Equilibrium Theory and Social Security

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I dedicate this thesis to God and my family.
For everything.

The centurion replied,
"Lord, I am not worthy to have
you come under my roof.
But just say the word,
and my servant will be healed".
Matthew 8:8

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Abstract

Overlapping generations models furnish a solid theoretical framework to study pay-as-you-go social security systems. We build on these equilibrium models in order to describe sustainable and optimal systems (*notional accounts* systems) subject to demographic and productivity fluctuations. Intra and intergenerational transfers, and possible absence of incentive compatibility are characterized. Also, we describe an equilibrium calculation method which yields Pareto optimal allocations and the role of a social security fund, which can lead to Pareto improvements over the system. Finally, we present a numerical analysis over the city and state level public servants social security system in Rio de Janeiro, which was awarded the XXIV National Treasury Prize.

Keywords: Overlapping Generations, Social Security, Notional Accounts
JEL: D15, D64, H55

Resumo

Modelos de gerações sobrepostas fornecem uma sólida base teórica para o estudo de sistemas previdenciários de repartição simples. Nós partimos destes modelos de equilíbrio para descrever sistemas sustentáveis e ótimos (sistemas de *contas nocionais*) sujeitos a flutuações demográficas e de produtividade. Transferências intra e intergeracionais, e possível ausência de compatibilidade de incentivos em tais sistemas são caracterizadas. Ainda, descrevemos um método para o cálculo de equilíbrios que implementam alocações Pareto-ótimas e o papel de um fundo de seguridade social, capaz de implementar melhorias de Pareto sobre o sistema. Por fim, nós apresentamos uma análise numérica dos Regimes Próprios de Previdência Social do município e do estado do Rio de Janeiro, a qual foi vencedora do XXIV Prêmio Tesouro Nacional.

Palavras-chave: Gerações Sobrepostas, Seguridade Social, Contas Nocionais

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1 Introduction

Pay-as-you-go social security systems have drawn a great deal of attention due to demographic contraction and low GDP growth rates in several countries. Under these conditions, financial and actuarial balance of the ongoing social security rules become compromised, leading to an unsustainable system, and reform efforts by Government start to emerge due to fiscal pressure. Then, two questions arise. How to define sustainable sets of social security rules? Which one, among all sustainable sets of social security rules, is best suited for the insured population?

In order to answer these questions, we built over the well grounded theoretical framework of general equilibrium overlapping generations models started by Paul Samuelson and Maurice Allais. We focus on consumption-loan models without uncertainty which have a close connection to social security rules applied to specific pay-as-you-go social security systems, the *notional accounts* ones.

As a first step, we define sustainable and optimal sets of social security rules, and show how equilibrium equations can be used to derive them. Next, we characterize the connection from demographic and productivity growth to the level of *per capita* contributions and benefits that can be implemented by a sustainable and optimal set of social security rules. One of our major results provides an analytical formula to solve equilibrium equations and allow for comparative statics analyses. Particularly, we show how future shocks on demographic and productivity growth rates impact current social security benefits, a phenomena we call *reverberation*.

We then integrate to the model a game-theoretical approach in order to characterize households behavior when allowed some degree of discretion vis-à-vis the set of social security rules. A real example of such discretion over retirement rules is given by systems in which households have different benefit values according to their chosen contribution level and retirement age. We show, particularly, that there are economies in which such degree of discretion will lead households to undersave due to a lack of incentive compatibility on social security rules. Also, we demonstrate that the existence of a social security fund is able to correct this undersaving behavior and, therefore, provides a Pareto improvement.

When solving equilibrium equations for heterogeneous populations, we demonstrate how intragenerational transfers are related to different groups demographic and productivity growth. Particularly, we show these transfers are not related to the *per capita* endowment of each group and, therefore, must be taken into account when defining the set of social security rules. Next, we characterize the role of compulsory savings when comparing economies with different endowment distributions between generations. We conclude that, for long run stable equilibria, real allocations obtained through compulsory savings can always be attained by a suitable endowment distribution.

Ending the first chapter, we provide an equilibrium calculation method for overlapping generations models with any number of periods. It relies on a long run optimal set of social security rules and uses a backward shift argument without imposing any restrictions on demographic or productivity growth rates dynamics. We also demonstrate how the method can be applied in order to define a selection criteria when one has multiple sustainable and Pareto optimal sets of social security rules.

The second chapter makes a brief review of a numerical analysis from Brazilian subnational entities social security systems that was awarded the XXIV National Treasury prize. There, we describe a methodology developed for financial and actuarial projections and also the results obtained when applying it to the city and state of Rio de Janeiro.

1.1 Literature review

Overlapping generations models have been the workhorse on general equilibrium theory for several fields, ranging from optimal growth theory to social security systems. It was independently developed by P. Samuelson (58) and M. Allais (48). Core results regarding equilibrium existence, indeterminacy and optimality were obtained by Y. Balasko, D. Cass and K. Shell (5; 6; 7; 8), L. Benveniste and D. Gale (10), M. Okuno and I. Zilcha (53), T. J. Kehoe and D. K. Levine (41), J. Geanakoplos and H. M. Polemarchakis (32), among others. The reader is invited to consult J. Geanakoplos (35) and M. Woodford (65) for a survey of these results.

Overlapping generations models[1] dealing explicitly with social security matters were developed by P. Samuelson (59), W. B. Arthur and G. McNicoll (12), J. B. Burbidge (13), F. Breyer (14), A. Imrohoroglu, S. Imrohoroglu and D. H. Joines (38; 39), among others. Also, simulation of overlapping generations models in order to describe transition paths after reforms and intergenerational risk sharing were performed by E. R. McGrattan and E. C. Prescott (49), L. J. Kotlikoff and J. Hasanhodzic (43).

Models dealing with uncertainty have formed a notorious research branch by S. R. Ayagari and D. Peled (4), S. Chattopadhyay and P. Gottardi (18), E. Sheshinski and Y. Weiss (61), G. Demange and G. Laroque (23; 24; 25; 26; 27), L. Forni (31), among others. Other important theoretical branch is the one which tackles overlapping generations and social security models with a game-theoretical approach, as in M. Kandori (40) and T. Cooley and J. Soares (19). Political mechanisms are explicitly dealt with in A. Alesina and G. Tabellini (1), and M. Boldrin and A. Rustichini (11).

Essays and discussions on demographic dynamics, social security reform, transition costs and comparison of funded and pay-as-you-go systems were done by J. Geanakoplos, O. S. Mitchell and S. Zeldes (33), P. R. Orszag and J. E. Stiglitz (55), P. Diamond (28) and the World Bank (66). Notional accounts social security systems are described in J. B. Williamson and M. Williams (64), R. F. Disney (30), M. Cichon (17) and World Bank (67).

Finally, the thesis builds upon and improves this literature mainly by providing a calculation method for equilibrium equations that yields sustainable and Pareto optimal notional accounts social security systems, by characterizing the intragenerational and intergenerational transfers on such systems and the Pareto improvements obtained through a social security fund.

[1] We may divide overlapping generations models in consumption-loan and production ones. The results of the thesis are derived for consumption-loan models without uncertainty.

2 An Equilibrium Analysis of Social Security

This chapter develops an analysis of social security based on equilibrium theory. Our baseline economy[2] is an overlapping generations one with an infinite number of periods starting at $t = 1$. Households (whose set is denoted by $\mathcal{H}$) live for two periods, except for the ones alive in the inception of the economy at $t = 1$ (who live for a single period), and are indexed by the period they are born. We denote the generation born in period $t \geq 0$ by

$$G_t = \{h \in \mathcal{H} \mid h \text{ is born in period } t\}$$

and by $L_t \in \mathbb{N}$, $t \geq 0$, the total number of households in generation G_t. Also, all households from generation G_t, $t \geq 0$, have a common endowment of $E_t \in \mathbb{R}_{++}$ units of a perishable consumption good during their first life period. Therefore, in period $t \geq 1$ there are $L_{t-1} + L_t$ households alive, with $L_t E_t$ accounting for the total consumption good in the economy. The demographic and productivity[3] dynamics are summarized by

$$\alpha_t^L = \frac{L_t}{L_{t-1}}$$
$$\alpha_t^E = \frac{E_t}{E_{t-1}}$$

for $t \geq 1$. Also, $\alpha_t = \alpha_t^L \alpha_t^E$, $t \geq 1$, is the compound rate of demographic and productivity growth. Households from generation G_t, $t \geq 0$, beside the common first-period endowment, also have a common utility function $U_t : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ which satisfies the following assumption.

Assumption 1. *For all* $t \geq 0$, U_t *is differentiable, strictly concave, strictly increasing in each coordinate and satisfies Inada's condition.*

For a given interperiod return rate $R \in \mathbb{R}_+$, the continuity and strict concavity of U_t imply that the savings demand $\phi_t(R, E_t)$, $t \geq 0$, is well-defined by the following optimization problem

$$\begin{aligned} \phi_t(R, E_t) = \arg \max_{\phi \in [0,1]} \quad & U_t(c_0, c_1) \\ \text{s.t.} \quad & c_0 = (1 - \phi) E_t \\ & c_1 = R \phi E_t \end{aligned}$$

The Maximum Theorem[4] and the stricty concavity of U_t imply $\phi_t : \mathbb{R}_+ \times \mathbb{R}_{++} \to [0, 1]$ is a continuous function. Also, $\phi_t(0, E_t) = 0$ since U_t is strictly increasing in its first coordinate.

Government maintains a pay-as-you-go social security system under which it manages transfers of the perishable consumption good from current young to old generations. The social security system assigns to each generation G_t, $t \geq 1$, a set of retirement rules $S_t = (\mathcal{C}_t, \mathcal{R}_t) \in [0, 1] \times \mathbb{R}_+$ specifying contribution and replacement rates. That is, households from generation G_t, $t \geq 1$, are initially required to contribute $\mathcal{C}_t E_t$ of the consumption good when young at period t and are entitled to receive $\mathcal{R}_t E_t$ when old at period $t+1$. Generation G_0 has assigned

[2] We follow the notation used by Okuno and Zilcha (53; 54) in their consumption-loan type models.

[3] If all generations G_t have a fixed *per capita* labour supply when young and each one has its own production technology for the perishable good, the endowment E_t can be seen as a productivity measure.

[4] Also known as Berge's Maximum Theorem, after Claude Berge (1959).

only a replacement rate[5], i.e., $S_0 = \mathcal{R}_0 \in \mathbb{R}_+$. In period $t \geq 1$, the social security system budget result is

$$\Delta_t = L_t \mathcal{C}_t E_t - L_{t-1} \mathcal{R}_{t-1} E_{t-1}$$

where $\Delta_t > 0$ implies a budget surplus over the social security system, $\Delta_t < 0$ a budget deficit and $\Delta_t = 0$ a balanced system in period t. Finally, since Government does not consume or produce any amount of the perishable good, it must define a proportion $\gamma_t \in [0,1]$, $t \geq 1$, in which a budget surplus or deficit in period t is divided among the young and old generations. That is, real contribution and replacement rates $(\mathcal{C}_t^*, \mathcal{R}_t^*)$, after the budget result is divided according to a proportion γ_t between the young and old in period t, are given by

$$\begin{aligned} \mathcal{C}_t^* &= \mathcal{C}_t - \gamma_t \frac{\Delta_t}{L_t E_t} \\ &= (1-\gamma_t)\mathcal{C}_t + \gamma_t \frac{\mathcal{R}_{t-1}}{\alpha_t} \end{aligned}$$

for $t \geq 1$[6], and

$$\begin{aligned} \mathcal{R}_t^* &= \mathcal{R}_t + (1-\gamma_{t+1}) \frac{\Delta_{t+1}}{L_t E_t} \\ &= \gamma_{t+1}\mathcal{R}_t + (1-\gamma_{t+1})\alpha_{t+1}\mathcal{C}_{t+1} \end{aligned}$$

for $t \geq 0$. Also, final consumption of generation G_t, $t \geq 1$, is $(c_0^t, c_1^t) = ((1-\mathcal{C}_t^*)E_t, \mathcal{R}_t^* E_t)$.

2.1 Sustainability and optimality of a social security system

In this section we describe how sustainable and optimal social security rules can be designed. When managing the social security system, Government must choose, among all possible sets of social security rules, one that is best suited for its households. In other words, given its economy $\mathcal{E} = \{(U_t, L_t, E_t)\}_{t\geq 0}$, Government must choose, among all possible sets of social security rules $\{S_t, \gamma_{t+1}\}_{t\geq 0}$, one that is best for its households according to some choice criteria. We define and motivate below three desirable characteristics for sets of social security rules which are later used as such choice criteria. The first one is sustainability.

Definition 2.1. *Given $\mathcal{E} = \{(U_t, L_t, E_t)\}_{t\geq 0}$, a set of social security rules $\{S_t, \gamma_{t+1}\}_{t\geq 0}$ is sustainable if all real contribution rates are affordable, i.e., if $\mathcal{C}_t^* \in [0,1]$, $\forall t \geq 1$.*

A sustainable set of rules is one that can be truly implemented in all periods after accounting for budget deficits of the social security system. An important feature is that sustainability is independent of households preferences, but is directly attached to the demographic and productivity dynamic, as can be seen from the next proposition.

Proposition 2.2. *If for some $t \geq 1$, $\gamma_t > 0$ and $\mathcal{R}_{t-1} > 0$, then there is a demographic or a productivity dynamic that turns the set of social security rules unsustainable.*

[5]Notice that the social security rules $\{S_t\}_{t\geq 0}$ are not necessarily attached to the demographic or productivity dynamics of the economy, neither to the households preferences.

[6]Generation G_0 retirement rules do not define a contribution rate.

Proof. The real contribution rate from generation G_t can be written as

$$\mathcal{C}_t^* \;=\; (1-\gamma_t)\mathcal{C}_t + \gamma_t \frac{\mathcal{R}_{t-1}}{\alpha_t^L \alpha_t^E}$$

for $t \geq 1$. Then, since $1-(1-\gamma_t)\mathcal{C}_t > 0$ and $\gamma_t \mathcal{R}_{t-1} > 0$, we have $\mathcal{C}_t^* > 1$ if, and only if,

$$\frac{\gamma_t \mathcal{R}_{t-1}}{1-(1-\gamma_t)\mathcal{C}_t} \;>\; \alpha_t^L \alpha_t^E$$

We conclude that for α_t^L or α_t^E sufficiently small the set of social security rules is unsustainable. □

We can read the previous proposition as a direct claim that social security rules should not be thought or implemented independently of the fundamentals of the economy, specifically of its demographic and productivity dynamic, if Government seeks for sustainable systems. Another important conclusion is that demography should not be seen as the unique factor defining the sustainability of a pay-as-you-go social security system. More accurate, indeed, is the concern with the compound demographic and productivity growth rate.

If the set of social security rules is sustainable the real contribution and replacement rates of generation G_t, $t \geq 1$, define an implicit real return rate Q_t^* with

$$Q_t^* = \frac{\mathcal{R}_t^*}{\mathcal{C}_t^*}$$

for $t \geq 1$. One important property of social security rules is that, for a fixed $\{\gamma_t\}_{t\geq 1}$, if $\{S_t\}_{t\geq 0}$ is sustainable then $\{\lambda S_t\}_{t\geq 0}$ is also sustainable, $\forall \lambda \in [0,1]$. For $\lambda > 1$, $\{\lambda S_t\}_{t\geq 0}$ may also be sustainable, as long as the affordability condition is satisfied, i.e., $\mathcal{C}_t^* \in [0,1]$, $\forall t \geq 1$. In any of such transformations, the value of the real return rates $\{Q_t^*\}_{t\geq 1}$ is kept constant so that only the absolute values of contribution and replacement rates are changed. The next definition characterizes the sets of social security rules under which all generations would agree on the absolute level of contributions and replacement rates.

Definition 2.3. *Given $\mathcal{E} = \{(U_t, L_t, E_t)\}_{t\geq 0}$, a sustainable set of social security rules $\{S_t, \gamma_{t+1}\}_{t\geq 0}$ is individually optimal if for all generations G_t, $t \geq 1$, $\mathcal{C}_t^* = \phi_t(Q_t^*, E_t)$.*

Individually optimal social security rules must take into account not only the demographic and productivity dynamic, as sustainable ones do, but also households preferences. Individual optimality means that households have no incentives to call for transformations over the absolute levels of contribution and replacement rates. The next example illustrates this point.

Example 2.4. *Let the economy be stationary, i.e., $U_t = U$, $E_t = E$ and $L_t = L$, $t \geq 0$. For all $\lambda \in (0,1)$, $S_0 = \lambda$, $S_t = (\lambda, \lambda)$, $t \geq 1$, defines a sustainable set of social security rules, where λ defines the absolute level of contribution and replacement rates. Notice that $\Delta_t = 0$, $t \geq 1$, and, therefore, $\{\gamma_t\}_{t\geq 1}$ is irrelevant in this case. Also, $\forall \lambda \in (0,1)$, $Q_t^* = 1$, $t \geq 1$. Although all sets of rules defined by $\lambda \in (0,1)$ are sustainable, only the one with $\lambda = \phi(1, E)$ is individually optimal. That is, for any $\lambda \neq \phi(1, E)$, households would prefer a different absolute level of contribution and replacement rates.*

A third desirable characteristic of social security rules is Pareto optimality.

Definition 2.5. *Given* $\mathcal{E} = \{(U_t, L_t, E_t)\}_{t\geq 0}$*, a sustainable and individually optimal set of social security rules* $\{S_t, \gamma_{t+1}\}_{t\geq 0}$ *is Pareto optimal if there is no other set of social security rules which yield a Pareto improvement, i.e.,* $\nexists\{S'_t, \gamma'_{t+1}\}_{t\geq 0}$ *such that* $U_t(c^t) \leq U_t(c^{t\prime})$[7], $\forall t \geq 0$*, with at least one strict inequality.*

Pareto optimality can be stated as the lack of possibility for reorganizing the social security system in a way that improves the condition of at least one generation, without imposing a welfare loss to any other. After identifying three desirable characteristics of social security rules (sustainability, individual optimality and Pareto optimality), it still remains the question on how to actually calculate such rules, i.e., given an economy $\mathcal{E} = \{(U_t, L_t, E_t)\}_{t\geq 0}$, how Government defines a sustainable, individually optimal and Pareto optimal set of social security rules $\{(S_t, \gamma_{t+1})\}_{t\geq 0}$?

A natural way to start is looking at the following equilibrium equations over savings

$$
\begin{aligned}
L_1 E_1 \phi_1(R_1, E_1) &= L_0 R_0 E_0 \phi_0(R_0, E_0) \\
L_2 E_2 \phi_2(R_2, E_2) &= L_1 R_1 E_1 \phi_1(R_1, E_1) \\
&[\ldots]
\end{aligned}
$$

where, in each period $t \geq 1$, savings demand from the young generation equate the return of savings from the previous period for the old generation[8] according to a sequence of return rates $R = \{R_t\}_{t\geq 0} \in \mathbb{R}_+^\infty$. Assuming that such equilibrium return rates sequence R exists and that Government can calculate it, one way of replicating the real allocations is by setting contribution rates as

$$\mathcal{C}_t = \phi_t(R_t, E_t)$$

for $t \geq 1$ and replacement rates as

$$\mathcal{R}_t = R_t \phi_t(R_t, E_t)$$

for $t \geq 0$. Under these social security rules the budget result of the system in period $t \geq 1$ is

$$
\begin{aligned}
\Delta_t &= L_t \mathcal{C}_t E_t - L_{t-1} \mathcal{R}_{t-1} E_{t-1} \\
&= L_t E_t \phi_t(R_t, E_t) - L_{t-1} R_{t-1} E_{t-1} \phi_{t-1}(R_{t-1}, E_{t-1}) \\
&= 0
\end{aligned}
$$

where the last equality comes from the equilibrium equations. Therefore, when dealing with social security rules that replicate real allocations obtained through equilibrium equations, the system is balanced in every period, i.e., $\Delta_t = 0$, $t \geq 1$. This makes the definition of $\{\gamma_t\}_{t\geq 1}$ irrelevant and $\mathcal{R}_t^* = \mathcal{R}_t$, $t \geq 0$, $\mathcal{C}_t^* = \mathcal{C}_t$ and $Q_t^* = R_t$, $t \geq 1$. Also, $Im(\phi_t) \subseteq [0, 1]$[9] and $\mathcal{C}_t^* = \phi_t(R_t, E_t)$, $t \geq 1$, imply that the set of rules is sustainable and individually optimal.

[7] When comparing welfare levels for generation G_0, we adopt a constrained utility function, $\bar{U}_0(\cdot) = U_0(c_0^0, \cdot)$, where $U_0 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is the original utility function and $c_0^0 \in \mathbb{R}_{++}$ is any fixed past consumption level.

[8] Although generation G_0 lives for a single period, it has a utility function defined over $\mathbb{R}_+ \times \mathbb{R}_+$ in order to have a well-defined savings demand $\phi_0(\cdot, \cdot)$. This formulation considers that in period $t = 1$, the inception of the economy, old age individuals behave as if they were entitled to some predefined return R_0 over their past savings. Also, this way of writing equilibrium equations will become clear with the results of Section 2.8.

[9] $Im(\cdot)$ represents the image set of a function.

Although calculating social security rules through equilibrium equations yield sustainable and individually optimal systems, they may not be Pareto optimal[10]. There is still one last remark to be made about calculating social security rules trough equilibrium equations. The budget constraint on the utility maximization problem that defines $\phi_t(R, E_t)$ can be writen as

$$c_0 + \frac{c_1}{R} = E_t$$

for $t \geq 0$. The left side corresponds to the discounted value of the consumption bundle under an interperiod return rate R, while the right side corresponds to the discounted value of the households endowment. Since contribution and replacement rates derived from equilibrium equations replicate such consumption bundle for every generation G_t, $t \geq 1$, we may interpret such set of social security rules as one that mimics an "account behavior" according to a given sequence of interperiod return rates $\{R_t\}_{t\geq 0}$, i.e., equating contributions and benefits present value. Pay-as-you-go systems structured according to these sets of rules are conventionally called *notional accounts social security system.*

2.2 The role of demography and productivity growth

The previous section has shown that the definition of sustainable, individually optimal and Pareto optimal sets of social security rules is closely related to the problem of solving equilibrium equations. In this section, we characterize the relation between demography, productivity and equilibrium return rates $R = \{R_t\}_{t\geq 0}$, while also providing a theorem on how to effectively calculate such sequence. The economy is our baseline one. We start by rewriting equilibrium equations using the definition of $\{\alpha_t\}_{t\geq 1}$ as

$$\alpha_t \phi_t(R_t, E_t) = R_{t-1}\phi_{t-1}(R_{t-1}, E_{t-1})$$

for $t \geq 1$. In order to analyse the impact of changes in both demography $\{L_t\}_{t\geq 0}$ and productivity $\{E_t\}_{t\geq 0}$ over equilibrium return rates, we make the following assumption.

