Compatible Poisson and Riemann Structures

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Abstract

We consider compatibility conditions between Poisson and Riemannian structures on smooth manifolds by means of a contravariant Levi-Civita connection. These include Riemann–Poisson structures (as defined by M. Boucetta), and (almost) Kälher–Poisson manifolds. Additionally, we study the geometry of the symplectic foliation. This is part of a joint work with Andrés Vargas(Universidad Javeriana, Bogotá)

1 Compatible geometric structures

In a symplectic manifold (M,ω) with a Riemann metric $\langle\cdot,\cdot\rangle$ there always exists an almost complex structure J so that

$$\omega(\cdot,\cdot)=\langle\cdot,J\cdot\rangle.$$

In this situation the triple $(\omega,\langle,\rangle J)$ is called compatible. If in addition we consider the Levi-Civita connection ∇^{LC} of the Riemann structure, the condition $\nabla^{LC}J=0$ makes the manifold M a **Kähler** manifold. A simple verification lead us to verify that $\nabla^{LC}J=0$ is equivalent to $\nabla^{LC}\omega=0$. This idea is the key point to introduce a new notion of compatibility condition between metric structure and Poisson structure in a manifod M in such a way that extends the Kähler case. As we will work with Poisson bivector it will be useful to consider a **cometric**

$$\langle X^\flat,Y^\flat\rangle:=\langle X,Y\rangle=X^b(Y)\quad\text{or}\quad\langle\alpha,\beta\rangle:=\langle\alpha^\sharp,\beta^\sharp\rangle=\alpha(\beta^\sharp)\tag{1}$$

instead of the usual Riemann metric $\langle -,-\rangle$ on M. Associated to the cometric we also can define the **contravariant Levi-Civita connection** $\nabla^\bullet$ via the Koszul identity:

$$2\langle\nabla^\alpha\beta,\gamma\rangle=\pi^\sharp(\alpha)\langle\beta,\gamma\rangle+\pi^\sharp(\beta)\langle\alpha,\gamma\rangle-\pi^\sharp(\gamma)\langle\alpha,\beta\rangle\tag{2}$$
$$+\langle[\alpha,\beta]_\pi,\gamma\rangle-\langle[\alpha,\gamma]_\pi,\beta\rangle-\langle[\beta,\gamma]_\pi,\alpha\rangle.\tag{3}$$

Also the notion of (almost) complex structure J must change for a contravariant version of it that also include a non-trivial kernel (just because π could be degenerated), for this consider $J\colon T^*M\to T^*M$ a bundle map satisfying

$$J_p^3+J_p=0$$

at every point $p\in M$. Such J is called **almost partially complex structure**.

Here we present some possible definition of *compatible condition* between metric and Poisson structure:

Definition 1

Let (M,π) a Poisson manifold with Riemanian structure $\langle -,-\rangle$ and associated cometric $\langle -,-\rangle$, and almost partially complex structure J. The couple $(\pi,\langle -,-\rangle)$ is called **orthogonally invariant** if it is preserved by the flow of vector fields orthogonal to the symplectic foliation, i.e.,

$$\mathscr{L}_X\pi=0\tag{4}$$

for all vector field X orthogonal to the leaves. A weaker condition of orthogonally invariant is the **Casimir invariant** where the last equation holds but just for gradiente vector fields $X=\nabla f$ for f a Casimir function.

The couple $(\pi,\langle -,-\rangle)$ is called **Riemann–Poisson** manifold if π is contravariantly parallel, i.e.,

$$\nabla^\bullet\pi=0,\tag{5}$$

where $\nabla^\bullet$ is the contravariant Levi-Civita connection (2) associated to $(\pi,\langle -,-\rangle)$, and with $\nabla^\bullet\pi(\alpha,\beta,\gamma):=(\nabla^\alpha\pi)(\beta,\gamma)=\pi^\sharp(\alpha)\pi(\beta,\gamma)-\pi(\nabla^\alpha\beta,\gamma)-\pi(\beta,\nabla^\alpha\gamma)$ for all $\alpha,\beta,\gamma\in\Omega^1(M)$.

