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Holonomy of singular foliations via paths

Singular Foliations A singular foliation $\mathcal{F}$ on a manifold M is a subsheaf of the sheaf of vector fields satisfying the following properties:

- $\mathcal{F}$ is involutive:
$$\forall X, Y \in \mathcal{F} \qquad [X, Y] \in \mathcal{F}$$
- $\mathcal{F}$ is locally finitely generated: For all sufficiently small open sets U,
$$\exists X_1, \dots, X_n \in \mathcal{F}(U) \qquad \mathcal{F}(U) = \langle X_1, \dots, X_n \rangle_{C^\infty_M}$$

Smooth Coefficients A time-dependent vector field $X(t)$ is said to have smooth coefficients in $\mathcal{F}$ if locally:

$$X(t)_p = \sum_{i=1}^{n} c^i(t,p) X_i$$

where $\{X_i\} \subset \mathcal{F}$ and $\{c^i\} \subset C^\infty(\mathbb{R} \times M, \mathbb{R})$.

Example Suppose $M = \mathbb{R}$ and $\mathcal{F} = \langle f(x)\frac{\partial}{\partial x} \rangle$ where $f(x) = 0$ for $x \leq 0$ and $f(x) > 0$ otherwise. Suppose $g(x)$ has the same property as f. Then $X(t) = t \cdot g(x - t)\frac{\partial}{\partial x}$ has smooth coefficients in $\mathcal{F}$ if and only if:

$$c(t,x) := \frac{t \cdot g(x - t^2)}{f(x)} \qquad x > 0$$

extends smoothly to $t \leq 0$.

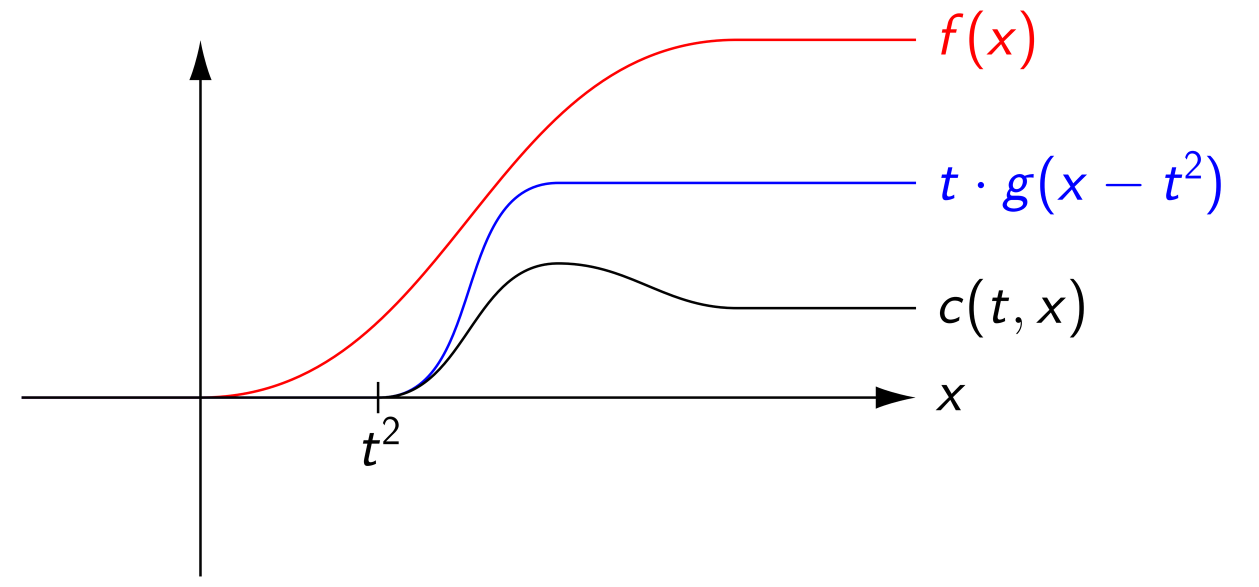

Figure: An illustration of the relationship between f, g and c

Foliation Paths An $\mathcal{F}$-path is a pair $(X(t), p(t))$ where $X(t)$ is a time-dependent vector field with smooth coefficients in $\mathcal{F}$ and $p(t)$ is an integral curve of X.

