Rigidity of Lie groupoid morphisms and subgroupoids from a cohomological perspective

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Abstract
The deformation cohomology of Lie groupoid morphisms, briefly discussed in [2], offers a framework to study deformations of morphisms of Lie groupoids. We remark several properties of this cohomology, such as its Morita invariance and its interpretation in terms of the adjoint representation of Lie groupoids. We also study how the deformation cohomology governs the deformations of morphisms, and we apply that to obtain rigidity properties of morphisms. Finally, we comment on some analogous results in relation to deformations of Lie subgroupoids. The theorems here generalize the existing results of Cárdenas and Struchiner on Lie group homomorphisms [1]. This poster is based on part of a joint work with I. Struchiner (USP-Brazil).

Notations and Background

- $\mathcal{H} \rightrightarrows N$ and $\mathcal{G} \rightrightarrows M$ are Lie groupoids.
- $\Phi : \mathcal{H} \to \mathcal{G}$ is a Lie groupoid morphism.

Definition 1. Let $f : N \to M$ be a smooth map between the unit spaces of $\mathcal{H}$ and $\mathcal{G}$. We say that a smooth map $\tau : N \to \mathcal{G}$ is a **gauge map with base** f (or *over* f) if $\tau(x) \in s_{\mathcal{G}}^{-1}(f(x))$.

Examples of gauge maps can be obtained by considering bisections of $\mathcal{G}$: if σ is a bisection of $\mathcal{G}$, $\sigma \circ f$ is a gauge map with base f.

- From the categorical viewpoint, two morphisms Φ and Ψ are naturally isomorphic if and only if the underlying natural transformation τ is a gauge map. Alternatively we say that Φ and Ψ are **gauge related**.

Example 1. *Let $P \to M$ be a trivial G-principal bundle.*
In this case, the parallel transport $\mathcal{T}$ of a flat principal connection on P determines a morphism of Lie groupoids $\Phi : \pi(M) \to G;\ [\gamma] \mapsto \mathcal{T}_\gamma(1_G)$.
Two flat principal connections on P are related by a gauge transformation if and only if the induced morphisms are related by a gauge map.

Definitions

Deformations

Definition 2. A **deformation of** Φ is a pair of smooth maps $\tilde{\Phi} : \mathcal{H} \times I \to \mathcal{G}$ and $\tilde{\phi} : N \times I \to M$ such that $\tilde{\Phi}(\cdot, 0) = \Phi$, $\tilde{\phi}(\cdot, 0) = \phi$, and for each $\varepsilon \in I$ the map

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\Phi_\varepsilon} & \mathcal{G} \\ \downdownarrows & & \downdownarrows \\ N & \xrightarrow[\phi_\varepsilon]{} & M \end{array}$$

is a Lie groupoid morphism, where $\Phi_\varepsilon = \tilde{\Phi}(\cdot, \varepsilon)$ and similarly $\phi_\varepsilon = \tilde{\phi}(\cdot, \varepsilon)$.

Example 2. *Let ω_0 be a multiplicative k-form on $\mathcal{G}$. A smooth family ω_ε of multiplicative k-forms gives rise to the deformation*

$$\begin{array}{ccc} \bigoplus^k T\mathcal{G} & \xrightarrow{\hat{\omega}_\varepsilon} & \mathbb{R} \\ \downdownarrows & & \downdownarrows \\ \bigoplus^k TM & \longrightarrow & \{*\} \end{array}$$

of ω_0 viewed as the Lie groupoid morphism $\bigoplus^k T\mathcal{G} \longrightarrow \mathbb{R}$.

Definition 3. Two deformations Φ_ε and Φ'_ε of Φ are **gauge equivalent** if there exists a smooth family $\tau_\varepsilon : N \to \mathcal{G}$ of gauge maps over ϕ_ε such that $\tau_0 = u_{\mathcal{G}} \circ \phi_0$ and

$$\Phi_\varepsilon(h) = \tau_\varepsilon(t(h))\Phi'_\varepsilon(h)\tau_\varepsilon(s(h))^{-1}, \tag{1}$$

for all ε in some open interval I containing 0, and all $h \in \mathcal{H}$.

A deformation will be called **gauge trivial** if it is gauge equivalent to the constant deformation.

- When the family τ_ε of gauge maps comes from a family of bisections σ_ε, the deformations are simply said to be **equivalent**. And a deformation equivalent to the constant one will be called **trivial**.