Assumption 2. *For all $t \geq 0$, ϕ_t is strictly increasing on its first argument and constant on its second, i.e., $\phi_t(R, E) = \phi_t(R, E')$, $\forall E, E' \in \mathbb{R}_{++}$.*

Therefore, savings demand is strictly increasing over interperiod return rates and is not affected by endowment levels[11], i.e., $\phi_t(R, E) = \phi_t(R)$, $\forall t \geq 0$. Assumption 2 is satisfied by Constant Relative Risk Aversion (CRRA) utility functions where, for $\beta_t \in \mathbb{R}_{++}$, $\theta_t \in (0, 1)$,

$$U_t(c_0, c_1) = \frac{c_0^{1-\theta_t}}{1-\theta_t} + \beta_t \frac{c_1^{1-\theta_t}}{1-\theta_t}$$

and savings demand[12] is given by

$$\phi_t(R) = 1 - \frac{1}{1 + \beta_t^{\frac{1}{\theta_t}} R^{\frac{1-\theta_t}{\theta_t}}}$$

[10] This point was first highlighted by Paul Samuelson (58).

[11] If one does not assume that savings demand is constant over its second argument, all results from this section remain valid but are no longer stated in terms of the compound demographic and productivity growth rates $\{\alpha_t\}_{t\geq 1}$, but only in terms of the demographic growth rates $\{\alpha_t^L\}_{t\geq 1}$.

[12] The logarithmic CRRA $U_t(c_0, c_1) = \log(c_0) + \beta_t \log(c_1)$ implies a constant savings demand $\phi_t(R, E) = \frac{\beta_t}{1+\beta_t}$.

Equilibrium equations can then be shortly written as

$$\alpha_t \phi_t(R_t) = R_{t-1}\phi_{t-1}(R_{t-1}) \tag{1}$$

for $t \geq 1$. Since $\phi_t(\cdot)$ is injective, $t \geq 0$, the following relation holds for any equilibrium return rates sequence $\{R_t\}_{t\geq 0}$

$$R_t \quad = \quad \phi_t^{-1}\left(\frac{R_{t-1}}{\alpha_t}\phi_{t-1}(R_{t-1})\right)$$

$t \geq 1$. Iterating the previous equation yields

$$R_t \quad = \quad \phi_t^{-1}\left(\frac{\prod_{i=0}^{t-1} R_i}{\prod_{i=1}^{t} \alpha_i}\phi_0(R_0)\right) \tag{2}$$

$t \geq 1$. The next propositions characterize the relation between the compound demographic and productivity growth rate $\{\alpha_t\}_{t\geq 1}$ and the sequence of equilibrium return rates $\{R_t\}_{t\geq 0}$ that is used to define a sustainable and optimal set of social security rules $\{S_t\}_{t\geq 0}$. To ease notation we use, when convenient, α for $\{\alpha_t\}_{t\geq 1}$ and R for $\{R_t\}_{t\geq 0}$.

Proposition 2.6. *Given $\{\alpha_t\}_{t\geq 1} \in \mathbb{R}_{++}^{\infty}$, there exists $\eta(\alpha) \in \mathbb{R}_+$ such that $\{R_t\}_{t\geq 0}$ is an equilibrium sequence if, and only if, $R_0 \in [0, \eta(\alpha)]$. Also, if $\alpha_t \geq \tilde{\alpha}_t > 0$, $t \geq 1$, then*
(i) $\eta(\alpha) \geq \eta(\tilde{\alpha})$;
(ii) $R_t \geq \tilde{R}_t$, $t \geq 0$, where $\{R_t\}_{t\geq 0}$ and $\{\tilde{R}_t\}_{t\geq 0}$ are the equilibrium return rates sequences with $R_0 = \eta(\alpha)$ and $\tilde{R}_0 = \eta(\tilde{\alpha})$.

Proof. Given $\{\alpha_t\}_{t\geq 1} \in \mathbb{R}_{++}^{\infty}$, injectivity of ϕ_t, $t \geq 0$, along with Equation 2, assures that every equilibrium return rates sequence $\{R_t\}_{t\geq 0}$ is uniquely defined by R_0. Also, $\phi_t(0) = 0$, $t \geq 0$, implies that $R_t = 0$, $t \geq 0$, is an equilibrium. Suppose $\{\bar{R}_t\}_{t\geq 0}$ is also an equilibrium with $\bar{R}_0 \in \mathbb{R}_{++}$ and let $R_0 \in (0, \bar{R}_0)$. Since ϕ_0 is strictly increasing

$$0 < \frac{R_0}{\alpha_1}\phi_0(R_0) < \frac{\bar{R}_0}{\alpha_1}\phi_0(\bar{R}_0) \implies \frac{R_0}{\alpha_1}\phi_0(R_0) \in Im(\phi_1)$$

So $R_1 = \phi_1^{-1}\left(\frac{R_0}{\alpha_1}\phi_0(R_0)\right)$ is well-defined and $0 < R_1 < \bar{R}_1$, since ϕ_1 is strictly increasing. By induction, the system of equations has a well-defined solution starting from R_0 and $R_t < \bar{R}_t$, $\forall t \geq 0$. Next, let $R_0^n \to R_0$, where R_0^n defines an equilibrium sequence, $n \geq 1$. To show that R_0 also defines an equilibrium sequence, notice that R_1^n is well-defined and that the continuity of ϕ_0 and ϕ_1^{-1} implies $R_1^n \to R_1$, where R_1 satisfies the first equilibrium equation. In a similar way, all other equations are satisfied by the limit points $R_t^n \to R_t$ and, therefore, $(R_t)_{t\geq 0}$ is an equilibrium return rates sequence. If we define the following set

$$I(\alpha) = \{R_0 \in \mathbb{R}_+ \mid R_0 \text{ defines an equilibrium sequence according to Equation 1}\}$$

then the first remark implies that every equilibrium return rates sequence is uniquely defined by an element of $I(\alpha)$ and $0 \in I(\alpha)$, the second remark implies that $I(\alpha)$ is an interval and the third that such interval is closed. Since ϕ_1 is bounded, the first equilibrium equation imposes an upper bound on all possible equilibrium values of the initial return rate. Then, since $I(\alpha)$ is bounded, there exists $\eta(\alpha) \in \mathbb{R}_+$ such that $I(\alpha) = [0, \eta(\alpha)]$.

For item (i), let $\tilde{R}_0 \in [0, \eta(\tilde{\alpha})]$ and notice that

$$\tilde{R}_1 = \phi_1^{-1}\left(\frac{\tilde{R}_0}{\tilde{\alpha}_1}\phi_0(\tilde{R}_0)\right) \geq \phi_1^{-1}\left(\frac{\tilde{R}_0}{\alpha_1}\phi_0(\tilde{R}_0)\right) = R_1$$

so that R_1 is well-defined for an initial return rate $R_0 = \tilde{R}_0$. Also,

$$\tilde{R}_2 = \phi_2^{-1}\left(\frac{\tilde{R}_1}{\tilde{\alpha}_2}\phi_1(\tilde{R}_1)\right) \geq \phi_2^{-1}\left(\frac{R_1}{\alpha_2}\phi_1(R_1)\right) = R_2$$

where the inequality comes from $\tilde{R}_1 \geq R_1$ and $\tilde{\alpha}_2 \leq \alpha_2$. By induction, the sequence defined after $\tilde{R}_0$ is well-defined and we conclude that $\tilde{R}_0 \leq \eta(\alpha)$. Since $\tilde{R}_0$ is arbitrary, $\eta(\tilde{\alpha}) \leq \eta(\alpha)$.

For item (ii), let $R_0 = \eta(\alpha)$ and $\tilde{R}_0 = \eta(\tilde{\alpha})$, so that item (i) implies $R_0 \geq \tilde{R}_0$. Suppose $R_k < \tilde{R}_k$ for some $k > 0$. Then $\alpha_{k+1} \geq \tilde{\alpha}_{k+1}$ implies that

$$R_{k+1} = \phi_{k+1}^{-1}\left(\frac{R_k}{\alpha_{k+1}}\phi_k(R_k)\right) < \phi_{k+1}^{-1}\left(\frac{\tilde{R}_k}{\tilde{\alpha}_{k+1}}\phi_k(\tilde{R}_k)\right) = \tilde{R}_{k+1}$$

and, by induction, we conclude that $R_i < \tilde{R}_i$, $i \geq k$. Let, indeed, $k > 0$ be the smallest value that satisfies this property. Then $R_i \geq \tilde{R}_i$, $i < k$. Also, define an auxiliary sequence $(P_t)_{t\geq 0}$, with $P_k = \tilde{R}_k$ and the remaining elements defined below. Since $\tilde{R}_k \geq P_k > R_k$, we have

$$\begin{aligned}
\tilde{R}_{k+1} &= \phi_{k+1}^{-1}\left(\frac{\tilde{R}_k}{\tilde{\alpha}_{k+1}}\phi_k(\tilde{R}_k)\right) \\
&\geq \phi_{k+1}^{-1}\left(\frac{P_k}{\alpha_{k+1}}\phi_k(P_k)\right) \\
&> \phi_{k+1}^{-1}\left(\frac{R_k}{\alpha_{k+1}}\phi_k(R_k)\right) \\
&= R_{k+1}
\end{aligned}$$

and the first inequality allows us to define

$$P_{k+1} = \phi_{k+1}^{-1}\left(\frac{P_k}{\alpha_{k+1}}\phi_k(P_k)\right)$$

Also, $\tilde{R}_{k+1} \geq P_{k+1} > R_{k+1}$. By induction, P_t is well-defined and satisfies equilibrium equations, $\forall t \geq k$, and $\tilde{R}_t \geq P_t > R_t$. To define P_t for $0 \leq t < k$ we must solve

$$\begin{aligned}
\alpha_1\phi_1(P_1) &= S_0\phi_0(P_0) \\
\alpha_2\phi_2(P_2) &= S_1\phi_1(P_1) \\
&[\ldots] \\
\alpha_k\phi_k(P_k) &= S_{k-1}\phi_{k-1}(P_{k-1})
\end{aligned}$$

Since the function $x \to x\phi_t(x)$ has its image over $[0, +\infty]$, $0 \leq t < k$, it is always possible to define P_t, $0 \leq t < k$, so that all previous equations hold true. Notice, however, that $x \to x\phi_t(x)$ is an strictly increasing function and, therefore, $P_k > R_k$ implies that $P_t > R_t$, $0 \leq t < k$. Particularly, $P_0 > R_0$. But this contradicts the definition of R_0, absurd. We conclude that

$R_t \geq \tilde{R}_t$, $t \geq 0$. □

Proposition 2.6 states, initially, that, for a given demographic and productivity dynamic $\{\alpha_t\}_{t\geq 1}$, equilibrium return rates sequence $\{R_t\}_{t\geq 0}$ is uniquely determined by its initial value R_0. Moreover, there is a continuum of possible initial values and, therefore, one may have an infinite number of sustainable and individually optimal sets of social security rules. There is, however, a Pareto dominant set of rules, namely the one obtained according to the maximum possible value $\eta(\alpha)$ for the initial return rate R_0. When analysing the impact of changes over $\{\alpha_t\}_{t\geq 1}$, a more favourable scenario in terms of the compound growth rates imply that the set of social security rules becomes able to sustain larger return rates for all generations, even if changes happen only at future periods. The next proposition imposes homogeneity over preferences in order to derive further properties of equilibrium return rates sequences.

Proposition 2.7. *Suppose preferences are homogeneous, i.e.,* $U_t = U$, $t \geq 0$, *and* $\{R_t\}_{t\geq 0}$ *is the equilibrium return rates sequence defined after* $R_0 = \eta(\alpha)$. *The following implications hold*
(i) If $\alpha_t \geq \delta > 0$, $t \geq 1$, *then* $R_t \geq \delta$, $t \geq 0$*;*
(ii) If $\{\alpha_t\}_{t\geq 1}$ *is a non-increasing sequence, then* $R_t \leq \alpha_{t+1}$, $t \geq 0$, *and the returns are, itself, a non-increasing sequence;*
(iii) If $\{\alpha_t\}_{t\geq 1}$ *is a non-decreasing sequence, then* $R_t \geq \alpha_{t+1}$, $t \geq 0$, *and the returns are, itself, a non-decreasing sequence;*
(iv) If $\alpha_t = \delta > 0$, $t \geq 1$, *then* $R_t = \delta$, $t \geq 0$.

Proof. Let $\phi_t = \phi$, $t \geq 0$. For (i), suppose $\exists t_0 \geq 0$ such that $R_{t_0} < \delta$. Then, define the sequence $\{P_t\}_{t\geq 0}$ by $P_{t_0} = \delta$ and

$$P_{t+1} = \phi^{-1}\left(\frac{P_t}{\alpha_{t+1}}\phi(P_t)\right)$$

for $t \geq t_0$. Notice that P_t, $t \geq t_0$, is well-defined since $P_{t_0} \leq \delta$ and if $P_t \leq \delta \leq \alpha_{t+1}$ the following implication hold

$$P_{t+1} = \phi^{-1}\left(\frac{P_t}{\alpha_{t+1}}\phi(P_t)\right) \leq \phi^{-1} \circ \phi(P_t) \implies P_{t+1} \leq P_t \leq \delta$$

In order to define P_t for $0 \leq t < t_0$ we must solve the following system of equations

$$\begin{aligned} \alpha_1\phi(P_1) &= P_0\phi(P_0) \\ \alpha_2\phi(P_2) &= P_1\phi(P_1) \\ &[\ldots] \\ \alpha_{t_0}\phi(\delta) &= P_{t_0-1}\phi(P_{t_0-1}) \end{aligned}$$

Since the function $x \to x\phi(x)$ has its image equal to $\mathbb{R}_+$, the above system of equations has a solution and $\{P_t\}_{t\geq 0}$ is a well-defined equilibrium return rates sequence. Notice, next, that the function $x \to x\phi(x)$ is strictly increasing and, therefore, $P_{t_0} = \delta > R_{t_0}$ implies $P_0 > R_0$. Absurd, since $R_0 = \eta(\alpha)$. We conclude that $R_t \geq \delta$, $t \geq 0$.

For (ii), let $\{\alpha_t\}_{t\geq 1}$ be a non-increasing sequence and suppose $R_0 > \alpha_1$. Since $\phi(R_1) = \frac{R_0}{\alpha_1}\phi(R_0)$ and ϕ is strictly increasing, we have $R_1 > R_0 > \alpha_1 \geq \alpha_2$. However, since $\phi(R_2) = \frac{R_1}{\alpha_2}\phi(R_1)$, this implies that $R_2 > R_1 > R_0 > \alpha_1 \geq \alpha_2 \geq \alpha_3$. But then, assuming $\{R_t\}_{t\geq 0}$ is well-defined, we would have

$$\left(\frac{R_0}{\alpha_1}\right)^t \phi(R_0) \leq \frac{\prod_{i=0}^{t-1} R_i}{\prod_{i=1}^{t} \alpha_i}\phi(R_0) = \phi(R_t) \to +\infty$$

absurd. Then $R_0 \leq \alpha_1$ and $R_1 \leq R_0$. Suppose now $R_1 > \alpha_2$, so that $R_2 > R_1 > \alpha_2 \geq \alpha_3$. Again, this would imply $R_3 > R_2 > R_1 > \alpha_2 \geq \alpha_3 \geq \alpha_4$. Assuming $\{R_t\}_{t\geq 0}$ is well-defined, we would have

$$\frac{R_0}{\alpha_1}\left(\frac{R_1}{\alpha_2}\right)^{t-1} \phi(R_0) \leq \frac{\prod_{i=0}^{t-1} R_i}{\prod_{i=1}^{t} \alpha_i}\phi(R_0) = \phi(R_t) \to +\infty$$

absurd. So $R_1 \leq \alpha_2$ and $R_2 \leq R_1$. By induction, we conclude that $R_t \leq \alpha_{t+1}$ and that $\{R_t\}_{t\geq 0}$ is also non-increasing.

For (iii), let $\{\alpha_t\}_{t\geq 1}$ be a non-decreasing sequence and define the following auxiliary sequences $\alpha^k = \{\alpha_t^k\}_{t\geq 1}$, $k \geq 1$, where $\alpha_t^k = \alpha_t$, $t < k$, and $\alpha_t^k = \alpha_k$, $t \geq k$. Notice then that item (ii) from Proposition 2.6 allows us to write that $R_t^k \leq R_t^{k+1} \leq R_t$, $t \geq 0$, where $\{R_t^k\}_{t\geq 0}$ is the equilibrium return rates sequence with $R_0^k = \eta(\alpha^k)$, $k \geq 1$. Also, for each sequence α^k, it is possible to construct an equilibrium return rates sequence $\bar{R}^k$ such that $\bar{R}_t^k = \alpha_k$, $t \geq k-1$. To see this, notice that for $t \geq k$

$$\bar{R}_t^k = \bar{R}_{t-1}^k = \alpha_t^k = \alpha_k \to \phi(\bar{R}_t^k) = \frac{\bar{R}_{t-1}^k}{\alpha_t^k}\phi(\bar{R}_{t-1}^k)$$

In order to find the values of $\bar{R}_t^k$, for $t < k$, one must solve

$$\begin{aligned} \alpha_1^k \phi(\bar{R}_1^k) &= \bar{R}_0^k \phi(\bar{R}_0^k) \\ &[\ldots] \\ \alpha_{k-1}^k \phi(\bar{R}_{k-1}^k) &= \bar{R}_{k-2}^k \phi(\bar{R}_{k-2}^k) \end{aligned}$$

The last equation can be written as

$$\alpha_{k-1}^k \phi(\alpha_k) = \bar{R}_{k-2}^k \phi(\bar{R}_{k-2}^k)$$

Since the function $x \to x\phi(x)$ has its image over $[0, +\infty)$, it is always possible to find $\bar{R}_0^k, \ldots, \bar{R}_{k-1}^k$, that satisfy all previous equilibrium equations. Then

$$\alpha_k = \bar{R}_{k-1}^k \leq R_{k-1}^k \leq R_{k-1}$$

for $k \geq 1$. So $\alpha_t \leq R_{t-1}$, $t \geq 1$, and the fact that R is a non-decreasing sequence follows directly from $\phi(R_t) = \frac{R_{t-1}}{\alpha_t}\phi(R_{t-1})$, $t \geq 1$.

Finally, for (iv), if $R_0 = \delta$, then $R_t = \delta$, $t \geq 1$, follows directly from Equation 2. If $R_0 > \delta$, then, if R_1 is well-defined, $R_1 > R_0$. Assuming the equilibrium sequence exists, then $R_{t+1} > R_t$, $t \geq 0$. But then

$$\left(\frac{R_0}{\delta}\right)^t \phi(R_0) < \frac{\prod_{i=0}^{t-1} R_i}{\prod_{i=1}^{t} \alpha_i}\phi(R_0) = \phi(R_t) \to +\infty$$

absurd. We conclude that $R_t = \delta$, $t \geq 0$. □

Proposition 2.7 states that if there is a positive lower bound $\delta > 0$ on compound growth rates $\{\alpha_t\}_{t\geq 1}$, then social security rules can sustain at least such return level. Also, in case of a non-increasing sequence of compound growth rates, the Pareto dominant return rates sequence is also non-increasing. Furthermore, such return rates sequence is bounded above by the sequence of compound growth rates. This is a direct consequence of equilibrium effects, since a fall on demographic and productivity growth does not only brings down the affordable return rates by itself, but also reduces the absolute level of savings each generation is willing to hold, depressing even more the returns. The same phenomena, but in the opposite direction, takes place when one faces a non-decreasing sequence. Finally, when preferences are homogeneous and there is a constant compound growth rate $\delta > 0$, the Pareto dominant equilibrium equates all return rates to such value, i.e., $R_t = \delta$, $t \geq 0$. A remarkable feature from this last result is that it does not depend on the actual households preferences. Particularly, this implies that Government may not need to know U in advance to structure the social security system. This point will be further developed in Sections 2.3 and 2.4.

Before stating the main result of this section, we need some extra notation and assumptions.

Assumption 3. *For all* $t \geq 0$, $\phi_t : \mathbb{R}_+ \to [0,1]$ *is concave.*

Assumption 3 is satisfied, for example, by CRRA utility functions. Since $\psi_t^{-1}(x) = x\phi_t(x)$ is strictly increasing with $\psi_t^{-1}(0) = 0$ and $Im(\psi_t^{-1}) = \mathbb{R}_+$, it has a well-defined inverse function $\psi_t : \mathbb{R}_+ \to \mathbb{R}_+$, $t \geq 0$. Define also $f_t = \alpha_t \phi_t \circ \psi_t$, $t \geq 1$. Assumptions 1 and 2 imply

$$\psi'(x) = \frac{1}{\psi(x)\phi'(\psi(x)) + \phi(\psi(x))} > 0$$
$$f'(x) = \frac{\alpha\phi'(\psi(x))}{\psi(x)\phi'(\psi(x)) + \phi(\psi(x))} > 0$$

where the index t was omitted to ease notation. The sign of second derivative of f is given by

$$\begin{aligned} f''(x) &\overset{\text{sign}}{=} [\psi(x)\phi'(\psi(x)) + \phi(\psi(x))]\phi''(\psi(x))\psi'(x)... \\ &\quad ... - \phi'(\psi(x))[\psi'(x)\phi'(\psi(x)) + \psi(x)\phi''(\psi(x))\psi'(x) + \phi'(\psi(x))\psi'(x)] \\ &= \phi(\psi(x))\phi''(\psi(x))\psi'(x) - 2\psi'(x)\phi'(\psi(x))^2 \\ &\overset{\text{sign}}{=} \phi(\psi(x))\phi''(\psi(x)) - 2\phi'(\psi(x))^2 \\ &< 0 \end{aligned}$$

where the last inequality derives from Assumption 3. One may conclude that f is concave and strictly increasing. Since $\lim_{x\to 0}\psi(x) = \psi(0) = 0$ and $\phi(0) = 0$ with ϕ concave, we may write

$$\lim_{x\to 0} f'(x) \geq \lim_{x\to 0} \frac{\alpha}{\psi(x) + \frac{\phi(\psi(x))}{\phi'(1)}} = +\infty$$

The next lemma proves the existence of a unique non-zero fixed-point of f.

