with $p = 1$. For arbitrary dimensions, existence can be deduced as a particular case of the results proved by Frank, Lenzmann and Silvestre in [31] (see also [30]) for the class of nonlocal equations:

$$\Psi + (-\Delta)^s\Psi - |\Psi|^r\Psi = 0, \text{ in } \mathbb{R}^d, \tag{2.92}$$

with $d \geqslant 1$, $s \in (0,1)$ and $0 < r < r_*(d,s)$ where

$$r_* = \begin{cases} \frac{4s}{d-2s} & \text{for } 0 < s < d/2, \\ +\infty & \text{for } s \geqslant d/2. \end{cases}$$

Let us now state some of the results derived in [31] concerning (2.92). To establish the existence of solutions, one can use the Weinstein classical approach which consists of determining the best constant C_{opt} in the Gagliardo–Nirenberg inequality

$$\int |u|^{r+2}\,dx \leqslant C_{opt}\left(\int |(-\Delta)^{s/2}u|^2\,dx\right)^{dr/4s}\left(\int |u|^2\,dx\right)^{(r+2)/2-dr/4s}, \tag{2.93}$$

so that C_{opt} is obtained by minimizing the functional

$$J(u) = \frac{\left(\int |(-\Delta)^{s/2}u|^2\,dx\right)^{dr/4s}\left(\int |u|^2\,dx\right)^{(r+2)/2-dr/4s}}{\int |u|^{r+2}\,dx} \tag{2.94}$$

defined for $u \in H^s(\mathbb{R}^d)$ with $u \neq 0$. Indeed, by methods of variational calculus (see Appendix D in [31]), it is seen that $C^{-1}_{opt} = \inf_{u\neq 0} J(u)$ is attained. Moreover, by computing $J'(u)$, it easily seen that any minimizer $\Psi \in H^s(\mathbb{R}^d)$ satisfy equation (2.92) after a suitable rescaling $\Psi \mapsto c_1\Psi(c_2\cdot)$ for some constants c_1 and c_2. Finally, the inequality $J(|u|) \leqslant J(u)$ implies that the minimizer Ψ can be chosen to be nonnegative.

On the other hand, uniqueness issues have been addressed in [31] for the class of nonlocal equations (2.92). They consider ground state solutions according to the following definition.

Definition 2.18. *Assume that $\Psi \in H^s(\mathbb{R}^d)$ is a real–valued solution of equation (2.92). Let L_+ denote the corresponding linearized operator given by*

$$L_+ = (-\Delta)^s + 1 - (\mathfrak{v}+1)|\Psi|^r \tag{2.95}$$

acting on $L^2(\mathbb{R}^d)$. We say that $\Psi \geqslant 0$ with $\Psi \neq 0$ is a ground state solution of equation (2.92) if L_+ has Morse index equal to 1; i.e., L_+ has exactly one strictly negative eigenvalue (counting multiplicity).

It is worth pointing out that if $\Psi \geqslant 0$ is a (local) minimizer of the functional $J(u)$, then L_+ has Morse index equal to 1 (see the comments after Definition 3.2 in [**31**]). In particular, any nonnegative minimizer Ψ of $J(u)$ is a ground state in the sense of the above definition (cf. [**30**]). Summarizing the result in [**31**], we have:

Theorem 2.19. *Let $d \geqslant 1$, $s \in (0,1)$ and $0 < r < r_*(d,s)$. Then*

(i) Existence: *There exists a minimizer $\Psi \in H^s(\mathbb{R}^d)$ for $J(u)$, which can be chosen a nonnegative function $\Psi \geqslant 0$ that solves equation (2.92).*

(ii) Symmetry, regularity, and decay: *If $\Psi \in H^s(\mathbb{R}^d)$ with $\Psi \geqslant 0$ and $\Psi \neq 0$ solves (2.92), then there exists some $x_0 \in \mathbb{R}^d$ such that $\Psi(\cdot - x_0)$ is radial, positive, and strictly decreasing in $|x - x_0|$. Moreover, the function Ψ belongs to $H^{2s+1}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ and it satisfies*

$$\frac{C_1}{1+|x|^{d+2s}} \leqslant \Psi(x) \leqslant \frac{C_2}{1+|x|^{d+2s}} \text{ for } x \in \mathbb{R}^d$$

with some constants $C_2 \geqslant C_1 > 0$ depending on s, d, r and Ψ.

(iii) Uniqueness. *The ground state solution $\Psi \in H^s(\mathbb{R}^d)$ for equation (2.92) is unique up to translation.*

Consequently, Theorem 2.19 establishes existence of solutions for (2.87) under the restrictions (i) and (ii) of Proposition 2.17. Indeed, considering the conditions (i) for simplicity, it can be assumed that (2.87) has the normalized form

$$-\varphi - (-\Delta)^{1/2}\varphi + \frac{1}{p+1}\varphi^{p+1} = 0, \tag{2.96}$$

which follows by scaling the variables as

$$u(x,t) = av(bx, dt), \tag{2.97}$$

where $a = |\mathfrak{v}|^{1/p}$, $b = |\mathfrak{v}|^{-1/2}$ and $d = 1/c$. Then, letting $\varphi(\cdot) = (p+1)^{1/p}\Psi(\cdot)$, where $\Psi \geqslant 0$ is given by Theorem 2.19 for $r = p < 2/(d-1)$, we find that φ solves (2.96). In conclusion, we deduce:

Corollary 2.20. *Let $d \geqslant 2$, $\mathfrak{v}c < 0$, and $p = \frac{k}{m} < 2/(d-1)$, where m is an odd positive integer and m and k are relative prime. Then equation (2.87) admits a non-trivial solution in $H^{1/2}(\mathbb{R}^d)$ that satisfies the properties stated in part (ii) of Theorem 2.19.*

Regarding stability, by scaling, we will restrict our attention to the case $c > 0$ and $\mathfrak{v} = -1$ in (2.87). We require the following quantities for our discussions

$$\mathcal{E}(u) = \frac{1}{2}\int |(-\Delta)^{1/4}u|^2 - \frac{2}{(p+1)(p+2)}u^{p+2}\,dx, \tag{2.98}$$

$$\mathcal{M}(u) = \frac{1}{2}\int u^2\,dx. \tag{2.99}$$

In virtue of the embedding $H^{1/2}(\mathbb{R}^d) \hookrightarrow L^{p+2}(\mathbb{R}^d)$ valid for all $p < 2/(d-1)$, we have that $H^{1/2}(\mathbb{R}^d)$ is a natural space to define $\mathcal{E}(\cdot)$ and $\mathcal{M}(\cdot)$. Now, we introduce the following function

$$\mathfrak{d}(c) := \mathcal{E}(\varphi_c) + c\mathcal{M}(\varphi_c), \tag{2.100}$$

where φ_c solves (2.87) for $c > 0$ and $\mathfrak{v} = -1$. In particular, we notice that

$$\varphi_c(\cdot) = c^{1/p}\varphi_1(c\cdot), \tag{2.101}$$

where φ_1 solves (2.96). It is well-known that the function $\mathfrak{d}(\cdot)$ is employed to study stability and instability of solitary waves (see for instance [6, Theorem 2.3], [8, Theorem 3.1 and Theorem 4.1], [22, Theorem 3.2] and [25]). It is expected that the solitary wave φ_c is stable when $\mathfrak{d}''(c) > 0$ and unstable if $\mathfrak{d}''(c) < 0$. In our case, multiplying (2.87) with $\mathfrak{v} = -1$ by φ_c yields

$$\mathcal{E}'(\varphi_c) + \mathcal{M}'(\varphi_c) = 0 \tag{2.102}$$

so that changing variables and employing (2.101)

$$\mathfrak{d}'(c) = \mathcal{M}(\varphi_c) = \frac{c^{2/p-d}}{2}\mathcal{M}(\varphi_1). \tag{2.103}$$

From this we infer

$$\mathfrak{d}''(c) = \frac{1}{2}\left(\frac{2-pd}{p}\right)c^{2/p-d-1}\mathcal{M}(\varphi_1). \tag{2.104}$$

Therefore it is seen

(i) $\mathfrak{d}''(c) > 0$ if and only if $p < 2/d$,

(ii) $\mathfrak{d}''(c) < 0$ if and only if $2/d < p < 2/(d-1)$. (The condition $p < 2/(d-1)$ assures existence).

Remark 2.1. *Unfortunately, the physically relevant case $p = 1$ and $d = 2$ satisfies $\mathfrak{d}''(c) = 0$, and so it still remains an open problem to determinate stability or instability. Additionally, the range of indexes p covered by the previous approach does not include an integer number.*

Based on the previous remark, we decided not to proceed into any more aspects concerning orbital stability/instability. However, after establishing a local well-posedness theory for (2.86), one can adapt the ideas in [4, 6, 22, 25, 58, 70] for instance, to obtain stability of solitary wave solutions for the case $p < 2/d$, and instability whenever $2/d < p < 2/(d-1)$.

2.6. A note on local unique continuation principles

This section is aimed to present some unique continuation principles for a family of generalized dispersive equations that incorporate the model (0.2). More precisely, the idea is to prove that if $u_1(x,t), u_2(x,t)$ are two suitable solutions of a dispersive equation for $(x,t) \in \mathbb{R}^d \times [0,T]$, such that there exists some non-empty open set $\Omega \subset \mathbb{R}^d \times [0,T]$ for which

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \Omega,$$

Then it follows $u_1(x,t) = u_2(x,t)$ for all $(x,t) \in \mathbb{R}^d \times [0,T]$.

Our analysis on this subject is inspired by the recent results deduced by Kenig, Pilod, Ponce, and Vega in [52]. We remark that some other unique continuation principles for (0.2) are established in Chapter 3.

In this section, we are interested in examine the unique continuation principle stated above for the family of equations:

$$\begin{cases} \partial_t u - \partial_{x_1} D^a u + u u_{x_1} = 0, & (x,t) \in \mathbb{R}^{d+1}, \quad a \in (-1,2]\setminus\{0\} \\ u(x,0) = u_0(x), \end{cases} \tag{2.105}$$

where the dimension $d \geqslant 2$, $D^a f = (-\Delta)^{a/2} f$ is defined by the Fourier symbol: $|\xi|^a \widehat{f}(\xi)$, $|\xi| = \sqrt{\xi_1^2 + \cdots + \xi_d^2}$. The case $a = 1$ coincides with (0.2), and $a = 2$ with the widely studied Zakharov-Kuznetsov equation (see [55, 89]). The equation (2.105) for $1 \leqslant a \leqslant 2$ can be regarded as a mathematical model to measure the effects of dispersion on all the variables between the physical relevant models $a = 1$ and $a = 2$. The cases $a \in (-1,1)\setminus\{0\}$ can be implemented as a mathematical equation to measure the effects of weak dispersion and nonlinearity in a higher dimensional model. We emphasized that $a = -1$ yields the equation:

$$\partial_t u + \mathcal{R}_1 u + u u_x = 0, \tag{2.106}$$

which can be seen as a mathematical extension of the Burgers-Hilbert equation (0.5).

Formally, real solutions of (2.105) satisfy three conservation laws:

$$\begin{aligned} I_a(u) &= \int u(x,t)\,dx, \\ M_a(u) &= \int u^2(x,t)\,dx, \\ H_a(u) &= \int \left|D^{a/2}u(x,t)\right|^2 - \frac{1}{3}u^3(x,t)\,dx. \end{aligned} \tag{2.107}$$

Reffering to well-posedness for (2.105), the best known local well-posedness conclusion for $1 \leqslant a < 2$ were established in [80], it was proved that (2.105) is LWP in $H^s(\mathbb{R}^d)$ $s > d/2 + 3/2 - a$ whenever $1 \leqslant a < 2$. Concerning the initial value problem for the Zakharov-Kuznetsov $a = 2$, in [53], it was shown LWP in $H^s(\mathbb{R}^2)$ $s > -1/4$ and Global well-posedness (GWP) in $L^2(\mathbb{R}^2)$, in [38] it was determined LWP in $H^s(\mathbb{R}^d)$ $s > (d-4)/2$ for $d \geqslant 3$, and GWP in $L^2(\mathbb{R}^3)$. As far as we know there are non-standard results addressing well-posedness issues for the dispersions $-1 < a < 1$ with $a \neq 0$.

Our main result is the following:

Theorem 2.21. *Let $a \in (-1,2)\setminus\{0\}$. Let u_1, u_2 be two real solutions of the IVP (2.105) such that*

$$u_1, u_2 \in C([0,T]; H^s(\mathbb{R}^d)) \cap C^1([0,T]; H^{s'}(\mathbb{R}^d)), \tag{2.108}$$

with $s > \max\{a+1, d/2+1\}$, $s' > \min\{s-(a+1), s-1\}$. If there exists a non-empty open set $\Omega \subset \mathbb{R}^d \times [0,T]$ such that

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \Omega,$$

then, $u_1(x,t) = u_2(x,t)$ for all $(x,t) \in \mathbb{R}^d \times [0,T]$.

The existence of solutions for (2.105) in the class (2.108) can be obtained by applying a parabolic regularization argument in the spirit of [45, 47] or [60, Chapter 10]. Notice that this technique does not consider the effects of dispersion to establish the existence of solutions.

We remark that some unique continuation properties of solutions to the Zakharov-Kuznetsov equation ($\alpha = 2$ in (2.105)) for dimensions $d = 2, 3$ have been studied in [9, 16, 17]. Roughly, in these references it was established that for two sufficiently regular solutions u_1, u_2, if $u_1 - u_2$ decays fast enough at two distinct times, it follows $u_1 \equiv u_2$.

A key argument in the proof of Theorem 2.21 is the following global uniqueness results for fractional Schrödinger equation established in [32, Theorem 1.2].

Theorem 2.22. *Let $a \in (0,2)$ and $f \in H^s(\mathbb{R}^d)$ for some $s \in \mathbb{R}$. If there exists an open (non-empty) set Θ such that*

$$(-\Delta)^{a/2} f(x) = f(x) = 0, \ \text{in } \mathcal{D}'(\Theta), \tag{2.109}$$

then $f \equiv 0$ in $H^s(\mathbb{R}^d)$.

In the last theorem, $\mathcal{D}'(\Theta)$ denotes the space of distributions on Θ, i.e., the space of continuous linear functional on $C_c^\infty(\Theta)$.

Remark 2.2. *The statement of Theorem 2.22 clearly extends to $a \in (0,\infty)\backslash 2\mathbb{Z}$. Indeed, writing $a = 2k + b$, where $k \in \mathbb{Z}^+ \cup \{0\}$, $b \in (0,2)$ and $g := (-\Delta)^k f$, by using that $(-\Delta)^k$ is a local operator we have*

$$g(x) = (-\Delta)^k f(x) = (-\Delta)^{a/2} f(x) = (-\Delta)^{b/2} g(x) = 0, \ \text{in } \mathcal{D}'(\Theta).$$

Thus, Theorem 2.22 establishes that $g \equiv 0$ in $H^{s-2k}(\mathbb{R}^d)$, and so $f \equiv 0$ in $H^s(\mathbb{R}^d)$.

Additionally, the conclusion of Theorem 2.22 holds for $a \in (-d/2, 0)\backslash 2\mathbb{Z}$ assuming that $f \in H^s(\mathbb{R}^d)$ for some $s \geqslant 0$ such that $\lfloor s \rfloor > |a|/2$. To see this, we first notice that by Hardy-Littlewood-Sobolev inequality $(-\Delta)^{a/2} f$ is a well-defined function in $L^p(\mathbb{R}^d)$, where $1/p = 1/2 + a/d$. Then, we let $g := (-\Delta)^{\lfloor s \rfloor + a/2} f$, so since $(-\Delta)^{\lfloor s \rfloor}$ is a local operator the desired conclusion follows by the previous result for positive fractional derivatives and by observing that

$$g(x) = (-\Delta)^{\lfloor s \rfloor}(-\Delta)^{a/2} f(x) = (-\Delta)^{\lfloor s \rfloor} f(x) = (-\Delta)^{|a|/2} g(x) = 0, \ \text{in } \mathcal{D}'(\Theta).$$

PROOF OF THEOREM 2.21. We define $w(x,t) := u_1(x,t) - u_2(x,t)$, then

$$\partial_t w - \partial_{x_1} D^a w + \partial_{x_1} u_1 w + u_2 \partial_{x_1} w = 0, \ (x,t) \in \mathbb{R}^d \times [0,T]. \tag{2.110}$$

Since u_1, u_2 are in the class (2.108), the equation (2.110) is satisfied in $H^{s'}(\mathbb{R}^d)$ and in consequence it is valid for almost every $(x,t) \in \mathbb{R}^d \times [0,T]$. Then it follows

$$\partial_{x_1} D^a w(x,t) = 0, \quad \text{a.e.} \quad (x,t) \in \Omega. \tag{2.111}$$

We emphasize that by Hardy-Littlewood-Sobolev inequality $D^a w(\cdot,t)$ is a well-defined function in $L^p(\mathbb{R}^d)$ with $p = 2$ if $a \in (0,2)$, and in $1 < p < \infty$ with $1/p = 1/2 + a/d$, if $a \in (-1,0)$.

According to (2.111) there exist t_0 and $\Theta \subset \mathbb{R}^d$ open non-empty such that $\Theta \times \{t_0\} \subset \Omega$ and $\partial_{x_1} D^a w(x,t_0) = 0$ for a.e. $x \in \Theta$.

Therefore, Theorem 2.22, Remark 2.2 and the fact that $\partial_{x_1} w(\cdot, t_0)$ is a continuous function yield $\partial_{x_1} w(x,t_0) = 0$ for all $x \in \mathbb{R}^d$, that is, $w(x_1, x', t_0)$ depends only on the variables $x' = (x_2, \dots, x_d)$. However, Fubini's Theorem implies that $w(\cdot, x', t_0) \in L^2(\mathbb{R})$ for almost every $x' \in \mathbb{R}^{d-1}$, so it must follows that $w(x,t_0) = 0$ for all $x \in \mathbb{R}^d$. This completes the proof. ☐

Remark 2.3. *The previous reasoning in the proof of Theorem 2.21 extends to the equation in* (2.105) *with a more general non-linearity. As a matter of fact, Theorem 2.21 applies to any pair of appropriate solutions u_1, u_2 of the IVP associated to the equation*

$$\partial_t u - \partial_{x_1} D^a u + F(u, \dots, \partial^\alpha u) = 0, \quad (x,t) \in \mathbb{R}^{d+1}, \tag{2.112}$$

where $a \in (-d/2, \infty)\backslash 2\mathbb{Z}$, $\alpha \in \mathbb{N}^d$ is a multi-index and $F(\cdot)$ is a regular enough function representing the non-linearity. In particular, taking $F(u, \partial_{x_1} u) = u^k \partial_{x_1} u$, $k \in \mathbb{Z}^+$, we deduce Theorem 2.21 for the IVP associated to (2.2).

Chapter 3

Study of the HBO equation in weighted spaces

In this chapter, we study the initial value problem (0.2) in weighted spaces. Our purpose is to establish local well-posedness results in weighted Sobolev spaces and to determine according to them some sharp unique continuation properties of the solution flow. In consequence, optimal decay rate for this model is determined. We remark that a key ingredient in our considerations is the deduction of a new commutator estimate involving Riesz transforms (See Proposition 3.8 below). The results stated in this chapter are contained in [76].

3.1. Statement of results

This work is intended to determine if for a given initial data in the Sobolev space $H^s(\mathbb{R}^d)$ with some additional decay at infinity (for instance polynomial), it is expected that the corresponding solution of (0.2) inherits this behavior. Such matter has been addressed before for the Benjamin-Ono equation in [27, 29], showing that in general polynomial type decay is not preserved by the flow of this model. As a consequence of our results, we shall determine that the same conclusion extends to the (0.2) equation.

Let us now state our results. Our first consequence is motivated from the fact that the weight function $\langle x\rangle^r = (1+|x|^2)^{r/2}$ is smooth with bounded derivatives when $r \in [0,1]$. This property allows us to consider well-posedness issues for a more general class of weights.

PROPOSITION 3.1. *Let ω be a smooth weight with all its first and second derivatives bounded. Then, the IVP (0.2) is locally well-posed in $H^s(\mathbb{R}^d) \cap L^2(\omega^2\, dx)$ for all $s > s_d$, where $s_2 = 5/3$ and $s_d = d/2 + 1/2$ for $d \geqslant 3$.*

The proof of Proposition 3.1 is similar in spirit to that in [18] for a two-dimension model. A remarkable difference is that our well-posedness results in Theorem 2.1 (see also [39]) enable us to prove Proposition 3.1 in Sobolev spaces of lower regularity compared with those obtained by implementing a parabolic regularization argument as in Lemma 2.10.

Next, we discuss LWP for the IVP (0.2) in the weighted Sobolev spaces $Z_{s,r}(\mathbf{R}^d)$ and $\dot{Z}_{s,r}(\mathbf{R}^d)$ defined by (1.1) and (1.2) respectively.

For the purpose of obtaining a relation between differentiability and decay in the spaces (1.1), we notice that the linear part of the equation (0.2) $\mathcal{L} = \partial_t - \mathcal{R}_1\Delta$ commutes with the operators

$$\Gamma_l = x_l + t\delta_{1,l}(-\Delta)^{1/2} + t\partial_{x_l}\mathcal{R}_1, \ \ l = 1, \dots, d,$$

where in this chapter $\delta_{1,l}$ will denote the Kronecker delta function with $\delta_{1,l} = 1$ if $l = 1$ and zero otherwise, thus one has

$$[\mathcal{L}, \Gamma_l] = \mathcal{L}\Gamma_l - \Gamma_l\mathcal{L} = 0.$$

For this reason, it is natural to study well-posedness in weighted Sobolev spaces $Z_{s,r}(\mathbb{R}^d)$ where the balancing between decay and regularity satisfies the relation, $r \leqslant s$.

Remark 3.1. *For the sake of brevity, from now on we shall state our results for the* (0.2) *equation only for dimensions two and three. Actually, it will be clear from our arguments that solutions of this model in the spaces* (3.1) *behave quite different in each of these dimensions. Nevertheless, following our ideas one can extend the ensuing conclusions to arbitrary even and odd dimensions.*

Theorem 3.2. *Consider $d = 2, 3$. Let $s > s_d$ where $s_2 = 5/3$ and $s_3 = 2$.*

(i) *If $r \in [0, d/2 + 2)$ with $r \leqslant s$, then the IVP associated to* (0.2) *is locally well-posed in $Z_{s,r}(\mathbb{R}^d)$.*
(ii) *If $r \in [0, d/2 + 3)$ with $r \leqslant s$, then the IVP associated to* (0.2) *is locally well-posed in $\dot{Z}_{s,r}(\mathbb{R}^d)$.*

The proof of Theorem 3.2 is adapted from the arguments used by Fonseca and Ponce in [**29**] and Fonseca, Linares and Ponce in [**28**]. Additional difficulties arise from extending these ideas to the (0.2) equation, since here we deal with a higher dimensional model involving Riesz transform operators. Among them, the commutator relation between $\mathcal{R}_1$ and a polynomial of a certain higher degree requires to infer weighted estimates for derivatives of negative order. In this regard, as a further consequence of the proof of Theorem 3.2 we deduce.

Corollary 3.3. *Consider $d = 2, 3$ and $r_0 \in [0, d/2)$. Let $u \in C([0, T]; \dot{Z}_{s,r}(\mathbb{R}^d))$ be a solution of the IVP* (0.2) *with $(d/2 + 2)^- \leqslant r \leqslant s$. Then*

$$|\nabla|^{-1}u \in C([0, T]; L^2(|x|^{2r_0}\, dx)).$$

Where the operator $|\nabla|^{-1}$ is defined by the Fourier multiplier $|\xi|^{-1} = (\xi_1^2 + \dots + \xi_d^2)^{-1/2}$. Next we state some continuation principles for the (0.2) equation.

Theorem 3.4. *Assume that $d = 2, 3$. Let u be a solution of the IVP associated to* (0.2) *such that $u \in C([0, T]; Z_{2^+,2}(\mathbb{R}^2))$ when $d = 2$ and $u \in C([0, T]; Z_{3,3}(\mathbb{R}^3))$ when $d = 3$. If there exist two different times $t_1, t_2 \in [0, T]$ for which*

$$u(\cdot, t_j) \in Z_{d/2+2,d/2+2}(\mathbb{R}^d),\ j = 1, 2 \quad \textit{then} \quad \widehat{u}_0(0) = 0.$$

In Theorem 3.4, $u \in Z_{2^+,2}(\mathbb{R}^2)$ means that $u \in H^{2^+}(\mathbb{R}^2) \cap L^2(|x|^4dx)$, where there exists a positive number $\epsilon \ll 1$ such that $u \in H^{2+\epsilon}(\mathbb{R}^2)$.

Theorem 3.5. *Suppose that $d = 2, 3$, $r_2 = 3$ and $r_3 = 4$. Let $u \in C([0, T]; \dot{Z}_{r_d,r_d}(\mathbb{R}^d))$ be a solution of the IVP associated to* (0.2)*. If there exist three different times $t_1, t_2, t_3 \in [0, T]$ such that*

$$u(\cdot, t_j) \in Z_{d/2+3,d/2+3}(\mathbb{R}^d),\ j = 1, 2, 3 \quad \textit{then} \quad u(x, t) = 0.$$

It is worth pointing out that the deduction of Theorems 3.4 and 3.5 is more involved in the odd dimension case, where the decay rates $d/2+2$ and $d/2+3$ are not integer numbers. Roughly speaking, transferring decay to regularity in the frequency domain, on this setting one has to deal with an extra $1/2$-fractional derivative to achieve these conclusions.

We remark that similar unique continuation properties have been established for the Benjamin-Ono equation in [29] and the dispersion generalized Benjamin-Ono equation in [28]. A difference in the present work is that our proof of Theorems 3.4 and 3.5 incorporates an extra weight in the frequency domain, which allows us to consider less regular solutions of (0.2) to reach these consequences.

Remarks. (i) *When $d=1$, the conclusion of Theorem 3.2 coincides with the decay rates showed for the Benjamin-Ono equation in* [29, Theorem 1]. *In this sense, our results can be regarded as a generalization of those derived by the Benjamin-Ono equation* (0.1). *As a matter of fact, Theorem 3.2 tells us that an increment in the dimension allows a $1/2$ larger decay with respect to the preceding setting.*

(ii) *The restrictions on the Sobolev regularity stated in Proposition 3.1 and Theorem 3.2 are imposed from the results in Theorem 2.1, which assure that under such considerations the solution $u(x,t)$ satisfies*

$$u \in L^1([0,T); W^{1,\infty}(\mathbb{R}^d)), \tag{3.1}$$

Since we employ energy estimates to establish LWP in $Z_{s,r}(\mathbb{R}^d)$, the property (3.1) *is essential to consider lower regularity solutions for our result in this chapter.*

(iii) *Theorem 3.4 shows that the decay $r=(d/2+2)^-$ is the largest possible for arbitrary initial data. In this regard Theorem 3.2 (i) is sharp. In addition, Theorem 3.4 shows that if $u_0 \in \mathbb{Z}_{s,r}(\mathbb{R}^d)$ with $d/2+2 \leqslant r \leqslant s$ and $\widehat{u_0}(0) \neq 0$, then the corresponding solution $u=u(x,t)$ verifies*

$$|x|^{(d/2+2)^-} u \in L^\infty([0,T]; L^2(\mathbb{R}^d)), \quad T>0.$$

Although, there does not exist a non-trivial solution u corresponding to data u_0 with $\widehat{u_0}(0) \neq 0$ with

$$|x|^{d/2+2} u \in L^\infty([0,T']; L^2(\mathbb{R}^d)), \ \textit{for some } T'>0.$$

(iv) *Theorem 3.5 shows that the decay $r=(d/2+3)^-$ is the largest possible in the spatial L^2-decay rate. As a result, Theorem 3.4 (ii) is sharp. In addition, Theorem 3.5 tells us that there are non-trivial solutions $u=u(x,t)$ such that*

$$|x|^{(d/2+3)^-} u \in L^\infty([0,T]; L^2(\mathbb{R}^d)), \qquad T>0$$

and it guarantees that there does not exist a non-trivial solution such that

$$|x|^{d/2+3} u \in L^\infty([0,T]; L^2(\mathbb{R}^d)), \quad \textit{for some } \ T'>0.$$

One may ask whether the assumption in Theorem 3.5 can be reduced to two different times $t_1<t_2$. In this respect, we have the following consequences.

Theorem 3.6. *Suppose that $d=2,3$, $r_2=3$ and $r_3=4$. Let $u \in C([0,T]; \dot{Z}_{r_d,r_d}(\mathbb{R}^d))$ be a solution of the IVP associated to* (0.2). *If there exist $t_1,t_2 \in [0,T]$, $t_1 \neq t_2$, such that*

$$u(\cdot,t_j) \in Z_{d/2+3,d/2+3}(\mathbb{R}^d),\ j=1,2,$$

and

$$\int x_1 u(x,t_1)\,dx = 0 \quad \text{or} \quad \int x_1 u(x,t_2)\,dx = 0,$$

then

$$u \equiv 0.$$

Theorem 3.7. *Suppose that $d = 2,3$, $r_2 = 3$ and $r_3 = 4$. Let $u \in C([0,T];\dot{Z}_{s,r_d}(\mathbb{R}^d))$ with $s \geqslant d/2+4$ be a nontrivial solution of the IVP associated to (0.2) such that*

$$u_0 \in \dot{Z}_{d/2+3,d/2+3}(\mathbb{R}^d) \quad \text{and} \quad \int x_1 u_0(x)\,dx \neq 0.$$

Let

$$t^* := -\frac{4}{\|u_0\|_{L^2}^2}\int x_1 u_0(x)\,dx.$$

If $t^* \in (0,T]$, then

$$u(t^*) \in \dot{Z}_{d/2+3,d/2+3}(\mathbb{R}^2).$$

Remarks. (i) *Theorem 3.6 tells us that the three times condition in Theorem 3.5 can be reduced to two times $t_1 \neq t_2$ provided that*

$$\int x_1 u(x,t_1)\,dx = 0 \quad \text{or} \quad \int x_1 u(x,t_2)\,dx = 0.$$

(ii) *Theorem 3.7 asserts that the condition of Theorem 3.5 in general cannot be reduced to two different times. In this sense the result of Theorem 3.6 is optimal.*

(iii) *In view of Theorem 3.7, we notice that the number of times involved in Theorems 3.4 and 3.5 is the same required to establish similar unique continuation properties for the Benjamin-Ono equation, see* [**29**, Theorem 1 and Theorem 2]. *Therefore, our conclusions on the (0.2) equation are again regarded as a generalization of their equivalents for the Benjamin-Ono model.*

Next we introduce the main ingredient behind the proof of Proposition 3.1 and Theorem 3.2. When dealing with energy estimates, motivated by the structure of the dispersion term in the equation (0.2), it is reasonable to try to find a commutator relation involving the Riesz transform, in such a way that when applied to a differential operator it redistributes the derivatives lowering the order of the operator. In this direction, we provide a new generalization of Calderón's first commutator estimate [**12**] in the context of the Riesz transform.

Proposition 3.8. *Let $\mathcal{R}_l$ be the usual Riesz transform in the direction $l = 1,\dots,d$. Consider $a \in C^\infty(\mathbb{R}^d)$ with $\partial^\gamma a \in L^\infty(\mathbb{R}^d)$ for all multi-index γ, and $f \in S(\mathbb{R}^d)$. Then for any $1 < p < \infty$, any multi-index α with $|\alpha| \geqslant 1$, there exists a constant c depending on α and p such that*

$$\Big\|\mathcal{R}_l(a\partial^\alpha f) - a\mathcal{R}_l\partial^\alpha f - \sum_{1\leqslant|\beta|<|\alpha|}\frac{1}{\beta!}\partial^\beta a D^\beta_{R_l}\partial^\alpha f\Big\|_{L^p} \leqslant c_{\alpha,p}\sum_{|\beta|=|\alpha|}\|\partial^\beta a\|_{L^\infty}\|f\|_{L^p}. \tag{3.2}$$

The operator $D^\beta_{R_l}$ is defined via its Fourier transform as

$$\widehat{D^\beta_{R_l} g}(\xi) = i^{-|\beta|}\partial_\xi^\beta\left(\frac{-i\xi_l}{|\xi|}\right)\widehat{g}(\xi). \tag{3.3}$$

In Proposition 3.8 the convention for the empty summation (such as $\sum_{1\leqslant|\beta|<1}$) is defined as zero. Consequently, when $|\alpha|=1$ we find

$$\|[\mathcal{R}_l,a]\partial^{\alpha}f\|_{L^p}\lesssim\|\nabla a\|_{L^\infty}\|f\|_{L^p}.$$

where

$$[\mathcal{R}_l,a]\partial^{\alpha}f=\mathcal{R}_l(a\partial^{\alpha}f)-a\mathcal{R}_l\partial^{\alpha}f.$$

Estimates of the form (3.2) are of interest on their own in Harmonic Analysis, see **[56]** for similar results and several applications dealing with homogeneous differential operators. The result of Proposition 3.8 may be of independent interest. Indeed, we believe that it could certainly be used to derive other properties for the (0.2) equation.

In the present work, (3.2) is essential to transfer derivatives to some weighted functions. Additionally, the operators $D^{\beta}_{R_l}$ defined by (3.3) are useful to represent commutator relations between the Riesz transform and polynomials.

We will begin by introducing some additional notation and preliminary estimates to be used in subsequent sections. In Section 3.3 we prove Proposition 3.1, and Theorems 3.2, 3.4, 3.5, 3.6 and 3.7 will be deduced in the following Sections 3.4, 3.5, 3.6, 3.7 and 3.8 respectively. We conclude this chapter with an appendix where we show the commutator estimate stated in Proposition 3.8.

3.2. Notation and preliminary estimates

Besides the considerations introduced in Section 1.2, we require of some additional considerations. But before we state these conclusions, it is worth to recall the result of Proposition 1.11 that implies that all the estimates involving the Riesz transform and the weights $\{w_n^{\theta}\}$ (see (1.20)) are independent of $n\in\mathbb{Z}^+$ for some appropriated values of θ.

A radial function $\phi\in C_c^{\infty}(\mathbb{R}^d)$, with $\phi(x)=1$ when $|x|\leqslant 1$ and $\phi(x)=0$ if $|x|\geqslant 2$ will appear several times in our arguments.

Now, we introduce some notation that will be convenient in the proof of Theorem 3.4 and Theorem 3.5. Given $k=1,\dots,d$ fixed, we define the operators F_j^k's as being:

$$F_j^k(t,\xi,f)=\partial^{j}_{\xi_k}\big(e^{it\xi_1|\xi|}f(\xi)\big) \tag{3.4}$$

for $j=1,2,3,4$. More precisely,

$$\begin{aligned}
F_1^k(t,\xi,f)&=\partial_{\xi_k}\big(it\xi_1|\xi|\big)e^{it\xi_1|\xi|}f(\xi)+e^{it\xi_1|\xi|}\partial_{\xi_k}f(\xi),\\
F_2^k(t,\xi,f)&=\partial^2_{\xi_k}\big(it\xi_1|\xi|\big)e^{it\xi_1|\xi|}f(\xi)+\partial_{\xi_k}\big(it\xi_1|\xi|\big)F_1^k(t,\xi,f)+F_1^k(t,\xi,\partial_{\xi_k}f),\\
F_3^k(t,\xi,f)&=\partial^3_{\xi_k}\big(it\xi_1|\xi|\big)e^{it\xi_1|\xi|}f(\xi)+2\partial^2_{\xi_k}\big(it\xi_1|\xi|\big)F_1^k(t,\xi,f)+\partial_{\xi_k}(it\xi_1|\xi|)F_2^k(t,\xi,f)\\
&\qquad+F_2^k(t,\xi,\partial_{\xi_k}f),\\
F_4^k(t,\xi,f)&=\partial^4_{\xi_k}\big(it\xi_1|\xi|\big)e^{it\xi_1|\xi|}f(\xi)+3\partial^3_{\xi_k}\big(it\xi_1|\xi|\big)F_1^k(t,\xi,f)+3\partial^2_{\xi_k}(it\xi_1|\xi|)F_2^k(t,\xi,f)\\
&\qquad+\partial_{\xi_k}(it\xi_1|\xi|)F_3^k(t,\xi,f)+F_3^k(t,\xi,\partial_{\xi_k}f).
\end{aligned} \tag{3.5}$$

Additionally, the operators $\widetilde{F}_j^k$, $j=1,2,3,4$ are defined according to (3.5) by the relations

$$\widetilde{F}_j^k(t,\xi,f)=e^{-it\xi_1|\xi|}F_j^k(t,\xi,f). \tag{3.6}$$

The following identities will be frequently considered in our arguments:

$$\begin{aligned}
&\partial_{\xi_k}(\xi_1|\xi|) = \delta_{1,k}|\xi| + \frac{\xi_1\xi_k}{|\xi|}, \qquad \partial^2_{\xi_k}(\xi_1|\xi|) = 2\delta_{1,k}\frac{\xi_k}{|\xi|} + \frac{\xi_1}{|\xi|} - \frac{\xi_1\xi_k^2}{|\xi|^3},\\
&\partial^3_{\xi_k}(\xi_1|\xi|) = 3\delta_{1,k}\frac{1}{|\xi|} - 3\delta_{1,k}\frac{\xi_k^2}{|\xi|^3} - 3\frac{\xi_1\xi_k}{|\xi|^3} + 3\frac{\xi_1\xi_k^3}{|\xi|^5},\\
&\partial^4_{\xi_k}(\xi_1|\xi|) = -12\delta_{1,k}\frac{\xi_k}{|\xi|^3} + 12\delta_{1,k}\frac{\xi_k^3}{|\xi|^5} - 3\frac{\xi_1}{|\xi|^3} + 18\frac{\xi_1\xi_k^2}{|\xi|^5} - 15\frac{\xi_1\xi_k^4}{|\xi|^7}.
\end{aligned} \tag{3.7}$$

Next we discuss some properties of the operators $D^{\beta}_{R_l}$ defined by (3.3). The following lemma is useful to estimate the L^2-norm of these operators.

Lemma 3.9. *Let α and β be multi-indexes and $f \in \dot{H}^{|\alpha|-|\beta|}(\mathbb{R}^d)$. Then there exist constants $c_\sigma \in \mathbb{R}$ such that*

$$D^{\beta}_{R_1}(\partial^{\alpha} f) = \sum_{\sigma} c_\sigma \mathcal{R}_\sigma(|\nabla|^{|\alpha|-|\beta|} f), \tag{3.8}$$

where the sum runs over all index $\sigma = (\sigma_1, \dots, \sigma_{|\alpha|+|\beta|+1})$ with integer components such that $1 \leqslant \sigma_j \leqslant d$, $j = 1, \dots, |\alpha| + |\beta| + 1$ and we denote by

$$\mathcal{R}_\sigma = \mathcal{R}_{\sigma_1} \cdots \mathcal{R}_{\sigma_{|\alpha|+|\beta|+1}}.$$

For instance, when $\alpha = 0$ and $|\beta| = 1$, say $\beta = e_k$, one has

$$D^{e_k}_{R_1} f = -\delta_{1,k}|\nabla|^{-1} f - \mathcal{R}_1\mathcal{R}_k\big(|\nabla|^{-1} f\big), \tag{3.9}$$

and so, letting now $\alpha = e_j$,

$$D^{e_k}_{R_1}\partial_{x_j} f = \delta_{1,k}\mathcal{R}_j f + \mathcal{R}_1\mathcal{R}_k\mathcal{R}_j f. \tag{3.10}$$

PROOF OF LEMMA 3.9 . An inductive argument yields the following identity

$$\partial^{\beta}\left(\frac{\xi_1}{|\xi|}\right) = \frac{P_\beta(\xi)}{|\xi|^{2|\beta|+1}}, \qquad \xi \neq 0, \tag{3.11}$$

where $P_\beta(\xi)$ is a homogeneous polynomial with real coefficients of order $|\beta| + 1$. Accordingly, we deduce the following point-wise identity

$$\mathcal{F}D^{\beta}_{R_1}(\partial^{\alpha} f)(\xi) = \frac{-1}{i^{|\beta|-|\alpha|-1}}\partial^{\beta}\left(\frac{\xi}{|\xi|}\right)\xi^{\alpha}\widehat{f}(\xi) = (-1)^{|\alpha|}\left(\frac{P_\beta(-i\xi)(-i\xi)^{\alpha}}{|\xi|^{|\alpha|+|\beta|+1}}\right)|\xi|^{|\alpha|-|\beta|}\widehat{f}(\xi). \tag{3.12}$$

The proof is now a consequence of the fact that the inverse Fourier transform of

$$\frac{P_\beta(-i\xi)(-i\xi)^{\alpha}}{|\xi|^{|\alpha|+|\beta|+1}}$$

can be written as a linear combination of the operators $\mathcal{R}_\sigma$, where $\sigma = (\sigma_1, \dots, \sigma_{|\alpha|+|\beta|+1})$ with $1 \leqslant \sigma_j \leqslant d$. □

As already mentioned, the operators $D^{\beta}_{R_l}$ are useful to express commutator relations between Riesz transforms and polynomials. More explicitly, for a given a multi-index $|\gamma|\geqslant 1$, we shall use the following point-wise estimate

$$[\mathcal{R}_l, x^{\gamma}]f=\sum_{0<\beta\leqslant\gamma}\binom{\gamma}{\beta}(-1)^{|\beta|+1}D^{\beta}_{R_l}(x^{\gamma-\beta}f), \tag{3.13}$$

valid for f regular enough with appropriated decay and satisfying for instance

$$\int x^{\beta}f(x)\,dx=0,\quad \text{for each } |\beta|<|\gamma|.$$

In particular, taking $\gamma=e_k$, $k=1,\dots,d$ and recalling (3.10), we obtain

$$[\mathcal{R}_1, x_k]\partial_{x_j}f=D^{e_k}_{R_1}\partial_{x_j}f=\delta_{1,k}\mathcal{R}_jf+\mathcal{R}_1\mathcal{R}_k\mathcal{R}_jf. \tag{3.14}$$

Now we state some consequences of Theorem 1.13.

Proposition 3.10. *Let $b\in(0,1)$. For any $t>0$*

$$\mathcal{D}^b(e^{ix_1|x|t})\lesssim(|t|^{b/2}+|t|^b|x|^b),\qquad x\in\mathbb{R}^d. \tag{3.15}$$

Proof. This result is proved following similar arguments as in [69]. □

By implementing Theorem 1.13, we deduce the following point-wise estimate:

Lemma 3.11. *Let $\theta\in(0,1)$, $l=0,1$ fixed and $P(x)$ be a homogeneous polynomial of degree $k\geqslant 0$ in $\mathbb{R}^d$. In addition, let $g\in L^{\infty}(\mathbb{R}^d)$ such that $|\cdot|^{-l}g,\nabla g\in L^{\infty}(\mathbb{R}^d)$. Then,*

$$\mathcal{D}^{\theta}\big(|\cdot|^{-k-l}P(\cdot)g\big)(\xi)\lesssim_k\big(\||\cdot|^{-l}g\|_{L^{\infty}}+\|\nabla g\|_{L^{\infty}}\big)\big(1+|\xi|^{-\theta}\big), \tag{3.16}$$

for all $\xi\neq 0$.

Proof. Let $l=0,1$, we write

$$\begin{aligned}\big(\mathcal{D}^{\theta}(|\cdot|^{-k-l}P(\cdot)g)\big)^2(\xi)&=\int\frac{\big||\xi|^{-k-l}P(\xi)g(\xi)-|\xi-\eta|^{-k-l}P(\xi-\eta)g(\xi-\eta)\big|^2}{|\eta|^{d+2\theta}}\,d\eta\\&=\int_{|\eta|\leqslant\min\{|\xi|/2,1\}}(\cdots)\,d\eta+\int_{|\eta|>\min\{|\xi|/2,1\}}(\cdots)\,d\eta\\&=:\mathcal{I}+\mathcal{II}.\end{aligned} \tag{3.17}$$

Given that $P(\xi)$ is a homogeneous polynomial of degree k, it is deduced

$$\begin{aligned}\mathcal{II}&\lesssim\||\cdot|^{-l}g\|_{L^{\infty}}^2\Big(\int_{|\eta|>|\xi|/2}\frac{1}{|\eta|^{d+2\theta}}\,d\eta+\int_{\min\{|\xi|/2,1\}<|\eta|\leqslant|\xi|/2}\frac{1}{|\eta|^{d+2\theta}}\,d\eta\Big)\\&\lesssim\||\cdot|^{-l}g\|_{L^{\infty}}^2\big(1+|\xi|^{-2\theta}\big).\end{aligned} \tag{3.18}$$

On the other hand, when $|\eta| \leqslant \min\{|\xi|/2, 1\}$, $|\eta - \xi| \sim |\xi|$ and so

$$
\begin{aligned}
&\Big||\xi|^{-k-l}P(\xi)g(\xi) - |\xi-\eta|^{-k-l}P(\xi-\eta)g(\xi-\eta)\Big| \\
&\quad\leqslant \Big||\xi|^{-k-l}P(\xi)\big(g(\xi)-g(\xi-\eta)\big)\Big| + \Big||\xi|^{-k-l}P(\xi) - |\xi-\eta|^{-k-l}P(\xi-\eta)\Big|\,|g(\xi-\eta)| \\
&\quad\lesssim \|\nabla g\|_{L^\infty}\,|\xi|^{-l}|\eta| + \sum_{j=0}^{k+l-1}\frac{|\eta||\xi|^{k+l-1-j}|\xi-\eta|^j|\xi|^k}{|\xi|^{k+l}|\xi-\eta|^{k+l}}|g(\xi-\eta)| \\
&\qquad + \sum_{j=0}^{k-1}\frac{|\eta||\xi|^{k-1-j}|\xi-\eta|^j}{|\xi-\eta|^{k+l}}|g(\xi-\eta)| \\
&\quad\lesssim \Big(\frac{\|\nabla g\|_{L^\infty}}{|\xi|^l} + \frac{\||\cdot|^{-l}g\|_{L^\infty}}{|\xi|}\Big)|\eta|.
\end{aligned}
\tag{3.19}
$$

Hence we get

$$
\begin{aligned}
\mathcal{I} &\lesssim \Big(\frac{\|\nabla g\|_{L^\infty}^2}{|\xi|^{2l}} + \frac{\||\cdot|^{-l}g\|_{L^\infty}^2}{|\xi|^2}\Big) \\
&\qquad\times\int_{|\eta|\leqslant\min\{|\xi|/2,1\}}\frac{1}{|\eta|^{d-2+2\theta}}\,d\eta \lesssim \big(\||\cdot|^{-l}g\|_{L^\infty}^2 + \|\nabla g\|_{L^\infty}^2\big)\big(1+|\xi|^{-2\theta}\big).
\end{aligned}
\tag{3.20}
$$

Gathering (3.18) and (3.20), we deduce (3.16). □

We are now in position to show the following result, which will be useful to deduce Theorems 3.4 and 3.5 in the three-dimensional setting.

PROPOSITION 3.12. *Let $g \in C_c^\infty(\mathbb{R}^3)$ and $P(x)$ a homogeneous polynomial of degree $k \geqslant 1$ in $\mathbb{R}^3$. Then*

$$
\Big\|\frac{P(\cdot)}{|\cdot|^k}fg\Big\|_{H^{1/2}} \lesssim_{k,g} \|f\|_{H^{(1/2)^+}}.
\tag{3.21}
$$

Furthermore, if m is an integer with $0 \leqslant m < k$,

$$
\Big\|\frac{P(\cdot)}{|\cdot|^m}fg\Big\|_{H^{1/2}} \lesssim_{k,m,g} \|f\|_{H^{1/2}}.
\tag{3.22}
$$

PROOF. Let us first prove (3.21). Consider a function $\tilde{g} \in C_c^\infty(\mathbb{R}^d)$ such that $\tilde{g}g = g$, then from (3.16) with $l = 0$, we have

$$
\begin{aligned}
\Big\|\frac{P(\cdot)}{|\cdot|^k}fg\Big\|_{H^{1/2}} &\lesssim \|fg\|_{L^2} + \|\mathcal{D}^{1/2}(|\cdot|^{-k}P(\cdot)fg)\|_{L^2} \\
&\lesssim \|fg\|_{L^2} + \|\mathcal{D}^{1/2}(|\cdot|^{-k}P(\cdot)\tilde{g})fg\|_{L^2} + \||\cdot|^{-k}P(\cdot)\tilde{g}\mathcal{D}^{1/2}(fg)\|_{L^2} \\
&\lesssim \|fg\|_{H^{1/2}} + \||\cdot|^{-1/2}fg\|_{L^2}.
\end{aligned}
\tag{3.23}
$$

Thus, the commutator relation (1.25) with $p_1 = 1$ and $p_2 = 2$ yields

$$
\|fg\|_{H^{1/2}} \lesssim \|fg\|_{L^2} + \|[D^{1/2}, g]f\|_{L^2} + \|gD^{1/2}f\|_{L^2} \lesssim_g \|f\|_{H^{1/2}}.
\tag{3.24}
$$

On the other hand, taking $0 < \epsilon < 1$, Hölder's inequality and Sobolev's embedding imply

$$
\||\cdot|^{-1/2}fg\|_{L^2} \lesssim \|f\|_{L^{3/(1-\epsilon)}}\||\cdot|^{-1/2}g\|_{L^{6/(1+2\epsilon)}} \lesssim_g \|D^{1/2+\epsilon}f\|_{L^2} \leqslant \|f\|_{H^{1/2+\epsilon}},
\tag{3.25}
$$

where we have used that $|\cdot|^{-1} \in L^{6/(1+2\epsilon)}_{loc}(\mathbb{R}^3)$. Thus incorporating the above estimates in (3.23), we get (3.21). To deduce (3.22), since $P(x)$ has degree k, there exist finite multi-indexes $\beta_1, \dots, \beta_l$ of order $k-m$ and homogeneous polynomials $P_{\beta_1}(x), \dots, P_{\beta_l}(x)$ of order m such that

$$\frac{P(x)}{|x|^m} = \sum_{j=1}^{l} \frac{P_{\beta_j}(x)}{|x|^m} x^{\beta_j}. \tag{3.26}$$

Therefore, since $k-m \geqslant 1$ and $x^{\beta_j} g$ is a smooth function with compact support for each j, arguing as in (3.23) and (3.24), we obtain

$$\begin{aligned}
\left\| \frac{P(\cdot)}{|\cdot|^m} fg \right\|_{H^{1/2}} &\lesssim \sum_{j=1}^{l} \| |x|^{-m} P_{\beta_j}(x) x^{\beta_j} fg \|_{H^{1/2}} \\
&\lesssim \sum_{j=1}^{l} \| x^{\beta_j} fg \|_{L^2} + \| |\cdot|^{-m} P_{\beta_j}(\cdot) \tilde{g} \mathcal{D}^{1/2}(x^{\beta_j} fg) \|_{L^2} + \| \mathcal{D}^{1/2}\big(|x|^{-m} P_{\beta_j}(x) \tilde{g}\big) x^{\beta_j} fg \|_{L^2} \\
&\lesssim \sum_{j=1}^{l} \| x^{\beta_j} fg \|_{H^{1/2}} + \| |x|^{-1/2} x^{\beta_j} fg \|_{L^2} \lesssim \|f\|_{H^{1/2}}.
\end{aligned}$$

The proof of the proposition is now completed. □

3.2.1. Approximation by smooth solutions. The results concerning local well-posedness for the IVP (0.2) in classical Sobolev spaces $H^s(\mathbb{R}^d)$ are fundamental in our arguments to extend the LWP result to the weighted domain. In this regard, part of the proof of Theorem 2.1 (see also [**39**, Proposition 5.10 and Lemma 5.9]) guarantees existence of solutions of (0.2) as the strong limit of smooth solutions in the class

$$C([0,T]; H^s(\mathbb{R}^d)) \cap L^1([0,T]; W^{1,\infty}(\mathbb{R}^d)),$$

whose initial data are mollified versions of u_0 in the sense of the Bona-Smith argument [**7**]. More precisely, for a given solution $u \in C\big([0,T]; H^s(\mathbb{R}^d)\big) \cap L^1\big([0,T]; W^{1,\infty}(\mathbb{R}^d)\big)$ provided by Theorem 2.1, there exists a sequence of smooth solutions of (0.2), $u_N \in C([0,T]; H^\infty(\mathbb{R}^d))$ $N \geqslant 1$, such that

$$\sup_{t\in[0,T]} \|u_N(t)\|_{H^s} \leqslant 2\,\|u_0\|_{H^s}, \tag{3.27}$$

and

$$u_N \to u \quad \text{in the sense of} \quad C\big([0,T]; H^s(\mathbb{R}^d)\big) \cap L^1\big([0,T]; W^{1,\infty}(\mathbb{R}^d)\big). \tag{3.28}$$

Therefore, (3.28) will be useful to perform rigorously weighted energy estimates at the $H^s(\mathbb{R}^d)$-level stated in Theorem 2.1, and then taking the limit $N \to \infty$ to deduce Proposition 3.1 and Theorem 3.2.

3.3. Well-posedness in $H^s(\mathbb{R}^d) \cap L^2(\omega^2\, dx)$

In this section we establish local well-posedness in the space $H^s(\mathbb{R}^d) \cap L^2(\omega^2\, dx)$, that is to say we deduce Proposition 3.1. We require the following result.

Lemma 3.13. *Let ω be a smooth weight with all its first and second derivatives bounded. Define*

$$\omega_\lambda(x) = \omega(x)e^{-\lambda|x|^2},\ x \in \mathbb{R}^d,\ \lambda \in (0,1). \tag{3.29}$$

Then, there exists a constant $c > 0$ independent of λ, such that

$$\|\partial^\alpha \omega_\lambda\|_\infty \leqslant c,$$

where α is a multi-index of order $1 \leqslant |\alpha| \leqslant 2$.

PROOF. The proof is similar to that in [18, Lemma 4.1]. ☐

Now, we proceed to deduce well-posedness in $H^s(\mathbb{R}^d) \cap L^2(\omega^2\, dx)$.

PROOF OF PROPOSITION 3.1. Given $u_0 \in H^s(\mathbb{R}^d) \cap L^2(\omega^2\, dx)$, from Theorem 2.1, there exist $T = T(\|u_0\|_{H^s}) > 0$, $u \in C([0,T]; H^s(\mathbb{R}^d))$ solution of (0.2) with initial datum u_0 and a smooth sequence of solutions $u_N \in C([0,T]; H^\infty(\mathbb{R}^d))$ with $u_N(0) \in L^2(\omega^2\, dx)$, satisfying (3.27) and (3.28). We shall prove the persistence property $u \in C([0,T]; L^2(\omega^2\, dx))$.

We first perform energy estimates for the regularized solutions $u_N \in C([0,T]; H^\infty(\mathbb{R}^d))$, $N \geqslant 1$. Let ω_λ be defined as in Lemma 3.13. Since ω_λ is bounded and u_N is smooth, we can multiply the equation (0.2) associated to u_N by $\omega_\lambda^2 u_N$ and then integrate on the spatial variable to deduce

$$\frac{d}{dt}\int (\omega_\lambda u_N)^2(t)\, dx - \int \omega_\lambda \mathcal{R}_1 \Delta u_N \omega_\lambda u_N\, dx + \int \omega_\lambda u_N \partial_{x_1} u_N \omega_\lambda u_N\, dx = 0. \tag{3.30}$$

The nonlinear term can be bounded as follows

$$\left|\int \omega_\lambda u_N \partial_{x_1} u_N \omega_\lambda u_N\, dx\right| \leqslant \|\nabla u_N\|_{L^\infty_x} \|\omega_\lambda u_N\|^2_{L^2_x}.$$

To control the factor involving the dispersion, we write

$$-\omega_\lambda \mathcal{R}_1 \Delta u_N = [\mathcal{R}_1, \omega_\lambda]\Delta u_N - \mathcal{R}_1(\omega_\lambda \Delta u_N) = [\mathcal{R}_1, \omega_\lambda]\Delta u_N - \mathcal{R}_1([\omega_\lambda, \Delta]u_N) - \mathcal{R}_1\Delta(\omega_\lambda u_N). \tag{3.31}$$

Since the Riesz transform $\mathcal{R}_1$ is an skew-symmetric operator it is seen that

$$-\int \mathcal{R}_1 \Delta(\omega_\lambda u_N)\omega_\lambda u_N\, dx = 0.$$

Thus, it remains to control the first two terms on the r.h.s of (3.31). In light of the commutator estimate (3.2), Lemma 3.9 and (3.27), we have

$$\begin{aligned}
\|[\mathcal{R}_1, \omega_\lambda]\Delta u_N\|_{L^2_x} &\lesssim \sum_{j=1}^d \|[\mathcal{R}_1, \omega_\lambda]\partial^2_{x_j} u_N\|_{L^2_x} \lesssim \sum_{|\beta|=2} \|\partial^\beta \omega_\lambda\|_{L^\infty} \|u_N\|_{L^2_x} + \sum_{j=1}^d \sum_{|\beta|=1} \|\partial^\beta \omega_\lambda D^\beta_{R_1} \partial^2_{x_j} u_N\|_{L^2_x} \\
&\lesssim \|u_N\|_{L^2_x} + \sum_{j=1}^d \sum_{|\beta|=1} \|\partial^\beta \omega_\lambda\|_{L^\infty} \|D^\beta_{R_1} \partial^2_{x_j} u_N\|_{L^2_x} \\
&\lesssim \|u_N\|_{L^\infty_T H^s} \lesssim \|u_0\|_{H^s},
\end{aligned}$$

where the implicit constant on the r.h.s of the above inequality is independent of λ by virtue of Lemma 3.13. On the other hand, the identity

$$[\omega_\lambda, \Delta]\, u_N = (\Delta \omega_\lambda)u_N - 2\nabla \omega_\lambda \cdot \nabla u_N$$

and (3.27) yield

$$\|\mathcal{R}_1([\omega_\lambda,\Delta]u_N)\|_{L^2_x} \lesssim \|\Delta\omega_\lambda\|_{L^\infty}\|u_N\|_{L^2_x} + \|\nabla\omega_\lambda\|_{L^\infty}\|\nabla u_N\|_{L^2_x}$$
$$\lesssim (\|\Delta\omega_\lambda\|_{L^\infty} + \|\nabla\omega_\lambda\|_{L^\infty})\|u_0\|_{H^s}.$$

Gathering all these estimates, there exist constants c_0 and c_1 (depending on the L^∞-norm of the weight w and its derivatives, and independent of λ) such that

$$\frac{d}{dt}\|\omega_\lambda u_N(t)\|_{L^2}^2 \leqslant c_0\|u_0\|_{H^s}\|\omega_\lambda u_N(t)\|_{L^2} + c_1\|\nabla u_N\|_{L^\infty}\|\omega_\lambda u_N(t)\|_{L^2}^2.$$

Consequently, in view of Gronwall's inequality we arrive at

$$\|\omega_\lambda u_N(t)\|_{L^2} \leqslant (\|\omega_\lambda u_N(0)\|_{L^2} + c_0\|u_0\|_{H^s}\,t)e^{c_1\int_0^t\|\nabla u_N(s)\|_{L^\infty}\,ds}. \tag{3.32}$$

From (3.28) and the fact that ω_λ is bounded, one can take the limit $N\to\infty$ in (3.32) to find

$$\|\omega_\lambda u(t)\|_{L^2} \leqslant (\|\omega_\lambda u_0\|_{L^2} + c_0\|u_0\|_{H^s}\,t)e^{c_1\int_0^t\|\nabla u(s)\|_{L^\infty}\,ds}$$
$$\leqslant (\|\omega u_0\|_{L^2} + c_0\|u_0\|_{H^s}\,t)e^{c_1\int_0^t\|\nabla u(s)\|_{L^\infty}\,ds}.$$

The above inequality and Fatou's lemma yield

$$\|\omega u(t)\|_{L^2} \leqslant (\|\omega u_0\|_{L^2} + c_0\|u_0\|_{H^s}\,t)e^{c_1\int_0^t\|\nabla u(s)\|_{L^\infty}\,ds}, \quad 0\leqslant t\leqslant T. \tag{3.33}$$

This shows that $u\in L^\infty([0,T];L^2(\omega^2\,dx))$. Let us prove that $u\in C([0,T];L^2(\omega^2\,dx))$. Firstly, we claim that $u:[0,T]\mapsto L^2(\omega^2\,dxdy)$ is weakly continuous. Indeed, for a given $g\in S(\mathbb{R}^d)$,

$$\begin{aligned}\left|\int\omega(u(s)-u(t))\omega g\,dx\right| &\leqslant \left|\int\omega(u(s)-u(t))(\omega-\omega_\lambda)g\,dx\right| + \left|\int\omega(u(s)-u(t))\omega_\lambda g\,dx\right|\\ &\lesssim \sup_{t\in[0,T]}\|wu(t)\|_{L^2}\|(\omega-\omega_\lambda)g\|_{L^2} + \|u(s)-u(t)\|_{L^2}\|\,\omega\omega_\lambda g\|_{L^2}.\end{aligned} \tag{3.34}$$

Therefore, since $\|\omega\omega_\lambda g\|_{L^2}\leqslant\|\omega^2 g\|_{L^2}<\infty$, by using that $g(\omega-\omega_\lambda)\to 0$ as $\lambda\to 0$ in $L^2(\mathbb{R}^d)$ (due to Lebesgue dominated convergence theorem), (3.33) and the fact that $u\in C([0,T];H^s(\mathbb{R}^d))$ for $s>0$, we can take $\lambda\to 0$ in (3.34) to deduce weak continuity.

On the other hand, the estimate (3.33) yields

$$\begin{aligned}\|\omega(u(t)-u_0)\|_{L^2}^2 &= \|\omega u(t)\|_{L^2}^2 + \|\omega u_0\|_{L^2}^2 - 2\int\omega u(t)\omega u_0\,dx\\ &\leqslant (\|\omega u_0\|_{L^2} + c_0 t)^2 e^{2c_1\int_0^t\|\nabla u(s)\|_{L^\infty}\,ds} + \|\omega u_0\|_{L^2}^2 - 2\int\omega u(t)\omega u_0\,dx.\end{aligned} \tag{3.35}$$

Clearly, weak continuity implies that the right-hand side of (3.35) goes to zero as $t\to 0^+$. This shows right continuity at the origin of the map $u:[0,T]\mapsto L^2(\omega^2\,dxdy)$. Fixing $\tau\in(0,T)$ and using that (0.2) is invariant under the transformations, $(x,t)\mapsto(x,t+\tau)$ and $(x,t)\mapsto(-x,\tau-t)$, right continuity at the origin entails continuity in all the interval $[0,T]$, in other words $u\in C([0,T];H^s(\mathbb{R}^d)\cap L^2(\omega^2\,dx))$.