Holonomy Lemma Suppose (X, p) is an $\mathcal{F}$-path and we have chosen germs of slices $S_{p(0)}, S_{p(1)} \subset M$ through $p(0)$ and $p(1)$. Then there exists a vector field $Z \in I_{p(1)}\mathcal{F}$ such that':

$$\Phi^1_Z \circ \Phi^1_X(S_{p(0)}) \subset S_{p(1)}$$

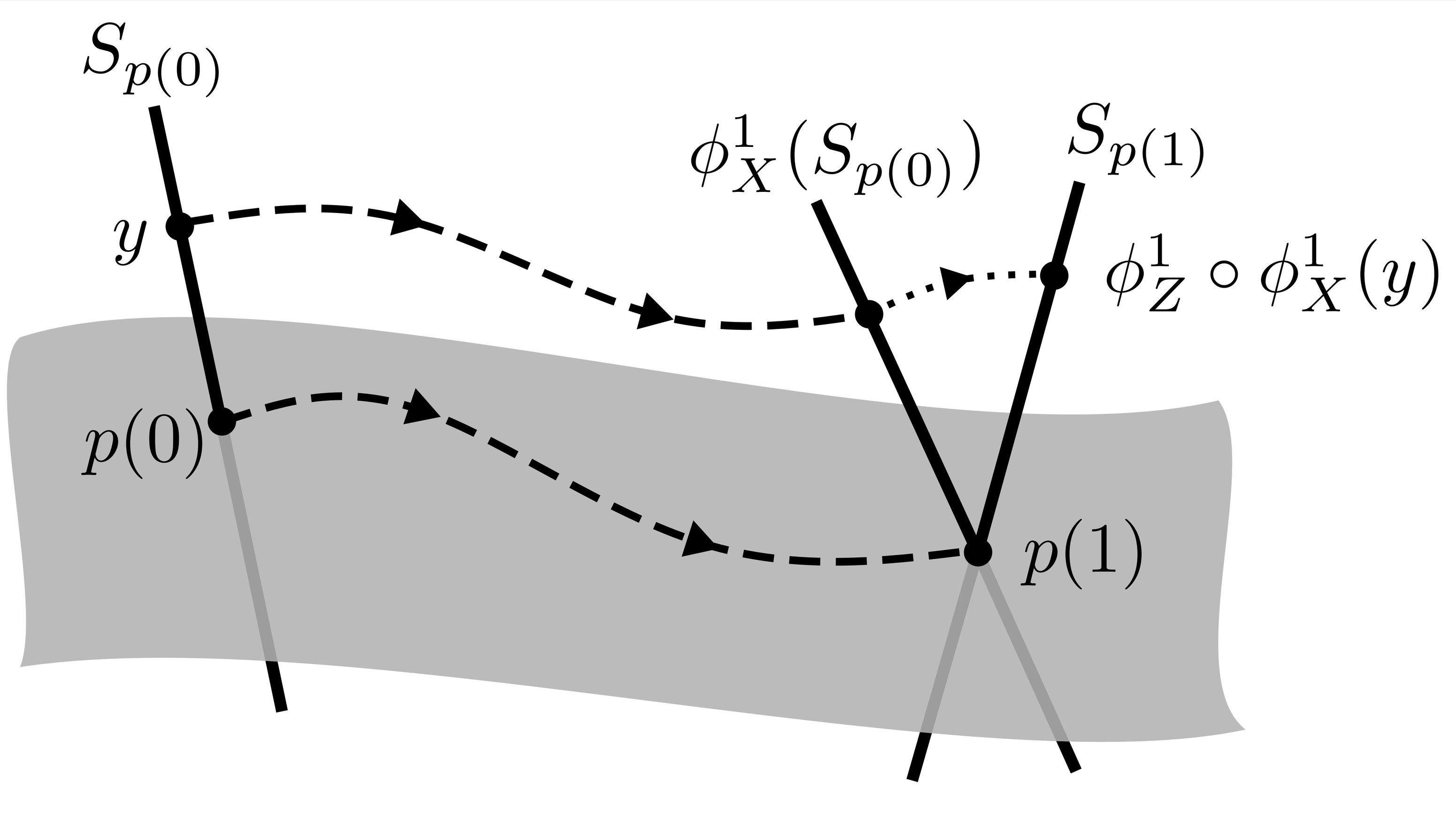

Figure: A visualization of the holonomy lemma. The dashed lines represent integral curves of X while the dotted lines represent integral curves of Z.

Definition of Holonomy The holonomy of an $\mathcal{F}$-path (X, p) relative to a choice of germs of slices $S_{p(0)}$, $S_{p(1)}$ is defined to be:

$$\mathrm{Hol}(X,p) := [\Phi^1_Z \circ \Phi^1_X] \in \frac{\mathrm{Diff}(S_0, S_1)}{\mathrm{Exp}(I_{p(1)}\mathcal{F})}$$

Remark

The set $\frac{\mathrm{Diff}(S_0,S_1)}{\mathrm{Exp}(I_{p(1)}\mathcal{F})}$ appeared previously in [2] as a way of defining holonomy transformations but in a different setting.

