Reduction of Equivariant Polyvector Fields and Ideas for Polydifferential Operators

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Motivation and Aim

A *star product* $\star$ on a Poisson manifold M is a bilinear associative product on $\mathscr{C}^\infty(M)[[\hbar]]$ with

$$f \star g = f \cdot g + \sum_{r=1}^{\infty} \hbar^r C_r(f,g)$$

for $f, g \in \mathscr{C}^\infty(M)[[\hbar]]$, where $C_1(f,g) - C_1(g,f) = \mathrm{i}\{f,g\}$ and where all C_r are bidifferential operators. We are interested in the following *symmetries*:

- **Classical setting (Marsden-Weinstein reduction):** Proper Hamiltonian action of Lie group G with 0 as regular value of the momentum map. If induced action on $C = J^{-1}(\{0\})$ is free, then $(M_{\mathrm{red}} = C/\mathrm{G}, \pi_{\mathrm{red}})$ is a Poisson manifold.
- **Quantized setting:** *Equivariant star products* $(\star, \boldsymbol{J})$ with quantum momentum maps. These pairs are Maurer-Cartan elements in the curved DGLA of equivariant polydifferential operators $(D^\bullet_{\mathfrak{g}}(M)[[\hbar]], \hbar^2\lambda, \partial_{\mathfrak{g}}, [-,-]_{\mathfrak{g}})$, see [3].

There exists an L_∞-quasi-isomorphism

$$F^{\mathfrak{g}} \colon T^\bullet_{\mathfrak{g}}(M)[[\hbar]] \rightsquigarrow D^\bullet_{\mathfrak{g}}(M)[[\hbar]],$$

whence equivariant star products are classified by equivalence classes of formal Maurer-Cartan elements $[\pi - J]$ in

$$T^\bullet_{\mathfrak{g}}(M)[[\hbar]] = (\mathrm{S}\mathfrak{g}^* \otimes T_{\mathrm{poly}}(M))^{\mathrm{G}}[[\hbar]]$$

with trivially extended Schouten bracket, zero differential and curvature $\hbar^2\lambda = \hbar^2 e^i \otimes (e_i)_M$.

$[Q, R] = 0$?

BRST reduction [1] gives star products $\star_{\mathrm{red}}$ on M_{red} out of equivariant ones, and from [2] one knows that star products on M_{red} are classified by equivalence classes of formal Maurer-Cartan elements $[\pi_{\mathrm{red}}]$ in the minimal DGLA $T_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]]$.

$\Longrightarrow$ **Aim:** Construct

$$\begin{array}{ccc} T_{\mathfrak{g}}(M)[[\hbar]] & \overset{F^{\mathfrak{g}}}{\rightsquigarrow} & D_{\mathfrak{g}}(M)[[\hbar]] \\ {\scriptstyle ??}\wr\downarrow & & \wr\downarrow{\scriptstyle ??} \\ T_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]] & \overset{F}{\rightsquigarrow} & D_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]] \end{array}$$

(at least on the level of Maurer-Cartan elements) and investigate its commutativity.

Reduction of Equivariant Polyvector Fields

We assume $M = C \times \mathfrak{g}^*$ and perform a Taylor series expansion in $\mathfrak{g}^*$-direction:

$$T^\bullet_{\mathrm{Tay}}(C \times \mathfrak{g}^*) = (\mathrm{S}\mathfrak{g}^* \otimes \prod_{i=0}^{\infty} (\mathrm{S}^i\mathfrak{g} \otimes \Lambda\mathfrak{g}^* \otimes T_{\mathrm{poly}}(C)))^{\mathrm{G}}$$

degrees: [2] [0] [1] $\deg_{\mathrm{poly}}$

The Poisson structure is of the form

$$\pi = \pi_{\mathrm{KKS}} - R + \pi_C$$

with

- $\pi_{\mathrm{KKS}} = \frac{1}{2} f_{ij}^k e_k \otimes e^i \wedge e^j \in (\mathfrak{g} \otimes \Lambda^2\mathfrak{g}^*)^{\mathrm{G}}$,
- $R = e^i \wedge (e_i)_C \in (\mathfrak{g}^* \wedge T^0_{\mathrm{poly}}(C))^{\mathrm{G}}$, and
- $\pi_C \in (\prod_{i=0}^\infty \mathrm{S}^i\mathfrak{g} \otimes T^1_{\mathrm{poly}}(C))^{\mathrm{G}}$.

