Lecture 2: Stability on triangulated categories

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Triangulated categories

Recall that a *triangulated category* $\mathscr{T}$ is an additive category (ie. Hom sets are abelian groups) equipped with a shift functor $A \mapsto A[1]$ and a collection of *distinguished triangles*

$$A \to B \to C \to A[1]$$

satisfying various axioms that essentially make distinguished triangles look like short exact sequences in abelian categories.

Main example: categories of complexes, homotopic categories, derived categories of abelian categories.

t-structures and hearts

A *t-structure* on $\mathscr{T}$ is a pair $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ of subcategories of $\mathscr{T}$ such that

- $X \in \mathscr{D}^{\leq 0} \implies X[1] \in \mathscr{D}^{\leq 0}$ and
  $Y \in \mathscr{D}^{\leq 0} \implies Y[-1] \in \mathscr{D}^{\leq 0}$;
- $X \in \mathscr{D}^{\leq 0}, Y \in \mathscr{D}^{\leq 0} \implies \mathrm{Hom}(X, Y) = 0$;
- for any $A \in \mathscr{T}$, there are $X \in \mathscr{D}^{\leq 0}$ and $Y \in \mathscr{D}^{\leq 0}$ and a triangle $X \to A \to Y \to X[1]$.

The *heart* $\mathscr{D}^{\heartsuit}$ of $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ is the full subcategory of $\mathscr{T}$ given by $\mathscr{D}^{\leq 0} \cap \mathscr{D}^{\geq 0}$.

Theorem

$\mathscr{D}^{\heartsuit}$ is an abelian category.

Basic example: the standard t-structure

Let $\mathcal{A}$ be an abelian category, and $D^*(\mathcal{A})$ be its derived category. Take:

$$\mathscr{D}^{\leq 0} := \{E \in D^*(\mathcal{A}) \mid \mathcal{H}^p(E) = 0 \text{ for } p > 0\}$$
$$\mathscr{D}^{\geq 0} := \{E \in D^*(\mathcal{A}) \mid \mathcal{H}^p(E) = 0 \text{ for } p < 0\}$$

All the axioms are satisfied and

$$\mathscr{D}^{\heartsuit} = \{E \in D^*(\mathcal{A}) \mid \mathcal{H}^p(E) = 0 \text{ for } p \neq 0\} = \mathcal{A}$$

as a subcategory of $D^*(\mathcal{A})$.

Stability conditions on triangulated categories: ingredients

A *stability condition* on a triangulated category $\mathscr{T}$ is a pair $\sigma = (\mathcal{A}, Z)$ consisting of the heart $\mathcal{A}$ of a t-structure on $\mathscr{T}$ and a central charge $Z : K_0(\mathcal{A}) \to \mathbb{C}$ on $\mathcal{A}$.

In addition, we assume that Z factors through a surjective group homomorphism $\tau : K_0(\mathcal{A}) \twoheadrightarrow \Lambda$, where Λ is a finite dimensional lattice equipped with a norm $||\cdot||$ on $\Lambda \otimes \mathbb{R}$.

(1) $\mathbf{Im}(Z(A)) \geq 0$ for every $A \in \mathcal{A}$, and
$\mathbf{Im}(Z(A)) \geq 0 \implies \mathbf{Re}(Z(A)) < 0$; set

$$\mu_Z(A) := -\frac{\mathbf{Re}(Z(A))}{\mathbf{Im}(Z(A))}$$

(2) Every $E \in \mathcal{A}$ admits a Harder–Narasimhan filtration;

(3) the *support property*

$$\inf\left\{ \frac{|Z(E)|}{||\tau(E)||} \mid E \in \mathcal{A} \text{ semistable} \right\} > 0$$

Basic example: sheaves on curves

Let X be a smooth projective <u>curve</u>, and let $\mathscr{T} = D^{\mathrm{b}}(X)$.

Take $\mathcal{A} = \mathcal{C}oh(X)$ as the heart of the standard t-structure on $D^{\mathrm{b}}(X)$.

Set $\Lambda = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z})$, so that γ is just the Chern character map; $||\cdot||$ can be the usual euclidian norm.

Set $Z(E) := -\deg(E) + \sqrt{-1}\,\mathrm{rk}(E)$.

The support property is trivially satisfied because $||\gamma(E)|| > 1$ for every sheaf E.

Revisiting the axioms: weak stability conditions

(1') $\mathbf{Im}(Z(A)) \geq 0$ for every $A \in \mathcal{A}$, and
$\mathbf{Im}(Z(A)) \geq 0 \implies \mathbf{Re}(Z(A)) \leq 0$

We say that $\sigma = (\mathcal{A}, Z)$ is a *weak stability condition* if it satisfies (1') instead of (1).

Example: usual μ-stability on $\mathcal{C}oh(X)$ when $\dim X \geq 2$.

Revisiting the axioms: support property

(3') There is a symmetric bilinear form Q on $\Lambda \otimes \mathbb{R}$ such that

(i) If $E \in \mathcal{A}$ is semistable, then $Q(\gamma(E), \gamma(E)) \geq 0$;

(ii) For a non zero $v \in \Lambda \otimes \mathbb{R}$ with $Z(v) = 0$, then $Q(\gamma(E), \gamma(E)) < 0$.

In fact, (3) $\iff$ (3').