The triple $(\pi,\langle -,-\rangle,J)$ is called an **almost Kähler–Poisson** manifold if at each $p\in M$ the structures $(\pi_p,\langle -,-\rangle_p,J_p)$ are linearly compatible, i.e for every $\alpha,\beta\in T^*M$ the relation

$$\pi(\alpha,\beta)=\langle J\alpha,\beta\rangle$$

is satisfied.

In addition, an almost Kähler–Poisson manifold $(\pi,\langle -,-\rangle,J)$ is called a **Kähler–Poisson** manifold if J is contravariantly parallel, i.e.,

$$\nabla^\bullet J=0,\tag{6}$$

where $\nabla^\bullet J(\alpha,\beta):=(\nabla^\alpha J)(\beta)=\nabla^\alpha(J\beta)-J(\nabla^\alpha\beta)$ for all $\alpha,\beta\in\Omega^1(M)$.

From a result by Boucetta[2, Thm. 1.3] it is possible to deduce that any Riemann–Poisson is always Casimir invariant. Furthermore, a direct computation shows that any almost Kähler–Poisson structure is Kähler–Poisson if and only if $\nabla^\bullet\pi=0$, which conclude that Kähler–Poisson are Riemann–Poisson. The three compatibilities are not equivalent as the following examples show:

Example

A direct computation shows that the constant Poisson structure $\pi=\partial_1\wedge\partial_2$ in $\mathbb{R}^3$ and the euclidean metric makes the couple $(\pi,\langle -,-\rangle)$ into a Riemann–Poisson structure. Now, for $k>0$ we consider in $\mathbb{R}^3$ the same Poisson bivector but the k-rescaled cometric $\langle -,-\rangle_k:=k\langle -,-\rangle$ and it is routine to verify that the contravariant Levi-Civita connection associated to $(\pi,\langle -,-\rangle_k)$ coincides with the one associated to $(\pi,\langle -,-\rangle)$, hence $(\mathbb{R}^3,\pi,\langle -,-\rangle_k)$ is Riemann–Poisson. Finally, note that $(\mathbb{R}^3,\pi,\langle -,-\rangle,J)$ is Kähler–Poisson, while $(\mathbb{R}^3,\pi,\langle -,-\rangle_k)$ is Riemann–Poisson but not Kähler–Poisson because there exists no partially complex structure J_k compatible with π and $\langle -,-\rangle_k$.

It worth to mention here that, in the symplectic case when π is invertible and $\omega:=\pi^{-1}$, a Kähler structure is equivalent to a Riemann–Poisson structure (see [1]). But in this same situation, the orthogonally invariance is not equivalent to the other two compatibilities as the following example shows:

Example

Note that for every Riemannian metric on M, the space $\Gamma(TM^\perp)=\{0\}$, so the condition in (4) is trivially true. But the usual known obstructions for the existence of Kähler structures yield us to the existence of non Riemann–Poisson (i.e. non Kähler) manifold that are transversaly invariant.

In addition to the claim that any Riemann–Poisson is also Casimir invariant, there is another Riemannina consequence on the bivector

Proposition 2

For any Riemann–Poisson structure $(\pi,\langle -,-\rangle)$ on M we get that the structure is divergence free, i.e.

$$\operatorname{div}\pi=0$$

where $\operatorname{div}\pi$ is the unique vector field so that $\operatorname{div}\pi(f)=\operatorname{div}X_f$ for all smooth function f.

The key remark here is that the divervenge free condition of π is independent of the contravariant parallelism of the bivector

Example

If $\langle -,-\rangle$ is a Riemannian metric on the symplectic manifold (M,ω) with associated Riemannian volume form v, there exists a nowhere vanishing function $\rho\in C^\infty(M)$ such that $v=\rho\,\omega^n$ and in this case we can derive that $\operatorname{div}\pi=0$ if and only if $d\rho=0$. Hence, the function $\phi:=\frac{1}{n}\log|\rho|$ turns the couple $(\pi,|\rho|^{\frac{2}{n}}\langle -,-\rangle)$ into othogonaly invariant and divergence free. On the other hand, the Kähler obstruction lead us to obtain a symplectic manifold that does not admit Riemann–Poisson structure (i.e. Kähler structure) but does admit $(\pi,|\rho|^{\frac{2}{n}}\langle -,-\rangle)$ orthogonal invariance and divergence free Poisson bivector.

