Lecture 4: Bridgeland stability on 3-folds

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Set up

X = smooth projective variety with $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$, with H ample.

Set $\omega = \alpha H$ and $B = \beta H$ with $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$, and consider the family of <u>weak</u> stability conditions $(\mathcal{B}^\beta, Z^{\mathrm{tilt}}_{\alpha,\beta})$ on $D^{\mathrm{b}}(X)$ constructed in the previous lecture:

$$\mathcal{B}^\beta(X) := \langle \mathcal{F}_\beta[1], \mathcal{T}_\beta \rangle$$

$$\mathcal{T}_\beta := \{E \in \mathcal{C}oh(X) \mid \forall\ E \twoheadrightarrow G \text{ satisfies } \mu_\beta(G) > 0\}, \text{ and}$$

$$\mathcal{F}_\beta := \{E \in \mathcal{C}oh(X) \mid \forall\ F \hookrightarrow E \text{ satisfies } \mu_\beta(F) \leq 0\}.$$

$$Z^{\mathrm{tilt}}_{\alpha,\beta}(B) := -\left(\mathrm{ch}^\beta_2(B) - \frac{1}{2}\alpha^2 \mathrm{ch}_0(B)\right) + \sqrt{-1}\, \mathrm{ch}^\beta_1(B)$$

for $B \in \mathcal{B}^\beta$

The idea of Bayer–Macri–Toda is to tilt $\mathcal{B}^\beta$ on the torsion pair

$$\mathcal{T}_{\alpha,\beta} := \{E \in \mathcal{B}^\beta(X) \mid \forall\ E \twoheadrightarrow G \text{ satisfies } \nu_{\alpha,\beta}(G) > 0\}, \text{ and}$$

$$\mathcal{F}_{\alpha,\beta} := \{E \in \mathcal{B}^\beta(X) \mid \forall\ F \hookrightarrow E \text{ satisfies } \nu_{\alpha,\beta}(F) \leq 0\}.$$

where $\nu_{\alpha,\beta}$ is the slope function for the central charge $Z^{\mathrm{tilt}}_{\alpha,\beta}$:

$$\nu_{\alpha,\beta}(B) := \begin{cases} \dfrac{\mathrm{ch}_2^\beta(B) - \alpha^2\,\mathrm{ch}_0(B)/2}{\mathrm{ch}_1^\beta(B)}, & \text{if } \mathrm{ch}_1^\beta(B) \neq 0; \\ +\infty, & \text{if } \mathrm{ch}_1^\beta(B) = 0. \end{cases}$$

The category $\mathcal{A}^{\alpha,\beta} := \langle \mathcal{F}_{\alpha,\beta}[1], \mathcal{T}_{\alpha,\beta} \rangle$ will be the heart of a t-structure on $D^{\mathrm{b}}(X)$.

Construction for 3-folds, II

BMT define the central charge, for objects $A \in \mathcal{A}^{\alpha,\beta}$, as follows

$$Z_{\alpha,\beta,s}(A) := -\operatorname{ch}_3^\beta(A) + (s+1/6)\alpha^2 \operatorname{ch}_1^\beta(A) + \\ + \sqrt{-1}\left(\operatorname{ch}_2^\beta(A) - \alpha^2 \operatorname{ch}_0(A)/2\right)$$

and prove that the pair $(\mathcal{A}^{\alpha,\beta}, Z_{\alpha,\beta,s})$ with $s > 0$ is a Bridgeland stability condition on X provided a certain *generalized Bogomolov inequality* is satisfied.

Generalized Bogomolov inequality, I

Recall that the usual Bogomolov inequality for μ-semistable sheaves

$$Q^{\mathrm{tilt}}(B) := \mathrm{ch}_1(B)^2 - 2\,\mathrm{ch}_0(B)\,\mathrm{ch}_2(B) \geq 0$$

was a key ingredient to prove that $(\mathcal{B}^\beta, Z^{\mathrm{tilt}}_{\alpha,\beta})$ satisfied the support property.

Similarly, every $\nu_{\alpha,\beta}$-semistable object $B \in \mathcal{B}^\beta$ must satisfy

$$Q_{\alpha,\beta}(B) = \alpha^2 Q^{\mathrm{tilt}}(B) + 4(\mathrm{ch}_2^\beta(B))^2 - 6\,\mathrm{ch}_1^\beta(B)\,\mathrm{ch}_3^\beta(B) \geq 0,$$

in order for the pair $(\mathcal{A}^{\alpha,\beta}, Z_{\alpha,\beta,s})$ to satisfy the support property.