The Deformation Complex $(C^*_{def}(\Phi), \delta)$

For $k \in \mathbb{Z}^+$, let $\mathcal{H}^{(k)}$ be the manifold of k-tuples of composable arrows of $\mathcal{H}$. The space $C^k_{def}(\Phi)$ consists of the maps $c : \mathcal{H}^{(k)} \to T\mathcal{G}$ such that

$$c(h_1, ..., h_k) \in T_{\Phi(h_1)}\mathcal{G} \quad \text{and} \quad ds_{\mathcal{G}}(c(h_1, ..., h_k)) \text{ does not depend on } h_1.$$

The differential of c is given by

$$\begin{aligned}(\delta c)(h_1, ..., h_{k+1}) : = & -d\overline{m}_{\mathcal{G}}(c(h_1h_2, h_3, ..., h_{k+1}), c(h_2, ..., h_{k+1}))+ \\ & + \sum_{i=2}^{k}(-1)^i c(h_1, ...h_ih_{i+1}, ...h_{k+1}) + (-1)^{k+1}c(h_1, ..., h_k),\end{aligned}$$

where $\overline{m}_{\mathcal{G}}$ is the division map of $\mathcal{G}$. For 0-cochains we define

$$C^0_{def}(\Phi) := \Gamma(\bar{\phi}^* A_{\mathcal{G}}) \qquad \text{and} \qquad \delta\alpha = \overrightarrow{\alpha} + \overleftarrow{\alpha},$$

where $\overrightarrow{\alpha}(h) = r_{\Phi(h)}(\alpha_{t(h)})$ and $\overleftarrow{\alpha}(h) = l_{\Phi(h)}(di(\alpha_{s(h)}))$.

Some Properties of $C^*_{def}(\Phi)$

- **Pullback of adjoint representation**

$$H^*_{def}(\Phi) \cong H^*(\mathcal{H}, \Phi^* Ad_{\mathcal{G}}).$$

This isomorphism also holds at the level of complexes after choosing a connection on $\mathcal{G}$.

- **Morita Invariance**

Let $F_1 : \mathcal{H}' \to \mathcal{H}$ and $F_2 : \mathcal{G} \to \mathcal{G}'$ be two **Morita maps**. Then

$$H^\bullet_{def}(\Phi) \cong H^\bullet(F_2 \circ \Phi \circ F_1).$$

- **Vanishing**

Assume that $\mathcal{H}$ is a *proper* groupoid. Then

$$H^k_{def}(\Phi) = 0, \text{ for } k \geq 2.$$

$$H^0_{def}(\Phi) \cong \Gamma(\phi^* \mathfrak{i}_{\mathcal{G}})^{inv} \text{ and } H^1_{def}(\Phi) \cong \Gamma(\phi^* \nu_{\mathcal{G}})^{inv}.$$

Results

Proposition 1 (Deformation Cocycles)**.** *Let Φ_ε be a deformation of $(\Phi_0, \phi_0) : \mathcal{H} \to \mathcal{G}$. Then, for each ε we obtain a 1-cocycle*

$$X_\varepsilon(h) = \frac{d}{d\varepsilon}\Phi_\varepsilon(h)$$

in the deformation complex $C^*_{def}(\Phi_\varepsilon)$ of Φ_ε.

Theorem 2 (Characterizing gauge-triviality)**.** *Let $(\Phi_\varepsilon, \phi_\varepsilon)$ be a deformation of $(\Phi_0, \phi_0) : (\mathcal{H} \rightrightarrows N) \longrightarrow (\mathcal{G} \rightrightarrows M)$. Then, Φ_ε is gauge trivial if and only if the family X_ε of cocycles is smoothly exact.*

Gauge Triviality $\Longleftrightarrow$ Cohomological Triviality.

Theorem 3 (Gauge-Rigidity)**.** *Let $\Phi : \mathcal{H} \longrightarrow \mathcal{G}$ be a Lie groupoid morphism. Assume that the groupoid $\mathcal{H}$ is proper and that $\mathcal{G}$ is transitive. Then any deformation of Φ is gauge-trivial.*

Under additional conditions we can obtain rigidity w.r.t. gauge transformations which **comes from bisections**.