Lemma 2.8. *Suppose* $h : \mathbb{R}_+ \to \mathbb{R}$ *is a differentiable, concave and upper bounded function with* $h(0) = 0$ *and* $\lim_{x\to 0} h'(x) = +\infty$. *Then* $\exists! K > 0$ *such that* $h(x) \geq x$, $x \leq K$, *and* $h(x) \leq x$, $x \geq K$.

Proof. Let $g(x) = h(x) - x$. Since $\lim_{x\to 0} h'(x) = +\infty$ and $h(0) = 0$, the Mean Value Theorem implies $\exists x_0 > 0$ such that $g(x_0), g'(x_0) \in \mathbb{R}_{++}$. Also, h bounded implies $\lim_{x\to+\infty} g(x) = -\infty$.

By Bolzano's Theorem (BT), $\exists K > x_0 > 0$ with $g(K) = 0$. Let $\Omega = \{K \in \mathbb{R}_{++} \mid g(K) = 0\}$ and $K_1 = \inf \Omega$, where the infimum is well-defined since Ω is bounded below and non-empty. The continuity of g implies $g(K_1) = 0$. The concavity of f implies that g is also concave, and so g' is a non-increasing function. Then $g'(x) \geq g'(x_0) > 0$, $x \leq x_0$, and $x_0 < K_1$. Suppose $\exists K_2 > K_1$, $K_2 \in \Omega$. By Rolle's Theorem, $\exists y_1 \in (0, K_1), y_2 \in (K_1, K_2)$ with $g'(y_1) = g'(y_2) = 0$. Since g' is non-increasing, we conclude that $g'(y) = 0$, $\forall y \in [y_1, y_2]$ and by the Second Fundamental Theorem of Calculus g is constant on $[y_1, y_2]$. Since $K_1 \in [y_1, y_2]$ we have $g(y) = g(K_1) = 0$, $\forall y \in [y_1, y_2]$. But then $g(y_1) = 0$ with $y_1 < K_1$, absurd. We conclude that $\exists! K \in \mathbb{R}_{++}$ satisfying $g(K) = 0$. Suppose $\exists 0 < x < K$ with $h(x) < x$, so that $g(x) < 0$. Since $g(x_0) > 0$, the BT implies $\exists z \in (x, x_0)$ (or (x_0, x) if $x_0 < x$) such that $g(z) = 0$. Since $z < K$, absurd. Then $h(x) \geq x$, $x \in [0, K]$. Suppose, next, $\exists x > K$ with $h(x) > x$, so that $g(x) > 0$. Since $\lim_{x \to +\infty} g(x) = -\infty$, the BT implies $\exists z > x$ such that $g(z) = 0$. Since $z > K$, absurd. Then $h(x) \leq x$, $x \in [K, +\infty]$. ☐

The fact that f satisfies all conditions from Lemma 2.8 allow us to define $K(f) > 0$ as its fixed-point.

Assumption 4. *For* $\Gamma = \{K \in \mathbb{R}_+ \mid \exists t \geq 1, K = K(f_t)\}$, *we have* $\inf \Gamma > 0$ *and* $\sup \Gamma < +\infty$. *Also, there exists* $\sigma > 0$, $\Sigma > 0$ *such that* $f_t'(x) \geq (1 + \Sigma)$, $\forall x \in (0, \sigma)$, $t \geq 1$.

Assumption 4 rules out pathological preferences behavior in which savings demand curves come arbitrarily close to the axis in $\mathbb{R}^2$ and also unbounded declines over the derivatives of f_t, $t \geq 1$. It is satisfied, for example, if there is only a finite number of different preferences. Finally, Assumption 5 below simply states that compound growth rates are upper bounded.

Assumption 5. *There exists* $M > 0$ *such that* $\alpha_t \leq M$, $t \geq 1$.

The next theorem provides an analytical formula for the Pareto dominant equilibrium.

Theorem 2.9. *Under the previous assumptions,* $\forall L > 0$, $\eta(\alpha) = \lim_{k \to \infty} \psi_0 \circ f_1 \circ \ldots \circ f_k(L)$.

Proof. For $L > 0$, define the sequence $\{x_t(L)\}_{t \geq 1}$ where

$$\begin{aligned} x_1(L) &= f_1(L) \\ x_2(L) &= f_1(f_2(L)) \\ x_3(L) &= f_1(f_2(f_3(L))) \\ &[\ldots] \end{aligned}$$

Suppose $L > \max\{\sup \Gamma, M\}$. Then, since $L \geq f_t(L)$ and f_t is strictly increasing, $t \geq 1$, we have

$$x_1(L) \geq x_2(L) \geq x_3(L) \geq \ldots$$

Also for $t \geq 1$

$$f_t(L) \geq f_t(\inf \Gamma) \geq \inf \Gamma > 0$$

and, therefore, $x_t(L) \geq \inf \Gamma$, $t \geq 1$. We conclude that the sequence $\{x_t(L)\}_{t \geq 1}$ has a well-defined limit, since is non-increasing and bounded below. Let $x(L) = \lim_{t \to \infty} x_t(L) \geq \inf \Gamma > 0$ and $L_2 > L_1 > \max\{\sup \Gamma, M\}$. Suppose $x(L_1) < x(L_2)$, so that $\exists k > 0$ such that, $\forall j \geq 0$, $x_k(L_1) < x(L_2) \leq x_{k+j}(L_2)$. Since all the f_t are increasing and bounded by M we have $L_1 <$

$f^{t+1}(f^{t+2}(...f^{t+j}(L_2)...)) < M$, absurd. Then $x(L_1) \geq x(L_2)$. But since $x_t(L_1) < x_t(L_2)$, $t \geq 1$, we conclude that $x(L_1) = x(L_2)$.

Next, suppose $0 < L < \min\{\inf \Gamma, \sigma\}$. Then, since $M \geq f_t(L) \geq L$ and f_t is strictly increasing, $t \geq 1$, we have

$$x_1(L) \leq x_2(L) \leq x_3(L) \leq ... \leq M$$

Let $x(L) = \lim_{t\to\infty} x_t(L) \leq M$ and $0 < L_1 < L_2 < \min\{\inf \Gamma, \sigma\}$. Suppose $x(L_1) < x(L_2)$, so that $\exists k > 0$ such that, $\forall j \geq 0$, $x_{k+j}(L_1) \leq x(L_1) < x_k(L_2)$. But then, since all the f_t are strictly increasing, we have $f_{k+1}(f_{k+2}(...f_{k+j}(L_1)...)) < L_2$, $j \geq 1$. Notice, however, that Assumption 4 implies

$$f_t(x) \geq (1 + \Sigma)x$$

for all $x \in (0, \sigma)$, $t \geq 1$. Let j_0 be the smallest integer such that $(1 + \Sigma)^{j_0} L_1 \geq \sigma$[13]. For $j > j_0$ we have

$$\begin{aligned} f_{k+1}(f_{k+2}(...f_{k+j}(L_1)...)) &\geq f_{k+1}(f_{k+2}(...f_{k+j-j_0}((1 + \Sigma)^{j_0} L_1)...)) \\ &\geq f_{k+1}(f_{k+2}(...f_{k+j-j_0}(\sigma)...)) \\ &\geq f_{k+1}(f_{k+2}(...f_{k+j-j_0}(L_2)...)) \\ &\geq L_2 \end{aligned}$$

absurd. Then, we must have $x(L_1) \geq x(L_2)$. But since $x_t(L_1) < x_t(L_2)$, $\forall t$, we conclude that $x(L_1) = x(L_2)$.

Next, if g is a concave and strictly increasing function, then, for $L_2 > L_1$, the following inequality holds

$$g(L_2) - g(L_1) = g'(z)(L_2 - L_1) \leq g'(L_1)(L_2 - L_1)$$

with $z \in (L_1, L_2)$. If $g(L_2) - g(L_1) \geq \rho > 0$ then

$$\rho \leq g'(L_1)(L_2 - L_1) \implies L_2 \geq \frac{\rho}{g'(L_1)} + L_1$$

Let $0 < L_1 < \min(\inf \Gamma, \sigma)$ and $L_2 > \max(\sup \Gamma, M)$, with $L_1 < L_2$. Then $x(L_1) \leq x(L_2)$. Suppose $x(L_1) + \rho = x(L_2)$, $\rho > 0$. By definition $x_k(\cdot)$, $k \geq 1$, is an strictly increasing and concave function with

$$\lim_{k\to\infty} x_k(L) = \lim_{k\to\infty} x_k(L_1) = x$$

for $0 < L \leq L_1$. Also, $\forall \varepsilon > 0, \exists k_0(L)$ such that $k \geq k_0(L)$ implies $x_k(L) > x - \varepsilon$. The Mean Value Theorem states that, for any given k, $\exists c_k \in (L, L_1)$ such that

$$x'_k(c_k) = \frac{x_k(L_1) - x_k(L)}{L_1 - L}$$

[13] $\Sigma > 0$ ensures that such value exists.

For $k \geq k_0(L)$, we have $x \geq x_k(L_1) > x_k(L) > x - \varepsilon$. The concavity of $x_k(\cdot)$ the implies

$$x_k'(L_1) < x_k'(c_k) < \frac{\varepsilon}{L_1 - L}$$

and since $x_k(\cdot)$ is strictly increasing, we conclude that $\lim_{k\to\infty} x_k'(L_1) = 0$. The fact that $\{x_k(L_1)\}_{k\geq 1}$ is non-decreasing and $\{x_k(L_2)\}_{k\geq 1}$ is non-increasing implies that

$$x_0(L_1) \leq x_1(L_1) \leq \ldots \leq x(L_1) < x(L_1) + \rho = x(L_2) \leq \ldots \leq x_1(L_2) \leq x_0(L_2)$$

We may use our previous result, with x_k instead of g, to write

$$x_k(L_2) - x_k(L_1) > \rho \implies L_2 \geq \frac{\rho}{x_k'(L_1)} + L_1$$

Taking the limit as $k \to +\infty$ implies $L_2 \geq +\infty$, absurd. We conclude that $x(L_1) = x(L_2)$. Let $L > 0$ be any value, and take $0 < L_1 < L < L_2$, with $L_1 < \min(\inf \Gamma, \sigma)$ and $L_2 > \max(\sup \Gamma, M)$. For all $k \geq 1$ we have

$$x_k(L_1) \leq x_k(L) \leq x_k(L_2)$$

and our previous result implies $\lim_{k\to\infty} x_k(L_1) = \lim_{k\to\infty} x_k(L_2) = x$. Taking the limit on both sides of the previous inequalities allow us to conclude that $\lim_{k\to\infty} x_k(L) = x$.

Next, define the return rates sequence $\{R_t\}_{t\geq 0}$ as

$$\begin{aligned}
R_0 &= \psi_0(x) \\
R_1 &= \phi_1^{-1}\left[\frac{1}{\alpha_1}\psi_0^{-1}(R_0)\right] \\
R_2 &= \phi_2^{-1}\left[\frac{1}{\alpha_2}\psi_1^{-1}(R_1)\right] \\
&[\ldots]
\end{aligned}$$

The sequence is well-defined since can also be written as

$$\begin{aligned}
R_0 &= \lim_{k\to\infty} \psi_0 \circ f_1 \circ \ldots \circ f_k(L) \\
R_1 &= \lim_{k\to\infty} \psi_1 \circ f_2 \circ \ldots \circ f_k(L) \\
R_2 &= \lim_{k\to\infty} \psi_2 \circ f_3 \circ \ldots \circ f_k(L) \\
&[\ldots]
\end{aligned}$$

where all limits exist due to our previous result and the continuity of ψ_t, $t \geq 0$. In order to show that $\{R_t\}_{t\geq 0}$ satisfies Equation 1, notice that for $t \geq 1$

$$\begin{aligned}
\alpha_t\phi_t(R_t) = R_{t-1}\phi_{t-1}(R_{t-1}) &\iff \alpha_t\phi_t(R_t) = \psi_{t-1}^{-1}(R_{t-1}) \\
&\iff \alpha_t\phi_t(\lim_{k\to\infty} \psi_t \circ f_{t+1} \circ \ldots) = \psi_{t-1}^{-1}(\lim_{k\to\infty} \psi_{t-1} \circ f_t \circ \ldots) \\
&\iff \lim_{k\to\infty} \alpha_t\phi_t \circ \psi_t \circ f_{t+1} \circ \ldots = \lim_{k\to\infty} \psi_{t-1}^{-1} \circ \psi_{t-1} \circ f_t \circ \ldots \\
&\iff \lim_{k\to\infty} f_t \circ f_{t+1} \circ \ldots \circ f_k(L) = \lim_{k\to\infty} f_t \circ \ldots \circ f_k(L)
\end{aligned}$$

It remains to show that $R_0 = \eta(\alpha)$. Suppose there is an equilibrium $\{P_t\}_{t\geq 0}$ with $P_0 > R_0$. Since $\{\alpha_t\}_{t\geq 1}$ is bounded by $M > 0$, Equation 1 implies that there is $\Pi > 0$ so that $P_t \leq \Pi$, $t \geq 0$. Also, notice that for $t \geq 1$

$$P_0 = \psi_0 \circ f_1 \circ \ldots \circ f_t \circ \psi_t^{-1}(P_t)$$

where $\psi_t^{-1}(P_t) = P_t \phi_t(P_t) \leq P_t \leq \Pi$. But then, since all the f_t are increasing, we have

$$\psi_0 \circ f_1 \circ \ldots \circ f_t(\Pi) \geq \psi_0 \circ f_1 \circ \ldots \circ f_t \circ \psi_t^{-1}(P_t) = P_0$$

Taking the limit as $t \to \infty$ we find $R_0 \geq P_0$, absurd. We conclude that $R_0 = \eta(\alpha)$. □

Theorem 2.9 is the central result of this section. It furnishes an analytical formula to find an equilibrium return rates sequence that yields a sustainable and individually optimal set of social security rules. Furthermore, the limit formula provides not only an equilibrium return rates sequence, but the Pareto dominant one. This is a fundamental step on the study of *notional accounts* systems, since in order to define the set of social security rules Government must not only care for finding solutions for equilibrium equations (in order to have sustainable and individually optimal systems), but also to adopt methods that furnish Pareto optimal sets of rules. This point is further discussed in Section 2.8. The next proposition states a direct consequence of the theorem.

Proposition 2.10. *Under the previous assumptions, the* t*th-partial derivative of the function* $\eta : \hat{\mathbb{R}}^\infty_{++} \to \mathbb{R}_+$[14] *defined by Proposition 2.6 is*

$$\frac{\partial \eta(\alpha)}{\partial \alpha_t} = \left[\prod_{i=1}^{t-1} \psi'_{i-1}(\alpha_i \phi_i(R_i))\alpha_i \phi'_i(R_i)\right] \psi'_{t-1}(\alpha_t \phi_t(R_t))\phi_t(R_t)$$

where $\alpha = \{\alpha_t\}_{t\geq 1} \in \hat{\mathbb{R}}^\infty_{++}$ *and* $\{R_t\}_{t\geq 0}$ *is the equilibrium return rates sequence with* $R_0 = \eta(\alpha)$.

Proof. Let $\{R_t\}_{t\geq 0}$ be the equilibrium return rates sequence defined after $R_0 = \eta(\alpha)$, with $\alpha = \{\alpha_t\}_{t\geq 1} \in \hat{\mathbb{R}}^\infty_{++}$. Theorem 2.9 allows us to write

$$R_t = \lim_{k\to\infty} \psi_t(\alpha_{t+1}\phi_{t+1}(\psi_{t+1}(\alpha_{t+2}\phi_{t+2}(\psi_{t+2}\ldots\alpha_{t+k}\phi_{t+k}(\psi_{t+k}(L))\ldots)$$

for $t \geq 0$, $L > 0$. Since $R_0 = \eta(\alpha)$, we may write

$$\eta(\alpha) = \psi_0(\alpha_1\phi_1(\psi_1(\alpha_2(\phi_2(\psi_2\ldots\alpha_t\phi_t(R_t)\ldots)$$

for $t \geq 1$. Using the chain rule, we obtain

$$\frac{\partial \eta(\alpha)}{\partial \alpha_t} = \left[\prod_{i=1}^{t-1} \psi'_{i-1}(\alpha_i \phi_i(R_i))\alpha_i \phi'_i(R_i)\right] \psi'_{t-1}(\alpha_t \phi_t(R_t))\phi_t(R_t)$$

□

Proposition 2.10 defines the marginal impact caused by changes on demographic and productivity compound growth rates over equilibrium return rates. It allows one to study, as in the example below, in which degree future changes affect present social security return rates.

[14] $\hat{\mathbb{R}}^\infty_{++}$ is the set of bounded sequences in $\mathbb{R}^\infty_{++}$.

Example 2.11. *Suppose a constant compound growth rate* $\alpha_t = \delta > 0$, $t \geq 1$, *and homogeneous households* $U_t(c_0, c_1) = 2\sqrt{c_0} + 2\sqrt{c_1}$, $t \geq 0$. *We drop index to ease notation. Savings demand* ϕ *and the auxiliary functions* ψ *are written,* $x \geq 0$, *as*

$$\begin{aligned} \phi(x) &= \frac{x}{1+x} \\ \psi(x) &= \frac{x + \sqrt{x^2 + 4x}}{2} \end{aligned}$$

and their derivative is

$$\begin{aligned} \phi'(x) &= \frac{1}{(1+x)^2} \\ \psi'(x) &= \frac{1}{2} + \frac{x+2}{2\sqrt{x^2+4x}} \end{aligned}$$

From item (iv) of Proposition 2.7, $\eta(\alpha) = \delta$. *Also,* $R_t = \delta$, $t \geq 1$. *Proposition 2.10 then implies*

$$\begin{aligned} \frac{\partial \eta(\alpha)}{\partial \alpha_t} &= \left[\prod_{i=1}^{t-1} \psi'_{i-1}(\alpha_i \phi_i(R_i)) \alpha_i \phi'_i(R_i) \right] \psi'_{t-1}(\alpha_t \phi_t(R_t)) \phi_t(R_t) \\ &= [\psi'(\delta\phi(\delta))\delta\phi'(\delta)]^{t-1} \psi'(\delta\phi(\delta))\phi(\delta) \end{aligned}$$

for $t \geq 1$. *Notice that the following relations hold*

$$\begin{aligned} \lim_{\delta \to 0} \psi'(\delta\phi(\delta))\delta\phi'(\delta) &= \frac{1}{2}, \ \lim_{\delta \to 0} \psi'(\delta\phi(\delta))\phi(\delta) = \frac{1}{2} \\ \lim_{\delta \to +\infty} \psi'(\delta\phi(\delta))\delta\phi'(\delta) &= 0, \ \lim_{\delta \to +\infty} \psi'(\delta\phi(\delta))\phi(\delta) = 1 \end{aligned}$$

so that $\delta \approx 0$ *implies*

$$\frac{\partial \eta(\alpha)}{\partial \alpha_t} \approx \frac{1}{2^t} \tag{3}$$

for $t \geq 1$. *Also,* $\delta >> 1$ *implies*

$$\frac{\partial \eta(\alpha)}{\partial \alpha_t} \approx \begin{cases} 1, \ t = 1 \\ 0, \ t \geq 2 \end{cases} \tag{4}$$

Since $\eta(\alpha) = \alpha_t = \delta$, $t \geq 1$, *we can write*

$$\frac{\partial \eta(\alpha)/\eta(\alpha)}{\partial \alpha_t/\alpha_t} = \frac{\partial \eta(\alpha)/\delta}{\partial \alpha_t/\delta} = \frac{\partial \eta(\alpha)}{\partial \alpha_t}$$

for $t \geq 1$, *and such value is the elasticity of the initial social security return rate relative to the compound growth rate from period* t. *Equation 3 implies that under a small growth scenario, i.e.,* $\delta \approx 0$, *a* 1% *change over the compound growth rate from period* 1 *increases the social security initial return rate by* 0.5%. *If the change happens later on the future, i.e.,* $t \geq 2$, *then one has a* 2^{-t}% *increase on the initial return rate. Under a high growth scenario, i.e.,* $\delta >> 1$, *the situation is different. Equation 4 implies that a* 1% *change over the compound growth rate from period* 1 *increases the social security initial return rate by the same* 1%, *so that the* *short-run increase is fully reverted to the initial return rate. However, if the change happens later on the future, then none of it reverberates to the initial return rate.*

There is yet a different interpretation of this result. Notice that one may read Equations 3 and 4 as if they furnished, for a fixed $t \geq 1$*, the elasticity of past social security return rates, i.e., the values of* $\frac{\partial R_0}{\partial \alpha_t}, \frac{\partial R_1}{\partial \alpha_t}, \ldots, \frac{\partial R_{t-1}}{\partial \alpha_t}$*, where* $\{R_t\}_{t\geq 0}$ *is the equilibrium return rates sequence defined by* $R_0 = \eta(\alpha)$*. In this way, Equation 3 can be rewritten as*

$$\frac{\partial R_k}{\partial \alpha_t} \approx \frac{1}{2^{t-k}}$$

for $0 \leq k \leq t-1$*, and Equation 4 as*

$$\frac{\partial R_k}{\partial \alpha_t} \approx \begin{cases} 1, & k = t-1 \\ 0, & 0 \leq k \leq t-2 \end{cases}$$

Therefore, in a high growth scenario, a 1% increase over period t *compound growth rate* α_t *affects only* R_{t-1} *by the same amount and has no effect over* R_k*, for* $0 \leq k \leq t-2$*. In a low growth scenario, however, 1% increase over period* t *compound growth rate* α_t *increases* R_{t-1} *by 0.5%,* R_{t-2} *by 0.25% and so on*[15]*. We conclude that the level of growth* δ *in the economy influences the degree in which future compound growth rates changes reverberate over the sequence of Pareto dominant equilibrium return rates* $\{R_t\}_{t\geq 0}$*.*

Finally, we state a corollary from Proposition 2.10 that synthesizes the *reverberation* phenomena highlighted in Example 2.11.

