Applications of graded manifolds to Poisson geometry

by

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Abstract

In this thesis we use graded manifolds to study Poisson and related higher geometries shedding light on many new aspects of their connection. The relation between graded manifolds and classical geometric objects is based on a new geometrization functor that allows us to think of graded manifolds as collections of vector bundles carrying some additional structure. As a consequence of this geometrization of graded manifolds, we prove a graded version of the Frobenius theorem about integrability of involutive distributions, classify odd degree symplectic Q-manifolds, relate the geometry of higher Courant algebroids to shifted cotangent bundles and semi-direct products of the adjoint representation (up to homotopy) of a Lie algebroid, and give a new description of the infinitesimal data of a multiplicative Dirac structure.

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l'encomana que m'has fet l'aigua li ha fet mal
doncs tira-li farina i se t'endurirà

A la llum de la de mediterrània que m'has vist créixer, a la roja terra dels ametllers i tarongers dels quals sóc fruit, a la tèbia sorra de la platja i el suau vent del mar que sempre em portes esperança. Però sobretot, a la gent, al poble que m'ha envoltat amb la seva estima, calor i força. Especialment agraït a aquelles quatre dones de la meva infantesa que, lligades per sempre mes a un temps passat, em van mostrar una forma diferent de veure les coses.

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Tristeza não tem fim
felicidade sim

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Contents

Chapter 1

Introduction

One of the most fruitful stories of cooperation in science involves mathematics and physics. The interactions between these two areas of knowledge have been crucial to the development of both. The present work is devoted to the study of the interplay between Poisson geometry, related geometric structures, and graded geometry; all these fields lie at the interface of mathematics and physics and illustrate their interactions.

The Poisson bracket naturally appeared in physics around 200 years ago as an operation on functions of the phase space of classical mechanical systems ("observables"), see [105]. During the 1970's, see e.g. [84], the concept of *Poisson manifold* was formalized as a smooth manifold M endowed with a bivector field $\pi \in \mathfrak{X}^2(M)$ satisfying

$$[\pi, \pi] = 0,$$

where $[\cdot, \cdot]$ is the Schouten bracket on multivector fields. Poisson structures have a rich geometry connected to many interesting areas of mathematics, such as symplectic geometry, integrable systems, foliations, Lie theory, quantization, etc.

A central question for physics is how to pass from classical to quantum mechanics. Regarding Poisson manifolds as phase spaces of classical mechanics leads to the problem of "quantization of Poisson manifolds". To give a rigorous mathematical formulation of quantization, Weinstein introduced in the 1980's the concept of *symplectic groupoid* $(\mathcal{G} \rightrightarrows M, \omega)$, see [127]. Since then, Lie groupoids and their infinitesimal counterparts, Lie algebroids, have become indispensable tools in the study of Poisson geometry.

In the 1990's the works of Courant [45] and Dorfman [50] in constrained mechanical systems introduced the bracket

$$[\![X + \alpha, Y + \beta]\!] = [X, Y] + \mathcal{L}_X \beta - i_Y d\alpha, \qquad X, Y \in \mathfrak{X}(M),\ \alpha, \beta \in \Omega(M)$$

on sections of the vector bundle $TM \oplus T^*M$. This bracket codifies the integrability of many geometric structures (distributions, bivector fields, two-forms...) and gives a unified framework for Poisson and pre-symplectic geometry. The abstraction of the properties of these objects give rise to *Courant algebroids* [85]. More recently, higher versions of this bracket in $TM \oplus \bigwedge^k T^*M$ have been considered, see [62, 65].

In the first half of the past century, the quantum revolution was occurring in physics. The new physical paradigm divides the matter particles in two types depending on their spin: bosons and fermions. While bosonic particles commute, the nature of fermions is different and they anticommute. Therefore fermionic particles can not be represented by coordinates on a smooth manifold. In order to give a rigorous mathematical treatment of these new particles and trying to formulate classical mechanics with them, in the 1970's Berezin and Leites [8] introduced the notion of *supermanifold*. Roughly speaking they added to a smooth manifold M new formal Grassmann variables in a way that the sheaf of functions on a supermanifold becomes

$$C^\infty(M)[\xi^1, \cdots, \xi^k] \quad \text{where} \quad \xi^i\xi^j = -\xi^j\xi^i.$$