Theorem 4 (Rigidity)**.** *Let $\Phi : \mathcal{H} \longrightarrow \mathcal{G}$ be a Lie groupoid morphism such that ϕ is an* injective immersion*. Assume that the groupoid $\mathcal{H}$ is compact and that $\mathcal{G}$ is transitive. Then any deformation of Φ is trivial.*

The proofs of these Theorems are based on both the Moser trick argument (exponential flow) and the properties of the deformation cohomology.

Examples of non-rigidity

1. *Assume that $\mathcal{G}$ is non-transitive.* Then there exist non gauge trivial deformations of Φ: Let $\gamma : I \to \mathcal{G}$ be a smooth curve on the unit space M which goes across the orbits of $\mathcal{G}$. Then, the family Φ_ε of constant morphisms $\Phi_\varepsilon(h) = \gamma(\varepsilon)$ is a non gauge trivial deformation of Φ_0.
2. *$\mathcal{H}$ non-proper.* Let $\mathcal{H} = \mathbb{R} \times \mathbb{R}$ be the bundle of Lie groups over $\mathbb{R}$, and $\mathcal{G} = \mathbb{R}$ viewed as a Lie groupoid over a point. Consider the morphism $\Phi(r, t) := r$, which is the identity over every fiber. Then, $\Phi_\varepsilon := (1 - \varepsilon) \cdot \Phi$ is a non gauge trivial deformation of Φ.

Remarks on Lie subgroupoids

Definition 4. A **deformation of the subgroupoid** $\iota : \mathcal{H} \hookrightarrow \mathcal{G}$ consists of a deformation $\widetilde{\mathcal{H}} \xrightarrow{\pi} I$ of the Lie groupoid $\mathcal{H}$ and a morphism $\widetilde{\iota} : \widetilde{\mathcal{H}} \to \mathcal{G}$ such that the restriction maps $\iota_\varepsilon : \mathcal{H}_\varepsilon \to \mathcal{G}$ to each fiber is an injective immersion, and $\iota_0 = \iota$.

$$\begin{array}{ccc} \mathcal{H}_\varepsilon & \xrightarrow{\iota_\varepsilon} & \mathcal{G} \\ \downdownarrows & & \downdownarrows \\ N_\varepsilon & \xrightarrow[j_\varepsilon]{} & M. \end{array}$$

A deformation $(\widetilde{\mathcal{H}}, \iota_\varepsilon)$ is called **trivial** if $\widetilde{\mathcal{H}} \cong \mathcal{H} \times I$ and, under this identification, every ι_ε is the conjugation of ι by a bisection σ_ε of $\mathcal{G}$.

The **deformation complex** $(C^*_{def}(\mathcal{H} \subset \mathcal{G}), \bar{\delta})$ is defined as the quotient complex of the *push forward* map

$$\iota_* : C^*_{def}(Id_{\mathcal{H}}) \to C^*_{def}(\iota), \quad \iota_* c(h_1, ..., h_k) := d\iota(c(h_1, ..., h_k)).$$

Facts:

- This deformation cohomology can be interpretated in terms of the *quotient adjoint representation*, it also verifies the properties of *Morita invariance* and *vanishing* for proper subgroupoids.
- There are also triviality and rigidity theorems for Lie subgroupoids. In particular:

Theorem 5 (Rigidity)**.** *Let $(\mathcal{H} \rightrightarrows N) \subset (\mathcal{G} \rightrightarrows M)$ be a compact Lie subgroupoid. If N is transversal to the orbits of $\mathcal{G}$, then any deformation of $\mathcal{H} \subset \mathcal{G}$ is trivial.*

Non-rigidity

- *N non-transversal.. and $\mathcal{H}$ regular.* The restriction subgroupoid $\mathcal{H}_{\mathcal{O}}$ over an orbit $\mathcal{O}$ can be deformed in a non-trivial way: take the family of orbits $\mathcal{O}_\varepsilon$ parametrized by a curve crossing the orbit $\mathcal{O}$. Then $\mathcal{H}_{\mathcal{O}_\varepsilon} \subset \mathcal{H}$ will not be a trivial deformation.

References

[1] Cristian Camilo Cárdenas and Ivan Struchiner. Stability of Lie group homomorphisms and Lie subgroups. *Journal of Pure and Applied Algebra*, 2019.

[2] Marius Crainic, João Nuno Mestre, and Ivan Struchiner. Deformations of Lie groupoids. *International Mathematics Research Notices*, 2018.