Palis Conjecture in a One-dimensional Scenario

Paulo Brandão

paulo@impa.br

impa Instituto de Matemática Pura e Aplicada

Abstract

We present some recent results of joint works together with Palis and Pinheiro about the classification and finiteness of attractors of interval maps with discontinuities.

Introduction

Attractors play a fundamental role in the study of dynamical systems for the understanding of future evolution of initial states, since many orbits converge to them in the future. Palis Global Conjecture says that there is a dense set $\mathcal{D}$ of dynamics such that any element of $\mathcal{D}$ has finitely many attractors whose union of basins of attraction has total probability.

For one-dimensional dynamics, due to the simplicity of the topology, one may ask if the finiteness of the number of attractors is a condition fulfilled by all maps, instead of only a dense subset of dynamics as conjectured for higher dimensions.

For smooth non-flat maps of the interval, Lyubich proved the finiteness of non-periodic attractors. Later, van Strien-Vargas sharpened the classification of these attractors for C^3 non-flat maps with a finite number of critical points.

Here we present some recent results together with Palis and Pinheiro about the classification and finiteness of attractors to interval maps with discontinuities.

We point out that maps of the interval with discontinuities naturally arise from vector fields. Indeed, piecewise C^r maps, $r \geq 1$, that are piecewise monotone and non-flat can be obtained as the quotient by stable manifolds of a Poincaré map of some C^r dissipative flow. In Figure below we have sketched a flow giving rise to a piecewise C^r one-dimensional map with two discontinuities.

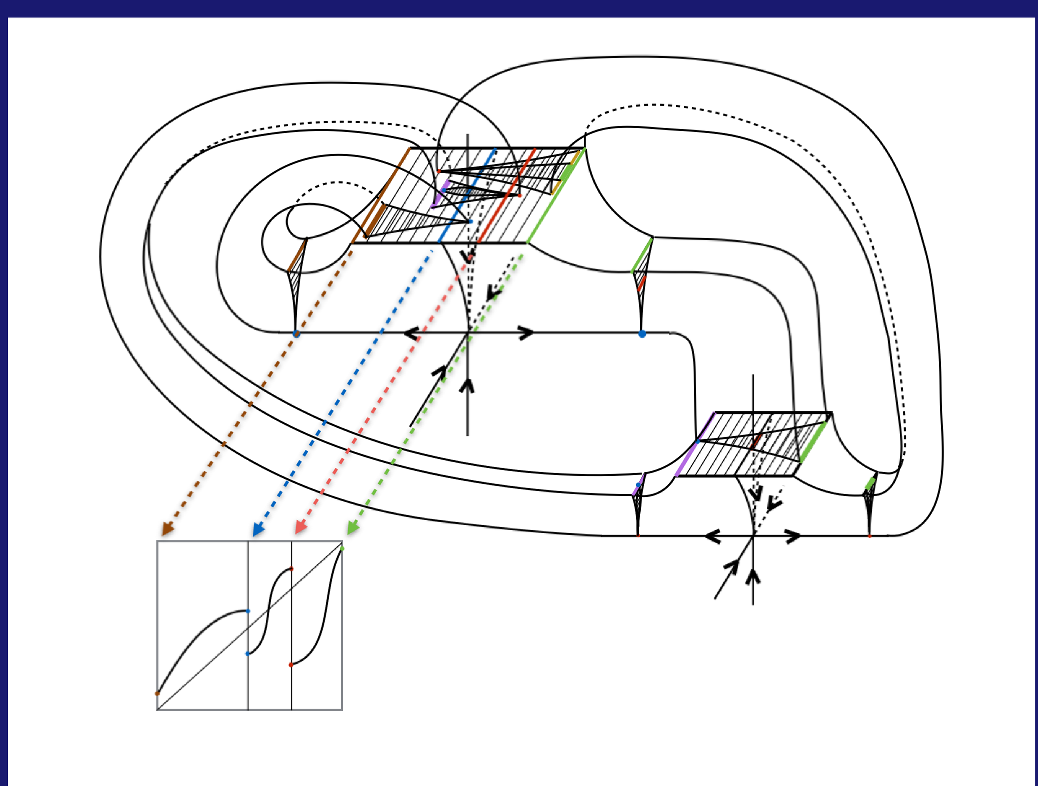

Figura 1: An example of a flow displaying two coupled singularities and inducing an one-dimensional map with two discontinuities.

Results

Theorem A ([BPP17]). *Let $f : [0,1] \to [0,1]$, be a non-flat C^2 local diffeomorphism in the whole interval, except for a finite set $\mathcal{C}_f \subset (0,1)$. Then, f admits only a finite number of non periodic-like attractors. Indeed, there is a finite collection of attractors $A_1, \cdots, A_n$, such that*

$$\text{Leb}\left(\beta_f(A_1) \cup \cdots \cup \beta_f(A_n) \cup \mathbb{B}_0(f) \cup \mathcal{O}_f^-(Per(f))\right) = 1,$$

where $\mathbb{B}_0(f)$ is the union of the basin of attraction of all attracting periodic-like orbits. Furthermore, $\omega_f(x) = A_j$ for almost every $x \in \beta_f(A_j)$ and every $j = 1, ..., n$.