In the 1990's the works of Vaintrob [120], Voronov [122], Ševera [113] and Roytenberg [108] described a refinement of this idea by introducing an additional $\mathbb{N}$-grading on the functions on a supermanifold. More concretely, $(M, C_{\mathcal{M}})$ is called a $\mathbb{N}$-*graded manifold of degree* n, or an n-*manifold* for short, if $C_{\mathcal{M}}$ is a sheaf of graded commutative algebras which locally looks like

$$(C_{\mathcal{M}})_{|U} = C^\infty(U) \otimes \mathrm{Sym}(\mathbf{V}), \quad \text{where } \mathbf{V} = \oplus_{i=1}^n V_i.$$

Since all graded manifolds considered here are non-negatively graded, we will simplify the terminology "$\mathbb{N}$-graded" by just "graded". These new objects turn out to codify many geometric structures appearing in Poisson geometry and lead to interesting new results. Examples of correspondences involving graded and classical objects include:

- Lie algebroids $\leftrightarrows$ Degree 1 Q-manifolds [120].
- Lie bialgebroids $\leftrightarrows$ Degree 1 PQ-manifolds [122].
- Poisson manifolds $\leftrightarrows$ Degree 1 symplectic Q-manifolds [113].
- Courant algebroids $\leftrightarrows$ Degree 2 symplectic Q-manifolds [108, 113].
- LA-groupoids $\leftrightarrows$ Degree 1 graded Q-groupoids [96].

This graded-geometric perspective to classical objects sheds light on many aspects that may seem non-evident or difficult to deal with. For example, the clearest way to define Lie algebroid morphisms is as morphisms of Q-manifolds, see [120], while Poisson and Courant structures can be both seen as graded symplectic Q-manifolds (in different degrees), so their reductions by symmetries can be expressed as usual Marsden-Weinstein reduction in the context of graded symplectic gometry, see [24, 40].

In this thesis, we will enlarge this dictionary to other objects that also appear naturally in the study of Poisson and related geometries. This will allow us to give different descriptions of "higher" geometric objects, including LA-Courant algebroids [82], multiplicative Dirac structures and their infinitesimal counterparts [103], higher Courant algebroids [130] and their Dirac structures, among others.

All the work is based on a new "geometrization functor" for graded manifolds (see Theorem 3.5), which expresses n-manifolds in terms of ordinary differential-geometric objects. Indeed, we find that the category of graded manifolds is equivalent to a category where objects are given by collections of vector bundles carrying some extra structure. This geometrization functor extends the known correspondences between 1-manifolds and vector bundles (see e.g.[108]), as well as 2-manifolds and pairs of vector bundles E_1, E_2 along with an injective vector-bundle map $\psi : E_1 \wedge E_1 \hookrightarrow E_2$ (as in [24]).

More concretely, given a graded manifold $(M, C_{\mathcal{M}})$, for each $i = 1, 2, \ldots$ there exists a vector bundle $E_i \to M$ such that $\Gamma E_i = C^i_{\mathcal{M}}$, where $C^i_{\mathcal{M}}$ is the subsheaf of $C_{\mathcal{M}}$ of elements of degree equal to i; moreover, the multiplication of functions on $C_{\mathcal{M}}$ induces maps

$$m_{ij} : \Gamma E_i \times \Gamma E_j \to \Gamma E_{i+j}.$$

If the manifold has degree n, then a simple observation is that all the relevant information about the graded manifold is encoded in the first n vector bundles E_i and the multiplication maps m_{ij} for $i + j \leq n$. Therefore we define a category capturing the key properties of these objects and prove that it is equivalent to the category of graded manifolds. We remark that the possibility of fully expressing graded manifolds in terms of classical geometric objects contrasts with the usual theory of supermanifolds, whose descriptions in terms of vector bundles is neither canonical nor an equivalence of categories (see e.g. [6, 9]).

The geometrization functor allows us to think of graded manifolds as a collection of classical vector bundles, hence we can translate graded-geometric concepts into vector bundles and vice-versa. In this work, we discuss many applications of this principle, including new ones. Our main contributions can be summarized as follows:

I) The Frobenius theorem for graded manifolds, see Chapter 4.

II) A classification of odd degree symplectic Q-manifolds, see Chapter 6.