The Holonomy Groupoid Two $\mathcal{F}$-paths are said to be holonomic if they have the same endpoints and their holonomies are equal relative to an arbitrary choice of slices. The set of all $\mathcal{F}$-paths modulo this equivalence relation is holonomy groupoid of $\mathcal{F}$.

$$\mathrm{Hol}(\mathcal{F}) := \frac{\mathcal{F}\text{-paths}}{\text{holonomy}} \rightrightarrows M$$

Variation of $\mathcal{F}$-paths Suppose $(X(t,s), p(t,s))$ is an $\mathcal{F}$-path for each fixed s and $p_0 = p(0,s)$ is constant in s. Then this is called a variation of $\mathcal{F}$-paths. The complement of (X, p) is defined to be:

$$Y(t,s)|_{\Phi^{t,s}_X(x)} := \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)\Big|_{u=s} \Phi^{t,u}_X(x)$$

where $\Phi^{t,s}_X$ is the flow of X in the t-direction.

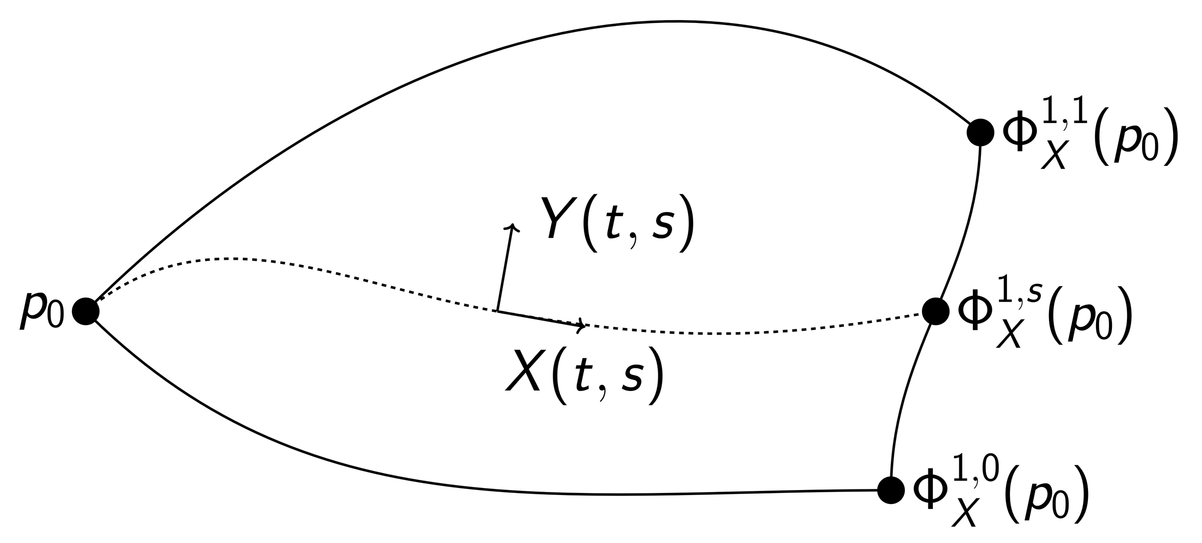

Figure: An illustration of the relationship between a variation X and its complement Y.

Definition of Homotopy Two $\mathcal{F}$-paths (X_0, p_0) and (X_1, p_1) are homotopic if there exists a variation $(X(t,s), p(t,s))$ interpolating them such that the complement of (X, p) satisfies:

$$Y(1,s) \in I_{p(1,s)}\mathcal{F}$$

The fundamental groupoid The fundamental groupoid of a foliation $\mathcal{F}$ is defined to be the set of $\mathcal{F}$-paths modulo the homotopy equivalence relation.

$$\Pi_1(\mathcal{F}) := \frac{\mathcal{F}\text{-paths}}{\text{homotopy}} \rightrightarrows M$$

The Main Theorems

- Homotopy $\Rightarrow$ Holonomy. 37.33pt
- The holonomy groupoid constructed above diffeomorphic to the one appearing in [1].
- The fundamental groupoid constructed above is the same as integrating the leaf-wise algebroids.

References

[1] Iakovos Androulidakis and Georges Skandalis. *The holonomy groupoid of a singular foliation*. J. Reine Angew. Math., 626:1–37, 2009.

[2] Iakovos Androulidakis and Marco Zambon. *Holonomy transformations for singular foliations*. Adv. Math., 256:348–397, 2014.