The continuous dependence on the initial data can be deduced from its equivalent in $H^s(\mathbb{R}^d)$ and employing the above arguments. The proof of Proposition 3.1 is now completed.

□

3.4. Well-posedness in $Z_{s,r}$ and $\dot{Z}_{s,r}$

In this section, we prove Theorem 3.2. When the decay parameter $r \in [0,1]$, the weight $\langle x\rangle^r$ satisfies the hypothesis of Proposition 3.1. Thereby, we may assume that $1 < r \leqslant s$.

Let $u \in C([0,T]; H^s(\mathbb{R}^d))$ be a solution of (0.2) with initial datum $u_0 \in Z_{s,r}(\mathbb{R}^d)$ provided by Theorem 2.1. We shall prove that $u \in L^\infty([0,T]; L^2(|x|^{2r}\,dx))$. Once we have established this conclusion, the fact that $u \in C([0,T]; L^2(|x|^{2r}\,dx))$ and the continuous dependence on the initial data follows by the same reasoning in the proof of Proposition 3.1.

We begin by giving a brief sketch of the proof. Let m be a non-negative integer, $0 \leqslant \theta \leqslant 1$ and write $r = m+1+\theta$. Consider $k = 1,2\dots,d$, multiplying (0.2) by $w_n^{2+2\theta}x_k^{2m}u$ (where w_n is given by (1.21)) and integrating in $\mathbb{R}^d$ we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int \big(w_n^{1+\theta}x_k^m u\big)^2(t)\,dx - \int w_n^{1+\theta}x_k^m\mathcal{R}_1\Delta u w_n^{1+\theta}x_k^m u\,dx \\ + \int w_n^{1+\theta}x_k^m u\partial_{x_1}u w_n^{1+\theta}x_k^m u\,dx = 0.\end{aligned} \tag{3.36}$$

Arguing recursively on the size of the parameter $r = m+1+\theta$, starting with $m = 0$, we will deduce from previous cases (decay $r \leqslant (m-1)+1+\theta$), that $u \in L^\infty([0,T]; Z_{s,r-1}(\mathbb{R}^d))$ and satisfies

$$\sup_{t\in[0,T]}\big(\|\langle x\rangle^{r-1}u(t)\|_{L^2} + \sum_{1\leqslant|\beta|\leqslant m}\|\langle x\rangle^{r-|\beta|}\partial^\beta u(t)\|_{L^2}\big) \leqslant C_1 \tag{3.37}$$

where C_1[1] depends on T, $\|u_0\|_{H^s}$, $\|\langle x\rangle^{r-1}u_0\|_{L^2}$ and $\int_0^T \|u(\tau)\|_{W^{1,\infty}(\mathbb{R}^d)}\,d\tau$. With the aim of (3.37), we proceed to estimate the last two term on the left-hand side of (3.36) to obtain a differential inequality, which after adding for $k = 1,\dots,d$ has the form

$$\frac{d}{dt}\big(\sum_{k=1}^d \|w_n^{1+\theta}x_k^m u\|_{L^2}^2\big) \leqslant K_1\big(\sum_{k=1}^d \|w_n^{1+\theta}x_k^m u\|_{L^2}^2\big)^{1/2} + K_2\big(\sum_{k=1}^d \|w_n^{1+\theta}x_k^m u\|_{L^2}^2\big) \tag{3.38}$$

for some positive constants K_1 and K_2. Then Gronwall's lemma shows

$$\sum_{k=1}^d \|w_n^{1+\theta}x_k^m u(t)\|_{L^2} \leqslant C_2$$

and so letting $n \to \infty$, one gets

$$\sup_{r\in[0,T]}\|\langle x\rangle^r u(t)\|_{L^2} \lesssim C_2, \tag{3.39}$$

where C_2 is independent of n, depends on T, $\|u_0\|_{H^s}$, $\|\langle x\rangle^r u_0\|_{L^2}$ and $\|u\|_{L^1_T W^{1,\infty}(\mathbb{R}^d)}$.

Therefore, we continue in this fashion, increasing $r = m+1+\theta$ and deducing (3.37) in each step to conclude the proof of Theorem 3.2 (i). This same procedure also provides a method to deduce Theorem 3.2 (ii). However, in this case the estimates for the integral equation (3.36) require of additional weighted bounds for derivatives of negative order, which will be deduced from the hypothesis $\widehat{u}(0) = 0$. This discussion encloses the scheme of the proof for Theorem 3.2.

[1]Since we rely on Gronwall's lemma to attain our estimates, one may expect that C_1 depends on $\left\|\langle x\rangle^{r-|\beta|}\partial^\beta u_0\right\|_{L^2}$ for each multi-index $1 \leqslant |\beta| \leqslant m$. However, the interpolation inequality (1.28) shows that these expressions are bounded by $\|u_0\|_{H^s}$ and $\|\langle x\rangle^{r-1}u_0\|_{L^2}$.

Next, we state the main considerations to get (3.37). As above, let $r = m + 1 + \theta$ with $m \geqslant 1$, consider a fixed integer $1 \leqslant l \leqslant m$ and a multi-index γ of order l. We use the (0.2) equation to obtain new equations

$$\partial_t(\partial^\gamma u) - \mathcal{R}_1 \Delta \partial^\gamma u + \partial^\gamma(u u_{x_1}) = 0. \tag{3.40}$$

After multiply (3.40) by $w_N^{2+2\theta} x_k^{2m-2|\gamma|} \partial^\gamma u$ and integrate over $\mathbb{R}^d$, it is deduced

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int \left(w_n^{1+\theta} x_k^{m-|\gamma|}\partial^\gamma u\right)^2(t)\,dx - \int w_n^{1+\theta} x_k^{m-|\gamma|} \mathcal{R}_1 \Delta \partial^\gamma u w_n^{1+\theta} x_k^{m-|\gamma|} \partial^\gamma u\,dx \\ + \int w_n^{1+\theta} x_k^{m-|\gamma|} \partial^\gamma (u\partial_{x_1} u) w_n^{1+\theta} x_k^{m-|\gamma|} \partial^\gamma u\,dx = 0. \end{aligned} \tag{3.41}$$

Estimating the above equivalences for all $k = 1, \dots, d$ and each multi-index γ with $|\gamma| = l$, we will deduce a closed differential inequality similar to (3.38), which yields $L^2(\langle x\rangle^{2r-2l}\,dx)$ bounds for all derivative of order $|\gamma| = l$. Then, adding for $l = 1, \dots, m$, (3.37) follows.

A first step to study (3.36) and (3.41) is to reduce our arguments to bound the dispersive terms corresponding to the second factors on the left-hand sides of these equations. Indeed, we first consider a fixed decay parameter $r = m + 1 + \theta$ for some nonnegative integer m and $\theta \in [0, 1]$. Then, the nonlinear part of (3.36) can be controlled as follows

$$\left|\int w_n^{1+\theta} x_k^m u \partial_{x_1} u w_n^{1+\theta} x_k^m u\,dx\right| \leqslant \|\nabla u\|_{L^\infty_x} \|w_n^{1+\theta} x_k^m u\|_{L^2_x}^2.$$

Since our local theory in $H^s(\mathbb{R}^d)$ assures that $u \in L^1((0,T); W^{1,\infty}(\mathbb{R}^d))$, the above expression leads to an appropriated bound after Gronwall's Lemma. Now, we proceed to bound the nonlinearity in (3.41). Here, $m \geqslant 1$ and we shall assume from previous steps that

$$\sup_{r\in[0,T]} \Big(\|\langle x\rangle^{r-2} u(t)\|_{L^2} + \sum_{1\leqslant|\beta|\leqslant m-1} \|\langle x\rangle^{r-1-|\beta|}\partial^\beta u(t)\|_{L^2}\Big) \leqslant C_3, \tag{3.42}$$

where the constant C_3 has the same dependence of C_1 in (3.37), after changing r by $r - 1$. We write

$$\begin{aligned} &\int w_n^{1+\theta} x_k^{m-|\gamma|} \partial^\gamma (u\partial_{x_1} u) w_n^{1+\theta} x_k^{m-|\gamma|} \partial^\gamma u\,dx \\ &\quad= \sum_{\gamma_1+\gamma_2=\gamma} c_{\gamma_1,\gamma_2} \int w_n^{1+\theta} x_k^{m-|\gamma|} \partial^{\gamma_1} u \partial^{\gamma_2}\partial_{x_1} u w_n^{1+\theta} x_k^{m-|\gamma|}\partial^\gamma u\,dx \\ &\quad= \sum_{\substack{\gamma_1+\gamma_2=\gamma \\ |\gamma_1|=0 \text{ or } |\gamma_1|=|\gamma|}} (\cdots) + \sum_{\substack{\gamma_1+\gamma_2=\gamma \\ |\gamma_1|=1}} (\cdots) + \sum_{\substack{\gamma_1+\gamma_2=\gamma \\ 2\leqslant|\gamma_1|\leqslant|\gamma|-1}} (\cdots) \\ &\quad=: B_1 + B_2 + B_3. \end{aligned} \tag{3.43}$$

We proceed to estimate the terms B_j, $j = 1, 2, 3$. Formally integrating by parts in the x_1 variable gives

$$B_1 = \frac{1}{2}\int w_n^{1+\theta} x_k^{m-|\gamma|}\partial^\gamma u \partial_{x_1} u w_n^{1+\theta} x_k^{m-|\gamma|} \partial^\gamma u\,dx - \int \partial_{x_1}\left(w_n^{1+\theta} x_k^{m-|\gamma|}\right) u \partial^\gamma u w_n^{1+\theta} x_k^{m-|\gamma|}\partial^\gamma u\,dx.$$

Then, when $|\gamma| = m$, using that $|\nabla w_n^{1+\theta}| \lesssim |w_n^\theta|$ with a constant independent of n, we find

$$|B_1| \lesssim \left(\|u\|_{L^\infty_x} + \|\nabla u\|_{L^\infty_x}\right) \|w_n^{1+\theta}\partial^\gamma u\|_{L^2_x}^2,$$

which is controlled by the local theory after Gronwall's lemma. Now, when $1 \leqslant |\gamma| < m$, the inequality (1.22) reveals that

$$|\partial_{x_1}(w_n^{1+\theta}x_k^{m-|\gamma|})| \lesssim \langle x\rangle^{m+1+\theta-1-|\gamma|},$$

with implicit constant independent of n, and so

$$|B_1| \lesssim \|u\|_{L^\infty_x} \|\langle x\rangle^{r-1-|\gamma|}\partial^\gamma u\|_{L^2_x}\|w_n^{1+\theta}x_k^{m-|\gamma|}\partial^\gamma u\|_{L^2_x} + \|\nabla u\|_{L^\infty_x}\|w_n^{1+\theta}x_k^{m-|\gamma|}\partial^\gamma u\|^2_{L^2_x}.$$

Since $1 \leqslant |\gamma| < m$, our assumption (3.42) shows that the above expression is controlled. This completes the estimate for B_1. Now, we consider B_2, in this case $|\gamma_1| = 1$, then $\partial^{\gamma_2}\partial_{x_1}$ has order $|\gamma|$ and so

$$|B_2| \lesssim \|\nabla u\|_{L^\infty_x}\sum_{|\beta|=|\gamma|}\|w_n^{1+\theta}x_k^{m-|\gamma|}\partial^\beta u\|_{L^2_x}\|w_n^{1+\theta}x_k^{m-|\gamma|}\partial^\gamma u\|_{L^2_x}.$$

Notice that the previous estimate is part of the differential inequality collected after adding (3.41) for all multi-index of fixed order $|\gamma|$. To control the last term, we use that

$$w_n^{1+\theta}|x_k|^{m-|\gamma|} \lesssim \langle x\rangle^{1+\theta+m-1-(|\gamma_2|+1)},$$

whenever $\gamma = \gamma_1 + \gamma_2$ and $2 \leqslant |\gamma_1|$. Then Sobolev's embedding gives,

$$\begin{aligned} |B_3| &\lesssim \sum_{\substack{\gamma_1+\gamma_2=\gamma\\ 2\leqslant|\gamma_1|\leqslant|\gamma|-1}} \|\partial^{\gamma_1}u\|_{L^\infty_x}\|w_n^{1+\theta}x_k^{m-|\gamma|}\partial^{\gamma_2}\partial_{x_1}u\|_{L^2_x}\|w_n^{1+\theta}x_k^{m-|\gamma|}\partial^\gamma u\|_{L^2_x} \\ &\lesssim \sum_{\substack{\gamma_1+\gamma_2=\gamma\\ 2\leqslant|\gamma_1|\leqslant|\gamma|-1}} \|J^{d/2+|\gamma_1|+\epsilon}u\|_{L^2_x}\|\langle x\rangle^{r-1-(|\gamma_2|+1)}\partial^{\gamma_2}\partial_{x_1}u\|_{L^2_x}\|w_n^{1+\theta}x_k^{m-|\gamma|}\partial^\gamma u\|_{L^2_x}, \end{aligned} \tag{3.44}$$

for any $\epsilon > 0$. Since $|\gamma_1| \leqslant m-1$, taking $0 < \epsilon < m+1+\theta-|\gamma_1|-d/2$ and recalling that the regularity $s \geqslant r = m+1+\theta$, we get

$$\|J^{d/2+|\gamma_1|+\epsilon}u\|_{L^2_x} \lesssim \|u\|_{H^s},$$

for all $|\gamma_1| \leqslant m-1$. Plugging this information in (3.44) and using (3.42), we get a controlled estimate for B_3. This completes the study of the non-linear term (3.43).

Thus matters are reduced to control the second term on the left-hand sides of (3.36) and (3.41). Since the estimate for the latter can be obtained from the former by changing the roles of u by $\partial^\gamma u$, we will mainly focus on the l.h.s of (3.36). Whence we write

$$\begin{aligned} &w_n^{1+\theta}x_k^m\mathcal{R}_1\Delta u \\ &\quad= w_n^{1+\theta}\mathcal{R}_1(x_k^m\Delta u) + w_n^{1+\theta}\left[x_k^m,\mathcal{R}_1\right]\Delta u \\ &\quad= w_n^{1+\theta}\mathcal{R}_1\Delta(x_k^m u) + w_n^{1+\theta}\mathcal{R}_1([x_k^m,\Delta]u) + w_n^{1+\theta}\left[x_k^m,\mathcal{R}_1\right]\Delta u \\ &\quad= \mathcal{R}_1(w_n^{1+\theta}\Delta(x_k^m u)) + [w_n^{1+\theta},\mathcal{R}_1]\Delta(x_k^m u) + w_n^{1+\theta}\mathcal{R}_1([x_k^m,\Delta]u) + w_n^{1+\theta}\left[x_k^m,\mathcal{R}_1\right]\Delta u \\ &\quad= \mathcal{R}_1\Delta(w_n^{1+\theta}x_k^m u) + \mathcal{R}_1([w_n^{1+\theta},\Delta](x_k^m u)) + [w_n^{1+\theta},\mathcal{R}_1]\Delta(x_k^m u) + w_n^{1+\theta}\mathcal{R}_1([x_k^m,\Delta]u) \\ &\qquad + w_n^{1+\theta}\left[x_k^m,\mathcal{R}_1\right]\Delta u \\ &\quad=: \mathcal{R}_1\Delta(w_n^{1+\theta}x_k^m u) + Q_1 + Q_2 + Q_3 + Q_4. \end{aligned} \tag{3.45}$$

To simplify our arguments, the same notation Q_j will be implemented for different parameters r previously fixed. Inserting $\mathcal{R}_1\Delta(w_n^{1+\theta}x_k^m u)$ in (3.36), one finds that its contribution is null since the Riesz transform defines a skew-symmetric operator. Accordingly, it remains to bound the Q_j-terms to deduce Theorem 3.2.

3.4.1. LWP in $Z_{s,r}(\mathbb{R}^d)$ for $r \in [0,3)$ if $d = 2$, and $r \in [0,3]$ when $d = 3$. We divide the proof into two main cases.

Case 1: $r \in [0,2]$. As discussed, when $r \in [0,1]$, LWP is a consequence of Theorem 3.1. Suppose that $r \geqslant 1$, so our conclusion is obtained from (3.36) with $m = 0$, $r = 1+\theta \in [1,2]$ with $0 \leqslant \theta \leqslant 1$. Notice that we do not require to deduce weighted estimates for derivatives. Besides, $Q_3 = Q_4 = 0$ in (3.45), which reduce our arguments to handle the terms Q_1 and Q_2.
We write

$$Q_1 = -\mathcal{R}_1(\Delta(w_n^{1+\theta})u + 2\nabla w_n^{1+\theta}\cdot\nabla u).$$

Then, the properties of the weight w_n in (1.22) lead to the following estimate

$$\|Q_1\|_{L^2_x} \lesssim \|w_n^{\theta}\nabla u\|_{L^2_x} \lesssim \|\nabla(w_n^{\theta}u)\|_{L^2_x} + \|\nabla w_n^{\theta} u\|_{L^2_x} \lesssim \|\nabla(w_n^{\theta}u)\|_{L^2_x} + \|u\|_{L^2_x}. \tag{3.46}$$

The interpolation inequality (1.28) shows

$$\|\nabla(w_n^{\theta}u)\|_{L^2_x} \lesssim \|J^1(w_n^{\theta}u)\|_{L^2_x} \lesssim \|w_n^{1+\theta}u\|_{L^2_x}^{\theta/(1+\theta)}\|J^{1+\theta}u\|_{L^2_x}^{1/(1+\theta)}. \tag{3.47}$$

Note that this imposes the condition $r = 1+\theta \leqslant s$. Applying in (3.47) Young's inequality and going back to (3.46), we bound Q_1. To estimate Q_2, we apply Proposition 3.8 to find

$$\|Q_2\|_{L^2_x} \leqslant \sum_{j=1}^{d}\|[w_n^{1+\theta},\mathcal{R}_1]\partial_{x_j}^2 u\|_{L^2_x} \lesssim \sum_{j=1}^{d}\sum_{|\beta|=2}\|\partial^{\beta}w_n^{1+\theta}\|_{L^\infty}\|u\|_{L^2_x} + \sum_{|\beta|=1}\|\partial^{\beta}w_n^{1+\theta}D_{R_1}^{\beta}\partial_{x_j}^2 u\|_{L^2_x}. \tag{3.48}$$

The second term on the r.h.s can be bounded by combining Proposition 1.11, (3.10) and (1.22) to obtain

$$\|\partial^{\beta}w_n^{1+\theta}D_{R_1}^{e_k}\partial_{x_j}^2 u\|_{L^2_x} \lesssim \delta_{1,k}\|w_n^{\theta}\mathcal{R}_j\partial_{x_j}u\|_{L^2_x} + \|w_n^{\theta}\mathcal{R}_1\mathcal{R}_k\mathcal{R}_j\partial_{x_j}u\|_{L^2_x} \lesssim \|w_n^{\theta}\partial_{x_j}u\|_{L^2_x}, \tag{3.49}$$

$0 < 2\theta < 2 \leqslant d$, which is controlled as in (3.47). Notice that the above argument fails when $\theta = 1$ in dimension $d = 2$ (since w_n^2 does not satisfies the $A_2(\mathbb{R}^2)$ condition), instead letting $\beta = e_l$, we use the identity (3.10) to write

$$\begin{aligned} w_n D_{R_1}^{e_l}\partial_{x_j}^2 u =&\delta_{1,l}w_n\mathcal{R}_j\partial_{x_j}u + w_n\mathcal{R}_1\mathcal{R}_l\mathcal{R}_j\partial_{x_j}u\\ =&\delta_{1,l}[w_n,\mathcal{R}_j]\partial_{x_j}u + \delta_{1,l}\mathcal{R}_j(w_n\partial_{x_j}u) + [w_n,\mathcal{R}_1]\partial_{x_j}\mathcal{R}_l\mathcal{R}_j u + \mathcal{R}_1([w_n,\mathcal{R}_l]\partial_{x_j}\mathcal{R}_j u)\\ &+\mathcal{R}_1\mathcal{R}_l([w_n,\mathcal{R}_j]\partial_{x_j}u) + \mathcal{R}_1\mathcal{R}_l\mathcal{R}_j(w_n\partial_{x_j}u). \end{aligned} \tag{3.50}$$

Hence, the decomposition (3.50) allows us to apply Proposition 3.8 with one derivative to get

$$\sum_{|\beta|=1}\|\partial^{\beta}w_N^2 D_{R_1}^{\beta}\partial_{x_j}^2 u\|_{L^2_x} \lesssim \sum_{l=1}^{2}\|w_n D_{R_1}^{e_l}\partial_{x_j}^2 u\|_{L^2_x} \lesssim \|u\|_{L^2_x} + \|w_n\partial_{x_j}u\|_{L^2_x}. \tag{3.51}$$

It is worth to notice that the above argument also establishes the bound (3.49) without the aim of Proposition 1.11. In this manner, the right-hand side of (3.49) and (3.51) can be estimated as in (3.47). Putting together these results in (3.48), we bound Q_2 by Gronwall's terms. Finally, inserting the above information in (3.36) with $m = 0$ yield the desire conclusion.

Case 2: $r \in (2,3)$ if $d = 2$ and $r \in (2,3]$ when $d = 3$. By setting $m = 1$ and $r = 2+\theta$, with $0 < \theta < 1$ if $d = 2$ and including $\theta = 1$ if $d = 3$, our conclusions are obtained from (3.36). We first claim that

$$\sup_{t\in[0,T]} \|\langle x\rangle^{r-1}\nabla u(t)\|_{L^2} \leqslant M, \tag{3.52}$$

with M depending on $\|u_0\|_{H^s}$, $\|\langle x\rangle^r u_0\|_{L^2}$ and T. This estimate is derived from (3.41) with $m = 1$ and γ of order 1. Hence, (3.52) is established by reapplying the same arguments in the previous case, substituting u by $\partial_{x_l} u$, $l = 1, \dots, d$ in each estimate. Notice that in this case, (3.47) is given by

$$\|J^1(w_n^\theta \partial_{x_l} u)\|_{L^2_x} \lesssim \|w_n^{1+\theta}\partial_{x_l} u\|_{L^2_x}^{\theta/(1+\theta)} \|J^{1+\theta}\partial_{x_l} u\|_{L^2_x}^{1/(1+\theta)} \lesssim \|w_n^{1+\theta}\partial_{x_l} u\|_{L^2_x}^{\theta/(1+\theta)} \|J^{2+\theta} u\|_{L^2_x}^{1/(1+\theta)},$$

which leads to a controlled expression after Young's inequality, since $\|w_n^{1+\theta}\partial_{x_l} u\|_{L^2_x}$ is part of the Gronwall's term to be estimated and $2+\theta \leqslant s$. It remains to study the factors Q_j in (3.45). To treat Q_1, we write

$$\begin{aligned}[w_n^{1+\theta}, \Delta](x_k u) &= -\Delta(w_n^{1+\theta})x_k u - 2\nabla(w_n^{1+\theta})\cdot\nabla(x_k u)\\ &= -\Delta(w_n^{1+\theta})x_k u - 2\partial_{x_k}(w_n^{1+\theta})u - 2x_k\nabla(w_n^{1+\theta})\cdot\nabla u.\end{aligned} \tag{3.53}$$

This expression and (1.22) imply

$$\|Q_1\|_{L^2_x} \lesssim \|\langle x\rangle u\|_{L^2_x} + \|w_n^{1+\theta}\nabla u\|_{L^2_x} \lesssim \|\langle x\rangle u\|_{L^2_x} + \|\langle x\rangle^{1+\theta}\nabla u\|_{L^2_x}. \tag{3.54}$$

Notice that $\|\langle x\rangle u\|_{L^2_x}$ is bounded by the preceding case and $\|\langle x\rangle^{1+\theta}\nabla u\|_{L^2_x}$ by (3.52). To deal with Q_2, we gather Proposition 3.8, Lemma 3.9 and (1.22) to find

$$\begin{aligned}\|Q_2\|_{L^2_x} &\lesssim \|x_k u\|_{L^2_x} + \sum_{|\beta|=1}\|\partial^\beta w_n^{1+\theta} D^\beta_{R_1}\Delta(x_k u)\|_{L^2_x} \lesssim \|x_k u\|_{L^2_x} + \|w_n^\theta D^\beta_{R_1}\Delta(x_k u)\|_{L^2_x}\\ &\lesssim \|\langle x\rangle u\|_{L^2_x} + \|w_n^\theta x_k \nabla u\|_{L^2_x},\end{aligned} \tag{3.55}$$

which is controlled due to (3.52). To estimate Q_3 we employ the following point-wise inequality

$$\begin{aligned}|Q_3| = |-2w_n^{1+\theta}\mathcal{R}_1\partial_{x_k} u| &\lesssim |w_n^\theta \mathcal{R}_1\partial_{x_k} u| + \sum_{l=1}^d |w_n^\theta x_l \mathcal{R}_1 \partial_{x_k} u|\\ &\lesssim |w_n^\theta \mathcal{R}_1\partial_{x_k} u| + \sum_{l=1}^d |w_n^\theta [x_l, \mathcal{R}_1]\partial_{x_k} u| + \sum_{l=1}^d |w_n^\theta \mathcal{R}_1(x_l\partial_{x_k} u)|,\end{aligned} \tag{3.56}$$

which hold since $w_n^{1+\theta} \lesssim w_n^\theta + |x| w_n^\theta$. Thus, recalling (3.14) to handle the second term on the r.h.s of (3.56) and using Proposition 1.11 with $0 \leqslant \theta < 1$ when $d = 2$, and with $0 \leqslant \theta \leqslant 1$ when $d = 3$, it is deduced that

$$\|Q_3\|_{L^2_x} \lesssim \|w_n^\theta \partial_{x_k} u\|_{L^2_x} + \|w_n^\theta u\|_{L^2_x} + \|w_n^\theta |x| \partial_{x_k} u\|_{L^2_x} \lesssim \|\langle x\rangle^\theta u\|_{L^2_x} + \|\langle x\rangle^{1+\theta}\nabla u\|_{L^2_x}$$

which is controlled by previous cases and (3.52). This complete the estimate for Q_3.

Next, we use the identity (3.14) to write Q_4 as

$$Q_4 = w_n^{1+\theta}[x_k, \mathcal{R}_1]\Delta u = -w_n^{1+\theta} D^{e_k}_{R_1}\Delta u.$$

Using again the inequality $w_n^{1+\theta} \lesssim w_n^{\theta} + |x|w_n^{\theta}$, we find

$$\|Q_4\|_{L^2_x} \lesssim \|w_n^{\theta} D^{e_k}_{R_1}\Delta u\|_{L^2_x} + \sum_{l=1}^{d} \|w_n^{\theta} x_l D^{e_k}_{R_1}\Delta u\|_{L^2_x}.$$

It is not difficult to see that for all $j = 1, \dots, d$

$$\begin{aligned} x_l D^{e_k}_{R_1}\partial^2_{x_j} u &= -D^{e_l+e_k}_{R_1}\partial^2_{x_j} u + D^{e_k}_{R_1}(x_l\partial^2_{x_j} u) \\ &= -D^{e_l+e_k}_{R_1}\partial^2_{x_j} u + D^{e_k}_{R_1}\partial_{x_j}(x_l\partial_{x_j} u) - \delta_{j,l} D^{e_k}_{R_1}\partial_{x_j} u. \end{aligned}$$

Thus, combining the above decomposition, Lemma 3.9 and Proposition 1.11 with $0 \leqslant \theta < 1$ if $d = 2$ or $0 \leqslant \theta \leqslant 1$ when $d = 3$, we obtain

$$\|Q_4\|_{L^2_x} \lesssim \|w_n^{\theta} u\|_{L^2_x} + \|w_n^{\theta}\nabla u\|_{L^2_x} + \sum_{l=1}^{d} \|w_n^{\theta} x_l \nabla u\|_{L^2_x}. \tag{3.57}$$

The above expression is controlled by previous cases and (3.52). This concludes the estimates for the factors Q_j.

Finally, gathering the above information in (3.36) with $m = 1$ and recalling our previous discussions, we have deduced Theorem 3.2 (i) when $d = 2$. In addition, when $d = 3$, we have shown that $u \in C([0, T]; Z_{r,s}(\mathbb{R}^3))$ with $r \in [0, 3]$, $s \geqslant r$.

3.4.2. LWP in $Z_{s,r}(\mathbb{R}^3)$, $r \in (3, 7/2)$. In this part we complete the proof of Theorem 3.2 (i) for $d = 3$. To obtain our estimates, we consider the differential equation (3.36) with $m = 2$, $0 \leqslant \theta < 1/2$, $r = 3 + \theta$ and $r \leqslant s$.

We start by deducing weighted estimates for derivatives of u. Considering (3.41) with $m = 2$ and γ of order 2, we can reapply the argument when the decay parameter r lies in the interval $(1, 2]$ to deduce

$$\sup_{t\in[0,T]} \sum_{|\beta|=2} \|\langle x\rangle^{r-2}\partial^{\beta} u(t)\|_{L^2} \leqslant M_0, \tag{3.58}$$

where M_0 depends on $\|u_0\|_{H^s}$, $\|\langle x\rangle^r u_0\|_{L^2}$ and T. Therefore, setting $m = 2$ and γ of order 1 in (3.41), the inequality (3.58) allows us to argue exactly as in the previous subsection to deduce

$$\sup_{t\in[0,T]} \|\langle x\rangle^{r-1}\nabla u(t)\|_{L^2} \leqslant M_1, \tag{3.59}$$

with M_1 depending on $\|u_0\|_{H^s}$, $\|\langle x\rangle^r u_0\|_{L^2}$ and T. Now we can proceed to estimate the terms Q_j defined by (3.45) with $m = 2$.

We can deduce a similar estimate as that of (3.53) dealing with x_k^2, then by employing (3.59), we derive a bound similar as the one in (3.54) to finally control Q_1. The estimate for Q_2 is achieved as in (3.55) employing Proposition 3.8, substituting x_k by x_k^2 and controlling the resulting factor by (3.59). The terms Q_3 and Q_4 can be controlled from the fact that $w_n^{2+2\theta}$ satisfies the hypothesis of Proposition 1.11 whenever $0 \leqslant \theta < 1/2$. Indeed, writing

$$Q_3 = -2w_n^{1+\theta}\mathcal{R}_1 u - 4w_n^{1+\theta}\mathcal{R}_1(x_k\partial_{x_k} u) \tag{3.60}$$

and employing identity (3.13) with $\beta = 2e_k$,

$$Q_4 = w_n^{1+\theta}[x_k^2, \mathcal{R}_1]\Delta u = w_n^{1+\theta} D^{2e_k}_{R_1}\Delta u - 2w_n^{1+\theta} D^{e_k}_{R_1}\Delta(x_k u) + 4w_n^{1+\theta} D^{e_k}_{R_1}\partial_{x_k} u.$$

Then Lemma 3.9 and Proposition 1.11 imply

$$\|Q_3\|_{L^2_x}+\|Q_4\|_{L^2_x}\lesssim\|w_n^{1+\theta}u\|_{L^2_x}+\|w_n^{1+\theta}x_k\nabla u\|_{L^2_x}\lesssim\|\langle x\rangle^{3/2}u\|_{L^2_x}+\|\langle x\rangle^{r-1}\nabla u\|_{L^2_x},\tag{3.61}$$

which is bounded by previous cases and (3.59). Whence inserting this bound in (3.36) yields the proof of Theorem 3.2 (i).

3.4.3. LWP in $\dot{Z}_{s,r}(\mathbb{R}^2)$, $r\in[3,4)$. Here we restrict our arguments to dimension $d=2$. Our conclusions are achieved from (3.36) by setting $m=2$, $0\leqslant\theta<1$ and so $r=3+\theta$. When the initial datum $u_0\in Z_{s,r}(\mathbb{R}^2)$, $3\leqslant r<4$ and $r\leqslant s$, we can repeat the arguments leading to (3.58) and (3.59) in dimension 3 to deduce

$$\sup_{t\in[0,T]}\sum_{1\leqslant|\gamma|\leqslant 2}\|\langle x\rangle^{r-|\gamma|}\partial^{\gamma}u(t)\|_{L^2}\leqslant M_0,\tag{3.62}$$

where M_0 depends on $\|u_0\|_{H^s}$, $\|\langle x\rangle^r u_0\|_{L^2}$ and T. On the other hand, when $\widehat{u}(0,t)=\widehat{u_0}(0)=0$ in $\mathbb{R}^2$, we claim

$$\sup_{t\in[0,T]}\|\langle x\rangle^{\theta}|\nabla|^{-1}u(t)\|_{L^2(\mathbb{R}^2)}\lesssim M_1\tag{3.63}$$

for all $0\leqslant\theta<1$ and M_1 depending on $\|u_0\|_{H^s}$, $\|\langle x\rangle^r u_0\|_{L^2}$ and T. Indeed, let $\phi\in C^{\infty}_c(\mathbb{R}^d)$ with $\phi\equiv 1$ when $|\xi|\leqslant 1$ and write

$$D^{\theta}_{\xi}\big(|\xi|^{-1}\widehat{u}(\xi)\big)=D^{\theta}_{\xi}\big(|\xi|^{-1}\widehat{u}(\xi)\phi\big)+D^{\theta}_{\xi}\big(|\xi|^{-1}\widehat{u}(\xi)(1-\phi)\big).\tag{3.64}$$

In sight of the zero mean assumption and Sobolev's embedding

$$||\xi|^{-1}\widehat{u}(\xi)|\lesssim\|\nabla\widehat{u}\|_{L^{\infty}_{\xi}}\lesssim\|\langle x\rangle^{2+\epsilon}u\|_{L^2_x}\tag{3.65}$$

for all $\epsilon>0$. Hence, from (1.26) and Lemma 3.11 one deduces

$$\begin{aligned}\|D^{\theta}_{\xi}\big(|\xi|^{-1}\widehat{u}(\xi)\big)\|_{L^2_{\xi}}&\lesssim\|D^{\theta}_{\xi}\big(|\xi|^{-1}\widehat{u}(\xi)\phi\big)\|_{L^2_{\xi}}+\||\xi|^{-1}\widehat{u}(\xi)(1-\phi)\|_{H^1_{\xi}}\\&\lesssim\||\xi|^{-1}\widehat{u}(\xi)\phi\|_{L^2_{\xi}}+\|\mathcal{D}^{\theta}_{\xi}\big(|\xi|^{-1}\widehat{u}(\xi)\phi\big)\|_{L^2_{\xi}}+\|\widehat{u}\|_{H^1_{\xi}}\|\nabla\phi\|_{L^{\infty}}\\&\lesssim\|\nabla\widehat{u}\|_{L^{\infty}_{\xi}}\|\phi\|_{L^2}+\|\mathcal{D}^{\theta}_{\xi}\big(|\xi|^{-1}\widehat{u}(\xi)\big)\phi\|_{L^2_{\xi}}+\||\xi|^{-1}\widehat{u}(\xi)\mathcal{D}^{\theta}_{\xi}\phi\|_{L^2_{\xi}}+\|\widehat{u}\|_{H^1_{\xi}}\\&\lesssim\|\nabla\widehat{u}\|_{L^{\infty}_{\xi}}\|\phi\|_{L^2}+\big(\|\nabla\widehat{u}\|_{L^{\infty}_{\xi}}+\||\xi|^{-1}\widehat{u}\|_{L^{\infty}_{\xi}}\big)\big(\||\xi|^{-\theta}\phi\|_{L^2}+\|\phi\|_{L^2}\big)+\|\widehat{u}\|_{H^1_{\xi}}.\end{aligned}\tag{3.66}$$

Consequently, the above estimate and (3.65) yield

$$\sup_{t\in[0,T]}\|\langle x\rangle^{\theta}|\nabla|^{-1}u(t)\|_{L^2}\lesssim\sup_{t\in[0,T]}\|\langle x\rangle^{2+\epsilon}u(t)\|_{L^2}.\tag{3.67}$$

Since the right-hand side of the above inequality is bounded by previous cases whenever $\epsilon<1$, the proof of (3.63) is now completed. In this manner, with the aim of (3.62) and (3.63) we proceed to estimate the terms Q_j given by (3.45) with $m=2$.

The analysis of Q_1 and Q_2 is obtained by implementing the same ideas leading to (3.54) and (3.55) respectively. To estimate Q_3, we write

$$Q_3=-2w_n^{1+\theta}\mathcal{R}_1u-4w_n^{1+\theta}\mathcal{R}_1(x_k\partial_{x_k}u)=2w_n^{1+\theta}\mathcal{R}_1u-4w_n^{1+\theta}\mathcal{R}_1(\partial_{x_k}(x_ku)),\tag{3.68}$$

then using that $w_n^{1+\theta} \lesssim w_n^{\theta} + w_n^{\theta}|x|$, it is not difficult to deduce a similar estimate to (3.56) to find

$$\|Q_3\|_{L^2_x} \lesssim \|\langle x\rangle^{\theta}|\nabla|^{-1}u\|_{L^2_x} + \|\langle x\rangle^{1+\theta}u\|_{L^2_x} + \|\langle x\rangle^{2+\theta}\nabla u\|_{L^2_x}.$$

Now, we detail which estimate requires the negative derivative in the above expression. Arguing as in (3.56) to study the first factor on the r.h.s of (3.68), we have

$$|w_n^{1+\theta}\mathcal{R}_1 u| \lesssim |w_n^{\theta}\mathcal{R}_1 u| + \sum_{l=1}^{d} |w^{\theta}[x_l, \mathcal{R}_1]u| + |w^{\theta}\mathcal{R}_1(x_l u)|.$$

Since $w^{\theta}[x_l, \mathcal{R}_1]u = -w^{\theta}D^{e_l}_{R_1}(u)$, Lemma 3.9 shows that this expression is bounded by $\|\langle x\rangle^{\theta}|\nabla|^{-1}u\|_{L^2_x}$. To study Q_4, we consider the identity

$$Q_4 = w_n^{1+\theta}\left[x_k^2, \mathcal{R}_1\right]\Delta u = w_n^{1+\theta}D^{2e_k}_{R_1}\Delta u - 2w_n^{1+\theta}D^{e_k}_{R_1}\Delta(x_k u) + 4w_n^{1+\theta}D^{e_k}_{R_1}\partial_{x_k}u. \tag{3.69}$$

Then using that $w_n^{1+\theta} \lesssim w_n^{\theta} + w_n^{\theta}|x|$, by a similar reasoning to the deduction of (3.57) we find

$$\|Q_4\|_{L^2_x} \lesssim \|\langle x\rangle^{\theta}|\nabla|^{-1}u\|_{L^2_x} + \|\langle x\rangle^{1+\theta}u\|_{L^2_x} + \|\langle x\rangle^{2+\theta}\nabla u\|_{L^2_x}. \tag{3.70}$$

Once again, it is worth pointing out which expressions require to consider negative derivatives following the ideas behind (3.57) to control Q_4. Indeed, this procedure yields the identities

$$x_l D^{2e_k}_{R_1}\Delta u = D^{2e_k+e_l}_{R_1}\Delta u + D^{2e_k}_{R_1}\Delta(x_l u) - 2D^{2e_k}_{R_1}\partial_{x_l}u \tag{3.71}$$

and

$$\begin{aligned} x_l D^{e_k}_{R_1}\partial_{x_k}u &= [x_l, D^{e_k}_{R_1}]\partial_{x_k}u + D^{e_k}_{R_1}(x_l\partial_{x_k}u) \\ &= -D^{e_l+e_k}_{R_1}\partial_{x_k}u + D^{e_k}_{R_1}\partial_{x_k}(x_l u) - \delta_{k,l}D^{e_k}_{R_1}u. \end{aligned} \tag{3.72}$$

Hence, we use Lemma 3.9 and Proposition 1.11 to get

$$\|w_n^{\theta}D^{2e_k+e_l}_{R_1}\Delta u\|_{L^2_x} + \|w_n^{\theta}D^{e_l+e_k}_{R_1}\partial_{x_k}u\|_{L^2_x} + \|D^{e_k}_{R_1}u\|_{L^2_x} \lesssim \|w_n^{\theta}|\nabla|^{-1}u\|_{L^2_x}.$$

Finally, from the previous conclusions we have completed the proof of Theorem 3.2 (ii) for the 2-dimensional case.

3.4.4. LWP in $\dot{Z}_{s,r}(\mathbb{R}^3)$, $r \in [7/2, 9/2)$. Here we assume that $r \in [7/2, 9/2)$ with $r \leqslant s$ and $\widehat{u_0}(0) = 0$. As usual, letting $r = 1 + m + \theta$, our estimates are derived from (3.36) with $m = 2$, $1/2 \leqslant \theta \leqslant 1$ when $r \in [7/2, 4]$, and setting $m = 3$, $0 \leqslant \theta < 1/2$ if $r \in (4, 9/2)$. By recurring arguments employing (3.41) and proceedings cases, starting with the derivatives of higher order and then descending to those of order 1, it is not difficult to observe

$$\sup_{t\in[0,T]} \sum_{1\leqslant|\beta|\leqslant m} \|\langle x\rangle^{r-|\beta|}\partial^{\beta}u(t)\|_{L^2} \leqslant M, \tag{3.73}$$

where M depends on $\|u_0\|_{H^s}$, $\|\langle x\rangle^r u_0\|_{L^2}$ and T. On the other hand, we claim

$$\sup_{t\in[0,T]} \|\langle x\rangle^{\tilde{\theta}}|\nabla|^{-1}u(t)\|_{L^2} \leqslant M, \tag{3.74}$$

for all $0 \leqslant \tilde{\theta} < 3/2$. As above, we let $\phi \in C_c^{\infty}(\mathbb{R}^3)$ such that $\phi(\xi) = 1$ when $|\xi| \leqslant 1$. We decompose according to

$$D^{\tilde{\theta}}_{\xi}\big(|\xi|^{-1}\widehat{u}(\xi)\big) = D^{\tilde{\theta}}_{\xi}\big(|\xi|^{-1}\widehat{u}(\xi)\phi\big) + D^{\tilde{\theta}}_{\xi}\big(|\xi|^{-1}\widehat{u}(\xi)(1-\phi)\big).$$

Given that $\tilde{\theta} \leqslant 2$, from Sobolev's embedding

$$\|D_{\xi}^{\tilde{\theta}}\big(|\xi|^{-1}\hat{u}(\xi)(1-\phi)\big)\|_{L^2_{\xi}} \lesssim \||\xi|^{-1}\hat{u}(\xi)(1-\phi)\|_{H^2_{\xi}} \lesssim \|\hat{u}\|_{H^2_{\xi}} \lesssim \|\langle x\rangle^2 u\|_{L^2_x}. \tag{3.75}$$

Consequently, it remains to estimate the L^2-norm of $D_{\xi}^{\tilde{\theta}}\big(|\xi|^{-1}\hat{u}(\xi)\phi(\xi)\big)$. The assumption $\hat{u}(0)=0$ along with Sobolev's embedding yield

$$||\xi|^{-1}\hat{u}(\xi)| \lesssim \|\nabla\hat{u}\|_{L^{\infty}_{\xi}} \lesssim \|\langle x\rangle^{5/2+\epsilon}u\|_{L^2_x}. \tag{3.76}$$

Let us suppose first that $0 \leqslant \tilde{\theta} \leqslant 1$, the above inequality then shows

$$\begin{aligned}\|D_{\xi}^{\tilde{\theta}}\big(|\xi|^{-1}\hat{u}(\xi)\phi\big)\|_{L^2_{\xi}} \lesssim \||\xi|^{-1}\hat{u}(\xi)\phi\|_{H^1_{\xi}} &\lesssim \||\xi|^{-1}\hat{u}(\xi)(\phi+\nabla\phi)\|_{L^2_{\xi}} + \big\||\xi|^{-2}\hat{u}\phi\big\|_{L^2_{\xi}} + \||\xi|^{-1}\nabla\hat{u}\phi\|_{L^2_{\xi}}\\ &\lesssim \|\nabla\hat{u}\|_{L^{\infty}_{\xi}}\big(\|\phi\|_{L^2}+\|\nabla\phi\|_{L^2}+\||\cdot|^{-1}\phi\|_{L^2}\big)\\ &\lesssim \|\langle x\rangle^{5/2+\epsilon}u\|_{L^2_x},\end{aligned}$$

where we have used that $|\xi|^{-1} \in L^2_{loc}(\mathbb{R}^3)$. This concludes (3.74) as soon as $0 \leqslant \tilde{\theta} \leqslant 1$. To deduce (3.74) when $1 < \tilde{\theta} < 3/2$, we let $0 < \theta^* < 1/2$ and equivalently we shall bound the L^2-norm of the expression

$$\begin{aligned}D_{\xi}^{\theta^*}\partial_{\xi_l}\big(|\xi|^{-1}\hat{u}(\xi)\phi\big) =& - D_{\xi}^{\theta^*}\big(|\xi|^{-3}\xi_l\hat{u}(\xi)\phi\big) - iD_{\xi}^{\theta^*}\big(|\xi|^{-1}\widehat{x_l u}(\xi)\phi\big)\\ &+ D_{\xi}^{\theta^*}\big(|\xi|^{-1}\hat{u}(\xi)\partial_{\xi_l}\phi\big),\end{aligned} \tag{3.77}$$

for all $l=1,2,3$. Since $\partial_{\xi_l}\phi$ is supported outside of the origin, the last term on the r.h.s of (3.77) is bounded as in (3.75). To control the remaining parts we require a preliminary result.

Lemma 3.14. *Let $\phi,\psi \in C_c^{\infty}(\mathbb{R}^3)$ and $0 < \theta^* < \frac{1}{2}$ fixed, then*

$$\|\phi\mathcal{D}^{\theta^*}(|\cdot|^{-1}\psi)\|_{L^2} \lesssim_{\theta^*,\phi,\psi} 1 \tag{3.78}$$

and

$$\|\phi D^{\theta^*}(|\cdot|^{-1}\psi)\|_{L^2} \lesssim_{\theta^*,\phi,\psi} 1. \tag{3.79}$$

PROOF. We write

$$\begin{aligned}&\|\phi\mathcal{D}^{\theta^*}(|\cdot|^{-1}\psi)(\xi)\|_{L^2}^2\\ &\quad= \int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|\phi(\xi)|^2\,\big||\xi|^{-1}\psi(\xi)-|\eta|^{-1}\psi(\eta)\big|^2}{|\xi-\eta|^{3+2\theta^*}}\,d\eta\,d\xi\\ &\quad\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|\phi(\xi)|^2}{|\xi|^2}\frac{|\psi(\xi)-\psi(\eta)|^2}{|\xi-\eta|^{3+2\theta^*}}\,d\eta\,d\xi + \int_{\mathbb{R}^3\times\mathbb{R}^3}|\phi(\xi)|^2\frac{\big||\xi|^{-1}-|\eta|^{-1}\big|^2|\psi(\eta)|^2}{|\xi-\eta|^{3+2\theta^*}}\,d\eta\,d\xi\\ &\quad= \widetilde{\mathcal{I}}+\widetilde{\mathcal{II}}.\end{aligned}$$

From (1.27) and the fact that $|\xi|^{-1}\phi(\xi) \in L^2(\mathbb{R}^3)$,

$$\widetilde{\mathcal{I}} \lesssim \||\cdot|^{-1}\phi\mathcal{D}^{\theta^*}\psi\|_{L^2}^2 \lesssim \big(\|\psi\|_{L^{\infty}}+\|\nabla\psi\|_{L^{\infty}}\big)^2\||\cdot|^{-1}\phi\|_{L^2}^2.$$

On the other hand, gathering together Fubinni's theorem, Hölder's inequality and Hardy-Littlewood-Sobolev inequality we find

$$\widetilde{\mathcal{I}\mathcal{I}} \lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\psi(\eta)|^2}{|\eta|^2}\frac{1}{|\xi-\eta|^{3-(2-2\theta*)}}\frac{|\phi(\xi)|^2}{|\xi|^2}\,d\eta\,d\xi \lesssim \||\eta|^{-2}|\psi(\eta)|^2\frac{1}{|\cdot|^{3-(2-2\theta*)}} * ||\cdot|^{-1}\phi(\cdot)|^2(\eta)\|_{L^1}$$
$$\lesssim \||\cdot|^{-1}\psi\|_{L^{2p}}^2\||\cdot|^{-1}\phi\|_{L^{2q}}^2,$$

where in order to control the above expression one must assure that $1<p,q<3/2$ with

$$\frac{1}{q}=\frac{5}{3}-\frac{1}{p}-\frac{2\theta^*}{3}, \qquad 0<\theta^*<1/2.$$

Note that $2/3<1/q<1$, if and only if, $(2-2\theta^*)/3<1/p<(3-2\theta^*)/3$, and since $2/3<1/p<1$, we get

$$\frac{2}{3}<\frac{1}{p}<\frac{3-2\theta^*}{3}.$$

Consequently, for fixed $\theta^*\in(0,\frac{1}{2})$, one can always find p assuming the above condition. This establishes (3.78). To prove the last assertion of the lemma, we use the commutator estimate (1.25) to find

$$\|\phi D^{\theta^*}(|\cdot|^{-1}\psi)\|_{L^2} \lesssim \|[D^{\theta^*},\phi]|\cdot|^{-1}\psi\|_{L^2}+\|D^{\theta^*}(\phi|\cdot|^{-1}\psi)\|_{L^2}$$
$$\lesssim \left(\||\cdot|^{\theta^*}\widehat{\phi}\|_{L^1}+\|\mathcal{D}^{\theta^*}\phi\|_{L^\infty}+\|\phi\|_{L^\infty}\right)\||\cdot|^{-1}\psi\|_{L^2}+\|\phi\mathcal{D}^{\theta^*}(|\cdot|^{-1}\psi)\|_{L^2}$$

which is bounded by (3.78). □

Now, we can estimate the first term on the r.h.s of (3.77). In view of the zero mean assumption and Sobolev's embedding we get

$$||\xi|^{-2}\xi_l\widehat{u}(\xi)| \lesssim \|\nabla\widehat{u}\|_{L^\infty_\xi} \lesssim \|\langle x\rangle^{5/2+\epsilon}u\|_{L^2_x}, \tag{3.80}$$

where we have set $\epsilon>0$ small to control the above expression by the result in Theorem 3.2 (i). Thus, let $\tilde{\phi}\in C_c^\infty(\mathbb{R}^2)$ with $\phi\tilde{\phi}=\phi$, combining (3.80), Lemmas 3.11 and (3.79) we get

$$\|D_\xi^{\theta^*}\left(|\xi|^{-3}\xi_l\widehat{u}(\xi)\tilde{\phi}(\xi)\phi(\xi)\right)\|_{L^2_\xi}$$
$$\lesssim \||\xi|^{-3}\xi_l\widehat{u}(\xi)\phi(\xi)\|_{L^2_\xi}+\||\xi|^{-1}\tilde{\phi}\mathcal{D}_\xi^{\theta^*}(|\xi|^{-2}\xi_l\widehat{u}\phi)\|_{L^2_\xi}+\||\xi|^{-2}\xi_l\widehat{u}\phi\mathcal{D}^{\theta^*}(|\cdot|^{-1}\tilde{\phi})\|_{L^2_\xi}$$
$$\lesssim \left(\||\cdot|^{-1}\phi\|_{L^2}+\||\cdot|^{-1-\theta^*}\tilde{\phi}\|_{L^2}+\|\phi\mathcal{D}^{\theta^*}(|\cdot|^{-1}\tilde{\phi})\|_{L^2}\right)\|\nabla\widehat{u}\|_{L^\infty_\xi}$$
$$\lesssim \|\langle x\rangle^{5/2+\epsilon}u\|_{L^2_x}.$$

To deal with the second term on the r.h.s of (3.77), we use Lemma 3.14 to find

$$\|D_\xi^{\theta^*}(|\cdot|^{-1}\widehat{x_l u}\phi)\|_{L^2_\xi} \lesssim \|[D_\xi^{\theta^*},\widehat{x_l u}\tilde{\phi}]|\cdot|^{-1}\phi\|_{L^2_\xi}+\|\widehat{x_l u}\tilde{\phi}D_\xi^{\theta^*}(|\cdot|^{-1}\phi)\|_{L^2_\xi}$$
$$\lesssim \||\cdot|^{\theta^*}(x_l u*\tilde{\phi}^\vee)\|_{L^1_x}\||\cdot|^{-1}\phi\|_{L^2}+\|\langle x\rangle^{5/2+\epsilon}u\|_{L^2_x}\|\tilde{\phi}D_\xi^{\theta^*}(|\cdot|^{-1}\phi)\|_{L^2}$$
$$\lesssim \||x|^{\theta^*}x_l u\|_{L^1_x}\|\tilde{\phi}^\vee\|_{L^1}\||\cdot|^{-1}\phi\|_{L^2}+\|x_l u\|_{L^1_x}\||\cdot|^{\theta^*}\tilde{\phi}^\vee\|_{L^1}\||\cdot|^{-1}\phi\|_{L^2}$$
$$+\|\langle x\rangle^{5/2+\epsilon}u\|_{L^2_x}\|\tilde{\phi}D_\xi^{\theta^*}(|\cdot|^{-1}\phi)\|_{L^2}$$
$$\lesssim \|\langle x\rangle^{5/2+\theta^*+\epsilon}u\|_{L^2_x},$$

where we have used that $\|\widehat{x_l u}\|_{L^\infty_x}\lesssim\|\langle x\rangle^{5/2+\epsilon}u\|_{L^2_x}$ for all $\epsilon>0$. Notice that when $0<\epsilon<1-\theta^*$, Theorem 3.2 (i) assures that the r.h.s of the above inequality is controlled. This shows that (3.77) is bounded for all $l = 1, 2, 3$, which establishes (3.74).

Proof of LWP in $\dot{Z}_{s,r}(\mathbb{R}^3)$, $r \in [7/2, 9/2)$. In light of (3.73), (3.74) and Proposition 1.11 with $0 \leqslant \theta \leqslant 1$, one can employ the same line of arguments leading to LWP in $\dot{Z}_{s,r}(\mathbb{R}^2)$, $r \in [3, 4)$ to deduce the same conclusion in $Z_{s,r}(\mathbb{R}^3)$ $r \in [7/2, 4]$, $r \leqslant s$ (the extension to $r = 4$ is given by the fact that w_n^2 satisfies the $A_2(\mathbb{R}^3)$ condition).

Accordingly, it remains to establish LWP when the decay parameter $r \in (4, 9/2)$. This conclusion is obtained from (3.36) with $m = 3$ and $0 < \theta < 1/2$. Under these restrictions, the estimates for Q_1, Q_2 and Q_3 follow from (3.73) and recurring arguments. Finally, in view of identity (3.13) with $\gamma = 3e_k$ and using that $w_N^{2+2\theta}$ satisfies the $A_2(\mathbb{R}^3)$ condition when $0 < \theta < 1/2$, it is seen that

$$\|Q_4\|_{L^2_x} \lesssim \|\langle x\rangle^{1+\theta}|\nabla|^{-1}u\|_{L^2_x} + \|\langle x\rangle^{2+\theta}u\|_{L^2_x} + \|\langle x\rangle^{r-1}\nabla u\|_{L^2_x}, \tag{3.81}$$

which is bounded by previous cases, (3.73) and (3.74). This completes the proof of the Theorem 3.2 (ii).

3.5. Unique continuation principle: two times condition

In this section we infer Theorem 3.4. We begin by introducing some notation and general considerations independent of the dimension to be applied in the proof of Theorem 3.4. We split F_3^k defined by (3.4) as

$$F^k_{3,1}(t,\xi,f) = \partial^3_{\xi_k}\big(it\xi_1|\xi|\big)e^{it\xi_1|\xi|}f(\xi), \text{ and } F^k_{3,2}(t,\xi,f) = F^k_3(t,\xi,f) - F^k_{3,1}(t,\xi,f). \tag{3.82}$$

In addition, we define $\widetilde{F}^k_{3,1}$ and $\widetilde{F}^k_{3,2}$ as in (3.6), that is,

$$\widetilde{F}^k_{3,l}(t,\xi,f) = e^{-it\xi_1|\xi|}F^k_{3,l}(t,\xi,f), \qquad l = 1, 2. \tag{3.83}$$

Without loss of generality we shall assume that $t_1 = 0 < t_2$, i.e., $u_0 \in Z_{d/2+2,d/2+2}(\mathbb{R}^d)$. Recalling (2.5), the solution of the IVP (0.2) can be represented by Duhamel's formula

$$u(t) = U(t)u_0 - \int_0^t U(t-t')u(t')\partial_{x_1}u(t')\,dt' \tag{3.84}$$

or equivalently via the Fourier transform

$$\widehat{u}(t) = e^{it\xi_1|\xi|}\widehat{u_0} - \frac{i}{2}\int_0^t e^{i(t-t')\xi_1|\xi|}\xi_1\widehat{u^2}(t')\,dt'.$$

By means of the notation introduced in (3.4) and (3.82), we have for $k = 1, 2$ that

$$\partial^3_{\xi_k}\widehat{u}(t) = \sum_{m=1}^{2} F^k_{3,m}(t,\xi,\widehat{u_0}) - \frac{i}{2}\int_0^t F^k_{3,m}(t-t',\xi,\xi_1\widehat{u^2})\,dt'. \tag{3.85}$$

Notice that $\partial^3_{\xi_k}\big(\xi_1|\xi|\big)$ is locally integrable in $\mathbb{R}^2$ but not square integrable at the origin. The idea is to use this fact to determinate that all terms in (3.85) except $F^k_{3,1}(t,\xi,\widehat{u_0})$ have the appropriate decay at a later time in dimension $d = 2$. When $d = 3$, we shall use that for $\phi \in C_c^\infty(\mathbb{R}^3)$, $\mathcal{D}^{1/2}_\xi(\partial^3_{\xi_k}\big(\xi_1|\xi|\big)\phi)(\xi) \notin L^2(\mathbb{R}^3)$ to reach the same conclusion. At the end, these facts lead to the proof of Theorem 3.4.

Next, we proceed to infer some estimates for $F^k_{3,l}(t,\xi,f)$ and $\widetilde{F}^k_{3,l}(t,\xi,f)$, assuming that f is a sufficiently regular function with enough decay and setting $0 \leqslant t \leqslant T$. Let $a, b \in \mathbb{R}$, in view of the identities (3.5) and (3.7), it is not difficult to deduce

$$\begin{aligned}\|\langle\xi\rangle^a \widetilde{F}^k_{3,2}(t,\xi,f)\|_{H^b_\xi} \lesssim_T & \sum_{m=0}^{3}\sum_{j=0}^{3-m} \|\langle\xi\rangle^a (\partial_{\xi_k}(it\xi_1|\xi|))^j \partial^m_{\xi_k} f\|_{H^b_\xi} \\ & + \sum_{m=0}^{1}\sum_{j=0}^{1-m} \|\langle\xi\rangle^a \partial^2_{\xi_k}(it\xi_1|\xi|)(\partial_{\xi_k}(it\xi_1|\xi|))^j \partial^m_{\xi_k} f\|_{H^b_\xi}.\end{aligned} \tag{3.86}$$

In particular, since our arguments in dimension $d=3$ require of localization in frequency with a function $\phi \in C^\infty_c(\mathbb{R}^3)$, the same reasoning yields

$$\begin{aligned}\|\langle\xi\rangle^a \widetilde{F}^k_{3,2}(t,\xi,f)\phi\|_{H^b_\xi} \lesssim & \sum_{m=0}^{3}\sum_{j=0}^{3-m} \|\langle\xi\rangle^a (\partial_{\xi_k}(it\xi_1|\xi|))^j \partial^m_{\xi_k} f\phi\|_{H^b_\xi} \\ & + \sum_{m=0}^{1}\sum_{j=0}^{1-m} \|\langle\xi\rangle^a \partial^2_{\xi_k}(it\xi_1|\xi|)(\partial_{\xi_k}(it\xi_1|\xi|))^j \partial^m_{\xi_k} f\phi\|_{H^b_\xi}.\end{aligned} \tag{3.87}$$

On the other hand, since (3.7) implies that $|\partial^l_{\xi_k}(\xi_1|\xi|)| \lesssim \langle\xi\rangle^{2-l}$, $l=1,2$, one can take $b=0$ in (3.86) to find

$$\|\langle\xi\rangle^a F^k_{3,2}(t,\xi,f)\|_{L^2_\xi} = \|\langle\xi\rangle^a \tilde{F}^k_{3,2}(t,\xi,f)\|_{L^2_\xi} \lesssim \sum_{m=0}^{3}\sum_{j=0}^{3-m} \|\langle\xi\rangle^{a+j}\partial^m_{\xi_k} f\|_{L^2_\xi}. \tag{3.88}$$

We can now return to the proof of Theorem 3.4. We divide our arguments according to the dimension.

3.5.1. Dimension $d=2$. In this case, we assume that $u \in C([0,T]; Z_{2^+,2}(\mathbb{R}^2))$ solves (0.2) with $u_0, u(t_2) \in Z_{3,3}(\mathbb{R}^2)$ for some $t_2 > 0$. Additionally, we take $k=1,2$ fixed. Recalling (3.85), we have

Claim 3.15. *The following estimate hold:*

$$F^k_{3,2}(t,\xi,\widehat{u_0}) - \frac{i}{2}\sum_{m=1}^{2}\int_0^t F^k_{3,m}(t-t',\xi,\xi_1\widehat{u^2})\,dt' \in L^2(\langle\xi\rangle^{-4}d\xi) \tag{3.89}$$

for all $t \in [0,T]$.

