Dirac structures & Nijenhuis tensors

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Introduction

In [7], F. Magri and C. Morosi defined Poisson-Nijenhuis (PN) and presymplectic-Nijenhuis (ΩN) structures in the context of integrable systems. We present the definition of Dirac-Nijenhuis structures, which generalizes these notions. Our definition differs from others found in the literature, e.g. given by Carinena-Grabowski-Marmo in [4] and Longguang-Baokang in [6], and it is based on geometric tools developed by Bursztyn and Drummond in their study of multiplicative structures on groupoids [3]. We study several aspects of Dirac-Nijenhuis structures, including their integration to presymplectic-Nijenhuis groupoids (extending the integration of Poisson-Nijenhuis structures due to Stienon-Xu [8]) and the relations with holomorphic Dirac geometry.

Dirac-Nijenhuis structures

Let M be a manifold and $r : TM \to TM$ be a Nijenhuis $(1,1)-$tensor field on M. For each $X \in \mathfrak{X}(M)$, we define the maps $D_X^{TM,r}$ and $D_X^{T^*M,r}$, which depend on r, as follows

$$D_X^{TM,r} : \Gamma(TM) \to \Gamma(TM), \quad Y \mapsto [Y, r(X)] - r[X,Y] \qquad D_X^{T^*M,r} : \Gamma(T^*M) \to \Gamma(T^*M), \quad \alpha \mapsto i_X d(r^*\alpha) - i_{r(X)}d\alpha$$

Let $\mathbb{T}M = TM \oplus T^*M$ be a Courant algebroid. We define the operator $\mathbb{D} : \Gamma(\mathbb{T}M) \to \Gamma(T^*M \otimes \mathbb{T}M)$ over $\mathbb{T}M$ by:

$$\mathbb{D}_X(Y,\alpha) = (D_X^{TM,r}(Y), D_X^{T^*M,r}(\alpha))$$

for $(Y,\alpha) \in \mathbb{T}M$ and $X \in \mathfrak{X}(M)$.

Definition of Dirac-Nijenhuis structures

Let $L \subset \mathbb{T}M$ be a Dirac structure and $r : TM \to TM$ be a Nijenhuis $(1,1)-$tensor field. The pair (L,r) is called a *Dirac-Nijenhuis* structure on $\mathbb{T}M$ if

1. $(r, r^*)(L) \subseteq L$
2. $\mathbb{D}_X(\Gamma(L)) \subset \Gamma(L), \ \forall X \in \mathfrak{X}(M)$.

In the next examples we see that Dirac-Nijenhuis structures generalize $PN-$structures and $\Omega N-$structures.

Example 1. Poisson-Nijenhuis structures

Let $\pi \in \mathfrak{X}^2(M)$ be a Poisson structure on a manifold M and $r : TM \to TM$ a Nijenhuis $(1,1)-$tensor field.

Definition 1. (Magri and Morosi in [7]) The pair (π, r) is called a *PN-structure* on M if:

1. $\pi^\sharp \circ r^* = r \circ \pi^\sharp$,
2. The tensor field $R(\pi, r)$ vanishes, where

$$R(\pi,r)(X,\alpha) = (\mathcal{L}_{\pi^\sharp(\alpha)} r)(X) - \pi^\sharp(\mathcal{L}_X(r^*\alpha) - \mathcal{L}_{r(X)}(\alpha)), \ X \in \mathfrak{X}(M), \alpha \in \Omega^1(M).$$

Proposition

Let $L = graph(\pi^\sharp)$ be a Dirac structure on $\mathbb{T}M$ and $r : TM \to TM$ be a Nijenhuis tensor. The pair (L,r) is a Dirac-Nijenhuis structure if and only if (π, r) is a PN-structure.

Example 2. Presymplectic-Nijenhuis structures

Let $\omega \in \Omega^2(M)$ be a closed $2-$form on a manifold M and $r : TM \to TM$ a Nijenhuis $(1,1)-$tensor field.

Definition 2. (Magri and Morosi in [7]) The pair (ω, r) is called a *ΩN-structure* on M if:

1. $\omega^\flat \circ r = r^* \circ \omega^\flat$,
2. The tensor field $S(\omega, r)$ vanishes, where $S(\omega,r)(X,Y) = d(\omega^\flat \circ r), \ X, Y \in \mathfrak{X}(M)$.

Proposition

Let $L = graph(\omega^\flat)$ be a Dirac structure on $\mathbb{T}M$ and $r : TM \to TM$ be a Nijenhuis tensor. The pair (L,r) is a Dirac-Nijenhuis structure if and only if (ω, r) is a ΩN-structure.

Remark. A important result which we prove is that each leaf in the foliation of the Dirac-Nijenhuis structure hold a ΩN-structure.

