Lecture 1: Stability on abelian categories

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Stability conditions

Let $\mathcal{A}$ be an abelian category; my personal favorites are $\mathcal{C}oh(X)$ and $\text{Rep}(Q)$.

A *stability condition* condition on $\mathcal{A}$ is the choice of a total pre-order $\preceq$ on the class of non zero objects of $\mathcal{A}$ such that, for every short exact sequence $0 \to A \to B \to C \to 0$ on $\mathcal{A}$ satisfies the *seesaw property*:

$$\begin{aligned} \text{either } A \preceq B &\iff A \preceq C \iff B \preceq C \\ \text{or } A \succeq B &\iff A \succeq C \iff B \succeq C \\ \text{or } A \asymp B &\iff A \asymp C \iff B \asymp C \end{aligned}$$

An nontrivial object A is (semi)stable if every nontrivial, proper sub-object $B \hookrightarrow A$ satisfies $B \prec (\preceq)\ A$.

Trivial example

Example

Any two non zero objects A and B satisfy $A \asymp B$. In this case, every object is semistable, and an object is stable if and only if it is simple (ie. it has no sub-objects).

In general, one can show that

$$\text{simple} \implies \text{stable} \implies \text{indecomposable}$$

Semistable objects might be decomposable.

Example: Mumford–Takemoto stability

Let X be a smooth projective curve, and set $\mathcal{A} = \mathcal{C}oh(X)$, the category of coherent sheaves on X.
The slope of a sheaf on X is defined as follows:

$$\mu(E) := \begin{cases} \deg(E)/\,\mathrm{rk}(E) & \text{if } \mathrm{rk}(E) \neq 0 \\ +\infty & \text{otherwise} \end{cases}$$

We then declare that $F \preceq E$ when $\mu(F) \leq \mu(E)$.

Not hard to check that the seesaw property is satisfied.

Example: King stability for quivers

Let $Q = (Q_0, Q_1)$ be a quiver, and set $\mathcal{A} = \mathrm{Rep}(Q)$, the category of linear representations of Q.

Let $n = \#Q_0$ and choose vectors $\theta \in \mathbb{Z}^n$ and $\alpha \in \mathbb{Z}^n_+$; define the slope of a representation R as follows:

$$\mu(R) = \frac{\theta \cdot \dim R}{\alpha \cdot \dim R}.$$

Again, we declare that $S \preceq R$ when $\mu(S) \leq \mu(R)$.

Just as in the previous example, one can also show that the seesaw property is satisfied.

Elementary properties

Pretty much all of the basic familiar properties of μ-stability for sheaves will generalize to arbitrary stability conditions on abelian categories.

For instance, let A and B be semistable objects with if $B \preceq A$. If $\phi : A \to B$ is a nonzero morphism, then:

- $A \asymp B$;
- B stable $\implies \phi$ is an epimorphism;
- A stable $\implies \phi$ is an monomorphism;
- A and B stable $\implies \phi$ is an isomorphism.

Moreover, assume that $\mathrm{Hom}(A, B)$ is always a finite dimensional vector space over an algebraically closed field κ. Then every stable object is a Schur object, i.e. $\mathrm{Hom}(A, A) = \kappa$.

Weakly artinian, weakly noetherian

Let $(\mathcal{A}, \preceq)$ be an abelian category equipped with a stability condition.

$(\mathcal{A}, \preceq)$ is *weakly artinian* if every descending chain

$$\cdots \hookrightarrow A_2 \hookrightarrow A_1 \hookrightarrow A_0 = B$$

of sub-objects of an object B satisfying $A_i \preceq A_{i+1}$ has to stabilize.

$(\mathcal{A}, \preceq)$ is *weakly noetherian* if every ascending chain

$$A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow B$$

of sub-objects of an object B satisfying both $A_i \succeq A_{i+1}$ and $A_i \preceq A_{i+1}$ has to stabilize.

Harder–Narasimhan filtrations

Theorem

If $(\mathcal{A}, \preceq)$ is weakly artinian and weakly noetherian, then every object B admits a unique filtration

$$0 = B_0 \hookrightarrow B_1 \hookrightarrow \cdots \hookrightarrow B_n = B$$

such that

1. *each factor $G_i := B_i / B_{i-1}$ is semistable;*
2. $G_1 \succ G_2 \succ \cdots \succ G_n$.