III) The relation of the graded cotangent bundles $T^*[k]A[1]$ with higher geometry, especially higher Courant and Dirac structures, see Chapter 6.

IV) Infinitesimal description of multiplicative Dirac structures via coisotropic Q-submanifolds of degree 2 PQ-manifolds, see Chapter 7.

The Frobenius theorem

The Frobenius theorem for smooth manifolds (see e.g. [126]) states that a distribution (i.e., a subbundle of the tangent bundle to a manifold) is integrable if and only if it is involutive. One possible formulation of this result asserts that a distribution is involutive if and only if, locally, it is spanned by coordinate vector fields, and this immediately implies that, through each point of the manifold, there is an integral submanifold to the distribution. By taking the collection of maximal integrals submanifolds, one then obtains a foliation.

We prove a Frobenius theorem for graded manifolds in Theorem 4.4. As a first step, we use the geometrization functor to characterize the vector fields on a graded manifold of non-positive degree in terms of the vector bundle data. As a result, we also obtain a description of distributions, which we then use to show that a distribution on a graded manifold is involutive if and only if, locally, it is spanned by coordinate vector fields. This result is analogous to the one proved for supermanifolds [30] or $\mathbb{Z}_2^n$-graded manifolds [46], though the techniques are rather different.

In order to complete the picture of Frobenius theorem, we then discuss integral submanifolds. We observe that there are shortcomings in their usual definition, so we propose a new, enhanced way to define integral submanifolds, which should refine the global description of distributions in [131] for graded manifolds of degree 1.

Apart from being a foundational theorem in the theory of graded manifolds (with applications e.g. to coisotropic reduction of Courant algebroids [24]), we are also interested in the Frobenius theorem because, in the classical case, it is a crucial step to prove the integration of Lie-algebroid morphisms to Lie-groupoid morphisms (Lie II), see e.g. [90]. Therefore we believe that the graded Frobenius theorem will be key also to prove an analogous integration result passing from morphisms of graded Lie algebroids to morphisms of graded Lie groupoids, which is currently in progress.

Odd degree symplectic Q-manifolds

Symplectic Q-manifolds are graded manifolds that are symplectic and carry a degree-1 symplectic vector field satisfying the homological condition

$$[Q, Q] = 2Q^2 = 0.$$

As we already mentioned before, symplectic Q-manifolds in degree 1 are in correspondence with ordinary Poisson manifolds, and in degree 2 they codify Courant algebroids, see [108, 113]. Therefore they are objects of central interest to us.

There is also a second context where graded symplectic Q-manifolds appear naturally. The celebrated paper [2], produced a new kind of topological quantum field theory (TQFT), nowadays known as AKSZ sigma models, where the space of fields are the maps between two supermanifolds and the target is a symplectic Q-manifold. The graded refinement of this theory was given in [109].

In this work, we see in Theorem 6.8 that given an n-manifold $\mathcal{M}$, its graded cotangent bundle $T^*[2n+1]\mathcal{M}$ is always an odd degree symplectic manifold and we can describe all their compatible Q-structures in terms of information coming from $\mathcal{M}$. In fact, the symplectic Q-structures on $T^*[2n+1]\mathcal{M}$ are equivalent to Q-structures on $\mathcal{M}$ or twisted versions of them. Therefore they give rise to L_∞-algebroids, in the terminology of [13].

Our second result for this part is Theorem 6.9, where we prove that any odd degree symplectic Q-manifold is symplectomorphic to a cotangent bundle as described before. Hence we obtain a classification of odd degree symplectic Q-manifolds. Theorem 6.9 can be seen as a graded version of the same result proven for odd symplectic supermanifolds in [111].

The geometry of the graded cotangent bundles $T^*[k]A[1]$

The works of Ševera and Roytenberg [108, 113] relate the geometry of the standard Courant algebroid $(TM \oplus T^*M, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], \rho)$ with the degree 2 symplectic Q-manifold $(T^*[2]T[1]M, \{\cdot,\cdot\}, Q_{dr})$. Given a vector bundle $A \to M$ we follow this idea and use the graded symplectic manifolds $T^*[k]A[1]$ to describe the higher Courant structures on $A \oplus \bigwedge^{k-1} A^*$, in such a way that the symplectic structure on $T^*[k]A[1]$ corresponds to the pairing on $A \oplus \bigwedge^{k-1} A^*$. The relation between $T^*[k]A[1]$ and the geometry of $A \oplus \bigwedge^{k-1} A^*$ has been suggested in previous works [15, 130], and here we give the precise correspondence.