2 The symplectic foliation and the compatibility

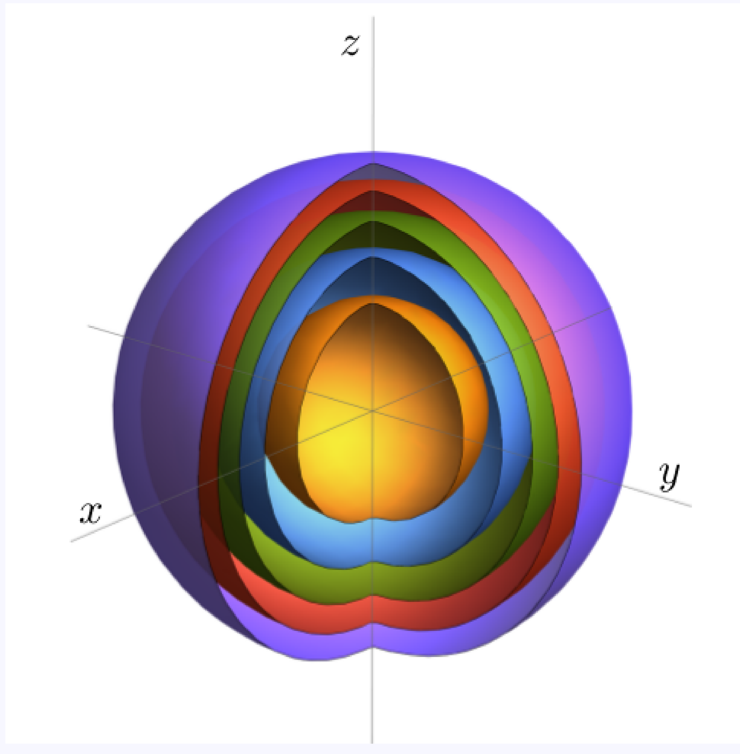
Before we state the main results on the foliation, we want to motivate the situation with the following example:

Example

Consider the dual Lie algebra $\mathfrak{so}_3^*(\mathbb{R})$ with its Lie–Poisson structure defined by the bivector

$$\pi_{\mathfrak{so}_3^*}:=z\partial_x\wedge\partial_y-y\partial_z\wedge\partial_x+x\partial_y\wedge\partial_z,$$

and its symplectic foliation



If $\langle -,-\rangle_{\mathbb{R}^3}$ denotes the standard three-dimensional Euclidean metric, a straightforward calculation shows that $\nabla^\bullet\pi_{\mathfrak{so}_3^*}=0$, so that the triple $(\mathfrak{so}_3^*(\mathbb{R}),\pi_{\mathfrak{so}_3^*},\langle -,-\rangle_{\mathbb{R}^3})$ is a Riemann–Poisson manifold. Moreover, the rescaled Poisson bivector $\tilde{\pi}:=r\pi_{\mathfrak{so}_3^*(\mathbb{R})}$, where the function $r\equiv r(x,y,z):=(x^2+y^2+z^2)^{\frac{1}{2}}$, turns out to be divergence free with respect to $\langle -,-\rangle_{\mathbb{R}^3}$ and orthogonally invariant. Moreover, the Poisson and Riemannian structure on the manifold $\mathfrak{so}_3^*(\mathbb{R})$ make that each leaf (of the sungular foliation) is a Kähler manifold.

Now, if we consider $\pi_{\text{reg}}:=\pi_{\mathfrak{so}_3^*}|_{\mathbb{R}^3\setminus\{o\}}$ as the restriction of the Lie–Poisson structure $\pi_{\mathfrak{so}_3^*}$ to $\mathbb{R}^3\setminus\{o\}$, we can endow it with the conformally Euclidean metric $\langle -,-\rangle_r:=r^{-1}\langle -,-\rangle_{\mathbb{R}^3}$ and the almost partial complex structure

$$J_{\text{reg}}:=z(\partial_x\otimes dy-\partial_y\otimes dx)-y(\partial_z\otimes dx-\partial_x\otimes dz)+x(\partial_y\otimes dz-\partial_z\otimes dy).$$

A straightforward computation lead us to conclude that $(\mathbb{R}^3\setminus\{o\},r\pi_{\text{reg}},J_{\text{reg}},\langle -,-\rangle_r)$ is a Kähler–Poisson manifold, where $\langle -,-\rangle_r$ denotes the cometric associated to $\langle -,-\rangle_r$. Furthermore, in this case the *regular* foliation also has Kähler leaves but in addition the Kähler metric is the restriction of $\langle -,-\rangle_r$ to each sphere and the foliation is Riemann foliation.