Let us suppose for the moment the conclusion of Claim 3.15, thus one has

$$\partial^3_{\xi_k}\widehat{u}(t) \in L^2(\langle\xi\rangle^{-4}d\xi) \text{ if and only if } F^k_{3,1}(t,\xi,\widehat{u_0}) \in L^2(\langle\xi\rangle^{-4}d\xi).$$

Let $\phi \in C^\infty_c(\mathbb{R}^2)$ with $\phi \equiv 1$ when $|\xi| \leqslant 1$. We divide $F^k_{3,1}(t,\xi,\widehat{u_0})$ as

$$\begin{aligned}F^k_{3,1}(t,\xi,\widehat{u_0}) =& \partial^3_{\xi_k}(it\xi_1|\xi|)(e^{it\xi_1|\xi|}-1)\widehat{u_0}(\xi)\phi + \partial^3_{\xi_k}(it\xi_1|\xi|)(\widehat{u_0}(\xi)-\widehat{u_0}(0))\phi \\ & + \partial^3_{\xi_k}(it\xi_1|\xi|)\widehat{u_0}(0)\phi + \partial^3_{\xi_k}(it\xi_1|\xi|)e^{it\xi_1|\xi|}\widehat{u_0}(\xi)(1-\phi) \\ =:& F^k_{3,1,1} + F^k_{3,1,2} + F^k_{3,1,3} + F^k_{3,1,4}.\end{aligned}$$

Since $|\partial^3_{\xi_k}(it\xi_1|\xi|)| \lesssim |\xi|^{-1}$, $\xi \neq 0$ and $t \leqslant T$ the mean value inequality shows that the L^2-norms of $F^k_{3,1,1}$ and $F^k_{3,1,4}$ are bounded by a constant (depending on T) times $\|\widehat{u_0}\|_{L^2}$. Moreover, Sobolev's embedding gives

$$\|F^k_{3,1,2}\|_{L^2_\xi} \lesssim \|\nabla\widehat{u_0}\|_{L^\infty_\xi}\|\phi\|_{L^2} \lesssim \|J^{2^+}_\xi \widehat{u_0}\|_{L^2_\xi} \lesssim \|\langle x\rangle^3 u_0\|_{L^2_x}. \tag{3.90}$$

Hence, we get

$$\partial^3_{\xi_k}\widehat{u}(t) \in L^2(\langle\xi\rangle^{-4}d\xi) \text{ if and only if } \partial^3_{\xi_k}(it\xi_1|\xi|)\widehat{u_0}(0)\phi(\xi) \in L^2(\langle\xi\rangle^{-4}d\xi).$$

Considering that $u(t_2) \in Z_{3,3}(\mathbb{R}^2)$, the above implication holds at $t_2 > 0$. At the same time, $|\partial^3_{\xi_k}(\xi_1|\xi|)|^2$ is not integrable at the origin, so it must be the case that $\widehat{u_0}(0) = 0$.

PROOF OF CLAIM 3.15. In view of (3.88) with $a = -2$ we find

$$\|\langle\xi\rangle^{-2}F^k_{3,2}(t,\xi,\widehat{u_0})\|_{L^2_\xi} \lesssim \|\langle\xi\rangle\widehat{u_0}\|_{L^2_\xi} + \|\partial_{\xi_k}\widehat{u_0}\|_{L^2_\xi} + \sum_{m=0}^{3}\|\langle\xi\rangle^{-1}\partial^m_{\xi_k}\widehat{u_0}\|_{L^2_\xi}. \tag{3.91}$$

Noticing that the r.h.s of (3.91) is bounded by $\|Ju_0\|_{L^2_x} + \|\langle x\rangle^3 u_0\|_{L^2_x}$, we complete the estimate for the homogeneous part of the integral equation. To control the integral term, replacing $\widehat{u_0}$ by $\widehat{u\partial_{x_1}u}$ in (3.91) and using (1.28), we observe that it is enough to show

$$u^2 \in L^\infty([0,T]; Z_{1,3}(\mathbb{R}^2)). \tag{3.92}$$

Indeed, $u\partial_{x_1}u \in H^1(\mathbb{R}^2)$ follows from the fact that $H^2(\mathbb{R}^2)$ is a Banach algebra. In addition, the hypothesis $u \in Z_{2^+,2}(\mathbb{R}^2)$ assures that there exists $\epsilon > 0$ such that $u \in H^{2+\epsilon}(\mathbb{R}^2)$, as a result (1.28) yields

$$\begin{aligned}\|\langle x\rangle^3 u^2\|_{L^2_x} &\lesssim \|\langle x\rangle u\|_{L^\infty_x}\|\langle x\rangle^2 u\|_{L^2_x} \lesssim \|J^{1+\epsilon/2}(\langle x\rangle u)\|_{L^2_x}\|\langle x\rangle^2 u\|_{L^2_x} \\ &\lesssim \|\langle x\rangle^2 u\|^{3/2}_{L^2_x}\|J^{2+\epsilon}u\|^{1/2}_{L^2_x}.\end{aligned} \tag{3.93}$$

This establishes (3.92) and consequently the proof of Claim 3.15. ☐

3.5.2. Dimension $d = 3$. We consider $u \in C([0,T]; Z_{3,3}(\mathbb{R}^3))$ solution of (0.2) with $u_0, u(t_2) \in Z_{7/2,7/2}(\mathbb{R}^2)$ for some $t_2 > 0$. Our arguments require localizing near the origin in Fourier frequencies by a function $\phi \in C^\infty_c(\mathbb{R}^3)$ with $\phi(\xi) = 1$ if $|\xi| \leqslant 1$. Thus, recalling (3.85) we have:

Claim 3.16. *Let $k = 1,2,3$. Then*

$$F^k_{3,2}(t,\xi,\widehat{u_0})\phi(\xi) - \frac{i}{2}\sum_{m=1}^{2}\int_0^t F^k_{3,m}(t-t',\xi,\xi_1\widehat{u^2})\phi(\xi)\,dt' \in H^{1/2}_\xi(\mathbb{R}^3) \tag{3.94}$$

for all $t \in [0,T]$.

Let us suppose for the moment that Claim 3.16 holds, then

$$\partial^3_{\xi_k}\widehat{u}(t)\phi \in H^{1/2}_\xi(\mathbb{R}^3) \text{ if and only if } F^k_{3,1}(t,\xi,\widehat{u_0})\phi \in H^{1/2}_\xi(\mathbb{R}^3).$$

We split $F^k_{3,1}$ as

$$\begin{aligned}F^k_{3,1}(t,\xi,\widehat{u_0})\phi =&\partial^3_{\xi_k}(it\xi_1|\xi|)(e^{it\xi_1|\xi|}-1)\widehat{u_0}(\xi)\phi + \partial^3_{\xi_k}(it\xi_1|\xi|)(\widehat{u_0}(\xi)-\widehat{u_0}(0))\phi + \partial^3_{\xi_k}(it\xi_1|\xi|)\widehat{u_0}(0)\phi \\ =&:F^k_{3,1,1} + F^k_{3,1,2} + F^k_{3,1,3}.\end{aligned}$$

The mean value inequality reveals

$$\|F^k_{3,1,1}\|_{H^{1/2}_\xi} \leqslant \|F^k_{3,1,1}\|_{H^1_\xi} \lesssim \|\langle\xi\rangle \widehat{u_0}\phi\|_{L^2_\xi} + \||\xi|\nabla_\xi(\widehat{u_0}\phi)\|_{L^2_\xi} \lesssim \|\widehat{u_0}\|_{H^1_\xi} \lesssim \|\langle x\rangle u_0\|_{L^2_x}, \tag{3.95}$$

and from Sobolev's embedding and the fact that $|\cdot|^{-1}\phi \in L^2(\mathbb{R}^3)$ one gets

$$\|F^k_{3,1,2}\|_{H^{1/2}_\xi} \lesssim \|F^k_{3,1,2}\|_{H^1_\xi} \lesssim \big(\|\phi\|_{H^1} + \||\cdot|^{-1}\phi\|_{L^2}\big)\|\nabla\widehat{u_0}\|_{L^\infty_\xi} \lesssim \|\langle x\rangle^3 u_0\|_{L^2_x}. \tag{3.96}$$

Hence,

$$\partial^3_{\xi_k}\widehat{u}(t)\phi \in H^{1/2}_\xi(\mathbb{R}^3) \text{ if and only if } \partial^3_{\xi_k}(it\xi_1|\xi|)\widehat{u_0}(0)\phi \in H^{1/2}_\xi(\mathbb{R}^3). \tag{3.97}$$

Letting $k=1$ in (3.97), we claim

$$\mathcal{D}^{1/2}_\xi\big(\partial^3_{\xi_1}(\xi_1|\xi|)\phi\big) \notin L^2(\mathbb{R}^3). \tag{3.98}$$

Consequently, since (3.97) holds for $t=t_2>0$, (3.98) imposes that $\widehat{u_0}(0)=0$. We now turn to the proof of (3.98). For a given $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, we denote by $\tilde{x}=(x_2,x_3)\in\mathbb{R}^2$. Let

$$F(\xi) := \partial^3_{\xi_1}(\xi_1|\xi|) = 3(\xi_2^2+\xi_3^2)^2/|\xi|^5 = 3|\tilde{\xi}|^4/|\xi|^5$$

and the region

$$\mathcal{P} := \left\{x\in\mathbb{R}^3 : |x| \leqslant 2^{1/4}|\tilde{x}|, \quad |x|\leqslant 1/16\right\}.$$

When $\xi\in\mathcal{P}$ and $4|\xi|\leqslant|\eta|\leqslant 1/2$, one has $|\xi-\eta|\geqslant 3|\xi|$ and $|\tilde{\xi}|^4\geqslant|\xi|^4/2$, from these deductions,

$$\begin{aligned}|F(\xi)-F(\xi-\eta)| &= \frac{3}{|\xi|^5|\xi-\eta|^5}\left||\xi-\eta|^5|\tilde{\xi}|^4 - |\xi|^5|\tilde{\xi}-\tilde{\eta}|^4\right| \\ &\geqslant \frac{3}{|\xi|^5|\xi-\eta|^5}\big(|\xi-\eta|^5|\tilde{\xi}|^4 - |\xi|^4|\xi-\eta|^5/3\big) \gtrsim |\xi|^{-1}.\end{aligned}$$

Hence,

$$\begin{aligned}\big(\mathcal{D}^{1/2}_\xi(\partial^3_{\xi_1}(\xi_1|\xi|)\phi)\big)^2(\xi)\chi_{\mathcal{P}}(\xi) &\geqslant \int_{4|\xi|\leqslant|\eta|\leqslant 1/2}\frac{|F(\xi)-F(\xi-\eta)|^2}{|\eta|^4}\,d\eta\,\chi_{\mathcal{P}}(\xi) \\ &\gtrsim \frac{1}{|\xi|^2}\int_{4|\xi|\leqslant|\eta|\leqslant 1/2}\frac{1}{|\eta|^4}\,d\eta\chi_{\mathcal{P}}(\xi) \gtrsim \frac{1}{|\xi|^3}\chi_{\mathcal{P}}(\xi),\end{aligned} \tag{3.99}$$

where $\chi_{\mathcal{P}}$ stands for the indicator function on the set $\mathcal{P}$. Therefore, given that $|\xi|^{-3/2}\chi_{\mathcal{P}}\notin L^2(\mathbb{R}^3)$, we get $\mathcal{D}^{1/2}_\xi(\partial^3_{\xi_1}(\xi_1|\xi|)\phi)\notin L^2(\mathbb{R}^3)$.

PROOF OF CLAIM 3.16. Letting $\tilde{\phi}\in C^\infty_c(\mathbb{R}^3)$ with $\tilde{\phi}\phi=\phi$, Proposition 3.10 yields

$$\begin{aligned}\|\mathcal{D}^{1/2}_\xi(F^k_j(t,\xi,f)\phi)\|_{L^2_\xi} &\lesssim \|\mathcal{D}^{1/2}_\xi(e^{it\xi_1|\xi|})\widetilde{F}^k_j(t,\xi,f)\phi\|_{L^2_\xi} + \|\mathcal{D}^{1/2}_\xi(\widetilde{F}^k_j(t,\xi,f)\phi)\|_{L^2_\xi} \\ &\lesssim \|\mathcal{D}^{1/2}_\xi(e^{it\xi_1|\xi|})\tilde{\phi}\|_{L^\infty_\xi}\|\widetilde{F}^k_j(t,\xi,f)\phi\|_{L^2_\xi} + \|\mathcal{D}^{1/2}_\xi(\widetilde{F}^k_j(t,\xi,f)\phi)\|_{L^2_\xi} \\ &\lesssim \|\widetilde{F}^k_j(t,\xi,f)\phi\|_{H^{1/2}_\xi}.\end{aligned} \tag{3.100}$$

Analogously, we bound the $H^{1/2}_\xi$-norm of $F^k_{3,2}(t,\xi,f)\phi$ by that of $\widetilde{F}^k_{3,2}(t,\xi,f)\phi$. Consequently the above computation reduces our arguments to bound (3.94) for the operators $\widetilde{F}_{3,m}$. Letting $f=\widehat{u_0}$ and $b=1/2$ in (3.87), repeated applications of Proposition 3.12 show

$$\|\widetilde{F}^k_{3,2}(t,\xi,\widehat{u_0})\phi\|_{H^{1/2}_\xi} \lesssim \sum_{m=0}^{3}\|\partial^m_{\xi_k}\widehat{u_0}\phi\|_{H^{1/2}_\xi} + \|\widehat{u_0}\|_{H^{(1/2)+}_\xi} + \|\partial_{\xi_k}\widehat{u_0}\|_{H^{(1/2)+}_\xi} \lesssim \|\langle x\rangle^{3+1/2}u_0\|_{L^2_x}. \tag{3.101}$$

On the other hand, employing (3.87) with $f=\xi_1\widehat{u^2}$ and $b=1/2$, it is deduced

$$\|\widetilde{F}^k_{3,2}(t-t',\xi,\xi_1\widehat{u^2})\phi\|_{H^{1/2}_{\xi}}\lesssim\sum_{m=0}^{3}\|\partial^m_{\xi_k}(\xi_1\widehat{u^2})\phi\|_{H^1_{\xi}}\lesssim\|\langle x\rangle^4u^2\|_{L^2_x}. \tag{3.102}$$

This expression is controlled since

$$u^2\in L^\infty([0,T];L^2(|x|^8\,dx)),$$

which holds arguing as in (3.93) employing complex interpolation (1.28). Finally, one can follow the ideas around (3.96) to bound $\|\widetilde{F}^k_{3,1}(t-t',\xi,\xi_1\widehat{u^2})\phi\|_{H^1_{\xi}}$ by the r.h.s of (3.102). The proof is now completed. □

3.6. Unique continuation principle: three times condition

We first discuss the main ideas leading to the proof of Theorem 3.5. By hypothesis, there exist three different times t_1,t_2 and t_3 such that

$$u(\cdot,t_j)\in Z_{d/2+3,d/2+3}(\mathbb{R}^d),\qquad j=1,2,3, \tag{3.103}$$

The equation in (0.2) yields the following identities,

$$\frac{d}{dt}\int x_l u(x,t)\,dx=\frac{\delta_{1,l}}{2}\|u(t)\|^2_{L^2}=\frac{\delta_{1,l}}{2}\|u_0\|^2_{L^2},\quad l=1,\dots,d \tag{3.104}$$

and hence

$$\int x_l u(x,t)\,dx=\int x_l u_0(x)\,dx+\frac{\delta_{1,l}}{2}\|u_0\|^2_{L^2},\quad l=1,\dots,d. \tag{3.105}$$

If we prove that there exist $\tilde{t}_1\in(t_1,t_2)$ and $\tilde{t}_2\in(t_2,t_3)$ such that

$$\int x_1u(x,\tilde{t}_j)\,dx=0,\ \text{for all } j=1,2,$$

in view of (3.104) with $l=1$, it follows that $u\equiv 0$. In this manner, assuming (3.103), we just need to show that there exists $\tilde{t}_1\in(t_1,t_2)$ such that

$$\int x_1u(x,\tilde{t}_1)\,dx=0.$$

Without loss of generality, we let $t_1=0<t_2<t_3$, that is,

$$u_0,u(t_j)\in Z_{d/2+3,d/2+3}(\mathbb{R}^d),\qquad j=2,3.$$

Next, we introduce some further notation and estimates to be used in the proof of Theorem 3.5. For a given $k=1,\dots,d$, recalling (3.5), we split F^k_4 as

$$F^k_4(t,\xi,f)=F^k_{4,1}(t,\xi,f)+F^k_{4,2}(t,\xi,f), \tag{3.106}$$

where

$$F^k_{4,1}(t,\xi,f)=\partial^4_{\xi_k}(it\xi_1|\xi|)e^{it\xi_1|\xi|}f(\xi)+4\partial^3_{\xi_k}(it\xi_1|\xi|)e^{it\xi_1|\xi|}\partial_{\xi_k}f(\xi).$$

In addition, we set

$$\widetilde{F}^k_{4,l}(t,\xi,f)=e^{-it\xi_1|\xi|}F^k_{4,l}(t,\xi,f),\qquad l=1,2. \tag{3.107}$$

We require to estimate the following differential equation obtained from (3.84),

$$\partial_{\xi_k}^4 \widehat{u}(t) = \sum_{m=1}^{2} F_{4,m}^k(t,\xi,\widehat{u_0}) - \frac{i}{2}\int_0^t F_{4,m}^k(t-t',\xi,\xi_1\widehat{u^2})\,dt', \tag{3.108}$$

for each $k = 1,\dots,d$. Now, we proceed to bound the terms F_j^k. To localize in frequency, taking $g \in L^\infty(\mathbb{R}^d)$, (3.5) gives

$$\begin{aligned} &\|\langle\xi\rangle^a \widetilde{F}_{4,2}^k(t,\xi,f)g\|_{H_\xi^b} \\ &\quad\lesssim \|\langle\xi\rangle^a \partial_{\xi_k}^3(it\xi_1|\xi|)\partial_{\xi_k}(it\xi_1|\xi|)fg\|_{H_\xi^b} + \sum_{m=0}^{4}\sum_{j=0}^{4-m}\|\langle\xi\rangle^a(\partial_{\xi_k}(it\xi_1|\xi|))^j\partial_{\xi_k}^m fg\|_{H_\xi^b} \\ &\qquad + \sum_{m=0}^{2}\Big(\sum_{j=0}^{2-m}\|\langle\xi\rangle^a\partial_{\xi_k}^2(it\xi_1|\xi|)(\partial_{\xi_k}(it\xi_1|\xi|))^j\partial_{\xi_k}^m fg\|_{H_\xi^b} \\ &\qquad\quad + \|\langle\xi\rangle^a(\partial_{\xi_k}^2(it\xi_1|\xi|))^j\partial_{\xi_k}^m fg\|_{H_\xi^b}\Big). \end{aligned} \tag{3.109}$$

In particular, setting $b=0$, $g=1$ and using that $|\partial_{\xi_k}^l(it\xi_1|\xi|)| \lesssim |\xi|^{2-l}$, $l=1,2$ and $|\partial_{\xi_k}^3(it\xi_1|\xi|)| \lesssim |\xi|^{-1}$, we have

$$\|\langle\xi\rangle^a F_{4,2}^k(t,\xi,f)\|_{L_\xi^2} = \|\langle\xi\rangle^a \widetilde{F}_{4,2}^k(t,\xi,f)\|_{L_\xi^2} \lesssim \sum_{m=0}^{4}\sum_{j=0}^{4-m}\|\langle\xi\rangle^{a+j}\partial_{\xi_k}^m f\|_{L_\xi^2}. \tag{3.110}$$

Additionally, when $f=\widehat{u_0}$, we define the operators

$$\begin{aligned} F_{4,1,1}^k(t,\xi,\widehat{u_0}(\xi)) &= \partial_{\xi_k}^4(it\xi_1|\xi|)(e^{it\xi_1|\xi|}-1)\widehat{u_0}(\xi), \\ F_{4,1,2}^k(t,\xi,\widehat{u_0}(\xi)) &= \sum_{|\beta|=2}\partial_{\xi_k}^4(it\xi_1|\xi|)R_\beta(\widehat{u_0},\xi)\xi^\beta\phi(\xi), \\ F_{4,1,3}^k(t,\xi,\widehat{u_0}(\xi)) &= \partial_{\xi_k}^4(it\xi_1|\xi|)\widehat{u_0}(\xi)(1-\phi(\xi)), \\ F_{4,1,4}^k(t,\xi,\widehat{u_0}(\xi)) &= 4\partial_{\xi_k}^3(it\xi_1|\xi|)(e^{it\xi_1|\xi|}-1)\partial_{\xi_k}\widehat{u_0}(\xi), \\ F_{4,1,5}^k(t,\xi,\widehat{u_0}(\xi)) &= 4\partial_{\xi_k}^3(it\xi_1|\xi|)(\partial_{\xi_k}\widehat{u_0}(\xi)-\partial_{\xi_k}\widehat{u_0}(0))\phi(\xi), \\ F_{4,1,6}^k(t,\xi,\widehat{u_0}(\xi)) &= 4\partial_{\xi_k}^3(it\xi_1|\xi|)\partial_{\xi_k}\widehat{u_0}(\xi)(1-\phi(\xi)), \end{aligned}$$

where $\phi \in C_c^\infty(\mathbb{R}^d)$ is radial such that $\phi = 1$ when $|\xi| \leqslant 1$ and

$$R_\beta(\widehat{u_0},\xi) = \frac{|\beta|}{\beta!}\int_0^1 (1-\nu)^{|\beta|-1}\partial^\beta\widehat{u_0}(\nu\xi)\,d\nu.$$

Consequently, when $\widehat{u}(0) = \widehat{u_0}(0) = 0$, it holds

$$\begin{aligned} F_{4,1}^k(t,\xi,\widehat{u_0}(\xi)) = \sum_{j=1}^{6} F_{4,1,j}^k(t,\xi,\widehat{u_0}(\xi)) + \partial_{\xi_k}^4(it\xi_1|\xi|)\nabla\widehat{u_0}(0)\cdot\xi\phi \\ + 4\partial_{\xi_k}^3(it\xi_1|\xi|)\partial_{\xi_k}\widehat{u_0}(0)\phi(\xi). \end{aligned} \tag{3.111}$$

Notice that (3.111) is still valid replacing $\widehat{u_0}$ by $\xi_1\widehat{u^2}$. We are now in position to prove Theorem 3.5. We divide our arguments according to the dimension.

3.6.1. Dimension $d = 2$. Suppose that $u \in C([0,T];\dot{Z}_{3,3}(\mathbb{R}^2))$ with $\widehat{u_0}, u(t_2) \in Z_{4,4}(\mathbb{R}^2)$. Under these considerations we have:

Claim 3.17. *We find the following estimate to hold:*

$$\sum_{j=1}^{6} F^k_{4,1,j}(t,\xi,\widehat{u_0}) - \frac{i}{2}\int_0^t F^k_{4,1,j}(t-t',\xi,\xi_1\widehat{u^2})\,dt' \in L^2(\mathbb{R}^2) \tag{3.112}$$

and

$$F^k_{4,2}(t,\xi,\widehat{u_0}) - \frac{i}{2}\int_0^t F^k_{4,2}(t-t',\xi,\xi_1\widehat{u^2})\,dt' \in L^2(\langle\xi\rangle^{-8}d\xi) \tag{3.113}$$

for all $t \in [0,T]$.

PROOF. We first prove (3.112). The mean value inequality shows that $F^k_{4,1,j}(t,\xi,\widehat{u_0}(\xi))$ is bounded by the L^2-norm of u_0 for all $j \neq 2,5$. We use Sobolev's embedding to find

$$\begin{aligned}
\|F^k_{4,1,2}(t,\xi,\widehat{u_0})\|_{L^2_\xi} &\lesssim \sum_{|\beta|=2} \|\int_0^1 (1-\nu)\partial^\beta\widehat{u_0}(\nu\xi)\phi\,d\nu\|_{L^2_\xi} \\
&\lesssim \sum_{|\beta|=2}\int_0^1 (1-\nu)\|\partial^\beta\widehat{u_0}(\nu\xi)\|_{L^4_\xi}\|\phi\|_{L^4}\,d\nu \\
&\lesssim \sum_{|\beta|=2}(\int_0^1 (1-\nu)\nu^{-1/2}\,d\nu)\|\partial^\beta\widehat{u_0}\|_{L^4_\xi}\|\phi\|_{L^4} \\
&\lesssim \sum_{|\beta|=2}\|D^{1/2}_\xi\partial^\beta\widehat{u_0}\|_{L^2_\xi} \lesssim \|\langle x\rangle^{2+1/2}u_0\|_{L^2_x}.
\end{aligned} \tag{3.114}$$

This argument provides the same bound for $F^k_{4,1,5}$, since one can write

$$F^k_{4,1,5}(t,\xi,\widehat{u_0}(\xi)) = 4\partial^3_{\xi_k}(it\xi_1|\xi|)\int_0^1 \nabla\partial_{\xi_k}\widehat{u_0}(\nu\xi)\cdot\xi\,\phi\,d\nu.$$

On the other hand, given that $u \in C([0,T];\dot{Z}_{3,3}(\mathbb{R}^2))$, it is possible to argue as in the deduction of (3.93) to find

$$u\partial_{x_1}u \in L^\infty([0,T];L^2(|x|^5\,dx)) \quad \text{and} \quad u^2 \in L^\infty([0,T];L^2(|x|^8\,dx)). \tag{3.115}$$

Thus, replacing $\widehat{u_0}$ by $\xi_1\widehat{u^2}$ in the preceding discussions and employing (3.115), we conclude (3.112).

Next we deduce (3.113). To estimate the homogeneous part, we employ (3.110) with $a = -4$ and $f = \widehat{u_0}$ to deduce

$$\|\langle\xi\rangle^{-4}F^k_{4,2}(t,\xi,\widehat{u_0}(\xi))\|_{L^2_\xi} \lesssim \|\widehat{u_0}\|_{L^2_\xi} + \sum_{m=0}^{4}\|\langle\xi\rangle^{-1}\partial^m_{\xi_k}\widehat{u_0}\|_{L^2_\xi}, \tag{3.116}$$

and so the above inequality is controlled after Plancherel's theorem by $\|\langle x\rangle^4 u_0\|_{L^2_x}$. Finally, replacing $\widehat{u_0}$ by $\xi_1\widehat{u^2}$ in (3.116), one can control the resulting expression by (3.115) and the fact $u\partial_{x_1}u \in H^3(\mathbb{R}^3)$. This completes the deduction of (3.113). □

Summing up we get

$$\partial^4_{\xi_k}\widehat{u}(t) \in L^2(\langle\xi\rangle^{-8}d\xi), \quad \text{if and only if}$$

$$\begin{aligned}
&t\partial^4_{\xi_k}(\xi_1|\xi|)\nabla\widehat{u_0}(0)\cdot\xi\phi(\xi) - \frac{i}{2}\int_0^t (t-t')\partial^4_{\xi_k}(\xi_1|\xi|)\nabla\big(\xi_1\widehat{u^2}\big)(0,t')\cdot\xi\phi(\xi)\,dt' \\
&+4t\partial^3_{\xi_k}(\xi_1|\xi|)\partial_{\xi_k}\widehat{u_0}(0)\phi(\xi) - 4\frac{i}{2}\int_0^t (t-t')\partial^3_{\xi_k}(\xi_1|\xi|)\partial_{\xi_k}\big(\xi_1\widehat{u^2}\big)(0,t')\phi(\xi)\,dt' \\
&\qquad\in L^2(\langle\xi\rangle^{-8}d\xi),
\end{aligned} \tag{3.117}$$

for fixed $t \geqslant 0$. Let us denote by

$$\mathcal{C}_l(t) := t\partial_{\xi_l}\widehat{u_0}(0) - \frac{i}{2}\int_0^t (t-t')\partial_{\xi_l}\big(\xi_1\widehat{u^2}\big)(0,t')\,dt', \qquad l=1,2. \tag{3.118}$$

The hypothesis at $t=t_2$, the fact that $\langle\xi\rangle \sim 1$ on the support of ϕ and (3.117) imply

$$\sum_{l=1}^{2}\mathcal{C}_l(t_2)\partial^4_{\xi_k}(\xi_1|\xi|)\xi_l\phi(\xi) + 4\mathcal{C}_k(t_2)\partial^3_{\xi_k}(\xi_1|\xi|)\phi(\xi) \in L^2(\mathbb{R}^2). \tag{3.119}$$

From this, we claim that

$$\mathcal{C}_1(t_2) = \mathcal{C}_2(t_2) = 0. \tag{3.120}$$

Let us first write $\mathcal{C}_1(t)$ in a more convenient way for our arguments. We have

$$\partial_{\xi_l}\widehat{u_0}(0) = -i\widehat{x_l u_0}(0) = -i\int x_l u_0(x)dx$$

and by (3.104),

$$\begin{aligned}
\partial_{\xi_l}\big((i\xi_1/2)\widehat{u^2}\big)(0,t') &= -i\widehat{x_l u\partial_{x_1}u}(0,t') = -i\int x_l u\partial_{x_1}u(x,t')\,dx \\
&= i\frac{\delta_{1,l}}{2}\|u(t')\|^2_{L^2} = i\delta_{1,l}\frac{d}{dt}\int x_l u(x,t)\,dx.
\end{aligned} \tag{3.121}$$

Integration by parts then gives

$$\begin{aligned}
\mathcal{C}_l(t) =& t\partial_{\xi_l}\widehat{u_0}(0) - \frac{i}{2}\int_0^t (t-t')\partial_{\xi_l}\big(\xi_1\widehat{u^2}\big)(0,t')\,dt' \\
=& -it\int x_l u_0(x)\,dx - i\delta_{1,l}\int_0^t (t-t')\frac{d}{dt'}\Big(\int x_l u(x,t')\,dx\Big)dt' \\
=& -it(1-\delta_{1,l})\int x_l u_0(x)\,dx - i\delta_{1,l}\int_0^t\int x_l u(x,t')\,dx\,dt'.
\end{aligned} \tag{3.122}$$

Let us suppose for the moment that (3.120) holds, as a result the equation (3.122) shows

$$0 = \mathcal{C}_1(t_2) = -i\int_0^{t_2}\int x_1 u(x,\tau)\,dx\,d\tau.$$

In this manner, the continuity of the application $\tau \mapsto \int x_1 u(x,\tau)\,dx$ assures that there exists a time $\tilde{t}_1 \in (0,t_2)$ at which this map vanishes. According to our reasoning at the beginning of this section, this concludes the proof of Theorem 3.5 when $d=2$.

We can now return to deduce (3.120). We set

$$G(\xi) := \sum_{l=1}^{2} i\mathcal{C}_l(t_2)\partial_{\xi_k}^4(\xi_1|\xi|)\xi_l + 4i\mathcal{C}_k(t_2)\partial_{\xi_k}^3(\xi_1|\xi|).$$

Given that $G(\nu\xi) = \nu^{-1}G(\xi)$, $\xi \neq 0$, $\nu > 0$, by changing to polar coordinates and recalling that ϕ is radial, we find

$$\|G(\xi)\phi(\xi)\|_{L^2}^2 \sim \Big(\int_{\mathbb{S}^1} |G(x)|^2\, dS(x)\Big) \int_0^\infty |\nu|^{-1}|\phi(\nu)|^2\, d\nu. \tag{3.123}$$

Since $|v|^{-1}\phi(\nu)$ is not integrable, (3.119) implies that $G \equiv 0$. However, the functions $\partial_{\xi_k}^4(\xi_1|\xi|)\xi_1$, $\partial_{\xi_k}^4(\xi_1|\xi|)\xi_2$ and $\partial_{\xi_k}^3(\xi_1|\xi|)$ are linear independent (on $\mathbb{R}$), so it must be the case that $\mathcal{C}_1(t_2) = \mathcal{C}_2(t_2) = 0$, which is (3.120).

3.6.2. Dimension $d = 3$. Here we assume that $u \in C([0,T];\dot{Z}_{4,4}(\mathbb{R}^3))$ with $u_0, u(t_2) \in Z_{9/2,9/2}(\mathbb{R}^3)$. Recalling the notation (3.111), we state:

Claim 3.18. *One has:*

$$\sum_{j=1}^{6} F_{4,1,j}^k(t,\xi,\widehat{u_0}) - \frac{i}{2}\int_0^t F_{4,1,j}^k(t-t',\xi,\xi_1\widehat{u^2})\phi(\xi)\, dt' \in H_\xi^1(\mathbb{R}^3). \tag{3.124}$$

and

$$\langle\xi\rangle^{-2}F_{4,2}^k(t,\xi,\widehat{u_0}) - \frac{i}{2}\int_0^t \langle\xi\rangle^{-2}F_{4,2}^k(t-t',\xi,\xi_1\widehat{u^2})\, dt' \in H_\xi^{1/2}(\mathbb{R}^3). \tag{3.125}$$

for all $t \in [0,T]$.

PROOF. We first establish (3.124). The mean value inequality, the fact that $|\xi|^{-1} \in L^2_{loc}(\mathbb{R}^3)$ and a similar reasoning to (3.95) and (3.96) establish

$$\|F_{4,1,j}^k(t,\xi,\widehat{u_0}(\xi))\|_{H_\xi^1} \lesssim \|\langle x\rangle^2 u_0\|_{L_x^2} + \|u_0\|_{H_x^2} \tag{3.126}$$

for all $j = 1,3,4,6$. An analogous argument to (3.114), making a change of variables and using Sobolev's embedding provides

$$\begin{aligned}\|F_{4,1,2}^k(t,\xi,\widehat{u_0}(\xi))\|_{H_\xi^1} &\lesssim \sum_{|\beta|=2} \big(\||\cdot|^{-1}\phi\|_{L^2} + \|\phi\|_{H^1}\big)\|R_\beta(\widehat{u_0},\xi)\|_{L_\xi^\infty} + \|\nabla R_\beta(\widehat{u_0},\xi)\phi\|_{L_\xi^2}\\ &\lesssim \sum_{|\beta|=2} \|\partial^\beta\widehat{u_0}\|_{L_\xi^\infty} + \sum_{|\beta|=2}\Big(\int_0^1 (1-\nu)\, d\nu\Big)\|\nabla\partial^\beta\widehat{u_0}\|_{L_\xi^3}\|\phi\|_{L^6}\\ &\lesssim \|\langle x\rangle^4 u_0\|_{L_x^2} + \sum_{|\beta|=2}\|D_\xi^{1/2}\nabla\partial^\beta\widehat{u_0}\|_{L_\xi^2} \lesssim \|\langle x\rangle^4 u_0\|_{L_x^2}.\end{aligned} \tag{3.127}$$

The estimate $F_{4,1,5}^k(t,\xi,\widehat{u_0}(\xi))$ is obtained in a similar fashion to $F_{4,1,2}^k(t,\xi,\widehat{u_0}(\xi))$. This concludes the considerations for the homogeneous part in (3.124). On the other hand, given that

$$u \in C([0,T];\dot{Z}_{4,4}(\mathbb{R}^3)),$$

by a similar reasoning to (3.93) one has

$$u\partial_{x_1}u \in L^\infty([0,T];\dot{Z}_{3,9/2}(\mathbb{R}^3)). \tag{3.128}$$

This enables us to change the roles of $\widehat{u_0}$ by $\xi_1\widehat{u^2}$ in the above estimates to conclude (3.124).

Let us now establish (3.125). The inequality (1.26) and Proposition 3.10 imply

$$\begin{aligned}\|\langle\xi\rangle^{-2}F^k_{4,2}(t,\xi,\widehat{u_0})\|_{H^{1/2}_\xi} \lesssim & \|\langle\xi\rangle^{-3/2}\widetilde{F}^k_{4,2}(t,\xi,\widehat{u_0})\|_{L^2_\xi}+\|\langle\xi\rangle^{-2}\widetilde{F}^k_{4,2}(t,\xi,\widehat{u_0})\phi\|_{H^{1/2}_\xi}\\ &+\|\langle\xi\rangle^{-2}\widetilde{F}^k_{4,2}(t,\xi,\widehat{u_0})(1-\phi)\|_{H^{1/2}_\xi}.\end{aligned}\tag{3.129}$$

We proceed then to estimate each term on the r.h.s of (3.129). From (3.110) with $a=-3/2$ we find

$$\begin{aligned}&\|\langle\xi\rangle^{-3/2}\widetilde{F}^k_{4,2}(t,\xi,\widehat{u_0})\|_{L^2_\xi}\\ &\quad\lesssim \sum_{m=0}^{4}\sum_{j=0}^{4-m}\|\langle\xi\rangle^{-3/2+j}\partial^m_{\xi_k}\widehat{u_0}\|_{L^2_\xi}\\ &\quad\lesssim \sum_{m=0}^{2}\sum_{j=0}^{1}(\cdots)+\sum_{m=3}^{4}\sum_{j=0}^{4-m}(\cdots)+\sum_{m=1}^{2}\sum_{j=2}^{4-m}(\cdots)+\sum_{j=2}^{4}\|\langle\xi\rangle^{-3/2+j}\widehat{u_0}\|_{L^2}\\ &\quad\lesssim \|J^4_\xi\widehat{u_0}\|_{L^2_\xi}+\sum_{m=1}^{2}\sum_{j=2}^{4-m}\|\langle\xi\rangle^{-3/2+j}\partial^m_{\xi_k}\widehat{u_0}\|_{L^2_\xi}+\|\langle\xi\rangle^{5/2}\widehat{u_0}\|_{L^2_\xi}.\end{aligned}\tag{3.130}$$

In view of the inequality $\|\langle\xi\rangle^{-3/2+j}\partial^m_{\xi_k}\widehat{u_0}\|_{L^2_\xi}\lesssim\|\partial^m_{\xi_k}(\langle\xi\rangle^{j-3/2}\widehat{u_0})\|_{L^2_\xi}+\|[\langle\xi\rangle^{j-3/2},\partial^m_{\xi_k}]\widehat{u_0}\|_{L^2_\xi}$ and complex interpolation,

$$\begin{aligned}&\sum_{m=1}^{2}\sum_{j=2}^{4-m}\|\langle\xi\rangle^{-3/2+j}\partial^m_{\xi_k}\widehat{u_0}\|_{L^2_\xi}\\ &\quad\lesssim \sum_{m=1}^{2}\sum_{j=2}^{4-m}\|J^m_\xi(\langle\xi\rangle^{j-3/2}\widehat{u_0})\|_{L^2_\xi}+\|\langle\xi\rangle^{5/2}\widehat{u_0}\|_{L^2_\xi}\\ &\quad\lesssim \sum_{m=1}^{2}\sum_{j=2}^{4-m}\|\langle\xi\rangle^{5/2}\widehat{u_0}\|^{(2j-3)/5}_{L^2_\xi}\|J^{5m/(8-2j)}_\xi\widehat{u_0}\|^{(8-2j)/5}_{L^2_\xi}+\|\langle\xi\rangle^{5/2}\widehat{u_0}\|_{L^2_\xi}\\ &\quad\lesssim \|J^{5/2}_\xi\widehat{u_0}\|_{L^2_\xi}+\|\langle\xi\rangle^{5/2}\widehat{u_0}\|_{L^2_\xi}.\end{aligned}\tag{3.131}$$

Plugging the above conclusion in (3.130) gives

$$\|\langle\xi\rangle^{-3/2}\widetilde{F}^k_{4,2}(t,\xi,\widehat{u_0})\|_{L^2_\xi}\lesssim\|J^4_\xi\widehat{u_0}\|_{L^2_\xi}+\|\langle\xi\rangle^{5/2}\widehat{u_0}\|_{L^2_\xi}\lesssim\|\langle x\rangle^4u_0\|_{L^2_x}+\|J^{5/2}u_0\|_{L^2_x}.\tag{3.132}$$

To treat the second term on the r.h.s of (3.129), in view of Proposition 1.14 with $h=\langle\xi\rangle^{-2}$, we shall estimate the $H^{1/2}_\xi(\mathbb{R}^3)$-norm of $\widetilde{F}^k_{4,2}(t,\xi,\widehat{u_0})\phi$. Therefore, setting $a=0$, $g=\phi$ and $b=1/2$ in (3.109), after repeated applications of Proposition 3.12 we find

$$\|\widetilde{F}^k_{4,2}(t,\xi,\widehat{u_0})\phi\|_{H^{1/2}_\xi}\lesssim\sum_{l=0}^{4}\|\partial^l_{\xi_k}\widehat{u_0}\|_{H^{1/2}_\xi}+\sum_{m=0}^{2}\|\partial^l_{\xi_k}\widehat{u_0}\|_{H^{(1/2)^+}_\xi}\lesssim\left\|\langle x\rangle^{9/2}u_0\right\|_{L^2_x}.\tag{3.133}$$

Next we deal with the remaining term on the r.h.s of (3.129). Let us first deduce some additional inequalities. Let $P(\xi)$ be a homogeneous polynomial of degree k with $1\leqslant k\leqslant 4$, l an integer number such that $0 \leqslant l \leqslant k$ and f a sufficiently regular function. Then if $k-l \leqslant 2$, from (1.27) we get

$$\begin{aligned}\|\mathcal{D}_\xi^{1/2}\big(\langle\xi\rangle^{-2}\frac{P(\xi)}{|\xi|^l}f(1-\phi)\big)\|_{L^2} \lesssim &\|\mathcal{D}_\xi^{1/2}\big(\langle\xi\rangle^{-2}\frac{P(\xi)}{|\xi|^l}(1-\phi)\big)\|_{L^\infty}\|f\|_{L^2} \\ &+\|\langle\xi\rangle^{-2}\frac{P(\xi)}{|\xi|^l}(1-\phi)\|_{L^\infty}\|\mathcal{D}_\xi^{1/2}f\|_{L^2} \lesssim \|f\|_{H_\xi^{1/2}},\end{aligned} \tag{3.134}$$

and when $k-l>2$,

$$\begin{aligned}\|\mathcal{D}_\xi^{1/2}\big(\langle\xi\rangle^{-2}\frac{P(\xi)}{|\xi|^l}f(1-\phi)\big)\|_{L^2} \lesssim &\|\mathcal{D}_\xi^{1/2}\big(\langle\xi\rangle^{l-k}\frac{P(\xi)}{|\xi|^m}(1-\phi)\big)\|_{L^\infty}\|\langle\xi\rangle^{k-l-2}f\|_{L^2} \\ &+\|\langle\xi\rangle^{l-k}\frac{P(\xi)}{|\xi|^l}(1-\phi)\|_{L^\infty}\|\mathcal{D}_\xi^{1/2}\big(\langle\xi\rangle^{k-l-2}f\big)\|_{L^2} \\ \lesssim &\|\langle\xi\rangle^{k-l-2}f\|_{H_\xi^{1/2}}.\end{aligned} \tag{3.135}$$

In consequence, letting $g=1-\phi$, $a=-2$ and $b=1/2$ in (3.109), after applying (3.134), (3.135) and (1.28) to the resulting inequality one has

$$\|\langle\xi\rangle^{-2}F_{4,2}^{\tilde{k}}(t,\xi,\widehat{u_0})(1-\phi)\|_{H_\xi^{1/2}} \lesssim \|\langle\xi\rangle^{5/2}\widehat{u_0}\|_{L^2_\xi}+\|J_\xi^{9/2}\widehat{u_0}\|_{L^2_\xi}\sim\|\langle x\rangle^{9/2}u_0\|_{L^2_x}+\|J^{5/2}u_0\|_{L^2_x}. \tag{3.136}$$

Finally, collecting (3.132), (3.133) and (3.136), we complete the analysis of the homogeneous part in (3.125). The estimate for the integral term is achieved by the same estimates applied to $\xi_1\widehat{u^2}$ in view of (3.128).

□

Summing up, we can conclude that

$$\begin{aligned}&\partial_{\xi_k}^4\widehat{u}(t)\in H_\xi^{1/2}(\mathbb{R}^3) \quad \text{implies} \\ &\qquad\langle\xi\rangle^{-2}\partial_{\xi_k}^4\widehat{u}(t)\in H_\xi^{1/2}(\mathbb{R}^3), \quad \text{which holds if and only if} \\ &\qquad\sum_{l=1}^3\mathcal{C}_l(t_2)\langle\xi\rangle^{-2}\partial_{\xi_k}^4(\xi_1|\xi|)\xi_l\phi(\xi)+4\mathcal{C}_k(t_2)\langle\xi\rangle^{-2}\partial_{\xi_k}^3(\xi_1|\xi|)\phi(\xi)\in H_\xi^{1/2}(\mathbb{R}^3),\end{aligned} \tag{3.137}$$

for fixed $t\geqslant 0$, where we have defined $\mathcal{C}_l(t)$ exactly as in (3.118) extending to $l=1,2,3$.
We now focus on (3.137) when $k=1$. Given $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$, we denoted by $\tilde{\xi}=(\xi_2,\xi_3)\in\mathbb{R}^2$ and

$$\begin{aligned}G(\xi):&=\sum_{l=1}^3 i\mathcal{C}_l(t_2)\partial_{\xi_1}^4(\xi_1|\xi|)\xi_l\langle\xi\rangle^{-2}+4i\mathcal{C}_1(t_2)\partial_{\xi_1}^3(\xi_1|\xi|)\langle\xi\rangle^{-2} \\ &=|\xi|^{-5}|\tilde{\xi}|^4\langle\xi\rangle^{-2}\left(-15\sum_{l=1}^3 i\mathcal{C}_l(t)|\xi|^{-2}\xi_1\xi_l+12i\mathcal{C}_1(t)\right).\end{aligned}$$

Whenever $\mathcal{C}_1(t)\neq 0$ for some $t>0$ fixed, we claim that

$$\mathcal{D}_\xi^{1/2}(G(\cdot)\phi)\notin L^2(\mathbb{R}^3). \tag{3.138}$$

Since (3.137) is valid at $t_2>0$ and $k=1$, once we have established (3.138), it must follow that

$$\mathcal{C}_1(t_2)=0. \tag{3.139}$$

This in turn allows us to proceed as in the previous subsection to infer Theorem 3.5 in the three-dimensional case. In this manner, it remains to prove claim (3.138). Suppose that for some $t > 0$, $\mathcal{C}_1(t) \neq 0$, we choose then a fixed constant K satisfying

$$0 < K \leqslant \min\left\{\frac{1}{15}, \frac{|\mathcal{C}_1(t)|}{15|\mathcal{C}_2(t)|}, \frac{|\mathcal{C}_1(t)|}{15|\mathcal{C}_3(t)|}\right\}$$

and we define

$$\mathcal{P}_K := \left\{x \in \mathbb{R}^3 : |x| \leqslant (1-K^2)^{-1/2}|\tilde{x}|\right\}. \tag{3.140}$$

Notice that when $x \in \mathcal{P}_K$, one has that $|x_1| \leqslant K|x|$ and so

$$\Big|15\sum_{l=1}^{3}\mathcal{C}_l(t)|x|^{-2}x_1x_l\Big| \leqslant 15\sum_{l=1}^{3}|\mathcal{C}_l(t)||x|^{-1}|x_1| \leqslant 3|\mathcal{C}_1(t)|.$$

In addition, let us consider

$$\xi \in \mathcal{P}_K \cap \{|\xi| \leqslant 1/16\}, \tag{3.141}$$

and for fixed ξ satisfying the above conditions, we take

$$\eta \in \mathcal{P}_K \cap \{4|\xi| \leqslant |\eta| \leqslant 1/2\}. \tag{3.142}$$

Therefore, for such ξ and η, one gets the following lower bound

$$9\langle\xi\rangle^{-2}|\mathcal{C}_1(t)|\frac{|\tilde{\xi}|^4}{|\xi|^5} \leqslant |G(\xi)|,$$

and since $|\xi_1 - \eta_1| \leqslant 2K|\xi - \eta|$,

$$|G(\xi-\eta)| \leqslant 18|C_1(t)|\frac{|\tilde{\xi}-\tilde{\eta}|^4}{|\xi-\eta|^5}.$$

Consequently, collecting the above estimates and using that $3|\xi|, 3|\eta|/4 \leqslant |\xi-\eta|$ and $(8/9)^2 \leqslant \langle\xi\rangle^{-2} \leqslant 1$, whenever (3.141) and (3.142) hold, we arrive at

$$\begin{aligned}|G(\xi) - G(\xi-\eta)| &\geqslant \frac{9|\mathcal{C}_1(t)|}{|\xi|^5|\xi-\eta|^5}\left((8/9)^2|\xi-\eta|^5|\tilde{\xi}|^4 - 2|\tilde{\xi}-\tilde{\eta}|^4|\xi|^5\right)\\ &\geqslant \frac{6|\mathcal{C}_1(t)|}{|\xi|^5}\left(\frac{2^5}{3^3}|\tilde{\xi}|^4 - |\xi|^4\right) \gtrsim_{K,|\mathcal{C}_1|} \frac{1}{|\xi|}.\end{aligned} \tag{3.143}$$

Then, (3.143) and the fact that $\phi \equiv 1$ when $|\xi| \leqslant 1$ yield

$$\begin{aligned}\left(\mathcal{D}_\xi^{1/2}(G(\cdot)\phi)\right)^2(\xi)\chi_{\mathcal{P}_K\cap\{|\xi|\leqslant 1/16\}}(\xi) &\geqslant \int_{\eta\in\mathcal{P}_k\cap\{4|\xi|\leqslant|\eta|\leqslant 1/2\}}\frac{|G(\xi)-G(\xi-\eta)|^2}{|\eta|^4}\,d\eta\,\chi_{\mathcal{P}_K\cap\{|\xi|\leqslant 1/16\}}(\xi)\\ &\gtrsim \frac{1}{|\xi|^2}\int_{\eta\in\mathcal{P}_k\cap\{4|\xi|\leqslant|\eta|\leqslant 1/2\}}\frac{1}{|\eta|^4}\,d\eta\,\chi_{\mathcal{P}_K\cap\{|\xi|\leqslant 1/16\}}(\xi)\\ &\gtrsim \frac{1}{|\xi|^3}\chi_{\mathcal{P}_K\cap\{|\xi|\leqslant 1/6\}}(\xi).\end{aligned}$$

Considering that $\frac{1}{|\xi|^{3/2}}\chi_{\mathcal{P}_K\cap\{|\xi|\leqslant 1/16\}} \notin L^2(\mathbb{R}^3)$, the last inequality establishes (3.138). The proof is now completed.

3.7. Reduction to two times condition

In this section we deduce Theorem 3.6. Without loss of generality we may assume that

$$t_1 = 0 \quad \text{and} \quad \int x_1 u_0(x)\, dx = 0. \tag{3.144}$$

Let us treat first the two-dimensional case. Collecting (3.117), (3.122) and (3.105), we have for $t_2 \neq 0$ that

$$\begin{aligned} &\partial_{\xi_k}^4 \widehat{u}(\cdot, t_2) \in L^2(\mathbb{R}^2) \quad \text{implies} \\ &\qquad \partial_{\xi_k}^4 \widehat{u}(\cdot, t_2) \in L^2(\langle \xi \rangle^{-4} d\xi), \quad \text{this holds if and only if} \\ &\qquad 0 = \int_0^{t_2} \int x_1 u(x, t')\, dx\, dt' = \frac{1}{2} \int_0^{t_2} t' \left\| u(t') \right\|_{L^2}^2 dt' = \frac{t_2^2}{4} \left\| u_0 \right\|_{L^2}^2, \end{aligned} \tag{3.145}$$

whenever $k = 1, 2$. A similar conclusion can be drawn for the three-dimensional case after gathering together (3.137), (3.139) and (3.105) to deduce

$$\begin{aligned} &\partial_{\xi_1}^4 \widehat{u}(\cdot, t_2) \in H^{1/2}(\mathbb{R}^3) \quad \text{implies} \\ &\qquad \langle \xi \rangle^{-2} \partial_{\xi_1}^4 \widehat{u}(\cdot, t_2) \in H^{1/2}(\mathbb{R}^3), \quad \text{which holds if and only if} \\ &\qquad 0 = \int_0^{t_2} \int x_1 u(x, t')\, dx\, dt' = \frac{1}{2} \int_0^{t_2} t' \left\| u(t') \right\|_{L^2}^2 dt' = \frac{t_2^2}{4} \left\| u_0 \right\|_{L^2}^2. \end{aligned} \tag{3.146}$$

3.8. Sharpness three times condition

This part concerns the proof of Theorem 3.7. Whenever $u \in C([0, T]; \dot{Z}_{s, r_d}(\mathbb{R}^d))$ with $r_2 = 3$, $r_3 = 4$ and $s \geqslant d/2 + 4$ one has

$$u \partial_{x_1} u \in L^\infty([0, T]; Z_{d/2+3, d/2+3}(\mathbb{R}^d)). \tag{3.147}$$

Setting $d = 2$, we can employ (3.147) to replace all the $L^2(\langle \xi \rangle^{-8}\, d\xi)$ estimates provided in the proof of Theorem 3.5 by their equivalents in the space $L^2(\mathbb{R}^2)$. This in turn yields

$$\begin{aligned} &\partial_{\xi_k}^4 \widehat{u}(\cdot, t) \in L^2(\mathbb{R}^2), \quad \text{if and only if} \\ &0 = \int_0^t \int x_1 u(x, t')\, dx\, dt' = \int_0^t \int x_1 u_0(x)\, dx + \frac{t'}{2} \left\| u_0 \right\|_{L^2}^2 dt' = 0, \quad \text{if and only if} \\ &\quad t \left(\int x_1 u_0(x)\, dx + \frac{t}{4} \left\| u_0 \right\|_{L^2}^2 \right) = 0, \end{aligned} \tag{3.148}$$

for each $k = 1, 2$. On the other hand, when $d = 3$, (3.147) establishes that all the estimates exhibited in the proof of Theorem 3.5 can be achieved directly in the space $H_\xi^{1/2}(\mathbb{R}^3)$ without the aim of the weight $\langle \xi \rangle^{-2}$. Consequently,

$$\begin{aligned} &\partial_{\xi_k}^4 \widehat{u}(\cdot, t) \in H^{1/2}(\mathbb{R}^3), \quad \text{if and only if} \\ &\int_0^t \int x_1 u(x, t')\, dx\, dt' = 0, \quad \text{if and only if} \\ &0 = \int_0^t \int x_1 u(x, t')\, dx\, dt' = \int_0^t \int x_1 u_0(x)\, dx + \frac{t'}{2} \left\| u_0 \right\|_{L^2}^2 dt' = 0, \quad \text{if and only if} \\ &\quad t \left(\int x_1 u_0(x)\, dx + \frac{t}{4} \left\| u_0 \right\|_{L^2}^2 \right) = 0, \end{aligned} \tag{3.149}$$

This completes the proof of the theorem.

3.9. Appendix: Commutator estimate for Riesz transform operators

This section is devoted to establishing Proposition 3.8.

PROOF OF PROPOSITION 3.8. Without loss of generality we shall deduce (3.2) for $\mathcal{R}_1$. By applying Bony's paraproduct decomposition we write

$$\begin{aligned}
&\mathcal{R}_1(a\partial^\alpha f) - a\mathcal{R}_1\partial^\alpha f - \sum_{1\leqslant|\beta|<|\alpha|}\frac{1}{\beta!}\partial^\beta a D^\beta_{R_1}\partial^\alpha f\\
&\quad= \sum_{N>0}\mathcal{R}_1(P_{<N/2}aP_N\partial^\alpha f) - P_{<N/2}aP_N\partial^\alpha\mathcal{R}_1 f - \sum_{1\leqslant|\beta|<|\alpha|}\frac{1}{\beta!}\partial^\beta P_{<N/2}aP_N D^\beta_{R_1}\partial^\alpha f\\
&\qquad+\sum_{N>0}\Big(\mathcal{R}_1(P_NaP_{<N/2}\partial^\alpha f + P_Na\widetilde{P}_N\partial^\alpha f) - (P_NaP_{<N/2}\partial^\alpha\mathcal{R}_1 f + P_Na\widetilde{P}_N\partial^\alpha\mathcal{R}_1 f)\\
&\qquad\qquad-\sum_{1\leqslant|\beta|<|\alpha|}\frac{1}{\beta!}(\partial^\beta P_NaP_{<N/2}D^\beta_{R_1}\partial^\alpha f + \partial^\beta P_Na\widetilde{P}_ND^\beta_{R_1}\partial^\alpha f)\Big)\\
&\quad=: \pi(lh) + \pi(hl+hh).
\end{aligned}$$

Here $\pi(lh)$ corresponds to the lower-higher frequencies and $\pi(hl+hh)$ combines the higher-lower and higher-higher iterations. We first estimate $\pi(lh)$. The Littlewood-Paley inequality asserts

$$\|\pi(lh)(f,g)\|_{L^p} \sim \left\|\big(P_M\pi(lh)(f,g)\big)_{l^2}\right\|_{L^p}.$$

Then by support considerations,

$$\begin{aligned}
P_M\pi(lh) &= \sum_{N\sim M} -\int i^{|\alpha|+1}\eta^\alpha\left(\frac{\xi_1+\eta_1}{|\xi+\eta|} - \frac{\eta_1}{|\eta|} - \sum_{1\leqslant|\beta|<|\alpha|}\frac{1}{\beta!}\partial^\beta\left(\frac{\eta_1}{|\eta|}\right)\xi^\beta\right)\\
&\qquad\qquad\times\psi_M(\xi+\eta)\psi_{<N/2}(\xi)\psi_N(\eta)\widehat{a}(\xi)\widehat{f}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta\\
&= \sum_{N\sim M}\sum_{|\beta|=|\alpha|}\sigma_{\beta,N}(D)(P_{<N/2}\partial^\beta a, P_N f),
\end{aligned}$$

Where, by the Taylor's expansion of the function $|x|^{-1}x_1$, we have defined for each multi-index β the bilinear operator $\sigma_{\beta,N}(D)$ as in (1.18) with associated symbol

$$\sigma_{\beta,N}(\xi,\eta) = -\frac{i|\beta|}{\beta!}\eta^\alpha\left(\int_0^1(1-\nu)^{|\beta|-1}\partial_x^\beta\left(\frac{x_1}{|x|}\right)(\eta+\nu\xi)\,d\nu\right)\psi_M(\xi+\eta)\phi^0_{<N/2}(\xi)\phi^1_N(\eta),$$

for some suitable bump functions satisfy: $\phi^0_{<N/2}(\cdot) = \phi^0(2^3N^{-1}\cdot)$, $\phi^1_N(\cdot) = \phi(N^{-1}\cdot)$ with $\phi^0\psi_0 = \psi_0$, $\phi^1\psi = \psi$, $\mathrm{dist}(\mathrm{supp}(\phi^1),0) > 0$ and such that $\phi^0_{<N/2}(\xi)\phi^1_N(\eta)$ is supported in the region $|\xi| \ll |\eta|$. [2].

Consequently, one can verify that $\sigma_{\beta,N} \in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ is compact supported outside of the origin in the region $|\xi| \ll |\eta|$ and it satisfies (1.17) uniformly on $\nu \in [0,1]$, for each $N \sim M$. Indeed,

[2]For instance one can take ϕ^0 supported on $B(0,2+\epsilon)$ with $\phi^0 \equiv 1$ on $B(0,1)$ and ϕ^1 supported on $\{x : 1/2-\epsilon \leqslant |x| \leqslant 2+\epsilon\}$ with $\phi^1 \equiv 1$ for $1/2 \leqslant |\xi| \leqslant 1$. Thus, for $\epsilon > 0$ sufficiently small ($\epsilon < 2/7$ is enough), $\phi^1_{<N/2}(\xi)\phi^2_N(\eta)$ is supported in the region $|\xi| \ll |\eta|$.

since $\sigma_{\beta,N}$ is supported in the region $|\xi| \ll |\eta|$ with $|\eta| \sim N$, we have $|\eta + t\xi| \sim |\eta|$ uniformly on $0 \leqslant t \leqslant 1$, which implies that $\sigma_{\beta,N}$ is smooth. To establish the decay property for the derivatives of the symbol, we denote by

$$g(\xi,\eta) := \int_0^1 (1-v)^{|\beta|-1} \partial_x^\beta \left(\frac{x_1}{|x|}\right)(\eta + v\xi)\, dv, \tag{3.150}$$

and let γ_1, γ_2 be arbitrary multi-indexes. Since $|\xi + \eta| \sim |\eta|$ and $\partial_\xi^{\gamma_1}\partial_\beta^{\gamma_2}\sigma_{\beta,N}(\xi,\eta)$ is a linear combination of terms of the form

$$\partial_\eta^{\gamma_{2,1}}(\eta^\alpha)\partial_\xi^{\gamma_{1,1}}\partial_\eta^{\gamma_{2,2}}(g(\xi,\eta))\partial_\xi^{\gamma_{1,2}}(\phi^0_{<N/2}(\xi))\partial_\eta^{\gamma_{2,3}}(\phi^1_N(\eta)) \tag{3.151}$$

where $\gamma_{1,1} + \gamma_{1,2} = \gamma_1$ and $\gamma_{2,1} + \gamma_{2,2} + \gamma_{2,3} = \gamma_2$ with $|\gamma_{2,1}| \leqslant |\alpha|$, we are reduced to show

$$|(3.151)| \lesssim |\eta|^{-|\gamma_1|-|\gamma_2|}.$$

To obtain this estimate, we use that

$$|\partial_\eta^{\gamma_{2,1}}(\eta^\alpha)| \lesssim |\eta|^{|\alpha|-|\gamma_{2,1}|},$$

for all $\eta \neq 0$, given that $|\gamma_{2,1}| \leqslant |\alpha|$. Hence, since $\left|\partial^\gamma\left(x_1|x|^{-1}\right)\right| \lesssim |x|^{-|\gamma|}$, we find

$$|\partial_\xi^{\gamma_{1,1}}\partial_\eta^{\gamma_{2,2}} g(\xi,\eta)| \lesssim |\eta|^{-|\alpha|-|\gamma_{1,1}|-|\gamma_{2,2}|}$$

uniformly on $0 \leqslant t \leqslant 1$. Thus gathering these results we arrive at

$$\begin{aligned}
|(3.151)| &\lesssim |\eta|^{-|\gamma_{1,1}|-|\gamma_{2,1}|-|\gamma_{2,2}|} N^{-|\gamma_{1,2}|} N^{-|\gamma_{2,3}|} |(\partial_\xi^{\gamma_{1,2}}\phi^0)_{<N/2}(\xi)(\partial_\eta^{\gamma_{2,3}}\phi^1)_N(\eta)| \\
&\lesssim |\eta|^{-|\gamma_1|-|\gamma_2|} |(|\cdot|^{|\gamma_{1,2}|+|\gamma_{2,3}|}\partial_\eta^{\gamma_{2,3}}\phi^1)(\eta/N)| \\
&\lesssim |\eta|^{-|\gamma_1|-|\gamma_2|}.
\end{aligned}$$

Thus $\sigma_{\beta,N}$ satisfies (1.17). These facts allow us to use the Fourier decomposition on a cube in $\mathbb{R}^d \times \mathbb{R}^d$ of side length CN for C large to deduce

$$\sigma_{\beta,N}(\xi,\eta) = \sum_{n_1,n_2 \in \mathbb{Z}^d} c_{n_1,n_2,N} e^{i(n_1\cdot\xi + n_2\cdot\eta)/CN}$$

where the Fourier coefficients $\{c_{n_1,n_2,N}\}$ are rapidly decreasing. After this we get

$$\sigma_{\beta,N}(D)(P_{<N/2}\partial^\beta a, P_N f)(x) = \sum_{n_1,n_2\in\mathbb{Z}^d} c_{n_1,n_2,N} P_{<N/2}\partial^\beta a(x - n_1/CN) P_N f(x - n_2/CN),$$

and so we arrive at

$$\begin{aligned}
&|P_M \pi(lh)(x)| \\
&\quad\lesssim \sum_{N\sim M}\sum_{|\beta|=|\alpha|}\sum_{n_1,n_2\in\mathbb{Z}^d} |c_{n_1,n_2,N}||P_{<N}\partial^\beta a(x-n_1/CN)P_N f(x-n_2/CN)|.
\end{aligned}$$

To control the above expression, we use Lemma 1.8 to find

$$|P_{<N/2}\partial^\beta a(x - n_1/CN)| \lesssim (1+|n_1|)^d \mathcal{M}(\partial^\beta a)(x),$$

and writing $\psi_N = \phi^1_N \psi_N$,

$$|P_N f(x - n_2/CN)| \lesssim (1+|n_2|)^d \mathcal{M}(P_N f)(x).$$

Gathering the above estimates with the decay of the coefficients $\{c_{n_1,n_2,N}\}$ yield

$$|P_M\pi(lh)(x)| \lesssim \sum_{N\sim M}\sum_{|\beta|=|\alpha|}\mathcal{M}(\partial^\beta a)(x)\mathcal{M}(P_N f)(x).$$

In this manner, the above display, Lemma 1.7 and the Littlewood-Paley inequality show

$$\|\pi(lh)(f,g)\|_{L^p} \lesssim \sum_{|\beta|=|\alpha|}\|\mathcal{M}(\partial^\beta a)\big(\mathcal{M}(P_M f)\big)_{L^2_M(\mathbb{Z})}\|_{L^p} \lesssim \sum_{|\beta|=|\alpha|}\|\partial^\beta a\|_{L^\infty}\|f\|_{L^p}.$$

It remains to derive a bound for $\pi(hl+hh)$. Notice that our previous considerations cannot be adapted to this case, since the support in frequency of $\pi(hl+hh)$ lies in the region $|\eta|\lesssim|\xi|$, where the line segment $\eta+\nu\xi$ can pass through the origin. Instead, we estimate separately each term in $\pi(hl+hh)$.