Corollary 2.12. *Under the previous assumptions, let* $\{R_t\}_{t\geq 0}$ *be the equilibrium return rates sequence with* $R_0 = \eta(\alpha)$*. Then*

$$\frac{\partial R_k}{\partial \alpha_t} = \left[\prod_{i=k+1}^{t-1} \psi'_{i-1}(\alpha_i \phi_i(R_i))\alpha_i \phi'_i(R_i)\right] \psi'_{t-1}(\alpha_t \phi_t(R_t))\phi_t(R_t)$$

for $0 \leq k \leq t-1$*. Also,*

$$\frac{\partial R_{t-1}}{\partial \alpha_t} = \psi'_{t-1}(\alpha_t \phi_t(R_t))\phi_t(R_t)$$

$$\frac{\partial R_k}{\partial \alpha_t} = \psi'_k(\alpha_{k+1}\phi_{k+1}(R_{k+1}))\alpha_{k+1}\phi'_{k+1}(R_{k+1})\frac{\partial R_{k+1}}{\partial \alpha_t}$$

for $0 \leq k \leq t-2$*.*

2.3 Intergenerational transfers on notional accounts systems

In this section we study incentive compatibility on sets of social security rules that allow for discretion of the insured households. The economy is our baseline one. We assume there are only two types of households preferences, A and B. Let $\phi_A, \phi_B : \mathbb{R}_+ \to [0,1]$ be the savings demand of each type, which, along with the corresponding utility functions, satisfy

[15] Notice that for t large enough the sum of the percentage increases is nearly equivalent since $\sum_{i=1}^{t} 2^{-i} \approx 1$.

all our previous assumptions. Assume, furthermore, that $\phi_A(R) \geq \phi_B(R)$, $\forall R \in \mathbb{R}_+$, so type A households are high savers and type B ones are low savers. Assume that endowments are constant over time, i.e., $E_t = E > 0$, $t \geq 0$. Also, demographic growth rate is constant, i.e., $\alpha_t^L = \delta > 0$, $t \geq 1$, and so $\alpha_t = \alpha_t^L \alpha_t^E = \delta$, $t \geq 1$. Also, let generation G_t type be given by $\lambda_t \in \{A, B\}$, $t \geq 0$. Equilibrium equations are then written as

$$\delta \phi_{\lambda_t}(R_t) = R_{t-1} \phi_{\lambda_{t-1}}(R_{t-1})$$

for $t \geq 1$. Before stating our first result, we will need the following assumption.

Assumption 6. *The function* $h_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$, $h_\lambda(x) = x\phi_\lambda(x)$, *is convex for* $\lambda \in \{A, B\}$.

Assumption 6 is satisfied, for example, by CRRA utility functions with $\theta \in [1/2, 1)$ and $\beta \in \mathbb{R}_{++}$. The next proposition describes equilibrium return rates sequences under this low and high savers dichotomy.

Proposition 2.13. *Under the previous assumptions, if* $\{R_t\}_{t\geq 0}$ *is an equilibrium return rates sequence then* $R_t \leq \delta$, *for all* $t \geq 0$ *such that* $\lambda_t = A$.

Proof. Define the following auxiliary functions

$$\begin{aligned}
H^{A,A}(x, y) &= \delta\phi_A(y) - x\phi_A(x) \\
H^{A,B}(x, y) &= \delta\phi_B(y) - x\phi_A(x) \\
H^{B,A}(x, y) &= \delta\phi_A(y) - x\phi_B(x) \\
H^{B,B}(x, y) &= \delta\phi_B(y) - x\phi_B(x)
\end{aligned}$$

Also, the following implicit ones

$$\begin{aligned}
H^{A,A}(x, f^{A,A}(x)) &= 0 \\
H^{A,B}(x, f^{A,B}(x)) &= 0 \\
H^{B,A}(x, f^{B,A}(x)) &= 0 \\
H^{B,B}(x, f^{B,B}(x)) &= 0
\end{aligned}$$

Equilibrium equations can, therefore, be written as

$$R_t = f^{\lambda_{t-1}, \lambda_t}(R_{t-1})$$

for $t \geq 1$. The previous definitions allow us to write

$$f^{A,A}(x) = [\phi_A]^{-1}\left(\frac{x\phi_A(x)}{\delta}\right)$$

for $x \geq 0$. Therefore, $f^{A,A}$ is an strictly increasing and convex function. Similar reasoning allow us to state that $f^{A,B}$, $f^{B,A}$ and $f^{B,B}$ are also strictly increasing and convex. The definitions also directly imply $f^{A,A}(0) = f^{A,B}(0) = f^{B,A}(0) = f^{B,B}(0) = 0$ and $f^{A,A}(\delta) = f^{B,B}(\delta) = \delta$. Next, we will prove the following inequality holds

$$f^{B,A}(x) \leq f^{\lambda,\lambda}(x) \leq f^{A,B}(x)$$

for $x \geq 0$, $\lambda \in \{A, B\}$. Since $\phi_A(x) \geq \phi_B(x)$, $\forall x \geq 0$, we may write

$$\begin{aligned}
H^{A,B}(x, f^{A,A}(x)) &= \delta\phi_B(f^{A,A}(x)) - x\phi_A(x) \\
&= \delta[\phi_B(f^{A,A}(x)) - \phi_A(f^{A,A}(x))] \\
&\leq 0 \\
H^{B,A}(x, f^{A,A}(x)) &= \delta\phi_A(f^{A,A}(x)) - x\phi_B(x) \\
&= x[\phi_A(x) - \phi_B(x)] \\
&\geq 0 \\
H^{A,B}(x, f^{B,B}(x)) &= \delta\phi_B(f^{B,B}(x)) - x\phi_A(x) \\
&= x[\phi_B(x) - \phi_A(x)] \\
&\leq 0 \\
H^{B,A}(x, f^{B,B}(x)) &= \delta\phi_A(f^{B,B}(x)) - x\phi_B(x) \\
&= \delta[\phi_A(f^{B,B}(x)) - \phi_B(f^{B,B}(x))] \\
&\geq 0
\end{aligned}$$

for $x \geq 0$. The inequality is then obtained directly from the definition of $f^{B,A}$ and $f^{A,B}$. Suppose w.l.o.g. that $\lambda_0 = A$ and that $R_0 > \delta$. If there is no generation G_t of type B, $t \geq 1$, then the equilibrium return rates sequence is defined by iterates of $f^{A,A}$. Since $f^{A,A}(\delta) = \delta$ and $R_0 > \delta$, the following relation holds

$$R_1 = f^{A,A}(R_0) \geq f^{A,A}(\delta) + \frac{df^{A,A}(\delta)}{dx}(R_0 - \delta) = \delta + \frac{df^{A,A}(\delta)}{dx}(R_0 - \delta)$$

More generally, R_t, $t \geq 1$, satisfies

$$R_t \geq \delta + \left[\frac{df^{A,A}(\delta)}{dx}\right]^t (R_0 - \delta)$$

Since $f^{A,A}(0) = 0$ and $f^{A,A}(\delta) = \delta$, the convexity of $f^{A,A}$ implies

$$\frac{df^{A,A}(\delta)}{dx} > 1$$

Therefore $\lim_{t\to+\infty} R_t = +\infty$, absurd. Then, suppose that generation G_k, $k \geq 1$, is the first one of type B, so that R_k is given by

$$R_k = f^{A,B}(f^{A,A})^{k-1}(R_0)$$

Since $f^{A,B}(x) \geq f^{A,A}(x) > x$, $\forall x > \delta$, then

$$R_k = f^{A,B}(f^{A,A})^{k-1}(R_0) \geq (f^{A,A})^k(R_0) > R_0 > \delta$$

If there is no next generation G_t, $t \geq k+1$, of type A, we may then use the initial reasoning for $f^{B,B}$ to conclude that this hypothesis is absurd. Let, therefore, generation G_l, $l \geq k+1$, be the first of type A after generation G_k. Then we may write

$$\begin{aligned} R_l &= f^{B,A}(f^{B,B})^{l-k-1}(R_k) \\ &> f^{B,A}(R_k) \\ &= f^{B,A}(f^{A,B}(f^{A,A})^{k-1}(R_0)) \\ &> f^{B,A}(f^{A,B}(R_0)) \end{aligned}$$

where the inequalities derive from the fact that iterates of $f^{B,B}$ and $f^{A,A}$ are strictly increasing when the initial point is larger then δ. The definition of $f^{B,A}$ applied to $f^{A,B}(\delta)$ implies

$$\delta\phi_A(f^{B,A}(f^{A,B}(\delta))) - f^{A,B}(\delta)\phi_B(f^{A,B}(\delta)) = 0$$

Also, the definition of $f^{A,B}(\delta)$ implies

$$\delta\phi_B(f^{A,B}(\delta)) - \delta\phi_A(\delta) = 0$$

so that we may write

$$\phi_A(f^{B,A}(f^{A,B}(\delta))) = \frac{f^{A,B}(\delta)}{\delta}\phi_A(\delta) \geq \phi_A(\delta)$$

where the last inequality comes from $f^{A,B}(\delta) \geq f^{A,A}(\delta) = \delta$. Since ϕ_A is strictly increasing we conclude that $f^{B,A}(f^{A,B}(\delta)) \geq \delta$ and, therefore, $R_l > \delta$. If there is no other type shift on future generations, or only a finite number of shifts, then we may apply our initial reasoning to conclude that our initial hypothesis, i.e., $R_0 > \delta$, leads to an absurd. If there is an infinite number of shifts between generations of type A and B then, since $f^{B,A} \circ f^{A,B}$ is increasing and convex, we may still apply our initial reasoning on such function since $f^{B,A}(f^{A,B}(\delta)) \geq \delta$ implies that the derivative of $f^{B,A} \circ f^{A,B}$ on δ is greater than 1. We conclude that $R_0 \leq \delta$ and, by a suitable translation of coordinates, that $R_t \leq \delta$, $\forall t \geq 0$ such that $\lambda_t = A$. □

Before coming to the proposition itself, notice that in a constant demographic growth economy with intergenerational homogeneity, i.e., $\lambda_t = \lambda$, $t \geq 0$, $\lambda \in \{A, B\}$, Proposition 2.7 states that the Pareto dominant return rates sequence is constant and given by $R_t = \delta$, $t \geq 0$. Therefore, if the set of social security rules is derived from such sequence, final utility for all generations in each case is given by

$$U^\lambda((1 - \phi_\lambda(\delta))E, \delta\phi_\lambda(\delta)E)$$

for $\lambda \in \{A, B\}$. Proposition 2.19 states that under intergenerational heterogeneity equilibrium return rates for high savers generations are upper bounded by the return rates level under intergenerational homogeneity δ, i.e., $R_t \leq \delta$, for all $t \geq 0$ such that $\lambda_t = A$, and therefore

$$U^{\lambda_t}((1 - \phi_{\lambda_t}(R_t))E, \delta\phi_{\lambda_t}(R_t)E) \leq U^{\lambda_t}((1 - \phi_{\lambda_t}(\delta))E, \delta\phi_{\lambda_t}(\delta)E)$$

for all $t \geq 0$ such that $\lambda_t = A$. Since the set of social security rules is derived from the Pareto dominant equilibrium return rates sequence, i.e., from $\{R_t\}_{t\geq 0}$ with $R_0 = \eta(\alpha)$ given by Proposition 2.6, we conclude that under intergenerational heterogeneity social security return rates can be smaller for high savers generations than they are under intergenerational homogeneity. This is a direct consequence of the overlapping structure. Since high savers contributions finance retirement of past generations, these generations tend to have larger return rates. However, when high savers retire, they need the current young generation to sustain their higher level of benefits. If, however, the current young generation is a low savers one, this will not happen under the expected return rate. This phenomena creates a transfer from high savers generations to low savers ones under intergenerational heterogeneity when return rates are compared with an intergenerational homogeneity scenario. The next example discusses the consequences of these transfers in more detail.

Example 2.14. *Let* $\lambda_{2t} = B$, $\lambda_{2t+1} = A$, $t \geq 0$, *so that equilibrium equations are given by*

$$\begin{aligned} \delta\phi_A(R_1) &= R_0\phi_B(R_0) \\ \delta\phi_B(R_2) &= R_1\phi_A(R_1) \\ \delta\phi_A(R_3) &= R_2\phi_B(R_2) \\ &[\ldots] \end{aligned}$$

Let $\{R_t\}_{t\geq 0}$ *be the Pareto dominant equilibrium given by Proposition 2.7, i.e.,* $R_0 = \eta(\alpha)$. *The symmetry of the overlapping structure allow us to state that* $R_{2t} = R_B$, $R_{2t+1} = R_A$, $t \geq 0$, *and therefore equilibrium equations become*

$$\begin{aligned} \delta\phi_A(R_A) &= R_B\phi_B(R_B) \\ R_A R_B &= \delta^2 \end{aligned}$$

Since $\phi_A(\delta) \geq \phi_B(\delta)$ *applying the Intermediate Value Theorem on suitable transformation of the first equation implies that* $\exists r \geq 0$ *such that* $R_A = \delta/(1+r)$ *and* $R_B = \delta(1+r)$. *If* $\phi_A(\delta) = \phi_B(\delta)$, *then* $r = 0$ *and* $R_A = R_B = \delta$. *Let, therefore,* $\phi_A(\delta) > \phi_B(\delta)$ *so that* $r > 0$. *If* $R_A\phi_A(R_A) \geq R_B\phi_B(R_B)$ *then*

$$\frac{R_A}{R_B} \geq \frac{\phi_B(R_B)}{\phi_A(R_A)} = \sqrt{\frac{R_A}{R_B}} \implies (1+r) \geq (1+r)^2$$

absurd. We conclude that if $\phi_A(\delta) > \phi_B(\delta)$ *then* $R_A < \delta < R_B$ *and* $R_A\phi_A(R_A) < R_B\phi_B(R_B)$. *Also, we have*

$$\frac{\phi_A(R_A)}{\phi_B(R_B)} = \frac{\delta}{R_A} = (1+r) \implies \phi_A(R_A) > \phi_B(R_B)$$

Therefore every household of type A *saves a higher amount than the ones of type B, i.e.,* $\phi_A(R_A) > \phi_B(R_B)$, *although has a lower absolute retirement benefit, i.e.,* $R_A\phi_A(R_A) < R_B\phi_B(R_B)$. *A direct implication is that*

$$u^\lambda((1-\phi_A(R_A))E, R_A\phi_A(R_A)E) < u^\lambda((1-\phi_B(R_B))E, R_B\phi_B(R_B)E)$$

for $\lambda \in \{A, B\}$. *We conclude that the consumption bundle implemented by the set of social security rules for low savers generations is unanimously strictly better than the bundle implemented for high savers ones because of the intergenerational transfers described above.*

Example 2.14 shows that intergenerational transfers, which can also be seen as subsidies, may lead to social security rules under which high savers generations have an unequivocally worse life-time consumption bundle than low savers ones. Particularly, this implies that high savers generations will always prefer the social security rules that Government offers to low savers generations. The next example shows a first attempt under which high savers and low savers generations own welfare evaluation of the set of social security rules are taken into account when defining such rules.

Example 2.15. *We build over the setting of Example 2.14. In this example, however, we assume that before Government defines the set of social security rules high and low savers generations make a single and unique claim about their own preference types. Such claims are represented by* $\tilde{\lambda}_A, \tilde{\lambda}_B \in \{A, B\}$*. Furthermore, after high and low savers make their claims, Government defines the set of social security rules based on them, i.e., assuming that* $\lambda_{2t} = \tilde{\lambda}_B$*,* $\lambda_{2t+1} = \tilde{\lambda}_A$*,* $t \geq 0$*, and letting* $\{R_t\}_{t\geq 0}$ *be the Pareto dominant return rates sequence defined by equilibrium equations after* $R_0 = \eta(\alpha)$*. We also assume* $\phi_A(\delta) > \phi_B(\delta)$*. Therefore, final utility levels for each generation given the vector of claims* $(\tilde{\lambda}_A, \tilde{\lambda}_B) \in \{A, B\}^2$ *is*

$$U^A(\tilde{\lambda}_A, \tilde{\lambda}_B) = \begin{cases} U^A((1-\phi_A(\delta))E, \delta\phi_A(\delta)E) & , \textit{if } (\tilde{\lambda}_A, \tilde{\lambda}_B) = (A, A) \\ U^A((1-\phi_A(R_A))E, R_A\phi_A(R_A)E) & , \textit{if } (\tilde{\lambda}_A, \tilde{\lambda}_B) = (A, B) \\ U^A((1-\phi_B(R_B))E, R_B\phi_B(R_B)E) & , \textit{if } (\tilde{\lambda}_A, \tilde{\lambda}_B) = (B, A) \\ U^A((1-\phi_B(\delta))E, \delta\phi_B(\delta)E) & , \textit{if } (\tilde{\lambda}_A, \tilde{\lambda}_B) = (B, B) \end{cases}$$

$$U^B(\tilde{\lambda}_A, \tilde{\lambda}_B) = \begin{cases} U^B((1-\phi_A(\delta))E, \delta\phi_A(\delta)E) & , \textit{if } (\tilde{\lambda}_A, \tilde{\lambda}_B) = (A, A) \\ U^B((1-\phi_B(R_B))E, R_B\phi_B(R_B)E) & , \textit{if } (\tilde{\lambda}_A, \tilde{\lambda}_B) = (A, B) \\ U^B((1-\phi_A(R_A))E, R_A\phi_A(R_A)E) & , \textit{if } (\tilde{\lambda}_A, \tilde{\lambda}_B) = (B, A) \\ U^B((1-\phi_B(\delta))E, \delta\phi_B(\delta)E) & , \textit{if } (\tilde{\lambda}_A, \tilde{\lambda}_B) = (B, B) \end{cases}$$

where R_A*,* R_B *are the return rates calculated in Example 2.14. The previous setting clearly assembles the normal form of a well-defined simultaneous move game. Next, we look for its Nash equilibrium. From Example 2.14 we have* $R_A < \delta < R_B$ *and*

$$\begin{aligned} \phi_A(R_A) &> \phi_B(R_B) \\ R_A\phi_A(R_A) &< R_B\phi_B(R_B) \end{aligned}$$

Notice that the following inequalities hold

$$\begin{aligned} U^B((1-\phi_A(\delta))E, \delta\phi_A(\delta)E) &< U^B((1-\phi_B(\delta))E, \delta\phi_B(\delta)E) \\ &< U^B((1-\phi_B(\delta))E, R_B\phi_B(\delta)E) \\ &< U^B((1-\phi_B(R_B))E, R_B\phi_B(R_B)E) \end{aligned}$$

where the first inequality comes from the definition of $\phi_B(\delta)$ *and the fact that* $\phi_B(\delta) < \phi_A(\delta)$*, the second from* $R_B > \delta$ *and the third from the definition of* $\phi_B(R_B)$ *and the fact that* ϕ_B *is strictly increasing. Also*

$$\begin{aligned} U^B((1-\phi_A(R_A))E, R_A\phi_A(R_A)E) &< U^B((1-\phi_B(R_A))E, R_A\phi_B(R_A)E) \\ &< U^B((1-\phi_B(R_A))E, \delta\phi_B(R_A)E) \\ &< U^B((1-\phi_B(\delta))E, \delta\phi_B(\delta)E) \end{aligned}$$

Therefore $\tilde{\lambda}_B = B$ *is an strictly dominant strategy. Then, the Nash equilibrium of this game is* *given by* $(\tilde{\lambda}_A, \tilde{\lambda}_B) = (A, B)$ *if*

$$U^A((1 - \phi_A(R_A))E, R_A \phi_A(R_A)E) > U^A((1 - \phi_B(\delta))E, \delta\phi_B(\delta)E)$$

and by $(\tilde{\lambda}_A, \tilde{\lambda}_B) = (B, B)$ *if*

$$U^A((1 - \phi_A(R_A))E, R_A \phi_A(R_A)E) < U^A((1 - \phi_B(\delta))E, \delta\phi_B(\delta)E)$$

The case of equality has both $(\tilde{\lambda}_A, \tilde{\lambda}_B) = (A, B)$ *and* $(\tilde{\lambda}_A, \tilde{\lambda}_B) = (B, B)$ *as Nash equilibria. Let* U^A *and* U^B *be logarithmic utility functions given by*

$$U^A(c_0, c_1) = \log c_0 + \frac{\theta_A}{1 - \theta_A} \log c_1$$
$$U^B(c_0, c_1) = \log c_0 + \frac{\theta_B}{1 - \theta_B} \log c_1$$

with $\theta_A, \theta_B \in (0, 1)$ *and* $\theta_A > \theta_B$. *Savings demand is written as*

$$\phi_\lambda(R) = \theta_\lambda$$

for $R \in \mathbb{R}_+$, $\lambda \in \{A, B\}$. *Also, we have the values of* R_A *and* R_B *given by*

$$R_A = \frac{\delta\theta_B}{\theta_A}$$
$$R_B = \frac{\delta\theta_A}{\theta_B}$$

From our previous discussion, $(\tilde{\lambda}_A, \tilde{\lambda}_B) = (A, B)$ *is a Nash equilibrium if, and only if,*

$$U^A((1 - \theta_A)E, \delta\theta_B E) \geq U^A((1 - \theta_B)E, \delta\theta_B E)$$

Since $(1 - \theta_A) < (1 - \theta_B)$, *we conclude that* $(\tilde{\lambda}_A, \tilde{\lambda}_B) = (A, B)$ *is never a Nash equilibrium. Therefore,* $(\tilde{\lambda}_A, \tilde{\lambda}_B) = (B, B)$ *is the only possible outcome and, so, truth-telling is never an optimal behavior for high savers generations.*

Example 2.15 brings the description of a game theoretical approach to the definition of the set of social security rules by Government. Existing notional accounts systems often allow for its insured households a degree of discretion over contribution levels or retirement age. Since optimal social security design must take into account the behavior of present and future generations when facing such choices, Example 2.15 attempts to model this dynamic using the optimality results derived in Section 2.2 and game theory. The results are somewhat surprising even after a thoughtful reading of Example 2.14. Notice that Example 2.14 has shown that

$$U^A((1 - \phi_A(R_A))E, R_A\phi_A(R_A)E) < U^A((1 - \phi_B(R_B))E, R_B\phi_B(R_B)E)$$

and, therefore, at first sight it seems high savers generations, i.e., the ones of type A, would certainly prefer that social security rules were defined as if there were only low savers generations and, so, no intergenerational heterogeneity. However, this conclusion ignores the equilibrium return rates own dependence over preferences. Example 2.15 has shown then that truth-telling can be a Nash equilibrium for high savers generations as long as

$$U^A((1-\phi_A(R_A))E, R_A\phi_A(R_A)E) > U^A((1-\phi_B(\delta))E, \delta\phi_B(\delta)E)$$

The limiting case where savings demands are constant over return rates shows a scenario where truth-telling is never a Nash equilibrium for the high savers generations. This happens because high-savers retirement benefits in this case do not depend on its own savings level. So the only effect of adopting a truth-telling behavior is to finance a larger retirement benefit to low savers generations, what is shown by the following inequality

$$U^A((1-\theta_A)E, \delta\theta_B E) < U^A((1-\theta_B)E, \delta\theta_B E)$$

Although the above results are significant they require strong assumptions. First, the simultaneous move game rely on the possibility of a joint decision of all generations (present and future ones) of a given type being taken at a single moment and unanimously accepted. This can only be reasonably assumed because of the recursive structure of the overlapping economy under study. Second, if Government relies on each generation claim of its own type, how would households know precisely the future structure of the overlapping economy in order to define the outcomes of the equilibrium return rates?