Generalized Bogomolov inequality, II

The generalized Bogomolov inequality has been shown to hold in various cases, by different authors, in the past 5 years: $\mathbb{P}^3$ (Macri), smooth quadric three-folds (Schmidt), abelian threefolds with $\mathrm{Pic} = \mathbb{Z}$ (Maciocia–Piyaratne and Bayer–Macri–Stelari), Fano 3-folds with $\mathrm{Pic} = \mathbb{Z}$, and the smooth quintic 3-fold (Li).

However, a counter-example was given by Schmidt in the case of $\mathbb{P}^3$ blown-up at a point.

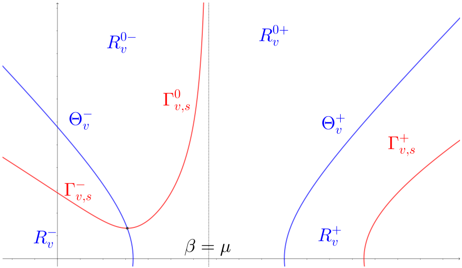
From now on, we take a 3-fold X on which the generalized Bogomolov inequality holds.

Issues

- The structure of numerical and actual walls is a lot more complicated... in particular, they may intersect one another at several points.
- The *large volume limit* also gets a lot more complicated, and depend on the way a path goes to infinity.

I will end the course presenting work done in collaboration with A. Maciocia.

Important curves: Θ_v and $\Gamma_{v,s}$



The blue hyperbola is the curve $\mathbf{Re}(Z^{\text{tilt}}_{\alpha,\beta}(v)) = 0$; we call it Θ_v. Together with the dotted vertical line, it divides the plane into 4 regions, labeled (from left to right) R_v^-, R_v^{0-}, R_v^{0+} and R_v^+.

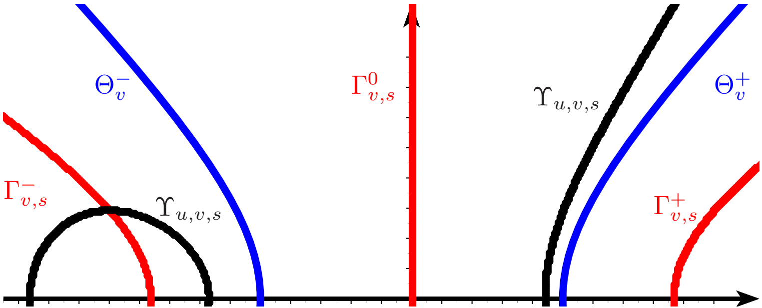
The red curve is given by $\mathbf{Re}(Z_{\alpha,\beta,s}(v)) = 0$; , which we call $\Gamma_{v,s}$. It may or may not cross Θ_v, either in its left or in its right branch.

Bounded and unbounded walls, I

- Numerical λ-walls can either be bounded or unbounded, and may have two connected components.
- The shape of numerical λ-walls may depend a lot on the value of the parameter s; $s = 1/3$ appears as a critical value in many situations.

Bounded and unbounded walls, II

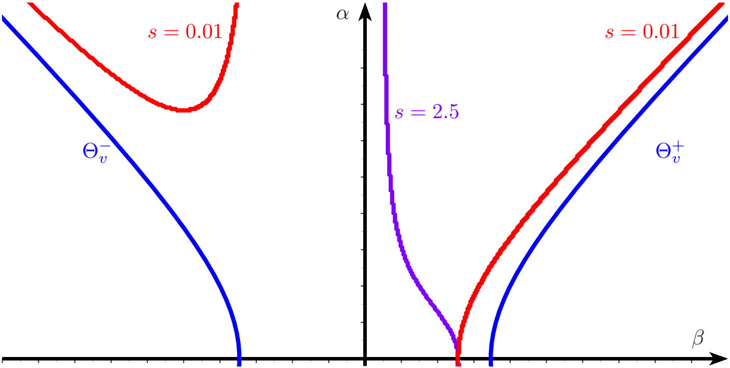
This numerical λ-wall $\Upsilon_{u,v,s}$ has two connected components, one bounded and the other unbounded.



Here, $v = (2, 0, -1, 0)$ (null correlation bundle on $\mathbb{P}^3$)
$u = (1, 0, -1, 1)$ (ideal sheaf of a line), and $s = 1/3$.

Bounded and unbounded walls, III

This picture shows the same numerical λ-wall for different values of s, one is connected while the other has two components.