Moreover, $[\pi, \pi] = 0$ gives

$$\frac{1}{2}[\pi_C, \pi_C] + (e_i)_C \wedge \mathrm{i}_{\mathrm{s}}(e^i)\pi_C = \frac{1}{2}[\pi_C, \pi_C] + \partial\pi_C = 0,$$

where

$$\partial = \mathrm{id} \otimes \mathrm{i}_{\mathrm{s}}(e^i) \otimes \mathrm{id} \otimes (e_i)_C \wedge \ .$$

Differential given by $[J, -] = -e^i \vee \otimes \mathrm{id} \otimes \mathrm{i}_{\mathrm{a}}(e_i) \otimes \mathrm{id}$ with the following cohomology:

Proposition *The cohomology of the Taylor series expansion is given by*

$$\mathrm{H}(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), [J, -]) \cong \left(\prod_{i=0}^\infty (\mathrm{S}^i\mathfrak{g} \otimes T_{\mathrm{poly}}(C)) \right)^{\mathrm{G}}.$$

Here ∂ induces a DGLA structure with the desired cohomology: One has

$$\mathrm{H}\left(\left(\prod_{i=0}^\infty (\mathrm{S}^i\mathfrak{g} \otimes T_{\mathrm{poly}}(C)) \right)^{\mathrm{G}}, \partial \right) \cong T_{\mathrm{poly}}(M_{\mathrm{red}}).$$

Finally, we can extend ∂ to all of $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$:

Proposition *The map $[\pi_{\mathrm{KKS}} - R, -]$ is a well-defined differential on $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$ that is explicitly given by*

$$[\pi_{\mathrm{KKS}} - R, \xi \otimes P \otimes \alpha \otimes X] = \xi \otimes \delta_{\mathrm{CE}}(P \otimes \alpha \otimes X) + \partial(\xi \otimes P \otimes \alpha \otimes X).$$

$\pi_{\mathrm{KKS}} - R - J$ is curved Maurer-Cartan element in $(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), \lambda, 0, [-,-])$.

$\Longrightarrow$ Twisting this structure yields a Lie algebra differential $[\pi_{\mathrm{KKS}} - R - J, -]$ on $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$ with curvature zero.

Models for Equivariant Polyvector Fields

Summarizing, we have:

$$\begin{array}{c} \left((\mathrm{S}\mathfrak{g}^* \otimes \prod_{i=0}^\infty (\mathrm{S}^i\mathfrak{g} \otimes \Lambda\mathfrak{g}^* \otimes T_{\mathrm{poly}}(C)))^{\mathrm{G}}, [\pi_{\mathrm{KKS}} - R - J, -], [-,-]\right) \\ \uparrow{\scriptstyle \iota} \\ \left(\left(\prod_{i=0}^\infty (\mathrm{S}^i\mathfrak{g} \otimes T_{\mathrm{poly}}(C))\right)^{\mathrm{G}}, \partial, [-,-] \right) \\ {\scriptstyle i_\infty}\wr\!\uparrow \quad \downarrow \\ (T_{\mathrm{poly}}(M_{\mathrm{red}}), 0, [-,-]) \end{array}$$

"contrav. Cartan model"

- All the straight arrows are DGLA morphisms given by projections resp. inclusions,
- the squiggly arrow is an L_∞-quasi-isomorphism (given by the Homotopy Transfer Thm.).