The inequality in (3'.i) plays the role of a *Bogomolov inequality*.

It is possible to have a weak stability condition that satisfies (3'), eg. usual μ-stability on $\mathcal{C}oh(X)$ when $\dim X \geq 2$.

Examples for higher dimensional varieties

Let X be a projective variety for which there are a quiver Q and an equivalence of triangulated categories $D^{\mathrm{b}}(X) \simeq D^{\mathrm{b}}(Q)$; eg. $X = \mathbb{P}^n$, also smooth quadrics, etc.

The standard t-structure on $D^{\mathrm{b}}(Q)$ induces a (non standard) t-structure on $D^{\mathrm{b}}(X)$.

Since $K_0(X) \simeq K_0(Q)$, one can use a central charge on $\mathrm{Rep}(Q)$ to induce a central charge on the heart on $D^{\mathrm{b}}(Q)$ pulled back from $D^{\mathrm{b}}(X)$.

Bridgeland's deformation theorem

Let $\mathrm{Stab}(\mathscr{T})$ be the set of stability conditions on the triangulated category $\mathscr{T}$. Consider the coarsest topology in $\mathrm{Stab}(\mathscr{T})$ for which the maps

$$\sigma : (\mathcal{A}, Z) \mapsto \begin{cases} Z \in \mathrm{Hom}(\Lambda, \mathbb{C}) \\ \mu_Z^+(E) \in \mathbb{R} \text{ for each } E \in \mathcal{A} \\ \mu_Z^-(E) \in \mathbb{R} \text{ for each } E \in \mathcal{A} \end{cases}$$

are continuous.

Theorem

The map $\mathscr{Z} : \mathrm{Stab}(\mathscr{T}) \to \mathrm{Hom}(\Lambda, \mathbb{C})$ *is a local homeomorphism. In particular,* $\mathrm{Stab}(\mathscr{T})$ *is a complex manifold of complex dimension* $\mathrm{rk}(\Lambda)$.

Two groups act naturally on Stab($\mathscr{T}$):

- $\mathrm{Aut}(\mathscr{T})$, auto-equivalences of $\mathscr{T}$,

$$\Phi \cdot (\mathcal{A}, Z) \mapsto (\Phi(\mathcal{A}), Z \circ \Phi_*)$$

where $\Phi_* : K_0(\mathscr{T}) \to K_0(\mathscr{T})$ is the induced group homomorphism.

- $\widetilde{GL^+(2,\mathbb{R})}$ universal cover of 2×2 matrices with positive determinant.

$$T \cdot (\mathcal{A}, Z) \mapsto (\mathcal{A}, T^{-1} \circ Z)$$

(full action is harder to summarize...)

Stab(X) for curves

Stab(X) has been fully described when X is a smooth projective curve:

- $\mathrm{Stab}(\mathbb{P}^1) = \mathbb{C}^2$;
- when $g(X) > 0$, $\mathrm{Stab}(X) = \mathbb{H} \times \mathbb{C} = \widetilde{GL^+(2,\mathbb{R})} \cdot \sigma_0$.

Here, σ_0 is just the usual μ-stability on the standard t-structure.

Numerical walls

How the moduli set of semistable objects varies when the stability condition changes?

Given vectors $u, v \in \Lambda$, a *numerical wall* for u and v is the set

$$\Upsilon_{u,v} := \{\sigma \in \mathrm{Stab}(X) \mid \mu_Z(u) = \mu_Z(v)\}.$$

This is a real submanifold of $\mathrm{Stab}(X)$ of real codimension 1.

$\Upsilon_{u,v}$ may not be connected or irreducible, and may have components of higher codimension.

Actual walls

Let σ_t be a path in $\mathrm{Stab}(X)$ where $t \in (-\epsilon, \epsilon)$; assume that σ_0 lies in a numerical wall $\Upsilon_{u,v}$.

Take an object $E \in \mathcal{A}_t$ for each t with $\gamma(E) = v$. Nothing may happen with E as one moves along σ_t and crosses the numerical wall:

- there may not exist objects $F \in \mathcal{A}_0$ with $\gamma(F) = u$;
- if such objects exist, there might not be a monomorphism $F \in E$ in $\mathcal{A}_0$.

An *actual wall* for v is the subset W_u, v of $\Upsilon_{u,v}$ consisting of those $\sigma \in \Upsilon_{u,v}$ such that there are objects $E, F \in \mathcal{A}_\sigma$ with $\gamma(F) = u$ and $\gamma(E) = v$, and a monomorphism $F \hookrightarrow E$ such that both F and E/F are σ-semistable.

Local finiteness

Bridgeland proved that $\mathrm{Stab}(X)$ has a reasonable wall and chamber structure, at least locally.

Theorem

Fix $v \in \Lambda$ and a compact set $K \subset \mathrm{Stab}(X)$.
There are only finitely many actual walls $\{W_{u_i,v}\}_{i=1}^{n}$ for v intersecting K, each of real codimension 1, and any connected component

$$C \subset K \setminus \bigcup_{i=1}^{n} W_{u_i,v}$$

has the following property: if E is σ-semistable for some $\sigma \in C$, then E is σ'-semistable for every $\sigma' \in C$.

✓ The moduli set $\mathcal{M}_\sigma(v)$ is constant in each chamber C.

See you tomorrow!