Using that $D^{2|\alpha|}=\sum_{|\gamma|=|\alpha|}c_\gamma\partial^\gamma\partial^\gamma$ for some constants $c_\gamma\in\mathbb{R}$, we can write

$$\begin{aligned}\pi(hl+hh) &= \sum_{N>0}\sum_{|\gamma|=|\alpha|}c_\gamma\mathcal{R}_1\big((D^{-2|\alpha|}\partial^\gamma P_N\partial^\gamma a)(P_{\leqslant 2N}\partial^\alpha f)\big)-c_\gamma(D^{-2|\alpha|}\partial^\gamma P_N\partial^\gamma a)(P_{\leqslant 2N}\partial^\alpha\mathcal{R}_1 f)\\ &\quad -c_\gamma\sum_{1\leqslant|\beta|<|\alpha|}\frac{1}{\beta!}\big(\partial^\beta(D^{-2|\alpha|}\partial^\gamma P_N\partial^\gamma a)(P_{\leqslant 2N}D^\beta_{R_1}\partial^\alpha f)\big)\\ &=: \sum_{|\gamma|=|\alpha|}c_\gamma\mathcal{R}_1\sigma^*_{1,\gamma}(D)(\partial^\gamma a,f)+c_\gamma\sigma^*_{1,\gamma}(D)(\partial^\gamma a,\mathcal{R}_1 f)+c_\gamma\sigma^*_{2,\gamma}(D)(\partial^\gamma a,f)\end{aligned}$$

where we have employed $P_{<N/2}+\widetilde{P}_N=P_{\leqslant 2N}$ and the operators $\sigma^*_{1,\gamma}(D)$ are defined through the symbols

$$\sigma^*_{1,\gamma}(\xi,\eta)=\sum_{N>0}i^{|\gamma|+|\alpha|}\frac{\xi^\gamma}{|\xi|^{2|\alpha|}}\eta^\alpha\psi_N(\xi)\psi_{\leqslant 2N}(\eta),$$

and

$$\sigma^*_{2,\gamma}(D)(\partial^\gamma a,f) = \sum_{N>0}\sum_{1\leqslant|\beta|<|\alpha|}\int\frac{(-1)^{|\alpha|+1}i}{\beta!}\partial^\beta\left(\frac{\eta_1}{|\eta|}\right)\frac{\xi^\beta\xi^\gamma}{|\xi|^{2|\alpha|}}\eta^\alpha\psi_N(\xi)\psi_{\leqslant 2N}(\eta)\widehat{\partial^\gamma a}(\xi)\widehat{f}(\eta)\,d\xi d\eta,$$

for each $|\gamma|=|\alpha|$. Using that $\sigma^*_{1,\gamma}(\xi,\eta)$ is supported in the region $|\eta|\lesssim|\xi|$, we can argue exactly as in the analysis of $\sigma^*_{\beta,N}$ above to prove that this operator satisfies the hypothesis of Proposition 1.6. Consequently, the L^p boundedness of the Riesz transform yields

$$\left\|\mathcal{R}_1\sigma^*_{1,\gamma}(D)(\partial^\gamma a,f)+\sigma^*_{1,\gamma}(D)(\partial^\gamma a,\mathcal{R}_1 f)\right\|_{L^p}\lesssim\|\partial^\gamma a\|_{L^\infty}\|f\|_{L^p},$$

for all $|\gamma|=|\alpha|$. On the other hand, we divide the operator $\sigma^*_{2,\gamma}(D)$ by choosing (fixed) multi-indexes $\alpha(k)$ with $1\leqslant k<|\alpha|$ satisfying, $\alpha(k)\leqslant\alpha$ and $|\alpha(k)|=k$. Then we write

$$\sigma^*_{2,\gamma}(D)(\partial^\gamma a,f)=\sum_{1\leqslant|\beta|<|\alpha|}\sigma^*_{2,\gamma,\beta}(D)(\partial^\gamma a,T_\beta f),$$

where for each $|\beta|=k$, $k=1,\dots,|\alpha|-1$ we have set

$$\sigma^*_{2,\gamma,\beta}(\xi,\eta)=\sum_{N>0}\frac{(-1)^{|\alpha|+1}i}{\beta!}\frac{\xi^\beta\xi^\gamma}{|\xi|^{2|\alpha|}}\eta^{\alpha-\alpha(k)}\psi_N(\xi)\psi_{\leqslant 2N}(\eta)$$

and the operators

$$T_\beta(f)(x)=\int\eta^{\alpha(k)}\partial^\beta\left(\frac{\eta_1}{|\eta|}\right)\widehat{f}(\eta)e^{ix\cdot\eta}\,d\eta.$$

One can verify that $\sigma^*_{2,\gamma,\beta}(\xi,\eta)$ satisfies the hypothesis of Proposition 1.6 for each $1 \leqslant |\beta| < |\alpha|$. Additionally, the classical Mikhlin multiplier theorem establishes that T_β defines a bounded operator from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$, whenever $1 < p < \infty$. Notice that this same fact can also be proved directly by observing that $\eta^{\alpha(k)}\partial^\beta\left(\frac{\eta_1}{|\eta|}\right)$ can be written as a linear combination of compositions of Riesz transform operators. Summarizing we conclude:

$$\begin{aligned}\|\sigma^*_{2,\gamma}(D)(\partial^\gamma a, f)\|_{L^p} &\lesssim \sum_{1\leqslant|\beta|<\alpha} \|\sigma^*_{2,\gamma,\beta}(\partial^\gamma a, T_\beta f)\|_{L^p} \lesssim \sum_{1\leqslant|\beta|<\alpha} \|\partial^\gamma a\|_{L^\infty} \left\|T_\beta f\right\|_{L^p} \\ &\lesssim \|\partial^\gamma a\|_{L^\infty} \|f\|_{L^p}.\end{aligned}$$

This completes the estimate for $\pi(hl + hh)$ and in consequence the proof of Proposition 1.6.

□

Chapter 4

Study of a model arising from capillary-gravity wave flows

This chapter is aimed to prove various well-posedness results in real, periodic and anisotropic weighted Sobolev spaces for the IVP (0.4) that arise in the study of capillary-gravity wave flows. To achieve these conclusions a key ingredient is the deduction of a fractional commutator estimate for the Hilbert transform (see Proposition 4.2 below). Additionally, in this chapter, we determinate some unique continuation principles that characterize the spatial behavior of solutions of (0.4). As a further consequence of our results, we derive new well-posedness conclusions for the Shrira equation that appears in the context of waves in shear flows. The contents of this chapter are also presented in [75].

4.1. Statement of results

Since some of our estimates depend on the direction of the variables, for now on we will denote the spatial variables by $(x,y) \in \mathbb{R}^2$. In this chapter, we will mainly work on the IVP (0.4) without distinguishing between the signs of the term $\pm\mathcal{H}_x\partial_y^2 u$. Firstly, to justify the quantity (0.8), we consider the spaces $X^s(\mathbb{R}^2)$ defined by all the tempered distributions such that

(4.1) $$\|f\|_{X^s} = \|J_x^s f\|_{L^2_{xy}} + \|D_x^{-1/2} f\|_{L^2_{xy}} + \|D_x^{-1/2}\partial_y f\|_{L^2_{xy}} < \infty.$$

Our first conclusion establishes local well-posedness in the spaces $H^s(\mathbb{R}^2)$ and $X^s(\mathbb{R}^2)$.

Theorem 4.1. *Let $s > 3/2$ and let $\mathfrak{X}^s(\mathbb{R}^2)$ be any of the spaces $H^s(\mathbb{R}^2)$ and $X^s(\mathbb{R}^2)$. Then for any $u_0 \in \mathfrak{X}^s(\mathbb{R}^2)$, there exist a time $T = T(\|u_0\|_{\mathfrak{X}^s})$ and a unique solution u to the IVP (0.4) in the class*

(4.2) $$C([0,T]; H^s(\mathbb{R}^2)) \cap L^1([0,T]; W^{1,\infty}(\mathbb{R}^2))$$

if $u_0 \in H^s(\mathbb{R}^2)$, or in

(4.3) $$C([0,T]; X^s(\mathbb{R}^2)) \cap L^1([0,T]; W_x^{1,\infty}(\mathbb{R}^2))$$

if $u_0 \in X^s(\mathbb{R}^2)$. Moreover, the flow map $u_0 \mapsto u(t)$ is continuous from $\mathfrak{X}^s(\mathbb{R}^2)$ to $\mathfrak{X}^s(\mathbb{R}^2)$.

The proof of Theorem 4.1 is adapted from the short-time Strichartz linear approach implemented by Kenig [50], and Linares, Pilod and Saut [59]. A novelty in the present work is the study of the operators $D_x^{-1/2}$ and $D_x^{-1/2}\partial_y$ which yields additional difficulties in contrast with the operator $\partial_x^{-1}\partial_y$ considered in the previous references. Among them, we required to deduce the following commutator relation:

PROPOSITION 4.2. *Let $1 < p < \infty$ and $0 \leqslant \alpha, \beta \leqslant 1$, $\beta > 0$ with $\alpha + \beta = 1$, then*

$$\|D_x^{\alpha}[\mathcal{H}_x, g]D_x^{\beta} f\|_{L^p(\mathbb{R})} \lesssim_{p,\alpha.\beta} \|\partial_x g\|_{L^\infty(\mathbb{R})}\|f\|_{L^p(\mathbb{R})}. \tag{4.4}$$

Proposition 4.2 can be regarded as a non-local version of Calderon's first commutator estimate deduced in [19, Lemma 3.1] and its extension to the BMO spaces in [56, Proposition 3.8] (see Proposition 1.5). This commutator is useful to perform energy estimates involving the operator $D_x^{-1/2}\partial_y$ and the nonlinearity in the equation in (0.4).

We remark that Theorem 4.1 improves the conclusion in [21] lowering the regularity in the Sobolev scale to $s > 3/2$ and obtaining well-posedness conclusion in spaces well-adapted to (0.8). Furthermore, we believe that these results could certainly be used to study existence and stability of solitary wave solutions, where one employs the quantity $E(u)$ (see for instance [24]).

Next, we present our result in the periodic setting.

Theorem 4.3. *Let $s > 3/2$. Then for any $u_0 \in H^s(\mathbb{T}^2)$, there exist $T = T(\|u_0\|_{H^s})$ and a unique solution u of the IVP (0.4) that belongs to*

$$C([0,T]; H^s(\mathbb{T}^2)) \cap F^s(T) \cap B^s(T).$$

Moreover, for any $0 < T' < T$, there exists a neighborhood $\mathcal{U}$ of u_0 in $H^s(\mathbb{T}^2)$ such that the flow map data-solution,

$$v \in \mathcal{U} \mapsto v \in C([0,T']; H^s(\mathbb{T}^2))$$

is continuous.

The function spaces $F^s(T)$ and $B^s(T)$ are defined in Section 4.4 below. Theorem 4.3 is proved by means of the short-time Fourier restriction norm method developed by Ionescu, Kenig and Tataru [44], see also [77, 90]. Mainly, this technique consists of an energy method combined with linear and nonlinear estimates in the short-time Bourgain's spaces $F^s(T)$ and their dual $\mathcal{N}^s(T)$ (see Section 4.4), where the former spaces enjoy the $X^{s,b}$ structure with localization in small time intervals whose length is of order 2^{-j}, $j \in \mathbb{Z}^+ \cup \{0\}$. We emphasize that up to our knowledge Theorem 4.3 seems to be the first non-standard result dealing with the periodic equation (0.4).

Regarding the quantity $E(u)$ in the periodic setting, we consider the Sobolev spaces

$$X^s(\mathbb{T}^2) = \{f \in H^s(\mathbb{T}^2) : \widehat{f}(0,n) = 0, \text{ for all } n \in \mathbb{Z}\} \tag{4.5}$$

equipped with the norm $\|f\|_{X^s(\mathbb{T}^2)} = \|f\|_{H^s(\mathbb{T}^2)}$. Then, since $X^s(\mathbb{T}^2)$ is a closed subspace of $H^s(\mathbb{T}^2)$, by replacing the spaces $H^s(\mathbb{T}^2)$ by $X^s(\mathbb{T}^2)$ in Section 4.4 below, the same proof of Theorem 4.3 yields:

Corollary 4.4. *Let $s > 3/2$. Then the IVP (0.4) is locally well-posed in $X^s(\mathbb{T}^2)$.*

Remarks. (i) *Our local theory is still not sufficient to reach the energy spaces* $X^1(\mathbb{K}^2)$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{T}\}$ *determined by* (0.8). *Thus we cannot implement the invariant* $E(u)$ *to obtain global solutions.*

(ii) *For the one-dimensional Benjamin-Ono equation* (0.1), *many authors, see* **[43, 64, 85]** *for instance, have applied the gauge transformation to establish local and global results. Unfortunately, we do not know if there exists such a gauge transformation for* (0.4). *Additionally, we do not know if there is a maximal norm estimate avalaible for solutions of the IVP* (0.4), *which would allow us to argue as in* **[51]** *to improve the results in Theorem 4.1.*

(iii) *Concerning* $\mathbb{R}^2$ *solutions of the IVP* (0.4), *we do not have a standard approach to derive bilinear estimates in the spaces* $F^s(T)$ *and* $N^s(T)$. *As a consequence, the short-time Fourier restriction norm method applied to this case leads the same regularity attained in Theorem 4.1. For this reason, we have proved Theorem 4.1 employing the short-time linear Strichartz approach instead, which also provides solutions in the class* $L^1([0,T]; W^{1,\infty}(\mathbb{R}^2))$. *The advantage of using this consequence lies in its application to methods based on energy estimates as the one we employ here to deduce well-posedness in weighted spaces.*

Next, we study LWP issues in the anisotropic weighted Sobolev spaces defined by (1.3) and (1.4).

To motivate our results, we observe that $x(\mathcal{H}_x u \pm \mathcal{H}_x \partial_y^2) f \in L^2(\mathbb{R}^2)$ requires the condition $\int f(x,y) e^{iy\eta}\, dxdy = 0$ for almost every η. Thus, formally transferring this idea to the equation in (0.4), we do not expect that in general solutions of this model propagate weights of arbitrary order in the x-variable. Indeed, the first weight we contemplate to propagate without any further assumption should be of order $|x|^{\alpha}$ for some $0 < \alpha < 1$. Therefore, answering this question we have the following theorem:

Theorem 4.5. (i) *If* $r_1 \in [0, 1/2)$ *and* $r_2 \geqslant 0$ *with* $s \geqslant \max\{(3/2)^+, r_2\}$, *then the IVP associated to* (0.4) *is locally well-posed in* $Z_{s,r_1,r_2}(\mathbb{R}^2)$.

(ii) *Let* $r_2 \geqslant 0$, $s \geqslant \max\{(3/2)^+, r_2\}$. *Then the IVP* (0.4) *is locally well-posed in the space*

$$ZH_{s,1/2,r_2}(\mathbb{R}^2) = \{f \in Z_{s,1/2,r_2}(\mathbb{R}^2) : \|f\|_{Z_{s,1/2,r_2}} + \||x|^{1/2}\mathcal{H}_x f\|_{L^2_{xy}} < \infty\}. \tag{4.6}$$

(iii) *If* $r_1 \in (1/2, 3/2)$ *and* $r_2 \geqslant 0$ *with* $s \geqslant \max\{(3/2)^+, r_2\}$, *then the IVP associated to* (0.4) *is locally well-posed in* $\dot{Z}_{s,r_1,r_2}(\mathbb{R}^2)$.

In particular, Theorem 4.5 shows that solutions of the IVP (0.4) admits weights of arbitrary order in the y-variables. The proof of these results follows the ideas of Fonseca, Linares and Ponce **[27, 28, 29]**. We emphasize that our conclusions involve further difficulties, since here we deal with anisotropic spaces in two spatial variables, and the x-spatial decay allowed by solutions of (0.4) does not even reach an integer number for arbitrary initial data. In this regard, in **[29**, Theorem 1], it was established that for general initial data, solutions of the Benjamin-Ono equation (0.1) propagate weights of order between $[0, 5/2)$, while solutions of (0.4) allow weights of order $[0, 1/2)$ in the x-variable. Finally, we remark that Theorem 4.5 improves the range of weights determined in the work of **[21]**, and we do not require the assumption $\partial_x^{-1} u \in H^s(\mathbb{R}^2)$.

Next, we state some unique continuation principles for solutions of the IVP (0.4).

Theorem 4.6. *Let $r_1 \in (1/4, 1/2)$, $r_2 \geqslant r_1$ and $s \geqslant \max\{\frac{2r_1}{(4r_1-1)^-}, r_2\}$. Let u be a solution of the IVP (0.4) such that $u \in C([0,T]; Z_{s,r_1,r_2}(\mathbb{R}^2)) \cap L^1([0,T]; W^{\infty}_{1,x}(\mathbb{R}^2))$. If there exist two different times $t_1 < t_2$ in $[0,T]$ for which*

$$u(\cdot, t_1) \in Z_{s,(1/2)^+,r_2}(\mathbb{R}^2) \text{ and } u(\cdot, t_2) \in Z_{s,1/2,r_2}(\mathbb{R}^2),$$

then $\widehat{u}(0,\eta,t) = 0$ for all $t \in [t_1, T]$ and almost every η.

Theorem 4.7. *$r_2 \geqslant r_1 = (3/2)^-$ and $s > \max\{3, r_2\}$. Let u be a solution of the IVP (0.4) such that $u \in C([0,T]; \dot{Z}_{s,r_1,r_2}(\mathbb{R}^2))$. If there exist two different times $t_1 < t_2$ in $[0,T]$ for which*

$$u(\cdot, t_1) \in Z_{s,(3/2)^+,r_2}(\mathbb{R}^2) \text{ and } u(\cdot, t_2) \in Z_{s,3/2,r_2}(\mathbb{R}^2),$$

Then the following identity holds true

$$2i\sin((1 \mp \eta^2)(t_2 - t_1))\partial_\xi \widehat{u}(0,\eta,t_1) = -\int_{t_1}^{t_2} \sin((1 \mp \eta^2)(t_2 - t'))\widehat{u^2}(0,\eta,t')\,dt', \tag{4.7}$$

for almost every $\eta \in \mathbb{R}$. In particular, if $u(\cdot, t_1) \in Z_{s,2^+,2^+}(\mathbb{R}^2)$ it holds

$$2\sin(t_2 - t_1)\int xu(x,y,t_1)\,dxdy = (\cos(t_2 - t_1) - 1)\int u_0^2(x,y)\,dxdy. \tag{4.8}$$

Remarks. (i) *Since the weight $|x|$ does not satisfy the A_2 condition (see [20, 84]) the assumption $\mathcal{H}_x u_0 \in L^2(|x|\,dxdy)$ subscribed in the space $ZH_{s,1/2,r_2}(\mathbb{R}^2)$ is necessary in our arguments. Moreover, for a function $u_0 \in Z_{s,1/2,r_2}(\mathbb{R}^2)$ the condition $\widehat{u_0}(0,\eta) = 0$ does not make sense in general. Besides by inspecting our arguments in Lemma 4.47 below and employing* [88, Theorem 4.3], *the hypothesis $\mathcal{H}_x u_0 \in L^2(|x|\,dxdy)$ can be replaced by the assumption that for a.e. η, the map $\xi \mapsto \widehat{u}_0(\xi,\eta)$ belongs to the $L^2(\mathbb{R})$-closure of the space of square integrable continuous odd functions.*

(ii) *Theorem 4.6 establishes that for arbitrary initial data in $Z_{s,r_1,r_2}(\mathbb{R}^2)$ with $r_2 \geqslant r_1$ and $r_1 \neq 1/2$, $(1/2)^-$ is the largest possible decay for solutions of the equation in (0.4) on the x-spatial variable. Consequently, for this regimen of indexes r_1, r_2, Theorem 4.5 (i) is sharp. However, it still remains an open problem to derive a similar conclusion for the cases $0 \leqslant r_2 < r_1$. Moreover, Theorem 4.6 shows that if $u_0 \in \mathbb{Z}_{s,r_1,r_2}(\mathbb{R}^2)$ with $r_2 \geqslant r_1 = (1/2)^+$, $s \geqslant \max\{\frac{2r_1}{(4r_1-1)^-}, r_2\}$ and $\widehat{u_0}(0,\eta) \neq 0$ for almost every η, then the corresponding solution $u = u(x,t)$ of the IVP (0.4) satisfies*

$$|x|^{(1/2)^-}u \in L^\infty([0,T]; L^2(\mathbb{R}^2)), \quad T > 0.$$

Although, there does not exist a non-trivial solution u corresponding to data u_0 with $\widehat{u_0}(0,\eta) \neq 0$ a.e. with

$$|x|^{1/2}u \in L^\infty([0,T']; L^2(\mathbb{R}^2)), \text{ for some } T' > 0.$$

(iii) *The condition $u(\cdot, t_1) \in Z_{s,2^+,2^+}(\mathbb{R}^2)$ in Theorem 4.7 can be relaxed assuming for instance*

$$u(\cdot, t_1) \in Z_{s,(3/2)^+,r_2}(\mathbb{R}^2) \text{ and } xu(x,y,t_1) \in L^1(\mathbb{R}^2).$$

In addition, (4.8) provides some unique continuation principles for solutions of the IVP (0.4). Indeed, if $(t_2 - t_1) = k\pi$ for some positive odd integer number k, then it must be the case that $u \equiv 0$. Besides, if there exists three times $t_1 < t_2 < t_3$ such that $u(\cdot, t_1) \in Z_{s,2^+,2^+}(\mathbb{R}^2)$, $u(\cdot, t_j) \in Z_{s,3/2,r_2}(\mathbb{R}^2)$, $j = 2, 3$ and

$$\sin(t_2 - t_1)(1 - \cos(t_3 - t_1)) \neq \sin(t_3 - t_1)(1 - \cos(t_2 - t_1)),$$

then $u \equiv 0$. *Accordingly, Theorem 4.7 establishes that for any initial data* $u_0 \in Z_{s,r_1,r_2}(\mathbb{R}^2)$, $r_2 \geqslant r_1 > 2$ *(or* $u_0 \in Z_{s,(3/2)^+,r_2}(\mathbb{R}^2)$ *with* $xu_0 \in L^1(\mathbb{R}^2)$*),* $s \geqslant \max\{3, r_2\}$ *the decay* $(3/2)^-$ *is the largest possible in the x-spatial variable. More precisely, if* $u_0 \in Z_{s,r_1,r_2}(\mathbb{R}^2)$, $r_2 \geqslant r_1 > 2$, $s \geqslant \max\{3, r_2\}$, *then the corresponding solution* $u = u(x,t)$ *of the IVP* (0.4) *satisfies*

$$|x|^{(3/2)^-} u \in L^\infty([0,T]; L^2(\mathbb{R}^2)), \qquad T > 0$$

and there does not exist a non-trivial solution with initial data u_0 *such that*

$$|x|^{3/2} u \in L^\infty([0,T']; L^2(\mathbb{R}^2)), \quad \text{for some } T' > 0.$$

All of the previous well-posedness conclusions were addressed by compactness method. As a matter of fact, we have that the local Cauchy problem for the equation (0.4) cannot be solved for initial data in any isotropic or anisotropic spaces by a direct contraction principle based on its integral formulation.

PROPOSITION 4.8. *Let* $s_1, s_2 \in \mathbb{R}$ *(resp.* $s \in \mathbb{R}$*). Then there does not exist a time* $T > 0$ *such that the Cauchy problem* (0.4) *admits a unique solution on the interval* $[0,T]$ *and such that the flow-map data-solution* $u_0 \mapsto u(t)$ *is* C^2*-differentiable from* $H^{s_1,s_2}(\mathbb{R}^2)$ *to* $H^{s_1,s_2}(\mathbb{R}^2)$ *(resp. from* $X^s(\mathbb{R}^2)$ *to* $X^s(\mathbb{R}^2)$*).*

We remark that a similar conclusion was derived before for the IVP (0.6) in [**23**]. Finally, we present our conclusions on the Shrira equation:

Theorem 4.9. *Let* $s > 3/2$, *then the IVP* (0.6) *is LWP in* $H^s(\mathbb{K}^2)$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{T}\}$ *and in the space* $\widetilde{X}^s(\mathbb{R}^2)$ *determining by the norm*

$$\|f\|_{\widetilde{X}^s} = \|J_x^s f\|_{L^2_{xy}} + \|D_x^{-1/2}\partial_y f\|_{L^2_{xy}}.$$

In addition, the results of Theorems 4.5 and 4.6 hold for the IVP (0.6)*. Moreover, the conclusion of Theorem 4.7 is also valid considering*

$$2i\sin(\eta^2(t_2 - t_1))\partial_\xi \widehat{u}(0,\eta,t_1) = -\int_{t_1}^{t_2} \sin(\eta^2(t_2 - t'))\widehat{u^2}(0,\eta,t')\, dt' \tag{4.9}$$

instead of (4.7)*. In particular, if* $\partial_\xi \widehat{u}(0,\eta,t_1) = 0$ *for a.e.* η*, then* $u \equiv 0$.

Consequently, Theorem 4.9 determines new well-posedness conclusion in the spaces $\widetilde{X}^s(\mathbb{R}^2)$ where the energy (0.9) makes sense. Besides, in the periodic setting, we obtain the same well-posedness result stated for the two-dimensional case in the work of Schippa [**81**, Theorem 1.2], that is, we deduced that (0.6) is LWP in $H^s(\mathbb{T}^2)$, $s > 3/2$. We remark that our results are provided by rather different considerations than those given in [**81**], where the author employed the setting of the periodic U^p-/V^p-spaces ([**36, 37**]) combined with key short-time bilinear Strichartz estimates (see Section 3 of the aforementioned reference). Certainly, we believe that these considerations can be adapted to (0.4).

Regarding weighted spaces, our conclusions extend the results in [**61**], since here we deal with less regular solutions, and we improve the x-spatial decay allowed by (4.9) to the interval $[0, 3/2)$. Actually, by increasing the required regularity, it is not difficult to adapt our result to solutions in anisotropic spaces $H^{s_1,s_2}(\mathbb{R}^2)$. We remark that our proof of well-posedness in $Z_{s,r_1,r_2}(\mathbb{R}^2)$ is applied directly to solutions in the space $H^s(\mathbb{R}^2)$, in contrast, in [**61**] the author first derive well-posedness in weighted spaces for solutions with the additional property $\partial_x^{-1}u \in H^s(\mathbb{R}^2)$.

We will begin by introducing some notation and preliminaries. Sections 4.3 and 4.4 are devoted to prove Theorem 4.1 and Theorem 4.3 respectively. Theorems 4.5, 4.6 and 4.7 will be deduced Section 4.5. Section 4.7 is aimed to prove Theorem 4.9. The ill-posedness result of Proposition 4.8 is deduced in Section 4.6. We conclude this chapter with an appendix where we show the commutator estimate in Proposition 4.2.

4.2. Notation

The Fourier variables of (x, y, t) are denoted by (ξ, μ, τ) and in the periodic case by (m, n, τ). Recalling the function ψ_0 satisfying (1.6) for $d = 1$, and the functions ψ_N, we define the projector operators in $L^2(\mathbb{R}^2)$ by the relations

$$\begin{aligned} \mathcal{F}(P_N^x(u))(\xi,\eta) &= \psi_N(\xi)\mathcal{F}(u)(\xi,\eta), \\ \mathcal{F}(P_{\leqslant N}^x(u))(\xi,\eta) &= \psi_{\leqslant N}(\xi)\mathcal{F}(u)(\xi,\eta), \end{aligned} \tag{4.10}$$

We will also employ the projectors (1.7) for $\mathbb{R}^2$. We set

$$\omega(\xi,\eta) = \operatorname{sign}(\xi) + \operatorname{sign}(\xi)\xi^2 \mp \operatorname{sign}(\xi)\eta^2, \tag{4.11}$$

and define the resonant function by

$$\Omega(\xi_1,\eta_1,\xi_2,\eta_2) := \omega(\xi_1+\xi_2,\eta_1+\eta_2) - \omega(\xi_1,\eta_1) - \omega(\xi_2,\eta_2). \tag{4.12}$$

4.3. Well-posedness in $H^s(\mathbb{R}^2)$ and $X^s(\mathbb{R}^2)$

This section is devoted to establish Theorem 4.1 in which we derive LWP for the IVP (0.4) in the spaces $H^s(\mathbb{R}^2)$ and $X^s(\mathbb{R}^2)$.

4.3.1. Preliminary estimates.

4.3.1.1. **Linear estimates.** This part is aimed to deduce some key linear estimates for the problem:

$$\begin{cases} \partial_t u + \mathcal{H}_x u - \mathcal{H}_x\partial_x^2 u \pm \mathcal{H}_x\partial_y^2 u = 0, & (x,y)\in\mathbb{R}^2,\ t\in\mathbb{R}, \\ u(x,0) = u_0, \end{cases} \tag{4.13}$$

where the solutions are given by

$$S(t)u_0(x,y) = \int e^{it\omega(\xi,\eta)+ix\xi+iy\eta}\widehat{u_0}(\xi,\eta)\,d\xi d\eta \tag{4.14}$$

and $\omega(\xi,\eta)$ as in (4.11). We have the following decay estimates:

Lemma 4.10. *Let $1 \leqslant p \leqslant 2$, then it holds*

$$\|S(t)f\|_{L^{p'}} \lesssim |t|^{-(2-p)/p}\|f\|_{L^p}. \tag{4.15}$$

PROOF. Let us prove first the case $p = 1$. We write

$$S(t)f(x,y) = I(\cdot,\cdot,t) * f(x,y), \tag{4.16}$$

where the semi-convergent integral is to be understood as

$$I(x,y,t) := \frac{1}{2\pi} \lim_{N,M\to\infty} \int_{\mathbb{R}^2} e^{it\omega(\xi,\eta)+ix\xi+iy\xi} \psi_0(\xi/N)\psi_0(\eta/M)\, d\xi d\eta, \tag{4.17}$$

and the limit is considered in the distributional sense. Employing the identity

$$\mathcal{F}(e^{i\delta x^2})(\xi,\eta) = \frac{\pi^{1/2}}{|\delta|^{1/2}} e^{-i\xi^2/4\delta} e^{i\frac{\pi}{4}\,\mathrm{sign}(\delta)}, \tag{4.18}$$

which is valid for any real number $\delta \neq 0$, and the fact that ψ_0 is an even function, we obtain

$$\begin{aligned}
&\lim_{M\to\infty} \int_{\mathbb{R}^2} e^{it\omega(\xi,\eta)+ix\xi+iy\xi} \psi_0(\xi/N)\psi_0(\eta/M)\, d\xi d\eta \\
&= \frac{\pi^{1/2}}{|t|^{1/2}} \int_0^\infty e^{it+it\xi^2 \pm \frac{iy^2}{t} \mp \frac{\pi}{4} i\,\mathrm{sign}(t)+ix\xi} \psi_0(\xi/N)\, d\xi \\
&\qquad + \frac{\pi^{1/2}}{|t|^{1/2}} \int_{-\infty}^0 e^{-it-it\xi^2 \mp \frac{iy^2}{t} \pm \frac{\pi}{4} i\,\mathrm{sign}(t)+ix\xi} \psi_0(\xi/N)\, d\xi \\
&= \frac{2\pi^{1/2}}{|t|^{1/2}} \Re\Big(\int_0^\infty e^{it+it\xi^2 \pm \frac{iy^2}{t} \mp \frac{\pi}{4} i\,\mathrm{sign}(t)+ix\xi} \psi_0(\xi/N)\, d\xi \Big).
\end{aligned} \tag{4.19}$$

Now, since the phase function $\phi(\xi) = t + t\xi^2 \pm \frac{y^2}{t} \mp \frac{\pi}{4}\,\mathrm{sign}(t) + x\xi$ satisfies, $\phi''(\xi) = 2t$, Van der Corput lemma (see Chapter VIII in [82]) yields

$$\Big| \int_0^\infty e^{it+it\xi^2 \pm \frac{iy^2}{t} \mp \frac{\pi}{4} i\,\mathrm{sign}(t)+ix\xi} \psi_0(\xi/N) \Big| \lesssim |t|^{-1/2}, \tag{4.20}$$

uniformly on $N > 1$. Then gathering (4.19) and (4.20), we find $|I(x,y,t)| \lesssim |t|^{-1}$. This result and (4.16) establish (4.15) when $p = 1$. Therefore, the preceding conclusion and the fact that $\|S(t)f\|_{L^2} = \|f\|_{L^2}$ allows us to use the Riesz-Thorin interpolation theorem to deduce (4.15), whenever $1 \leqslant p \leqslant 2$. □

By means of Lemma 4.10 and the Stein-Tomas argument, we deduce the following space-time norms for solutions of (4.13).

Lemma 4.11. *The following estimate holds*

$$\|S(t)f\|_{L^q_t L^p_{xy}} \lesssim \|f\|_{L^2}, \tag{4.21}$$

whenever $2 \leqslant p, q \leqslant \infty$, $q > 2$ *and* $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$.

Notice that the endpoint Strichartz estimate corresponding to $(q,p) = (2,\infty)$ is not stated in the preceding lemma, as a consequence we need to lose a little bit of regularity in order to control this norm.

Corollary 4.12. *For each* $T > 0$ *and* $\delta > 0$, *there exists* $\kappa_\delta \in (0, 1/2)$ *such that*

$$\|S(t)f\|_{L^2_T L^\infty_{xy}} \lesssim T^{\kappa_\delta} \left\| J^\delta f \right\|_{L^2} \tag{4.22}$$

where the implicit constant depends on δ.

PROOF. Taking p sufficiently large such that $\delta > 2/p$, Sobolev embedding and (4.11) yield

$$\|S(t)f\|_{L^2_T L^\infty} \lesssim_\delta T^{\frac{q-2}{2q}} \left\|S(t)J^\delta f\right\|_{L^q_T L^p_{xy}} \lesssim_\delta T^{\frac{q-2}{2q}} \left\|J^\delta f\right\|_{L^2_{xy}}. \tag{4.23}$$

This completes the proof. □

In addition, we require the following refined Strichartz estimate, which has been proved in a different context (see [11, 50, 59]).

Lemma 4.13. *Let $0 < \delta \leqslant 1$ and $T > 0$. Then there exist $\kappa_\delta \in (\frac{1}{2}, 1)$ and $\delta > 0$ such that*

$$\begin{aligned}\|v\|_{L^1_T L^\infty_{xy}} \lesssim_\delta T^{\kappa_\delta}\big(&\sup_{[0,T]}\|J_x^{1/2+2\delta}v(t)\|_{L^2_{xy}} + \sup_{[0,T]}\|J_x^{1/2+\delta}D_y^\delta v(t)\|_{L^2_{xy}} \\ &+ \int_0^T (\|J_x^{-1/2+2\delta}F(\cdot,t')\|_{L^2_{xy}} + \|J_x^{-1/2+\delta}D_y^\delta F(\cdot,t')\|_{L^2_{xy}})dt'\big),\end{aligned} \tag{4.24}$$

whenever v solves

$$\partial_t v + \mathcal{H}_x v - \mathcal{H}_x\partial_x^2 v \pm \mathcal{H}_x\partial_y^2 v = F. \tag{4.25}$$

PROOF. In view of Corollary 4.12, (4.24) is deduced following the same reasoning in the proof of Lemma 2.7 (see also, [59, Lemma 4.11] and [50, Lemma 1.7]). □

4.3.1.2. **Energy estimates.** Denoting by $X^\infty(\mathbb{R}^2) = \bigcap_{s\geqslant 0} X^s(\mathbb{R}^2)$, we have:

Lemma 4.14. *Let $s > 0$. Consider $T > 0$ and $u \in C([0,T]; H^\infty(\mathbb{R}^d))$ be a solution of the IVP (0.4). Then, there exists a positive constant c_0 such that*

$$\|u\|^2_{L^\infty_T H^s} \leqslant \|u_0\|^2_{H^s} + c_0\|\nabla u\|_{L^1_T L^\infty_x}\|u\|^2_{L^\infty_T H^s}. \tag{4.26}$$

Moreover, if $u \in C([0,T]; X^\infty(\mathbb{R}^d))$ solves the IVP (0.4), then there exists a constant $\widetilde{c}_0 > 0$ such that

$$\|u\|^2_{L^\infty_T X^s} \leqslant \|u_0\|^2_{X^s} + \widetilde{c}_0(\|u\|_{L^1_T L^\infty_{xy}} + \|\partial_x u\|_{L^1_T L^\infty_{xy}})\|u\|^2_{L^\infty_T X^s}. \tag{4.27}$$

PROOF. The estimates of the norms $\|J^s(\cdot)\|_{L^2_{xy}}$ and $\|J^s_x(\cdot)\|_{L^2_{xy}}$ are deduced applying the standard energy method implementing Lemma 1.1. This procedure was done in Lemma 2.8. However, we also invite the reader to see [11, Lemma 4.1] for the former norm and [50, Lemma 1.3] for the latter. This establishes (4.26).

Now, to deal with the component $\|D_x^{-1/2}(\cdot)\|_{L^2_{xy}}$ in the $X^s(\mathbb{R}^2)$-norm, we apply $D_x^{-1/2}$ to the equation in (0.4), we multiply then by $D_x^{-1/2}u$ and integrate in space to deduce

$$\frac{1}{2}\frac{d}{dt}\|D_x^{-1}u(t)\|^2_{L^2} = -\frac{1}{2}\int D_x^{-1/2}\partial_x(u^2)D_x^{-1/2}u\,dxdy, \tag{4.28}$$

where we have used that the operator $\mathcal{H}_x - \mathcal{H}_x\partial_x^2 \pm \mathcal{H}_x\partial_y^2$ is skew-symmetric. To estimate the integral term above, we write $\partial_x = -\mathcal{H}_x D_x$ to find

$$\begin{aligned}\left|\int D_x^{-1/2}\partial_x(u^2)D_x^{-1/2}u\,dxdy\right| = \left|\int \mathcal{H}_x(u^2)u\,dxdy\right| &= \left|\int u^2\mathcal{H}_x u\,dxdy\right| \\ &\lesssim \|u\|_{L^\infty_{xy}}\|u\|^2_{L^2_{xy}},\end{aligned} \tag{4.29}$$

Going back to (4.28), the last display shows

$$\frac{d}{dt}\|D_x^{-1}u(t)\|_{L^2_{xy}}^2 \lesssim \|u\|_{L^\infty_{xy}}\|u\|_{X^s}^2. \tag{4.30}$$

To control the norm $\|D_x^{-1/2}\partial_y(\cdot)\|_{L^2_{xy}}$, we apply $D_x\partial_y^{-1/2}$ to the equation in (0.4), multiplying the resulting expression by $D_x^{-1/2}\partial_y u$ and integrating in space, we deduce

$$\frac{1}{2}\frac{d}{dt}\left\|D_x^{-1}\partial_y u(t)\right\|_{L^2}^2 = -\frac{1}{2}\int D_x^{-1/2}\partial_y\partial_x(u^2)D_x^{-1/2}\partial_y u\,dxdy. \tag{4.31}$$

Once again, decomposing $\partial_x = -\mathcal{H}_x D_x$ and using that $\mathcal{H}_x$ is skew-symmetric, we get

$$\begin{aligned}\int D_x^{-1/2}\partial_y\partial_x(u^2)D_x^{-1/2}\partial_y u\,dxdy &= -\int \mathcal{H}_x\partial_y(u^2)\partial_y u\,dxdy\\ &= -\int([\mathcal{H}_x,u]\partial_y u)\partial_y u\,dxdy\\ &= \int(D_x^{1/2}[\mathcal{H}_x,u]D_x^{1/2}(D_x^{-1/2}\partial_y u))D_x^{-1/2}\partial_y u\,dxdy.\end{aligned} \tag{4.32}$$

Then the Cauchy-Schwarz inequality and Proposition 4.2 yield

$$\begin{aligned}&\left|\int(D_x^{1/2}[\mathcal{H}_x,u]D_x^{1/2}(D_x^{-1/2}\partial_y u))D_x^{-1/2}\partial_y u\,dxdy\right|\\ &\qquad\lesssim \|\|D_x^{1/2}[\mathcal{H}_x,u]D_x^{1/2}(D_x^{-1/2}\partial_y u)\|_{L^2_x}\|_{L^2_y}\|D_x^{-1/2}\partial_y u\|_{L^2_{xy}}\\ &\qquad\lesssim (\|u\|_{L^\infty_{xy}}+\|\partial_x u\|_{L^\infty_{xy}})\|D_x^{-1/2}\partial_y u\|_{L^2_{xy}}^2,\end{aligned} \tag{4.33}$$

and so we arrive at

$$\frac{d}{dt}\|D_x^{-1}\partial_y u(t)\|_{L^2_{xy}}^2 \lesssim (\|u\|_{L^\infty}+\|\partial_x u\|_{L^\infty})\|D_x^{-1/2}\partial_y u\|_{L^2_{xy}}^2. \tag{4.34}$$

Gathering all the above estimates for the components of the $X^s(\mathbb{R}^2)$-norm completes the proof. □

Next, we derive *a priori* estimates for the norms $\|u\|_{L^1_TL^\infty_{xy}}$ and $\|\nabla u\|_{L^1_TL^\infty_{xy}}$ in $H^s(\mathbb{R}^2)$, $s > 3/2$, and $\|u\|_{L^1_TL^\infty_{xy}}$ and $\|\partial_x u\|_{L^1_TL^\infty_{xy}}$ in $X^s(\mathbb{R}^2)$, $s > 3/2$.

Lemma 4.15. *Let $s > 3/2$ fixed.*

(i) *Consider $u \in C([0,T];H^\infty(\mathbb{R}^2))$ solution of the IVP (0.4). Then, there exist $\kappa_\delta \in (\frac{1}{2},1)$ and $c_s > 0$ such that*

$$h_1(T) := \|u\|_{L^1_TL^\infty_{xy}} + \|\nabla u\|_{L^1_TL^\infty_{xy}}. \tag{4.35}$$

satisfies

$$h_1(T) \leqslant c_s T^{\kappa_\delta}(1+h_1(T))\,\|u\|_{L^\infty_TH^s}. \tag{4.36}$$

(ii) *Assume that $u \in C([0,T];X^\infty(\mathbb{R}^2))$ solves the IVP (0.4). Then, there exist $\kappa_\delta \in (\frac{1}{2},1)$ and $c_s > 0$ such that*

$$h_2(T) := \|u\|_{L^1_TL^\infty_{xy}} + \|\partial_x u\|_{L^1_TL^\infty_{xy}}. \tag{4.37}$$

satisfies

$$h_2(T) \leqslant c_s T^{\kappa_\delta}(1+h_2(T))\,\|u\|_{L^\infty_TX^s}. \tag{4.38}$$

PROOF. Let us first deduce (i). In this case, we assume that $u \in C([0,T];H^{\infty}(\mathbb{R}^2))$ solves the IVP (0.4). We begin with the norm $\|\nabla u\|_{L^1_T L^\infty_x}$. Taking $F = -\nabla(u\partial_x u) = -\frac{1}{2}\partial_x \nabla(u^2)$ in (4.25) and using Lemma 4.13, we deduce

$$\|\nabla u\|_{L^1_T L^\infty_{xy}} \lesssim_\delta T^{\kappa_\delta}\big(\sup_{[0,T]}\|J^{3/2+2\delta}u(t)\|_{L^2} + \int_0^T \|J^{1/2+2\delta}(u\nabla u)(t')\|_{L^2}\,dt'\big), \tag{4.39}$$

where $0<\delta<s/2-3/4$. Our choice of δ implies that the first term on the right-hand side of (4.39) satisfies

$$\sup_{t\in[0,T]}\|J^{3/2+2\delta}u(t)\|_{L^2_{xy}} \leqslant \|u\|_{L^\infty_T H^s}. \tag{4.40}$$

On the other hand, applying Lemma 1.2 we find

$$\begin{aligned}
&\|J^{1/2+2\delta}(u\nabla u)\|_{L^2_{xy}}\\
&\quad\lesssim \|J^{1/2+2\delta}(u\partial_x u)\|_{L^2_{xy}} + \|J^{1/2+2\delta}(u\partial_y u)\|_{L^2_{xy}}\\
&\quad\lesssim \|u\|_{L^\infty_{xy}}\big(\|J^{1/2+2\delta}\partial_x u\|_{L^2_{xy}} + \|J^{1/2+2\delta}\partial_y u\|_{L^2_{xy}}\big) + \big(\|\partial_x u\|_{L^\infty_{xy}} + \|\partial_y u\|_{L^\infty_{xy}}\big)\|J^{1/2+2\delta}u\|_{L^2_{xy}}\\
&\quad\lesssim \Big(\|u\|_{L^\infty_{xy}} + \|\nabla u\|_{L^\infty_{xy}}\Big)\|u\|_{L^\infty_T H^s}.
\end{aligned} \tag{4.41}$$

Plugging (4.40) and (4.41) in (4.39), we arrive at

$$\|\nabla u\|_{L^1_T L^\infty_x} \lesssim T^{\kappa_\delta}(1+h_1(T))\,\|u\|_{L^\infty_T H^s}\,, \tag{4.42}$$

for some $\kappa_\delta \in (\frac{1}{2},1)$. Setting $F=-u\partial_x u$ in (4.25) and applying (4.24) and Lemma 1.11, the estimate for $\|u\|_{L^1_T L^\infty_{xy}}$ is obtained in a similar fashion as above. It is worth to notice that the resulting bound for this case can be controlled by the norm $\|u\|_{L^\infty_T H^{s-1}}$. This completes the deduction of (4.36).

Next, we proceed to deduce (ii) following the arguments in **[50]** and **[59]**. Here we assume that $u \in C([0,T],X^\infty(\mathbb{R}^2))$. In view of Lemma 4.13 with $F=-\partial_x(u\partial_x u)$, we find

$$\begin{aligned}
\|\partial_x u\|_{L^1_T L^\infty_{xy}} \lesssim_\delta T^{\kappa_\delta}\big(&\sup_{[0,T]}\|J_x^{3/2+2\delta}u(t)\|_{L^2_{xy}} + \sup_{[0,T]}\|J_x^{3/2+\delta}D_y^\delta u(t)\|_{L^2_{xy}}\\
&+\int_0^T \big(\|J_x^{1/2+2\delta}(u\partial_x u)(t')\|_{L^2_{xy}} + \|J_x^{1/2+\delta}D_y^\delta(u\partial_x u)(t')\|_{L^2_{xy}}\big)\,dt'\big).
\end{aligned} \tag{4.43}$$

We will derive bounds for each factor on the right-hand side of the above equation. Taking $\delta>0$ small such that $\frac{3}{2}\big(\frac{1+\delta}{1-\delta}\big)<s$, Young's inequality yields

$$(1+|\xi|)^{3/2+\delta}|\eta|^\delta \lesssim \big((1+|\xi|)^{3/2+\delta}|\xi|^{\delta/2}\big)^{1/(1-\delta)} + |\eta||\xi|^{-1/2} \lesssim (1+|\xi|)^s + |\eta||\xi|^{-1/2}. \tag{4.44}$$

By taking the same $\delta>0$ as above, the previous inequality and Plancherel's identity show

$$\begin{aligned}
&\sup_{[0,T]}\big(\|J_x^{3/2+2\delta}u(t)\|_{L^2_{xy}} + \|J_x^{3/2+\delta}D_y^\delta u(t)\|_{L^2_{xy}}\big)\\
&\qquad\lesssim \sup_{[0,T]}\big(\|J_x^s u(t)\|_{L^2_{xy}} + \|D_x^{-1/2}\partial_y u(t)\|_{L^2_{xy}}\big)\\
&\qquad\lesssim \|u\|_{L^\infty_T X^s}.
\end{aligned} \tag{4.45}$$

This completes the estimate for the first two terms on the right-hand side of (4.43). Next we deal with the third factor on the r.h.s of (4.43). An application of (1.11) allow us to deduce

$$\begin{aligned}\|J_x^{1/2+2\delta}(u\partial_x u)\|_{L^2_{xy}} &= \|\|J_x^{1/2+2\delta}(u\partial_x u)\|_{L^2_x}\|_{L^2_y}\\ &\lesssim \|\|u\|_{L^\infty_x}\|J_x^{1/2+2\delta}\partial_x u\|_{L^2_x}+\|\partial_x u\|_{L^\infty_x}\|J_x^{1/2+2\delta}u\|_{L^2_x}\|_{L^2_y}\\ &\lesssim (\|u\|_{L^\infty_{xy}}+\|\partial_x u\|_{L^\infty_{xy}})\|J^s_x u\|_{L^\infty_T L^2_{xy}},\end{aligned} \tag{4.46}$$

which holds for $0<\delta<\min\{1/2, s/2-3/4\}$. Using that $\|J^s_x u\|_{L^\infty_T L^2_{xy}} \leqslant \|u\|_{L^\infty_T X^s}$, integrating (4.46) between $[0,T]$ completes the analysis of $\int_0^T \|J_x^{1/2+2\delta}(u\partial_x u)(t')\|_{L^2_{xy}}\,dt'$. Next we decompose the last term on the r.h.s of (4.43) as follows

$$\begin{aligned}&\int_0^T \|J_x^{1/2+\delta}D_y^\delta(u\partial_x u)(t')\|_{L^2_{xy}}\,dt'\\ &\qquad\lesssim \int_0^T \|D_y^\delta(u\partial_x u)(t')\|_{L^2_{xy}}\,dt' + \int_0^T \|D_x^{1/2+\delta}D_y^\delta(u\partial_x u)(t')\|_{L^2_{xy}}\,dt'\\ &\qquad=: \mathcal{I}+\mathcal{II}.\end{aligned} \tag{4.47}$$

The fractional Leibniz's rule (1.10) shows

$$\begin{aligned}\mathcal{I} &= \int_0^T \|\|D_y^\delta(u\partial_x u)(t')\|_{L^2_y(\mathbb{R})}\|_{L^2_x(\mathbb{R})}\\ &\lesssim \int_0^T (\|u(t')\|_{L^\infty_{xy}}\|D^\delta_y\partial_x u(t')\|_{L^2_{xy}}+\|\partial_x u(t')\|_{L^\infty_{xy}}\|D^\delta_y u(t')\|_{L^2_{xy}})\,dt'.\end{aligned} \tag{4.48}$$

Therefore, from the point-wise estimate

$$|\xi|^l|\eta|^\delta = |\xi|^{l+\delta/2}(|\xi|^{-1/2}|\eta|)^\delta \lesssim (1+|\xi|)^{\frac{2l+\delta}{2(1-\delta)}}+|\xi|^{-1/2}|\eta|, \tag{4.49}$$

valid for $l=0,1$ and $0<\delta<1$ small satisfying $\frac{2l+\delta}{2(1-\delta)}<s$, we can apply Plancherel's identity to find

$$\begin{aligned}\mathcal{I} &\lesssim \Big(\int_0^T \|u(t')\|_{L^\infty_{xy}}+\|\partial_x u(t')\|_{L^\infty_{xy}}\,dt'\Big)\big(\|J^s_x u\|_{L^\infty_T L^2_{xy}}+\|D_x^{-1/2}\partial_y u\|_{L^\infty_T L^2_{xy}}\big)\\ &\lesssim \Big(\int_0^T \|u(t')\|_{L^\infty_{xy}}+\|\partial_x u(t')\|_{L^\infty_{xy}}\,dt'\Big)\|u\|_{L^\infty_T X^s}.\end{aligned} \tag{4.50}$$

On the other hand, employing Lemma 1.3, we further decompose $\mathcal{II}$ as follows

$$\begin{aligned}\mathcal{II} &\lesssim \int_0^T \|u(t')\|_{L^\infty_{xy}}\|D_x^{3/2+\delta}D^\delta_y u(t')\|_{L^2_{xy}}\,dt' + \int_0^T \|\partial_x u(t')\|_{L^\infty_{xy}}\|D_x^{1/2+\delta}D^\delta_y u(t')\|_{L^2_{xy}}\,dt'\\ &\quad+\int_0^T \|D_x^{1/2+\delta}u(t')\|_{L^\infty_{xy}}\|\partial_x D^\delta_y u(t')\|_{L^2_{xy}}\,dt' + \int_0^T \|\partial_x D_x^{1/2+\delta}u(t')\|_{L^{q_1}_{xy}}\|D^\delta_y u(t')\|_{L^{p_1}_{xy}}\,dt'\\ &= \mathcal{II}_1+\mathcal{II}_2+\mathcal{II}_3+\mathcal{II}_4,\end{aligned} \tag{4.51}$$

where $\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{2}$. Since the norms $\|D_x^{3/2+\delta}D^\delta_y u\|_{L^2_{xy}}, \|D_x^{1/2+\delta}D^\delta_y u\|_{L^2_{xy}} \leqslant \|J_x^{3/2+\delta}D^\delta_y u\|_{L^2_{xy}}$, we use (4.44) with $\frac{3}{2}\big(\frac{1+\delta}{1-\delta}\big)<s$ and Plancherel's identity to infer

$$\mathcal{II}_1+\mathcal{II}_2 \lesssim \Big(\int_0^T \|u(t')\|_{L^\infty_{xy}}+\|\partial_x u(t')\|_{L^\infty_{xy}}\,dt'\Big)\|u\|_{L^\infty_T X^s}.$$

To deal with $\mathcal{II}_3$, we let $0<\delta<1/2$ small satisfying $\frac{l+\delta}{2(1-\delta)}<s$, then we employ (1.13) to control the norm $\|D_x^{1/2+\delta}u\|_{L^\infty_{xy}}$. The estimate for $\|\partial_x D_y^{\delta}u\|_{L^2_{xy}}$ is a consequence of Plancherel's identity and (4.49) with $l=1$. Summarizing, it follows

$$\mathcal{II}_3 \lesssim \Big(\int_0^T \|u(t')\|_{L^\infty_{xy}}+\|\partial_x u(t')\|_{L^\infty_{xy}}\,dt'\Big)\|u\|_{L^\infty_T X^s}.$$

Next, by employing (1.14), (1.15) in Lemma 1.4, it is seen that

$$\begin{aligned}\mathcal{II}_4 &\lesssim \|\partial_x D_x^{1/2+\delta}u\|_{L^{s_1}_T L^{q_1}_{xy}}\|D_y^{\delta}u\|_{L^{r_1}_T L^{p_1}_{xy}}\\ &\lesssim \|\partial_x u\|^{\theta}_{L^1_T L^\infty_{xy}}\|J_x^{3/2+\delta_0}u\|^{1-\theta}_{L^\infty_T L^2_{xy}}\|u\|^{1-\theta}_{L^1_T L^\infty_{xy}}\big(\|D_y^{1/2}u\|_{L^\infty_T L^2_{xy}}+\|u\|_{L^\infty_T L^2_{xy}}\big)^{\theta},\end{aligned}$$

for some $0<\delta\ll 1$ and $0<\delta_0<s-3/2$ fixed. Given that

$$|\eta|^{1/2}=|\xi|^{1/4}(|\xi|^{-1/2}|\eta|)^{1/2}\lesssim |\xi|^{1/2}+|\xi|^{-1/2}|\eta|\lesssim (1+|\xi|)^s+|\xi|^{-1/2}|\eta|,$$

Plancherel's identity yields

$$\|D_y^{1/2}u\|_{L^\infty_T L^2_{xy}}+\|u\|_{L^\infty_T L^2_{xy}}\lesssim \|u\|_{L^\infty_T X^s}. \tag{4.52}$$

From this we get

$$\mathcal{II}_4 \lesssim \Big(\int_0^T \|u(t')\|_{L^\infty_{xy}}+\|\partial_x u(t')\|_{L^\infty_{xy}}\,dt'\Big)\|u\|_{L^\infty_T X^s}.$$

According to (4.51), this completes the estimate of $\mathcal{II}$. Collecting the bounds derived for $\mathcal{I}$ and $\mathcal{II}$, we obtain

$$\|\partial_x u\|_{L^1_T L^\infty_{xy}}\lesssim T^{\kappa_\delta}(1+h_2(T))\|u\|_{L^\infty_T X^s}.$$

To deal with $\|u\|_{L^1_T L^\infty_{xy}}$, we apply Lemma 4.13 with $F=-u\partial_x u=-\frac{1}{2}\partial_x(u^2)$ to get

$$\begin{aligned}\|u\|_{L^1_T L^\infty_{xy}}\lesssim_\delta T^{\kappa_\delta}\Big(&\sup_{[0,T]}\|J_x^{1/2+2\delta}u(t)\|_{L^2_{xy}}+\sup_{[0,T]}\|J_x^{1/2+\delta}D_y^{\delta}u(t)\|_{L^2_{xy}}\\ &+\int_0^T(\|J_x^{1/2+2\delta}(u^2)(t')\|_{L^2_{xy}}+\|J_x^{1/2+\delta}D_y^{\delta}(u^2)(t')\|_{L^2_{xy}})\,dt'\Big).\end{aligned}$$

From (4.45) it is deduced

$$\sup_{[0,T]}\big(\|J_x^{1/2+\delta}u(t)\|_{L^2_{xy}}+\|J_x^{1/2+\delta}D_y^{\delta}u(t)\|_{L^2_{xy}}\big)\lesssim \|u\|_{L^\infty_T X^s}. \tag{4.53}$$

On the other hand, applying (1.11) we find

$$\begin{aligned}\int_0^T(\|J_x^{1/2+2\delta}(u^2)(t')\|_{L^2_{xy}}+\|J_x^{1/2+\delta}D_y^{\delta}(u^2)(t')\|_{L^2_{xy}})\,dt') &\lesssim \int_0^T\|J^{1/2+2\delta}(u^2)(t')\|_{L^2_{xy}}\,dt'\\ &\lesssim \int_0^T\|u(t')\|_{L^\infty}\|J^{1/2+2\delta}u(t')\|_{L^2_{xy}}\,dt'.\end{aligned}$$

Taking $0<\delta<1/16$, Young's inequality establishes

$$|\eta|^{1/2+2\delta}=|\xi|^{(1+4\delta)/4}(|\xi|^{-1/2}|\eta|)^{(1+4\delta)/2}\lesssim (1+|\xi|)^{\frac{1+4\delta}{2(1-4\delta)}}+|\xi|^{-1/2}|\eta|, \tag{4.54}$$

thus an application of Plancherel's identity reveals

$$\begin{aligned}
&\int_0^T \|u(t')\|_{L^\infty}\|J^{1/2+2\delta}u(t')\|_{L^2_{xy}}\,dt' \\
&\lesssim \int_0^T \|u(t')\|_{L^\infty}\big(\|J_x^{1/2+2\delta}u(t')\|_{L^2_{xy}}+\|u(t')\|_{L^2_{xy}}+\|D_y^{1/2+2\delta}u(t')\|_{L^2_{xy}}\big)dt' \\
&\lesssim \Big(\int_0^T \|u(t')\|_{L^\infty}\,dt'\Big)\,\|u\|_{L^\infty_T X^s}.
\end{aligned} \tag{4.55}$$

Therefore, estimates (4.53) and (4.55) show that

$$\|u\|_{L^1_T L^\infty_{xy}} \lesssim_\delta T^{\kappa_\delta}(1+h_2(T))\|u\|_{L^\infty_T X^s}.$$

The proof is completed. □

Additionally, we require to control the norm $\|\partial_x^2 u\|_{L^1_T L^\infty_{xy}}$. This estimate will be useful to close the argument in the proof of Theorem 4.1 for space $X^s(\mathbb{R}^2)$

Lemma 4.16. *Let $T > 0$ and $u \in C([0,T]; X^\infty(\mathbb{R}^2))$ be a solution of the IVP (0.4). Then for all $s > 3/2$, there exist $\kappa_\delta \in (\frac{1}{2}, 1)$ and c_s such that*

$$\|\partial_x^2 u\|_{L^1_T L^\infty_{xy}} \leqslant c_s T^{\kappa_s}(1+h_2(T))\|u\|_{L^\infty_T X^{s+1}} + c_s T^{\kappa_s}\|\partial_x^2 u\|_{L^1_T L^\infty_{xy}}\|u\|_{L^\infty_T X^s},$$

where $h_2(T)$ is given as (4.37).

PROOF. Applying Lemma 4.13 with $F = -\partial_x\big(\partial_x u \partial_x u + u\partial_x^2 u\big)$, the proof of Lemma 4.16 follows the same arguments in the deduction of Lemma 4.15 (ii). □

4.3.2. LWP in $H^s(\mathbb{R}^2)$ and $X^s(\mathbb{R}^2)$, $s > 3/2$. This subsection concerns the deduction of Theorem 4.1. We begin by obtaining some a priori estimates.

4.3.2.1. **A priori estimates.** In this part we determine some key a priori estimates for smooth solutions. Our result relay on existence of smooth solutions for the IVP (0.4). To achieve this conclusion in the spaces $X^s(\mathbb{R}^2)$, we require the following lemma.

