The Palis and Viana Conjectures

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Abstract

We discuss conjectures by Palis and by Viana which have played a central role in the development of deep results in dynamics over the last several decades, and mention some recent results.

Physical measures and the Palis conjecture

Let M be a Riemannian manifold, m be the normalized Lebesgue measure on M, and $f : M \to M$ be a map. The fundamental problem of the theory of Dynamical Systems can be defined as the description of the orbits $\mathcal{O}^+ := \{x_i\}_{i=0}^{\infty}$ of points $x_0 \in M$, where $x_i := f^i(x_0)$. Describing such orbits can of course mean many things, including their geometrical or topological properties, how they vary with the initial condition x_0 and sometimes even how they vary under perturbations of the map f. Of particular interest is the statistical description of orbits through the sequence of probability measures

$$\mu_n(x) := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{x_i}.$$

For each $n \geq 1$, the measure μ_n is uniformly distributed on the first n points of the orbit of x. If the sequence $\mu_n(x)$ converges to some probability measure μ then we can say that μ describes the asymptotic statistics of the orbit of x, since for n large, $\mu_n(x) \approx \mu$ and thus the distribution of the first n points of the orbit is well approximated by μ. Of particular interest is the situation where this sequence converges to the same measure μ for "many" initial conditions. This motivates the following definitions.

Definition 1. *The **basin** of a probability measure μ is the set*

$$\mathcal{B}_\mu := \{x \in M : \mu_n(x) \to \mu\}$$

*and we say that μ is a **physical measure** if $m(\mathcal{B}_\mu) > 0$.*

There are many examples of physical measures. If f is a contraction mapping then the measure δ_p on the fixed point is a physical measure. If μ is invariant and ergodic then Birkhoff's Ergodic Theorem implies that $\mu(\mathcal{B}_\mu) = 1$ and thus if $\mu \ll m$ then this implies $m(\mathcal{B}_\mu) > 0$ and so μ is a physical measure. Ergodic Sinai-Ruelle-Bowen measures are a particularly important class of measures introduced in the 1970's, which have been proved to be physical despite not being absolutely continuous with respect to Lebesgue. Not all dynamical systems, however, have physical measures. Their existence may fail in such simple cases as the identity map or in more sophisticated examples where the sequences $\mu_n(x)$ fail to converge t for Lebesgue a.e. x. Moreover there are examples, such as for Newhouse diffeomorphisms, which have infinitely many physical measures.

Palis Conjecture. *For **most** dynamical systems there exist (a finite number of) probability measures*

$$\mu_1, \ldots, \mu_\ell \quad \textit{such that} \quad m\left(\bigcup_{i=1}^{\ell} \mu_i\right) = 1 \qquad (1)$$

There are of course several ways to define the notion of "most" dynamical systems. A natural formulation is in terms of full Lebesgue measure for generic one-parameter families, but we refer to [3] for a discussion and precise formulation.

Hyperbolicity and the Viana Conjecture

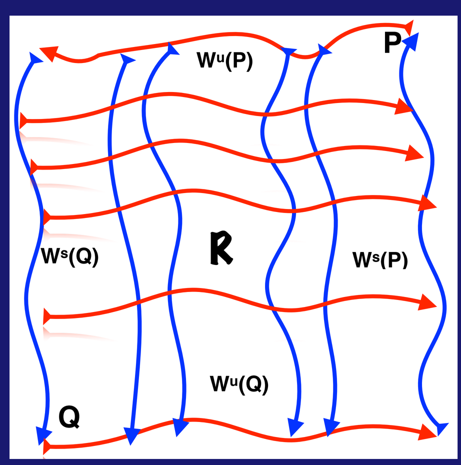
A natural approach to the Palis Conjecture is to determine as large as possible a class of systems which satisfy (1), and a first step in this direction is to determine as large as possible a class of systems which at least admit some physical measure. This is one of the motivation of the following conjecture formulated in [4]

Viana Conjecture. *If f has **non-zero Lyapunov exponents** then it admits a physical measure.*

The notion of "non-zero Lyapunov exponents" is used here to mean that f is **hyperbolic**, in some sense, and this has many possible subtle but significant variations. In the strongest form, that of **Uniform Hyperbolicity**, the conjecture was proved by Sinai, Ruelle, and Bowen, in the 1970s (they essentially proved (1) in this setting). Since then there have been a string of results for increasingly more general forms of hyperbolicity, and quite recently two papers [1, 2] which arguably come quite close to giving a full positive answer to Viana's conjecture. The paper [1], coauthored with V. Climehaga and Y. Pasin, gives a very concrete geometric construction of a "Young Tower" under some very general hyperbolicity assumptions, albeit in the restricted setting of two-dimensional manifolds, while [2] gives a quite different and more abstract construction of "Markov partitions" which however works in arbitrary dimension. In both cases the existence of physical measures can then be deduced under suitable recurrence conditions.

Young Towers for surface diffeomorphisms

We assume f is a C^2 diffeomorphism of a compact surface M, and Λ is a *hyperbolic set* (in an extremely general sense, see [1]). By classical theory, every point $x \in \Lambda$ admits local stable and unstable manifolds V^s_x, V^u_x which, in this generality, do not have a size which is uniformly bounded below.



A subset $\Gamma \subseteq \Lambda$ is a "rectangle" if it has local product structure, i.e. for every $x, y \in \Gamma$ the intersection $V^s_x \cap V^u_y$ is a single point which also belongs to Λ. Rectangles are extremely natural geometric structures which exist inside any non-degenerate hyperbolic set.

Definition 2. *A rectangle Γ is:*

1. ***Fat** if $m(\bigcup_{x\in\Gamma} V^s_x 0) > 0$*
2. ***Recurrent** if every $x \in \Gamma$ returns to Γ with positive frequency*
3. ***Nice** if it is contained in a region $\hat{\Gamma}$ bounded by the local stable and unstable manifolds of $p, q \in \Gamma$ such that for all $x \in \Gamma$, $n \geq 0$,*

$$f^n(V^s_x) \cap int(\hat{\Gamma}) = \emptyset \quad \textit{and} \quad f^{-n}(V^u_x) \cap int(\hat{\Gamma}) = \emptyset.$$

Theorem ([1]). *f admits an SRB physical measure if and only if it admits a fat, recurrent, nice rectangle.*

We note that the existence of such a rectangle given an SRB measure is not completely trivial but relatively unsuprising. The main result is that such a simple and natural geometric structure is sufficient fort the existence of an SRB measure.

Referências

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