Symplectic Q-structures on $T^*[k]A[1]$ (i.e., degree 1 homological symplectic vector fields) are always determined by degree $k+1$ functions θ on $T^*[k]A[1]$ satisfying the classical master equation $\{\theta, \theta\} = 0$. Using the derived bracket construction (see e.g. [77, 108]), we see that such functions give rise to brackets on sections of $A \oplus \bigwedge^{k-1} A^*$ that are compatible with the pairing. For $k = 2$, these functions have been geometrically classified in [78]. For $k = 3$ we present a similar result in Theorem 6.29, see also [61, 69], and for $k > 3$ we show that such functions have a particularly simple description: they are the same as a Lie algebroid structure on $A \to M$ together with $H \in \Gamma \bigwedge^{k+1} A^*$ such that $d_A H = 0$, see Theorem 6.30. In this last case, the bracket on $A \oplus \bigwedge^{k-1} A^*$ is given by the usual formula:

$$[\![a + \omega, b + \eta]\!]_H = [a, b] + \mathcal{L}_a \eta - i_b d\omega - i_b i_a H.$$

Once compatible Q-structures on $T^*[k]A[1]$ are understood, we define higher Dirac structures on $(A \oplus \bigwedge^{k-1} A^*, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!]_H)$ as the lagrangian Q-submanifolds of $(T^*[k]A[1], \{\cdot, \cdot\}, \theta_H)$, and we provide their classical description in Corollaries 6.19 and 6.20, and Theorem 6.36. Just as for ordinary Dirac structures, these higher Dirac structures encode interesting geometric objects. Examples include Nambu tensors [53, 117, 125], k-plectic structures [29, 107] and foliations. In the particular case when $A = TM$, $H = 0$, and the higher Dirac structure has support on the whole M, we recover the notion of a (regular) Nambu-Dirac structure as defined by Hagiwara in [64].

Finally we relate the geometry of the symplectic Q-manifolds $(T^*[k]A[1], \{\cdot, \cdot\}, \theta_H)$ with twisted versions of the semi-direct product of the Lie algebroid $(A \to M, [\cdot, \cdot], \rho)$ with their coadjoint representation up to homotopy as defined in [116], see Corollary 6.42, and make some final remarks on the relations with AKSZ and the integration of these Q-manifolds.

IM-Dirac structures

The work [20] introduced the concept of *multiplicative Dirac structures*, which are, roughly, Dirac structures over Lie groupoids satisfying a compatibility condition with the groupoid structure. These objects naturally unify, e.g., Poisson groupoids [128] and presymplectic groupoids [25]. The question of giving an infinitesimal description of multiplicative Dirac structure has been addressed in [72, 82, 103]. Here we will use graded geometry to present a new description of these infinitesimal objects.

We already mentioned the correspondence between Courant algebroids and degree 2 symplectic Q-manifolds from [108, 113]. In [82], an extension of this correspondence is proposed, relating *Courant groupoids* with degree 2 symplectic Q-groupoids, objects defined in [96]. Moreover, under this correspondence, *multiplicative Dirac structures* are sent to lagrangian Q-subgroupoids.

For classical symplectic groupoids $(G \rightrightarrows M, \omega)$, it is well known that M inherits a Poisson structure and that lagrangian subgroupoids in G have coisotropic submanifolds in M as their units, see [31, 43]. Moreover these coisotropic submanifolds have all the infinitesimal information of the lagrangian subgroupoid, [31].

In view of this result, we use our geometrization to give in Theorem 7.1 a geometric description of degree 2 PQ-manifolds and their coisotropic Q-submanifolds, see also [26]; we call the corresponding geometric objects *degenerate Courant algebroids* and *IM-Dirac structures*, respectively. We then relate, in Corollary 7.11, these objects to morphic Dirac structures on LA-Courant algebroids, as introduced in [82, 104].