Lemma 4.17. *Let $s \geqslant 4$. Then it holds*

$$(\|u\|_{L^\infty_{xy}} + \|\partial_x u\|_{L^\infty_{xy}}) \lesssim \|u\|_{X^s}, \tag{4.56}$$

$$\Big|\int D_x^{-1/2}\partial_y^l(u\partial_x u)D_x^{-1/2}\partial_y^l u\,dxdy\Big| \lesssim (\|u\|_{L^\infty_{xy}} + \|\partial_x u\|_{L^\infty_{xy}})\|D_x^{-1/2}\partial_y^l u\|^2_{L^2_{xy}}, \tag{4.57}$$

for every $l = 0, 1$.

PROOF. We first notice that (4.57) is deduced by applying the same reasoning in (4.32) and (4.33), which relays on Proposition 4.2. Next, to deduce (4.56), we use Sobolev embedding in the variables x and y to get

$$\|\partial_x u\|_{L^\infty_{xy}} \lesssim \|J_x^{1/2+\epsilon}J_y^{1/2+\epsilon}\partial_x u\|_{L^2_{xy}} \lesssim \|J_x^{3/2+\epsilon}u\|_{L^2_{xy}} + \|J_x^{3/2+\epsilon}D_y^{1/2+\epsilon}u\|_{L^2_{xy}} \lesssim \|u\|_{X^s},$$

for any $0 < \epsilon \ll 1$ and $s \geqslant 4$, where we have used a similar estimate as in (4.54) and Plancherel's identity to estimate $\|J_x^{3/2+\epsilon}D_y^{1/2+\epsilon}u\|_{L^2}$. Since this same reasoning also applies to $\|u\|_{L^\infty_{xy}}$, we obtain (4.56). The proof is completed. □

Whenever $s > 2$, local well-posedness for the IVP (0.4) in $H^s(\mathbb{R}^2)$ follows from a parabolic regularization. This procedure was applied in [21] for the IVP (0.4) establishing LWP in $H^s(\mathbb{R}^2)$ for all $s > 2$.

Furthermore, by employing Lemma 4.17, it is possible to apply a parabolic regularization argument adapting the ideas in [21], [60, Chapter 10] or [47, Section 6.2] to obtain local well-posedness for the IVP (0.4) in $X^s(\mathbb{R}^2)$, $s \geqslant 4$. Summarizing the preceding discussion we have:

Lemma 4.18. *Let $s \geqslant 4$ and $\mathfrak{X}^s(\mathbb{R}^2)$ be any of the spaces $H^s(\mathbb{R}^2)$ and $X^s(\mathbb{R}^2)$. Then for any $u_0 \in \mathfrak{X}^s(\mathbb{R}^2)$, there exist $T = T(\|u_0\|_{\mathfrak{X}^s}) > 0$ and a unique solution $u \in C([0,T]; \mathfrak{X}^s(\mathbb{R}^d))$ of the IVP (0.4). In addition, the flow-map $u_0 \mapsto u(t)$ is continuous in the $\mathfrak{X}^s$-norm.*

As in the case of Proposition 2.11, the proof of Lemma 4.18 also provides existence of smooth solutions and a blow-up criterion. More precisely, let $u_0 \in \mathfrak{X}^\infty(\mathbb{R}^2)$, where $\mathfrak{X}^\infty(\mathbb{R}^2)$ is any of the spaces $H^\infty(\mathbb{R}^2)$ and $X^\infty(\mathbb{R}^2)$, then there exists a solution $u \in C([0,T^*); \mathfrak{X}^\infty(\mathbb{R}^2))$ to (0.4), where T^* is the maximal time of existence of u satisfying $T^* > T(\|u\|_{\mathfrak{X}^4}) > 0$ and the following blow-up alternative holds true

$$\lim_{t\to T^*} \|u(t)\|_{\mathfrak{X}^4} = \infty, \tag{4.58}$$

if $T^* < \infty$.

Next, we state some key *a priori* estimates.

Lemma 4.19. *Let $s \in (3/2, 4]$.*

(i) *Then there exists $A_s > 0$, such that for all $u_0 \in H^\infty(\mathbb{R}^2)$, there is a solution*
$$u \in C([0,T^*); H^\infty(\mathbb{R}^2))$$
of the IVP (0.4) where $T^* = T^*(\|u_0\|_{H^4}) > (1 + A_s \|u_0\|_{H^s})^{-2}$. Moreover, there exists a constant $K_0 > 0$ such that
$$\|u\|_{L^\infty_T H^s} \leqslant 2\|u_0\|_{H^s},$$
and
$$h_1(T) = \|u\|_{L^1_T L^\infty_{xy}} + \|\nabla u\|_{L^1_T L^\infty_{xy}} \leqslant K_0,$$
whenever $T \leqslant (1 + A_s\|u_0\|_{H^s})^{-2}$.

(ii) *Additionally, there exists $A_s > 0$, such that for all $u_0 \in X^\infty(\mathbb{R}^2)$, there is a solution $u \in C([0,T^*); X^\infty(\mathbb{R}^2))$ of the IVP (0.4) where $T^* = T^*(\|u_0\|_{X^4}) > (1 + A_s\|u_0\|_{X^s})^{-2}$. Moreover, there exists a constant $K_0 > 0$ such that*
$$\|u\|_{L^\infty_T X^s} \leqslant 2\|u_0\|_{X^s},$$
and
$$h_2(T) = \|u\|_{L^1_T L^\infty_{xy}} + \|\partial_x u\|_{L^1_T L^\infty_{xy}} \leqslant K_0,$$
whenever $T \leqslant (1 + A_s\|u_0\|_{X^s})^{-2}$.

PROOF. In view of Lemmas 4.14, 4.15, 4.18 and the blow-up criteria (4.58) applied to the H^4-norm or the X^4-norm respectively, the proof is obtained by the same reasoning in the deduction of Lemma 2.12, we also refer to [59, Lemma 5.3]. □

Now we can prove the existence of solutions.

4.3.2.2. **Existence of solution.** This part is devoted to establish the existence part of Theorem 4.1. We will employ the Bona-Smith argument [7]. Recalling the notation introduced in (1.7) and (4.10), we have:

Lemma 4.20. *Let $0 \leqslant \sigma \leqslant s$ and $M, N \in \mathbb{D} = \{2^l : l \in \mathbb{Z}^+ \cup \{0\}\}$ such that $M \geqslant N$. Assume that $u_0 \in H^s(\mathbb{R}^2)$, then*

$$N^{\sigma} \left\| J^{s-\sigma}(P_{\leqslant N}u_0 - P_{\leqslant M}u_0) \right\|_{L^2_{xy}} \underset{N\to\infty}{\to} 0, \tag{4.59}$$

for all $0 \leqslant \sigma \leqslant s$. Moreover, if $u_0 \in X^s(\mathbb{R}^2)$, then

$$N^{\sigma} \left\| J_x^{s-\sigma}(P^x_{\leqslant N}u_0 - P^x_{\leqslant M}u_0) \right\|_{L^2_{xy}} \underset{N\to\infty}{\to} 0 \tag{4.60}$$

and

$$\left\| D_x^{-1/2}\partial_y(P^x_{\leqslant N}u_0 - P^x_{\leqslant M}u_0) \right\|_{L^2_{xy}} \underset{N\to\infty}{\to} 0, \tag{4.61}$$

for each $0 \leqslant \sigma \leqslant s$.

PROOF. By support considerations we observe

$$|\langle(\xi,\eta)\rangle^{s-\sigma}(\psi_0(|(\xi,\eta)|/N) - \psi_0(|(\xi,\eta)|/M))\widehat{u}_0(\xi,\eta)|^2 \lesssim N^{-2(s-\sigma)}|\langle(\xi,\eta)\rangle^{s}\widehat{u}_0(\xi,\eta)|^2,$$

and

$$|\langle\xi\rangle^{s-\sigma}(\psi_0(\xi/N) - \psi_0(\xi/M))\widehat{u}_0(\xi,\eta)|^2 \lesssim N^{-2(s-\sigma)}|\langle\xi\rangle^{s}\widehat{u}_0(\xi,\eta)|^2.$$

Integrating the above expression, using Plancherel's identity and Lebesgue dominated convergence theorem when $\sigma = s$, we have that (4.59), (4.60) hold true. A closely similar argument provides (4.61). □

Now, we consider $s \in (3/2, 4]$ fixed. We have the following conclusions according to the spaces $H^s(\mathbb{R}^2)$ and $X^s(\mathbb{R}^2)$.

Initial data $u_0 \in H^s(\mathbb{R}^2)$. For each dyadic number $N \in \mathbb{D}$, Lemma 4.19 assures the existence of a time

$$T = (1 + A_s \|u_0\|_{H^s})^{-2} \tag{4.62}$$

(for some constant $A_s > 0$), independent of N and solutions $u_N \in C([0,T]; H^{\infty}(\mathbb{R}^2))$ of (0.4) with initial data $P_{\leqslant N}u_0$, such that

$$\|u_N\|_{L^{\infty}_t H^s} \leqslant 2\|u_0\|_{H^s} \tag{4.63}$$

and

$$K := \sup_{N\in\mathbb{D}} \left\{ \|u_N\|_{L^1_T L^{\infty}_{xy}} + \|\nabla u_N\|_{L^1_T L^{\infty}_{xy}} \right\} < \infty. \tag{4.64}$$

Let $M, N \in \mathbb{D}$, $M \geqslant N$, we set $w_{N,M} := u_N - u_M$, hence $w_{N,M}$ solves the equation

$$\partial_t w_{N,M} + \mathcal{H}_x w_{N,M} - \mathcal{H}_x\partial_x^2 w_{N,M} \pm \mathcal{H}_x\partial_y^2 w_{N,M} + \frac{1}{2}\partial_x((u_N + u_M)w_{N,M}) = 0, \tag{4.65}$$

with initial condition $w_{N,M}(0) = P_{\leqslant N}u_0 - P_{\leqslant M}u_0$. Therefore standard energy estimates, (4.64) and (4.60) reveal

$$N^s \|w_{N,M}\|_{L^{\infty}_T L^2_{xy}} \lesssim e^{cK}(N^s \|P_N u_0 - P_M u_0\|_{L^2}) \underset{N\to\infty}{\to} 0.$$

Thus, interpolating the last result with (4.59) yields

$$N^{s-\sigma}\left\|J^{\sigma}w_{N,M}\right\|_{L_T^\infty L_{xy}^2} \leqslant N^{s-\sigma}\left\|J^{s}w_{N,M}\right\|_{L_T^\infty L_{xy}^2}^{\sigma/s}\left\|w_{N,M}\right\|_{L_T^\infty L_{xy}^2}^{1-\sigma/s} \underset{N\to\infty}{\to} 0, \tag{4.66}$$

whenever $0 \leqslant \sigma < s$.

__Initial data $u_0 \in X^s(\mathbb{R}^2)$.__ For each dyadic number $N \in \mathbb{D}$, Lemma 4.19 assures existence of a time $T = (1 + \tilde{A}_s \left\|u_0\right\|_{X^s})^{-2}$ (for some constant $\tilde{A}_s > 0$) independent of N and smooth solutions $v_N \in C([0,T]; X^\infty(\mathbb{R}^2))$ of (0.4) with initial data $P^x_{\leqslant N}u_0$ such that

$$\left\|v_N\right\|_{L_T^\infty X^s} \leqslant 2\left\|u_0\right\|_{X^s} \tag{4.67}$$

and

$$K_1 := \sup_{N\in\mathbb{D}}\left\{\left\|v_N\right\|_{L_T^1 L_{xy}^\infty} + \left\|\partial_x v_N\right\|_{L_T^1 L_{xy}^\infty}\right\} < \infty. \tag{4.68}$$

Additionally, we combine Lemma 4.16, (4.67) and (4.68) to infer

$$\left\|\partial_x^2 v_N\right\|_{L_T^1 L_{xy}^\infty} \lesssim \left\|v_N\right\|_{L_T^\infty X^{s+1}} \tag{4.69}$$

provided that A_s is chosen large enough. Now, let $M, N \in \mathbb{D}$, since $\widetilde{w}_{N,M} = v_N - v_M$ satisfies (4.65), employing similar energy estimates leading to (4.27) together with (4.60), we deduce

$$N^{s-\sigma}\left\|J_x^{\sigma}(v_N - v_M)\right\|_{L_T^\infty L_{xy}^2} \underset{N\to\infty}{\to} 0, \tag{4.70}$$

whenever $0 \leqslant \sigma < s$.

According to the preceding discussions, when $u_0 \in H^s(\mathbb{R}^2)$, we shall prove that $\{u_N\}_{N\in\mathbb{D}}$ is a Cauchy sequence in $C([0,T]; H^s(\mathbb{R}^2)) \cap L^1([0,T], W^{1,\infty}(\mathbb{R}^2))$.

Additionally, in the case $u_0 \in X^s(\mathbb{R}^2)$, we will establish that $\{v_N\}_{N\in\mathbb{D}}$ is a Cauchy sequence in $C([0,T]; X^s(\mathbb{R}^2)) \cap L^1([0,T], W_x^{1,\infty}(\mathbb{R}^2))$. We first obtain some estimates for $\{u_N\}$ and $\{v_N\}$ in the $\|\cdot\|_{L_T^1 W^{1,\infty}}$ and $\|\cdot\|_{L_T^1 W_x^{1,\infty}}$ norms respectively.

Lemma 4.21. *Let $M, N \in \mathbb{D}$, $M \geqslant N$.*

(i) *If $u_0 \in H^s(\mathbb{R}^2)$, $s \in (3/2, 4]$, then*

$$N\left\|u_N - u_M\right\|_{L_T^1 L_{xy}^\infty} + \left\|\nabla(u_N - u_M)\right\|_{L_T^1 L_{xy}^\infty} \underset{N\to\infty}{\to} 0, \tag{4.71}$$

provided that $T = T(\left\|u_0\right\|_{H^s}) > 0$ in (4.62) is chosen sufficiently small.

(ii) *If $u_0 \in X^s(\mathbb{R}^2)$, $s \in (3/2, 4]$, then*

$$\left\|v_N - v_M\right\|_{L_T^1 L_{xy}^\infty} \underset{N\to\infty}{=} o(N^{-1}) + O(N^{-1}\left\|D_x^{-1/2}\partial_y(v_N - v_M)\right\|_{L_T^\infty L_{xy}^2}) \tag{4.72}$$

and

$$\left\|\partial_x(v_N - v_M)\right\|_{L_T^1 L_{xy}^\infty} \underset{N\to\infty}{=} o(1) + O(\left\|D_x^{-1/2}\partial_y(v_N - v_M)\right\|_{L_T^\infty L_{xy}^2}), \tag{4.73}$$

for a time $T = (1 + \widetilde{A}\left\|u_0\right\|_{X^s})^{-2}$ sufficiently small.

PROOF. We first prove (4.71). Since $w_{N,M} = u_N - u_M$ satisfies (4.65), we employ Lemma 4.13 with $F = -\frac{1}{2}(u_N + u_M)w_{N,M}$ to get

$$\left\|u_N - u_M\right\|_{L_T^1 L_{xy}^\infty} \lesssim T^{1/2}\Big(\left\|J^{1/2+2\delta}w_{N,M}\right\|_{L_T^\infty L_{xy}^2} + \int_0^T \left\|J^{1/2+2\delta}((u_N + u_M)w_{N,M})(t')\right\|_{L_{xy}^2} dt'\Big), \tag{4.74}$$

where $0<\delta<s/2-3/4$ is fixed. It follows from (4.68) and our choice of δ,

$$N\|J^{1/2+2\delta}w_{N,M}\|_{L^\infty_T L^2_{xy}} \to 0, \tag{4.75}$$

as $N\to\infty$. Now, applying (1.11), we estimate the second term on the r.h.s of (4.74) as

$$\begin{aligned}
&T^{1/2}\int_0^T \|J^{1/2+2\delta}((u_N+u_M)w_{N,M})(t')\|_{L^2_{xy}}\,dt'\\
&\lesssim T^{1/2}\int_0^T \|J^{1/2+2\delta}(u_N+u_M)(t')\|_{L^2_{xy}}\|w_{N,M}(t')\|_{L^\infty_{xy}}\\
&\qquad\qquad + \|(u_N+u_M)(t')\|_{L^\infty_{xy}}\|J^{1/2+2\delta}w_{N,M}(t')\|_{L^2_{xy}}\,dt'\\
&\lesssim T^{1/2}\left(\|u_N\|_{L^\infty_T H^s}+\|u_M\|_{L^\infty_T H^s}\right)\|w_{N,M}\|_{L^1_T L^\infty_{xy}}\\
&\qquad\qquad + T^{1/2}\left(\|u_N\|_{L^1_T L^\infty_{xy}}+\|u_M\|_{L^1_T L^\infty_{xy}}\right)\|J^{1/2+2\delta}w_{N,M}\|_{L^\infty_T L^2_{xy}}.
\end{aligned} \tag{4.76}$$

In virtue of (4.64) and (4.75), the second term on the right-hand side of (4.76) satisfies the required decay as $N\to\infty$. Now taking T sufficiently small such that $T^{1/2}\|u_0\|_{H^s}\ll 1$ with respect to the implicit constant in (4.76) (which is independent of N and depends on s), we can absorb the first term on the r.h.s of (4.76) by the first term on the left-hand side of (4.74). This establishes

$$N\|u_N-u_M\|_{L^1_T L^\infty_{xy}} \underset{N\to\infty}{=} 0$$

On the other hand, using Lemma 4.13 with $F=-\frac{1}{2}\nabla((u_N+u_M)w_{N,M})$, it is seen that

$$\begin{aligned}
\|\nabla(u_N-u_M)\|_{L^1_T L^\infty_{xy}} \lesssim T^{1/2}\Big(&\|J^{3/2+2\delta}w_{N,M}\|_{L^\infty_T L^2_{xy}}\\
&+\int_0^T \|J^{1/2+2\delta}\partial_x((u_N+u_M)w_{N,M})(t')\|_{L^2_{xy}}\,dt'\Big),
\end{aligned} \tag{4.77}$$

where $0<\delta<s/2-3/4$. Clearly, (4.66) and our choice of $\delta>0$ implies that $\|J^{3/2+2\delta}w_{N,M}\|_{L^\infty_T L^2_{xy}}\to 0$ as $N\to\infty$. In virtue of (1.11) and arguing as in (4.76), we estimate the second term on the r.h.s of (4.77) as follows

$$\begin{aligned}
&\int_0^T \|J^{1/2+2\delta}\partial_x((u_N+u_M)w_{N,M})(t')\|_{L^2_{xy}}\,dt'\\
&\lesssim \left(\|u_N\|_{L^\infty_T H^s}+\|u_M\|_{L^\infty_T H^s}\right)\|w_{N,M}\|_{L^1_T L^\infty_{xy}}\\
&\qquad + \left(\|u_N\|_{L^1_T L^\infty_{xy}}+\|u_M\|_{L^1_T L^\infty_{xy}}\right)\|J^{3/2+2\delta}w_{N,M}\|_{L^\infty_T L^2_{xy}}.
\end{aligned} \tag{4.78}$$

The last display, (4.63) and the fact that $\|u_N-u_M\|_{L^1_T L^\infty_{xy}}, \|J^{3/2+2\delta}w_{N,M}\|_{L^\infty_T L^2_{xy}}\to 0$ as $N\to\infty$ complete the deduction of (4.75).

Next, we proceed to estimate (4.72) and (4.73). Since both of these estimates are inferred as in the proof of Lemma 4.15, we will only deduce (4.72). Let us denote by $\widetilde{w}_{N,M}=v_N-v_M$, then $\widetilde{w}_{N,M}$ satisfies (4.65) with $\widetilde{w}_{N,M}(0)=P^x_{\leqslant N}u_0-P^x_{\leqslant M}u_0$. Applying Lemma 4.13 with $F=$

$-\frac{1}{2}\partial_x((v_N+v_M)\widetilde{w}_{N,M}))$ we get

$$\begin{aligned}
&\|v_N - v_M\|_{L^1_T L^\infty_{xy}} \\
&\quad \lesssim_\delta T^{1/2}\big(\sup_{[0,T]}\|J_x^{1/2+2\delta}\widetilde{w}(t)\|_{L^2_{xy}}+\sup_{[0,T]}\|J_x^{1/2+\delta}D_y^\delta\widetilde{w}(t)\|_{L^2_{xy}} \\
&\quad +\int_0^T(\|J_x^{1/2+2\delta}((v_N+v_M)\widetilde{w}_{N,M})(t')\|_{L^2_{xy}}+\|J_x^{1/2+\delta}D_y^\delta((v_N+v_M)\widetilde{w}_{N,M})(t')\|_{L^2_{xy}})\,dt'\big) \\
&\quad =: T^{1/2}(\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4),
\end{aligned} \tag{4.79}$$

for some $0<\delta<\delta_0$ with δ_0 to be determined during the proof. Now we estimate each of the factors $\mathcal{I}_j$.

In view of (4.70), it follows

$$N\mathcal{I}_1 \underset{N\to\infty}{\to} 0,$$

whenever $0<\delta<\delta_0<s/2-3/4$. To study $\mathcal{I}_2$, we employ Young's inequality to derive

$$(1+|\xi|)^{1/2+\delta}|\eta|^\delta \lesssim N^{\frac{\delta}{1-\delta}}(1+|\xi|)^{\frac{1+3\delta}{2(1-\delta)}}+N^{-1}|\eta||\xi|^{-1/2}. \tag{4.80}$$

Plancherel's identity shows

$$\mathcal{I}_2 \lesssim N^{\frac{\delta}{1-\delta}}\|J_x^{\frac{1+3\delta}{2(1-\delta)}}\widetilde{w}_{N,M}\|_{L^\infty_T L^2_{xy}}+N^{-1}\|D_x^{-1/2}\partial_y\widetilde{w}_{N,M}\|_{L^\infty_T L^2_{xy}}. \tag{4.81}$$

Therefore, choosing $0<\delta<\delta_0<1$, where δ_0 is small satisfying $\frac{1+5\delta_0}{2(1-\delta_0)}<s-1$, we have from (4.70) and (4.81) that

$$\mathcal{I}_2 \underset{N\to\infty}{=} o(N^{-1})+O(N^{-1}\|D_x^{-1/2}\partial_y\widetilde{w}_{N,M}\|_{L^\infty_T L^2_{xy}}).$$

Next, we follow the arguments in (4.46) employing (1.11) to deduce

$$\begin{aligned}
\mathcal{I}_3 \lesssim&(\|J_x^{1/2+2\delta}v_N\|_{L^\infty_T L^2_{xy}}+\|J_x^{1/2+2\delta}v_M\|_{L^\infty_T L^2_{xy}})\|v_N-v_M\|_{L^1_T L^\infty_{xy}} \\
&+(\|v_N\|_{L^1_T L^\infty_{xy}}+\|v_M\|_{L^1_T L^\infty_{xy}})\|J_x^{1/2+2\delta}(v_N-v_M)\|_{L^\infty_T L^2_{xy}}.
\end{aligned}$$

Then the above inequality, (4.67), (4.68) and (4.70) show

$$\mathcal{I}_3 \underset{N\to\infty}{=} O(\|u_0\|_{X^s}\|v_N-v_M\|_{L^1_T L^\infty_{xy}})+o(N^{-1}),$$

for all $0<\delta<\delta_0$, where $\delta_0<s/2-3/4$. Now, we divide the remaining term $\mathcal{I}_4$ as follows

$$\begin{aligned}
\mathcal{I}_4 &\lesssim \int_0^T\|D_y^\delta((v_N+v_M)\widetilde{w}_{N,M})(t')\|_{L^2_{xy}}\,dt'+\int_0^T\|D_x^{1/2+\delta}D_y^\delta((v_N+v_M)\widetilde{w}_{N,M})(t')\|_{L^2_{xy}}\,dt' \\
&=: \mathcal{I}_{4,1}+\mathcal{I}_{4,2}.
\end{aligned}$$

By employing the fractional Leibniz's rule (1.10) in the y-variable we get

$$\begin{aligned}
\mathcal{I}_{4,1} \lesssim&(\|D_y^\delta v_N\|_{L^\infty_T L^2_{xy}}+\|D_y^\delta v_M\|_{L^\infty_T L^2_{xy}})\|v_N-v_M\|_{L^1_T L^\infty_{xy}} \\
&(\|v_N\|_{L^1_T L^\infty_{xy}}+\|v_M\|_{L^1_T L^\infty_{xy}})\|D_y^\delta(v_N-v_M)\|_{L^\infty_T L^2_{xy}}.
\end{aligned}$$

Consequently, choosing $0<\delta<\delta_0<1$ such that $\frac{3\delta_0}{2(1-\delta_0)}<s$, we use (4.67) and a similar argument to (4.80) to find

$$\mathcal{I}_{4,1} \underset{N\to\infty}{=} O(\|u_0\|_{X^s}\|v_N-v_M\|_{L^1_T L^\infty_{xy}})+o(N^{-1})+O(N^{-1}\|D_x^{-1/2}\partial_y(v_N-v_M)\|_{L^\infty_T L^2_{xy}}).$$

On the other hand, from Lemma 1.3 it is deduced

$$\begin{aligned}\mathcal{I}_{4,2} \lesssim &(\|v_N\|_{L^1_T L^\infty_{xy}} + \|v_M\|_{L^1_T L^\infty_{xy}})\|D_x^{1/2+\delta} D_y^\delta (v_N - v_M)\|_{L^\infty_T L^2_{xy}} \\ &+ (\|D_x^{1/2+\delta} D_y^\delta v_N\|_{L^\infty_T L^2_{xy}} + \|D_x^{1/2+\delta} D_y^\delta v_M\|_{L^\infty_T L^2_{xy}})\|v_N - v_M\|_{L^1_T L^\infty_{xy}} \\ &+ (\|D_x^{1/2+\delta} v_N\|_{L^1_T L^\infty_{xy}} + \|D_x^{1/2+\delta} v_M\|_{L^1_T L^\infty_{xy}})\|D_y^\delta (v_N - v_M)\|_{L^\infty_T L^2_{xy}} \\ &+ (\|D_y^\delta v_N\|_{L^{r_1}_T L^{p_1}_{xy}} + \|D_y^\delta v_M\|_{L^{r_1}_T L^{p_1}_{xy}})\|D_x^{1/2+\delta}(v_N - v_M)\|_{L^{s_1}_T L^{q_1}_{xy}} \\ &=: \mathcal{I}_{4,2,1} + \mathcal{I}_{4,2,2} + \mathcal{I}_{4,2,3} + \mathcal{I}_{4,2,4},\end{aligned}$$

where $1 < r_1, s_1 < \infty$ and $2 < p_1, q_1 < \infty$ satisfy the conditions of Lemma 1.4 (ii). An application of (4.80) shows

$$\mathcal{I}_{4,2,1} \underset{N\to\infty}{=} o(N^{-1}) + O(N^{-1}\|D_x^{-1/2}\partial_y(v_N - v_M)\|_{L^\infty_T L^2_{xy}}),$$

for each $0 < \delta < \delta_0 < 1$, where δ_0 is small satisfying $\frac{1+5\delta_0}{2(1-\delta_0)} < s - 1$. Now, we combine estimate (4.45) and (4.67) to derive

$$\mathcal{I}_{4,2,2} \lesssim \|u_0\|_{X^s}\|v_N - v_M\|_{L^1_T L^\infty_{xy}}.$$

Additionally, by employing (1.13) and identity (4.49) with $l = 0$, it is not difficult to see

$$\mathcal{I}_{4,2,3} \underset{N\to\infty}{=} o(N^{-1}) + O(N^{-1}\|D_x^{-1/2}\partial_y(v_N - v_M)\|_{L^\infty_T L^2_{xy}}),$$

for all $0 < \delta < \delta_0 < 1$ and $\frac{\delta_0}{2(1-\delta_0)} < s$. Finally, gathering together estimates (1.14), (1.15), (4.67) and (4.68) we deduce

$$\mathcal{I}\mathcal{I}_{4,2,4} \lesssim K^{1-\theta}\|u_0\|^\theta_{X^s}\|v_N - v_M\|^\theta_{L^1_T L^\infty_{xy}}\|J_x^{1/2+\delta_0}(v_N - v_M)\|^{1-\theta}_{L^1_T L^\infty_{xy}},$$

so that Young's inequality and (4.70) yield

$$\mathcal{I}_{4,2,4} \underset{N\to\infty}{=} o(N^{-1}) + O(\|u_0\|_{X^s}\|v_N - v_M\|_{L^1_T L^\infty_{xy}}),$$

for all $0 < \delta \leqslant \delta_0$ and $0 < \delta_0 \ll 1$ given by Lemma 1.4. Collecting all of the preceding estimates

$$\mathcal{I}_4 \underset{N\to\infty}{=} o(N^{-1}) + O(\|u_0\|_{X^s}\|v_N - v_M\|_{L^1_T L^\infty_{xy}}) + O(N^{-1}\|D_x^{-1/2}\partial_y(v_N - v_M)\|_{L^\infty_T L^2_{xy}}).$$

Plugging the previous estimates for the terms $\mathcal{I}_j$, $j = 1, \dots, 4$ in (4.79), we obtain

$$\|v_N - v_M\|_{L^1_T L^\infty_{xy}} \underset{N\to\infty}{=} o(N^{-1}) + O(T^{1/2}\|u_0\|_{X^s}\|v_N - v_M\|_{L^1_T L^\infty_{xy}}) + O(N^{-1}\|D_x^{-1/2}\partial_y(v_N - v_M)\|_{L^\infty_T L^2_{xy}}).$$

This completes the deduction of (4.72) provided that $T = (1 + \widetilde{A}\|u_0\|_{X^s})^{-2}$ is chosen sufficiently small. □

Next, we shall prove that $\{u_N\}$ is a Cauchy sequence in $C([0,T]; H^s(\mathbb{R}^2))$ and $\{v_N\}$ is a Cauchy sequence in $C([0,T]; X^s(\mathbb{R}^2))$.

PROPOSITION 4.22. *Let $M, N \in \mathbb{D}$, $M \geqslant N$.*

(i) *If $u_0 \in H^s(\mathbb{R}^2)$, $s \in (3/2, 4]$, then*

$$\|u_N - u_M\|_{L^\infty_T H^s} \underset{N\to\infty}{\to} 0. \tag{4.82}$$

(ii) *If $u_0 \in X^s(\mathbb{R}^2)$, $s \in (3/2, 4]$, then*

$$\|J_x^s(v_N - v_M)\|_{L^\infty_T L^2_{xy}} + \|D_x^{-1/2}(v_N - v_M)\|_{L^\infty_T L^2_{xy}} + \|D_x^{-1/2}\partial_y(v_N - v_M)\|_{L^\infty_T L^2_{xy}} \underset{N\to\infty}{\to} 0. \tag{4.83}$$

PROOF. Let us first deduce (4.82). We apply J^s to (4.65), rewriting the nonlinearity as

$$\frac{1}{2}\partial_x((u_N+u_M)w_{N,M}) = u_M\partial_x w_{N,M} + \partial_x u_N w_{N,M}.$$

and then multiplying by $J^s w_{N,M}$ and integrating the resulting expression in space, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|J^s(u_N-u_M)(t)\|_{L^2_{xy}}^2 &= -\int J^s(u_M\partial_x w_{N,M})J^s w_{N,M}\,dxdy - \int J^s(\partial_x u_N w_{N,M})J^s w_{N,M}\,dxdy \\ &=: -(\mathcal{I}+\mathcal{II}).\end{aligned}$$

Integrating by parts,

$$\mathcal{I} = \int [J^s,u_M]\partial_x w_{N,M} J^s w_{N,M}\,dxdy - \frac{1}{2}\int \partial_x u_M (J^s w_{N,M})^2\,dxdy,$$

this implies together with Lemma 1.1 ,

$$\begin{aligned}|\mathcal{I}| &\lesssim \|[J^s,u_M]\partial_x w_{N,M}\|_{L^2_{xy}}\|J^s w_{N,M}\|_{L^2_{xy}} + \|\nabla u_M\|_{L^2_{xy}}\|J^s w_{N,M}\|_{L^2_{xy}}^2 \\ &\lesssim \|J^s u_M\|_{L^2_{xy}}\|\nabla w_{N,M}\|_{L^\infty_{xy}}\|J^s w_{N,M}\|_{L^2_{xy}} + \|\nabla u_M\|_{L^\infty_{xy}}\|J^s w_{N,M}\|_{L^2_{xy}}^2.\end{aligned} \tag{4.84}$$

On the other hand,

$$\mathcal{II} = \int [J^s,w_{N,M}]\partial_x u_N J^s w_{N,M}\,dxdy + \int w_{N,M}(\partial_x J^s u_N)J^s w_{N,M}\,dxdy,$$

then Lemma 1.1 gives

$$\begin{aligned}|\mathcal{II}| \lesssim&\|[J^s,w_{N,M}]\partial_x u_N\|_{L^2_{xy}}\|J^s w_{N,M}\|_{L^2_{xy}} + \|w_{N,M}\|_{L^\infty_{xy}}\|J^{s+1}u_N\|_{L^2_{xy}}\|J^s w_{N,M}\|_{L^2_{xy}} \\ \lesssim&\|\nabla w_{N,M}\|_{L^\infty_{xy}}\|J^s u_N\|_{L^2_{xy}}\|J^s w_{N,M}\|_{L^2_{xy}} + \|\nabla u_N\|_{L^\infty_{xy}}\|J^s w_{N,M}\|_{L^2_{xy}}^2 \\ &+ \|w_{N,M}\|_{L^\infty_{xy}}\|J^{s+1}u_N\|_{L^2_{xy}}\|J^s w_{N,M}\|_{L^2_{xy}}.\end{aligned} \tag{4.85}$$

To control $\|J^{s+1}u_N\|_{L^2}$, we employ the fact that u_N solves the IVP (0.4) and standard energy estimates relaying on Lemma 1.1 to find

$$\|J^{s+1}u_N(t)\|_{L^2_{xy}} \leqslant e^{c\|\nabla u_N\|_{L^1_T L^\infty_{xy}}}\|J^{s+1}P_{\leqslant N}u_0\|_{L^2_{xy}} \lesssim Ne^{cK}\|J^s u_0\|_{L^2_{xy}}, \tag{4.86}$$

where we have also used Gronwall's inequality and (4.64). Therefore, gathering (4.84)-(4.86), (4.63) and (4.64), we find

$$\|J^s(u_N(t)-u_M)\|_{L^\infty_T L^2_{xy}} \lesssim e^{cK}\big(\|J^s(P_{\leqslant N}u_0 - P_{\leqslant M}u_0)\|_{L^2_{xy}} + \|\nabla w_{N,M}\|_{L^1_T L^\infty_{xy}} + N\|w_{N,M}\|_{L^1_T L^\infty_{xy}}\big) \underset{N\to\infty}{\to} 0,$$

which holds in virtue of (4.71). This completes the deduction of (4.82).

Next, we prove (4.83). Replacing J^s by J^s_x and ∇ by ∂_x and using (4.67), (4.68) and the inequality

$$\|J_x^{s+1}v_N\|_{L^\infty_T L^2_{xy}} \lesssim \|J_x^{s+1}P^x_{\leqslant N}u_0\|_{L^2_{xy}} \lesssim N\|J^s_x u_0\|_{L^2_{xy}}, \tag{4.87}$$

the estimate for the norm $\|J^s_x(v_N(t)-v_M)(t)\|_{L^2_{xy}}$ follows the same arguments in the deduction of (4.68). Now, setting $\widetilde{w}_{N,M} = v_N - v_M$, we have that $\widetilde{w}_{N,M}$ solves the equation (4.65) with initial condition $\widetilde{w}_{N,M}(0) = P^x_{\leqslant N}u_0 - P^x_{\leqslant M}u_0$. Applying $D_x^{-1/2}$ to this equation and then multiplying by $D_x^{-1/2}\widetilde{w}_{N,M}$ and integrating in space, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|D_x^{-1/2}\widetilde{w}_{N,M}(t)\|^2_{L^2_{xy}} &= -\int D_x^{-1/2}\partial_x((v_N+v_M)\widetilde{w}_{N,M})D_x^{-1/2}\widetilde{w}_{N,M}\,dxdy\\ &= -\int D_x^{-1}\partial_x((v_N+v_M)\widetilde{w}_{N,M})\widetilde{w}_{N,M}\,dxdy.\end{aligned}$$

Writing $\partial_x = -\mathcal{H}_x D_x$ and using that $\mathcal{H}_x$ determines a skew-symmetric operator, it is seen

$$\begin{aligned}&\int D_x^{-1}\partial_x((v_N+v_M)\widetilde{w}_{N,M})\widetilde{w}_{N,M}\,dxdy\\ &\qquad= \frac{1}{2}\int [\mathcal{H}_x, v_N+v_M]\widetilde{w}_{N,M}\widetilde{w}_{N,M}\,dxdy\\ &\qquad= \frac{1}{2}\int (D_x^{1/2}[\mathcal{H}_x, v_N+v_M]D_x^{1/2}(D_x^{-1/2}\widetilde{w}_{N,M}))D_x^{-1/2}\widetilde{w}_{N,M}\,dxdy,\end{aligned}$$

so that Proposition 4.2 applied to the x-variable gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|D_x^{-1/2}\widetilde{w}_{N,M}(t)\|^2_{L^2_{xy}} &\lesssim \|D_x^{1/2}[\mathcal{H}_x, v_N+v_M]D_x^{1/2}(D_x^{-1/2}\widetilde{w}_{N,M})\|_{L^2_{xy}}\|D_x^{-1/2}\widetilde{w}_{N,M}\|_{L^2_{xy}}\\ &\lesssim (\|v_N+v_M\|_{L^\infty_{xy}} + \|\partial_x(v_N+v_M)\|_{L^\infty_{xy}})\|D_x^{-1/2}\widetilde{w}_{N,M}\|^2_{L^2_{xy}}.\end{aligned}$$

Therefore, the preceding estimate, Gronwall's inequality and (4.68) imply

$$\|D_x^{-1/2}(v_N-v_M)\|_{L^\infty_T L^2_{xy}} \lesssim e^{cK_1}\|D_x^{-1/2}(P^x_{\leqslant N}u_0 - P^x_{\leqslant M}u_0)\|_{L^2_{xy}} \underset{N\to\infty}{\to} 0.$$

Finally, we proceed to estimate the norm $\|D_x^{-1/2}\partial_y(v_N-v_M)\|_{L^\infty_T L^2_{xy}}$. Since $\widetilde{w}_{N,M} = v_N - v_M$ solves (4.65), we apply $D_x^{-1/2}\partial_y$ to this equation multiplying by $D_x^{-1/2}\partial_y\widetilde{w}_{N,M}$, then integrating in space it follows

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|D_x^{-1/2}\partial_y\widetilde{w}_{N,M}(t)\|^2_{L^2_{xy}}\\ &\quad= -\int D_x^{-1/2}\partial_y\partial_x((v_N+v_M)\widetilde{w}_{N,M})D_x^{-1/2}\partial_y\widetilde{w}_{N,M}\,dxdy\\ &\quad= \int \mathcal{H}_x((v_N+v_M)\partial_y\widetilde{w}_{N,M})\partial_y\widetilde{w}_{N,M}\,dxdy + \int \mathcal{H}_x(\partial_y(v_N+v_M)\widetilde{w}_{N,M})\partial_y\widetilde{w}_{N,M}\,dxdy\\ &\quad=: \mathbb{I} + \mathbb{II},\end{aligned}\tag{4.88}$$

where we have employed the decomposition $\partial_x = -\mathcal{H}_x D_x$. Since $\mathcal{H}_x$ is a skew-symmetric operator, we have

$$\begin{aligned}\mathbb{I} &= \frac{1}{2}\int [\mathcal{H}_x, v_N+v_M]\partial_y\widetilde{w}_{N,M}\partial_y\widetilde{w}_{N,M}\,dx\\ &= \frac{1}{2}\int (D_x^{1/2}[\mathcal{H}_x, v_N+v_M]D_x^{1/2}(D_x^{-1/2}\partial_y\widetilde{w}_{N,M}))(D_x^{-1/2}\partial_y\widetilde{w}_{N,M})\,dx,\end{aligned}$$

then in view of Proposition 4.2 it follows

$$\begin{aligned}|\mathbb{I}| &\lesssim \|D_x^{1/2}[\mathcal{H}_x, v_N+v_M]D_x^{1/2}(D_x^{-1/2}\partial_y\widetilde{w}_{N,M})\|_{L^2_{xy}}\|D_x^{-1/2}\partial_y\widetilde{w}_{N,M}\|_{L^2_{xy}}\\ &\lesssim (\|v_N+v_M\|_{L^\infty_{xy}} + \|\partial_x(v_N-v_M)\|_{L^\infty_{xy}})\|D_x^{-1/2}\partial_y\widetilde{w}_{N,M}\|^2_{L^2_{xy}}.\end{aligned}\tag{4.89}$$

On the other hand, we use Hölder's inequality to find

$$|\mathbb{II}| \lesssim \|\widetilde{w}_{N,M}\|_{L^\infty_{xy}}\|\partial_y\widetilde{w}_{N,M}\|^2_{L^2_{xy}} + \|\partial_y v_N\|_{L^2_{xy}}\|\widetilde{w}_{N,M}\|_{L^\infty_{xy}}\|\partial_y\widetilde{w}_{N,M}\|_{L^2_{xy}}. \tag{4.90}$$

According to the above estimate, we are led to bound the norms $\|\partial_y v_N\|_{L^2_{xy}}$ and $\|\partial_y\widetilde{w}_{N,M}\|_{L^2_{xy}}$. Thus, given that v_N satisfies the equation in (0.4), integrating by parts it follows that

$$\frac{1}{2}\frac{d}{dt}\|\partial_y v_N(t)\|^2_{L^2_{xy}} = -\int \partial_y(v_N\partial_x v_N)\partial_y v_N\,dxdy = -\frac{1}{2}\int \partial_x v_N(\partial_y v_N)^2\,dxdy.$$

Then Gronwall's inequality and (4.68) yield

$$\|\partial_y v_N(t)\|_{L^2_{xy}} \leqslant e^{cK_1}\|\partial_y P^x_{\leqslant N}u_0\|_{L^2_{xy}},$$

since $\|\partial_y P^x_{\leqslant N}u_0\|_{L^2_{xy}} \lesssim N^{1/2}\|D_x^{-1/2}\partial_y u_0\|_{L^2_{xy}}$, we have

$$\|\partial_y v_N\|_{L^\infty_T L^2_{xy}} \lesssim N^{1/2}. \tag{4.91}$$

On the other hand, from the fact that $\widetilde{w}_{N,M}$ solves (4.65) and integrating by parts, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_y\widetilde{w}_{N,M}\|^2_{L^2_{xy}} &= -\frac{1}{2}\int \partial_x\partial_y((v_N+v_M)w_{N,M})\partial_y w_{N,M}\,dxdy\\ &= -\int \partial_x\partial_y v_N w_{N,M}\partial_y w_{N,M}\,dxdy - \int \partial_y v_N\partial_x w_{N,M}\partial_y w_{N,M}\,dxdy\\ &\quad -\frac{1}{2}\int \partial_x v_M(\partial_y w_{N,M})^2\,dxdy\\ &=: \mathbb{III}_1 + \mathbb{III}_2 + \mathbb{III}_3.\end{aligned} \tag{4.92}$$

To estimate $\mathbb{III}_1$, we employ that v_N solves the equation in (0.4) to get

$$\frac{1}{2}\frac{d}{dt}\|\partial_y\partial_x v_N(t)\|^2_{L^2_{xy}} = -\frac{3}{2}\int \partial_x v_N(\partial_y\partial_x v_N)^2\,dxdy - \int \partial_x^2 v_N\partial_y v_N\partial_y\partial_x v_N\,dxdy.$$

From this estimate and (4.91), it is seen

$$\frac{1}{2}\frac{d}{dt}\|\partial_y\partial_x v_N(t)\|^2_{L^2_{xy}} \lesssim \|\partial_x v_N\|_{L^\infty_{xy}}\|\partial_y\partial_x v_N\|^2_{L^2_{xy}} + \|\partial_x^2 v_N\|_{L^\infty_{xy}}\|\partial_y v_N\|_{L^2_{xy}}\|\partial_y\partial_x v_N\|_{L^2_{xy}}.$$

Then, in view of (4.67)-(4.69), (4.87), (4.91) and Gronwall's inequality

$$\|\partial_y\partial_x v_N\|_{L^\infty_T L^2_{xy}} \lesssim e^{cK_1}\big(\|\partial_y\partial_x P^x_{\leqslant N}u_0\|_{L^2_{xy}} + N^{1/2}\|\partial_x^2 v_N\|_{L^1_T L^\infty_{xy}}\big) \lesssim N^{3/2},$$

where we used that $\|\partial_y\partial_x P^x_{\leqslant N}u_0\|_{L^2_{xy}} \lesssim N^{3/2}\|D_x^{-1/2}\partial_y u_0\|_{L^2_{xy}}$. Consequently, the previous estimate allows us to deduce

$$\mathbb{III}_1 \lesssim N^{3/2}\|\widetilde{w}_{N,M}\|_{L^\infty_{xy}}\|\partial_y\widetilde{w}_{N,M}\|_{L^2_{xy}}.$$

Now, by using (4.91) and Hölder's inequality,

$$\mathbb{III}_2 + \mathbb{III}_3 \lesssim N^{1/2}(\|\partial_x v_N\|_{L^\infty_{xy}} + \|\partial_x v_M\|_{L^\infty_{xy}})\|\partial_y\widetilde{w}_{N,M}\|_{L^2_{xy}} + \|\partial_x v_M\|_{L^\infty_{xy}}\|\partial_y\widetilde{w}_{N,M}\|^2_{L^2_{xy}}.$$

Thus, inserting the above estimates in (4.92), applying Gronwall's inequality together with (4.67), (4.68) and (4.72) reveal

$$\begin{aligned}\|\partial_y\widetilde{w}_{N,M}\|_{L^\infty_T L^2_{xy}} \lesssim e^{cK_1}\big(&\|\partial_y(P^x_{\leqslant N}u_0 - P^x_{\leqslant M}u_0)\|_{L^2_{xy}} + N^{3/2}\|\widetilde{w}_{N,M}\|_{L^1_T L^\infty_{xy}}\\ &+ N^{1/2}(\|\partial_x v_N\|_{L^1_T L^\infty_{xy}} + \|\partial_x v_M\|_{L^1_T L^\infty_{xy}})\big) \lesssim N^{1/2} + N^{3/2}\|\widetilde{w}_{N,M}\|_{L^1_T L^\infty_{xy}}.\end{aligned} \tag{4.93}$$

Going back to $\mathbb{III}$, we plug (4.91) and (4.93) in (4.90) to obtain

$$|\mathbb{III}| \lesssim N\|\widetilde{w}_{N,M}\|_{L^\infty_{xy}} + N^2\|\widetilde{w}_{N,M}\|_{L^1_T L^\infty_{xy}}\|\widetilde{w}_{N,M}\|_{L^\infty_{xy}}. \tag{4.94}$$

Now, collecting (4.89), (4.94) in (4.88),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|D_x^{-1/2}\partial_y\widetilde{w}_{N,M}(t)\|^2_{L^2_{xy}} \lesssim &(\|v_N+v_M\|_{L^\infty_{xy}} + \|\partial_y(v_N+v_M)\|_{L^\infty_{xy}})\|D_x^{-1/2}\partial_y\widetilde{w}_{N,M}\|^2_{L^2_{xy}}\\ &+ N\|\widetilde{w}_{N,M}\|_{L^\infty_{xy}} + N^2\|\widetilde{w}_{N,M}\|_{L^1_T L^\infty_{xy}}\|\widetilde{w}_{N,M}\|_{L^\infty_{xy}}.\end{aligned}$$

Then, applying Gronwall's inequality to the last display, together with (4.67), (4.68) and (4.72) (provided that $T = (1+\widetilde{A}_s\|u_0\|_{X^s})^{-2}$ is chosen sufficiently small) yield

$$\|D_x^{-1/2}\partial_y\widetilde{w}_{N,M}\|_{L^\infty_T L^2_{xy}} \lesssim e^{cK_1}\big(\|D_x^{-1/2}\partial_y(P^x_{\leqslant N}u_0 - P^x_{\leqslant M}u_0)\|_{L^2_{xy}} + o(1)\big) \underset{N\to\infty}{\to} 0.$$

This completes the proof of (4.83). □

We deduce from part (i) in Lemma 4.21 and Proposition 4.22 that $\{u_N\}$ has a limit u in the class

$$C([0,T];H^s(\mathbb{R}^2)) \cap L^1([0,T];W^{1,\infty}(\mathbb{R}^2)).$$

Then, since u_N solves the integral equation

$$u_N(t) = S(t)P_{\leqslant N}u_0 - \frac{1}{2}\int_0^t S(t-t')\partial_x(u_N(t'))^2\,dt', \tag{4.95}$$

taking the limit when $N\to\infty$, we find that u satisfies the integral equation in $C([0,T];H^{s-1}(\mathbb{R}^2))$. This establish that u solves the IVP (0.4). On the other hand, from part (ii) in Lemma 4.21 and Proposition 4.22, we find that $\{v_N\}$ has a limit v in the class

$$C([0,T];X^s(\mathbb{R}^2)) \cap L^1([0,T];W^{1,\infty}_x(\mathbb{R}^2)).$$

Thus, implementing the integral equation (4.95) and taking the limit in the class $C([0,T];J^2X^s(\mathbb{R}^2))$, where $J^2X^s(\mathbb{R}^2) = \{f\in S'(\mathbb{R}^2) : J^{-2}f \in X^s\}$ with norm $\|f\|_{J^2X^s} = \|J^{-2}f\|_{X^s}$, we can argue as above to deduce that v solves the IVP (0.4). This completes the existence part of Theorem 4.1.

4.3.2.3. **Uniqueness and continuous dependence.** Let us first establish uniqueness when the initial data lies in $H^s(\mathbb{R}^2)$, i.e., in the class $C([0,T];H^s(\mathbb{R}^2)\cap L^1([0,T];W^{1,\infty}(\mathbb{R}^2))$. Let u_1 and u_2 be two solutions of the IVP (0.4) with initial data $u_1(0)=u_{1,0}$ and $u_2(0)=u_{2,0}$. We define

$$K := \max\{\|\nabla u_1\|_{L^1_T L^\infty_{xy}}, \|\nabla u_2\|_{L^1_T L^\infty_{xy}}\}.$$

We have that $w = u_1 - u_2$ solves

$$\partial_t w + \mathcal{H}_x w - \mathcal{H}_x\partial_x^2 w \pm \mathcal{H}_x\partial_y^2 w + \frac{1}{2}\partial_x((u_1+u_2)w) = 0, \tag{4.96}$$

with initial condition $w(0) = u_{1,0}-u_{2,0}$. We multiply (4.96) by w and integrate by part to obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2_{xy}} = -\frac{1}{4}\int \partial_x(u_1-u_2)w^2\,dxdy.$$

Then, Gronwall's inequality shows

$$\|u_1-u_2\|_{L^2_{xy}} \leqslant \exp\big(c(\|\partial_x u_1\|_{L^1_T L^\infty_{xy}} + \|\partial_x u_2\|_{L^1_T L^\infty_{xy}})\big)\|u_{1,0}-u_{2,0}\|_{L^2_{xy}} \leqslant e^{cK}\|u_{1,0}-u_{2,0}\|_{L^2_{xy}}.$$

This inequality yields the uniqueness result when $u_{1,0}=u_{2,0}$. Implementing the same argument above, we deduce uniqueness in the class $C([0,T];X^s(\mathbb{R}^2)\cap L^1([0,T];W^{1,\infty}_x(\mathbb{R}^2))$.

Continuous dependence in $H^s(\mathbb{R}^2)$ is arguing as in the proof of Theorem 2.1. Analogously, continuous dependence in $X^s(\mathbb{R}^2)$ is proved by this same property in Lemma 4.18 and the ideas in [59].

4.4. Study of the equation in $H^s(\mathbb{T}^2)$

4.4.1. Functions spaces and additional notation. We will follow the notation in [44] (see also, [77, 79, 80, 90]). Recalling that $\mathbb{D} = \{2^l : l \in \mathbb{Z}^+ \cup \{0\}\}$, for a given $N_1, N_2 \in \mathbb{D}$, we define $N_1 \vee N_2 = \max(N_1, N_2)$ and $N_1 \wedge N_2 = \min(N_1, N_2)$.

Now, for each $N \in \mathbb{D}\backslash\{1\}$ we set

$$I_N = \{m \in \mathbb{Z}^+ : N/2 \leqslant |m| < N\}$$

and $I_1 = \{0\}$. For the sake of simplicity, we will employ the same symbols in (1.7) to define their periodic equivalents. However, this convection will be limited to this section. Thus, we define the projector operators in $L^2(\mathbb{T}^2 \times \mathbb{R})$ by the relation

$$\mathcal{F}(P_N(u))(m,n,\tau) = \mathbb{1}_{I_N}(|(m,n)|)\mathcal{F}(u)(m,n,\tau), \tag{4.97}$$

for all $m, n \in \mathbb{Z}$ and $\tau \in \mathbb{R}$.

Given a dyadic number N, we define the operator $P_{\leqslant N}u$ by the Fourier multiplier $\mathbb{1}_{I_{\leqslant N}}(|(m,n)|)$, where $I_{\leqslant N} = \bigcup_{M \leqslant N} I_M$ with M dyadic. We also set $P_{>M}u = (I - P_{\leqslant M})u$.

Since we do not require to divide the lower modulations terms, recalling the function ψ_0 defining (1.7) for $d = 1$ and the functions ψ_N, we will denote by $\varphi_1 = \psi_0$, $\varphi_N = \psi_N$ for all N dyadic with $N > 1$. For a time $T_0 \in (0,1)$, let $N_0 \in \mathbb{D}$ be the greatest dyadic number such that $N_0 \leqslant 1/T_0$. Let $N \in \mathbb{D}$ and $b \in [0, 1/2]$, we define the dyadic $X^{s,b}$-type normed spaces

$$\begin{aligned} X_N^b = X_N^b(\mathbb{Z}^2 \times \mathbb{R}) = \{f \in L^2(\mathbb{Z}^2 \times \mathbb{R}) : \mathbb{1}_{I_N}(|(m,n)|)f = f \text{ and} \\ \|f\|_{X_N^b} = N_0^b \|\varphi_{\leqslant N_0}(\tau - \omega(m,n)) \cdot f\|_{L^2_{m,n,\tau}} \\ + \sum_{L > N_0} L^b \|\varphi_L(\tau - \omega(m,n)) \cdot f\|_{L^2_{m,n,\tau}} < \infty\}. \end{aligned} \tag{4.98}$$

We will denote by X_N the space $X_N^{1/2}$. Next, we introduce the spaces F_N^b according to X_N^b uniformly on time intervals of size N^{-1}:

$$F_N^b := \{f \in C(\mathbb{R}; L^2(\mathbb{T}^2)) : P_N f = f,\ \|f\|_{F_N^b} := \sup_{t_N \in \mathbb{R}} \|\mathcal{F}(\varphi_1(N(\cdot - t_N))f)\|_{X_N^b} < \infty\} \tag{4.99}$$

and

$$\begin{aligned} \mathcal{N}_N := \{f \in C(\mathbb{R}; L^2(\mathbb{T}^2)) \ : P_N f = f, \\ \|f\|_{\mathcal{N}_N} := \sup_{t_N \in \mathbb{R}} \||\tau + \omega(m,n) + iN|^{-1}\mathcal{F}(\varphi_1(N(\cdot - t_N))f)\|_{X_N}\}. \end{aligned} \tag{4.100}$$

Let $T \in (0, T_0]$ and Y_N be any of the spaces F_N^b or $\mathcal{N}_N$, we set

$$Y_N(T) := \{f \in C([0,T]; L^2(\mathbb{T}^2)) : \|f\|_{Y_N(T)} < \infty\}$$

equipped with the norm:

$$\|f\|_{Y_N(T)} := \inf\{\|\tilde{f}\|_{Y_N} : \ \tilde{f} \in Y_N,\ \tilde{f} \equiv f \text{ on } [0,T]\}.$$

Then for a given $s \geqslant 0$, we define the spaces $F^{s,b}(T)$ and $\mathcal{N}^s(T)$ from their frequency localized version $F_N^b(T)$ and $\mathcal{N}(T)$ by using the Littlewood-Paley decomposition in the following manner

$$F^{s,b}(T) := \{f \in C([0,T]; H^s(\mathbb{T}^2)),\ \|f\|_{F^{s,b}(T)}^2 = \sum_{N \in \mathbb{D}} (N^{2s} + N_0^{2s}) \|P_N f\|_{F_N^b(T)}^2 < \infty\} \tag{4.101}$$

and

$$\mathcal{N}^s(T) := \{f \in C([0,T]; H^s(\mathbb{T}^2)),\ \|f\|_{\mathcal{N}^s(T)}^2 = \sum_{N \in \mathbb{D}} (N^{2s} + N_0^{2s}) \|P_N f\|_{\mathcal{N}_N(T)}^2 < \infty\}. \tag{4.102}$$

Next, we define the associated energy spaces $B^s(T)$ endowed with the norm

$$\begin{aligned} B^s(T) := \{f \in C([0,T]; H^s(\mathbb{T}^2)),\ \|f\|_{B^s(T)}^2 &= \|P_{\leqslant N_0} f(0)\|_{H^s}^2 \\ &\quad + \sum_{N > N_0} \sup_{t_N \in [0,T]} \|P_N f(t_N)\|_{H^s}^2 < \infty\}. \end{aligned} \tag{4.103}$$

In the subsequent considerations F_N and $F^s(T)$ will denote the spaces above with parameter $b = 1/2$.

4.4.1.1. **Basic properties.** Now we collect some basic properties of the spaces X_N^b and $F_N^b(T)$. These results have been deduced in different contexts in [35, 44, 79, 78, 90].

Lemma 4.23. *Let $N \in \mathbb{D}$ and $b \in (0, 1/2]$. Then*

$$\|f_N\|_{L^2_{m,n} L^q_\tau} \lesssim \|f_N\|_{X_N^b}$$

where $q = 2/(1+2b)$, $f_N \in X_N^b$ and the implicit constant is independent of $N_0 \geqslant 1$.

PROOF. We decompose f_N according to its modulations to derive

$$\begin{aligned} \|f_N\|_{L^2_{m,n} L^q_\tau} &\leqslant \sum_{L > N_0} \|\varphi_L(\tau - \omega(m,n)) f_N\|_{L^2_{m,n} L^q_\tau} + \|\varphi_{L \leqslant N_0}(\tau - \omega(m,n)) f_N\|_{L^2_{m,n} L^q_\tau} \\ &\lesssim \sum_{L > N_0} L^b \|\widetilde{\varphi}_L(\tau - \omega(m,n)) \langle \tau - \omega(m,n)\rangle^{-b} \varphi_L(\tau - \omega(m,n)) f_N\|_{L^2_{m,n} L^q_\tau} \\ &\quad + N_0^b \|\widetilde{\varphi}_{\leqslant N_0}(\tau - \omega(m,n)) \langle \tau - \omega(m,n)\rangle^{-b} \varphi_{\leqslant N_0}(\tau - \omega(m,n)) f_N\|_{L^2_{m,n} L^q_\tau}, \end{aligned} \tag{4.104}$$

where $\widetilde{\varphi}_L$ and $\widetilde{\varphi}_{\leqslant N_0}$ are two adapted functions to the support of φ_L and $\varphi_{\leqslant N_0}$ respectively. Now, since $1/q = 1/2 + b$, we apply Cauchy-Schwarz in the time variable to obtain

$$\begin{aligned} &\|f_N\|_{L^2_{m,n} L^q_t \tau} \\ &\lesssim \sum_{L > N_0} L^b \| \|\widetilde{\varphi}_L(\tau - \omega(m,n)) \langle \tau - \omega(m,n)\rangle^{-b}\|_{L^{1/b}_\tau} \|\varphi_L(\tau - \omega(m,n)) f_N\|_{L^2_\tau} \|_{L^2_{m,n}} \\ &\quad + N_0^b \| \|\widetilde{\varphi}_{\leqslant N_0}(\tau - \omega(m,n)) \langle \tau - \omega(m,n)\rangle^{-b}\|_{L^{1/b}_\tau} \|\varphi_L(\tau - \omega(m,n)) f_N\|_{L^2_\tau} \|_{L^2_{m,n}} \\ &\lesssim \|f_N\|_{X_N^b}, \end{aligned} \tag{4.105}$$

where we have used that $\|\widetilde{\varphi}_L \langle \tau \rangle^{-b}\|_{L^{1/b}_\tau}, \|\widetilde{\varphi}_{\leqslant N_0} \langle \tau \rangle^{-b}\|_{L^{1/b}_\tau} \lesssim_b 1$ with involved constant independent of L. ☐

Lemma 4.24. *Let $N \in \mathbb{D}$, $b \in (0, 1/2]$, $f_N \in X_N^b$ and $h \in L^2(\mathbb{R})$ satisfying*

$$|\widehat{h}(\tau)| \lesssim \langle \tau \rangle^{-4}.$$

Then for any $\widetilde{N_0} \in \mathbb{D}$, $\widetilde{N_0} \geqslant N_0$ *and* $t_0 \in \mathbb{R}$,

$$\widetilde{N_0}^{\,b} \left\| \varphi_{\leqslant \widetilde{N_0}}(\tau - \omega(m,n)) \mathcal{F}(h(\widetilde{N_0}(t-t_0))\mathcal{F}^{-1}(f_N)) \right\|_{L^2_{m,n,\tau}} \lesssim \|f_N\|_{X^b_N}, \tag{4.106}$$

and

$$\sum_{L > \widetilde{N_0}} L^b \left\| \varphi_L(\tau - \omega(m,n)) \mathcal{F}(h(\widetilde{N_0}(t-t_0))\mathcal{F}^{-1}(f_N)) \right\|_{L^2_{m,n,\tau}} \lesssim \|f_N\|_{X^b_N}. \tag{4.107}$$

The implicit constants above are independent of $N_0 \geqslant 1$.