Proposition *The inclusion ι is a quasi-isomorphism of DGLAs which gives the desired L_∞-quasi-isomorphism*

$$T_{\mathrm{red}} \colon T_{\mathrm{Tay}}(C \times \mathfrak{g}^*) \rightsquigarrow T_{\mathrm{poly}}(M_{\mathrm{red}}),$$

resp. an L_∞-morphism $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]] \to T_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]]$ in the formal setting with rescaled differential $[\hbar(\pi_{\mathrm{KKS}} - R) - J, -]$.

$\Longrightarrow$ Formal MC elements $\hbar(\pi - J')$ in $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]]$ satisfy:

- $\pi_{\mathrm{KKS}} - R + \pi$ is an invariant formal Poisson structure, and
- $J + \hbar J'$ is formal momentum map w.r.t. $\pi_{\mathrm{KKS}} - R + \pi$.

Equivariant Polydifferential Operators and Formality

From now on $F^{\mathfrak{g}}$ denotes formality depending on lift of invariant torsion-free connection on C and canonical flat one on $C \times \mathfrak{g}^*$. Twisting by $\hbar(\pi_{\mathrm{KKS}} - R) - J$ gives

$$(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]], [\hbar(\pi_{\mathrm{KKS}} - R) - J, -]) \xrightarrow{F'} (D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]], [\star_D - J, -]),$$

where

$$D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]] = \left(\mathrm{S}\mathfrak{g}^* \otimes \prod_{i=0}^\infty (\mathrm{S}^i\mathfrak{g} \otimes T(\mathrm{S}\mathfrak{g}^*) \otimes D_{\mathrm{poly}}(C)) \right)^{\mathrm{G}} [[\hbar]].$$

Here

$$\star_D = \mu + \sum_{k=1}^\infty \frac{\hbar^k}{k!} F_k^D(\pi_{\mathrm{KKS}} - R, \dots, \pi_{\mathrm{KKS}} - R)$$

is the strongly invariant star product induced by Dolgushev's formality.

Proposition *The product $\star_D$ is equivalent to the product on the universal enveloping algebra $\mathrm{U}_\hbar(C \times \mathfrak{g})$ of the action Lie algebroid with rescaled bracket by $-\mathrm{i}\hbar$.*

Next steps and ideas:

- Describe reduction of polydifferential operators similarly as for vector fields. Can one use here the equivalence to the universal enveloping algebra?
- We have:

$$\begin{array}{ccc} \mathsf{Def}(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]]) & \xrightarrow{F',\ \cong} & \mathsf{Def}(D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]]) \\ {\scriptstyle T_{\mathrm{red}}}\downarrow & & \downarrow{\scriptstyle D_{\mathrm{red}}} \\ \mathsf{Def}(T_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]]) & \xrightarrow{F^{\mathrm{red}},\ \cong} & \mathsf{Def}(D_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]]), \end{array}$$

  where

  - Def denotes the set of equivalence classes of formal Maurer-Cartan elements,
  - F^{red} is the formality with respect to the restricted connection on M_{red},
  - $D_{\mathrm{red}} \colon (\star, \boldsymbol{J}) \mapsto \star_{\mathrm{red}}$ denotes reduction of star products from [1] .

  $\Longrightarrow$ Compare for $[\hbar(\pi_{\mathrm{KKS}} - R + \pi_C) - J]$ on $C \times \mathfrak{g}^*$ the product $\star_{\mathrm{red}}$ with

$$\mu + \sum_{k=1}^\infty \frac{\hbar^k}{k!} F_k^{\mathrm{red}}(\pi_{\mathrm{red}}, \dots, \pi_{\mathrm{red}})$$

References

[1] Bordemann, M., Herbig, H.-C., Waldmann, S.: *BRST Cohomology and Phase Space Reduction in Deformation Quantization.* Commun. Math. Phys. **210** (2000), 107–144.

[2] Dolgushev, V. A.: *Covariant and equivariant formality theorems.* Adv. Math. **191** (2005), 147–177.

[3] Esposito, C., de Kleijn, N., Schnitzer, J.: *A proof of Tsygan's formality conjecture for Hamiltonian actions.* Preprint **arXiv:1812.00403** (2018), 9 pages.