In order to develop a definition of truth-telling behavior, or incentive compatibility, over social security rules that is not subject to the comments above and, therefore, does not depend on any specific structure for the overlapping economy as the previous simultaneous move game approach did, we will need the definitions[16] below.

Definition 2.16. *The sequence* $\zeta = \{\zeta_t\}_{t\geq 2}$ *is a preferences forecast if* $\zeta_t \in \{A, B\}$, $t \geq 2$. ζ *is a perfect preferences forecast if* $\zeta_t = \lambda_t$, $t \geq 2$.

Definition 2.17. *A function* $\Psi : (\tilde{\lambda}_0, \tilde{\lambda}_1, \zeta) \to (\{S_t, \gamma_{t+1}\}_{t\geq 0}) \in \mathbb{R}_+ \times ([0,1] \times \mathbb{R}_+)^\infty \times [0,1]^\infty$, *with* $\tilde{\lambda}_0, \tilde{\lambda}_1 \in \{A, B\}$ *and* ζ *a preferences forecast, is called a social security design if its image is contained on the set of sustainable social security rules.*

Our standard social security design function is the one that assigns, for each sequence $(\tilde{\lambda}_0, \tilde{\lambda}_1, \zeta)$, the following set of social security rules $\{S_t, \gamma_{t+1}\}_{t\geq 0}$, with $S_0 = \mathcal{R}_0 = R_0\phi_0(R_0)$ and

$$S_t = (\mathcal{C}_t, \mathcal{R}_t) = (\phi_t(R_t), R_t\phi_t(R_t))$$

for $t \geq 1$, where $\{R_t\}_{t\geq 0}$ is the Pareto dominant equilibrium return rates sequence obtained according to the claimed preferences of current generations alive, i.e., $\tilde{\lambda}_0$ and $\tilde{\lambda}_1$, and the preferences forecast ζ. Also, $\gamma_t = 0$, $t \geq 1$. We call this social security design function an *equilibrium design function.*

Definition 2.18. *A social security design function* Ψ *is incentive compatible if* [17]

$$U^{\lambda_t}(\Psi(\lambda_{t-1}, \lambda_t, \{\lambda_i\}_{i\geq t+1})) \geq U^{\lambda_t}(\Psi(\lambda_{t-1}, x, \{\lambda_i\}_{i\geq t+1}))$$

for $x \neq \lambda_t$, $x \in \{A, B\}$, $t \geq 1$.

[16]Definitions 2.16, 2.17 and 2.18 assume a two periods overlapping generations economy with two types of preferences, but can be directly extended to any finite number of periods and preferences.

[17]There is a slight notation abuse when we use $U^{\lambda_t}(\Psi(\lambda_{t-1}, \lambda_t, \{\lambda_i\}_{i\geq t+1}))$ to denote an indirect utility function for a given set of social security rules.

Definition 2.18 states that a social security design function is incentive compatible if, under a perfect preferences forecast, young generations in every period $t \geq 1$ have no incentives to hide their true type. Therefore, if Government is able to make a perfect preferences forecast and the social security design function is incentive compatible, then if young generations are able to claim their own preferences according to this commonly known design function and preferences forecast, they will always adopt a truth-telling behavior and will never deviate throughout time from the initial setting of social security rules.

Proposition 2.19. *Under the previous assumptions, the equilibrium design function is not incentive compatible.*

Proof. Suppose, w.l.o.g., $\lambda_1 = A$ and notice that second period *per capita* consumption of generation G_1 is given by $\alpha_2 \phi^{\lambda_2}(R_2)$, where R_2 is given by

$$R_2 = \lim_{k \to \infty} \psi_{\lambda_2}(\alpha_{\lambda_3} \phi_{\lambda_3}(\psi_{\lambda_3}(\alpha_{\lambda_4} \phi_{\lambda_4}(\psi_{\lambda_4} ... \alpha_{\lambda_{t+k}} \phi_{\lambda_{t+k}}(\psi_{\lambda_{t+k}}(L))...)$$

for $L > 0$, according to Theorem 2.9. Therefore, the second period *per capita* consumption of generation G_1 does not depend on λ_1. Since U^A is strictly increasing on its first argument we conclude that

$$U^A(\Psi(\lambda_0, B, \{\lambda_i\}_{i \geq 2})) > U^A(\Psi(\lambda_0, A, \{\lambda_i\}_{i \geq 2}))$$

and, therefore, the *equilibrium design function* is not incentive compatible. □

Proposition 2.19 states that on two periods overlapping generations economies households have an incentive to undersave if Government follows an *equilibrium design function* to define the set of social security rules after households claims. Furthermore, one can notice that if there are more than two types of preferences, such design function always leads to the smallest possible savings schedule[18].

2.4 Social security fund and Pareto improvements

In this section we describe how the existence of a social security fund can lead to Pareto improvements over the social security system. The economy is the two periods overlapping generations one of Section 2.3. Briefly, there are two types of households preferences A and B, with savings demand satisfying $\phi_A(R) \geq \phi_B(R)$, $\forall R \in \mathbb{R}_+$. Generation G_t type is given by $\lambda_t \in \{A, B\}$, $t \geq 0$. Also, there is a constant compound demographic and productivity growth rate given by $\alpha_t = \delta_L \delta_E = \delta$, $t \geq 1$. Assume the following set of social security rules $\{S_t\}_{t \geq 0}$, with $S_0 = \mathcal{R}_0 = \delta \phi_{\lambda_0}(\delta)$ and

$$S_t = (\mathcal{C}_t, \mathcal{R}_t) = (\phi_{\lambda_t}(\delta), \delta \phi_{\lambda_t}(\delta))$$

for $t \geq 1$. Following Section 2.1, the budget result of the system in period $t \geq 1$ is given by

$$\begin{aligned} \Delta_t &= L_t \mathcal{C}_t E_t - L_{t-1} \mathcal{R}_{t-1} E_{t-1} \\ &= L_t \phi_{\lambda_t}(\delta) E_t - L_{t-1} \delta \phi_{\lambda_{t-1}}(\delta) E_{t-1} \end{aligned}$$

[18] The result, however, strongly relies on the assumption of two periods for the overlapping economy.

Finally, let Ω_T be the present value of all budget results Δ_t of the social security system up to period $T \geq 1$ at a discount rate of δ, i.e.,

$$\Omega_T = \sum_{t=1}^{T} \frac{\Delta_t}{\delta^{t-1}}$$

for $T \geq 1$. Therefore, if Government is able to borrow and lend at a rate δ, Ω_T can be seen as the necessary value to be held in a social security fund at period $t = 1$ in order to implement the set of social security rules $\{S_t\}_{t\geq 0}$ up to period $t = T$[19].

Proposition 2.20. *Under the previous assumptions, $\exists M \geq 0$ such that $\Omega_T + M \geq 0$, $\forall T \geq 1$.*

Proof. Let $T \geq 1$. The definitions of Ω_T and Δ_t, $1 \leq t \leq T$, imply

$$\begin{aligned} \Omega_T &= \sum_{t=1}^{T} \frac{\Delta_t}{\delta^{t-1}} \\ &= \sum_{t=1}^{T} \frac{L_t \phi_{\lambda_t}(\delta) E_t - L_{t-1} \delta \phi_{\lambda_{t-1}}(\delta) E_{t-1}}{\delta^{t-1}} \\ &= \sum_{t=1}^{T} \delta L_0 \phi_{\lambda_t}(\delta) E_0 - L_0 \delta \phi_{\lambda_{t-1}}(\delta) E_0 \\ &= -L_0 \delta \phi_{\lambda_0}(\delta) E_0 + L_0 \delta \phi_{\lambda_T}(\delta) E_0 \\ &= L_0 E_0 \delta (\phi_{\lambda_T}(\delta) - \phi_{\lambda_0}(\delta)) \end{aligned}$$

Let $M = -\inf_{T\geq 1} L_0 E_0 \delta(\phi_{\lambda_T}(\delta) - \phi_{\lambda_0}(\delta))$, so that

$$\Omega_T + M \geq 0$$

for $T \geq 1$. We conclude that M has the claimed property, and that its value depends on λ_0. Also, if $\phi_{\lambda_0}(\delta) = \inf_{T\geq 1} \phi_{\lambda_T}(\delta)$ then $M = 0$. □

Proposition 2.20 states that the discounted value of all budget results of the social security system that implements the constant growth allocation for all generations G_t, $t \geq 1$, no matter their type ordering, is bounded when the discount rate is $\delta > 0$. Furthermore, such value may be zero[20] and can be seen as the initial amount to be held on a social security fund in order to keep the social security system balanced in every period. The result can be directly extended for the case with a finite number of preferences or for overlapping economies with more than two periods. The next example shows that the adoption of a social security fund may lead to Pareto improvements.

Example 2.21. *We build over the setting of Example 2.15. In this case, however, Government defines the set of social security rules $\{S_t\}_{t\geq 0}$ after claims $\tilde{\lambda}_A, \tilde{\lambda}_B \in \{A, B\}$ according to a direct assignment of the constant growth allocations, i.e., $S_0 = \delta\phi_{\tilde{\lambda}_B}(\delta)$, $S_{2t} = (\delta\phi_{\tilde{\lambda}_B}(\delta), \delta\phi_{\tilde{\lambda}_B}(\delta))$*

[19] In this section, the existence of a fund to cope with deficits of the social security system makes the definition of $\{\gamma_t\}_{t\geq 1}$ irrelevant.

[20] The value of the fund depends on the initial setting of preferences for generations alive at the initial period $t = 1$. It will be minimum if such setting is composed by low savers and early retirees, which implicate the lowest possible total benefit level in $t = 1$.

and $S_{2t-1} = (\delta\phi_{\tilde{\lambda}_A}(\delta), \delta\phi_{\tilde{\lambda}_A}(\delta))$, $t \geq 1$. *Therefore, final utility levels for each generation given the vector of claims* $(\tilde{\lambda}_A, \tilde{\lambda}_B) \in \{A, B\}^2$ *is*

$$U^A(\tilde{\lambda}_A, \tilde{\lambda}_B) = \begin{cases} U^A((1-\phi_A(\delta))E, \delta\phi_A(\delta)E) & , \text{ if } (\tilde{\lambda}_A, \tilde{\lambda}_B) = (A, A), (A, B) \\ U^A((1-\phi_B(\delta))E, \delta\phi_B(\delta)E) & , \text{ if } (\tilde{\lambda}_A, \tilde{\lambda}_B) = (B, B), (B, A) \end{cases}$$

$$U^B(\tilde{\lambda}_A, \tilde{\lambda}_B) = \begin{cases} U^B((1-\phi_A(\delta))E, \delta\phi_A(\delta)E) & , \text{ if } (\tilde{\lambda}_A, \tilde{\lambda}_B) = (A, A), (B, A) \\ U^B((1-\phi_B(\delta))E, \delta\phi_B(\delta)E) & , \text{ if } (\tilde{\lambda}_A, \tilde{\lambda}_B) = (B, B), (A, B) \end{cases}$$

The definition of $\phi_A(\delta)$ *and* $\phi_B(\delta)$ *imply that the unique Nash equilibrium is* $(\tilde{\lambda}_A, \tilde{\lambda}_B) = (A, B)$. *Notice that, under this set of rules, final utility levels after the Nash equilibrium are given by* $U^A((1-\phi_A(\delta))E, \delta\phi_A(\delta)E)$ *and* $U^B((1-\phi_B(\delta))E, \delta\phi_B(\delta)E)$. *Example 2.15 shows that, under the previous set of rules, final utility levels after the Nash equilibrium* $(\tilde{\lambda}_A, \tilde{\lambda}_B) = (B, B)$ *are given by* $U^A((1-\phi_B(\delta))E, \delta\phi_B(\delta)E)$ *and* $U^B((1-\phi_B(\delta))E, \delta\phi_B(\delta)E)$. *Since*

$$U^A((1-\phi_A(\delta))E, \delta\phi_A(\delta)E) > U^A((1-\phi_B(\delta))E, \delta\phi_B(\delta)E)$$

and the utility of type B generations is kept unchanged, the adoption of a social security fund leads to a Pareto improvement.

The table below shows, for $E = \delta = 1$, the final transfers that the social security system manages in Example 2.21 when adopting a social security fund (SSF) starting with $M = 0$. Column SSF brings end of period balance of the fund, i.e., after transfers are done.

Period	SSF	G_0	G_1	G_2	G_3	G_4
1	$\phi_A(1) - \phi_B(1)$	$-\phi_B(1)$	$+\phi_A(1)$	-	-	-
2	0	-	$-\phi_A(1)$	$+\phi_B(1)$	-	-
3	$\phi_A(1) - \phi_B(1)$	-	-	$-\phi_B(1)$	$+\phi_A(1)$	-
4	0	-	-	-	$-\phi_A(1)$	$+\phi_B(1)$
5	$\phi_A(1) - \phi_B(1)$	-	-	-	-	$-\phi_B(1)$

Notice that the amount of money transfers to the old in each period $\phi_A(1)$ or $\phi_B(1)$ can be much larger than the value that actually is transferred from one period to the next via the social security fund $\phi_A(1) - \phi_B(1)$. This remark is directly attached to the actual possibility of guaranteeing a return level of δ on these transfers trough the social security fund. Finally, using the same reasoning from Example 2.21, we can state that the design function that implements the constant growth allocations is incentive compatible.

2.5 Intragenerational transfers on notional accounts systems

In this section we drop our initial assumption of homogeneous households in generation G_t, $t \geq 0$, and allow for intragenerational heterogeneity. We do so in order to better understand the effects of heterogeneous households when calculating sets of social security rules according to equilibrium equations. Therefore, we modify our baseline economy by assuming that each generation G_t, $t \geq 0$, can be divided in two groups of households, X and Y. Each group has a common utility function $U^i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ that satisfies Assumption 1 and savings demand ϕ_i that satisfies Assumption 2, $i \in \{X, Y\}$. Also, generation G_t is formed by $L_t^i \in \mathbb{N}$ households from type i, with a first period endowment of $E^i_t \in \mathbb{R}_{++}$ units of the perishable consumption good, $i \in \{X, Y\}$. Equilibrium equations are written as

$$L^X_t E^X_t \phi_X(R_t) + L^Y_t E^Y_t \phi_Y(R_t) \quad = \quad R_{t-1}(L^X_{t-1} E^X_{t-1} \phi_X(R_{t-1}) + L^Y_{t-1} E^Y_{t-1} \phi_Y(R_{t-1}))$$

for $t \geq 1$. Finally, we make a constant compound growth rate assumption for each group.

Assumption 7. *There exists* $\alpha_X, \alpha_Y > 0$*, such that* $\dfrac{L^X_t E^X_t}{L^X_{t-1} E^X_{t-1}} = \alpha_X$ *and* $\dfrac{L^Y_t E^Y_t}{L^Y_{t-1} E^Y_{t-1}} = \alpha_Y$*,* $t \geq 1$*.*

Proposition 2.22. *Under the previous assumptions, if* $\alpha_X > \alpha_Y$ *then Pareto dominant equilibrium return rates* $\{R_t\}_{t \geq 0}$ *satisfy* $\alpha_Y \leq R_t \leq \alpha_X$*,* $t \geq 0$*. Also,* $\lim_{t \to \infty} R_t = \alpha_X$*.*

Proof. Since $\alpha_X = \dfrac{L^X_t E^X_t}{L^X_{t-1} E^X_{t-1}}$ and $\alpha_Y = \dfrac{L^Y_t E^Y_t}{L^Y_{t-1} E^Y_{t-1}}$, $t \geq 1$, we may write equilibrium equations as

$$\alpha^t_X L^X_0 E^X_0 \phi_X(R_t) + \alpha^t_Y L^Y_0 E^Y_0 \phi_Y(R_t) \quad = \quad R_{t-1}(\alpha^{t-1}_X L^X_0 E^X_0 \phi_X(R_{t-1}) + \alpha^{t-1}_Y L^Y_0 E^Y_0 \phi_Y(R_{t-1}))$$

for $t \geq 1$. If $R_0 > \alpha_X$ then

$$\alpha_X L^X_0 E^X_0 \phi_X(R_0) + \alpha_Y L^Y_0 E^Y_0 \phi_Y(R_0) \quad < \quad R_0(L^X_0 E^X_0 \phi_X(R_0) + L^Y_0 E^Y_0 \phi_Y(R_0))$$

and so $R_1 > R_0 > \alpha_X > \alpha_Y$. By induction, $R_t > R_{t-1} > \alpha_X > \alpha_Y$, $t \geq 1$. If $\lim_{t \to \infty} R_t = R < \infty$, then making $t \to \infty$ on the equation below

$$L^X_0 E^X_0 \phi_X(R_t) + \left(\frac{\alpha_Y}{\alpha_X}\right)^t L^Y_0 E^Y_0 \phi_Y(R_t) \quad = \quad R_{t-1}\left[\frac{L^X_0 E^X_0}{\alpha_X} \phi_X(R_{t-1}) + \frac{L^Y_0 E^Y_0}{\alpha_X}\left(\frac{\alpha_Y}{\alpha_X}\right)^{t-1} \phi_Y(R_{t-1})\right]$$

allow us to write

$$L^X_0 E^X_0 \phi_X(R) = \frac{R}{\alpha_X} L^X_0 E^X_0 \phi_X(R) \implies R = \alpha_X$$

absurd. If $\lim_{t \to \infty} R_t = \infty$, equilibrium equations are violated. We conclude that $R_0 \leq \alpha_X$. If $R_0 = \alpha_Y$ we have

$$\alpha_X L^X_0 E^X_0 \phi_X(R_0) + \alpha_Y L^Y_0 E^Y_0 \phi_Y(R_0) \quad > \quad R_0(L^X_0 E^X_0 \phi_X(R_0) + L^Y_0 E^Y_0 \phi_Y(R_0))$$

so $R_1 < R_0 = \alpha_Y$. By induction, the sequence of return rates is well-defined and so the Pareto dominant equilibrium must satisfy $\alpha_Y \leq R_0 \leq \alpha_X$. Truncation of the economy in period $t \geq 1$ allow us to conclude that $\alpha_Y \leq R_t \leq \alpha_X$, $t \geq 0$. To demonstrate that $\lim_{t \to \infty} R_t = \alpha_X$, first write

$$L^X_0 E^X_0 \phi_X(R_t) + \left(\frac{\alpha_Y}{\alpha_X}\right)^t L^Y_0 E^Y_0 \phi_Y(R_t) \quad = \quad \left[\prod_{i=0}^{t-1} \frac{R_i}{\alpha_X}\right]\left[L^X_0 E^X_0 \phi_X(R_0) + L^Y_0 E^Y_0 \phi_Y(R_0)\right]$$

Suppose there is a convergent subsequence $\{R_{\sigma(t)}\}_{t \geq 0}$ with $\lim_{t \to \infty} R_{\sigma(t)} = R < \alpha_X$. Then

$$\begin{aligned} L^X_0 E^X_0 \phi_X(R_{\sigma(t)}) + \left(\frac{\alpha_Y}{\alpha_X}\right)^{\sigma(t)} L^Y_0 E^Y_0 \phi_Y(R_{\sigma(t)}) \quad &= \quad \left[\prod_{i=0}^{\sigma(t)-1} \frac{R_i}{\alpha_X}\right]\left[L^X_0 E^X_0 \phi_X(R_0) + L^Y_0 E^Y_0 \phi_Y(R_0)\right] \\ &< \quad \left[\prod_{i=0}^{t-1} \frac{R_{\sigma(i)}}{\alpha_X}\right]\left[L^X_0 E^X_0 \phi_X(R_0) + L^Y_0 E^Y_0 \phi_Y(R_0)\right] \end{aligned}$$

Taking the limit on both sides we obtain $\phi_X(R) = 0$. Since

$$\phi_X(R) = 0 \implies R = 0$$

and $R_{\sigma(t)} \geq \alpha_Y$, $t \geq 0$, absurd. We conclude that $\lim_{t\to\infty} R_t = \alpha_X$. □

Proposition 2.22 characterizes the behavior of the Pareto dominant return rates sequence $\{R_t\}_{t\geq 0}$, where all its elements are in the interval $[\alpha_Y, \alpha_X]$ and tend to α_X in the long run. Also, it is straightforward that R_0 can be arbitrarily close to α_Y as long as $L_0^Y E_0^Y >> L_0^X E_0^X$. An important implication of this result emerges from the comparison with the one stated on item (iv) of Proposition 2.7. According to such item, Government may define the following sustainable and individually optimal set social security rules, with $\mathcal{R}_0^i = \alpha_i \phi_i(\alpha_i)$ and

$$\begin{aligned} \mathcal{C}_t^i &= \phi_i(\alpha_i) \\ \mathcal{R}_t^i &= \alpha_i \phi_i(\alpha_i) \end{aligned}$$

for $t \geq 1$, $i \in \{X, Y\}$. Notice there are two different values of return rates in each period. Proposition 2.22, however, furnishes a different set of sustainable and individually optimal rules, with $\mathcal{R}_0^i = R_0 \phi_i(R_0)$ and

$$\begin{aligned} \mathcal{C}_t^i &= \phi_i(R_t) \\ \mathcal{R}_t^i &= R_t \phi_i(R_t) \end{aligned}$$

for $t \geq 1$, $i \in \{X, Y\}$. In order to obtain a common return rate value in each period for the social security system, households from group X end up with a smaller return than the one that would be affordable given their specific compound growth rate, i.e., $R_t \leq \alpha_X$, $t \geq 0$, while households from group Y end up with a higher one. In other words, when looking for a common return rate, the social security system creates an implicit transfer, or subsidy, from group X, the one with larger compound demographic and productivity growth rate, to group Y. Clearly, there is a welfare loss for group X when adopting the second set of social security rules. Also, the direction of such transfer is defined by $\alpha_X > \alpha_Y$ and, therefore, is not dependent on any relation between E_X^t and E_Y^t, $t \geq 0$. One could have, for example, $E_Y^t >> E_X^t$, $t \geq 0$, and a transfer from group X to group Y. This implication has direct relation to notional accounts systems since their social security rules are based on common return rates (as the ones obtained in Proposition 2.22) calculated after aggregate measures of demographic and productivity growth.