PROOF. We first deduce (4.106). From Hölder's inequality and Young's inequality it is seen

$$\begin{aligned} &\widetilde{N_0}^{\,b} \|\varphi_{\leqslant \widetilde{N_0}}(\tau - \omega(m,n)) \mathcal{F}(h(N_0(t-t_0))\mathcal{F}^{-1}(f_N))\|_{L^2_{m,n,\tau}} \\ &\qquad \lesssim \widetilde{N_0}^{\,b-1} \|\varphi_{\leqslant \widetilde{N_0}}\|_{L^{p_1}} \left\| \|e^{i\tau' t_0} \widehat{h}(\widetilde{N_0}^{\,-1}\tau') *_\tau (f_N)\|_{L^{p_2}_\tau} \right\|_{L^2_{m,n}} \\ &\qquad \lesssim \widetilde{N_0}^{\,b-1} \|\varphi_{\leqslant N_0}\|_{L^{p_1}} \|\widehat{h}(\widetilde{N_0}^{\,-1}\tau)\|_{L^{q_1}_\tau} \|f_N\|_{L^2_{m,n}L^q_\tau}, \end{aligned}$$

where $\frac{1}{q} = \frac{1}{2} + b$,

$$\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \text{and} \quad \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q} - 1 = \frac{1}{q_1} + b - \frac{1}{2}. \tag{4.108}$$

It is not difficult to see that for fixed $b \in (0, 1/2]$, one can always find indexes $2 \leqslant p_1, p_2 \leqslant \infty$, $1 \leqslant q_1 \leqslant \infty$ assuming the conditions displayed above. Therefore, given that $1 = \frac{1}{p_1} + \frac{1}{q_1} + b$ and that $\|\varphi_{\leqslant \widetilde{N_0}}\|_{L^{p_1}} \sim \widetilde{N_0}^{\,1/p_1}$ and $\|\widehat{h}(\widetilde{N_0}^{\,-1}\tau)\|_{L^{q_1}_\tau} \sim \widetilde{N_0}^{\,1/q_1}$, we deduce

$$\widetilde{N_0}^{\,b} \|\varphi_{\leqslant \widetilde{N_0}}(\tau - \omega(m,n)) \mathcal{F}(h(\widetilde{N_0}(t-t_0))\mathcal{F}^{-1}(f_N))\|_{L^2_{m,n,\tau}} \lesssim \|f_N\|_{L^2_{m,n}L^q_\tau}.$$

The estimate above and Lemma 4.23 complete the deduction of (4.106). Let us treat (4.107). Decomposing f_N by modulations, it follows

$$\begin{aligned} &\sum_{L > \widetilde{N_0}} L^b \left\| \varphi_L(\tau - \omega(m,n)) \mathcal{F}(h(\widetilde{N_0}(t-t_0))\mathcal{F}^{-1}(f_N)) \right\|_{L^2_{m,n,\tau}} \\ &\quad \lesssim \sum_{L > \widetilde{N_0}} \sum_{L_1 \geqslant N_0} L^b \widetilde{N_0}^{\,-1} \left\| \varphi_L(\tau - \omega(m,n)) (e^{t\tau' t_0} \widehat{h}(\widetilde{N_0}^{\,-1}\tau)) *_\tau (\eta_{L_1}(\tau - \omega(m,n)) f_N) \right\|_{L^2_{m,n,\tau}} \\ &\quad =: \sum_{L > \widetilde{N_0}} \sum_{N_0 \leqslant L_1 < L/10} (\cdots) + \sum_{L > \widetilde{N_0}} \sum_{L_1 \geqslant (N_0 \vee (L/10))} (\cdots) =: \mathcal{I} + \mathcal{II}, \end{aligned}$$

where we set $\eta_{N_0}(\tau) = \varphi_{\leqslant N_0}(\tau)$ and $\eta_{L_1}(\tau) = \varphi_{L_1}(\tau)$ for $L_1 > N_0$. To estimate $\mathcal{I}$, we notice that $|\tau - \tau'| \sim L$, since $|\tau - \omega(m,n)| \sim L$ and $|\tau' - \omega(m,n)| \lesssim L_1 \leqslant L/10$ in the support of each integral in the summation. Consequently, this fact, Hölder's inequality and Young's inequality imply

$$\begin{aligned} \mathcal{I} &\lesssim \sum_{L > \widetilde{N_0}} \sum_{N_0 \leqslant L_1 < L/10} L^{b-3/2} \widetilde{N_0}^{\,-1} \| \|\varphi_L(\tau - \omega(m,n)) (|| \cdot |^{3/2} \widehat{h}(\widetilde{N_0}^{\,-1}\tau)|) \\ &\qquad\qquad\qquad *_\tau (|\eta_{L_1}(\tau - \omega(m,n)) f_N|) \|_{L^2_\tau} \|_{L^2_{m,n}} \\ &\lesssim \sum_{L > \widetilde{N_0}} \sum_{N_0 \leqslant L_1 < L/10} L^{b-3/2} \widetilde{N_0}^{\,-1} \|\varphi_L\|_{L^{p_1}_\tau} \| | \cdot |^{3/2} \widehat{h}(\widetilde{N_0}^{\,-1} \cdot)\|_{L^{q_1}_\tau} \|\eta_{L_1}(\tau - \omega(m,n)) f_N\|_{L^2_{m,n}L^q_\tau}, \end{aligned}$$

where the indexes p_1, q_1, q satisfy the relations (4.108) and $\frac{1}{q} = \frac{1}{2} + b$. Now since $\|\varphi_L\|_{L_\tau^{p_1}} \sim L^{1/p_1}$, $\| |\cdot|^{3/2}\widehat{h}(\widetilde{N_0}^{-1}\cdot)\|_{L_\tau^{q_1}} \sim \widetilde{N_0}^{3/2+1/q_1}$ and $1/p_1 + b - 3/2 \leqslant -1/2$, we can sum over L to get

$$\begin{aligned}\mathcal{I} &\lesssim \sum_{L>\widetilde{N_0}} \sum_{N_0 \leqslant L_1 < L/10} L^{1/p_1+b-3/2}\widetilde{N_0}^{1/2+1/q_1}\|\eta_{L_1}(\tau-\omega(m,n))f_N\|_{L^2_{m,n}L^q_\tau} \\ &\lesssim \widetilde{N_0}^{-1+1/p_1+1/q_1+b}\big(\|\varphi_{\leqslant N_0}(\tau-\omega(m,n))f_N\|_{L^2_{m,n}L^q_\tau} + \sum_{L_1>N_0}\|\varphi_{L_1}(\tau-\omega(m,n))f_N\|_{L^2_{m,n}L^q_\tau}\big),\end{aligned}$$

which is controlled as in (4.104) and (4.105), since $-1+1/p_1+1/q_1+b=0$. The remaining term is controlled by Young's inequality in the following manner

$$\begin{aligned}\mathcal{II} &\lesssim \sum_{L_1 \geqslant (N_0 \vee (\widetilde{N_0}/10))} L_1^b \widetilde{N_0}^{-1}\| \| |\widehat{h}(\widetilde{N_0}^{-1}\tau')| *_\tau |\eta_{L_1}(\tau-\omega(m,n))f_N| \|_{L^2_\tau}\|_{L^2_{m,n}} \\ &\lesssim \sum_{L_1 \geqslant N_0} L_1^b \widetilde{N_0}^{-1}\|\widehat{h}(\widetilde{N_0}^{-1}\cdot)\|_{L^1}\|\eta_{L_1}(\tau-\omega(m,n))f_N\|_{L^2_{m,n},\tau} \lesssim \|f_N\|_{X_N^b}.\end{aligned} \tag{4.109}$$

This estimate completes the proof of the lemma. □

Additionally, we require the next result.

Lemma 4.25. *Let $N \in \mathbb{D}$, $b \in (0,1/2]$ and $I \subset \mathbb{R}$ a bounded interval. Then*

$$\sup_{L\in\mathbb{D}} L^b\|\varphi_L(\tau-\omega(m,n))\mathcal{F}(\mathbb{1}_I(t)f)\|_{L^2} \lesssim \|\mathcal{F}(f)\|_{X_N^b}$$

for all f whose Fourier transform is in X_N^b and the implicit constant is independent of $N_0 \geqslant 1$.

PROOF. We decompose $\widehat{f}$ by modulations according to

$$\begin{aligned}L^b\|\varphi_L(\tau-\omega(m,n))\mathcal{F}(\mathbb{1}_I f)\|_{L^2} &\lesssim \sum_{L_1\geqslant N_0} L^b\,\|\varphi_L(\tau-\omega(m,n))\mathcal{F}(\mathbb{1}_I) * |\eta_{L_1}(\tau-\omega(m,n))\mathcal{F}(f)|\|_{L^2_{m,n,\tau}} \\ &=: \sum_{N_0\leqslant L_1 < L/10}(\cdots) + \sum_{L_1 \geqslant (N_0\vee(L/10))}(\cdots) =: \mathcal{I}+\mathcal{II},\end{aligned}$$

where we set $\eta_{N_0}(\tau) = \varphi_{\leqslant N_0}(\tau)$ and $\eta_{L_1}(\tau) = \varphi_{L_1}(\tau)$ for $L_1 > N_0$. When $L < L/10$, we have $|\tau-\tau'| \sim L$, thus applying Hölder inequality and then Young's inequality it is seen

$$\mathcal{I} \lesssim \sum_{N_0\leqslant L_1<L/10} L^{b-m}\|\varphi_L\|_{L^{p_1}}\| |\cdot|^m\widehat{\mathbb{1}}_I\|_{L^{q_1}}\|\eta_{L_1}(\tau-\omega(m,n))\mathcal{F}(f)\|_{L^2_{m,n}L^q_\tau} \tag{4.110}$$

where $\frac{1}{q} = \frac{1}{2}+b$, $0 < m \leqslant 1$ and p_1, p_2, q_1 satisfy (4.108). If $b = 1/2$, we take $m = 1$, $p_1 = q = 2$ and $q_1 = \infty$, and when $0 < b < 1/2$, we set $m = b$, $p_1 = b^{-1}$ and $q_1 = (1-2b)^{-1}$. Notice that under any of these restrictions, $\| |\cdot|^m\widehat{\mathbb{1}}_I\|_{L^{q_1}} < \infty$, since $|\widehat{\mathbb{1}}_I(\tau)| \lesssim \langle\tau\rangle^{-1}$. Consequently,

$$\mathcal{I} \lesssim \sum_{N_0\leqslant L_1<L/10} L_1^b\|\eta_{L_1}(\tau-\omega(m,n))\mathcal{F}(f)\|_{L^2_{m,n}L^q_\tau}.$$

Arguing as in (4.105), the last display yields to the desired bound. The estimate for $\mathcal{II}$ is obtained following a similar reasoning as in (4.109). The proof is completed. □

We will require the following lemma to obtain time factors in the energy estimates.

Lemma 4.26. *Let $T \in (0, T_0)$ and $0 \leqslant b < 1/2$. Then for any $f \in F_N(T)$,*

$$\|f\|_{F_N^b(T)} \lesssim T^{(1/2-b)^-} \|f\|_{F_N(T)}$$

where the implicit constant is independent of N, N_0 and T_0.

PROOF. The proof follows the same arguments in [35, Lemma 3.4]. □

Next, we recall the embedding $F^s(T) \hookrightarrow C([0,T]; H^s(\mathbb{T}^2))$, $s > 0$, $T \in (0, T_0]$ established in [90, 44].

Lemma 4.27. *Let $T \in (0, T_0]$, then*

$$\sup_{t \in [0,T]} \|u(t)\|_{H^s} \lesssim \|u\|_{F^s(T)},$$

whenever $u \in F^s(T)$ and the implicit constant is independent of $N_0 \geqslant 1$,

We also need the following linear estimate which is deduced from the arguments in [44, Proposition 3.2] (see also [90, Proposition 6.2]).

PROPOSITION 4.28. *Assume that $T \in (0, T_0]$, $s \geqslant 0$ and $u, v \in C([0,T]; H^\infty(\mathbb{T}^2))$ with*

$$\partial_t u + \mathcal{H}_x u - \mathcal{H}_x \partial_x^2 u \pm \mathcal{H}_x \partial_y^2 u = v, \qquad \text{on } \mathbb{T}^2 \times [0,T).$$

Then

$$\|u\|_{F^s(T)} \lesssim \|u\|_{B^s(T)} + \|v\|_{\mathcal{N}^s(T)}, \tag{4.111}$$

where the implicit constant is independent of T_0.

To obtain a priori estimates for smooth solutions we need the following lemma.

Lemma 4.29. *Let $s \geqslant 0$, $v \in C([0, T_0]; H^\infty(\mathbb{T}^2))$. Then the mapping $T \to \|v\|_{\mathcal{N}^s(T)}$ is increasing and continuous on $[0, T_0]$ and*

$$\lim_{T \to 0} \|v\|_{\mathcal{N}^s(T)} \to 0. \tag{4.112}$$

PROOF. The proof follows the same line of arguments in [90, Lemma 6.3]. □

4.4.2. L^2 Bilinear estimates. Next, we obtain the crucial L^2 bilinear estimates, which will be applied to obtain both the short time estimates and energy estimates in the subsequent subsections.

Let $N, L \in \mathbb{D}$, we define

$$D_{N,L} = \{(m,n,\tau) \in \mathbb{Z}^2 \times \mathbb{R} : |(m,n)| \in I_N \text{ and } |\tau - \omega(m,n)| \leqslant L\}$$

PROPOSITION 4.30. *Assume that $N_i, L_i \in \mathbb{D}$ and $f_i : \mathbb{Z}^2 \times \mathbb{R} \to \mathbb{R}^+$ functions supported in D_{N_i,L_i} for $i = 1,2,3$.*

(1) *It holds that*

$$\int_{\mathbb{Z}^2 \times \mathbb{R}} (f_1 * f_2) \cdot f_3 \lesssim N_{min} L_{min}^{1/2} \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2}. \tag{4.113}$$

(2) *Suppose that* $N_{min} \ll N_{max}$. *If* $(N_i, L_i) = (N_{min}, L_{max})$ *for some* $i \in \{1,2,3\}$, *then*

$$\int_{\mathbb{Z}^2\times\mathbb{R}} (f_1 * f_2)\cdot f_3 \lesssim N_{max}^{-1/2} N_{min}^{1/2} L_{max}^{1/2} (N_{max}^{1/2} \vee L_{min}^{1/2}) \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2}, \tag{4.114}$$

otherwise

$$\int_{\mathbb{Z}^2\times\mathbb{R}} (f_1 * f_2)\cdot f_3 \lesssim N_{max}^{-1/2} N_{min}^{1/2} L_{med}^{1/2} (N_{max}^{1/2} \vee L_{min}^{1/2}) \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2}. \tag{4.115}$$

(3) *If* $N_{min} \sim N_{max}$,

$$\int_{\mathbb{Z}^2\times\mathbb{R}} (f_1 * f_2)\cdot f_3 \lesssim L_{max}^{1/2} (N_{max}^{1/2} \vee L_{med}^{1/2}) \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2}. \tag{4.116}$$

Before proving Proposition 4.30, we require the following elementary result (see [79]).

Lemma 4.31. *Let* I, J *be two intervals in* $\mathbb{R}$, *and* $\varphi : J \to \mathbb{R}$ *a* C^1 *function with* $\inf_{x\in J} |\varphi'(x)| > 0$. *Suppose that* $\{n \in J \cap \mathbb{Z}, \varphi(n) \in I\} \neq \emptyset$. *Then*

$$\#\{n \in J \cap \mathbb{Z}, \varphi(n) \in I\} \lesssim 1 + \frac{|I|}{\inf_{x\in J} |\varphi'(x)|}.$$

PROOF OF PROPOSITION 4.30. We notice that

$$\int_{\mathbb{Z}^2\times\mathbb{R}} (f_1 * f_2)\cdot f_3 = \int_{\mathbb{Z}^2\times\mathbb{R}} (\widetilde{f_1} * f_3)\cdot f_2 = \int_{\mathbb{Z}^2\times\mathbb{R}} (\widetilde{f_2} * f_3)\cdot f_1 =: \mathcal{I}. \tag{4.117}$$

Lest us first establish (1). In view of the above display we can assume that $L_1 = L_{min}$. Let $f_i^{\#}(m,n,\tau) = f_i(m,n,\tau+\omega(m,n))$, then $f_i^{\#}$ is supported in

$$D^{\#}_{N_i,L_i} = \left\{(m,n,\tau) \in \mathbb{R}^3 : |(m,n)| \in I_{N_i} \text{ and } |\tau| \leqslant L_i\right\},$$

and $\|f_i^{\#}\|_{L^2} = \|f_i\|_{L^2}$, $i = 1,2,3$, we find

$$\begin{aligned} \mathcal{I} &= \int_{\mathbb{Z}^2\times\mathbb{R}} (f_1 * f_2)\cdot f_3 \\ &= \sum_{m_1,n_1,m_2,n_2} \int f_1^{\#}(m_1,n_1,\tau_1) f_2^{\#}(m_2,n_2,\tau_2) \\ &\qquad\qquad \times f_3^{\#}(m_1+m_2, n_1+n_2, \tau_1+\tau_2+\Omega(m_1,n_1,m_2,n_2))\, d\tau_1 d\tau_2. \end{aligned} \tag{4.118}$$

Thus applying the Cauchy-Schwarz inequality in the τ_2 variable and then in τ_1 reveals

$$\begin{aligned} \mathcal{I} &\leqslant \sum_{m_1,n_1,m_2,n_2} \int |f_1^{\#}(m_1,n_1,\tau_1)| \left\|f_2^{\#}(m_2,n_2,\cdot)\right\|_{L^2_\tau} \left\|f_3^{\#}(m_1+m_2,n_1+n_2,\cdot)\right\|_{L^2_\tau} d\tau_1 \\ &\leqslant L_1^{1/2} \sum_{m_1,n_1,m_2,n_2} \left\|f_1^{\#}(m_1,n_1,\cdot)\right\|_{L^2_\tau} \left\|f_2^{\#}(m_2,n_2,\cdot)\right\|_{L^2_\tau} \left\|f_3^{\#}(m_1+m_2,n_1+n_2,\cdot)\right\|_{L^2_\tau}. \end{aligned} \tag{4.119}$$

In this manner, the same procedure displayed above applied to the spatial variables on the r.h.s of (4.119) yields (4.113).

Next we deduce part (2). By (4.117), we shall assume that $N_{min} \sim N_2$ and $L_1 \geqslant L_3$, that is $N_2 \ll N_1 \sim N_3$. Let us consider the domains:

$$\begin{aligned}
A_1 &= (\mathbb{Z}^4 \times \mathbb{R}^2)\setminus \bigcup_{j=2}^{4} A_j,\\
A_2 &= \Big\{(m_1,n_1,m_2,n_2,\tau_1,\tau_2) \in \mathbb{Z}^4 \times \mathbb{R}^2 \, : \, m_1m_2 < 0 \text{ and } |m_1| > |m_2|\Big\},\\
A_3 &= \Big\{(m_1,n_1,m_2,n_2,\tau_1,\tau_2) \in \mathbb{Z}^4 \times \mathbb{R}^2 \, : \, m_1m_2 < 0 \text{ and } |m_1| = |m_2|\Big\},\\
A_4 &= \Big\{(m_1,n_1,m_2,n_2,\tau_1,\tau_2) \in \mathbb{Z}^4 \times \mathbb{R}^2 \, : \, m_1 = 0 \text{ or } m_2 = 0\Big\}.
\end{aligned} \tag{4.120}$$

Accordingly, we divide I given by (4.118) as

$$\mathcal{I} = \sum_{j=1}^{4} \mathcal{I}^j, \tag{4.121}$$

where $\mathcal{I}^j$ corresponds to the restriction of $\mathcal{I}$ to the domain A_j. Notice that the regions A_3 and A_4 consist of the cases where at least one of the variables m_1, m_2 or $m_1 + m_2$ is null. We divide our arguments according to the partitions $\mathcal{I}^j$.

Estimate for $\mathcal{I}^1$. By support considerations it must follows that $m_2(m_1 + m_2) > 0$, or equivalently, $\text{sign}(m_2) = \text{sign}(m_1 + m_2)$. Thus the resonant function is given by

$$\begin{aligned}
\Omega(m_1,n_1,m_2,n_2) = &\,\text{sign}(m_2)(m_1^2 + 2m_1m_2) \mp \text{sign}(m_2)(n_1^2 + 2n_1n_2)\\
&- \text{sign}(m_1) - \text{sign}(m_1)m_1^2 \pm \text{sign}(m_1)n_1^2.
\end{aligned} \tag{4.122}$$

In this manner, we divide $A_1 = A_{1,1} \cup A_{1,2}$, where $A_{1,1}$ consists of the elements A_1 satisfying that $m_2 > 0$ and $A_{1,2}$ for which $m_2 < 0$. Thus, we find

$$\Big|\frac{\partial}{\partial m_2}\Omega(m_1,n_1,m_2,n_2)\Big| \sim |m_1| \text{ and } \Big|\frac{\partial}{\partial n_2}\Omega(m_1,n_1,m_2,n_2)\Big| \sim |n_1| \tag{4.123}$$

in each of the regions $A_{1,1}$ and $A_{1,2}$. Now, since $|(m_1,n_1)| \sim N_1$ in the support of $\mathcal{I}^1$, we further divide the region of integration according to the cases where $|\frac{\partial}{\partial m_2}\Omega(m_1,n_1,m_2,n_2)| \sim N_1$ and $|\frac{\partial}{\partial n_2}\Omega(m_1,n_1,m_2,n_2)| \sim N_1$, namely

$$\mathcal{I}^1 = \sum_{k=1}^{2} \int_{A_{1,k}\cap\{|m_1|\sim N_1\}} (f_1 * f_2)\cdot f_3 + \int_{A_{1,k}\cap\{|m_1|\ll N_1,\,|n_1|\sim N_1\}} (f_1 * f_2)\cdot f_3. \tag{4.124}$$

To estimate the first sum on the r.h.s of (4.124), we use that $|\tau_1+\tau_2+\Omega(m_1,n_1,m_2,n_2)| \leqslant L_3$, (4.123) and Lemma 4.31, together with the Cauchy-Schwarz inequality in the m_2 variable to find

$$
\begin{aligned}
&\sum_{k=1}^{2}\int_{A_{1,k}\cap\{|m_1|\sim N_1\}}(f_1 * f_2)\cdot f_3 \\
&\quad\lesssim \sum_{|m_1|\sim N_1, n_1, n_2}(1+L_3^{1/2}/N_1^{1/2})\int |f_1^{\#}(m_1,n_1,\tau_1)| \\
&\qquad\qquad \times \|f_2^{\#}(m_2,n_2,\tau_2)f_3^{\#}(m_1+m_2,n_1+n_2,\tau_1+\tau_2+\Omega(m_1,n_1,m_2,n_2))\|_{L^2_{m_2}}d\tau_1 d\tau_2 \\
&\quad\lesssim \sum_{n_2}(1+L_3^{1/2}/N_1^{1/2})\int \|f_1^{\#}\|_{L^2}\|f_3^{\#}\|_{L^2}\|f_2^{\#}(\cdot,n_2,\tau_2)\|_{L^2_m}d\tau_2 \\
&\quad\lesssim L_2^{1/2}N_2^{1/2}(1+L_3^{1/2}/N_1^{1/2})\|f_1^{\#}\|_{L^2}\|f_2^{\#}\|_{L^2}\|f_3^{\#}\|_{L^2},
\end{aligned} \tag{4.125}
$$

where the penultimate estimate follows from Cauchy-Schwarz in m_1, n_1, τ_1, and the last line is obtained by the Cauchy-Schwarz inequality in n_2, τ_2. The estimate for the second sum on the r.h.s of (4.124) is deduced changing the roles of m_2 and n_2 in the preceding argument. This completes the study of $\mathcal{I}^1$.

Estimate for $\mathcal{I}^2$. In this case $\operatorname{sign}(m_1) = -\operatorname{sign}(m_2)$ and $m_1(m_1+m_2) > 0$, in other words $\operatorname{sign}(m_1) = \operatorname{sign}(m_1+m_2)$. From these restrictions we get

$$\Omega(m_1,n_1,m_2,n_2) = \operatorname{sign}(m_1)(2m_1m_2+2m_2^2) \mp \operatorname{sign}(m_1)(2n_1n_2+2n_2^2) + \operatorname{sign}(m_1).$$

We write $A_2 = A_{2,1}\cup A_{2,2}$, where $A_{2,1}$ is the set of all the elements in A_2 for which $m_1 > 0$, and $A_{2,2}$ consists of those with $m_1 < 0$. Consequently, in each of the sets $A_{2,1}, A_{2,2}$ it holds

$$|\frac{\partial}{\partial m_2}\Omega(m_1,n_1,m_2,n_2)| \sim |2m_1+4m_2| \quad \text{and} \quad |\frac{\partial}{\partial n_2}\Omega(m_1,n_1,m_2,n_2)| \sim |2n_1+4n_2|. \tag{4.126}$$

Now, since $|(m_1,n_1)| \sim N_1$, $|(m_2,n_2)| \sim N_2$ with $N_2 \ll N_1$, (4.126) establishes that in each of the regions defined by $\mathcal{I}^2$ restricted to $A_{2,1}, A_{2,2}$, either

$$|\frac{\partial}{\partial m_2}\Omega(m_1,n_1,m_2,n_2)| \sim N_1 \quad \text{or} \quad |\frac{\partial}{\partial n_2}\Omega(m_1,n_1,m_2,n_2)| \sim N_1.$$

In consequence, we can further divide $\mathcal{I}^2$ as in (4.124) to apply a similar argument to (4.125), which ultimately leads to the desired estimate for $\mathcal{I}^2$.

Estimate for $\mathcal{I}^3$ and $\mathcal{I}^4$. In these cases both regions of integration can be bounded directly by means of the Cauchy-Schwarz inequality without any further consideration on the resonant function. Indeed, in the support of I_3, we have that $m_2 = -m_1$ and so

$$
\begin{aligned}
\mathcal{I}^3 &= \sum_{m_1\neq 0, n_1, n_2} \int f_1^\#(m_1,n_1,\tau_1) f_2^\#(-m_1,n_2,\tau_2) \\
&\qquad\qquad \times f_3^\#(0,n_1+n_2,\tau_1+\tau_2+\Omega(m_1,-m_1,n_1,n_2))\, d\tau_1 d\tau_2 \\
&\lesssim \sum_{n_1,n_2} \int \|f_1^\#(\cdot,n_1,\tau_1)\|_{L^2_m} \|f_2^\#(\cdot,n_2,\tau_2)\|_{L^2_m} \\
&\qquad\qquad \times |f_3^\#(0,n_1+n_2,\tau_1+\tau_2+\Omega(m_1,-m_1,n_1,n_2))|\, d\tau_1 d\tau_2 \\
&\lesssim \sum_{n_2} \int \|f_2^\#(\cdot,n_2,\tau_2)\|_{L^2_m} \|f_1^\#\|_{L^2} \|f_3^\#\|\, d\tau_2 \\
&\lesssim L_2^{1/2} N_2^{1/2} \|f_2^\#(\cdot,n_2,\tau_2)\|_{L^2_m} \|f_1^\#\|_{L^2} \|f_3^\#\|\, d\tau_2,
\end{aligned}
\tag{4.127}
$$

where we have employed the embedding $L^2(\mathbb{Z}^2) \subset L^\infty(\mathbb{Z}^2)$, together with consecutive applications of the Cauchy-Schwarz inequality starting with m_1, then with n_1, τ_1 and finally with n_2, τ_2. This completes the estimate for $\mathcal{I}^3$. On the other hand, we can divide the support of $\mathcal{I}^4$ in two parts for which at least one of the variables among m_1 and m_2 is not considered in the summation. This in turn allows us to perform some simple modifications to the previous argument to bound $\mathcal{I}_4$ by the r.h.s of (4.127).

Collecting the estimates for $\mathcal{I}^j$, $j = 1,2,3.4$, we complete the deduction of (2). Now we proceed to infer (3). In virtue of (4.117), we shall assume that $L_2 = L_{min}$ and $L_3 = L_{max}$. We write

$$\mathcal{I} = \sum_{j=1}^{4} \widetilde{\mathcal{I}}^j,$$

where $\widetilde{\mathcal{I}}^j$ corresponds to the restriction of $\mathcal{I}$ (given by (4.118)) to the domain A_j determined by (4.120). The estimate for $\widetilde{\mathcal{I}}^1$ is obtained by employing the reasoning in $\mathcal{I}^1$. Indeed, by using that $L_3 = L_{max}$ and $L_2 = L_{min}$ and $|(m_1,n_1)| \sim N_1 \sim N_{max}$, it is deduced

$$\widetilde{\mathcal{I}}^1 \lesssim N_{max}^{1/2} L_{min}^{1/2} (1 + L_{max}^{1/2}/N_{max}^{1/2}) \|f_1^\#\|_{L^2} \|f_2^\#\|_{L^2} \|f_3^\#\|_{L^2}.$$

The estimates for $\widetilde{\mathcal{I}}^3$ and $\widetilde{\mathcal{I}}^4$ can be controlled in a similar fashion as (4.127) without considering the behavior of the resonant function and employing only the Cauchy-Schwarz inequality to find

$$\widetilde{\mathcal{I}}^3 + \widetilde{\mathcal{I}}^4 \lesssim N_{max}^{1/2} L_{max}^{1/2} \|f_1^\#\|_{L^2} \|f_2^\#\|_{L^2} \|f_3^\#\|_{L^2}.$$

In the case of $\widetilde{\mathcal{I}}^2$, the derivatives in the m_2 and n_2 directions of Ω could vanish in the support of the integral. Instead, we will employ the remaining directions to deduce the desired estimates. Indeed, splitting $A_2 = A_{2,1} \cup A_{2,2}$, where $A_{2,1} = A_2 \cap \{m_1 > 0\}$, $A_{2,1} = A_2 \cap \{m_1 < 0\}$, we get

$$|\frac{\partial}{\partial m_1}\Omega(m_1,n_1,m_2,n_2)| \sim |m_2| \quad \text{and} \quad |\frac{\partial}{\partial n_1}\Omega(m_1,n_1,m_2,n_2)| \sim |n_2|, \tag{4.128}$$

in each of the regions $A_{2,1}$ and $A_{2,2}$. Employing (4.128) and similar considerations to (4.125) for the variables m_1 and m_2 we deduce

$$\widetilde{\mathcal{I}}^2 \lesssim N_{max}^{1/2} L_{med}^{1/2} (1 + L_{max}^{1/2}/N_{max}^{1/2}) \|f_1^\#\|_{L^2} \|f_2^\#\|_{L^2} \|f_3^\#\|_{L^2}.$$

This completes the deduction of (3). □

By duality and Proposition 4.30, we obtain the following L^2 bilinear estimates.

Corollary 4.32. *Let $N_1, N_2, N_3, L_1, L_2, L_3 \in \mathbb{D}$ be dyadic numbers and $f_i : \mathbb{R}^3 \to \mathbb{R}_+$ supported in D_{N_i,L_i} for $i = 1, 2$.*

(1) *It holds that*

$$\|\mathbb{1}_{D_{N_3,L_3}}(f_1 * f_2)\|_{L^2} \lesssim N_{min} L_{min}^{1/2} \|f_1\|_{L^2} \|f_2\|_{L^2} . \tag{4.129}$$

(2) *Suppose that $N_{min} \ll N_{max}$. If $(N_i, L_i) = (N_{min}, L_{max})$ for some $i \in \{1, 2, 3\}$, then*

$$\|\mathbb{1}_{D_{N_3,L_3}}(f_1 * f_2)\|_{L^2} \lesssim N_{max}^{-1/2} N_{min}^{1/2} L_{max}^{1/2} (N_{max}^{1/2} \vee L_{min}^{1/2}) \|f_1\|_{L^2} \|f_2\|_{L^2} , \tag{4.130}$$

otherwise

$$\|\mathbb{1}_{D_{N_3,L_3}}(f_1 * f_2)\|_{L^2} \lesssim N_{max}^{-1/2} N_{min}^{1/2} L_{med}^{1/2} (N_{max}^{1/2} \vee L_{min}^{1/2}) \|f_1\|_{L^2} \|f_2\|_{L^2} . \tag{4.131}$$

(3) *If $N_{min} \sim N_{max}$,*

$$\|\mathbb{1}_{D_{N_3,L_3}}(f_1 * f_2)\|_{L^2} \lesssim L_{max}^{1/2} (N_{max}^{1/2} \vee L_{med}^{1/2}) \|f_1\|_{L^2} \|f_2\|_{L^2} . \tag{4.132}$$

4.4.3. Short time bilinear estimates. In this subsection, we derive the bilinear estimates for the equation and the difference of solutions.

Proposition 4.33. *Let $s \geqslant s_0 \geqslant 1$, $T \in (0, T_0]$, then*

$$\|\partial_x(uv)\|_{\mathcal{N}^s(T)} \lesssim T_0^{1/4} \big(\|u\|_{F^{s_0}(T)} \|v\|_{F^s(T)} + \|v\|_{F^{s_0}(T)} \|u\|_{F^s(T)} \big), \tag{4.133}$$

$$\|\partial_x(uv)\|_{\mathcal{N}^0(T)} \lesssim T_0^{1/4} \|u\|_{F^0(T)} \|v\|_{F^{s_0}(T)}, \tag{4.134}$$

for all $u, v \in F^s(T)$ and where the implicit constants are independent of T_0.

We split the proof of Proposition 4.33 in the following technical lemmas.

Lemma 4.34 (*Low × High → High*). *Let $N, N_1, N_2 \in \mathbb{D}$ satisfying $N_1 \ll N \sim N_2$. Then,*

$$\|P_N(\partial_x(u_{N_1} v_{N_2}))\|_{\mathcal{N}_N} \lesssim N_1^{1/2} \|u_{N_1}\|_{F_{N_1}} \|v_{N_2}\|_{F_{N_2}}, \tag{4.135}$$

whenever $u_{N_1} \in F_{N_1}$ and $v_{N_2} \in F_{N_2}$.

Proof. We use the definition of the space $\mathcal{N}_N$ to find

$$\|P_N(\partial_x(u_{N_1} v_{N_2}))\|_{\mathcal{N}_N} \lesssim \sup_{t_N \in \mathbb{R}} \||\tau + \omega(m,n) + iN|^{-1} N \mathbb{1}_{\{|(m,n)| \sim N\}} f_{N_1} * g_{N_2}\|_{X_N}$$

where

$$\begin{aligned} f_{N_1} &= |\mathcal{F}(\varphi_1(N(\cdot - t_N)) u_{N_1})|, \\ g_{N_2} &= |\mathcal{F}(\widetilde{\varphi}_1(N(\cdot - t_N)) v_{N_2})|, \end{aligned} \tag{4.136}$$

with $\widetilde{\varphi}_1 \varphi_1 = \varphi_1$. Now we define

$$\begin{aligned} f_{N_1,(N \vee N_0)} &= \varphi_{\leqslant (N \vee N_0)}(\tau - \omega(m,n)) f_{N_1}(m,n,\tau), \\ f_{N_1,L} &= \varphi_L(\tau - \omega(m,n)) f_{N_1}(m,n,\tau), \end{aligned} \tag{4.137}$$

for $L > (N \vee N_0)$, and we set similarly $g_{N_2,(N\vee N_0)}$ and $g_{N_2,L}$. Therefore, from the definition of the spaces X_N, (4.130) and (4.131), we find

$$
\begin{aligned}
&\|P_N(\partial_x(u_{N_1}v_{N_2}))\|_{\mathcal{N}_N} \\
&\quad\lesssim \sup_{t_N\in\mathbb{R}} \sum_{L,L_1,L_2\geqslant(N\vee N_0)} NL^{-1/2}\|\mathbb{1}_{D_{N,L}}\cdot(f_{N_1,L_1}*g_{N_2,L_2})\|_{L^2} \\
&\quad\lesssim \sup_{t_N\in\mathbb{R}} \sum_{L,L_1,L_2\geqslant(N\vee N_0),\,L_1=L_{max}} NL^{-1/2}N^{-1/2}N_1^{1/2}L_1^{1/2}L_{min}^{1/2}\|f_{N_1,L_1}\|_{L^2}\|g_{N_2,L_2}\|_{L^2} \\
&\qquad + \sup_{t_N\in\mathbb{R}} \sum_{L,L_1,L_2\geqslant(N\vee N_0),\,L_1<L_{max}} NL^{-1/2}N^{-1/2}N_1^{1/2}L_{med}^{1/2}L_{min}^{1/2}\|f_{N_1,L_1}\|_{L^2}\|g_{N_2,L_2}\|_{L^2} \\
&\quad\lesssim \sup_{t_N\in\mathbb{R}} N_1^{1/2}\sum_{L\geqslant N}(N/L)^{1/2}\Big(\sum_{L_1\geqslant(N\vee N_0)}L_1^{1/2}\|f_{N_1,L_1}\|_{L^2}\Big)\Big(\sum_{L_2\geqslant(N\vee N_0)}L_2^{1/2}\|g_{N_2,L_2}\|_{L^2}\Big),
\end{aligned}
\tag{4.138}
$$

where, since $|\tau+\omega(m,n)+iN|^{-1}\leqslant N^{-1}$, the sum over $N_0\leqslant L<(N\vee N_0)$ on the left-hand of (4.138) can be controlled by the right-hand side of this inequality. In this manner the above expression and Lemma 4.24 complete the deduction of the lemma. □

Lemma 4.35 (*High × High → High*). *Let $N, N_1, N_2 \in \mathbb{D}$ satisfying $N \sim N_1 \sim N_2 \gg 1$. Then,*

$$
\|P_N(\partial_x(u_{N_1}v_{N_2}))\|_{\mathcal{N}_N} \lesssim N^{(1/2)^+}\|u_{N_1}\|_{F_{N_1}}\|v_{N_2}\|_{F_{N_2}},
\tag{4.139}
$$

whenever $u_{N_1}\in F_{N_1}$ and $v_{N_2}\in F_{N_2}$.

PROOF. Following the same arguments and notation in the proof of Lemma 4.34, we write

$$
\begin{aligned}
\|P_N(\partial_x(u_{N_1}v_{N_2}))\|_{\mathcal{N}_N} &\lesssim \sup_{t_N\in\mathbb{R}} \sum_{L,L_1,L_2\geqslant(N\vee N_0)} NL^{-1/2}\|\mathbb{1}_{D_{N,L}}\cdot(f_{N_1,L_1}*g_{N_2,L_2})\|_{L^2} \\
&= \sup_{t_N\in\mathbb{R}} \Big(\sum_{\substack{L,L_1,L_2\geqslant(N\vee N_0)\\ L\leqslant(L_1\wedge L_2)}} NL^{-1/2}(\cdots) + \sum_{\substack{L,L_1,L_2\geqslant(N\vee N_0)\\ L>(L_1\wedge L_2)}} NL^{-1/2}(\cdots)\Big).
\end{aligned}
\tag{4.140}
$$

To estimate the first term on the right-hand side of (4.140), we employ (4.132) and the restrictions $(N\vee N_0)\leqslant L\leqslant(L_1\wedge L_2)$ to find

$$
NL^{-1/2}\|\mathbb{1}_{D_{N,L}}\cdot(f_{N_1,L_1}*g_{N_2,L_2})\|_{L^2} \lesssim N^{1/2}(N/L)^{-1/2}(L_1^{1/2}\|f_{N_1,L_1}\|_{L^2})(L_2^{1/2}\|g_{N_2,L_2}\|_{L^2}).
\tag{4.141}
$$

Thus, we add the above expression over $L, L_1, L_2 \geqslant (N\vee N_0)$ with $L\leqslant(L_1\wedge L_2)$ and then we apply Lemma 4.24 to the resulting expression to obtain the desired bound. Next we deal with the second sum on the right-hand side of (4.140). Interpolating (4.129) and (4.132) it is seen

$$
\begin{aligned}
&NL^{-1/2}\|\mathbb{1}_{D_{N,L}}\cdot(f_{N_1,L_1}*g_{N_2,L_2})\|_{L^2} \\
&\qquad\lesssim N^{2-\theta}L^{-1/2}L_{min}^{(1-\theta)/2}L_{max}^{\theta/2}L_{med}^{\theta/2}L_1^{-1/2}L_2^{-1/2}(L_1^{1/2}\|f_{N_1,L_1}\|_{L^2})(L_2^{1/2}\|g_{N_2,L_2}\|_{L^2}),
\end{aligned}
\tag{4.142}
$$

for all $\theta\in[0,1]$ and $L>(L_1\wedge L_2)$. Under these considerations, either $L_1 = L_{min}$ or $L_2 = L_{min}$, which implies

$$
L^{-1/2}L_{min}^{(1-\theta)/2}L_{max}^{\theta/2}L_{med}^{\theta/2}L_1^{-1/2}L_2^{-1/2} \leqslant L_{min}^{-\theta/2}L_{max}^{-1/2+\theta/2}L_{med}^{-1/2+\theta/2}.
$$

Then, plugging the last display in (4.142) and recalling that $N \leqslant L_i, N \leqslant L$, we get

$$NL^{-1/2}\|\mathbb{1}_{D_{N,L}} \cdot (f_{N_1,L_1} * g_{N_2,L_2})\|_{L^2} \lesssim N^{1-\theta/2}(N/L)^{1/2-\theta/2}(L_1^{1/2}\|f_{N_1,L_1}\|_{L^2})(L_2^{1/2}\|g_{N_2,L_2}\|_{L^2}). \tag{4.143}$$

Therefore, taking θ sufficiently close to 1, we sum (4.143) over $L, L_1, L_2 \geqslant (N \vee N_0)$ with $L \geqslant (L_1 \wedge L_2)$ and then we apply Lemma 4.24 to control the second term on the r.h.s of (4.140). This completes the proof of the lemma. □

Lemma 4.36 (*High × High → Low*). *Let $N, N_1, N_2 \in \mathbb{D}$ satisfying $N \ll N_1 \sim N_2$. Then,*

$$\|P_N(\partial_x(u_{N_1}v_{N_2}))\|_{\mathcal{N}_N} \lesssim N^{(1/2)^+}\log(N_{max})\|u_{N_1}\|_{F_{N_1}}\|v_{N_2}\|_{F_{N_2}}, \tag{4.144}$$

whenever $u_{N_1} \in F_{N_1}$ and $v_{N_2} \in F_{N_2}$.

PROOF. Following the same notation employed in the proof of Lemma 4.34, we have

$$\begin{aligned} \|P_N(\partial_x(u_{N_1}v_{N_2}))\|_{\mathcal{N}_N} &\lesssim \sup_{t_N\in\mathbb{R}} \sum_{L,L_1,L_2\geqslant(N\vee N_0)} NL^{-1/2}\|\mathbb{1}_{D_{N,L}} \cdot (f_{N_1,L_1} * g_{N_2,L_2})\|_{L^2} \\ &= \sup_{t_N\in\mathbb{R}} \Big(\sum_{L,L_1,L_2\geqslant(N\vee N_0),\, L=L_{max}} NL^{-1/2}(\cdots) + \sum_{L,L_1,L_2\geqslant(N\vee N_0),\, L<L_{max}} NL^{-1/2}(\cdots)\Big). \end{aligned} \tag{4.145}$$

To estimate the first term on the r.h.s of (4.145), we use (4.130) to deduce

$$NL^{-1/2}\|\mathbb{1}_{D_{N,L}} \cdot (f_{N_1,L_1} * g_{N_2,L_2})\|_{L^2} \lesssim N^{3/2}N_1^{-1/2}(N_1^{1/2} \vee L_{min}^{1/2})\|f_{N_1,L_1}\|_{L^2}\|g_{N_2,L_2}\|_{L^2}, \tag{4.146}$$

where $L, L_1, L_2 \geqslant (N \vee N_0)$, $L = L_{max}$. These restrictions imply, $N_1^{-1/2}(N_1^{1/2} \vee L_{min}^{1/2})L_1^{-1/2}L_2^{-1/2} \lesssim N^{-1/2}L_{med}^{-1/2}$, then when $L_{med} \sim L = L_{max}$, we have

$$N^{3/2}N_1^{-1/2}(N_1^{1/2} \vee L_{min}^{1/2})\|f_{N_1,L_1}\|_{L^2}\|g_{N_2,L_2}\|_{L^2} \lesssim N^{1/2}(N/L)^{1/2}(L_1^{1/2}\|f_{N_1,L_1}\|_{L^2})(L_2^{1/2}\|g_{N_2,L_2}\|_{L^2}). \tag{4.147}$$

Now, when $L_{med} \ll L$, we use the bound,

$$N^{3/2}N_1^{-1/2}(N_1^{1/2} \vee L_{min}^{1/2})\|f_{N_1,L_1}\|_{L^2}\|g_{N_2,L_2}\|_{L^2} \lesssim N^{1/2}(L_1^{1/2}\|f_{N_1,L_1}\|_{L^2})(L_2^{1/2}\|g_{N_2,L_2}\|_{L^2}). \tag{4.148}$$

By support considerations it must follows that $L \sim |\Omega| \lesssim N_1^2$, whenever $L_{med} \ll L$, this implies that summing over L in (4.148) yields a factor of order $\log(N_1)$. This remark completes the estimate for the first sum in (4.145). The remaining sum in (4.145) is bounded directly by (4.131) in the following manner

$$\begin{aligned} NL^{-1/2}&\|\mathbb{1}_{D_{N,L}} \cdot (f_{N_1,L_1} * g_{N_2,L_2})\|_{L^2} \\ &\lesssim N^{3/2}L^{-1/2}N_1^{-1/2}L_{med}^{1/2}(N_1^{1/2} \vee L_{min}^{1/2})\|f_{N_1,L_1}\|_{L^2})(L_2^{1/2}\|g_{N_2,L_2}\|_{L^2}) \\ &\lesssim N^{1/2}(N/L)^{1/2}(L_1^{1/2}\|f_{N_1,L_1}\|_{L^2})(L_2^{1/2}\|g_{N_2,L_2}\|_{L^2}), \end{aligned}$$

where we used that $N_1^{-1/2}(N_1^{1/2} \vee L_{min}^{1/2}) \leqslant N^{-1/2}L_{min}^{1/2}$, for $L_1, L_2, L_3 \geqslant (N \vee N_0)$ and $L < L_{max}$. The proof of the lemma is now completed. □

Lemma 4.37 (*Low × Low → Low*). *Let $N, N_1, N_2 \in \mathbb{D}$ satisfying $N, N_1, N_2 \ll 1$. Then,*

$$\|P_N(\partial_x(u_{N_1}v_{N_2}))\|_{\mathcal{N}_N} \lesssim \|u_{N_1}\|_{F_{N_1}}\|v_{N_2}\|_{F_{N_2}}, \tag{4.149}$$

whenever $u_{N_1} \in F_{N_1}$ and $v_{N_2} \in F_{N_2}$.

PROOF. Following a similar reasoning as in the proof of Lemma 4.34, we notice that it is enough to establish

$$NL^{-1/2}\|\mathbb{1}_{D_{N,L}} \cdot (f_{N_1,L_1} * g_{N_2,L_2})\|_{L^2} \lesssim L^{-1/2}(L_1^{1/2}\|f_{N_1,L_1}\|_{L^2})(L_2^{1/2}\|g_{N_2,L_2}\|_{L^2}), \tag{4.150}$$

for $L, L_1, L_2 \geqslant N_0$, where we define $f_{N_1,N_0} = \varphi_{\leqslant N_0}(\tau - \omega(m,n))f_{N_1}(m,n,\tau)$ and $f_{N_1,L_1} = \varphi_{L_1}(\tau - \omega(m,n))f_{N_1}(m,n,\tau)$ $L_1 > N_0$ with f_{N_1} as in (4.136) and similarly we set g_{N_2,L_2} with $L_2 \geqslant N_0$. In this manner, we have that (4.150) is a direct consequence of (4.129) and the fact that $N, N_1, N_2 \lesssim 1$. □

We are in conditions to prove Proposition 4.33.

PROOF OF PROPOSITION 4.33. We will only deduce (4.133), since (4.134) is obtained by the same reasoning. For each $N_1, N_2 \in \mathbb{D}$, we choose extensions u_{N_1}, v_{N_2} of $P_{N_1}u$ and $P_{N_2}v$ satisfying $\|u_{N_1}\|_{F^s} \leqslant 2\|P_{N_1}u\|_{F^s_{N_1}(T)}$ and $\|v_{N_2}\|_{F^s} \leqslant 2\|P_{N_2}v\|_{F^s_{N_2}(T)}$. By the definition of the space $\mathcal{N}^s(T)$ and Minkowski inequality we have

$$\|\partial_x(uv)\|_{\mathcal{N}_N(T)} \lesssim \sum_{j=1}^{5}\Big(\sum_{N\geqslant 1}(N^{2s}+N_0^{2s})\Big(\sum_{(N_1,N_2)\in A_j}\|P_N(\partial_x(u_{N_1}v_{N_2}))\|_{\mathcal{N}_N}\Big)^2\Big)^{1/2} =: \sum_{j=1}^{5} S_j,$$

where

$$\begin{aligned}
A_1 &= \{(N_1,N_2)\in\mathbb{D}^2 : N_1 \ll N \sim N_2\},\\
A_2 &= \{(N_1,N_2)\in\mathbb{D}^2 : N_2 \ll N \sim N_1\},\\
A_3 &= \{(N_1,N_2)\in\mathbb{D}^2 : N \sim N_1 \sim N_2 \gg 1\},\\
A_4 &= \{(N_1,N_2)\in\mathbb{D}^2 : N \ll N_1 \sim N_2\},\\
A_5 &= \{(N_1,N_2)\in\mathbb{D}^2 : N \sim N_1 \sim N_2 \lesssim 1\}.
\end{aligned}$$

To estimate S_1, we use Lemma 4.34, the fact that $N_1^{1/2+\epsilon} \lesssim T_0^{1/4}(N_1^{3/4+\epsilon}+N_0^{3/4+\epsilon})$ for $\epsilon > 0$ small enough and the definition of $F^s(T)$ to derive

$$\begin{aligned}
S_1 &\lesssim T_0^{1/4}\Big(\sum_{N\geqslant 1}(N^{2s}+N_0^{2s})\Big(\sum_{N_1\ll N}N_1^{-\epsilon}(N_1^{3/4+\epsilon}+N_0^{3/4+\epsilon})\|u_{N_1}\|_{F_{N_1}}\|v_N\|_{F_N}\Big)^2\Big)^{1/2}\\
&\lesssim T_0^{1/4}\|u\|_{F^{s_0}(T)}\|v\|_{F^s(T)}.
\end{aligned}$$

The estimate for S_2 is obtained symmetrically as above. Next, we use Lemma 4.35 and that $N^{(1/2)^+} \lesssim T_0^{1/4}(N^{(3/4)^+}+N_0^{(3/4)^+})$ to obtain

$$S_3 \lesssim T_0^{1/4}\Big(\sum_{N\geqslant 1}(N^{2s}+N_0^{2s})(N^{(3/4)^+}+N_0^{(3/4)^+})\|u_N\|_{F_N}^2\|v_N\|_{F_N}^2\Big)^{1/2} \lesssim T_0^{1/4}\|u\|_{F^{s_0}(T)}\|v\|_{F^s(T)}.$$

Let $0 < \epsilon \ll 1$ fixed, then Lemma 4.36 and the Cauchy-Schwarz inequality yield

$$\begin{aligned}
S_4 &\lesssim T_0^{1/4}\Big(\sum_{N\geqslant 1}N^{-\epsilon}\Big(\sum_{N\ll N_1,N_2}N_1^{-\epsilon/2}N_2^{-\epsilon/2}(N^s+N_0^s)(N_{max}^{3/4+4\epsilon}+N_0^{3/4+4\epsilon})\|u_{N_1}\|_{F_{N_1}}\|v_{N_2}\|_{F_{N_2}}\Big)^2\Big)^{1/2}\\
&\lesssim T_0^{1/4}\|u\|_{F^{s_0}(T)}\|v\|_{F^s(T)},
\end{aligned}$$

which holds given that $N^{1/2+2\epsilon}\log(N_{max}) \lesssim T_0^{1/4}N_1^{-\epsilon/2}N_2^{-\epsilon/2}(N_{max}^{3/4+4\epsilon}+N_0^{3/4+4\epsilon})$. The estimate for S_5 follows from Lemma 4.37 and similar considerations as above. This concludes the deduction of (4.133). □

4.4.4. Energy estimates. This subsection is devoted to derive all the estimates required to control the B^s-norm of regular solutions and the difference of solutions.

Lemma 4.38. *Let $s_0 > 1/2$, then there exists $\nu > 0$ small enough such that for $T \in (0, T_0]$ it holds that*

$$\left| \int_{\mathbb{T}^2\times[0,T]} u_1u_2u_3 \right| \lesssim T^\nu N_{min}^{s_0}\prod_{i=1}^3 \|u_i\|_{F_{N_i}(T)}, \tag{4.151}$$

for each function $u_i \in F_{N_i}(T)$, $i = 1,2,3$.

PROOF. In view of (4.117) we will assume that $N_1 \leqslant N_2 \leqslant N_3$. Let $\widetilde{u}_i \in F_{N_i}$ extensions of u_i to $\mathbb{R}$ such that $\|\widetilde{u}_i\|_{F_{N_i}} \leqslant 2\|u_i\|_{F_{N_i}(T)}$, $i = 1,2,3$. Now let $h : \mathbb{R} \to \mathbb{R}$ be a smooth function supported in $[-1,1]$ such that

$$\sum_{k\in\mathbb{Z}} h^3(x-k) = 1, \qquad \forall x \in \mathbb{R}.$$

Then, we write

$$\begin{aligned}\left| \int_{\mathbb{T}^2\times[0,T]} u_1u_2u_3 \right| &\lesssim \sum_{|k|\lesssim N_3}\int_{\mathbb{Z}^2\times\mathbb{R}} |\mathcal{F}(h(N_3t-k)\mathbb{1}_{[0,T]}\widetilde{u}_3)| \\ &\qquad \cdot \left(|\mathcal{F}(h(N_3t-k)\mathbb{1}_{[0,T]}\widetilde{u}_1)|\right) * \left(|\mathcal{F}(h(N_3t-k)\mathbb{1}_{[0,T]}\widetilde{u}_2)|\right) \\ &=: \sum_{\mathcal{A}}(\cdots) + \sum_{\mathcal{B}}(\cdots),\end{aligned} \tag{4.152}$$

where

$$\mathcal{A} = \{k\in\mathbb{Z} : h(N_3t-k)\mathbb{1}_{[0,T]} = h(N_3t-k)\},$$
$$\mathcal{B} = \{k\in\mathbb{Z} : h(N_3t-k)\mathbb{1}_{[0,T]} \neq h(N_3t-k) \text{ and } h(N_3t-k)\mathbb{1}_{[0,T]} \neq 0\}.$$

Let us estimate first the sum over $\mathcal{A}$ in (4.152). Recalling the dyadic number N_0 defining the spaces X_N^b, we denote by

$$f^k_{N_i,N_3\vee N_0} = \varphi_{\leqslant(N_3\vee N_0)}(\tau-\omega)|\mathcal{F}(h(N_3t-k)\widetilde{u}_3)|,$$
$$f^k_{N_i,L} = \varphi_{L_3}(\tau-\omega)|\mathcal{F}(h(N_3t-k)\widetilde{u}_3)|,$$

for each $i = 1,2,3$, $L > (N_3\vee N_0)$ and $k\in\mathcal{A}$. Now since there are at most N_3T integers in $\mathcal{A}$, we employ (4.114) and (4.115) when $N_1 \ll N_3$, or (4.116) if $N_1 \sim N_3$ to deduce that

$$\begin{aligned}\mathcal{I}_{\mathcal{A}} &\lesssim \sum_{|k|\in\mathcal{A}}\sum_{L_1,L_2,L_3\geqslant(N_3\vee N_0)}\int_{\mathbb{Z}^2\times\mathbb{R}} (f^k_{N_1,L_1} * f^k_{N_2,L_2})\cdot f^k_{N_3,L_3} \\ &\lesssim N_1^{1/2}T\sup_{k\in\mathcal{A}}\prod_{i=1}^3\sum_{L_i\geqslant(N_3\vee N_0)} L_i^{1/2}\|f^k_{N_i,L_i}\|_{L^2} \lesssim N_1^{1/2}T\prod_{i=1}^3\|\widetilde{u}_i\|_{F_{N_i}},\end{aligned} \tag{4.153}$$

where the last line above follows from (4.106) and (4.107).

Next we deal with the sum over $\mathcal{B}$ in (4.152). We consider $b \in (0,1/2)$ fixed and let

$$g^k_{N_i,L} := \varphi_{L_i}(\tau-\omega)|\mathcal{F}(h(N_3t-k)\mathbb{1}_{[0,T]}\widetilde{u}_i|,$$

for each $i=1,2,3$, $L \in \mathbb{D}$ and $k \in \mathcal{B}$. We treat first the case $N_1 \ll N_3$. Since $\#\mathcal{B} \lesssim 1$, we have

$$\begin{aligned}\mathcal{I}_{\mathcal{B}} &\lesssim \sup_{k\in\mathcal{B}} \sum_{L_1,L_2,L_3\geqslant 1} \int_{\mathbb{Z}^2\times\mathbb{R}} (g^k_{N_1,L_1} * g^k_{N_2,L_2}) \cdot g^k_{N_3,L_3} \\ &\lesssim \sup_{k\in\mathcal{B}} \Big(\sum_{L_2,L_3\leqslant L_1} \int_{\mathbb{Z}^2\times\mathbb{R}} (\cdots) + \sum_{L_1,L_2,L_3,\, L_1<L_{max}} \int_{\mathbb{Z}^2\times\mathbb{R}} (\cdots) \Big) =: \sup_{k\in\mathcal{B}} \left(\mathcal{I}^{1,k}_{\mathcal{B}} + \mathcal{I}^{2,k}_{\mathcal{B}}\right).\end{aligned}$$

From (4.114) and the fact that $N_3^{-1/2}(N_3^{1/2} \vee L_{min}^{1/2})L_{min}^{-1/2} \leqslant (N_3^{-1/2} \vee L_{min}^{-1/2}) \leqslant 1$, we get

$$\begin{aligned}\mathcal{I}^{1,k}_{\mathcal{B}} &\lesssim \sum_{L_2,L_3\leqslant L_1} N_3^{-1/2}N_1^{1/2}L_{max}^{1/2}(N_3^{1/2} \vee L_{min}^{1/2})\|g^k_{N_1,L_1}\|_{L^2}\|g^k_{N_2,L_2}\|_{L^2}\|g^k_{N_3,L_3}\|_{L^2} \\ &\lesssim \sum_{L_2,L_3\leqslant L_1} N_1^{1/2}L_{max}^{1/2}L_{min}^{1/2}\|g^k_{N_1,L_1}\|_{L^2}\|g^k_{N_2,L_2}\|_{L^2}\|g^k_{N_3,L_3}\|_{L^2}.\end{aligned} \tag{4.154}$$

In the regions where $L_{med} \sim L_{max}$, we use Lemmas 4.25 and 4.26, together with the fact that $\|\widetilde{u}_i\|_{F_{N_i}} \leqslant 2\|u_i\|_{F_{N_i}(T)}$ to deduce

$$\begin{aligned}&\sup_{k\in\mathcal{B}} \sum_{\substack{L_2,L_3\leqslant L_1,\\ L_{med}\sim L_{max}}} N_1^{1/2}L_{med}^{-b}L_{max}^{1/2}L_{med}^{b}L_{min}^{1/2}\|g^k_{N_1,L_1}\|_{L^2}\|g^k_{N_2,L_2}\|_{L^2}\|g^k_{N_3,L_3}\|_{L^2} \\ &\qquad\qquad \lesssim N_1^{1/2}\Big(\sum_{L_1,L_2,L_3} L_{max}^{-b}\Big)T^{(1/2-b)-}\prod_{i=1}^{3}\|u_i\|_{F_{N_i}(T)}.\end{aligned} \tag{4.155}$$

Now we deal with the case $L_{med} \ll L_{max}$. Interpolating the right-hand side of (4.154) with the bound derived for $\mathcal{I}^{1,k}_{\mathcal{B}}$ using (4.113) instead of (4.114), we find for all $\theta \in [0,1)$ that

$$\begin{aligned}&\sup_{k\in\mathcal{B}} \sum_{\substack{L_2,L_3\leqslant L_1,\\ L_{med}\ll L_{max}}} N_1^{1-\theta/2}L_{max}^{\theta/2}L_{min}^{1/2}\|g^k_{N_1,L_1}\|_{L^2}\|g^k_{N_2,L_2}\|_{L^2}\|g^k_{N_3,L_3}\|_{L^2} \\ &= \sup_{k\in\mathcal{B}} \sum_{\substack{L_1=L_{max},\\ L_{med}\ll L_{max}}} N_1^{1-\theta/2}L_{med}^{-b}L_{max}^{-(1-\theta)/2}L_{max}^{1/2}L_{med}^{b}L_{min}^{1/2}\|g^k_{N_1,L_1}\|_{L^2}\|g^k_{N_2,L_2}\|_{L^2}\|g^k_{N_3,L_3}\|_{L^2} \\ &\lesssim N_1^{1-\theta/2}\Big(\sum_{L_1,L_2,L_2} L_{max}^{-(1-\theta)/2}L_{med}^{-b}\Big)T^{(1/2-b)-}\prod_{i=1}^{3}\|u_i\|_{F_{N_i}(T)}.\end{aligned} \tag{4.156}$$

Therefore, the estimate for $\sup_{k\in\mathcal{B}} \mathcal{I}^{1,k}_{\mathcal{B}}$ is now a consequence of (4.155) and (4.156). On the other hand, we can implement (4.115) and the same ideas dealing with (4.155) to derive the following bound

$$\begin{aligned}\sup_{k\in\mathcal{B}} \mathcal{I}^{2,k}_{\mathcal{B}} &\lesssim \sup_{k\in\mathcal{B}} \sum_{L_1,L_2,L_3,L_1<L_{max}} N_1^{1/2}L_{max}^{-b}L_{max}^{b}L_{med}^{1/2}L_{min}^{1/2}\|g^k_{N_1,L_1}\|_{L^2}\|g^k_{N_2,L_2}\|_{L^2}\|g^k_{N_3,L_3}\|_{L^2} \\ &\lesssim N_1^{1/2}\Big(\sum_{L_1,L_2,L_2} L_{max}^{-b}\Big)T^{(1/2-b)-}\prod_{i=1}^{3}\|u_i\|_{F_{N_i}(T)}.\end{aligned}$$

This completes the analysis of $\mathcal{I}_{\mathcal{B}}$ in the region $N_1 \ll N_3$. Next we treat the case $N_1 \sim N_2$. Interpolating (4.113) and (4.116), we obtain for all $\theta \in [0,1]$ that

$$\begin{aligned}\mathcal{I}_{\mathcal{B}} &\lesssim \sup_{k\in\mathcal{B}} \sum_{L_1,L_2,L_3} (L_{max}^{-\theta/2}(N_1^{1/2} \vee L_{med}^{1/2})^{1-\theta} N_1^{\theta} L_{med}^{-1/2} L_{min}^{\theta/2-b}) L_{max}^{1/2} L_{med}^{1/2} L_{min}^{b} \\ &\qquad\qquad \times \|g^k_{N_1,L_1}\|_{L^2} \|g^k_{N_2,L_2}\|_{L^2} \|g^k_{N_3,L_3}\|_{L^2} \\ &\lesssim \sup_{k\in\mathcal{B}} N_1^{1/2+\theta/2} \sum_{L_1,L_2,L_3} L_{max}^{-\theta/2} (L_{med}^{-\theta/2} L_{min}^{\theta/2-b}) L_{max}^{1/2} L_{med}^{1/2} L_{min}^{b} \|g^k_{N_1,L_1}\|_{L^2} \|g^k_{N_2,L_2}\|_{L^2} \|g^k_{N_3,L_3}\|_{L^2}.\end{aligned} \tag{4.157}$$

Therefore, taking θ sufficiently small and employing a similar reasoning to (4.156), the estimate for $\mathcal{I}_{\mathcal{B}}$ when $N_1 \sim N_3$ is a consequence of (4.157). Gathering all the previous results we obtain (4.151) for $\nu = 1/2 - b$. □

Lemma 4.39. *Assume that $s_0 > 3/2$, $N_1 \ll N$, then there exists $\nu > 0$ such that for $T \in (0,1]$,*

$$\left| \int_{\mathbb{T}^2\times\mathbb{R}} P_N(\partial_x u P_{N_1} v) P_N u \, dxdydt \right| \lesssim_{s_0} T^{\mu} N_{min}^{s_0} \|v\|_{F_{N_1}(T)} \sum_{N_2 \sim N} \|u\|^2_{F_{N_2}(T)},$$

whenever $v \in F_{N_1}(T)$ and $u \in F_{N_2}(T)$.