Theorem

If $(\mathcal{A}, \preceq)$ *is weakly artinian and weakly noetherian, then every semistable object* B *admits a filtration*

$$0 = B_0 \hookrightarrow B_1 \hookrightarrow \cdots \hookrightarrow B_n = B$$

such that

1. *each factor* $G_i := B_i / B_{i-1}$ *is stable;*
2. $G_1 \asymp G_2 \asymp \cdots \asymp G_n$.

The Jordan–Holder filtration is not unique, but the associated graded object $\bigoplus_{i=1}^{n} G_i$ is.

Numerical stability conditions, I

Let $Z : K_0(\mathcal{A}) \to \mathbb{C}$ be an additive group homomorphism such that $\mathbf{Im}(Z(A)) \geq 0$ for every object A, and $\mathbf{Re}(Z(A)) < 0$ whenever $\mathbf{Im}(Z(A)) = 0$.

This is usually called a *central charge* for the category $\mathcal{A}$. Set

$$\mu_Z(A) := -\frac{\mathbf{Re}(Z(A))}{\mathbf{Im}(Z(A))}$$

the *slope* or *phase* associated to the central charge Z; division by 0 is defined to be $+\infty$, as in the case of usual μ-stability described above.

It induces, as in the two examples discussed above, a stability condition on $\mathcal{A}$

$$A \preceq B \iff \mu_Z(A) \leq \mu_Z(B).$$

Numerical stability conditions, II

For the usual μ-stability on $\mathcal{C}oh(X)$, one would take

$$Z(E) := -\deg(E) + \sqrt{-1}\,\mathrm{rk}(E)$$

However, this is not quite a central charge as defined above when $\dim X \geq 2$, since $Z(\mathcal{O}_p) = 0$.

Therefore, it is useful to consider a *weaker* version:
$Z : K_0(\mathcal{A}) \to \mathbb{C}$ is an additive group homomorphism such that $\mathbf{Re}(Z(A)) \leq 0$ whenever $\mathbf{Im}(Z(A)) = 0$.

More generally, one may consider central charges taking values on a totally ordered $\mathbb{R}$-vector space Δ such that for $a \in \mathbb{R}$ and $v \in \Delta$, then $a > 0,\ v > 0 \implies a \cdot v > 0,\ \ -v < 0$.

Polynomial stability conditions

As an example, take $P : K_0(\mathcal{A}) \to \mathbb{R}[t]$ such that for any object A, the polynomial $P_A(t)$ has positive leading coefficient.

For instance, $P_E(t)$ can be the Hilbert polynomial of a sheaf E on a polarized projective variety.

Given $P_A(t) = \sum_{k=1}^{n} a_k t^k$ and $P_b(t) = \sum_{k=1}^{n} b_k t^k$, let
$\Lambda(A, B) := (\lambda_{m,m-1}, \ldots, \lambda_{m,0}, \lambda_{m-1,m-2}, \ldots, \lambda_{1,0})$
be the 2×2 minors of the matrix of coefficients

$$\begin{pmatrix} a_m & \cdots & a_0 \\ b_m & \cdots & b_0 \end{pmatrix}$$

Define: $A \preceq B \iff \Lambda(A, B) \geq_{\text{lex}} 0$

- Usual Gieseker stability for sheaves on a polarized projective variety!

S-equivalence

Two semistable objects are *S-equivalent* when the associated graded objects coming from their Jordan–Holder filtrations are isomorphic.

Clearly, two stable objects are S-equivalent if and only if they are isomorphic.

Moduli sets of stable objects

Fix a class $\gamma \in K_0(\mathcal{A})$ and consider the sets

$$\mathcal{A}(\gamma)^{sst} := \{A \in \mathcal{A} \mid [A] = \gamma,\ A \text{ semistable}\}$$
$$\text{and } \mathcal{M}(\gamma) := \mathcal{A}(\gamma)^{sst}/\text{S-eqv}$$

- Does the moduli set $\mathcal{M}(\gamma)$ admits some kind of geometric structure (stack, scheme, projective variety)?

Sheaves, quivers: $\mathcal{A}(\gamma)$ can be given the structure of a projective variety, and S-equivalence is translated into a group action; then the Hilbert-Mumford criterion in GIT can be translated into a numerical stability condition in $\mathcal{A}$!!

More on Joyce's series *Configurations in abelian categories, I-IV*.

See you tomorrow!