2.6 Compulsory savings

In this section we analyse the effects of compulsory savings over equilibrium outcomes. We slightly change our baseline economy and notation. Let E_t be the total endowment[21] of perishable good in period $t \geq 1$. Also, let $\omega_t \in [0, 1]$ be the fraction of E_t that is hold by the old generation in period $t \geq 1$. Since generations are indexed by their birth period, generation G_t is entitled a bundle of (E_t^y, E_t^o), where

$$\begin{aligned} E_t^y &= (1 - \omega_t) E_t \\ E_t^o &= \omega_{t+1} E_{t+1} \end{aligned}$$

[21] In our baseline economy E_t represents the first period endowment of generation G_t, $t \geq 0$.

$t \geq 1$. Households from generation G_t have their preferences represented by $U_t : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $t \geq 0$, which satisfies Assumption 1. Finally, let $\tau_t \in [0,1]$ represent a compulsory savings restriction, i.e., a minimum value for generation G_t savings, $t \geq 1$. Therefore, savings demand is obtained according to

$$\begin{aligned} \phi_t(R, \tau_t) = \arg \max_{\phi \in [\tau_t, 1]} \quad & U_t(c_0, c_1) \\ \text{s.t.} \quad & c_0 = (1 - \phi) E_t^y \\ & c_1 = R \phi E_t^y + E_t^o \end{aligned}$$

for $t \geq 1$[22]. To ease notation we omitted E_t^y and E_t^o (or, equivalently, E_t, E_{t+1}, ω_t and ω_{t+1}) from the arguments of ϕ_t, $t \geq 1$. Let $L_t \in \mathbb{N}$, $t \geq 0$, and $\alpha_t^L = \frac{L_t}{L_{t-1}}$, $t \geq 1$, so that equilibrium equations can be written as

$$\alpha_t^L \phi_t(R_t) E_t^y = R_{t-1} \phi_{t-1}(R_{t-1}) E_{t-1}^y$$

for $t \geq 1$. Final consumption of generation G_t is given by

$$c^t = ((1 - \phi_t(R_t, \tau_t)) E_t^y, R_t \phi_t(R_t, \tau_t) E_t^y + E_t^o) \tag{5}$$

for $t \geq 1$. Finally, for a given pair of sequences $\omega = \{\omega_t\}_{t \geq 1} \in [0,1]^\infty$ and $\tau = \{\tau_t\}_{t \geq 1} \in [0,1]^\infty$, we call $\mathcal{E}(\omega, \tau)$ the economy described above. Clearly, $\mathcal{E}(0,0)$[23] is our baseline economy. For a given equilibrium return rates sequence $\{R_t\}_{t \geq 0}$, define the following set

$$\Omega(R) = \{t \geq 1 \mid \phi_t(R_t, \tau_t) \neq \phi_t(R_t, 0)\}$$

which identifies all generations G_t that sustain a level of savings superior to the one they would be naturally willing to because of the compulsory savings restriction represented by τ_t. It is immediate to see that $\phi_t(R_t, \tau_t) = \tau_t$, $\forall t \in \Omega(R)$. Next, we define a long run stable equilibrium for an economy $\mathcal{E}(\omega, \tau)$.

Definition 2.23. *Let $\{R_t\}_{t \geq 0}$ be an equilibrium return rates sequence of $\mathcal{E}(\omega, \tau)$, for a given $\omega, \tau \in [0,1]^\infty$. Then $\{R_t\}_{t \geq 0}$ is long run stable if $\#\Omega(R) < \infty$.*

Definition 2.23 states that an equilibrium return rates sequence $\{R_t\}_{t \geq 0}$ is long run stable (or, shortly, *LRSE*) if only a finite number of generations sustain a level of savings superior to the one they would be willing to sustain without the compulsory savings restriction. One way to interpret such definition is to suppose that Government defines the compulsory savings policy, i.e., defines $\{\tau_t\}_{t \geq 1}$. Also, suppose that for any generation G_t with $t \in \Omega(R)$, there is a small $\delta > 0$ probability of abandoning the compulsory savings policy due to households dissatisfaction. Then, if $\#\Omega(R) = \infty$ the policy will eventually be abandoned with probability 1 by Borel-Cantelli Lemma. The equilibrium is not, therefore, long run stable.

[22]Let $E_0^y \in \mathbb{R}_{++}$ and $\tau_0 \in [0,1)$ be any fixed value in order to derive ϕ_0.

[23]We denote, when convenient, the sequence of zeros in $[0,1]^\infty$ by 0.

Let $\varepsilon > 0$ be any fixed value. The following sets will be needed for the statement of the next proposition.

$$\begin{aligned}
\Sigma_{0,0} &= \{c \in [\varepsilon, +\infty)^\infty \mid \exists \{R_t\}_{t\ge 0} \textit{ LRSE} \text{ of } \mathcal{E}(0,0) \text{ s.t. } \{c^t\}_{t\ge 0} \text{ satisfies Eq. 5}\} \\
\Sigma_{\omega,0} &= \{c \in [\varepsilon, +\infty)^\infty \mid \exists \omega \in [0,1]^\infty, \{R_t\}_{t\ge 0} \textit{ LRSE} \text{ of } \mathcal{E}(\omega,0) \text{ s.t. } \{c^t\}_{t\ge 0} \text{ satisfies Eq. 5}\} \\
\Sigma_{0,\tau} &= \{c \in [\varepsilon, +\infty)^\infty \mid \exists \tau \in [0,1]^\infty, \{R_t\}_{t\ge 0} \textit{ LRSE} \text{ of } \mathcal{E}(0,\tau) \text{ s.t. } \{c^t\}_{t\ge 0} \text{ satisfies Eq. 5}\} \\
\Sigma_{\omega,\tau} &= \{c \in [\varepsilon, +\infty)^\infty \mid \exists \omega, \tau \in [0,1]^\infty, \{R_t\}_{t\ge 0} \textit{LRSE} \text{ of } \mathcal{E}(\omega,\tau) \text{ s.t. } \{c^t\}_{t\ge 0} \text{ satisfies Eq. 5}\}
\end{aligned}$$

Notice that $\Sigma_{0,0}$ describes all consumption schedules of $\mathcal{E}(0,0)$ that are consistent with some equilibrium return rates sequence $\{R_t\}_{t\ge 0}$. Also, $\Sigma_{\omega,0}$ describes all consumption schedules of $\mathcal{E}(\omega,0)$, $\omega \in [0,1]^\infty$, i.e., after an endowment redistribution, that are consistent with some equilibrium return rates sequence $\{R_t\}_{t\ge 0}$. The descriptions of $\Sigma_{0,\tau}$ and $\Sigma_{\omega,\tau}$ are analogous. It is immediate from the definitions that $\Sigma_{0,0} \subseteq \Sigma_{\omega,0} \subseteq \Sigma_{\omega,\tau}$ and $\Sigma_{0,0} \subseteq \Sigma_{0,\tau} \subseteq \Sigma_{\omega,\tau}$. Finally, we assume an uniform upper bound over the endowments.

Assumption 8. *There is $M > 0$ such that $E_t \le M$, $t \ge 1$.*

We may state now the result of this section.

Proposition 2.24. *Under the previous assumptions, $\Sigma_{\omega,\tau} \subseteq \Sigma_{\omega,0}$.*

Proof. Let $\{c^t\}_{t\ge 0} \in \Sigma_{\omega,\tau}$, where ω, $\tau \in [0,1]^\infty$ define the economy $\mathcal{E}(\omega,\tau)$ and $\{R_t\}_{t\ge 0}$ is the corresponding long run stable equilibrium (*LRSE*). Equilibrium equations allow us to write

$$\alpha^L_{t+1}(E^y_{t+1} - c_0^{t+1}) = R_t(E^y_t - c_0^t)$$

for $t \ge 1$. Since $c_0^{t+1} \ge 0$, $t \ge 1$, we have

$$E^y_{t+1} > E^y_{t+1} - c_0^{t+1} = \prod_{i=1}^{t} \frac{R_i}{\alpha^L_{i+1}}(E^y_1 - c_0^1)$$

for $t \ge 1$. Also, $E_t \le M$, $t \ge 1$, implies

$$\sup_{t\ge 1} \prod_{i=1}^{t} \frac{R_i}{\alpha^L_{i+1}} < +\infty$$

Next, for $t \in \Omega(R)$, notice that Lagrange conditions for the utility maximization problem of generation G_t allow us to write

$$\frac{1}{R_t} \ge \frac{\partial U_t(c_0^t, c_1^t)/\partial c_1}{\partial U_t(c_0^t, c_1^t)/\partial c_0} > 0$$

where the last inequality comes from Assumption 1. Define the return rates sequence $\{P_t\}_{t\ge 0}$ by

$$P_t = \begin{cases} \dfrac{\partial U_t(c_0^t, c_1^t)/\partial c_0}{\partial U_t(c_0^t, c_1^t)/\partial c_1} & , t \in \Omega(R) \\ R_t & , t \notin \Omega(R) \end{cases}$$

for $t \ge 1$. P_0, at this point, remains undetermined. Next, we must show that $\exists \tilde{\omega} \in [0,1]^\infty$ such that $\{c_t\}_{t\ge 0}$ is the consumption schedule supported by $\{P_t\}_{t\ge 0}$ on the economy $\mathcal{E}(\tilde{\omega}, 0)$.

Notice that if $\tilde{E}^y_t$ and $\tilde{E}^o_t$, $t \geq 1$, satisfy the following equation

$$\tilde{E}^y_t + \frac{\tilde{E}^o_t}{P_t} = E^y_t + \frac{E^o_t}{P_t}$$

then the definition of P_t implies that

$$(c^t_0, c^t_1) = \arg \max_{(c_0,c_1)\in\mathbb{R}^2_+} \quad U_t(c_0, c_1)$$
$$\text{s.t.} \quad c_0 + \frac{c_1}{P_t} = \tilde{E}^y_t + \frac{\tilde{E}^o_t}{P_t}$$

Notice that $(\bar{E}^t_0, \bar{E}^t_1) = (c^t_0, c^t_1)$ is a possible solution. However, it leads to an autarkic equilibrium. In order to make savings demand non-null in each period, we must look for solutions with the following form

$$(\tilde{E}^y_t, \tilde{E}^o_t) = (c^t_0 + \delta^t, c^t_1 - \delta^t P_t)$$

with δ^t, $t \geq 1$. Since the increases in endowment for the old generation must equate the decrease for the young in each period $t \geq 2$, if generation G_1 has a *per capita* increase of $\delta > 0$, i.e., $\delta_1 = \delta > 0$, then

$$\delta^t = \prod_{i=1}^{t-1} \frac{P_i}{\alpha_{i+1}} \delta$$

for $t \geq 2$. Since only a finite number of return rates have changed when passing from $\{R_t\}_{t\geq0}$ to $\{P_t\}_{t\geq0}$ the following implication holds

$$\sup_{t\geq1} \prod_{i=1}^{t} \frac{R_i}{\alpha_{i+1}} < \infty \implies \sup_{t\geq1} \prod_{i=1}^{t} \frac{P_i}{\alpha_{i+1}} < \infty$$

We can then state that $\forall \varepsilon > 0, \exists \tilde{\delta}$ such that $\delta \in (0, \tilde{\delta})$ implies $\delta^t \in (0, \frac{\varepsilon}{2})$, $t \geq 1$. Since $(c^t_0, c^t_1)_{t\geq0}$ have, by definition, a positive lower bound $\varepsilon > 0$, $\exists \tilde{\delta} > 0$ such that $\delta \in (0, \tilde{\delta})$ implies

$$c^t_1 - \prod_{i=1}^{t-1} \frac{P_i}{\alpha_{i+1}} \delta > \frac{\varepsilon}{2} > 0$$

for $t \geq 1$. Let $\delta \in (0, \tilde{\delta})$ and define

$$\tilde{\omega}_t = 1 - \frac{c^t_0 + \delta^t}{E_t}$$

for $t \geq 1$, so that all endowments are strictly positive. It is clear, by its definition, that $\{\tilde{\omega}_t\}_{t\geq1}$ implements the desired endowment distribution. It remains to define P_0 in order to satisfy the equilibrium equation given by

$$\alpha_1(\tilde{E}^y_1 - c^1_0) = \alpha_1 \delta = P_0 \phi_0(P_0) E^y_0$$

Since the function $x \to x\phi_0(x)$ has an image set equal to $\mathbb{R}_+$, there is always a solution for such equation. We have defined then $\tilde{\omega} \in [0,1]^\infty$ and a long run stable equilibrium $\{P_t\}_{t\geq 0}$ from $\mathcal{E}(\tilde{\omega}, 0)$ that implements $\{c^t\}_{t\geq 0}$. Since $\{c^t\}_{t\geq 0} \in \Sigma_{\omega,0}$, we conclude that $\Sigma_{\omega,\tau} \subseteq \Sigma_{\omega,0}$. □

Proposition 2.24 states that on two periods overlapping generations economies every allocation that is achieved trough a LRSE under compulsory savings restrictions, can also be achieved without such restrictions by a suitable change on each generation endowment. The result can also be read as a claim stating that compulsory savings restrictions only matter if households endowment distribution do not request, under rational behavior, a minimum savings level.

2.7 A result on Pareto optimality

In this section we drop our initial assumption of a two periods overlapping generations economy and extend it to an $(n+1)$ periods one, $n \geq 1$. Households live for $n+1$ periods, except the ones alive in the inception of the economy at $t = 1$. Generations are indexed by the period they are born and there is no intragenerational heterogeneity. Therefore, for every generation G_t, $t \geq -n+1$, there is a common utility function $U_t : \mathbb{R}_+^{n+1} \to \mathbb{R}$ representing households preferences over life-time consumption bundles. We assume U_t satisfies Assumption 1, $t \geq -n+1$. Also, generation G_t has $L_t \in \mathbb{N}$ households with a life-time endowment given by $E_t \in \mathbb{R}_+^{n+1}$, $t \geq -n+1$. Following Sections 2.1 and 2.2 define, for life-time return rates $R \in \mathbb{R}_+^n$, savings demand by

$$\begin{aligned}
\phi^t(R) = \arg\max_{\phi \in \mathbb{R}^n} \quad & U_t(c) \\
\text{s.t.} \quad & c_i \geq 0,\ 0 \leq i \leq n \\
& c_0 + \phi_0 = E_0 \\
& c_i + \phi_i = E_i + R_i \phi_{i-1},\ 1 \leq i \leq n-1 \\
& c_n = E_n + R_n \phi_{n-1}
\end{aligned}$$

for $t \geq -n+1$. Notice that ϕ_i^t represents the absolute level of savings (and not relative level as in Section 2.1) in the $(i+1)$th-period of life of generation G_t, $0 \leq i \leq n-1$, $t \geq -n+1$. Also, we allow for borrowing, i.e., $\phi_i^t \in \mathbb{R}$. Equilibrium equations are written as

$$\sum_{i=1}^{n} L_{t+i-n+1} \phi_{n-i}^{t+i-n+1}(R_{t+i-n+1}, \ldots, R_{t+i}) = R_t \sum_{i=0}^{n-1} L_{t+i-n+1} \phi_{n-1-i}^{t+i-n+1}(R_{t+i-n+1}, \ldots, R_{t+i})$$

for $t \geq 0$. Before stating our main result of this section we make the following assumption.

Assumption 9. *There is a constant demographic growth $\delta > 0$, a common endowment bundle $E \in \mathbb{R}_+^{n+1}$ and no intergenerational heterogeneity, i.e., $\delta = \frac{L_{t+1}}{L_t}$, $E_t = E$ and $U_t = U$, $t \geq -n+1$. Also, utility function $U : \mathbb{R}_+^{n+1} \to \mathbb{R}$ is time-separable, i.e., $U(c_0, \ldots, c_n) = \sum_{i=0}^{n} u_i(c_i)$.*

Under Assumption 9 equilibrium equations become

$$\begin{aligned}
\sum_{i=1}^{n} \delta^i \phi_{n-i}(R_{t+i-n+1}, \ldots, R_{t+i}) &= R_t \sum_{i=0}^{n-1} \delta^i \phi_{n-1-i}(R_{t+i-n+1}, \ldots, R_{t+i}) \\
&= R_t \sum_{i=1}^{n} \delta^{i-1} \phi_{n-i}(R_{t+i-n}, \ldots, R_{t+i-1})
\end{aligned}$$

for $t \geq 0$. It is clear, therefore, that the constant growth sequence $R_t = \delta$, $t \geq -n+1$, is an equilibrium return rates sequence. The next proposition states that it also implements a Pareto optimal allocation[24].

Proposition 2.25. *Under the previous assumptions, the constant growth equilibrium* $R_t = \delta$, $\forall t \geq -n+1$, *is Pareto optimal.*

Proof. First, notice that $\forall \beta \in (0,1)$ the solution of the following maximization problem leads to a Pareto optimal allocation[25]

$$\begin{array}{lc} \max & \sum_{t=0}^{\infty} \beta^t U_{t-n+1}(c^{t-n+1}) \\ \text{s.t.} & c^{t-n+1} \geq 0, \forall t \geq 0 \\ & \sum_{i=0}^{n} L_{t+i-n+1}(c_{n-i}^{t+i-n+1} - E_{n-i}^{t+i-n+1}) = 0, \forall t \geq 0 \end{array}$$

Assume $n = 2$ to ease notation. Since $U(c_0, c_1, c_2) = u_0(c_0) + u_1(c_1) + u_2(c_2)$, Assumption 9 allow us to write the previous problem as

$$\begin{array}{lc} \max & \sum_{t=0}^{\infty} \beta^t \left[u_2(c_2^{t-1}) + \beta u_1(c_1^t) + \beta^2 u_0(c_0^{t+1}) \right] \\ \text{s.t.} & c^{t-1} \geq 0, \forall t \geq 0 \\ & c_2^{t-1} + \delta c_1^t + \delta^2 c_0^{t+1} = E_2 + \delta E_1 + \delta^2 E_0, \forall t \geq 0 \end{array}$$

Since the objective function is separable and the restrictions are independent we conclude that the optimal solution is given by $c_0^t = x_0(\beta)$, $\forall t \geq 1$, $c_1^t = x_1(\beta)$, $\forall t \geq 0$, and $c_2^t = x_2(\beta)$, $\forall t \geq -1$, where $x(\beta) = (x_0(\beta), x_1(\beta), x_2(\beta))$ is the solution of the following problem

$$\begin{array}{lc} \max & u_2(x_2) + \beta u_1(x_1) + \beta^2 u_0(x_0) \\ \text{s.t.} & x_0, x_1, x_2 \geq 0 \\ & x_2 + \delta x_1 + \delta^2 x_0 = E_2 + \delta E_1 + \delta^2 E_0 \end{array}$$

Next, notice that the constant growth equilibrium $R_t = \delta$, $t \geq -1$, implies that each household from generation G_t, $t \geq 1$, faces the following utility maximization problem

$$\begin{array}{lc} \max & U(c_0^t, c_1^t, c_2^t) \\ \text{s.t.} & c_0^t, c_1^t, c_2^t \geq 0 \\ & c_0^t + \phi_0^t = E_0 \\ & c_1^t + \phi_1^t = E_1 + \delta \phi_0^t \\ & c_2^t = E_2 + \delta \phi_1^t \end{array}$$

[24] This result was also derived under different assumptions by others, like Okuno and Zilcha (53).