If $(A \to M, [\cdot,\cdot], \rho)$ is a Lie algebroid, we have that $(TA \oplus T^*A; A, TM \oplus A^*; M)$ defines an LA-Courant algebroid that corresponds to the degree 2 PQ-manifold $(T[1]A^*[1], \{\cdot,\cdot\}, Q_{dr})$. In Section 7.2 we define the associated degenerate Courant algebroid, that we denote by $\mathfrak{D}_{ca}(A)$, and show how the previously known cases of IM 2-forms [21, 23], Lie bialgebroids [90] and IM-foliations [47, 51, 74] give rise to IM-Dirac structures in $\mathfrak{D}_{ca}(A)$.

Finally, in the case when the Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ integrates to a source simply-connected Lie groupoid $G \rightrightarrows M$, we establish in Theorem 7.20 an integration result between IM-Dirac structures in $\mathfrak{D}_{ca}(A)$ and multiplicative Dirac structures in $TG \oplus T^*G \rightrightarrows TM \oplus A^*$.

This thesis is organized in the following manner. In the preliminary Chapter 2, we introduce graded manifolds and define the objects over graded manifolds that we will use in the rest of this work, such as vector fields, Q-structures, submanifolds, and graded Poisson brackets. This chapter provides basic definitions and fixes our notation.

In Chapter 3 we present the geometrization functor for graded manifolds. This is the fundamental tool that allows us to connect graded manifolds to Poisson geometry. The first section of this Chapter 3 presents the category of coalgebra bundle as well as the geometrization functor, while in the other sections we give a geometric interpretation of the other geometric objects introduced in Chapter 2.

In Chapter 4 we prove the graded version of the Frobenius theorem, Theorem 4.4. For that, we start by defining distributions on graded manifolds and give their geometric interpretation. We then present a proof of the theorem and discuss integral submanifolds for distributions.

In Chapter 5 we give a graded-geometric viewpoint to two aspects of Lie algebroids: the first one is defining a homology theory for Lie algebroids, and the second one are semi-direct products of Lie algebroids with 2-term representations up to homotopy. The second part of this chapter will be used in Chapter 6.

In Chapter 6 we study graded cotangent bundles. In the first section we consider cotangent bundles $T^*[k]\mathcal{M}$ over arbitrary graded manifolds and give a classification of their Q-structures in the case $k = 2n + 1$, where n is the degree of $\mathcal{M}$; we then prove that any odd degree symplectic manifold is symplectomorphic to such graded cotangent bundles. In the other sections we study graded cotangent bundles over degree 1 manifolds: $T^*[k]A[1]$. We relate these manifolds to the geometry of higher Courant algebroids and semi-direct products.

In Chapter 7 we introduce *degenerate Courant algebroids* and *IM-Dirac structures* and prove that they are equivalent to LA-Courant algebroids and morphic Dirac structures. In Section 7.2 we explain carefully the example $(T[1]A^*[1], \{\cdot,\cdot\}, Q_{dr})$ where $(A \to M, [\cdot,\cdot], \rho)$ is a Lie algebroid and prove an integration result for multiplicative Dirac structures on the standard Courant algebroid. The last section of this chapter explains the role of graded symplectic Q-groupoids in our description.

Finally we have two appendices. In Appendix A we treat other aspects of graded manifolds that we do not cover in Chapter 2, including vector bundles over graded manifolds, differential forms and Cartan calculus, as well as graded groupoids and graded algebroids. Appendix B is about double vector bundles. The first part recalls general facts about double vector bundles and structures defined over them. The second part follows [48, 70, 82] and recalls another geometrization functor that goes from 2-manifolds to double vector bundles with suitable properties. In particular, this second part is used in Chapter 7.

The following diagram provides the main relations between the chapters of this thesis:

$$
\begin{tikzcd}
 & \text{C. 2} \arrow[r] \arrow[d] & \text{A} \arrow[d] \\
\text{C. 4} & \text{C. 3} \arrow[l] \arrow[r] \arrow[dl] \arrow[d] & \text{C. 7} \\
\text{C. 6} & \text{C. 5} \arrow[l] & \text{B} \arrow[l] \arrow[u]
\end{tikzcd}
$$

Notation

Here we summarize our notation:

