Lecture 3: Bridgeland stability on surfaces

Marcos Jardim

UFF, Niterói
06 February 2020

Torsion pairs

Let $\mathcal{A}$ be an abelian category.

A *torsion pair* $(\mathcal{F}, \mathcal{T})$ on $\mathcal{A}$ consists of two full additive subcategories such that

- $\mathrm{Hom}(T, F) = 0$ whenever $T \in \mathcal{T}$ and $F \in \mathcal{F}$;
- for each $E \in \mathcal{A}$, there are $T \in \mathcal{T}$ and $F \in \mathcal{F}$ such that

$$0 \to T \to E \to F \to 0.$$

- The exact sequence is unique.

Basic example: for $\mathcal{A} = \mathcal{C}oh(X)$, take

$$\mathcal{T} = \{\text{torsion sheaves}\} \quad \text{and} \quad \mathcal{F} = \{\text{torsion free sheaves}\}$$

Tilting on a torsion pair

Theorem

Let $\mathcal{A} \subset \mathscr{T}$ be the heart of a t-structure. If $(\mathcal{F}, \mathcal{T})$ is a torsion pair on $\mathcal{A}$, then the full subcategory of $\mathscr{T}$ given by

$$\mathcal{A}^{\#} := \left\{ E \in \mathscr{T} \;\middle|\; \begin{array}{c} \mathcal{H}^{p}_{\mathcal{A}}(E) = 0, \ p \neq -1, 0 \\ \mathcal{H}^{-1}_{\mathcal{A}}(E) \in \mathcal{F} \\ \mathcal{H}^{0}_{\mathcal{A}}(E) \in \mathcal{T} \end{array} \right\}$$

is also the heart of a t-structure on $\mathscr{T}$.

- $\mathcal{A}^{\#} = \langle \mathcal{F}[1], \mathcal{T} \rangle$.

Stability conditions for surfaces: construction, I

Let X be a smooth projective variety over $\mathbb{C}$

Fix an ample divisor ω and a divisor B, both in $\mathrm{NS}(X) \otimes \mathbb{R}$.

Given a sheaf E on X, defined

$$\mu_{\omega,B}(E) := \begin{cases} \dfrac{\omega \cdot \mathrm{ch}_1(E)}{\mathrm{ch}_0(E)} - \dfrac{\omega \cdot B}{\omega^2} & \text{if } \mathrm{ch}_0(E) \neq 0 \\ \\ +\infty, & \text{otherwise} \end{cases}$$

and consider the following subcategories of $\mathcal{C}oh(X)$

$$\mathcal{T}_{\omega,B} := \{E \in \mathcal{C}oh(X) \mid \forall E \twoheadrightarrow G,\ \mu_{\omega,B}(G) > 0\}$$

$$\mathcal{F}_{\omega,B} := \{E \in \mathcal{C}oh(X) \mid \forall F \hookrightarrow E,\ \mu_{\omega,B}(F) \leq 0\}$$

One can check that $(\mathcal{F}_{\omega,B}, \mathcal{T}_{\omega,B})$ is a torsion pair in $\mathcal{C}oh(X)$.

So we set $\mathcal{C}oh_{\omega,B}(X) := \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B} \rangle$. Take $\Lambda := K_{\mathrm{num}}(X)$, and $\gamma : K_0(X) \twoheadrightarrow K_{\mathrm{num}}(X)$ to be the Chern character map.

The final ingredient is the central charge on $\mathcal{C}oh_{\omega,B}(X)$:

$$Z_{\omega,B} := \left(\mathrm{ch}_2^B(E) - \omega^2\,\mathrm{ch}_0(E)/2\right) + \sqrt{-1}\left(\omega \cdot \mathrm{ch}_1^B(E)\right)$$

where we use the *twisted Chern character* $\mathrm{ch}^B(E) = e^{-B} \cdot \mathrm{ch}(E)$:

$$\mathrm{ch}_1^B(E) = \mathrm{ch}_1(E) - \mathrm{ch}_0(E) \cdot B \ \text{ and}$$
$$\mathrm{ch}_2^B(E) = \mathrm{ch}_2(E) - \mathrm{ch}_1(E) \cdot B + B^2\,\mathrm{ch}_0(E)/2.$$

Assume $\dim X = 2$.

Theorem

The pair $(\mathcal{C}oh_{\omega,B}(X), Z_{\omega,B})$ *is a Bridgeland stability condition on* $D^{\mathrm{b}}(X)$*, and the map*

$$\mathrm{Amp}(X) \times NS(X) \otimes \mathbb{R} \longrightarrow \mathrm{Stab}(X)$$

$$(\omega, B) \mapsto (\mathcal{C}oh_{\omega,B}(X), Z_{\omega,B})$$

is a continuous embedding.

- Main ingredients: Harder–Narasimhan filtration for μ-stability in $\mathcal{C}oh(X)$; the Bogomolov inequality for μ-semistable sheaves.

Stability conditions in the upper half plane

Fixing $(\omega, B) \in \mathrm{Amp}(X) \times NS(X) \otimes \mathbb{R}$ primitive, consider the family of stability conditions induced by $(\alpha \cdot \omega, \alpha\beta \cdot B)$ where $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}^{+} = \mathbb{H}$.