[25] There is a slight notation abuse when we use $U_{t-n+1}(c^{t-n+1})$ without mentioning that such utility functions are actually truncated for generations G_t, $t < 1$, since the economy starts in $t = 1$ and past consumption at $t < 1$ do not enter on the welfare evaluation (although past periods are considered to solve equilibrium equations).

which can also be written as

$$\begin{array}{rc} \max & u_2(c_2^t) + u_1(c_1^t) + u_0(c_0^t) \\ \text{s.t.} & c_0^t, c_1^t, c_2^t \geq 0 \\ & c_2^t + \delta c_1^t + \delta^2 c_0^t = E_2 + \delta E_1 + \delta^2 E_0 \end{array}$$

We conclude that the optimal solution is given by $c^t = x(1) = (x_0(1), x_1(1), x_2(1))$, $t \geq 1$. Since restrictions are independent of the value of $\beta \in (0,1)$ we have

$$U(x(1)) > U(x(\beta))$$

for all $\beta \in (0,1)$. Also,

$$\lim_{\beta \to 1} x(\beta) = x(1)$$

Suppose that $c^t = x(1)$, $\forall t \geq -1$[26], is not a Pareto optimal allocation and let $\{y^t\}_{t \geq -1}$ be a feasible allocation that Pareto dominates it, i.e., $U_t(x(1)) \leq U_t(y^t)$, $\forall t \geq -1$, with at least one strict inequality. Therefore, the following inequality holds $\forall \beta \in (0,1)$

$$\sum_{t=0}^{\infty} \beta^t U_{t-1}(y^{t-1}) > \sum_{t=0}^{\infty} \beta^t U_{t-1}(x(1))$$

Also $\exists \varepsilon > 0$, $\rho \in (0,1)$ such that $\forall \beta \in (\rho, 1)$

$$\sum_{t=0}^{\infty} \beta^t U_{t-1}(y^{t-1}) > \sum_{t=0}^{\infty} \beta^t U_{t-1}(x(1)) + \varepsilon$$

The feasibilty of $\{y^t\}_{t \geq -1}$ and the definition of $x(\beta)$, $\beta \in (0,1)$, imply that

$$\sum_{t=0}^{\infty} \beta^t U_{t-1}(x(\beta)) \geq \sum_{t=0}^{\infty} \beta^t U_{t-1}(y^{t-1})$$

Next, notice that

$$\begin{aligned} \lim_{\beta \to 1-} \sum_{t=2}^{\infty} \beta^t [U_{t-1}(x(1)) - U_{t-1}(x(\beta))] &= \lim_{\beta \to 1-} \sum_{t=2}^{\infty} \beta^t [U(x(1)) - U(x(\beta))] \\ &= \lim_{\beta \to 1-} \beta^2 \frac{U(x(1)) - U(x(\beta))}{1 - \beta} \end{aligned}$$

If we let $f(\beta) = U(x(\beta))$ and extend the definition of $x(\beta)$ for $\beta > 1$ in the direct way, we conclude that $f(1) \geq f(\beta)$, $\forall \beta \in \mathbb{R}_+$. Then the first order conditions for a local maximum allow

[26] To ease notation we write $c^{-1} = x(1)$ instead of $c^{-1} = c_2^{-1} = x_2(1) \in \mathbb{R}_+$ and $c^0 = x(1)$ instead of $c^0 = (c_1^0, c_2^0) = (x_1(1), x_2(1)) \in \mathbb{R}_+^2$. The same holds when writting $U^{-1}(x(1))$ and $U^0(x(1))$.

us to conclude that $f'(1) = 0$ and

$$
\begin{aligned}
\lim_{\beta\to 1-}\sum_{t=2}^{\infty}\beta^t[U_{t-1}(x(1)) - U_{t-1}(x(\beta))] &= \lim_{\beta\to 1-}\beta^2\frac{U(x(1)) - U(x(\beta))}{1-\beta} \\
&= \lim_{h\to 0+}(1-h)^2\frac{f(1) - f(1-h)}{h} \\
&= f'(1) \\
&= 0
\end{aligned}
$$

However, for $\beta \in (\rho, 1)$, our previous inequality implies

$$
\begin{aligned}
\varepsilon &< \sum_{t=0}^{\infty}\beta^t U_{t-1}(x(\beta)) - \sum_{t=0}^{\infty}\beta^t U_{t-1}(x(1)) \\
&= [u_2(x_2(\beta)) - u_2(x_2(1))] + \beta[u_2(x_2(\beta)) - u_2(x_2(1)) + \ldots \\
&\quad \ldots + u_1(x_1(\beta)) - u_1(x_1(1))] + \sum_{t=2}^{\infty}\beta^t[U_{t-1}(x(\beta)) - U_{t-1}(x(1))]
\end{aligned}
$$

Since $\lim_{\beta\to 1} x(\beta) = x(1)$ we have all terms on the right side converging to zero, absurd. We conclude that the allocation defined by $R_t = \alpha$, $\forall t \geq -1$, is Pareto optimal. □

Proposition 2.25 gives conditions under which constant growth equilibrium is Pareto optimal. The importance of such result will be clear in Section 2.8 since the characterization of Pareto optimal return rates sequences is a central point for the equilibrium calculation method described there. Another important implication is given by the next example.

Example 2.26. *We use the framework of Proposition 2.25. Let $n = 2$, $\delta = 1$, $E_t = (E, 0, 0)$, with $E > 0$, $\forall t \geq -1$, and $U(c_0, c_1, c_2) = \log(c_0) + (1-\theta)\log(c_1) + \theta\log(c_2)$, $\theta \in (0, 1)$, so that savings demand can be written, for a given sequence of return rates $\{R_t\}_{t\geq -1}$, as*

$$
\begin{aligned}
\phi_0^t(R_t, R_{t+1}) &= \frac{E}{2} \\
\phi_1^t(R_t, R_{t+1}) &= \frac{\theta R_t E}{2}
\end{aligned}
$$

for $t \geq -1$. Equilibrium equations are

$$
\phi_0^{t+1}(R_{t+1}, R_{t+2}) + \phi_1^t(R_t, R_{t+1}) = R_t[\phi_0^t(R_t, R_{t+1}) + \phi_1^{t-1}(R_{t-1}, R_t)]
$$

for $t \geq 0$. Using the previous equations for savings demand we have

$$
1 + \theta R_t = R_t[1 + \theta R_{t-1}]
$$

for $t \geq 0$. Or, equivalently

$$
R_t = \frac{1}{(1-\theta) + \theta R_{t-1}}
$$

for $t \geq 0$. Then, for every $R_{-1} > 0$ there is a well-defined equilibrium return rates sequence

$\{R_t\}_{t\geq -1}$ *given by the iterates of the following function* $f : \mathbb{R}_+ \to \mathbb{R}_+$

$$f(x) = \frac{1}{(1-\theta) + \theta x}$$

i.e., $R_t = f^{t+1}(R_{-1})$*, for* $t \geq 0$*. Let* $g = f \circ f$*, so that*

$$\begin{aligned} g(x) &= \frac{(1-\theta) + \theta x}{(1-\theta)^2 + \theta + (1-\theta)\theta x} \\ &= \frac{1}{1-\theta}\left[1 - \frac{\theta}{(1-\theta)^2 + \theta + (1-\theta)\theta x}\right] \end{aligned}$$

Then g *is strictly increasing and concave, with* $g(1) = 1$ *and* $\lim_{x\to\infty} g(x) = \frac{1}{1-\theta}$*. If* $R < 1$*, then* $g^k(R) \to 1$ *is a monotonically increasing sequence, and if* $R > 1$*, then* $g^k(R) \to 1$ *monotonically decreasing. We conclude that for every* $R_{-1} \in \mathbb{R}_{++}$ *there is a well-defined equilibrium sequence* $\{R_t\}_{t\geq -1}$*, where each subsequence* $\{R_{2k-1}\}_{k\geq 0}$*,* $\{R_{2k}\}_{k\geq 0}$ *converges monotonically, depending on the value of* R_{-1}*, and, therefore,* $R_{-1} \neq 1$ *implies the equilibrium return rates sequence will cycle around its long-term equilibrium value of* 1*.*

Example 2.26 shows that when the number of periods each generation lives increases there is also an increase on the dimension of the set of possible equilibrium return rates sequences that are not Pareto dominated. When $n = 1$, Proposition 2.6 states that there is a single Pareto dominant equilibrium sequence. In the case of Example 2.26, however, the set of equilibrium sequences $R = \{R_t\}_{t\geq -1}$ is given by

$$\Gamma = \{R \in \mathbb{R}_+^\infty \mid R_{-1} > 0 \text{ and } R_t = f^{t+1}(R_{-1})\}$$

Let $U(R_t, R_{t+1})$, $t \geq -1$, be the indirect utility function of generation G_t when facing return rates R_t, R_{t+1}, $t \geq -1$. The cycling behavior for $R_{-1} \neq 1$ described in the end of Example 2.26 implies that utility values also cycle and converge on the long term, i.e., $\lim_{t\to\infty} U(R_t, R_{t+1}) = U(1, 1)$. This is a direct consequence of Proposition 2.25, stated formally in the next corollary.

Corollary 2.27. *Under the previous assumptions, let* $\{R_t\}_{t\geq -n+1}$ *be any equilibrium return rates sequence. Then* $U(R_t, ..., R_{t+n-2}) \leq U(\delta, ..., \delta)$ *for infinitely many* $t \geq -n+1$*.*

Proof. Let $U(R_t, ..., R_{t+n-2})$ be the indirect utility function of generation G_t when facing life-time return rates $\{R_t, ..., R_{t+n-2}\} \in \mathbb{R}_+^{n-1}$, $t \geq -n+1$. Since the constant growth return rates sequence is Pareto optimal, there are two possibilities. Either $\exists t_0 \geq -n+1$ such that $U(R_{t_0}, ..., R_{t_0+n-2}) < U(\delta, ..., \delta)$ or $U(R_t, ..., R_{t+n-2}) = U(\delta, ..., \delta)$, $\forall t \geq -n+1$. In both cases, $\exists t_0 \geq -n+1$ such that $U(R_{t_0}, ..., R_{t_0+n-2}) \leq U(\delta, ..., \delta)$. Notice that $P_t = R_{t+k}$, $t \geq -n+1$, also defines an equilibrium return rates sequence, $k \geq 0$. Let $k = t_0 + n$. Then it is possible to find, according to our previous argument, $\exists l \geq -n+1$ such that $U(P_l, ..., P_{l+n-2}) \leq U(\delta, ..., \delta)$. Therefore, defining $t_1 = l + t_0 + n$ allow us to state that $t_1 \geq t_0 + 1$ and $U(R_{t_1}, ..., P_{t_1+n-2}) \leq U(\delta, ..., \delta)$. We may proceed by induction in order to define a strictly increasing sequence $\{t_i\}_{i\geq 0}$ such that $U(R_{t_i}, ..., R_{t_i+n-2}) \leq U(\delta, ..., \delta)$, $i \geq 0$. We conclude that $U(R_t, ..., R_{t+n-2}) \leq U(\delta, ..., \delta)$ for infinitely many $t \geq -n+1$. □

An important consequence from Corollary 2.27 is the following. Let

$$\Gamma = \{R \in \mathbb{R}_+^\infty \mid R \text{ is an equilibrium return rates sequence}\}$$

as in our previous discussion. Since there is no intergenerational heterogeneity a reasonable criteria for Government to choose among different possible equilibrium return rates sequences that are not Pareto ranked is the following max min welfare problem

$$\max_{R\in\Gamma}\min_{t\geq 1} U(R_t, \dots, R_{t+n-2}) \tag{6}$$

that equates the utility of all generations born at or after the initial period $t = 1$[27]. Corollary 2.27 allows us to state that the constant growth return rates sequence is the solution.

2.8 Backward shifts as an equilibrium calculation method

In this section we develop a method for calculating optimal equilibrium return rates sequence for $n+1$ periods overlapping generations economies, $n \geq 1$, that later on will be used to derive sets of social security rules. We adopt the framework of Section 2.7. Briefly, households live for $n+1$ periods, except the ones alive in the inception of the economy at $t = 1$. Generation G_t, $t \geq -n+1$, is characterized by the total number of households $L_t \in \mathbb{N}$, a time-separable utility function $U_t : \mathbb{R}_+^{n+1} \to \mathbb{R}$[28] and a life-time endowment $E_t \in \mathbb{R}_+^{n+1}$. Savings demand ϕ^t are derived according to the utility maximization problem described in Section 2.7 and equilibrium equations are given by

$$\sum_{i=1}^{n} L_{t+i-n+1}\phi_{n-i}^{t+i-n+1}(R_{t+i-n+1}, \dots, R_{t+i}) = R_t \sum_{i=0}^{n-1} L_{t+i-n+1}\phi_{n-1-i}^{t+i-n+1}(R_{t+i-n+1}, \dots, R_{t+i}) \tag{7}$$

for $t \geq 0$. Let $\mathcal{E}_1 = \mathcal{E}(\{U_t, L_t, E_t\}_{t\geq -n+1})$ be the economy described above. We define the chopped economy $\mathcal{E}_T$ by

$$\mathcal{E}_T = \mathcal{E}(\{U_t, L_t, E_t\}_{t\geq -n+T})$$

for $T \geq 2$. To ease notation, the next results will be stated for $n = 2$.

Proposition 2.28. *Let $\{x^t\}_{t\geq -1}$ be a Pareto optimal allocation in $\mathcal{E}_1$. Then $\{x^t\}_{t\geq T-2}$ is a Pareto optimal allocation in the chopped economy $\mathcal{E}_T$, $\forall T \geq 2$.*

Proof. First, notice that $x^{-1} = x_2^{-1} \in \mathbb{R}_+$, $x^0 = (x_1^0, x_2^0) \in \mathbb{R}_+^2$ and $x^t = (x_0^t, x_1^t, x_2^t) \in \mathbb{R}_+^3$, $t \geq 1$, satisfy

$$x_2^{t-1} + x_1^t + x_0^{t+1} = E_2^{t-1} + E_1^t + E_0^{t+1}$$

for $t \geq 0$, since $\{x^t\}_{t\geq -1}$ is feasible in $\mathcal{E}_1$. Let $T = 2$ and suppose $\{x^t\}_{t\geq 0}$[29] is not Pareto optimal in the chopped economy. Then $\exists \{y^t\}_{t\geq 0}$ which Pareto dominates $\{x^t\}_{t\geq 0}$, i.e., $U_t(y^t) \geq$

[27] Notice that all generations G_t for $t \geq 1$ have the same utility functions. For $t < 1$, utility functions are truncated due to shorter life time.

[28] Although generations G_t, $1 > t \geq -n+1$, do not live for $n+1$ periods, their utility function is defined over $\mathbb{R}_+^{n+1}$ in order to derive the savings demand functions that enter Equation 7. Since utilities are assumed to be time-separable, i.e., $U_t(c^t) = \sum_{i=0}^{n} u_i^t(c_i^t)$, we make a slight notation abuse when writing $U_t(c^t)$, $1 > t \geq -n+1$, in all results from this section where welfare comparisons between consumption bundles are made. This happens because, for example, instead of writing $U_{-n+1}(c^{-n+1})$ the most precise form would be $u_n^{-n+1}(c_n^{-n+1})$ since generation G_{-n+1} lives only for one period and, therefore, this brings the implicit assumption that consumption bundles are properly dimensioned, i.e., $c^{-n+1} \in \mathbb{R}_+$ and not $c^{-n+1} \in \mathbb{R}_+^{n+1}$. Notice that time separability allow us to make these comparisons regardless of past consumption values.

[29] There is a slight notation abuse when writing x^0 for $\mathcal{E}_2$, since its original definition for $\mathcal{E}_1$ implies $x^0 =$

$U_t(x^t)$, $t \geq 0$, with at least one strict inequality. Define the sequence $\{z^t\}_{t\geq -1}$ on the original economy $\mathcal{E}_1$ by $z^{-1} = z_2^{-1} = x_2^{-1}$, $z^0 = (z_1^0, z_2^0) = (x_1^0, y_2^0)$, $z^1 = (z_0^1, z_1^1, z_2^1) = (x_0^1, y_1^1, y_2^1)$ and $z^t = y^t$, $\forall t \geq 2$, so that

$$
\begin{aligned}
L_{-1}z_2^{-1} + L_0 z_1^0 + L_1 z_0^1 &= L_{-1}x_2^{-1} + L_0 x_1^0 + L_1 x_0^1 \\
&= L_{-1}E_2^{-1} + L_0 E_1^0 + L_1 E_0^1 \\
L_{t-1}z_2^{t-1} + L_t z_1^t + L_{t+1} z_0^{t+1} &= L_{t-1}y_2^{t-1} + L_t y_1^t + L_{t+1} y_0^{t+1} \\
&= L_{t-1}E_2^{t-1} + L_t E_1^t + L_{t+1} E_0^{t+1}
\end{aligned}
$$

for $t \geq 1$. We conclude that $\{z^t\}_{t\geq -1}$ is feasible on $\mathcal{E}_1$. Also,

$$
\begin{aligned}
U_{-1}(z^{-1}) \geq U_{-1}(x^{-1}) &\iff u_2^{-1}(z_2^{-1}) \geq u_2^{-1}(x_2^{-1}) \\
&\iff u_2^{-1}(x_2^{-1}) \geq u_2^{-1}(x_2^{-1}) \\
U_0(z^0) \geq U_0(x^0) &\iff u_1^0(z_1^0) + u_2^0(z_2^0) \geq u_1^0(x_1^0) + u_2^0(x_2^0) \\
&\iff u_1^0(x_1^0) + u_2^0(y_2^0) \geq u_1^0(x_1^0) + u_2^0(x_2^0) \\
&\iff u_2^0(y_2^0) \geq u_2^0(x_2^0) \\
&\iff U_0(y^0) \geq U_0(x^0) \\
U_1(z^1) \geq U_1(x^1) &\iff u_0^1(z_0^1) + u_1^1(z_1^1) + u_2^1(z_2^1) \geq u_0^1(x_0^1) + u_1^1(x_1^1) + u_2^1(x_2^1) \\
&\iff u_0^1(x_0^1) + u_1^1(y_1^1) + u_2^1(y_2^1) \geq u_0^1(x_0^1) + u_1^1(x_1^1) + u_2^1(x_2^1) \\
&\iff u_1^1(y_1^1) + u_2^1(y_2^1) \geq u_1^1(x_1^1) + u_2^1(x_2^1) \\
&\iff U_1(y^1) \geq U_1(x^1) \\
U_t(z^t) \geq U_t(x^t) &\iff U_t(y^t) \geq U_t(x^t)
\end{aligned}
$$

for $t \geq 2$. Therefore, $\{y^t\}_{t\geq 0}$ Pareto dominates $\{x^t\}_{t\geq 0}$ in $\mathcal{E}_2$ if, and only if, $\{z^t\}_{t\geq -1}$ Pareto dominates $\{x^t\}_{t\geq -1}$ in $\mathcal{E}_1$. Since $\{x^t\}_{t\geq -1}$ is Pareto optimal on $\mathcal{E}_1$, absurd. We conclude that $\{x^t\}_{t\geq 0}$ is Pareto optimal in the chopped economy $\mathcal{E}_2$. The general result for $T \geq 2$ is obtained by induction. □

Proposition 2.28 implies that Pareto optimality is an invariant property when one deals with chopped economies. The result is fairly intuitive since absence of Pareto optimality in overlapping generations economies often comes from the possibility of rearranging consumption bundles in a way that brings consumption from the "far future". Therefore, an allocation does not loses such property when the economy is chopped at any point in time. The next definition brings a weaker notion of optimality, called short-run Pareto optimality.

Definition 2.29. *Let $\{x^t\}_{t\geq -1}$ be a feasible allocation in $\mathcal{E}_1$. Then $\{x^t\}_{t\geq -1}$ is short-run Pareto optimal if there is no $\{y^t\}_{t\geq -1}$ feasible such that $\exists t_0 > -1$ with $y^t = x^t$, $t \geq t_0$, and $U_t(y^t) \geq U_t(x^t)$, $t \geq -1$, with at least one strict inequality.*

Short-run Pareto optimality states that it is not possible to obtain a Pareto improvement only rearranging a finite number of consumption bundles[30]. The importance of this definition is made clear on the proposition below.

$(x_1^0, x_2^0) \in \mathbb{R}_+^2$. When reading x^0 as an allocation of G_0 in $\mathcal{E}_2$, therefore, one must consider $x^0 = x_2^0 \in \mathbb{R}_+$. An analogous shift happens with x^1.

[30]Proposition 2.28 also applies to short-run Pareto optimal allocations.

Proposition 2.30. *Let* $\{R_t\}_{t\geq -1} \in \mathbb{R}^{\infty}_{++}$ *be an equilibrium return rates sequence, i.e.,*

$$L_{t+1}\phi_0^{t+1}(R_{t+1}, R_{t+2}) + L_t\phi_1^t(R_t, R_{t+1}) \quad = \quad R_t[L_t\phi_0^t(R_t, R_{t+1}) + L_{t-1}\phi_1^{t-1}(R_{t-1}, R_t)]$$

for $t \geq 0$. *Then* $\{R_t\}_{t\geq -1} \in \mathbb{R}^{\infty}_{++}$ *is short-run Pareto optimal.*

Proof. Let $\{x^t\}_{t\geq -1}$ be the equilibrium allocation and suppose $\{R_t\}_{t\geq -1} \in \mathbb{R}^{\infty}_{++}$ is not short-run Pareto optimal. Then, $\exists\{y^t\}_{t\geq -1}$ feasible and $t_0 > -1$ such that $y^t = x^t$, $t \geq t_0$, and $U_t(y^t) \geq U_t(x^t)$, $t \geq -1$, with at least one strict inequality. We assume $t_0 \geq 2$, otherwise it is straightforward that $x^t = y^t$, $t \geq -1$, absurd. Feasibility condition in period $t = t_0 + 1$ is written

$$L_{t_0+1}y_0^{t_0+1} + L_{t_0}y_1^{t_0} + L_{t_0-1}y_2^{t_0-1} = L_{t_0+1}x_0^{t_0+1} + L_{t_0}x_1^{t_0} + L_{t_0-1}x_2^{t_0-1}$$

Since $y^{t_0} = x^{t_0}$ and $y^{t_0+1} = x^{t_0+1}$, we conclude that $y_2^{t_0-1} = x_2^{t_0-1}$. Therefore, the following equivalence holds

$$U_{t_0-1}(y^{t_0-1}) \geq U_{t_0-1}(x^{t_0-1}) \iff u_0^{t_0-1}(y_0^{t_0-1}) + u_1^{t_0-1}(y_1^{t_0-1}) \geq u_0^{t_0-1}(x_0^{t_0-1}) + u_1^{t_0-1}(x_1^{t_0-1})$$

Let $\{\alpha_t\}_{-1\leq t\leq t_0-1} \in \mathbb{R}^{t_0+1}_{++}$ and define the following function $\mathcal{H} : \mathbb{R}_+ \times \mathbb{R}^2_+ \times \mathbb{R}^{3(t'-2)}_+ \times \mathbb{R}^2_+ \to \mathbb{R}$

$$\mathcal{H}(h^{-1}, h^0, h^1, ..., h_0^{t_0-1}, h_1^{t_0-1}) = \sum_{t=-1}^{t_0-2} \alpha_t U_t(h^t) + \alpha_{t_0-1}u_0^{t_0-1}(h_0^{t_0-1}) + \alpha_{t_0-1}u_1^{t_0-1}(h_1^{t_0-1})$$

where $h^{-1} = h_2^{-1} \in \mathbb{R}_+$, $h^0 = (h_1^0, h_2^0) \in \mathbb{R}^2_+$, $h^t = (h_0^t, h_1^t, h_2^t) \in \mathbb{R}^3_+$, $0 < t < t_0 - 1$, and $h_0^{t_0-1}, h_1^{t_0-1} \in \mathbb{R}_+$. The definition of $\{y^t\}_{t\geq -1}$ implies

$$\mathcal{H}(y^{-1}, y^0, y^1, ..., y_0^{t_0-1}, y_1^{t_0-1}) > \mathcal{H}(x^{-1}, x^0, x^1, ..., x_0^{t_0-1}, x_1^{t_0-1})$$

Next, define the following maximization problem

$$\begin{aligned} \max_{h\geq 0} \quad & \mathcal{H}(h^{-1}, h^0, h^1, ..., h_0^{t_0-1}, h_1^{t_0-1}) \\ \text{s.t.} \quad & L_{t-1}h_2^{t-1} + L_t h_1^t + L_{t+1}h_0^{t+1} = L_{t-1}x_2^{t-1} + L_t x_1^t + L_{t+1}x_0^{t+1}, \; t_0 - 2 \geq t \geq 0 \\ & L_{t_0-2}h_2^{t_0-2} + L_{t_0-1}h_1^{t_0-1} = L_{t_0-2}x_2^{t_0-2} + L_{t_0-1}x_1^{t_0-1} \end{aligned}$$

Inada's condition imply that the optimum is an interior point. Also, the strict concavity of $\mathcal{H}$ implies that first order conditions are necessary and sufficient. Notice, next,

$$\nabla\mathcal{H} = (\alpha_{-1}\nabla U_{-1}, \alpha_0\nabla U_0, ..., \alpha_{t_0-1}u_0^{t_0-1\prime}, \alpha_{t_0-1}u_1^{t_0-1\prime})$$

where arguments were omitted to ease notation. Since x^t, $t \geq -1$, was derived according to utility maximization under return rates R_t and R_{t+1}, we can assume, w.l.o.g.[31], that $\nabla U_{-1}(x^{-1}) = u_2^{-1\prime}(x_2^{-1}) = 1$, $\nabla U_0(x^0) = (1, \frac{1}{R_1})$, $\nabla U_t(x^t) = (1, \frac{1}{R_t}, \frac{1}{R_t R_{t+1}})$, $1 \leq t < t_0 - 1$, and $(u_0^{t_0-1\prime}, u_1^{t_0-1\prime}) = (1, \frac{1}{R_{t_0-1}})$. The equilibrium allocation clearly satisfies the restrictions of the maximization problem. First order conditions are satisfied if $\exists\{\lambda_t\}_{t_0-1\geq t\geq 0} \in \mathbb{R}^{t_0}$ so that

$$\nabla\mathcal{H}(x) = \lambda_0(L_{-1}, L_0, 0, L_1, ...) + \lambda_1(0, 0, L_0, 0, L_1, 0, L_2, ...) + ... + \lambda_{t_0-1}(0, ...L_{t_0-2}, 0, L_{t_0-1})$$

[31] If necessary one can make an affine transformation over each utility function in order to satisfy the assumption.