PROOF. We divide the integral expression in the following manner

$$\begin{aligned}&\int_{\mathbb{T}^2\times\mathbb{R}} P_N(\partial_x u P_{N_1} v) P_N u \, dxdydt \\ &\qquad = \int_{\mathbb{T}^2\times\mathbb{R}} \partial_x P_N u P_{N_1} v P_N u + \int_{\mathbb{T}^2\times\mathbb{R}} P_N(\partial_x u P_{N_1} v) P_N u - \partial_x P_N u P_{N_1} v P_N u \\ &\qquad = \mathcal{I} + \mathcal{II}.\end{aligned} \tag{4.158}$$

Integrating by parts and using (4.151), the first term on the right-hand side of the above expression satisfies

$$\mathcal{I} \lesssim T^{\nu} N_1^{(3/2)^+} \|v\|_{F_{N_1}(T)} \|u\|^2_{F_N(T)}. \tag{4.159}$$

The estimate for $\mathcal{II}$ is deduced arguing as in [**44**, Lemma 6.1], for completeness we shall show this procedure. We consider extensions $\widetilde{u}$ and $\widetilde{v}$ of u and v respectively such that $\|\widetilde{u}\|_{F_N} \leqslant 2\|u\|_{F_N(T)}$ and $\|\widetilde{v}\|_{F_{N_1}} \leqslant 2\|v\|_{F_N}$, then we write

$$\mathcal{II} = \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}^2\times\mathbb{R}} \left(P_N(\partial_x \widetilde{u}^k P_{N_1} \widetilde{v}^k) - \partial_x P_N \widetilde{u}^k P_{N_1} \widetilde{v}^k\right) P_N \widetilde{u}^k, \tag{4.160}$$

where $\widetilde{u}^k = \mathbb{1}_{[0,T]} h(Nt-k)\widetilde{u}$ and $\widetilde{v}^k = \mathbb{1}_{[0,T]} h(Nt-k)\widetilde{v}$ with h defined as in the proof of Lemma 4.38. In addition, we consider smooth partitions $\widetilde{P}_N$ defined by $\mathcal{F}(\widetilde{P}_N\phi)(m,n) = \eta_N(m,n)\widehat{\phi}(m,n)$ with $\eta_N : \mathbb{R}^2 \to \mathbb{R}$ smooth compact supported in $\{|(\xi,\eta)| \sim N\}$ with the property that $\widetilde{P}_N P_N = P_N$. Then we have

$$\begin{aligned}&\mathcal{F}\left([P_N(\partial_x \widetilde{u}^k P_{N_1} \widetilde{v}^k) - \partial_x P_N \widetilde{u}^k P_{N_1} \widetilde{v}^k]\right)(m,n) \\ &\qquad = \sum_{m_1,n_1} K(m,n,m_1,n_1) \mathcal{F}(\widetilde{u}^k)(m-m_1, n-n_1) \mathcal{F}(|\nabla| P_{N_1} \widetilde{v}^k)(m_1,n_1)\end{aligned} \tag{4.161}$$

where using that $|(m-m_1,n-n_1)|\sim N$, we set

$$K(m,n,m_1,n_1) = (m-m_1)\left\{\frac{\eta_N(m,n)-\eta_N(m-m_1,n-n_1)}{|(m_1,n_1)|}\right\}\widetilde{\eta}_{N_1}(m_1,n_1)\sum_{N_2\sim N}\eta_{N_2}(m-m_1,n-n_1)$$

with $\widetilde{\eta}_{N_1}\eta_{N_1}=\eta_{N_1}$. Since

$$|K(m,n,m_1,n_1)|\lesssim \widetilde{\eta}_{N_1}(m_1,n_1)\sum_{N_2\sim N}\widetilde{\eta}_{N_2}(m-m_1,n-n_1),$$

we can combine the last inequality, (4.161) and (4.160), to employ the same arguments leading to (4.151) to obtain the desired estimate for $\mathcal{II}$. This completes the proof of the lemma. □

PROPOSITION 4.40. *Let $T\in(0,T_0]$ and $s\geqslant s_0>3/2$. Then for any $u\in C([0,T];H^{\infty}(\mathbb{T}^2))$ solution of the IVP (0.4) on $[0,T]$,*

$$\|u\|^2_{B^s(T)}\lesssim \|u_0\|^2_{H^s}+T^{\nu}\|u\|_{F^{s_0}(T)}\|u\|^2_{F^s(T)}. \tag{4.162}$$

PROOF. According to the definition of the spaces $B^s(T)$ and the fact that u solves the IVP (0.4), it is enough to derive a bound for the following expression

$$N^{2s}\|P_N u(t_N)\|^2_{L^2}\lesssim N^{2s}\|P_N u_0\|^2_{L^2}+N^{2s}\left|\int_{\mathbb{T}^2\times[0,T]}P_N(u\partial_x u)P_N u\,dxdydt\right|, \tag{4.163}$$

for $N\geqslant N_0$. Now we split the estimate of the integral term above according to the following iterations: *High* × *Low* → *High*,

$$\int_{\mathbb{T}^2\times[0,T]}P_N(\partial_x u P_{N_1}u)P_N u\,dxdydt,\ \text{where } N_1\ll N, \tag{4.164}$$

Low × *High* → *High*,

$$\int_{\mathbb{T}^2\times[0,T]}P_N(\partial_x P_{N_1}u P_{N_2}u)P_N u\,dxdydt,\ \text{where } N_1\ll N_2\sim N, \tag{4.165}$$

High × *High* → *High*,

$$\int_{\mathbb{T}^2\times[0,T]}P_N(\partial_x P_{N_1}u P_{N_2}u)P_N u\,dxdydt,\ \text{where } N\sim N_1\sim N_2, \tag{4.166}$$

and *High* × *High* → *Low*,

$$\int_{\mathbb{T}^2\times[0,T]}P_N(\partial_x P_{N_1}u P_{N_2}u)P_N u\,dxdydt,\ \text{where } N\ll N_1\sim N_2. \tag{4.167}$$

In view of Lemma 4.39, the *High* × *Low* → *High* iteration satisfies

$$(4.164)\lesssim T^{\nu}N_1^{(3/2)^+}\|P_{N_1}u\|_{F_{N_1}(T)}\sum_{N_2\sim N}\|P_{N_2}u\|^2_{F_{N_2}(T)}. \tag{4.168}$$

Summing the above expression over N and $N_1\ll N$, we can modify the power of $N_1^{(3/2)^+}$ by an small factor to apply the Cauchy-Schwarz inequality in the sum over N_1. Next, we apply the same inequality for the sum over N_1, obtaining (4.162) for this case. Recalling (4.159) in the proof of Lemma 4.39, we notice that the *Low* × *High* → *High* iteration satisfies the same estimate on the r.h.s of (4.168).

Next we apply (4.151) to control the $High \times High \to High$ iterations as follows

$$(4.166) \lesssim T^{\nu} N^{(3/2)^+} \|P_N u\|_{F_N(T)} \|P_{N_1} u\|_{F_{N_1}(T)} \|P_{N_2} u\|_{F_{N_2}(T)}. \tag{4.169}$$

Since $N \sim N_1 \sim N_2$, we can increase the power in $N^{(3/2)^+}$ by a small factor to apply the Cauchy-Schwarz inequality separately in each of the sums N, N_1, N_2 to derive the desired result. The estimate for $High \times High \to Low$ is obtained by (4.151) and a similar reasoning to the iteration $High \times High \to High$. This completes the estimate for the r.h.s of (4.163) and thus the deduction of (4.162). □

We also require the following result to deal with the difference of solutions.

PROPOSITION 4.41. *Let $T \in (0, T_0]$, $s \geq s_0 > 3/2$. Consider $u, v \in C([0,T]; H^{\infty}(\mathbb{T}^2))$ solutions of the IVP (0.4) with initial data $u_0, v_0 \in H^{\infty}(\mathbb{T}^2)$ respectively, then*

$$\|u-v\|_{B^0(T)}^2 \lesssim \|u_0-v_0\|_{L^2}^2 + T^{\nu}\big(\|u-v\|_{F^{s_0}(T)}\|u-v\|_{F^0(T)}^2 + \|v\|_{F^{s_0}(T)}\|u-v\|_{F^0(T)}^2\big), \tag{4.170}$$

and

$$\begin{aligned}\|u-v\|_{B^s(T)}^2 \lesssim \|u_0-v_0\|_{H^s}^2 + T^{\nu}\big(&\|v\|_{F^{s_0}(T)}\|u-v\|_{F^s(T)}^2 + \|u-v\|_{F^{s_0}(T)}\|u-v\|_{F^s(T)}\|v\|_{F^s(T)} \\ &+ \|v\|_{F^{(s+3/2)^+}(T)}\|u-v\|_{F^s(T)}\|u-v\|_{F^0(T)}\big),\end{aligned} \tag{4.171}$$

where the implicit constants are independent of T_0.

PROOF. We shall employ a similar reasoning to the proof of Proposition 4.40. Letting $w = u - v$, we find that w solves the equation:

$$\begin{cases} \partial_t w + \mathcal{H}_x w - \mathcal{H}_x \partial_x^2 w \pm \mathcal{H}_x \partial_y^2 w + \frac{1}{2}\partial_x((u+v)w) = 0, \\ w(x,0) = u_0 - v_0. \end{cases} \tag{4.172}$$

Let $\widetilde{s} \in \{0, s\}$. The definition of the $B^{\widetilde{s}}(T)$-norm and the fact that w solves (4.172) yield

$$\begin{aligned}\|w\|_{B^{\widetilde{s}}(T)}^2 &\lesssim \|P_{\leq N_0} w(0)\|_{H^{\widetilde{s}}}^2 + \sum_{N > N_0} \sup_{t_N} N^{\widetilde{s}} \|P_N w(t_N)\|_{L^2}^2 \\ &\lesssim \|w(0)\|_{H^{\widetilde{s}}} + \sum_{N > N_0} N^{2\widetilde{s}} \left| \int_{\mathbb{T}^2 \times [0,T]} P_N(w\partial_x w + v\partial_x w + \partial_x v w) P_N w \, dxdydt \right|.\end{aligned} \tag{4.173}$$

Then, we are reduced to estimate the integral term on the right-hand side of the last inequality. Arguing as in the proof of Proposition 4.40, applying Lemmas 4.38 and 4.39, we obtain

$$\begin{aligned}\sum_{N > N_0} &\left| \int_{\mathbb{T}^2 \times [0,T]} P_N(w\partial_x w + v\partial_x w) P_N w \, dxdydt \right| \\ &\lesssim T^{\nu}\big(\|w\|_{F^{(3/2)^+}(T)}\|w\|_{F^0(T)}^2 + \|v\|_{F^{(3/2)^+}(T)}\|w\|_{F^0(T)}^2\big)\end{aligned} \tag{4.174}$$

and

$$\begin{aligned}\sum_{N > N_0} N^{2s} &\left| \int_{\mathbb{T}^2 \times [0,T]} P_N(w\partial_x w + v\partial_x w) P_N w \, dxdydt \right| \\ &\lesssim T^{\nu}\big(\|v\|_{F^{(3/2)^+}(T)}\|w\|_{F^s(T)}^2 + \|w\|_{F^{(3/2)^+}(T)}\|w\|_{F^s(T)}\|v\|_{F^s(T)}\big),\end{aligned} \tag{4.175}$$

where we emphasize that the last term on the right-hand side of (4.175) appears from the estimate dealing with the *Low* × *High* → *High* iteration and Lemma 4.38 since in this case

$$N^{2\tilde{s}}\left|\int_{\mathbb{T}^2\times[0,T]} P_N(\partial_x P_{N_1}wP_{N_2}v)P_N w\,dxdydt\right| \lesssim T^{\nu}N_1^{(3/2)^+}N^{2\tilde{s}}\|P_{N_1}w\|_{F_{N_1}(T)}\|P_{N_2}v\|_{F_{N_2}(T)}\|P_N w\|_{F_N(T)},$$

with $N_1 \ll N \sim N_2$. It remains to estimate the integral involving $v\partial_x w$ in (4.173). We divide our considerations as in the proof of Proposition 4.40 according to the iterations: *High* × *Low* → *High*, *Low* × *High* → *High*, *High* × *High* → *High* and *High* × *High* → *Low*. Notice that in this case we cannot apply Lemma 4.39 to control the *High* × *Low* → *High* iteration. We use instead Lemma 4.38 to find for $N_1 \ll N$ that

$$\begin{aligned} N^{2\tilde{s}}\Big|\int_{\mathbb{T}^2\times[0,T]} P_N(\partial_x vP_{N_1}w)P_N w\,dxdydt\Big| \\ \lesssim T^{\nu}\sum_{N_2\sim N} N_1^{(1/2)^+}N_2N^{2\tilde{s}}\|P_{N_1}w\|_{F_{N_1}(T)}\|P_{N_2}v\|_{F_{N_2}(T)}\|P_N w\|_{F_N(T)}. \end{aligned} \tag{4.176}$$

Summing (4.176) over N and $N_1 \ll N$, we use that $N_1^{(1/2)^+}N_2N^{2\tilde{s}} \lesssim N_2^{(3/2)^++\tilde{s}}N^{\tilde{s}}N_1^{-\epsilon}$ for $0<\epsilon \ll 1$ to apply the Cauchy-Schwarz inequality on the sum over N_1 and then on N to control the resulting expression by (4.170) if $\tilde{s}=0$, or (4.171) if $\tilde{s}=s$. The other iterations are treated as in the proof of Proposition 4.40, and their resulting bounds are the same displayed on the right-hand sides of (4.174) and (4.175) when $\tilde{s}=0$ and $\tilde{s}=s$ respectively. The proof of the proposition is now completed.

□

4.4.5. LWP in $H^s(\mathbb{T}^2)$, $s>3/2$. Here we prove Theorem 4.3. We shall implement similar considerations as in [**44**, **90**] to prove Theorem 4.3. We begin by recalling the local well-posedness result for smooth initial data, which can be deduced applying the parabolic regularization argument (see [**46**, Theorem 2.1]) with the periodic Kato-Ponce estimate in [**42**].

Theorem 4.42. *Let $u_0 \in H^{\infty}(\mathbb{T}^2)$. Then there exist $T>0$ and a unique $u \in C([0,T];H^3(\mathbb{T}^2))$ solution of the IVP (0.4). Moreover, the existence time $T=T(\|u_0\|_{H^3})$ is a non-increasing function of $\|u_0\|_{H^3}$ and the flow-map is continuous.*

We divide the proof of Theorem 4.3 in the following main parts.

4.4.5.1. **A priori estimates for smooth solutions.**

Proposition 4.43. *Let $s>3/2$ and $R>0$. Then there exists $T=T(R)>0$, such that for all $u_0 \in H^{\infty}(\mathbb{T}^2)$ satisfying $\|u_0\|_{H^s} \leqslant R$, then the corresponding solution u of the IVP (0.4) given by Theorem 4.42 is in the space $C([0,T];H^{\infty}(\mathbb{T}^2))$ and satisfies*

$$\sup_{t\in[0,T]}\|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s}. \tag{4.177}$$

Proof. We consider $s>3/2$ fixed and u_0 as in the statement of the proposition. In virtue of Theorem 4.42, there exist $T'=T'(\|u_0\|_{H^3}) \in (0,1]$ and $u \in C([0,T'];H^{\infty}(\mathbb{T}^2))$ solution of the IVP

(0.4) with initial data u_0. Then for a given $T_0 \in (0,1]$ to be chosen later, we collect the estimates (4.111), (4.133) and (4.162) to find for each $s_1 \geqslant s \geqslant s_0 > 3/2$ that

$$\begin{cases} \|u\|_{F^{s_1}(T)} \lesssim \|u\|_{B^{s_1}(T)} + \|\partial_x(u^2)\|_{\mathcal{N}^{s_1}(T)} \\ \|\partial_x(u^2)\|_{\mathcal{N}^{s_1}(T)} \lesssim T_0^{1/4}\|u\|_{F^{s_0}(T)}\|u\|_{F^{s_1}(T)}, \\ \|u\|_{B^{s_1}(T)} \lesssim \|u_0\|_{H^{s_1}} + T_0^{\nu}\|u\|_{F^{s_0}(T)}^{1/2}\|u\|_{F^{s_1}(T)}, \end{cases} \tag{4.178}$$

where $0 < T \leqslant (T' \wedge T_0)$. We emphasize that our arguments indicate that the implicit constants in (4.178) and $\nu > 0$ are independent of $T_0 \in (0,1]$ and in consequence of the definition of the spaces involved (which depend on $N_0 \leqslant T_0^{-1}$). Letting $s_1 = s = s_0$ and $\Gamma_s(T) = \|u\|_{B^s(T)} + \|\partial_x(u^2)\|_{\mathcal{N}^s(T)}$, (4.178) yields

$$\Gamma_s(T) \lesssim \|u_0\|_{H^s} + T_0^{1/4}\Gamma_s(T)^2 + T_0^{\nu}\,\Gamma_s(T)^{3/2}. \tag{4.179}$$

Considering now $s_1 = 3$, $s_0 = s$ in (4.178), we also find

$$\|u\|_{F^3(T)} \lesssim \|u_0\|_{H^3} + T_0^{1/4}\Gamma_s(T)\|u\|_{F^3(T)} + T_0^{\nu}\Gamma_s(T)^{1/2}\|u\|_{F^3(T)}. \tag{4.180}$$

Since the mapping $T \mapsto \|u\|_{B^s(T)}$ is decreasing and continuous with $\lim_{T\to 0}\|u\|_{B^s(T)} \lesssim \|u\|_{H^s}$, from (4.112) it follows that

$$\lim_{T\to 0}\Gamma_s(T) \lesssim \|u_0\|_{H^s}, \tag{4.181}$$

where the implicit constant is independent of T_0 and the definition of the spaces involved. Thus, we can choose $T_0 = T_0(R) > 0$ sufficiently small, such that $T_0^{1/4}R + T_0^{\nu}R^{1/2} \ll 1$ (according to the constants in (4.179) and (4.181)). Then, for this time and the associated spaces $F^s(T), \mathcal{N}^s(T), B^s(T)$, we can apply a bootstrap argument relaying on (4.179), (4.181) and the continuity of $\Gamma_s(T)$ to obtain $\Gamma_s(T) \lesssim \|u_0\|_{H^s}$, for any $0 < T \leqslant T_0$. Consequently, Lemma 4.27 reveals

$$\sup_{t\in[0,(T'\wedge T_0)]}\|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s}.$$

Therefore, up to choosing T_0 smaller at the beginning of the argument, from (4.180) we infer

$$\sup_{t\in[0,(T'\wedge T_0)]}\|u(t)\|_{H^3} \lesssim \|u_0\|_{H^3}.$$

In this manner, the last display and Theorem 4.42 allow us to extend u, if necessary, to the whole interval $[0, T_0(R)]$. This completes the proof of the proposition. □

4.4.5.2. **L^2-Lipschitz bounds and uniqueness.** Let $u, v \in C([0,T']; H^s(\mathbb{T}^2))$ be two solutions of the IVP (0.4) defined on $[0,T']$ with initial data $u_0, v_0 \in H^s(\mathbb{T}^2)$ such that $u, v \in F^s(T,T') \cap \mathcal{N}^s(T,T')$, where we denote by $F^s(T,T')$ and $B^s(T,T')$ the spaces defined at time T' and $0 < T \leqslant T'$. Notice that this implies that $u, v \in F^s(T,T_0) \cap \mathcal{N}^s(T,T_0)$, whenever $0 < T \leqslant T_0 \leqslant T'$. We collect (4.111), (4.134) and (4.170) to get

$$\begin{cases} \|u-v\|_{F^0(T,T_0)} \lesssim \|u-v\|_{B^0(T,T_0)} + \|\partial_x((u+v)(u-v))\|_{\mathcal{N}^0(T,T_0)}, \\ \|\partial_x((u+v)(u-v))\|_{\mathcal{N}^0(T,T_0)} \lesssim T_0^{1/4}(\|u\|_{F^s(T,T_0)} + \|v\|_{F^s(T,T_0)})\|u-v\|_{F^0(T,T_0)}, \\ \|u-v\|_{B^0(T,T_0)} \lesssim \|u_0-v_0\|_{L^2} + T_0^{\nu}(\|u\|_{F^s(T,T_0)} + \|v\|_{F^s(T,T_0)})^{1/2}\|u-v\|_{F^0(T,T_0)}. \end{cases} \tag{4.182}$$

Let $R > 0$, satisfying $\sup_{t\in[0,T']}(\|u(t)\|_{H^s} + \|v(t)\|_{H^s}) \leqslant R$. Following a similar reasoning as in the proof of Proposition 4.43, there exists a time $T_0 = T_0(R) > 0$ sufficiently small, for which $T_0^{1/4}R + T_0^{\nu}R^{1/2} \ll 1$ with respect to the constants in (4.182) and $\|u\|_{F^s(T,T_0)}, \|v\|_{F^s(T,T_0)} \lesssim R$. Consequently, (4.182) and Lemma 4.27 yield

$$\sup_{t\in[0,T]} \|u(t) - v(t)\|_{L^2} \lesssim \|u - v\|_{F^0(T,T_0)} \lesssim \|u_0 - v_0\|_{L^2},$$

for any $0 < T \leqslant T_0$. Thus, if $u_0 = v_0$, the last equation reveals that $u = v$ on $[0, T_0]$. Since T_0 depends on $R = R(\sup_{t\in[0,T']}(\|u(t)\|_{H^s} + \|v(t)\|_{H^s}))$, we can employ the same spaces to repeat this procedure a finite number of times obtaining uniqueness in the whole interval $[0, T']$.

4.4.5.3. **Existence.** Let $R > 0$ and $3/2 < s < 3$ fixed. For a given $u_0 \in H^s(\mathbb{T}^2)$ with $\|u_0\|_{H^s} \leqslant R$, we consider a sequence $(u_{0,n}) \subset H^\infty(\mathbb{T}^2)$ converging to u_0 in $H^s(\mathbb{T}^2)$, such that $\|u_{0,n}\|_{H^s} \leqslant R$. We denote by $\Phi(u_{0,n})$ the solution of the IVP (0.4) with initial data $u_{0,n}$ determined by Theorem 4.42. Therefore, according to Proposition 4.43, there exists $T' = T'(R) > 0$, such that $\Phi(u_{0,n}) \in C([0, T']; H^\infty(\mathbb{T}^2))$ and (4.177) holds. We shall prove that $(\Phi(u_{0,n}))$ defines a Cauchy sequence in $C([0, T]; H^s(\mathbb{T}^2))$ for some $0 < T \leqslant T'$. To this aim, we will proceed as in [**44**, **90**].

For a fixed $M > 0$ and $n, l \geqslant 0$ integers, we have

$$\begin{aligned}\sup_{t\in[0,T]} \|\Phi(u_{0,n})(t) - \Phi(u_{0,l})(t)\|_{H^s} \leqslant \sup_{t\in[0,T]} \big(&\|\Phi(u_{0,n})(t) - \Phi(P_{\leqslant M}u_{0,n})(t)\|_{H^s} \\ &+ \|\Phi(P_{\leqslant M}u_{0,n})(t) - \Phi(P_{\leqslant M}u_{0,l})(t)\|_{H^s} \\ &+ \|\Phi(u_{0,l})(t) - \Phi(P_{\leqslant M}u_{0,l})(t)\|_{H^s}\big),\end{aligned} \tag{4.183}$$

for all $0 < T < T'$. Using Sobolev embedding and (4.177), we get

$$\begin{aligned}\|\partial_x\big(\Phi(P_{\leqslant M}u_{0,n}) + \Phi(P_{\leqslant M}u_{0,l})\big)(t)\|_{L^\infty_x} &\lesssim \|\Phi(P_{\leqslant M}u_{0,n})(t)\|_{H^3} + \|\Phi(P_{\leqslant M}u_{0,l})(t)\|_{H^3} \\ &\lesssim \|P_{\leqslant M}u_{0,n}\|_{H^3} + \|P_{\leqslant M}u_{0,l}\|_{H^3}.\end{aligned}$$

Then, the standard energy method and the above inequality show that the second term on the right-hand side of (4.183) is controlled as follows

$$\sup_{t\in[0,T]} \|\Phi(P_{\leqslant M}u_{0,n})(t) - \Phi(P_{\leqslant M}u_{0,l})(t)\|_{H^s} \leqslant C(M)\|u_{0,n} - v_{0,l}\|_{H^s}, \tag{4.184}$$

for each $0 < T < T'$ and some constant $C(M) > 0$ depending on M. Therefore, it remains to estimate the first and last term in (4.183). By symmetry of the argument, we will restrict our considerations to study the former term. To simplify notation, let us denote by $u := \Phi(u_{0,n})$, $v := \Phi(P_{\leqslant M}u_{0,n})$ and $w = u - v$, then taking $T_0 \in (0, T']$, we gather (4.111), (4.133) and (4.171) to find

$$\begin{cases}\|w\|_{F^s(T)} \lesssim \|w\|_{B^s(T)} + \|\partial_x((u+v)w)\|_{\mathcal{N}^s(T)}, \\ \|\partial_x((u+v)w)\|_{\mathcal{N}^s(T)} \lesssim T_0^{1/4}(\|u+v\|_{F^s(T)}\|w\|_{F^s(T)}), \\ \|w\|_{B^s(T)} \lesssim \|u_{0,n} - P_{\leqslant M}u_{0,n}\|_{H^s} + T_0^{\nu}(\|v\|_{F^s(T)}^{1/2}\|w\|_{F^s(T)} + \|v\|_{F^{s'}(T)}^{1/2}\|w\|_{F^s(T)}^{1/2}\|w\|_{F^0(T)}^{1/2}),\end{cases} \tag{4.185}$$

for all $0 < T \leqslant T_0$, and where $s + 3/2 < s' < 2s$ is fixed. The above set of inequalities reveal

$$\|w\|_{F^s(T)} \lesssim \|u_{0,n} - P_{\leqslant M}u_{0,n}\|_{H^s} + (T_0^{1/2}(\|u\|_{F^s(T)} + \|v\|_{F^s(T)}) + T_0^{\nu}\|v\|_{F^s(T)}^{1/2})\|w\|_{F^s(T)} + T_0^{\nu}\|w\|_{F^s(T)}^{1/2}\|v\|_{F^{s'}(T)}^{1/2}\|w\|_{F^0(T)}^{1/2}. \tag{4.186}$$

Repeating the arguments in the proof of Proposition 4.43, using (4.178) with $s_1 = s'$ and $s_0 = s$, we choose $T_0 = T_0(R) < T'$ small so that

$$\|v\|_{F^{s'}(T)} \lesssim \|P_{\leqslant M}u_{0,n}\|_{H^{s'}}, \quad 0 < T \leqslant T_0,$$

and such that, employing (4.182) and similar considerations in the uniqueness part above,

$$\|w\|_{F^0(T)} \lesssim \|u_{0,n} - P_{\leqslant M}u_{0,n}\|_{L^2}, \quad 0 < T \leqslant T_0.$$

Furthermore, we can choose T_0 smaller, if necessary, to assure that $T_0^{1/2}R + T_0^{\nu}R^{1/2} \ll 1$ with respect to the implicit constant in (4.186). Then gathering these estimates in (4.186), we get

$$\begin{aligned}\|w\|_{F^s(T)} &\lesssim \|u_{0,n} - P_{\leqslant M}u_{0,n}\|_{H^s} + \|P_{\leqslant M}u_{0,n}\|_{H^{s'}}^{1/2}\|u_{0,n} - P_{\leqslant M}u_{0,n}\|_{L^2}^{1/2} \\ &\lesssim \|P_{\geqslant M}u_{0,n}\|_{H^s} + M^{s'-2s}\|P_{>M}u_{0,n}\|_{H^s}^{1/2}\|P_{>M}u_{0,n}\|_{H^s}^{1/2},\end{aligned}$$

where, given that $s < s' < 2s$, we have used that $\|P_{\leqslant M}u_{0,n}\|_{H^{s'}} \lesssim M^{s'-s}\|P_{\leqslant M}u_{0,n}\|_{H^s}$. From the above inequality and Lemma 4.27, we arrive at

$$\sup_{t\in[0,T]} \|\Phi(u_{0,n})(t) - \Phi(P_{\leqslant M}u_{0,n})(t)\|_{H^s} \lesssim (1 + \|P_{>M}u_{0,n}\|_{H^s}^{1/2})\|P_{>M}u_{0,n}\|_{H^s}^{1/2}, \tag{4.187}$$

where $0 < T \leqslant T_0$. Therefore, according to our previous discussion, this completes the estimate for the first and third terms on the r.h.s of (4.183). Noticing that for n, l large, $\|P_{>M}u_{0,n}\|_{H^s}$, $\|P_{>M}u_{0,l}\|_{H^s} \leqslant 2\|P_{>M}u_0\|_{H^s}$, we can take M large in (4.187), and then n, l large in (4.184), obtaining that $(\Phi(u_{0,n}))$ is a Cauchy sequence in $C([0,T]; H^s(\mathbb{T}^2))$ for a fixed time $0 < T \leqslant T_0$.

Since each of the elements in the sequence $(\Phi(u_{0,n}))$ solve the integral equation associated to (0.4) in $C([0,T]; H^{s-1}(\mathbb{T}^2))$, we find that the limit of this sequence is in fact a solution of the IVP (0.4) with initial data u_0. This completes the existence part.

4.4.5.4. **Continuity of the flow-map.** It is not difficult to obtain the continuity of the flow-map from the same property for smooth solutions in Theorem 4.42 and the preceding arguments. We refer to **[90]** for a more detailed discussion.

4.5. Well-posedness results in weighted spaces

This section is aimed to establish Theorem 4.5. We will start by introducing some preliminary results.

4.5.1. Notation and additional results. We shall employ the notation introduced in Section 1.2. We consider the approximations $\{w_n\}$ defined in (1.20) for $d = 1$. To explicitly show the dependence on the spatial variables x, y in our estimates, we will denote by $w_{n,x}(x) = w_n(x)$ and $w_{n,y}(y) = w_n(y)$.

Since we are interested in performing energy estimates with the weights w_n and then taking the limit $n \to \infty$, we must assure that all the computations involving the Hilbert transform and the aforementioned weights are independent of the parameter n. In this direction we have:

PROPOSITION 4.44. *For any $\theta \in (-1,1)$ and any $n \in \mathbb{Z}^+$, the Hilbert transform is bounded in $L^2(w_n^{\theta}(x)\,dx)$ with a constant depending on θ but independent of n.*

Proposition 4.44 was stated before in [**29**, Proposition 1]. We require the identity

$$[H_x, x]f = 0 \text{ if and only if } \int_{\mathbb{R}} f(x)\,dx = 0. \tag{4.188}$$

We will employ the characterization of the spaces $L^p_s(\mathbb{R}^d) = J^{-s}L^p(\mathbb{R}^d)$ determined in Theorem 1.13. Additionally, its consequences (1.26), (1.27) and Lemma 1.15 will be constantly employed in our considerations.

Additionally, we require the following result which is proved in much the same way as in [**69**].

PROPOSITION 4.45. *Let $p \in (1,\infty)$. If $f \in L^p(\mathbb{R})$ such that there exists $x_0 \in \mathbb{R}$ for which $f(x_0^+)$, $f(x_0^-)$ are defined and $f(x_0^+) \neq f(x_0^-)$, then for any $\delta > 0$, $\mathcal{D}^{1/p} f \notin L^p_{loc}(B(x_0,\delta))$ and consequently $f \notin L^p_{1/p}(\mathbb{R})$.*

Proposition 4.45 is useful to determine our unique continuation conclusions in Theorems 4.6 and 4.7.

PROPOSITION 4.46. *Let $b \in (0,1)$. For any $t > 0$*

$$\mathcal{D}^b(e^{ix|x|t}) \lesssim (|t|^{b/2} + |t|^b|x|^b), \qquad x \in \mathbb{R} \tag{4.189}$$

and

$$\mathcal{D}^b(e^{i\,\mathrm{sign}(x)t \mp i\,\mathrm{sign}(x)\eta^2 t}) \lesssim |x|^{-b}, \qquad x \in \mathbb{R}\setminus\{0\}, \tag{4.190}$$

for all $\eta \in \mathbb{R}$.

PROOF. Estimate (4.189) follows from the same arguments in [**69**]. On the other hand, since $|e^{i\,\mathrm{sign}(x)t \mp i\,\mathrm{sign}(x)\eta^2 t} - e^{i\,\mathrm{sign}(y)t \mp i\,\mathrm{sign}(y)\eta^2 t}| = 0$ whenever $\mathrm{sign}(y) = \mathrm{sign}(x)$, we perform a change of variables to find

$$\begin{aligned}\mathcal{D}^b(e^{i\,\mathrm{sign}(x)t \mp i\,\mathrm{sign}(x)\eta^2 t}) &= \Big(\int_{\mathbb{R}} \frac{|e^{i\,\mathrm{sign}(x)t \mp i\,\mathrm{sign}(x)\eta^2 t} - e^{i\,\mathrm{sign}(y)t \mp i\,\mathrm{sign}(y)\eta^2 t}|^2}{|x-y|^{1+2b}}\,dy\Big)^{1/2} \\ &\lesssim \Big(\int_{y \geqslant |x|} \frac{1}{|y|^{1+2b}}\,dy\Big)^{1/2} \sim |x|^{-b}.\end{aligned}$$

This completes the deduction of (4.190). $\square$

The following result will be useful to study the behavior of solutions of (0.4) in $L^2(|x|^{2r}\,dxdy)$, whenever $r \in (1/2, 1]$.

Lemma 4.47. *Let $1/2 < s \leqslant 1$ and $f \in H^s(\mathbb{R})$ such that $f(0) = 0$. Then, $\|\mathrm{sign}(\xi) f\|_{H^s} \lesssim \|f\|_{H^s}$.*

PROOF. Since the case $s = 1$ can be easily verified, we will restrict our considerations to the case $1/2 < s < 1$. We first notice that the same argument in the deduction of (4.190) establishes

$$\mathcal{D}^s(\text{sign}(x)) \sim |x|^{-s}.$$

Thus, an application of (1.27) and the previous result reduces our analysis to prove

$$\||\cdot|^{-s} f\|_{L^2} \lesssim \|f\|_{H^s}. \tag{4.191}$$

However, the preceding estimate is a consequence of [88, Proposition 3.2] and the assumption $f(0) = 0$. □

Now we are in the condition to prove Theorem 4.5

4.5.2. Well-posedness in Z_{s,r_1,r_2} and $\dot{Z}_{s,r_1,r_2}$. This part is dedicated to prove Theorem 4.5. In view of Theorem 4.1, for a given $u_0 \in Z_{r_1,r_2,s}(\mathbb{R}^2) = H^s(\mathbb{R}^2) \cap L^2((|x|^{2r_1} + |y|^{2r_2})\,dxdy)$ there exist $T = T(\|u_0\|_{H^s}) > 0$ and

$$u \in C([0,T]; H^s(\mathbb{R}^2)) \cap L^1([0,T]; W^{1,\infty}(\mathbb{R}^2))$$

solution of the IVP (0.4). Let $0 \leqslant K < \infty$ defined by

$$K = \|u\|_{L^\infty_T H^s} + \|u\|_{L^1_T L^\infty_{xy}} + \|\nabla u\|_{L^1_T L^\infty_{xy}}. \tag{4.192}$$

In what follows, we will assume that u is sufficiently regular to perform all the computations required in this section. Indeed, the proof of Theorem 4.1 establishes that there exists a smooth sequence of solutions $u_N \in C([0,T]; H^\infty(\mathbb{R}^2))$ with $u_N(0) \in H^\infty(\mathbb{R}^2) \cap L^2((|x|^{2r_1} + |y|^{2r_2})\,dxdy)$, satisfying $u_N(0) \to u_0$ in the $Z_{r_1,r_2,s}(\mathbb{R}^2)$ topology, and such that (4.63) and (4.64) hold. Thus, applying our arguments to the sequence u_N and then taking the limit $N \to \infty$ yield the required assumption on u.

4.5.2.1. **LWP in Z_{s,r_1,r_2}, $r_1 \in [0,1/2)$, $r_2 \geqslant 0$.** Here we deduce Theorem 4.5 (i). Let us first prove the persistence property $u \in C([0,T]; L^2((|x|^{2r_1} + |y|^{2r_2})\,dxdy))$. We begin by deriving some estimates in the spaces $L^2(|x|^{2r_1}\,dxdy)$ and $L^2(|y|^{2r_2}\,dxdy)$.

Estimate for $L^2(|x|^{2r_1}\,dxdy)$. Here, $0 < r_1 < 1/2$ fixed. We apply $\mathcal{H}_x$ to the equation in (0.4) to find

$$\partial_t \mathcal{H}_x u - u + \partial_x^2 u \mp \partial_y^2 u + \mathcal{H}_x(u\partial_x u) = 0, \tag{4.193}$$

multiplying then by $\mathcal{H}_x u\, w_{n,x}^{2r_1}$ and integrating in space, we infer

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathcal{H}_x u(t)\, w_{n,x}^{r_1}\|_{L^2_{xy}}^2 &- \int u\mathcal{H}_x u\, w_{n,x}^{2r_1}\,dxdy + \int \partial_x^2 u \mathcal{H}_x u\, w_{n,x}^{2r_1}\,dxdy \\ &\mp \int \partial_y^2 u \mathcal{H}_x u\, w_{n,x}^{2r_1}\,dxdy + \int \mathcal{H}_x(u\partial_x u)\mathcal{H}_x u\, w_{n,x}^{2r_1}\,dxdy = 0.\end{aligned} \tag{4.194}$$

Multiplying the equation in (0.4) by $uw_{N,x}^{2r_1}$ and then integrating in space, it is seen that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)w_{n,x}^{r_1}\|_{L^2_{xy}}^2 &+ \int \mathcal{H}_x u u w_{n,x}^{2r_1}\,dxdy - \int \mathcal{H}_x \partial_x^2 u u w_{n,x}^{2r_1}\,dxdy \\ &\pm \int \mathcal{H}_x \partial_y^2 u w_{n,x}^{2r_1}\,dxdy + \int u\partial_x u u w_{n,x}^{2r_1}\,dxdy = 0.\end{aligned} \tag{4.195}$$

Adding the differential inequalities (4.194) and (4.195), after integrating by parts in the y variable we deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|u(t)w_{n,x}^{r_1}\|_{L^2_{xy}}^2+\|\mathcal{H}_x u(t)\,w_{n,x}^{r_1}\|_{L^2_{xy}}^2\right)&=\int(\mathcal{H}_x\partial_x^2 uu-\partial_x^2 u\mathcal{H}_x u)w_{n,x}^{2r_1}\,dxdy\\&\quad-\int(u\partial_x uu+\mathcal{H}_x(u\partial_x u)\mathcal{H}_x u)w_{n,x}^{2r_1}\,dxdy\\&=:Q_1+Q_2.\end{aligned}\tag{4.196}$$

Now, since $0<r_1<1/2$, $|\partial_x w_{n,x}^{2r_1}|\lesssim w_{n,x}^{r_1}$ with implicit constant independent of n, integrating by parts and using the Cauchy-Schwarz inequality we find

$$\begin{aligned}|Q_1|&=\left|\int\partial_x\mathcal{H}_x uu\partial_x w_{n,x}^{2r_1}\,dxdy-\int\partial_x u\mathcal{H}_x u\,\partial_x w_{n,x}^{2r_1}\,dxdy\right|\\&\lesssim\|\partial_x u\|_{L^\infty_T L^2_{xy}}\|uw_{n,x}^{r_1}\|_{L^2_{xy}}+\|\partial_x u\|_{L^\infty_T L^2_{xy}}\|\mathcal{H}_x u\,w_{n,x}^{r_1}\|_{L^2_{xy}}.\end{aligned}$$

Notice that the norm $\|\partial_x u\|_{L^\infty_T L^2_{xy}}$ is controlled by (4.192). Next, since $0<r_1<1/2$, Proposition 4.44 shows

$$\|\mathcal{H}_x(u\partial_x u)w_{n,x}^{r_1}\|_{L^2_{xy}}=\|\|\mathcal{H}_x(u\partial_x u)w_{n,x}^{r_1}\|_{L^2_x}\|_{L^2_y}\lesssim\|u\partial_x u\,w_{n,x}^{r_1}\|_{L^2_{xy}}\lesssim\|\partial_x u\|_{L^\infty_{xy}}\|u\,w_{n,x}^{r_1}\|_{L^2_{xy}}.$$

Hence, we employ Hölder's inequality to get

$$\begin{aligned}|Q_2|&\leqslant\|\partial_x u\|_{L^\infty_{xy}}\|uw_{n,x}^{r_1}\|_{L^2_{xy}}^2+\|\mathcal{H}_x(u\partial_x u)w_{n,x}^{r_1}\|_{L^2_{xy}}\|\mathcal{H}_x u\,w_{n,x}^{r_1}\|_{L^2_{xy}}\\&\lesssim\|\partial_x u\|_{L^\infty_{xy}}\|uw_{n,x}^{r_1}\|_{L^2_{xy}}^2+\|\partial_x u\|_{L^\infty_{xy}}\|uw_{n,x}^{r_1}\|_{L^2_{xy}}\|\mathcal{H}_x u\,w_{n,x}^{r_1}\|_{L^2_{xy}}.\end{aligned}$$

Thus, gathering the previous estimates,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u(t)w_{n,x}^{r_1}\|_{L^2_{xy}}^2+\|\mathcal{H}_x u(t)\,w_{n,x}^{r_1}\|_{L^2_{xy}}^2\right)\\&\qquad\lesssim\left(\|uw_{n,x}^{r_1}\|_{L^2_{xy}}^2+\|\mathcal{H}_x uw_{n,x}^{r_1}\|_{L^2_{xy}}^2\right)^{1/2}+\|\partial_x u\|_{L^\infty_{xy}}\left(\|uw_{n,x}^{r_1}\|_{L^2_{xy}}^2+\|\mathcal{H}_x uw_{n,x}^{r_1}\|_{L^2_{xy}}^2\right).\end{aligned}\tag{4.197}$$

Estimate for the $L^2(|y|^{2r_2}\,dxdy)$. In this case, $r_2>0$ is arbitrary. Multiplying the equation in (0.4) by $uw_{n,y}^{r_2}$ and integrating in space yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)w_{n,y}^{r_2}\|_{L^2_{xy}}^2&=-\int\mathcal{H}_x uw_{n,y}^{r_2}uw_{n,y}^{r_2}\,dxdy+\int\mathcal{H}_x\partial_x^2 uw_{n,y}^{r_2}uw_{n,y}^{r_2}\,dxdy\\&\quad\mp\int\mathcal{H}_x\partial_y^2 uw_{n,y}^{r_2}uw_{n,y}^{r_2}\,dxdy-\int u\partial_x uw_{n,y}^{r_2}uw_{n,y}^{r_2}\,dxdy\\&=:A_1+A_2+A_3+A_4.\end{aligned}\tag{4.198}$$

Since the weight function $w_{n,y}^{r_2}=w_{n,y}^{r_2}(y)$ does not depend on x, writing $\mathcal{H}_x uw_{n,y}^{r_2}=\mathcal{H}_x(uw_{n,y}^{r_2})$ and using that $\mathcal{H}_x$ determines a skew-symmetric operator, we have that $A_1=0$. Similarly, integrating by parts on the x variable and writing $\mathcal{H}_x\partial_x uw_{n,y}^{r_2}=\mathcal{H}_x(\partial_x uw_{n,y}^{r_2})$, it follows that $A_2=0$.

Now, integrating by parts and using that $\mathcal{H}_x$ is skew-symmetric, it is not difficult to see that

$$|A_3|=\left|2\int\mathcal{H}_x\partial_y u\partial_y w_{n,y}^{r_2}uw_{n,y}^{r_2}\,dxdy\right|\lesssim\|\partial_y u\partial_y w_{n,y}^{r_2}\|_{L^2_{xy}}\|uw_{n,y}^{r_2}\|_{L^2_{xy}}.\tag{4.199}$$

From the fact that $|\partial_y^l w_{n,y}^{r_2}| \lesssim w_{n,y}^{r_2-l}$, $l=1,2$ with a constant independent of n and (4.192), it follows

$$\|\partial_y u \partial_y w_{n,y}^{r_2}\|_{L^2_{xy}} \lesssim \|\partial_y u\|_{L^\infty_T L^2_{xy}} \lesssim K, \tag{4.200}$$

whenever $0 < r_2 \leqslant 1$. Now, if $r_2 > 1$, the identity $\partial_y u \partial_y w_{n,y}^{r_2} = \partial_y (u \partial_y w_{n,y}^{r_2}) - u \partial_y^2 w_{n,y}^{r_2}$ shows

$$\|\partial_y u \partial_y w_{n,y}^{r_2}\|_{L^2_{x,y}} \lesssim \|J_y(u w_{n,y}^{r_2-1})\|_{L^2_{xy}} + \|u \partial_y^2 w_{n,y}^{r_2}\|_{L^2_{xy}} \lesssim \|J_y(u w_{n,y}^{r_2-1})\|_{L^2_{xy}} + \|u w_{n,y}^{r_2}\|_{L^2_{xy}}. \tag{4.201}$$

To estimate the last expression, choosing $\mathfrak{r} = r_2^{-1}$, $a = b = r_2$ in (1.28) and applying Young's inequality

$$\begin{aligned} \|J_y(u w_{n,y}^{r_2-1})\|_{L^2_{xy}} &= \|\|J_y(u w_{n,y}^{r_2-1})\|_{L^2_y}\|_{L^2_x} \lesssim \|\|u w_{n,y}^{r_2}\|_{L^2_y}^{(r_2-1)/r_2} \|J_y^{r_2} u\|_{L^2_y}^{1/r_2}\|_{L^2_x} \\ &\lesssim \|u w_{n,y}^{r_2}\|_{L^2_{xy}} + \|J^{r_2} u\|_{L^2_{xy}}. \end{aligned} \tag{4.202}$$

Thus, choosing $s > \max\{3/2, r_2\}$, (4.200)-(4.202) and (4.192) imply

$$|A_3| \lesssim \|u w_{n,y}^{r_2}\|_{L^2_{xy}} + \|u w_{n,y}^{r_2}\|_{L^2_{xy}}^2.$$

Finally,

$$|A_4| \lesssim \|\partial_x u\|_{L^\infty_{xy}} \|u w_{n,y}^{r_2}\|_{L^2_{xy}}^2.$$

Plugging the estimates for A_j, $j = 1, \dots, 4$ in (4.198) yields

$$\frac{1}{2}\frac{d}{dt}\|u(t) w_{n,y}^{r_2}\|_{L^2_{xy}}^2 \lesssim \|u w_{n,y}^{r_2}\|_{L^2_{xy}} + (1 + \|\partial_x u\|_{L^\infty_{xy}})\|u w_{n,y}^{r_2}\|_{L^2_{xy}}^2. \tag{4.203}$$

This completes the analyze for the $L^2(|y|^{2r_2}\, dxdy)$-norm;

Now, we collect the estimates derived for the norms $L^2(|x|^{2r_1}\, dxdy)$ and $L^2(|y|^{2r_2}\, dxdy)$ to conclude Theorem 4.5 (i). Letting

$$g(t) = \|u(t) w_{n,x}^{r_1}\|_{L^2_{xy}}^2 + \|\mathcal{H}_x u(t)\, w_{n,x}^{r_1}\|_{L^2_{xy}}^2 + \|u(t) w_{n,y}^{r_2}\|_{L^2_{xy}}^2,$$

the inequalities (4.197) and (4.203) assure that there exists some constant c_0 independent of n such that

$$\frac{d}{dt} g(t) \leqslant c_0 g(t)^{1/2} + c_0(1 + \|\partial_x u\|_{L^\infty_{xy}}) g(t).$$

Then, Gronwall's inequality reveals

$$\begin{aligned} &\|u(t) w_{n,x}^{r_1}\|_{L^2_{xy}}^2 + \|\mathcal{H}_x u(t)\, w_{n,x}^{r_1}\|_{L^2_{xy}}^2 + \|u(t) w_{n,y}^{r_2}\|_{L^2_{xy}}^2 \\ &\quad \leqslant \big((\|u_0 w_{n,x}^{r_1}\|_{L^2_{xy}}^2 + \|\mathcal{H}_x u_0 w_{n,x}^{r_1}\|_{L^2_{xy}}^2 + \|u_0 w_{n,y}^{r_2}\|_{L^2_{xy}}^2)^{1/2} + c_0 t/2 \big)^2 e^{c_0 t + c_0 \int_0^t \|\nabla u(s)\|_{L^\infty_{xy}} ds} \\ &\quad \leqslant \big((\|u_0 \langle x\rangle^{r_1}\|_{L^2_{xy}}^2 + \|\mathcal{H}_x u_0 \langle x\rangle^{r_1}\|_{L^2_{xy}}^2 + \|u_0 \langle y\rangle^{r_2}\|_{L^2_{xy}}^2)^{1/2} + c_0/2 \big)^2 e^{c_0 t + c_0 \int_0^t \|\nabla u(s)\|_{L^\infty_{xy}} ds}. \end{aligned}$$

Thus, taking $n \to \infty$ in the previous estimate yields

$$\begin{aligned} &\|u(t)\langle x\rangle^{r_1}\|_{L^2_{xy}}^2 + \|\mathcal{H}_x u(t)\langle x\rangle^{r_1}\|_{L^2_{xy}}^2 + \|u(t)\langle y\rangle^{r_2}\|_{L^2_{xy}}^2 \\ &\leqslant \big((\|u_0 \langle x\rangle^{r_1}\|_{L^2_{xy}}^2 + \|\mathcal{H}_x u_0 \langle x\rangle^{r_1}\|_{L^2_{xy}}^2 + \|u_0 \langle y\rangle^{r_2}\|_{L^2_{xy}}^2)^{1/2} + c_0 t/2 \big)^2 e^{c_0 t + c_0 \int_0^t \|\nabla u(s)\|_{L^\infty_{xy}} ds}. \end{aligned} \tag{4.204}$$

This shows that $u \in L^\infty([0,T]; L^2(|x|^{2r_1} + |y|^{2r_2}\, dxdy))$. Now, we shall prove that

$$u \in C([0,T]; L^2(|x|^{2r_1} + |y|^{2r_2}\, dxdy)).$$

Firstly, since $u \in C([0,T];H^s(\mathbb{R}^2))$, it is not difficult to see that $u:[0,T]\mapsto L^2(|x|^{2r_1}+|y|^{2r_2}\,dxdy)$ is weakly continuous. The same is true for the map $\mathcal{H}_x u(t)$ on $L^2(|x|^{2r_1}\,dxdy)$. On the other hand, (4.204) implies

$$
\begin{aligned}
&\|(u(t)-u_0)\langle x\rangle^{r_1}\|_{L^2_{xy}}^2+\|\mathcal{H}_x(u(t)-u_0)\langle x\rangle^{r_1}\|_{L^2_{xy}}^2+\|(u(t)-u_0)\langle y\rangle^{r_2}\|_{L^2_{xy}}^2\\
&\quad=\|u(t)\langle x\rangle^{r_1}\|_{L^2_{xy}}^2+\|\mathcal{H}_x u(t)\langle x\rangle^{r_1}\|_{L^2_{xy}}^2+\|u(t)\langle y\rangle^{r_2}\|_{L^2_{xy}}^2+\|u_0\langle x\rangle^{r_1}\|_{L^2_{xy}}^2+\|\mathcal{H}_x u_0\langle x\rangle^{r_1}\|_{L^2_{xy}}^2\\
&\qquad+\|u_0\langle y\rangle^{r_2}\|_{L^2_{xy}}^2-2\int u(t)u_0\langle x\rangle^{2r_1}\,dxdy-2\int \mathcal{H}_x u(t)\mathcal{H}_x u_0\langle x\rangle^{2r_1}\,dxdy\\
&\qquad-2\int u(t)u_0\langle y\rangle^{2r_2}\,dxdy\\
&\quad\leqslant\Big(\big(\|u_0\langle x\rangle^{r_1}\|_{L^2_{xy}}^2+\|\mathcal{H}_x u_0\langle x\rangle^{r_1}\|_{L^2_{xy}}^2+\|u_0\langle y\rangle^{r_2}\|_{L^2_{xy}}^2\big)^{1/2}+c_0t/2\Big)^2 e^{c_0t+c_0\int_0^t\|\nabla u(s)\|_{L^\infty_{xy}}ds}\\
&\qquad+\|u_0\langle x\rangle^{r_1}\|_{L^2_{xy}}^2+\|\mathcal{H}_x u_0\langle x\rangle^{r_1}\|_{L^2_{xy}}^2+\|u_0\langle y\rangle^{r_2}\|_{L^2_{xy}}^2-2\int u(t)u_0\langle x\rangle^{2r_1}\,dxdy\\
&\qquad-2\int \mathcal{H}_x u(t)\mathcal{H}_x u_0\langle x\rangle^{2r_1}\,dxdy-2\int u(t)u_0\langle y\rangle^{2r_2}\,dxdy.
\end{aligned}
\tag{4.205}
$$

Clearly, weak continuity implies that the right-hand side of (4.205) goes to zero as $t\to 0^+$. This shows right continuity at the origin of the map $u:[0,T]\mapsto L^2(|x|^{2r_1}+|y|^{2r_2}\,dxdy)$. Taking any $\tau\in(0,T)$ and using that the equation in (0.4) is invariant under the transformations: $(x,y,t)\mapsto(x,y,t+\tau)$ and $(x,y,t)\mapsto(-x,-y,\tau-t)$, right continuity at the origin yields continuity to the whole interval $[0,T]$, in other words, $u\in C([0,T];L^2(|x|^{2r_1}+|y|^{2r_2}\,dxdy))$.

The continuous dependence on the initial data follows from this property in $H^s(\mathbb{R}^2)$ and the same reasoning above. This completes the proof of Theorem 4.5 (i).

4.5.2.2. **Persistence property and LWP in $\dot{Z}_{s,r_1,r_2}$, $r_1\in[1/2,3/2)$ $r_2\geqslant 0$.** Here it is established Theorem 4.5 parts (ii) and (iii). Let $u\in C([0,T];H^s(\mathbb{R}^2))\cap L^1([0,T];W^{1,\infty}(\mathbb{R}^2))$ solution of the IVP (0.4) with initial data u_0 satisfying the hypothesis of Theorem 4.5 (ii) or (iii) provided by Theorem 4.1. Since we have already established that solutions of the IVP (0.4) preserve arbitrary polynomial decay in the y-variable, we will restrict our considerations to deduce $u,\mathcal{H}_x u\in L^\infty([0,T];L^2(|x|^{2r_1}\,dxdy))$, $r_1\geqslant 1/2$. Once this has been done, following the arguments in (4.205), we will have that $u,\mathcal{H}_x u\in C([0,T];L^2(|x|^{2r_1}\,dxdy))$. Moreover, the continuous dependence on the spaces $ZH_{s,1/2,r_2}(\mathbb{R}^2)$ and $\dot{Z}_{s,r_1,r_2}(\mathbb{R}^2)$, $r_1>1/2$ follows by applying the same energy estimates for the difference of solutions.

To assure the persistence property in $\dot{Z}_{s,r_1,r_2}(\mathbb{R}^2)$ for Theorem 4.5 (iii), we require the following claim:

Claim 4.48. *Let $r_1\in(1/2,3/2)$, $s>3/2$ fixed and*

$$u\in C([0,T],H^s(\mathbb{R}^2))\cap L^1([0,T];W^{1,\infty}_x(\mathbb{R}^2))\cap L^\infty([0,T];L^2(|x|^{2r_1}\,dxdy))$$

be a solution of the IVP (0.4). Assume that $\widehat{u}(0,\eta)=\widehat{u_0}(0,\eta)=0$ for a.e η. Then, $\widehat{u}(0,\eta,t)=0$ for every $t\in[0,T]$ and almost every $\eta\in\mathbb{R}$.

PROOF. Since u solves the integral equation associated to (0.4), taking its Fourier transform we find

$$\widehat{u}(\xi,\eta,t) = e^{i\omega(\xi,\eta)t}\widehat{u_0}(\xi,\eta) - \frac{i\xi}{2}\int_0^t e^{i\omega(\xi,\eta)(t-t')}\widehat{u^2}(\xi,\eta,t')\,dt', \tag{4.206}$$

where $\omega(\xi,\eta)$ is defined by (4.11). Now, the assumptions imposed on the solution show

$$u^2 \in L^1([0,T];L^2(|x|^{2r_1}\,dxdy)).$$

Hence, the above conclusion, Fubini's theorem and Sobolev's embedding on the ξ-variable determines $\widehat{u}(\xi,\eta,t)$ and $\int_0^t e^{i\omega(\xi,\eta)(t-t')}\widehat{u^2}(\xi,\eta,t')\,dt'$ are continuous on ξ for every $t \in [0,T]$ and almost every η. From this, (4.206) yields the desired result. □

We begin by considering the case $1/2 \leqslant r_1 \leqslant 1$. We employ the differential equation (4.196) with the present restrictions on r_1. This reduces our considerations to bound the terms Q_1 and Q_2 defined in (4.196) for this case. Thus, integrating by parts yields

$$\begin{aligned}|Q_1| &= \Big| -\int \partial_x\mathcal{H}_x u u\partial_x w_{n,x}^{2r_1}\,dxdy + \int \partial_x u\mathcal{H}_x u\,\partial_x w_{n,x}^{2r_1}\,dxdy\Big| \\ &\lesssim \|\partial_x u\|_{L^2_{xy}}\|uw_{n,x}^{r_1}\|_{L^2_{xy}} + \|\partial_x u\|_{L^2_{xy}}\|\mathcal{H}_x uw_{n,x}^{r_1}\|_{L^2_{xy}},\end{aligned}$$

where, given that $1/2 \leqslant r_1 \leqslant 1$, we have used $|\partial_x w_{n,x}^{2r_1}| \lesssim |w_{n,x}^{r_1}|$. On the other hand,

$$\begin{aligned}Q_2 &= -\int u^2\partial_x uw_{n,x}^{2r_1} - \frac{1}{2}\int \mathcal{H}_x(\partial_x u^2)\mathcal{H}_x uw_{n,x}^{2r_1}\,dxdy \\ &= -\int u^2\partial_x uw_{n,x}^{2r_1} - \frac{1}{2}\int [w_{n,x}^{r_1},\mathcal{H}_x]\partial_x u^2\mathcal{H}_x uw_{n,x}^{r_1}\,dxdy - \frac{1}{2}\int \mathcal{H}_x(\partial_x u^2 w_{n,x}^{r_1})\mathcal{H}_x uw_{n,x}^{r_1}\,dxdy.\end{aligned}$$

Hence, Proposition 1.5 and Hölder's inequality allow us to deduce

$$\begin{aligned}|Q_2| \lesssim &\|\partial_x u\|_{L^\infty_{xy}}\|uw_{n,x}^{r_1}\|^2_{L^2_{xy}} + \|\partial_x w_{n,x}^{r_1}\|_{L^\infty_{xy}}\|u\|_{L^\infty_{xy}}\|u\|_{L^2_{xy}}\|\mathcal{H}_x uw_{n,x}^{r_1}\|_{L^2_{xy}} \\ &+ \|\partial_x u\|_{L^\infty_{xy}}\|uw_{n,x}^{r_1}\|_{L^2_{xy}}\|\mathcal{H}_x uw_{n,x}^{r_1}\|_{L^2_{xy}}.\end{aligned}$$

Combining the estimates for Q_1 and Q_2, we will obtain the same differential inequality (4.197) adapted for this case. Consequently, this estimate, Gronwall's inequality and the assumption $\mathcal{H}u_0 \in L^2(|x|\,dxdy)$ imply $u, \mathcal{H}_x u \in L^\infty([0,T];L^2(|x|\,dxdy))$. The proof of Theorem 4.5 (ii) is completed.

On the other hand, under the hypothesis of Theorem 4.5 (iii), the fact that $\widehat{u_0}(0,\eta) = 0$ a.e η and Lemma 4.47 assure that $\mathcal{H}_x u_0 \in L^2(|x|^{2r_1}\,dxdy)$ for $1/2 < r_1 \leqslant 1$. Then Gronwall's inequality and the differential inequality (4.197) for this case yield $u \in L^\infty([0,T];L^2(|x|^{2r_1}\,dxdy))$, whenever $1/2 < r_1 \leqslant 1$. This consequence and Claim 4.48 complete the LWP results in $\dot{Z}_{s,r_1,r_2}(\mathbb{R}^2)$, $1/2 < r_1 \leqslant 1$.

