On topological entropy for set-valued maps

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Summary

In this work we define and study the topological entropy of set-valued dynamical systems. Actually, we obtain two entropies based on separated and spanning sets. Some properties of these entropies resembling the single-valued case will be obtained.

Definitions

Let X be a metric space and $f : X \to 2^X$. For all $n \in \mathbb{N}^+$ define $d_n : X \times X \to \mathbb{R}^+$ by

$$d_n(x, y) = \inf \left\{ \max_{0 \le i \le n-1} d(x_i, y_i) \right\} \tag{1}$$

where $\inf$ is taken on sequences $(x_i)_{i=0}^n, (y_i)_{i=0}^n$ satisfying $x_0 = x, y_0 = y, x_{i+1} \in f(x_i)$ and $y_{i+1} \in f(y_i)$ for all i with $0 \le i \le n-1$.

The ϵ-balls centered at $x \in X$ with respect to d_n, are $B_n[x, \epsilon] = \{y \in X : d_n(x, y) \le \epsilon\}$.

Given $\epsilon > 0$ and $F \subset X$ we say that F is (n, ϵ)-*separated* (for f) if $B_n[x, \epsilon] \cap F = \{x\}, \quad \forall x \in F$.

Define $s(n, \epsilon) = \sup\{card(F) : F \text{ is } (n, \epsilon)\text{-separated}\}$. and

$$h_{se}(f, \epsilon) = \limsup_{n \to \infty} \frac{\log s(n, \epsilon)}{n}.$$

Observe that the limit

$$\lim_{\epsilon \to 0} h_{se}(f, \epsilon) = \sup_{\epsilon > 0} h_{se}(f, \epsilon)$$

exists and the following definition is given.

Definition 1. *The* separated topological entropy *of f is defined by*

$$h_{se}(f) = \lim_{\epsilon \to 0} h_{se}(f, \epsilon).$$

Given $n \in \mathbb{N}^+$, $\epsilon > 0$ and $E \subset X$ we say that E is (n, ϵ)-*spanning* (for f) if

$$X = \bigcup_{x \in E} B_n[x, \epsilon].$$

Define $r(n, \epsilon) = \min\{card(E) : E \text{ is } (n, \epsilon)\text{-spanning}\}$ and

$$h_{sp}(f, \epsilon) = \limsup_{n \to \infty} \frac{\log r(n, \epsilon)}{n}.$$

Similarly to the separated case

$$\lim_{\epsilon \to 0} h_{sp}(f, \epsilon) = \sup_{\epsilon > 0} h_{sp}(f, \epsilon)$$

exists and the following definition is given.

Definition 2. *The* spanning topological entropy *of f is defined by*

$$h_{sp}(f) = \lim_{\epsilon \to 0} h_{sp}(f, \epsilon).$$

Main Results

Theorem 1. *Let f be a set-valued map on a metric space X. If $X = \bigcup_{i=1}^m A_i$ where each A_i is an invariant set of f, then $h_{se}(f) = \max_{1 \le i \le m} h_{se}(f|_{A_i})$.*

Theorem 2. *Both the separated and spanning entropies reverse the inclusion order on the set of set-valued maps.*

Recall that a *selection* of a set-valued map $f : X \to 2^X$ is any map $s : X \to X$ satisfying $s(x) \in f(x)$ for all $x \in X$. Selections always exist under the axiom of choice.

Corollary 3. *If f is a set-valued map on a metric space, then $h_{se}(f) \le h(s)$ for every selection s of f.*

Theorem 4. *The spanning entropy is less than or equal to the separated entropy.*

The next property gives a sufficient condition for the separated and spanning entropies of a set-valued dynamical system to be equal.

Theorem 5. *Both the separated and spanning entropies coincide when the maps d_n in (1) are metrics for all n large.*

For the natural notion of topological conjugacy and equivalent metrics.

Theorem 6. *The separated and spanning entropies are invariant under topological conjugacy. Moreover, both entropies are independent from uniformly equivalent metrics.*

Remark 7. *Theorem 6 implies that the topological entropy is also an invariant for any single-valued map, whether continuous or not. This extends the single-valued result Ciklová.*

Theorem 8. *Every uniformly continuous set-valued map f of a metric space satisfies*

$$h_*(f) \le h_*(f^k) \le k \cdot h_*(f) \qquad \forall k \in \mathbb{N}^+ \text{ where } * = sp, se.$$

We say that a set-valued map f of a metric space X is *equicontinuous* if for every $\epsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$ there are sequences $(x_i)_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$ such that $x_0 = x$, $y_0 = y$, $x_{i+1} \in f(x_i)$, $y_{i+1} \in f(y_i)$ and $d(x_i, y_i) < \epsilon$ for every $i \in \mathbb{N}$. This definition is the natural extension of the corresponding definition in the single-valued case. Another related concept but in the set-valued setting is a definition by Maschler & Peleg.

The last property is a generalization of a well known fact in the single-valued case.

Theorem 9. *Both separated and spanning entropies vanish for equicontinuous set-valued maps on compact metric spaces.*

Conclusions

We used the single-valued approach of separated and spanning sets to define the spanning and separated topological entropies for set-valued maps. We proved that these entropies satisfy some properties resembling to the single-valued case. These include:

- sub-additivity property in Theorem 1 (similar to the single-valued case),
- they reverse natural inclusion order's for set-valued maps (not available in the single-valued case),
- the spanning entropy is less than the separated one (and that they both coincide when the induced semimetrics are metrics),
- they are topological invariants (similar to the single-valued case),
- that they satisfy a power inequality closely related to the power formula in the single-valued case,
- they both vanish for equicontinuous set-valued maps (again as in the single-valued case).

Acknowledgments

This work was partially supported by FONDECYT-Peru CG-176-2014, by FONDECYT project 11121598, CONICYT (Chile) and Pos-Doutorado Verão, IMPA, Rio de Janeiro, Brazil.

For references see: Discrete and Continuous Dynamical Systems Series B Volume 20, Number 10, December 2015 doi:10.3934/dcdsb.2015.20.3461 pp. 3461?3474