This gives a copy of the upper half plane $\mathbb{H}$ embedded in Stab(X).

Note that $\mu_{\alpha\cdot\omega,\alpha\beta\cdot B} = \alpha \cdot \mu_{\omega,\beta\cdot B}$, so the heart $\mathcal{A}_{\alpha\cdot\omega,\alpha\beta\cdot B}$ does not depend on α.

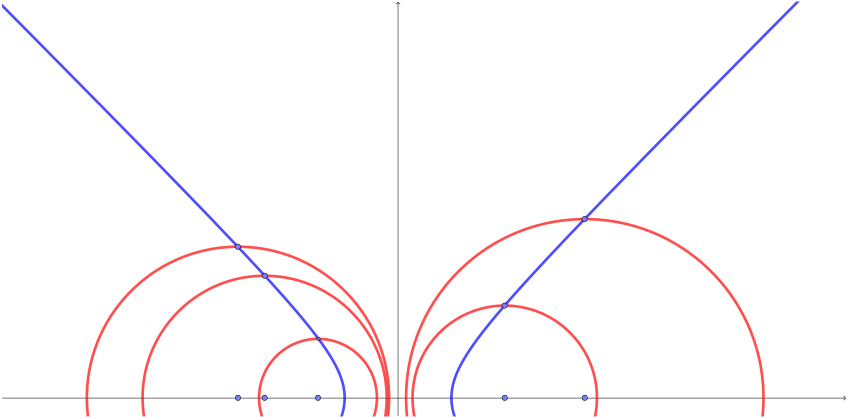
- Notation: $\sigma_{\alpha,\beta} = (\mathcal{C}oh^{\beta}(X), Z_{\alpha,\beta})$.

Take $v \in K_{\mathrm{num}}(X)$ for which there is a μ_ω-semistable sheaf E with $\mathrm{ch}(E) = v$.

Theorem

1. *Numerical walls for v in $\mathbb{H}$ are either semicircles centered in the β-axis, or the vertical line $\beta = \mu(v)$.*
2. *Distinct numerical walls for v do not intersect one another.*
3. *The top point of a non vertical numerical wall for v lie along the hyperbola $\mathbf{Re}(Z_{\alpha,\beta}(v)) = 0$.*
4. *If a single point of a numerical wall for v is actual, then the whole numerical wall is actual.*
5. *Then there is an actual wall for v which contains in its interior every other actual wall for v.*

Picturing walls in $\mathbb{H}$



Twisted Gieseker stability

Twisted Hilbert polynomial (cf. Riemann–Roch, $n = \dim X$):

$$P_{\omega,B}(E,t) := \int_X \mathrm{ch}^B(E) \cdot e^{t\omega} \cdot \mathrm{td}(X) = \sum_{k=0}^{n} a_k(E,B,\omega) t^k$$

We say that E is B-twisted Gieseser semistable w.r.t. ω if every subsheaf $F \hookrightarrow E$ satisfies

$$\frac{P_{\omega,B}(F,t)}{a_n(F,B,\omega)} < (\leq)\ \frac{P_{\omega,B}(E,t)}{a_n(E,B,\omega)} \text{ for } t \gg 0.$$

The large volume limit

Theorem

- *Let $E \in D^{\mathrm{b}}(X)$ be an object satisfying $\mathrm{ch}_1^B(E) \cdot \omega > 0$. E is a B-twisted Gieseker semistable sheaf if and only if there is an $\alpha_0 > 0$ such that E is $\sigma_{\alpha,\beta}$-semistable for every $\alpha > \alpha_0$.*
- *Let $E \in D^{\mathrm{b}}(X)$ be an object satisfying $\mathrm{ch}_1^B(E) \cdot \omega < 0$. E is a B-twisted Gieseker semistable sheaf if and only if there is an $\alpha_0 > 0$ such that $E^\vee[1]$ is $\sigma_{\alpha,\beta}$-semistable for every $\alpha > \alpha_0$.*

- There is a *Gieseker chamber* beyond the largest actual wall in the upper half plane $\mathbb{H}$.

When $\dim X \geq 3$, the pair $(\mathcal{C}oh_{\omega,B}(X), Z_{\omega,B})$ is a weak stability condition on $D^{\mathrm{b}}(X)$ satisfying the support property.

However, the *large volume limit* is a little different, considering a *truncated* version of the twisted Poincaré polynomial (ignoring coefficients of t^k for $k < \dim X - 2$).

Moduli spaces

After the construction of $\mathrm{Stab}(X)$, studying the moduli spaces of Bridgeland semistable is the main motivation to study stability conditions.

Given $\sigma = (\mathcal{A}, Z)$, let $\widehat{\mathcal{M}}_\sigma(v)$ be set of σ-semistable objects E with $\mathrm{ch}(E) = v$, and $\mathcal{M}_\sigma(v)$ be set of S-equivalence classes.

When σ is a (weak) stability condition constructed via tilting as above, then $\widehat{\mathcal{M}}_\sigma(v)$ is an algebraic stack of finite type over $\mathbb{C}$ [Toda, Piyaratne–Toda].

For certain surfaces (like $K3$, $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$) one can go further and show that $\mathcal{M}_\sigma(v)$ is a projective scheme.

See you tomorrow!