We took $v = (2, 0, -3, 0)$ (stable rank 2 sheaf on $\mathbb{P}^3$) and $u = (0, 0, 1, -1)$ (structure sheaf of a line).

Structure of numerical λ-walls

- A numerical λ-wall $\Upsilon_{u,v,s}$ bounded if and only if $\mu(v) \neq \mu(u)$.

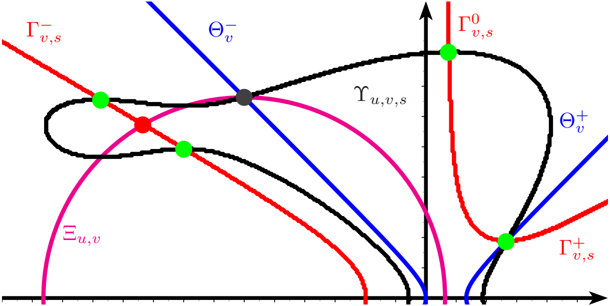
- Unbounded connected components of numerical λ-walls for v never cross Θ_v, and stay within R_v^0.

- Unbounded connected components never intersect one another.

- Suppose a Chern character v satisfies the Bogomolov inequality and $v_0 \neq 0$. Any connected bounded component of a numerical λ-wall in $R_{v,s}^-$ for some $s \geq 1/3$ intersects $\Gamma_{v,s}^-$.

- If $\Gamma_{v,s}$ intersects Θ_v, then there are vanishing $\nu-$ and $\lambda-$walls containing the point of intersection in its interior.

A cool wall



$v = (3, 1, 0, -1)$, $u = (0, 1, -3, 7)$, and $s = 1/3$.

Asymptotic λ-stability

Let $\gamma : (0, \infty) \to \mathbb{R} \times \mathbb{R}^+$ be an unbounded path.

An object $A \in D^{\mathrm{b}}(X)$ is *asymptotically λ-(semi)stable along γ* if the following two conditions hold for a given $s > 0$:

- there is $t_0 > 0$ such that $A \in \mathcal{A}^{\gamma(t)}$ for every $t > t_0$;
- for every sub-object $F \hookrightarrow A$ within $\mathcal{A}^{\gamma(t)}$ with $t > t_0$, there is $t_1 > t_0$ such that $\lambda_{\gamma(t),s}(F) < \ (\leq)\ \lambda_{\gamma(t),s}(A)$ for $t > t_1$.

One can show that asymptotically λ-semistable objects with $\mathrm{ch}_0(A) \neq 0$ satisfy the usual Bogomolov inequality $Q^{\mathrm{tilt}}(A) \geq 0$.

Let v be a numerical Chern character with $v_0 \neq 0$.

For each $s > 0$, we have:

- An object $A \in D^{\mathrm{b}}(X)$ with $\mathrm{ch}(A) = v$ is asymptotically λ-(semi)stable along $\Gamma^{-}_{v,s}$ if and only if A is a Gieseker (semi)stable sheaf.
- An object $A \in D^{\mathrm{b}}(X)$ is asymptotically $\lambda_{\alpha,\beta,s}$-(semi)stable objects along $\Gamma^{+}_{v,s}$ if and only if $A^{\vee}$ is a Gieseker (semi)stable sheaf.

A path $\gamma(t) = (\alpha(t), \beta(t))$ is called an *unbounded* Θ^-*-curve* if

$$\lim_{t\to\infty} \beta(t) = -\infty \ \text{ and } \ \lim_{t\to\infty} \frac{\dot{\alpha}(t)}{\dot{\beta}(t)} > -1.$$

That is, $\gamma(t)$ is asymptotically bounded by Θ_v^-

Similarly, we say that $\gamma(t) = (\alpha(t), \beta(t))$ is *an unbounded* Θ^+*-curve* if $\gamma^*(t) := (\alpha(t), -\beta(t))$ is an unbounded Θ^--curve

Asymptotic λ-stability on $R_v^{\pm}$

Let v be a numerical Chern character with $v_0 \neq 0$.

For each $\underline{s \geq 1/3}$, we have:

- An object $A \in D^{\mathrm{b}}(X)$ with $\mathrm{ch}(A) = v$ is asymptotically λ-(semi)stable along an unbounded Θ^--curve if and only if A is a Gieseker (semi)stable sheaf.