Both situation presented in the above example are general facts when we assume any of the compatibilities in Definition 1. For the statement of the theorem we should recall the definition of bundle like metric: A Riemannian metric $\langle -,-\rangle$ is called a **bundle-like** metric with respect to a foliation $\mathscr{F}=\{L_p:p\in M\}$ if the orthogonal part $\langle -,-\rangle_{T\mathscr{F}^\perp}$ of its decomposition satisfies the conditions:

1. $\operatorname{Ker}(V_p\mapsto\langle V_p,-\rangle_{T\mathscr{F}^\perp p})=T_pL_p$, for all $p\in M$, and
2. $\mathscr{L}_X\langle -,-\rangle_{T\mathscr{F}^\perp}=0$, for any vector field X tangent to the foliation.

Theorem 3,

Let $(M,\pi,\langle -,-\rangle)$ be a Poisson and Riemannian manifold. Then:

1. If the Poisson bivector π is orthogonally invariant and the structure $(M,\pi,\langle -,-\rangle)$ satisfies that the splitting $\langle -,-\rangle=\langle -,-\rangle^T+\langle -,-\rangle^\perp$ is real analytic, then the symplectic foliation of $(M,\pi,\langle -,-\rangle)$ is a singular Riemannian foliation.
2. If the Poisson bivector π is regular and Casimir invariant, then $\langle -,-\rangle$ is a bundle-like metric, and endowed with the transverse metric $\langle -,-\rangle^\perp$ the symplectic foliation of $(M,\pi,\langle -,-\rangle^\perp)$ is a Riemannian foliation.
3. If $(M,\langle -,-\rangle,\pi)$ is Riemann–Poisson, each leaf admits a Kähler metric compatible with its symplectic structure.
4. If in addition, $(M,\langle -,-\rangle,\pi,J)$ is Kähler–Poisson then the Kähler metric on each leaf is precisely the restricted one from M.

Remark

A key difference between both compatibilities arises in the properties of the symplectic foliation. Indeed, it follows from the main proposition in [3] that, when the partially complex structure exists, the symplectic foliation has to be regular. Hence, all Kähler–Poisson manifolds have a regular symplectic foliation. In contrast, there exist Riemann–Poisson manifolds with non-regular symplectic foliations (see example above)., thus they are not Kähler–Poisson.

To complement the remark we state the following proposition relating regularity of the foliation and Kählerian structures on the leaves:

Proposition 4

A regular Poisson manifold (M,π) admits a Kähler–Poisson structure if and only if its canonical symplectic foliation is a Riemannian foliation with Kählerian leaves.

Finally, we close the exposition on foliation by giving two interesting results, one conserning on the involutivity of the distribuition $(\operatorname{Ker}\pi^\sharp)^\perp$:

Theorem 5,

The following conditions are equivalent:

1. $\mathscr{L}_X\pi=0$ for all $X\in\Gamma(T\mathscr{F}^\perp)$.
2. $(\operatorname{Ker}\pi^\sharp)^\perp$ is involutive with respect to $[-,-]_\pi$.
3. The connection $\nabla^\bullet$ is independent of the transversal cometric $\langle -,-\rangle^\perp$

and a second result is aboutthe second fundamental form of the leaves:

Proposition 6

Orthogonally invariance of a couple $(\pi,\langle -,-\rangle)$ on a manifold M implies that the trace of the second fundamental form of each leaf vanishes, i.e each leave is *minimal. If in addition M is compact, then $(\pi,\langle -,-\rangle)$ is Riemann–Poisson if and only if it is Kähler–Poisson for some partially complex structure J compatible with π and $\langle -,-\rangle$.*

References

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