Now, we assume that $1 < r_1 < 3/2$. We write $r_1 = 1+\theta$ with $0 < \theta < 1/2$. By the previous step, the fact that $\widehat{u}(0,\eta,t) = 0$ for all $t \in [0,T]$ and almost every $\eta \in \mathbb{R}$ and identity (4.188), we have that $u, \mathcal{H}_x u \in C([0,T];L^2(|x|^2\,dxdy))$. Thus, we multiply the equation in (0.4) by $ux^2w_{n,x}^{2\theta}$ and

(4.193) by $\mathcal{H}_x u x^2 w_{n,x}^{2\theta}$, then integrating in space and adding the resulting expressions reveal

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u(t)xw_{n,x}^{\theta}\|_{L^2_{xy}}^2+\|\mathcal{H}_x u(t)\,xw_{n,x}^{\theta}\|_{L^2_{xy}}^2)&=\int(\mathcal{H}_x\partial_x^2 uu-\partial_x^2 u\mathcal{H}_x u)x^2w_{n,x}^{2r_1}\,dxdy\\&\quad-\int(u\partial_x uu+\mathcal{H}_x(u\partial_x u)\mathcal{H}_x u)x^2w_{n,x}^{2r_1}\,dxdy\\&=:\widetilde{Q}_1+\widetilde{Q}_2.\end{aligned}\tag{4.207}$$

Integrating by parts on the x-variable,

$$\begin{aligned}\widetilde{Q}_1=&-2\Big(\int\mathcal{H}_x\partial_x uu\,xw_{n,x}^{2\theta}\,dxdy-\int\partial_x u\mathcal{H}_x u\,xw_{n,x}^{2\theta}\,dxdy\Big)\\&-\Big(\int\mathcal{H}_x\partial_x uux^2\partial_x w_{n,x}^{2\theta}\,dxdy-\int\partial_x u\mathcal{H}_x ux^2\partial_x w_{n,x}^{2\theta}\,dxdy\Big)=:\widetilde{Q}_{1,1}+\widetilde{Q}_{1,2}.\end{aligned}$$

The Cauchy-Schwarz inequality and Proposition 4.44 determine

$$\begin{aligned}|\widetilde{Q}_{1,1}|&\lesssim\|\mathcal{H}_x\partial_x uw_{n,x}^{\theta}\|_{L^2_{xy}}\|u\,xw_{n,x}^{\theta}\|_{L^2_{xy}}+\|\partial_x uw_{n,x}^{\theta}\|_{L^2_{xy}}\|\mathcal{H}_x u\,xw_{n,x}^{\theta}\|_{L^2_{xy}}\\&\lesssim(\|J_x(uw_{n,x}^{\theta})\|_{L^2_{xy}}+\|u\|_{L^2_{xy}})(\|u\,xw_{n,x}^{\theta}\|_{L^2_{xy}}+\|\mathcal{H}_x u\,xw_{n,x}^{\theta}\|_{L^2_{xy}}).\end{aligned}\tag{4.208}$$

By complex interpolation (1.28) with $\mathfrak{r}=1/(1+\theta)$, we argue as in (4.202), using that $|w_{n,x}^{1+\theta}|\lesssim w_{n,x}^{\theta}+|x|w_{n,x}^{\theta}$ to deduce $\|J_x(uw_{n,x}^{\theta})\|_{L^2_{xy}}\lesssim\|uw_{n,x}^{\theta}\|_{L^2_{xy}}+\|uxw_{n,x}^{\theta}\|_{L^2_{xy}}+\|J^{1+\theta}u\|_{L^2_{xy}}$. This last estimate, the fact that $u\in C([0,T];L^2(|x|^{2r}\,dxdy))$, $0\leqslant r\leqslant 1$ and (4.208) complete the study of $\widetilde{Q}_{1,1}$.

On the other hand, since $|x^2\partial_x w_{n,x}^{2\theta}|\lesssim w_{n,x}^{1+2\theta}$ with implicit constant independent of n, the estimate for $\widetilde{Q}_{1,2}$ follows the same ideas employed to estimate $\widetilde{Q}_{1,1}$.

Finally, identity (4.188) and Proposition 4.44 show

$$|\widetilde{Q}_2|\lesssim\|\partial_x u\|_{L^\infty_{xy}}\|u\,xw_{n,x}^{\theta}\|_{L^2_{xy}}^2+\|\partial_x u\|_{L^\infty_{xy}}\|u\,xw_{n,x}^{\theta}\|_{L^2_{xy}}\|\mathcal{H}_x u\,xw_{n,x}^{\theta}\|_{L^2_{xy}}.$$

Noticing that (4.188) implies $\mathcal{H}_x u_0\,xw_{m,x}=\mathcal{H}_x(xu_0)w_{n,x}^{\theta}\in L^2(\mathbb{R}^2)$. Thus, we can employ recurrent arguments combining the previous estimates for $\widetilde{Q}_1$, $\widetilde{Q}_2$, (4.207) and Gronwall's inequality to conclude $u\in L^\infty(|x|^{2r_1}\,dxdy)$, whenever $1<r_1<3/2$. The proof of Theorem 4.5 (iii) is completed.

4.5.3. Two times condition in Z_{s,r_1,r_2}. Here we establish Theorem 4.6. Without loss of generality we shall assume that $t_1=0$, i.e., $u_0\in Z_{s,(1/2)^+,r_2}(\mathbb{R}^2)$ and $u(t_2)\in Z_{s,1/2,r_2}(\mathbb{R}^2)$. So that $u\in C([0,T];Z_{s,r_1,r_2}(\mathbb{R}^2))\cap L^1([0,T];W^{\infty}_{1,x}(\mathbb{R}^2))$, where $r_1\in(1/4,1/2)$, $r_2\geqslant r_1$ and $s\geqslant\max\{\frac{2r_1}{(4r_1-1)^-},r_2\}$. The solution of the IVP (0.4) can be represented by Duhamel's formula

$$u(t)=S(t)u_0-\int_0^t S(t-t')u\partial_x u(t')\,dt'.\tag{4.209}$$

Since our arguments require localizing near the origin, we consider a function $\phi\in C_c^\infty(\mathbb{R})$ such that $\phi(\xi)=1$ when $|\xi|\leqslant 1$. Then taking the Fourier transform to the integral equation (4.209), we have

$$\widehat{u}(\xi,\eta,t)\phi(\xi)=e^{i\omega(\xi,\eta)t}\widehat{u_0}(\xi,\eta)\phi(\xi)-\int_0^t e^{i\omega(\xi,\eta)(t-t')}\widehat{uu_x}(\xi,\eta,t')\phi(\xi)\,dt',\tag{4.210}$$

where, recalling (4.11), $\omega(\xi,\eta)=\operatorname{sign}(\xi)+\operatorname{sign}(\xi)\xi^2\mp\operatorname{sign}(\xi)\eta^2$.

Claim 4.49. *Let* $0 < \epsilon \ll 1$ *Then it holds*

$$J_{\xi}^{1/2+\epsilon}\Big(\int_0^t e^{i\omega(\xi,\eta)(t-t')}\widehat{uu_x}(\xi,\eta,t')\phi(\xi)\,dt'\Big) \in L^{\infty}([0,T];L^2(\mathbb{R}^2)). \tag{4.211}$$

Let us assume for the moment that Claim 4.49 holds, then

$$J_{\xi}^{1/2}\big(\widehat{u}(\xi,\eta,t)\phi(\xi)\big) \in L^2(\mathbb{R}^2) \quad \text{if and only if} \quad J_{\xi}^{1/2}\big(e^{i\omega(\xi,\eta)t}\widehat{u_0}(\xi,\eta)\phi(\xi)\big) \in L^2(\mathbb{R}^2). \tag{4.212}$$

We first notice that since $u_0 \in L^2(|x|^{1^+}\,dxdy)$, Fubini's theorem and Sobolev embedding on the ξ-variable determines that $\widehat{u_0}(\xi,\eta)$ is continuous in ξ for almost every $\eta \in \mathbb{R}$. Therefore, given that (4.212) holds at $t = t_2$, Fubini's theorem shows that $J_{\xi}^{1/2}\big(e^{i\omega(\xi,\eta)t_2}\widehat{u_0}(\xi,\eta)\phi(\xi)\big) \in L^2(\mathbb{R})$ for almost every $\eta \in \mathbb{R}$, then an application of Proposition 4.45 imposes that $\widehat{u_0}(0,\eta,t) = 0$ for almost every η. From this fact, the integral equation (4.210) and Claim 4.49, we deduce that $\widehat{u}(0,\eta,t) = 0$ for all $t \geqslant 0$ and almost every η.

PROOF OF CLAIM 4.49. In virtue of Theorem 1.13,

$$\begin{aligned}&\|J_{\xi}^{1/2+\epsilon}\Big(\int_0^t e^{i\omega(\xi,\eta)(t-t')}\widehat{uu_x}(\xi,\eta,t')\phi(\xi)\,dt'\Big)\|_{L^2_{\xi\eta}}\\ &\qquad\lesssim \int_0^T \|\phi\|_{L^{\infty}_{\xi}}\|\widehat{uu_x}(t')\|_{L^2_{\xi\eta}}\,dt' + \int_0^T \|\mathcal{D}_{\xi}^{1/2+\epsilon}\big(e^{i\omega(\xi,\eta)(t-t')}\widehat{uu_x}(t')\phi(\xi)\big)\|_{L^2_{\xi\eta}}\,dt'.\end{aligned} \tag{4.213}$$

To estimate the r.h.s of the last inequality, we decompose $\omega(\xi,\eta) = \omega_1(\xi,\eta) + \omega_2(\xi,\eta)$ where $\omega_1(\xi,\eta) := \operatorname{sign}(\xi) \mp \operatorname{sign}(\xi)\eta^2$. Then, writing $\widehat{uu_x}(\xi) = i\xi\widehat{u^2}(\xi)$ and using (1.26) and Proposition 4.46,

$$\begin{aligned}&\|\mathcal{D}_{\xi}^{1/2+\epsilon}\big(e^{i\omega(\xi,\eta)(t-t')}\widehat{uu_x}(\xi,\eta,t)\phi(\xi)\big)\|_{L^2_{\xi\eta}}\\ &\quad\lesssim \|\mathcal{D}_{\xi}^{1/2+\epsilon}(e^{i\omega_1(\xi,\eta)(t-t')})\widehat{uu_x}\phi(\xi)\|_{L^2_{\xi\eta}} + \|\mathcal{D}_{\xi}^{1/2+\epsilon}(e^{i\omega_2(\xi,\eta)(t-t')})\widehat{uu_x}\phi(\xi)\|_{L^2_{\xi\eta}}\\ &\qquad\quad + \|\mathcal{D}_{\xi}^{1/2+\epsilon}(\widehat{uu_x}\phi(\xi))\|_{L^2_{\xi\eta}}\\ &\quad\lesssim_T \big(\||\xi|^{-1/2-\epsilon}\widehat{uu_x}\|_{L^2_{\xi\eta}} + \|\widehat{uu_x}\|_{L^2_{\xi\eta}} + \||\xi|^{1/2+\epsilon}\widehat{uu_x}\|_{L^2_{\xi\eta}}\big)\|\phi\|_{L^{\infty}_{\xi}} + \|\mathcal{D}_{\xi}^{1/2+\epsilon}(\xi\phi)\widehat{u^2}\|_{L^2_{\xi\eta}}\\ &\qquad + \|\xi\phi\mathcal{D}_{\xi}^{1/2+\epsilon}(\widehat{u^2})\|_{L^2_{\xi\eta}}\\ &\quad\lesssim_T \|J_x^{1/2-\epsilon}(u^2)\|_{L^2_{xy}} + \|uu_x\|_{L^2_{xy}} + \|J_x^{1/2+\epsilon}(uu_x)\|_{L^2_{xy}} + \|\langle x\rangle^{1/2+\epsilon}u^2\|_{L^2_{xy}}\\ &\quad\lesssim_T \big(\|u\|_{L^{\infty}_{xy}} + \|\partial_x u\|_{L^{\infty}_{xy}}\big)\|J_x^{3/2+\epsilon}u\|_{L^2_{xy}} + \|\langle x\rangle^{1/4+\epsilon/2}u\|^2_{L^4_{xy}},\end{aligned} \tag{4.214}$$

where the last line is obtained by (1.11). We employ complex interpolation (1.28) to deduce

$$\begin{aligned}\|\langle x\rangle^{1/4+\epsilon/2}u\|_{L^4_{xy}} \lesssim \|\langle (x,y)\rangle^{1/4+\epsilon/2}u\|_{L^4_{xy}} &\lesssim \|J^{1/2}\big(\langle (x,y)\rangle^{1/4+\epsilon/2}u\big)\|_{L^2_{xy}}\\ &\lesssim \|\langle (x,y)\rangle^{r_1}u\|^{(1+2\epsilon)/4r_1}_{L^2_{xy}}\|J^s u\|^{(4r_1-1-2\epsilon)/4r_1}_{L^2_{xy}},\end{aligned} \tag{4.215}$$

where $s \geqslant \max\{\frac{2r_1}{(4r_1-1)^-}, r_2\}$. Hence, (4.213), (4.214) and (4.215) yield

$$\begin{aligned}&\|J_{\xi}^{1/2+\epsilon}\Big(\int_0^t e^{i\omega(\xi,\eta)(t-t')}\widehat{uu_x}(\xi,\eta,t')\,\phi(\xi)dt'\Big)\|_{L^2_{\xi\eta}}\\ &\qquad\lesssim_T \big(1 + \|u\|_{L^1_T L^{\infty}_{xy}} + \|\partial_x u\|_{L^1_T L^{\infty}_{xy}}\big)\big(1 + \|u\|_{L^{\infty}_T H^s} + \|\langle (x,y)\rangle^{r_1}u\|_{L^{\infty}_{xy}L^2_{xy}}\big)^2.\end{aligned}$$

This completes the proof of Claim 4.49. ∎

4.5.4. Three times condition in $\dot{Z}_{s,r_1,r_2}$. This part concerns the deduction of Theorem 4.7. Here we assume that $u \in C([0,T]; Z_{s,r_1,r_2}(\mathbb{R}^2))$, $s > \max\{3, r_2\}$, $r_2 \geqslant r_1 = 3/2 - \epsilon$, where $0 < \epsilon < 3/20$. Without loss of generality, we let $t_1 = 0 < t_2$, that is, $u_0 \in Z_{s,(3/2)^+,r_2}(\mathbb{R}^2)$ and $u(\cdot, t_2) \in Z_{s,3/2,r_2}(\mathbb{R}^2)$. Taking the Fourier transform in (4.209) and differentiating on the ξ variable yield

$$\begin{aligned}\frac{\partial}{\partial \xi}\widehat{u}(\xi,\eta,t) =&2it|\xi|e^{i\omega(\xi,\eta)t}\widehat{u_0}(\xi,\eta) + e^{i\omega(\xi,\eta)t}\partial_\xi \widehat{u_0}(\xi,\eta)\\ &- 2i\int_0^t e^{i\omega(\xi,\eta)(t-t')}(t-t')|\xi|\widehat{uu_x}(\xi,\eta,t')\,dt' - \frac{i}{2}\int_0^t e^{i\omega(\xi,\eta)(t-t')}\widehat{u^2}(\xi,\eta,t')\,dt'\\ &- \frac{i}{2}\int_0^t e^{i\omega(\xi,\eta)(t-t')}\xi\,\partial_\xi\widehat{u^2}(\xi,\eta,t')\,dt',\end{aligned} \tag{4.216}$$

where $\omega(\xi,\eta) = \operatorname{sign}(\xi) + \operatorname{sign}(\xi)\xi^2 \mp \operatorname{sign}(\xi)\eta^2$ and we have used that $\widehat{u}_0(0,\eta) = \widehat{uu_x}(0,\eta) = 0$ together with the identity

$$\partial_\xi e^{i\omega(\xi,\eta)t} = 2i\sin((1\mp\eta^2)t)\delta_0^\xi + 2it|\xi|e^{i\omega(\xi,\eta)t},$$

setting $(\delta_0^\xi\phi)(\xi,\eta) = \phi(0,\eta)$.

Claim 4.50. *It holds that*

$$\begin{aligned}J_\xi^{1/2}\Big(t|\xi|e^{i\omega(\xi,\eta)t}\widehat{u_0}(\xi,\eta) - &\int_0^t e^{i\omega(\xi,\eta)(t-t')}(t-t')|\xi|\widehat{uu_x}(\xi,\eta,t')\,dt'\\ &- \frac{1}{4}\int_0^t e^{i\omega(\xi,\eta)(t-t')}\xi\,\partial_\xi\widehat{u^2}(\xi,\eta,t')\,dt'\Big) \in L^\infty([0,T];L^2(\mathbb{R}^2)).\end{aligned}$$

PROOF. We first deal with the term provided by the homogeneous part of the integral equation. We use Theorem 1.13, (1.26) and Proposition 4.46 to find

$$\begin{aligned}\|J_\xi^{1/2}(|\xi|e^{i\omega(\xi,\eta)t}\widehat{u_0})\|_{L^2_{\xi\eta}} &\lesssim \||\xi|\widehat{u_0}\|_{L^2_{\xi\eta}} + \|\mathcal{D}_\xi^{1/2}(|\xi|e^{i\omega(\xi,\eta)t}\widehat{u_0})\|_{L^2_{\xi\eta}}\\ &\lesssim \||\xi|\widehat{u_0}\|_{L^2_{\xi\eta}} + \||\xi|^{1/2}\widehat{u_0}\|_{L^2_{\xi\eta}} + \|\mathcal{D}_\xi^{1/2}(|\xi|\widehat{u_0})\|_{L^2_{\xi\eta}}.\end{aligned} \tag{4.217}$$

To estimate the last term on the r.h.s of the above expression, we use (1.26), (1.27), Plancherel's identity and Young's inequality to find

$$\begin{aligned}\|\mathcal{D}_\xi^{1/2}(|\xi|\widehat{u_0})\|_{L^2_{\xi\eta}} = \|\mathcal{D}_\xi^{1/2}(\frac{|\xi|}{\langle\xi\rangle}\langle\xi\rangle\widehat{u_0})\|_{L^2_{\xi\eta}} &\lesssim \|\|J_\xi^{1/2}(\langle\xi\rangle\widehat{u_0})\|_{L^2_\xi}\|_{L^2_\eta}\\ &\lesssim \|\|\langle\xi\rangle^{3/2}\widehat{u_0}\|_{L^2_\xi}^{2/3}\|J_\xi^{3/2}\widehat{u_0}\|_{L^2_\xi}^{1/3}\|_{L^2_\eta}\\ &\lesssim \|J_x^{3/2}u_0\|_{L^2_{xy}} + \|\langle x\rangle^{3/2}u_0\|_{L^2_{xy}},\end{aligned} \tag{4.218}$$

where we have also used (1.28) with $\mathfrak{r} = 1/3$, $a = b = 3/2$. Collecting (4.217) and (4.218), we complete the analysis of $\|J_\xi^{1/2}(|\xi|e^{i\omega(\xi,\eta)t}\widehat{u_0})\|_{L^2_{\xi\eta}}$. Next, we shall prove that

$$uu_x \in L^\infty([0,T];H_x^{3/2}(\mathbb{R}^2)) \cap L^\infty([0,T];L^2(|x|^3dxdy)). \tag{4.219}$$

where $H_x^s(\mathbb{R}^2)$ is defined by the norm $\|f\|_{H_x^s} = \|J_x^s f\|_{L^2}$. Once this has been established, according to the reasoning in (4.217) and (4.218) it will follow

$$J_\xi^{1/2}\Big(\int_0^t e^{i\omega(\xi,\eta)(t-t')}(t-t')|\xi|\widehat{uu_x}(\xi,\eta,t')\,dt'\Big) \in L^\infty([0,T];L^2(\mathbb{R}^2)).$$

Indeed, (1.11) and Sobolev's embedding show $\|uu_x\|_{H^{3/2}_x} \lesssim \|u\|^2_{H^s}$, whenever $s \geqslant 5/2$. Now, complex interpolation (1.28), Young's inequality and Sobolev's embedding determine

$$\begin{aligned}\|\langle x\rangle^{3/2}uu_x\|_{L^2_{xy}} &\lesssim \|\langle x\rangle^{1/2}u^2\|_{L^2_{xy}} + \|J_x(\langle x\rangle^{3/2}u^2)\|_{L^2_{xy}} \\ &\lesssim \|u\|_{L^\infty_{xy}}\|\langle x\rangle^{1/2}u\|_{L^2_{xy}} + \|\|\langle x\rangle^{9/4}u^2\|^{2/3}_{L^2_x}\|J^3_x(u^2)\|^{1/3}_{L^2_x}\|_{L^2_y} \\ &\lesssim \|J^3u\|_{L^2}\|\langle x\rangle^{1/2}u\|_{L^2_{xy}} + \|\langle x\rangle^{9/4}u^2\|_{L^2_{xy}} + \|J^3_x(u^2)\|_{L^2_{xy}}.\end{aligned} \tag{4.220}$$

Since $H^3(\mathbb{R}^2)$ is a Banach algebra, $\|J^3_x(u^2)\|_{L^2_{xy}} \lesssim \|J^3(u^2)\|_{L^2_{xy}} \lesssim \|u\|^2_{H^3}$, so it remains to derive a bound for the second term on the right hand side of equation (4.220). Let $0 < \epsilon < 3/20$, applying Sobolev's embedding and complex interpolation we find

$$\begin{aligned}\|\langle x\rangle^{9/4}u^2\|_{L^2_{xy}} \lesssim \|\langle (x,y)\rangle^{9/8}u\|^2_{L^4_{xy}} &\lesssim \|J^{1/2}(\langle (x,y)\rangle^{9/8}u)\|^2_{L^2_{xy}} \\ &\lesssim \|\langle (x,y)\rangle^{3/2-\epsilon}u\|^{\frac{18}{12-8\epsilon}}_{L^2_{xy}}\|J^{\frac{6-4\epsilon}{3-8\epsilon}}u\|^{\frac{6-16\epsilon}{12-8\epsilon}}_{L^2_{xy}}.\end{aligned} \tag{4.221}$$

Notice that since $0 < \epsilon < 3/20$, $\|J^{\frac{6-4\epsilon}{3-8\epsilon}}u\|_{L^2_{xy}} \leqslant \|J^3u\|_{L^2_{xy}}$. Plugging (4.221) in (4.220), we complete the deduction of (4.219). To prove the remaining estimate, i.e.,

$$J^{1/2}_\xi\Big(\int_0^t e^{i\omega(\xi,\eta)(t-t')}\xi\,\partial_\xi\widehat{u^2}(\xi,\eta,t')\,dt'\Big) \in L^\infty([0,T];L^2(\mathbb{R}^2)),$$

we write $\frac{\partial}{\partial\xi}\widehat{u^2} = \widehat{-ixu^2}$, then according to (4.217) and (4.218), it is enough to show

$$xu^2 \in L^\infty([0,T];H^{3/2}_x(\mathbb{R}^2)) \cap L^\infty([0,T];L^2(|x|^3dxdy)). \tag{4.222}$$

To this aim, after some computations applying Theorem 1.13 and property (1.26), we employ complex interpolation and Young's inequality to show

$$\begin{aligned}\|J^{1/2}_x(xu^2)\|_{L^2_{xy}} &\lesssim \|xu^2\|_{L^2_{xy}} + \|J^{1/2}_x(u^2)\|_{L^2_{xy}} + \|J^{3/2}_x(\langle x\rangle u^2)\|_{L^2_{xy}} \\ &\lesssim \|u\|_{L^\infty_{xy}}\|\langle x\rangle u\|_{L^2_{xy}} + \|u\|_{L^\infty_{xy}}\|J^{1/2}_x u\|_{L^2_{xy}} + \|\|\langle x\rangle^{9/4}u^2\|^{4/9}_{L^2_x}\|J^{27/10}_x(u^2)\|^{5/9}_{L^2_x}\|_{L^2_y} \\ &\lesssim \|J^3u\|_{L^2_{xy}}\|\langle x\rangle u\|_{L^2_{xy}} + \|J^3u\|^2_{L^2_{xy}} + \|\langle x\rangle^{9/4}u^2\|_{L^2_{xy}}.\end{aligned}$$

Recalling (4.221), we conclude that $xu^2 \in L^\infty([0,T];H^{3/2}_x(\mathbb{R}^2))$. Finally, since $u \in C([0,T];H^s(\mathbb{R}^2))$, $s > \max\{3, r_2\}$, there exists some $0 < \delta < 1$ such that $3+\delta < s$, then we have

$$\begin{aligned}\|\langle x\rangle^{3/2}xu^2\|_{L^2_{xy}} \lesssim \|\langle x\rangle^{5/4}u\|^2_{L^4_{xy}} &\lesssim \|J^{1/2}(\langle (x,y)\rangle^{5/4}u)\|^2_{L^2_{xy}} \\ &\lesssim \|\langle (x,y)\rangle^{3/2-\epsilon}u\|^{\frac{10}{6-4\epsilon}}_{L^2_{xy}}\|J^{\frac{3-2\epsilon}{1-4\epsilon}}u\|^{\frac{2-8\epsilon}{6-4\epsilon}}_{L^2_{xy}}.\end{aligned} \tag{4.223}$$

Now, taking $0 < \epsilon \ll 1$ such that $\frac{3-2\epsilon}{1-4\epsilon} \leqslant 3+\delta < s$, (4.223) shows that $xu^2 \in L^\infty([0,T];L^2(|x|^3dxdy))$. This in turn verifies (4.222). □

Consequently, from (4.216) and Claim 4.50, it follows:

$$\begin{aligned}&J^{1/2}_\xi\partial_\xi\widehat{u}(\xi,\eta,t) \in L^2(\mathbb{R}^2) \text{ if and only if} \\ &\qquad J^{1/2}_\xi\Big(e^{i\omega(\xi,\eta)t}\partial_\xi\widehat{u_0}(\xi,\eta) - \frac{i}{2}\int_0^t e^{i\omega(\xi,\eta)(t-t')}\widehat{u^2}(\xi,\eta,t')\,dt'\Big) \in L^2(\mathbb{R}^2).\end{aligned} \tag{4.224}$$

Now, since (4.223) establishes that $\widehat{u^2} \in H^{1^+}(\mathbb{R}^2)$, Sobolev's embedding determines that $\widehat{u^2}$ can be regarded as a continuous function on the ξ and η variables. Additionally, since $\partial_\xi\widehat{u_0} \in H^{(1/2)^+}_\xi(\mathbb{R}^2)$, Fubinni's theorem and Sobolev's embedding shows that $\partial_\xi \widehat{u_0}(\xi,\eta)$ is continuous in ξ for almost every $\eta \in \mathbb{R}$. Thus, given that (4.224) holds at $t = t_2$, we gather the preceding discussions and Proposition 4.45 to get

$$\begin{aligned} &e^{i(1\mp\eta^2)t_2}\partial_\xi \widehat{u_0}(0,\eta) - \frac{i}{2}\int_0^{t_2} e^{i(1\mp\eta^2)(t_2-t')}\widehat{u^2}(0,\eta,t')\,dt' \\ &= e^{-i(1\mp\eta^2)t_2}\partial_\xi \widehat{u_0}(0,\eta) - \frac{i}{2}\int_0^{t_2} e^{-i(1\mp\eta^2)(t_2-t')}\widehat{u^2}(0,\eta,t')\,dt' \end{aligned}$$

so that

$$2i\sin((1\mp\eta^2)t_2)\partial_\xi \widehat{u_0}(0,\eta) = -\int_0^{t_2}\sin((1\mp\eta^2)(t_2-t'))\widehat{u^2}(0,\eta,t')\,dt', \tag{4.225}$$

for almost every $\eta \in \mathbb{R}$. This completes the deduction of identity (4.7). Now, recalling that the quantity $M(u) = \|u(t)\|_{L^2}$ is invariant for solution of the equation in (0.4), and that $\eta \mapsto \widehat{u^2}(0,\eta,t)$ determines a continuous map, we take $\eta \to 0$ in (4.225) to find

$$\begin{aligned} J_\xi^{1/2}\partial_\xi \widehat{u}(\xi,\eta,t_2) \in L^2(\mathbb{R}^2) \text{ and } \eta \mapsto \partial_\xi \widehat{u_0}(0,\eta) \text{ continuous at the origin imply} \\ 2i\sin(t_2)\partial_\xi \widehat{u_0}(0,0) = (\cos(t_2)-1)\|u_0\|_{L^2_{xy}}^2. \end{aligned} \tag{4.226}$$

Therefore, in the case that $u_0 \in Z_{s,2^+,2^+}(\mathbb{R}^2)$, (4.226) yield identity (4.8).

4.6. Lack of C^2-regularity flow-map data solution

Here we prove that the flow-map data solution determined by the IVP (0.4) is not of class C^2 at the origin of the spaces $H^{s_1,s_2}(\mathbb{R}^2)$ and $X^s(\mathbb{R}^2)$. We will consider the former case since the same considerations also work for $X^s(\mathbb{R}^2)$ instead. Following the reasoning in **[66]**, **[23]** and **[39]**, it is enough to establish that

$$\left\| \int_0^t S(t-t')\partial_x\big((S(t')\phi_1)(S(t')\phi_2)\big)\,dt' \right\|_{H^{s_1,s_2}} \lesssim \|\phi_1\|_{H^{s_1,s_2}}\|\phi_2\|_{H^{s_1,s_2}}$$

does not hold for arbitrary $\phi_1, \phi_2 \in H^{s_1,s_2}(\mathbb{R}^2)$, $s_1, s_2 \in \mathbb{R}$ and $0 < t < T$. Indeed, we will construct two sequences of functions, $\phi_{1,N}$ and $\phi_{2,N}$, such that

$$\|\phi_{1,N}\|_{H^{s_1,s_2}}, \|\phi_{2,N}\|_{H^{s_1,s_2}} \leqslant C \tag{4.227}$$

and

$$\lim_{N\to\infty} \|S(t-t')\partial_x\big(S(t')\phi_{1,N})(S(t')\phi_{2,N})\big)\,dt'\|_{H^{s_1,s_2}} = \infty. \tag{4.228}$$

We define $\phi_{1,N}$ and $\phi_{2,N}$ via their Fourier transforms as

$$\begin{cases} \widehat{\phi_{1,N}}(\xi) = \gamma^{-3/4}N^{-s_1}\chi_{\mathcal{A}_1}(\xi), & \text{with } \mathcal{A}_1 = [N, N+\gamma]\times[\gamma^{1/2}/2, \gamma^{1/2}], \\ \widehat{\phi_{2,N}}(\xi) = \gamma^{-3/4}\chi_{\mathcal{A}_2}(\xi), & \text{with } \mathcal{A}_2 = [-4\gamma, -3\gamma]\times[\gamma^{1/2}/2, \gamma^{1/2}] \end{cases}$$

where $N \gg 1$, $\gamma = N^{-(1+\epsilon)}$ and $0 < \epsilon < 1/3$. Notice that $\phi_{1,N}$ and $\phi_{2,N}$ satisfy (4.227). To estimate the integral term, we take the Fourier transform with respect to the space variable to find

$$\begin{aligned}\widehat{I_N}(\xi,\eta,t) &:= \left\{\int_0^t S(t-t')\partial_x\big(S(t')\phi_{1,N})(S(t')\phi_{2,N})\big)\,dt'\right\}^{\wedge}(\xi,\eta)\\ &= -\int_{K_{(\xi,\eta)}} \xi e^{it\omega(\xi,\eta)}\frac{e^{-i\Omega(\xi_1,\eta_1,\xi_2,\eta_2)t}-1}{\Omega(\xi_1,\eta_1,\xi_2,\eta_2)}\widehat{\phi_{1,N}}(\xi-\xi_1,\eta-\eta_1)\widehat{\phi_{2,N}}(\xi_1,\eta_1)\,d\xi_1 d\eta_1\end{aligned} \tag{4.229}$$

where we employed the notation introduced in (4.11) and (4.12) with $(\xi,\eta) = (\xi_1+\xi_2,\eta_1+\eta_2)$ and

$$K_{(\xi,\eta)} := \left\{\eta \in \mathbb{R}^2 \,:\, (\xi-\xi_1,\eta-\eta_1) \in \mathcal{A}_1,\ (\xi_1,\eta_1) \in \mathcal{A}_2\right\}.$$

When $(\xi-\xi_1,\eta-\eta_1) \in \mathcal{A}_1$, $(\xi_1,\eta_1) \in \mathcal{A}_2$, we have that $\widehat{I}_N(\xi,\eta)$ is supported in

$$\mathcal{A}_3 = [N-4\gamma, N-2\gamma] \times [\gamma^{1/2}, 2\gamma^{1/2}].$$

Then one has

$$\Omega(\xi_1,\eta_1,\xi_2,\eta_2) = 1 + 2\xi\xi_1 \mp 2\eta\eta_1. \tag{4.230}$$

and so, since $\xi\xi_1 < 0$, $|\xi\xi_1| \sim \gamma N$ and $|\eta\eta_1| \sim \gamma$,

$$\Omega(\xi_1,\eta_1,\xi_1,\eta_2) \sim (1-\gamma N) \sim 1. \tag{4.231}$$

From this we get

$$\begin{aligned}\Re\left(\frac{1-e^{i\Omega(\xi_1,\eta_1,\xi_2,\eta_2)t}}{\Omega(\xi_1,\eta_1,\xi_2,\eta_2)}\right) = \frac{1-\cos(\Omega(\xi_1,\eta_1,\xi_2,\eta_2)t)}{\Omega(\xi_1,\eta_1,\xi_2,\eta_2)} &\gtrsim \int_0^{t/2}\sin(w)\,dw\\ &= |1-\cos(t/2)|,\end{aligned} \tag{4.232}$$

where $0 < t < 1$ and N is large such that $1/2 \leqslant \Omega(\xi_1,\eta_1,\xi_2,\eta_2)$. From (4.232) and $|K_{(\xi,\eta)}| \sim \gamma^{3/2}$, we infer

$$|\widehat{I_N}(\xi,\eta,t)|\chi_{\mathcal{A}_3}(\xi) \gtrsim \frac{N\gamma^{3/2}}{N^{s_1}\gamma^{3/2}}|1-\cos(t/2)|\,\chi_{\mathcal{A}_3}(\xi),$$

which yields

$$\|I_N(t)\|_{H^{s_1,s_2}} \gtrsim N\gamma^{3/4}|1-\cos(t/2)| = N^{1/4-3\epsilon/4}|1-\cos(t/2)|,$$

for $0 < t < 1$, above we used that $(1+|\eta|^2)^{s_2} \sim 1$ with involved constants independent of $s_2 \in \mathbb{R}$. Given that $0 < \epsilon < 1/3$, the above display shows that (4.228) holds. The proof is now completed.

4.7. Results on the Shrira equation

This section is aimed to briefly indicate the modifications needed to prove Theorem 4.9. We first recall that (0.6) is LWP in the space $H^s(\mathbb{R}^2)$, $s > 3/2$ by the results established in [**11**]. To prove well-posedness in the space $\widetilde{X}^s(\mathbb{R}^2)$ determined by the norm

$$\|f\|_{\widetilde{X}^s} = \|J_x^s f\|_{L^2_{xy}} + \|D_x^{-1/2}\partial_y f\|_{L^2_{xy}},$$

the key ingredient is the refined Strichartz estimate deduced in [**11**]:

Lemma 4.51. *The results of Lemma 4.13 hold for solutions of the IVP (0.6).*

Once the above lemma has been established, the proof of LWP in $\widetilde{X}^s(\mathbb{R}^2)$ follows the same line of arguments leading to the conclusion of Theorem 4.1. Actually, this case does not require to estimate the norm $\|D_x^{-1/2}u\|_{L^2_{xy}}$, which slightly simplifies our arguments. We emphasize that Lemma 4.17 assure existence of solutions of the IVP (0.6) in the space $\widetilde{X}^\infty(\mathbb{R}^2) = \bigcap_{s\geqslant 0}\widetilde{X}^s(\mathbb{R}^2)$. Consequently, it follows that (0.6) is LWP in $\widetilde{X}^s(\mathbb{R}^2)$, $s > 3/2$.

On the other hand, setting

$$\widetilde{\omega}(\xi,\eta) = \operatorname{sign}(\xi)\xi^2 + \operatorname{sign}(\xi)\eta^2,$$

the resonant function determined by the equation in (0.6) is given by

$$\widetilde{\Omega}(\xi_1,\eta_1,\xi_2,\eta_2) = \widetilde{\omega}(\xi_1+\xi_2,\eta_1+\eta_2) - \widetilde{\omega}(\xi_1,\eta_2) - \widetilde{\omega}(\xi_2,\eta_2).$$

Then, it is not difficult to see:

PROPOSITION 4.52. *The results in Proposition 4.30 are valid replacing the set $D_{N,L}$ by*

$$\widetilde{D}_{N,L} = \left\{(m,n,\tau) \in \mathbb{Z}^2 \times \mathbb{R} : |(m,n)| \in I_N \text{ and } |\tau - \widetilde{\omega}(m,n)| \leqslant L\right\},$$

whenever $N, L \in \mathbb{D}$.

This in turn allow us to follow the same reasoning leading to the deduction of Theorem 4.3 to conclude that the IVP (0.6) is LWP in $H^s(\mathbb{T}^2)$, $s > 3/2$.

Concerning well-posedness in weighted spaces, here we replace equation (4.193) by

$$\partial_t \mathcal{H}_x u + \partial_x^2 u + \partial_y^2 u + \mathcal{H}_x(u\partial_x u) = 0.$$

Then, by employing the identity above, we can adapt the arguments in the proof of Theorem 4.5 to obtain the same well-posedness conclusion in anisotropic spaces for the equation in (0.6). Besides, the arguments in Proposition 4.46 show

$$\mathcal{D}^b(e^{i\operatorname{sign}(x)\eta^2 t}) \lesssim |x|^{-b}, \quad x \in \mathbb{R}\backslash\{0\},$$

whenever $b \in (0,1)$ fixed and for all $\eta \in \mathbb{R}$. Thus, the previous estimate allows us to deduce Theorems 4.6 and 4.7 in a similar fashion. However, instead of (4.7) we get

$$2i\sin(\eta^2(t_2-t_1))\partial_\xi \widehat{u}(0,\eta,t_1) = -\int_{t_1}^{t_2} \sin(\eta^2(t_2-t'))\widehat{u^2}(0,\eta,t')\,dt',$$

for almost $\eta \in \mathbb{R}$. This encloses the discussion leading Theorem 4.9.

4.8. Appendix: Fractional commutator estimate for the Hilbert transform

In this part we deduce the estimate (4.4).

PROOF OF PROPOSITION 4.2. When $\beta = 1$, by writing $D_x = \mathcal{H}_x\partial_x$ and using that $\mathcal{H}_x$ determines a bounded operator in L^p, we have that (4.4) follows from Proposition 1.5.

We will assume that $0 < \alpha, \beta < 1$ with $\alpha + \beta = 1$. We write

$$\begin{aligned} &D_x^\alpha[\mathcal{H}_x, g]D_x^\beta f(x) \\ &\quad = -i\int |\xi_1+\xi_2|^\alpha|\xi_2|^\beta\big(\operatorname{sign}(\xi_1+\xi_2) - \operatorname{sign}(\xi_2)\big)\widehat{g}(\xi_1)\widehat{f}(\xi_2)e^{ix\cdot(\xi_1+\xi_2)}\,d\xi_1 d\xi_2, \end{aligned} \tag{4.233}$$

then neglecting the null measure sets where $\xi_1+\xi_2=0$ or $\xi_2=0$, we observe that the integral in (4.233) is not null only when $(\xi_1+\xi_2)\xi_2<0$, in order words, when $|\xi_2|<|\xi_1|$. Thus, by Bony's paraproduct decomposition we find

$$
\begin{aligned}
D_x^{\alpha}[\mathcal{H}_x,g]D_x^{\beta}f =&\mathcal{H}_x\Big(\sum_{N>0}D^{\alpha}(P_N^xgP_{\ll N}^xD_x^{\beta}f)\Big)-\sum_{N>0}D^{\alpha}(P_N^xgP_{\ll N}^x\mathcal{H}_xD_x^{\beta}f)\\
&+\mathcal{H}_x\Big(\sum_{N>0}D^{\alpha}(P_N^xg\widetilde{P}_N^xD_x^{\beta}f)\Big)-\sum_{N>0}D^{\alpha}(P_N^xg\widetilde{P}_N^x\mathcal{H}_xD_x^{\beta}f)\\
=:&\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3+\mathcal{A}_4,
\end{aligned}
\tag{4.234}
$$

where $P_{\ll N}^xf=\sum_{M\ll N}P_M^xf$ and $\widetilde{P}_N^xf=\sum_{M\sim N}P_M^xf$. We proceed to estimate each of the factors $\mathcal{A}_j$, $j=1,\dots,4$. Since $\alpha+\beta=1$, $\beta>0$, and the Hilbert transform determines a bounded operator in L^p, by the Littlewood-Paley inequality and support considerations we have

$$
\begin{aligned}
\|\mathcal{A}_1\|_{L^p}\lesssim \Big\|\Big(P_M^x\Big(\sum_{N>0}D^{\alpha}(P_N^xgP_{\ll N}^xD_x^{\beta}f)\Big)\Big)_{l_M^2}\Big\|_{L^p}&\lesssim\Big\|\Big(\sum_{N\sim M}D^{\alpha}P_M^x(P_N^xgP_{\ll N}^xD_x^{\beta}f)_{l_M^2}\Big\|_{L^p}\\
&\lesssim\sum_{L\sim 1}\Big\|\Big(\overline{P}_{LN}^x(\overline{P}_N^x\partial_xgN^{-\beta}P_{\ll N}^xD_x^{\beta}f\Big)_{l_N^2}\Big\|_{L^p},
\end{aligned}
\tag{4.235}
$$

for some adapted projections $\overline{P}_N^x$ supported in frequency on the set $|\xi|\sim N$, and with $L\sim 1$ dyadic. Now, by employing Lemma 1.8, we deduce

$$
|\overline{P}_{LN}^x(\overline{P}_N^x\partial_xgN^{-\beta}P_{\ll N}^xD_x^{\beta}f)(x)|\lesssim\mathcal{M}(\overline{P}_N^x\partial_xgN^{-\beta}P_{\ll N}^xD_x^{\beta}f)(x).
\tag{4.236}
$$

Inserting the above expression on the r.h.s of (4.235), applying Lemmas 1.7 and 1.8, we get

$$
\begin{aligned}
\|\mathcal{A}_1\|_{L^p}\lesssim\|(\overline{P}_N^x\partial_xgN^{-\beta}P_{\ll N}^xD_x^{\beta}f)_{l_N^2}\|_{L^p}&\lesssim\|\mathcal{M}(\partial_xg)(N^{-\beta}P_{\ll N}^xD_x^{\beta}f)_{l_N^2}\|_{L^p}\\
&\lesssim\|\partial_xg\|_{L^\infty}\|(N^{-\beta}P_{\ll N}^xD_x^{\beta}f)_{l_N^2}\|_{L^p}.
\end{aligned}
\tag{4.237}
$$

To estimate the preceding inequality, we write $P_N^x=\overline{P}_N^xP_N^x$, then employing Lemma 1.8, it follows

$$
|N^{-\beta}P_{\ll N}^xD_x^{\beta}f(x)|\leqslant N^{-\beta}\sum_{M\ll N}\Big|M^{\beta}P_M^xf(x)\Big|\lesssim\sum_{1\ll L}L^{-\beta}\mathcal{M}(P_{N/L}^xf)(x),
$$

so that

$$
(N^{-\beta}P_{\ll N}^xD_x^{\beta}f)_{l_N^2}\lesssim(\mathcal{M}(P_N^xf))_{l_N^2}.
\tag{4.238}
$$

Hence, plugging (4.238) in (4.237), by Fefferman-Stein inequality and Littlewood-Paley inequality, we conclude

$$
\|\mathcal{A}_1\|_{L^p}\lesssim\|\partial_xg\|_{L^\infty}\|f\|_{L^p}.
\tag{4.239}
$$

Now, replacing f by $\mathcal{H}_xf$ in the arguments above, we derive the same estimate in (4.239) for the term $\mathcal{A}_2$.

A similar reasoning yields the desired estimate for $\mathcal{A}_3$. Indeed, since $\alpha+\beta=1$, $\alpha>0$, by Littlewood-Paley inequality

$$
\|\mathcal{A}_3\|_{L^p}\lesssim\Big\|\Big(\sum_{N\gtrsim M}M^{\alpha}N^{-\alpha}\overline{P}_M^x(\overline{P}_N^x\partial_xg\overline{\widetilde{P}}_N^x\widetilde{P}_N^xf)\Big)_{l_M^2}\Big\|_{L^p}.
\tag{4.240}
$$

Then Lemma 1.8 shows

$$\begin{aligned}(4.241)\qquad \big(\sum_{N\gtrsim M} M^{\alpha}N^{-\alpha}\overline{P}^x_M(\overline{P}^x_N\partial_x g\overline{\widetilde{P}}^x_N\widetilde{P}^x_N f)\big)_{l^2_M} &\lesssim \big(\sum_{L\gtrsim 1} L^{-\alpha}\mathcal{M}(\overline{P}^x_{LM}\partial_x g\overline{\widetilde{P}}^x_{LM}\widetilde{P}^x_{LM}f)\big)_{l^2_M}\\ &\lesssim \big(\mathcal{M}(\overline{P}^x_N\partial_x g\overline{\widetilde{P}}^x_N\widetilde{P}^x_N f)\big)_{l^2_N}.\end{aligned}$$

Thus, the preceding estimates and Lemma 1.7 reveal

$$(4.242)\qquad \|\mathcal{A}_3\|_{L^p} \lesssim \|\mathcal{M}(\partial_x g)(\mathcal{M}(P^x_N f)_{l^2_N})\|_{L^p} \lesssim \|\partial_x g\|_{L^\infty}\|f\|_{L^p}.$$

The estimate for $\mathcal{A}_4$ follows from the same arguments employed to analyse $\mathcal{A}_3$. The proof of Proposition 4.2 is completed. □

Bibliography

[1] Abdelouhab, L., Bona, J., Felland, M., and Saut, J.-C. Nonlocal models for nonlinear, dispersive waves. *Physica D: Nonlinear Phenomena 40*, 3 (1989), 360–392.

[2] Abramyan, L. A., Stepanyants, Y. A., and Shrira, V. I. Multidimensional solitons in shear flows of the boundary-layer type. *Sov. Phys. Dokl 37*, 12 (1992), 575–578.

[3] Akers, B., and Milewski, P. A. A Model Equation for Wavepacket Solitary Waves Arising from Capillary-Gravity Flows. *Studies in Applied Mathematics 122*, 3 (2009), 249–274.

[4] Albert, J. Concentration compactness and the stability of solitary-wave solutions to nonlocal equations. *Contemporary Mathemathics 221* (1999), 1–29.

[5] Biello, J., and Hunter, J. K. Nonlinear Hamiltonian waves with constant frequency and surface waves on vorticity discontinuities. *Communications on Pure and Applied Mathematics 63*, 3 (2010), 303–336.

[6] Bona, J. L., and Liu, Y. Instability of solitary-wave solutions of the 3-dimensional Kadomtsev-Petviashvili equation. *Adv. Differential Equations 7*, 1 (2002), 1–23.

[7] Bona, J. L., and Smith, R. The Initial-Value Problem for the Korteweg-de Vries Equation. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences 278*, 1287 (1975), 555–601.

[8] Bona, J. L., Souganidis, P. E., and Strauss, W. A. Stability and Instability of Solitary Waves of Korteweg-de Vries Type. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences 411*, 1841 (1987), 395–412.

[9] Bustamante, E., Isaza, P., and Mejía, J. On uniqueness properties of solutions of the zakharov–kuznetsov equation. *Journal of Functional Analysis 264*, 11 (2013), 2529–2549.

[10] Bustamante, E., Urrea, J. J., and Mejía, J. Periodic Cauchy problem for one two-dimensional generalization of the Benjamin-Ono equation in Sobolev spaces of low regularity. *Nonlinear Analysis 188* (2019), 50–69.

[11] Bustamante, E., Urrea, J. J., and Mejía, J. The Cauchy problem for a family of two-dimensional fractional Benjamin-Ono equations. *Communications on Pure & Applied Analysis 18*, 3 (2019), 1177–1203.

[12] Calderón, A. P. Commutators of Singular Integral Operators. *Proc. Natl. Acad. Sci. USA. 53*, 5 (1965), 1092–1099.

[13] Case, K. M. Properties of the Benjamin-Ono equation. *Journal of Mathematical Physics 20*, 5 (1979), 972–977.

[14] Coifman, R., and Meyer, Y. On Commutators of Singular Integrals and Bilinear Singular Integrals. *Transactions of the American Mathematical Society 212* (1975), 315–331.

[15] Coifman, R., and Meyer, Y. *Au-delà des opérateurs pseudo-différentiels*. Astérisque 57, Société Mathématique de France, 1978.

[16] Cossetti, L. On the unique continuation property of solutions to the two-dimensional Zakharov-Kuznetsov equation. *arXiv e-prints* (2019), arXiv:1907.13377.

[17] Cossetti, L., Fanelli, L., and Linares, F. Uniqueness results for Zakharov-Kuznetsov equation. *Communications in Partial Differential Equations 44*, 6 (2019), 504–544.

[18] Cunha, A., and Pastor, A. The IVP for the Benjamin-Ono-Zakharov-Kuznetsov equation in weighted Sobolev spaces. *Journal of Mathematical Analysis and Applications 417*, 2 (2014), 660 – 693.

[19] DAWSON, L., MCGAHAGAN, H., AND PONCE, G. On the Decay Properties of Solutions to a Class of Schrödinger Equations. *Proceedings of the American Mathematical Society 136*, 6 (2008), 2081–2090.

[20] DUOANDIKOETXEA, J. *Fourier Analysis*. Crm Proceedings & Lecture Notes. American Mathematical Soc., 2001.

[21] DUQUE GÓMEZ, O. *Sobre una versión bidimensional de la ecuación Benjamin-Ono generalizada*. PhD thesis, Universidad Nacional de Colombia-Bogotá, 2014.

[22] ESFAHANI, A., AND PASTOR, A. Instability of solitary wave solutions for the generalized BO–ZK equation. *Journal of Differential Equations 247*, 12 (2009), 3181 – 3201.

[23] ESFAHANI, A., AND PASTOR, A. Ill-posedness results for the (generalized) Benjamin-Ono-Zakharov-Kuznetsov equation. *Proceedings of the American Mathematical Society 139*, 3 (2011), 943–956.

[24] ESFAHANI, A., AND PASTOR, A. Two dimensional solitary waves in shear flows. *Calculus of Variations and Partial Differential Equations 57*, 4 (2018), 102.

[25] ESFAHANI, A., PASTOR, A., AND BONA, J. L. Stability and decay properties of solitary-wave solutions to the generalized BO–ZK equation. *Adv. Differential Equations 20*, 9/10 (2015), 801–834.

[26] FEFFERMAN, C., AND STEIN, E. M. Some Maximal Inequalities. *American Journal of Mathematics 93*, 1 (1971), 107–115.

[27] FONSECA, G., LINARES, F., AND PONCE, G. The IVP for the Benjamin-Ono equation in weighted Sobolev spaces II. *Journal of Functional Analysis 262*, 5 (2012), 2031 – 2049.

[28] FONSECA, G., LINARES, F., AND PONCE, G. The IVP for the dispersion generalized Benjamin-Ono equation in weighted Sobolev spaces. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis 30*, 5 (2013), 763 – 790.

[29] FONSECA, G., AND PONCE, G. The IVP for the Benjamin-Ono equation in weighted Sobolev spaces. *Journal of Functional Analysis 260*, 2 (2011), 436–459.

[30] FRANK, R. L., AND LENZMANN, E. Uniqueness of non-linear ground states for fractional Laplacians in $\mathbb{R}$. *Acta Mathematica 210*, 2 (2013), 261–318.

[31] FRANK, R. L., LENZMANN, E., AND SILVESTRE, L. Uniqueness of Radial Solutions for the Fractional Laplacian. *Communications on Pure and Applied Mathematics 69*, 9 (2016), 1671–1726.

[32] GHOSH, T., SALO, M., AND UHLMANN, G. The Calderón problem for the fractional Schrödinger equation. *arXiv e-prints* (2016), arXiv:1609.09248, to appear in Analysis & PDE.

[33] GRAFAKOS, L. *Modern Fourier Analysis*. Graduate Texts in Mathematics. Springer New York, 2014.

[34] GRAFAKOS, L., AND OH, S. The Kato-Ponce Inequality. *Communications in Partial Differential Equations 39*, 6 (2014), 1128–1157.

[35] GUO, Z., AND OH, T. Non-Existence of Solutions for the Periodic Cubic NLS below L^2. *International Mathematics Research Notices 2018*, 6 (2018), 1656–1729.

[36] HADAC, M., HERR, S., AND KOCH, H. Well-posedness and scattering for the KP-II equation in a critical space. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis 26*, 3 (2009), 917–941.

[37] HADAC, M., HERR, S., AND KOCH, H. Erratum to "Well-posedness and scattering for the KP-II equation in a critical space" [Ann. I. H. Poincaré–AN 26 (3) (2009) 917–941]. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis 27*, 3 (2010), 971–972.

[38] HERR, S., AND KINOSHITA, S. Subcritical well-posedness results for the Zakharov-Kuznetsov equation in dimension three and higher. *arXiv e-prints* (2020), arXiv:2001.09047.

[39] HICKMAN, J., LINARES, F., RIAÑO, O., ROGERS, K., AND WRIGHT, J. On a Higher Dimensional Version of the Benjamin–Ono Equation. *SIAM Journal on Mathematical Analysis 51*, 6 (2019), 4544–4569.

[40] HUNTER, J. K., IFRIM, M., TATARU, D., AND WONG, T. K. Long time solutions for a Burgers-Hilbert equation via a modified energy method. *Proceedings of the American Mathematical Society 143*, 8 (2015), 3407–3412.

[41] IONESCU, A., AND KENIG, C. Local and global wellposedness of periodic KP-I equations. *Annals of Mathematics Studies*, 163 (2007), 181–211.

[42] IONESCU, A., AND KENIG, C. Local and global wellposedness of periodic KP-I equations. *Ann. of Math. Stud. 163* (2007), 181–211.

[43] IONESCU, A. D., AND KENIG, C. E. Global Well-Posedness of the Benjamin-Ono Equation in Low-Regularity Spaces. *Journal of the American Mathematical Society 20*, 3 (2007), 753–798.

[44] IONESCU, A. D., KENIG, C. E., AND TATARU, D. Global well-posedness of the KP-I initial-value problem in the energy space. *Inventiones mathematicae 173*, 2 (2008), 265–304.

[45] IÓRIO, R. J. On the Cauchy problem for the Benjamin-Ono equation. *Communications in Partial Differential Equations 11*, 10 (1986), 1031–1081.

[46] IÓRIO, R. J., AND NUNES, W. V. L. On equations of KP-type. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics 128*, 4 (1998), 725–743.

[47] IORIO, JR, R. J., AND IORIO, V. D. M. A. *Fourier Analysis and Partial Differential Equations*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2001.

[48] KATO, T., AND PONCE, G. Commutator estimates and the Euler and Navier-Stokes equations. *Communications on Pure and Applied Mathematics 41*, 7 (1988), 891–907.

[49] KAUP, D., LAKOBA, T., AND MATSUNO, Y. Complete integrability of the Benjamin-Ono equation by means of action-angle variables. *Physics Letters A 238*, 2 (1998), 123–133.

[50] KENIG, C. E. On the local and global well-posedness theory for the KP-I equation. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis 21*, 6 (2004), 827 – 838.

[51] KENIG, C. E., AND KOENIG, K. D. On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations. *Mathematical Research Letters 10*, 6 (2003), 879–895.

[52] KENIG, C. E., PILOD, D., PONCE, G., AND VEGA, L. On the unique continuation of solutions to non-local non-linear dispersive equations. *arXiv e-prints* (2020), arXiv:2002.04339.

[53] KINOSHITA, S. Global Well-posedness for the Cauchy problem of the Zakharov-Kuznetsov equation in 2D. *arXiv e-prints* (2019), arXiv:1905.01490.

[54] KOCH, H., AND TZVETKOV, N. On the local well-posedness of the Benjamin-Ono equation in $H^s(\mathbb{R})$. *International Mathematics Research Notices 2003*, 26 (2003), 1449–1464.

[55] LAEDKE, E. W., AND SPATSCHEK, K. H. Nonlinear ion-acoustic waves in weak magnetic fields. *The Physics of Fluids 25*, 6 (1982), 985–989.

[56] LI, D. On Kato-Ponce and Fractional Leibniz. *Revista Matemática Iberoamericana 35*, 1 (2019), 23–100.

[57] LINARES, F., PANTHEE, M., ROBERT, T., AND TZVETKOV, N. On the periodic Zakharov-Kuznetsov equation. *Discrete & Continuous Dynamical Systems-A 39*, 6 (2019), 3521–3533.

[58] LINARES, F., PILOD, D., AND SAUT, J.-C. Remarks on the orbital stability of ground state solutions of fKdV and related equations. *Adv. Differential Equations 20*, 9/10 (2015), 835–858.

[59] LINARES, F., PILOD, D., AND SAUT, J.-C. The Cauchy Problem for the Fractional Kadomtsev-Petviashvili Equations. *SIAM J. Math. Analysis 50* (2018), 3172–3209.

[60] LINARES, F., AND PONCE, G. *Introduction to Nonlinear Dispersive Equations*. Universitext. Springer New York, 2015.

[61] LIZARAZO, J. *El problema de Cauchy de la clase de ecuaciones de dispersión generalizada de Benjamin-Ono bidimensionales*. PhD thesis, Universidad Nacional de Colombia-Bogotá, 2018.

[62] MARIŞ, M. On the Existence, Regularity and Decay of Solitary Waves to a Generalized Benjamin-Ono Equation. *Nonlinear Anal. 51*, 6 (2002), 1073–1085.

[63] MOLINET, L. Global well-posedness in L2 for the periodic Benjamin-Ono equation. *American Journal of Mathematics 130*, 3 (2008), 635–683.

[64] MOLINET, L., AND PILOD, D. The Cauchy problem for the Benjamin-Ono equation in L^2 revisited. *Anal. PDE 5*, 2 (2012), 365–395.

[65] MOLINET, L., AND RIBAUD, F. Well-posedness results for the generalized Benjamin-Ono equation with small initial data. *Journal de Mathématiques Pures et Appliquées 83*, 2 (2004), 277–311.

[66] MOLINET, L., SAUT, J., AND TZVETKOV, N. Ill-Posedness Issues for the Benjamin-Ono and Related Equations. *SIAM Journal on Mathematical Analysis 33*, 4 (2001), 982–988.

[67] MUSCALU, C., PIPHER, J., TAO, T., AND THIELE, C. Bi-parameter paraproducts. *Acta Math. 193*, 2 (2004), 269–296.

[68] MUSCALU, C., AND SCHLAG, W. *Classical and Multilinear Harmonic Analysis*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2013.

[69] NAHAS, J., AND PONCE, G. On the Persistent Properties of Solutions to Semi-Linear Schrödinger Equation. *Communications in Partial Differential Equations 34*, 10 (2009), 1208–1227.

[70] PAVA, J. *Nonlinear Dispersive Equations: Existence and Stability of Solitary and Periodic Travelling Wave Solutions*. Mathematical surveys and monographs. American Mathematical Society, 2009.

[71] PELINOVSKY, D. E., AND SHRIRA, V. I. Collapse transformation for self-focusing solitary waves in boundary-layer type shear flows. *Physics Letters A 206*, 3 (1995), 195 – 202.

[72] PELINOVSKY, D. E., AND SHRIRA, V. I. Collapse transformation for self-focusing solitary waves in boundary-layer type shear flows. *Physics Letters A 206*, 3 (1995), 195 – 202.

[73] PETERMICHL, S. The sharp weighted bound for the Riesz transforms. *Proc. Amer. Math. Soc 136*, 4 (2008), 1237 – 1249.

[74] PONCE, G. On the global well-posedness of the Benjamin-Ono equation. *Differential Integral Equations 4*, 3 (1991), 527–542.

[75] RIAÑO, O. Well-posedness for a two-dimensional dispersive model arising from capillary-gravity flows. *arXiv e-prints* (2020), arXiv:2005.09184.

[76] RIAÑO, O. G. The IVP for a higher dimensional version of the Benjamin-Ono equation in weighted Sobolev spaces. *arXiv e-prints* (2019), arXiv:1908.07079.

[77] RIBAUD, F., AND VENTO, S. Local and global well-posedness results for the Benjamin-Ono-Zakharov-Kuznetsov equation. *Discrete & Continuous Dynamical Systems-A 37*, 1 (2017), 449–483.

[78] ROBERT, T. Global well-posedness of partially periodic KP-I equation in the energy space and application. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis 35*, 7 (2018), 1773 – 1826.

[79] ROBERT, T. On the Cauchy problem for the periodic fifth-order KP-I equation. *Differential Integral Equations 32*, 11/12 (2019), 679–704.

[80] SCHIPPA, R. On the Cauchy problem for higher dimensional Benjamin-Ono and Zakharov-Kuznetsov equations. *arXiv e-prints* (2019), arXiv:1903.02027.

[81] SCHIPPA, R. On short-time bilinear Strichartz estimates and applications to the Shrira equation. *Nonlinear Analysis 198* (2020), 111910.

[82] STEIN. *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*. Monographs in harmonic analysis. Princeton University Press, 1993.

[83] STEIN, E. *Singular Integrals and Differentiability Properties of Functions (PMS-30)*. Princeton Mathematical Series. Princeton University Press, 1970.

[84] STEIN, E. M. The characterization of functions arising as potentials. *Bull. Amer. Math. Soc. 67*, 1 (1961), 102–104.

[85] TAO, T. Global well-posedness of the Benjamin-Ono equation in $H^1(\mathbb{R})$. *Journal of Hyperbolic Differential Equations 01*, 01 (2004), 27–49.

[86] TAO, T. *Nonlinear Dispersive Equations: Local and Global Analysis*. Conference Board of the Mathematical Sciences. Regional conference series in mathematics. American Mathematical Society, 2006.

[87] VORONOVICH, V. V., AND SHRIRA, V. I. Internal wave–shear flow resonance and wave breaking in the subsurface layer. In *Nonlinear instability analysis*, vol. II of *Advances in fluid mechanics*. 28 WIT Press, 2001, pp. 133–177.

[88] YAFAEV, D. Sharp Constants in the Hardy-Rellich Inequalities. *Journal of Functional Analysis 168*, 1 (1999), 121–144.

[89] ZAKHAROV, V., AND KUZNETSOV, E. Three-dimensional solitons. *Soviet Physics JETP 39*, 2 (1974), 285–286.

[90] ZHANG, Y. Local well-posedness of KP-I initial value problem on torus in the Besov space. *Communications in Partial Differential Equations 41*, 2 (2016), 256–281.