Let $\lambda_0 = \frac{1}{L_{-1}}$ and

$$\lambda_t = \frac{1}{L_{-1}} \prod_{i=1}^{t} \frac{1}{R_i}$$

for $1 \le t \le t_0$. Also, let $\alpha_{-1} = 1$, $\alpha_0 = \frac{L_0}{L_{-1}}$, $\alpha_1 = \frac{L_1}{L_{-1}}$ and

$$\alpha_t = \frac{L_t}{L_{-1}} \prod_{i=1}^{t-1} \frac{1}{R_i}$$

for $2 \le t \le t_0 - 1$. Notice then that for such values the equilibrium allocation satisfies the first order condition. For example, checking the last coordinate of the gradient yields

$$\alpha_{t_0-1} u_1^{t_0-1\prime}(x_1^{t_0-1\prime}) = \lambda_{t_0-1} L_{t_0-1} \iff \frac{1}{L_{-1}} \prod_{i=1}^{t_0-1} \frac{1}{R_i} L_{t_0-1} = \frac{L_{t_0-1}}{L_{-1}} \prod_{i=1}^{t_0-2} \frac{1}{R_i} \frac{1}{R_{t_0-1}}$$

Therefore, the equilibrium allocation is the unique optimum and

$$\mathcal{H}(y^{-1}, y^0, y^1, ..., y_0^{t_0-1}, y_1^{t_0-1}) < \mathcal{H}(x^{-1}, x^0, x^1, ..., x_0^{t_0-1}, x_1^{t_0-1})$$

absurd. We conclude $\{x^t\}_{t \ge -1}$ is short-run Pareto Optimal. □

Proposition 2.30 states that every allocation obtained from an equilibrium return rates sequence is short-run Pareto optimal. Next, we state, under our current notation[32], a well established result based on the theory of efficiency prices due to the work of Cass(15), Benveniste-Gale(10), Balasko-Shell(6) and Okuno-Zilcha(53).

Theorem 2.31. *(Cass, 1972, Benveniste-Gale, 1975, Balasko-Shell, 1980, Okuno-Zilcha, 1980) If households have uniformly strictly concave utilities and demographic growth rates and per capita endowments are uniformly bounded, then an equilibrium return rates sequence* $\{R_t\}_{t \ge -1}$ *is Pareto optimal if, and only if,*

$$\sum_{t=1}^{\infty} \frac{\prod_{i=1}^{t} R_i}{L_t} = \infty$$

Theorem 2.31 provides a complete characterization of Pareto optimal equilibrium return rates sequence[33]. For example, if the two periods overlapping economies in Section 2.1 satisfy the assumptions of Theorem 2.31 then every equilibrium return rates sequence $\{R_t\}_{t \ge 0}$ with $\lim_{t \to \infty} R_t = 0$ is not Pareto optimal. Now we are able to state the main result of this section.

Proposition 2.32. *Under the previous assumption, let* $\{R_t\}_{t \ge -1} \in \mathbb{R}_{++}^{\infty}$ *be a short-run Pareto optimal (or Pareto optimal) equilibrium return rates sequence defined after equilibrium equa-*

[32]The statement of the theorem in Geanakoplos (35) was based on present value prices and not on interperiod return rates. Also, it assumed a uniformly bounded aggregate endowment.

[33]One must pay close attention to the fact that values of R_{-1} and R_0 are not taken into account when determining Pareto optimality, as long as they satisfy equilibrium equations. This happens because utility functions used to define optimal consumption bundles are not the ones used to derive final utility for generations alive in the first period $t = 1$.

tions given by

$$\begin{aligned}
L_1\phi_0^1(R_1,R_2)+L_0\phi_1^0(R_0,R_1) &= R_0[L_0\phi_0^0(R_0,R_1)+L_{-1}\phi_1^{-1}(R_{-1},R_0)]\\
L_2\phi_0^2(R_2,R_3)+L_1\phi_1^1(R_1,R_2) &= R_1[L_1\phi_0^1(R_1,R_2)+L_0\phi_1^0(R_0,R_1)]\\
L_3\phi_0^3(R_3,R_4)+L_2\phi_1^2(R_2,R_3) &= R_2[L_2\phi_0^2(R_2,R_3)+L_1\phi_1^1(R_1,R_2)]\\
&[...]
\end{aligned}$$

Suppose that a backward shift through equilibrium equations is possible in order to define an extended economy that starts in period $t=0$*, i.e., there exists* $L_{-2}\in\mathbb{N}$*,* $U_{-2}:\mathbb{R}_+^3\to\mathbb{R}$*,* $E_{-2}\in\mathbb{R}_+^3$ *and* $R_{-2}\in\mathbb{R}_{++}$ *such that*

$$\begin{aligned}
L_0\phi_0^0(R_0,R_1)+L_{-1}\phi_1^{-1}(R_{-1},R_0) &= R_{-1}[L_{-1}\phi_0^{-1}(R_{-1},R_0)+L_{-2}\phi_1^{-2}(R_{-2},R_{-1})]\\
L_1\phi_0^1(R_1,R_2)+L_0\phi_1^0(R_0,R_1) &= R_0[L_0\phi_0^0(R_0,R_1)+L_{-1}\phi_1^{-1}(R_{-1},R_0)]\\
L_2\phi_0^2(R_2,R_3)+L_1\phi_1^1(R_1,R_2) &= R_1[L_1\phi_0^1(R_1,R_2)+L_0\phi_1^0(R_0,R_1)]\\
L_3\phi_0^3(R_3,R_4)+L_2\phi_1^2(R_2,R_3) &= R_2[L_2\phi_0^2(R_2,R_3)+L_1\phi_1^1(R_1,R_2)]\\
&[...]
\end{aligned}$$

Then $\{R_t\}_{t\geq -2}$ *is short-run Pareto optimal (or Pareto optimal) in the extended economy.*

Proof. It follows directly from Proposition 2.30 and Theorem 2.31. □

Proposition 2.32 describes a method for calculating short-run Pareto optimal (or Pareto optimal) equilibrium return rates sequences. It is based on backward shifts under equilibrium equations departing from a given short-run Pareto optimal (or Pareto optimal) sequence after a future time period. Before providing an example to better illustrate this point, there is an important connection between this result and Theorem 2.9. Let $\mathcal{E}_1=\mathcal{E}(\{U_t,L_t,E_t\}_{t\geq 0})$ be a two periods overlapping generations economy as the ones in Section 2.1 satisfying the assumptions of Theorem 2.9 and Proposition 2.32. Then, equilibrium equations are written

$$\alpha_t\phi_t(R_t)=R_{t-1}\phi_{t-1}(R_{t-1})$$

for $t\geq 1$. For a given time-separable utility $V:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ and compound growth rate $\delta=\delta^L\delta^E>0$, define $\mathcal{E}_1^k=\mathcal{E}(\{\tilde{U}_t,\tilde{L}_t,\tilde{E}_t\}_{t\geq 0})$ where

$$(\tilde{U}_t,\tilde{L}_t,\tilde{E}_t)=\begin{cases}(U_t,L_t,E_t),\ t\leq k\\(V,(\delta^L)^{t-k}L_k,(\delta^E)^{t-k}E_k),\ t>k\end{cases}$$

for $k\geq 1$. Notice that equilibrium equations for $\mathcal{E}_1^k$ are written

$$\begin{aligned}
\alpha_t\phi_t(R_t) &= R_{t-1}\phi_{t-1}(R_{t-1}),\ \text{for } t\leq k\\
\delta\phi(R_t) &= R_{t-1}\phi(R_{t-1}),\ \text{for } t>k
\end{aligned}$$

where ϕ is the savings demand associated with utility function V. Therefore, the constant growth is clearly an equilibrium of the constrained economy at period $T=k$, $\mathcal{E}_k^k$, which is Pareto optimal according to Proposition 2.25. According to Proposition 2.32, the following value of R_0 obtained through backward shifts defines a Pareto optimal equilibrium return rates sequence on $\mathcal{E}_1^k$

$$R_0^k = \psi_0(\alpha_1 \phi_1(\psi_1(...\psi_{k-1}(\delta\phi(\delta)...))))$$

Then, by calling $L = \delta\phi(\delta)$, Theorem 2.9 allows us to state that

$$\lim_{k\to\infty} R_0^k = R_0$$

where R_0 defines the Pareto optimal equilibrium in $\mathcal{E}_1$. We may see this as a result stating that economies $\mathcal{E}_1^k$ (which have a simpler calculation of their optimal return rates sequence due to the backward shift method described in Proposition 2.32) correctly "approximate" $\mathcal{E}_1$ when $k \to \infty$, and so can be used to calculate the Pareto optimal equilibrium return rates sequence of this economy. Furthermore, this result does not depend on V or δ. Although the convergence result of Theorem 2.9 was only proven for two periods overlapping economies, i.e., $n = 1$, the backward shift method implied by Proposition 2.32 can always be used when one is able to calculate, after any future time period, a short-run Pareto optimal or Pareto optimal equilibrium on overlapping economies with $n > 1$. The next example shows one of these cases and illustrates how the method can be used to make comparative statics analysis over optimal social security rules.

Example 2.33. *We slightly change Example 2.26. Let* $n = 2$, $E_t = (E, 0, 0)$, $E > 0$, *and* $U_t(c) = U(c) = \log(c_0) + (1-\theta)\log(c_1) + \theta\log(c_2)$, $\theta \in (0,1)$, $\forall t \geq -1$. *Also, let* $\delta_S > 0$ *and* $\delta_L > 0$ *be the short-run and the long-run demographic growth rates, i.e.,*

$$\frac{L_t}{L_{t-1}} = \begin{cases} \delta_S, & t \leq 1 \\ \delta_L, & t > 1 \end{cases}$$

Equilibrium equations are written as

$$L_{t+1}\phi_0^{t+1}(R_{t+1}, R_{t+2}) + L_t\phi_1^t(R_t, R_{t+1}) \quad = \quad R_t[L_t\phi_0^t(R_t, R_{t+1}) + L_{t-1}\phi_1^{t-1}(R_{t-1}, R_t)]$$

for $t \geq 0$. *Savings demand are given by*

$$\phi_0^t(R_t, R_{t+1}) = \phi_0(R_t, R_{t+1}) = \frac{E}{2}$$
$$\phi_1^t(R_t, R_{t+1}) = \phi_1(R_t, R_{t+1}) = \frac{\theta R_t E}{2}$$

for $t \geq -1$. *Therefore, we have*

$$\begin{aligned} \delta_S^2 + \delta_S\theta R_0 &= R_0[\delta_S + \theta R_{-1}] \\ \delta_L\delta_S + \delta_S\theta R_1 &= R_1[\delta_S + \theta R_0] \\ \delta_L^2 + \delta_L\theta R_t &= R_t[\delta_L + \theta R_{t-1}] \end{aligned}$$

for $t \geq 2$. *Notice that the previous equations furnish a well-defined equilibrium return rates sequence* $\{R_t\}_{t\geq -1}$, *for every* $R_{-1} > 0$. *However, under the same reasoning applied on the welfare evaluation of Problem 6, in order to have Pareto optimal allocation with minimum possible volatility over utility values* U_t, $t \geq 1$, *we must have*

$$R_t = \delta_L$$

for $t \geq 1$. *Applying the backward shift method from Proposition 2.32 to calculate* R_0 *and* R_{-1} *we obtain*

$$R_{-1} = R_0 = \delta_S$$

Then, we can make a comparative statics exercise in order to evaluate welfare impacts of demographic changes. Generation G_0 *utility is given by*

$$\begin{aligned} U_0(c_1^0, c_2^0) &= (1-\theta)\log(c_1^0) + \theta\log(c_2^0) \\ &= (1-\theta)\log\left[\frac{(1-\theta)\delta_S E}{2}\right] + \theta\log\left[\frac{\theta\delta_L\delta_S E}{2}\right] \end{aligned}$$

Therefore, $\frac{\partial U_0}{\partial \delta_S} = \frac{1}{\delta_S}$ *and* $\frac{\partial U_0}{\partial \delta_L} = \frac{\theta}{\delta_L}$. *Particularly,*

$$\left.\frac{\partial U_0}{\partial \delta_L}\right|_{\delta_L=\delta} = \theta \left.\frac{\partial U_0}{\partial \delta_S}\right|_{\delta_S=\delta}$$

We conclude that if $\delta_S = \delta_L = \delta > 0$ *then a* 1% *increase on the long-term growth rate* δ_L *increases generation* G_0 *welfare by a fraction* θ *when compared to a* 1% *increase on the short-term growth rate* δ_S.

Example 2.33 illustrates how Proposition 2.32 can be used to define a selection criteria among different possible equilibrium return rates sequences. In this case, the criteria was the sequence that implied minimum volatility over final households utilities once population growth rates reached an stable level long run value δ_L. Also, the equilibrium calculation method derived from Proposition 2.32 allows for comparative statics analysis to be performed in order to realize welfare evaluations.

2.9 Concluding remarks

Sections 2.1 described the relation between equilibrium equations and the design of sustainable and optimal pay-as-you-go social security systems called *notional accounts* ones. Theorem 2.9 in Section 2.2 provided an analytical formula for solving equilibrium equations on two periods overlapping generations economies. The validity of such result for larger number of periods remains an open question.

Sections 2.3 and 2.4 characterize intergenerational transfers and the role of a social security fund for obtaining Pareto improvements on the social security system when compatibility of incentives is considered. The effects of different possible retirement ages over these results remain an open question, although it seems reasonable to conjecture that the dichotomy between high and low savers is also present when dealing with late and early retirees. Also, possible Pareto improvements related to the social security fund when uncertainty enters the model leave room for further research.

Finally, the equilibrium calculation method defined in Section 2.8 depends on the knowledge of an equilibrium return rates sequence after a given future time period. If the economy becomes stationary at any future time period, the answer is given by the result in Section 2.7. If not, however, one must again look for extensions of Theorem 2.9.

3 A Numerical Analysis of Social Security

This chapter makes a brief review of the numerical analysis from Brazilian subnational entities social security systems that was awarded the XXIV National Treasury Prize. The analysis is not based on the theoretical results derived in the previous chapter. Instead, it aimed on establishing a methodology for financial and actuarial projections and managing of subnational entities social security systems according to current Brazilian legislation. The reader is invited to consult the original portuguese version of the monograph (47) for further details.

3.1 Brazil's framework

Letting the private sector aside, Brazil's social security is organized in two different systems. First, the *Regime Geral de Previdência Social* (RGPS) inscribed in article 201 of the Constitution, which is a pay-as-you-go system open for all Brazilian citizens. Second, the *Regime Próprio de Previdência Social* (RPPS) inscribed in article 40, which is a two tier or purely pay-as-you-go system for public servants. Every entity of the Federation can institute and manage the RPPS of its own public servants, which is maintained with their contributions and the ones from the entity Treasure[34].

After the promulgation of the 1988 Constitution, the number of RPPS in the country skyrocketed: from 251 in 1988 to 2123 in 2018. In parallel, successive reform efforts[35] were made in order to approximate the set of social security rules of the RPPS to the ones of the RGPS and reverse the scenario of financial and actuarial unsustainability.

An audit conducted by the *Tribunal de Contas da União*[36] in 2015 aimed at detailing the financial and actuarial situation of the RPPS from all subnational entities. On its conclusion, it asserted that the RPPS were financially and actuarially unbalanced, having reserves to face at most one year of their owed benefits. Also, the total actuarial deficit from the RPPS summed R$ 4 trillions (corresponding to 66% of the 2015 Brazilian GDP[37]), what highlighted the systemic risk and the fiscal crisis that could unroll from the unbalance of the RPPS. The National Treasury Secretariat emitted a report in 2018 stating that from 2012 to 2017 there was an average increase of 25% on the number of retired public servants from all Brazilian states. Therefore, it was pressing to tackle the RPPS reforms not only due to the financial and actuarial unbalances but also because of current shortfalls on state level active public servants. In 2019, Constitutional Amendment nº 103/2019 reformed the set of social security rules of the RPPS in order to control the expansion of the financial and actuarial deficits. Although its appliance for subnational entities was not immediate, several ones followed the reform effort afterwards due to adverse fiscal scenario.

Under this framework, the first objective of our numerical analysis was to build a replicable methodology for evaluating financial and actuarial balance of subnational entities RPPS. Lack of such common methodology allows for a great degree of discretion when reporting financial and actuarial results and, therefore, hinders comparisons and allows for data distortion. The second objective was to evaluate the impact for the city and state of Rio de Janeiro of the changes proposed by the Constitutional Amendment nº 103/2019. Also, it aimed at characterizing the effects of adopting a two tier structure over a purely pay-as-you-go one.

[34] If the RPPS is unbalanced the entity Treasure is usually responsible for covering the deficit.

[35] Constitutional Amendments nº 20/1998 and 41/2003, and several other infraconstitutional laws.

[36] An administrative court responsible for the oversight of public expenditure on federal level.

[37] Brazil's 2015 GDP was R$ 5,996 trillions according to *Instituto Brasileiro de Geografia e Estatística* (IBGE).

3.2 Numerical analysis

The numerical analysis was based on commonly collected data for all RPPS and the current social security legislation. The goal was to build a model that described the time evolution of the number of active, inactive and pensioners of the RPPS in order to be able to forecast the amount of contributions received and benefits payed in each subsequent year. It was shown how to estimate mortality and invalidity curves, salary progression, permanence allowance[38], public servants turnover and average entrance age on the job market. Estimates from the RPPS of the state and the city of Rio de Janeiro were used as examples. Also, the assumptions that supported the calculations were formally stated, e.g., *perfect replacement*[39].

Next, forecasts of the total number of inactive and pensioners were done, together with the evolution of the RPPS deficit and the transition costs related to the adoption of a two tier structure for the social security system. The same forecasts were repeated assuming the changes proposed by Constitutional Amendment nº 103/2019 were fulfilled by the state and the city of Rio de Janeiro RPPS. Such changes were capable to reduce the upward trend over the system deficit and also to conduct it over the long run to a level lower than the current one for both state and city, although the RPPS contributions will still be insufficient to cover all of the systems obligations.

Other than this, the results were fundamentally different for the state and for the city of Rio de Janeiro. Although on the short run the city has a smaller unbalance than the state, this scenario is reverted over the long run. The short run results derive mainly from the initial composition of each RPPS, i.e., the number of active, inactive and pensioners when they were created, and to different hiring schedules of new public servants. On the other hand, the long-run result is mainly due to career and gender differences over the insured population, and higher invalidity rates for the city public servants. Also, transition costs become larger under Constitutional Amendment nº 103/2019 rules since in this case the original purely pay-as-you-go system becomes less unbalanced.

It was also shown how to integrate to the projections an evolution on life expectancy that is coherent with the phenomena of *rectangularization*. Another analytical refinement was the evolution of the composition of the active public servants population according to forecasts of public services demand. Then, it was described how one could apply the proposed methodology to manage the hiring of new public servants and also to ensure compliance with legal provisions. The need for risk measures when using volatile assets to build reserves was presented based on an analytical example and the case of oil royalties used by the state of Rio de Janeiro to underestimate the actuarial deficit of its RPPS.

Finally, it was argued for the need of a social security fund in order to isolate the RPPS accounting from the entity one. The fund is necessary in order to correctly manage financial fluctuations that the system may have even when actuarially balanced. Also, it was highlighted the fact that poorly designed extraordinary contributions to finance the transition deficit after the Constitutional Amendment nº 103/2019 could concentrate a disproportionate burden on low income active public servants.

[38]When a public servant fulfills all legal requirements to retire but decides to continue working he receives a permanence allowance.

[39]The assumption of *perfect replacement* states that every active public servant that, for any reason, moves out of service is immediately replaced by another one with the same characteristics (sex, career, age of admission, marital status etc).

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