- An object $A \in D^{\mathrm{b}}(X)$ is asymptotically $\lambda_{\alpha,\beta,s}$-(semi)stable objects along an unbounded Θ^+-curve if and only if $A^\vee$ is a Gieseker (semi)stable sheaf.

Victor Pretti is currently studying the case $v_0 = 0$, and a similar result holds.

Asymptotic λ-stability on R_v^0

If $A \in D^{\mathrm{b}}(X)$ is an asymptotically λ-semistable object along the vertical line $\{\beta = \overline{\beta}\}$ with $\mathrm{ch}_0(A) \neq 0$, then:

- $\mathcal{H}^{-2}(A) = 0$;
- $\dim \mathcal{H}^0(A) \leq 1$, and every sheaf quotient $\mathcal{H}^0(A) \twoheadrightarrow P$ (including $\mathcal{H}^0(A)$ itself) satisfies
$$\frac{\mathrm{ch}_3(P)}{\mathrm{ch}_2(P)} \geq \frac{6s+1}{3}\left(\mu(A) - \overline{\beta}\right) + \overline{\beta}$$
whenever $\mathrm{ch}_2(P) \neq 0$;
- $\mathcal{H}^{-1}(A) = \tilde{A}$ is μ-semistable, and every sub-object $F \hookrightarrow A$ with $\mu(F) = \mu(A)$ satisfies
$$\frac{3s-1}{3}\delta_{20}(F,A)(\mu(A) - \overline{\beta}) + \frac{1}{2}(\delta_{20}(F,A) - \delta_{30}(F,A)) \leq 0;$$
- $\tilde{A}^{**}/\tilde{A}$ has pure dimension 1, and every subsheaf $R \hookrightarrow \tilde{A}^{**}/\tilde{A}$ (including $\tilde{A}^{**}/\tilde{A}$ itself) satisfies
$$\frac{\mathrm{ch}_3(R)}{\mathrm{ch}_2(R)} \leq \frac{6s+1}{3}\left(\mu(A) - \overline{\beta}\right) + \overline{\beta};$$
- if U is a sheaf of dimension at most 1 and $u : U \to A_{00}$ is a non-zero morphism that lifts to a monomorphism $\tilde{u} : U \hookrightarrow A$ within $\mathcal{A}^{\alpha,\overline{\beta}}$ for every $\alpha \gg 0$, then U also satisfies the previous inequality.

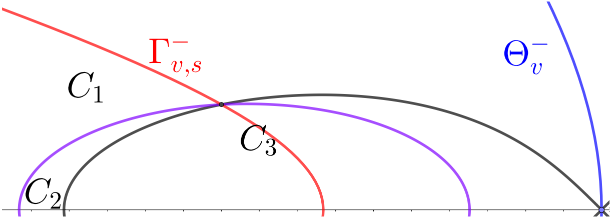
Case study: null correlation sheaves on $\mathbb{P}^3$

Let $v = (2, 0, -1, 0)$ be the numerical Chern character corresponding to null correlation sheaves on $\mathbb{P}^3$, and fix $s = 1/3$. The region R_v^- is divided into three stability chambers C_i within which the $\lambda_{\alpha,\beta,s}$-stable objects are described as follows:

(C_1) null correlation sheaves;

(C_2) nontrivial extensions of a semistable torsion free sheaf K with $\mathrm{ch}(K) = (2, 0, -2, 2)$ by $\mathcal{O}_L(-1)$, where L is a line;

(C_3) no stable objects.



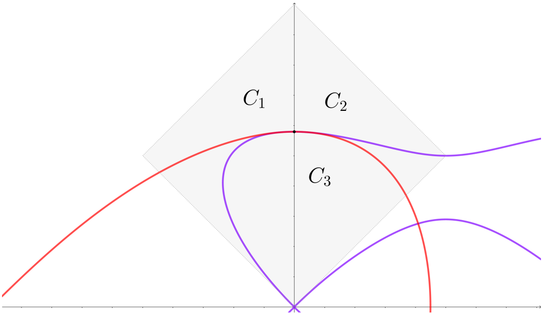
Case study: null correlation sheaves on $\mathbb{P}^3$

For each $s > 0$, there is a neighbourhood of the point $(\alpha = 1/\sqrt{6s+1}, \beta = 0)$ which is divided into exactly three stability chambers C_i; within which the $\lambda_{\alpha,\beta,s}$-stable objects are described as follows:

(C_1) shifted null correlation sheaves $E[1]$;

(C_2) their dual objects $E^{\vee}[1]$;

(C_3) no stable objects.



Thanks!

Thanks!