Note that, if $Sf(x) < 0$ for $x \in [0,1] \setminus \mathcal{C}_f$ in the theorem above, where $Sf(x)$ is the Schwarzian derivative of f at x, then f must have a bounded number of periodic-like attractors and $\text{Leb}(\mathcal{O}_f^-(\text{Per}(f))) = 0$. Hence, f has a finite numbers of attractors. That is,

Corollary. *Let $f : [0,1] \to [0,1]$, be a C^3 local diffeomorphism with negative Schwarzian derivative in the whole interval, except for a finite set $\mathcal{C}_f \subset (0,1)$. Then, there is a finite collection of attractors $A_1, \cdots, A_n$, such that*

$$\text{Leb}(\beta_f(A_1) \cup \cdots \cup \beta_f(A_n)) = 1.$$

Furthermore, for almost all points x, we have $\omega_f(x) = A_j$ for some $j = 1, \cdots, n$.

Below (Theorem B) we classify the possible attractors that appears in Theorem A. For that, recall that a cycle of intervals is a transitive finite union of non-trivial closed intervals. This is a common type of attractor for one-dimensional maps. Indeed, the support of any ergodic absolutely continuous invariant measure is always a cycle of intervals.

Theorem B ([BPP17]). *Let $f : [0,1] \to [0,1]$ be a non-flat C^2 local diffeomorphism in the whole interval, except for a finite set $\mathcal{C}_f \subset (0,1)$, and let $\mathcal{V}_f = \{f(c_\pm)\,;\, c \in \mathcal{C}_f\}$. If A_j is one of the attractors given by Theorem A then A_j is either a cycle of intervals or a Cantor set of the form $A_j = \bigcup_{v \in \mathbb{V}_j} \omega_f(v)$ for some $\mathbb{V}_j \subset \mathcal{V}_f \setminus \mathcal{O}_f^-(\mathcal{C}_f)$ with $v \in \omega_f(v)$ $\forall\, v \in \mathbb{V}_j$.*

We say that the orbit $\mathcal{O}_f^+(x)$ of a point $x \in \beta_f(A)$ is ***asymptotically inaccessible*** if given any $p \notin A$ there is a $n_0 \geq 0$ such that $\{(1-t)p + tf^n(x)\,;\, t \in (0,1)\}$ contains a point of A for every $n \geq n_0$.

An attracting Cantor set A has an ***asymptotically inaccessible basin of attraction*** if the forward orbit of almost every point in $\beta_f(A)$ is asymptotically inaccessible. This is equivalent to say that $\#(\mathcal{O}_f^+(x) \cap J) < \infty$ for almost every $x \in \beta_f(A)$ and every gap J of A.

Theorem C ([BPP19]). *Let f be a non-flat piecewise C^2 map and A an attracting Cantor set of f. For almost every point $x \in \beta_f(A)$, either x belongs to a wandering interval or the forward orbit of x is asymptotically inaccessible.*

As a non-flat C^2 maps of the interval do not admit wandering intervals, we get the following corollary.

Corollary. *The basin of attraction of any attracting Cantor set for a non-flat C^2 map is always asymptotically inaccessible.*

Theorem D ([BPP]). *Let f be a C^3 contracting Lorenz map $f : [0,1] \setminus \{c\} \to [0,1]$, $c \in (0,1)$, with negative Schwarzian derivative. If f does not have periodic-like attractors, then f has an attractor A such that $\omega_f(x) = A$ for almost every $x \in [0,1]$. In particular, $\text{Leb}(\beta_f(A)) = 1$.*

Furthermore, f can have at most two periodic-like attractors. If f has a single periodic attractor, its basin of attraction has full Lebesgue measure. In the case that f has two periodic-like attractors, the union of their basins of attraction has full Lebesgue measure.

If f does not have periodic-like attractors, then A is either a cycle of intervals or a transitive Cantor set.

If A is a Cantor set, then $A = \omega_f(c_-)$ or $\omega_f(c_+)$. Moreover, if f is non-flat and A is a Cantor set, then c_- and $c_+ \in A = \omega_f(c_-) = \omega_f(c_+)$.

Referências

[BPP17] Brandão, P.; Palis, J.; Pinheiro, V..*On the finiteness of attractors for piecewise maps of the interval.* ETDS, v1, 2017.

[BPP19] Brandão, P.; Palis, J.; Pinheiro, V..*On the Statistical Attractors and Attracting Cantor Sets for Piecewise Smooth Maps.* Springer Proc. in Math. & Stat.. Springer Int. Pub., v285, 2019.

[BPP] Brandão, P.; Palis, J.; Pinheiro, V..*On the Finiteness of Attractors for One-Dimensional Maps with Discontinuities* (Preprint).