Example 3. Holomorphic Dirac structures

Let $r : TM \to TM$ be a complex structure on a manifold M. Recall that we have the identification $\mathbb{T}M \xrightarrow{\Phi} T_{10} \oplus (T_{10})^*$, where $\Phi(X,\alpha) = \frac{1}{2}(x - ir(x), \alpha - ir^*(\alpha))$, and that $T_{10} \oplus (T_{10})^*$ carries a structure of holomorphic Courant algebroid.

Definition 3. (Gualtieri in [5]) A maximal isotropic and involutive holomorphic subbundle $\mathcal{L} \subset T_{10} \oplus (T_{10})^*$ is called a holomorphic Dirac structure.

Proposition

The pair (L,r) is a Dirac-Nijenhuis structure on $\mathbb{T}M$ if and only if $\Phi(L)$ is a holomorphic Dirac structure on $T_{10} \oplus (T_{10})^*$.

This extends the characterization of holomorphic Poisson structures as PN-structures.

Hierarchy of Dirac-Nijenhuis structures

Any Dirac-Nijenhuis structure (L,r) on $\mathbb{T}M$ is equipped with two hierarchies, both are defined by the recursive relation

$$L_{1,n} := (r^n, id)(L) \qquad \text{and} \qquad L_{2,n} := (id, (r^*)^n)(L),$$

for all $n \in \mathbb{N}$. The first element in both hierarchies (when $n = 0$) is $L_{1,0} = L_{2,0} = L$. Under the condition $ker(r^n, id)|_L = 0$, we obtain that $L_{1,n}$ is a Dirac structure on $\mathbb{T}M$. Also, if $ker((r^*)^n \circ \mu) \cap ker\rho = 0$, then $L_{2,n}$ is a Dirac structure on $\mathbb{T}M$.

Integration

In [8], Stienon and Xu integrate Poisson-Nijenhuis structures to symplectic-Nijenhuis Lie groupoids, and in [2], Bursztyn, Crainic, Weinstein and Zhu integrate Dirac structures to presymplectic Lie groupoids. Motivated by these results, we prove the theorem of integration of Dirac-Nijenhuis structures.

Theorem (Integration of Dirac-Nijenhuis structures)

Let $(\mathcal{G}, \omega)$ be the source 1-connected presymplectic Lie groupoid integrating a Dirac structure L in $\mathbb{T}M$. There is a one-to-one correspondence between Dirac-Nijenhuis structures (L,r) on $\mathbb{T}M$ and $\Omega N-$structures (ω, J) on $\mathcal{G}$, where $J \in \Omega^1(\mathcal{G}, T\mathcal{G})$ is multiplicative.

Idea of proof

Let $\mathcal{G} \rightrightarrows M$ be a source 1-connected Lie groupoid and $A \to M$ be its Lie algebroid. If $(\mathcal{G}, \omega)$ is presymplectic Lie groupoid integrating L, we have an isomorphism of vector bundles $(\rho, \mu) : A \to L \subset \mathbb{T}M$, where $\rho : A \to TM$ is the anchor map and $\mu : A \to T^*M$ is the infinitesimal part of presymplectic form $\omega \in \Omega^2(\mathcal{G})$ (see [1]). Let $J \in \Omega^1(\mathcal{G}, T\mathcal{G})$ be a multiplicative Nijenhuis $(1,1)-$tensor with infinitesimal components (D, l, r). Under this assumptions, we have the following lemma

Lemma

$$\left.\begin{array}{c} \omega^\flat \circ J = J^* \circ \omega^\flat \\ d(\omega^\flat \circ J) = 0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} \mu \circ l = r^* \circ \mu \\ \mu(D_X(a)) = D_X^{T^*M,r}(\mu(a)) \end{array}\right.$$

If (ω, J) is a $\Omega N-$structure on $(\mathcal{G}, \omega)$, by the lemma we have that $l = (r, r^*) : L \to L$ and $D = \mathbb{D} : \Gamma(L) \to \Gamma(T^*M \otimes L)$. Moreover, (L,r) is a Dirac-Nijenhuis structure. Conversely, if (L,r) is a Dirac-Nijenhuis structure, then $(\mathbb{D}, (r, r^*), r)$ is an IM $(1,1)-$tensor on L. We consider the multiplicative $(1,1)-$tensor $J \in \Omega^1(\mathcal{G}, T\mathcal{G})$ associated to $(\mathbb{D}, (r, r^*), r)$ and, by the lemma, (ω, J) is a $\Omega N-$structure on $(\mathcal{G}, \omega)$.

Remark. Recall that a complex Lie groupoid $(\mathcal{G}, J)$ is a Lie groupoid $\mathcal{G} \rightrightarrows M$ with a multiplicative $(1,1)-$tensor $J \in \Omega^1(\mathcal{G}, T\mathcal{G})$ such that $J^2 = -id$ and $\mathcal{N}_J = 0$.

Corollary

Holomorphic Dirac structures (L,r) are integrated by holomorphic presymplectic Lie groupoids $(\mathcal{G}, \omega, J)$.

References

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