- Smooth manifolds are denoted by M, N, while for graded manifolds we use calligraphic letters $\mathcal{M}, \mathcal{N}$ (if the graded manifold is given in terms of vector bundles, we use also notations like $E[1]$ or $T[1]A[1]$).
- Morphisms between graded manifolds are denoted by $\Psi = (\psi, \psi^\sharp) : \mathcal{M} = (M, C_\mathcal{M}) \to \mathcal{N} = (N, C_\mathcal{N})$ where $\psi : M \to N$ is a smooth map and $\psi^\sharp : C_\mathcal{N} \to \psi_* C_\mathcal{M}$ is a degree preserving morphism of sheaves.
- Given a graded manifold $\mathcal{M} = (M, C_\mathcal{M})$ and $U \subseteq M$ open then the restriction of $\mathcal{M}$ to the open is denoted by $\mathcal{M}_{|U} = (U, (C_\mathcal{M})_{|U})$.
- Vector bundles are denoted by $E \to M, A \to N$, while for graded vector bundles we use $\mathbf{E} \to M, \mathbf{A} \to N$, where $\mathbf{E} = \oplus_i E_i$.
- Given $(\varphi, \phi) : (E \to M) \to (F \to N)$ a vector bundle morphism, then the induced map on sections is denoted by $\phi^\sharp : \Gamma F^* \to \varphi_* \Gamma E^*$.
- Given a vector bundle $E \to M$, $\mathbb{A}_E \to M$ denotes its Atiyah algebroid.
- If $(E \to N) \subseteq (F \to M)$ is a subbundle, then its annihilator is denoted by $(E^\circ \to N) \subseteq (F^* \to M)$.

- Given a coalgebra bundle $(\mathbf{E}^*, \mu)$, we denote by $m : \Gamma\mathbf{E} \times \Gamma\mathbf{E} \to \Gamma\mathbf{E}$ the dual algebra structure, by $\widetilde{\mathbf{E}}^* = \ker \mu$ and the pure functions of degree k are denoted by $\widetilde{E}_k = E_k / \oplus_{i+j=k} m(E_i, E_j)$.
- If we say that a graded manifold has coordinates $\{e^{j_i}\}$, we always mean that $|e^{j_i}| = i$, where $|\cdot|$ denotes the degree.
- $(D; A, B; M)$ will always denote a double vector bundle where A is in the bottom left position and B in the top right position.

Chapter 2

Graded manifolds

This chapter contains the basic definitions concerning the main object of this thesis: (non-negatively) graded manifolds, a.k.a. $\mathbb{N}$-graded manifolds. Graded manifolds first appeared as supermanifolds endowed with an additional grading, see [120, 122]. Our definition is different from that one and follows the works [39, 94, 108], where graded manifolds are defined intrinsically. The goal of this preliminary chapter is to give a brief treatment of the topic that will be used in subsequent chapters; other aspects of graded manifolds are explained in Appendix A. Many of our definitions are parallel to the ones given for supermanifolds, see e.g. [9, 30].

2.1 The category $\mathcal{GM}^n$

Given a non-negative integer $n \in \mathbb{N}$, a *graded manifold of degree n* (or simply *n-manifold*) is a ringed space $\mathcal{M} = (M, C_{\mathcal{M}})$ where M is a smooth manifold and $C_{\mathcal{M}}$ is a sheaf of graded commutative algebras such that $\forall p \in M$, $\exists\, U$ open around p such that

$$(C_{\mathcal{M}})_{|U} \cong C^\infty(M)_{|U} \otimes \operatorname{Sym} \mathbf{V} \tag{2.1}$$

as sheaves of graded commutative algebras; here $\operatorname{Sym}\mathbf{V}$ denotes the graded symmetric vector space of a graded vector space $\mathbf{V} = \oplus_{i=1}^n V_i$. The manifold M is known as the *body* of $\mathcal{M}$ and $f \in C^k_{\mathcal{M}}$ is called a *homogeneous function of degree k*; we write $|f| = k$ for the degree of f. Given $p \in M$ we denote by $C_{\mathcal{M}_p}$ the *stalk* of $C_{\mathcal{M}}$ at p, and given $f \in C_{\mathcal{M}}$ we denote by $\mathbf{f}$ its class on $C_{\mathcal{M}_p}$.

We say that $\mathcal{M}$ has *dimension* $m_0|\cdots|m_n$ if $m_0 = \dim M$ and $m_i = \dim V_i$, and its *total dimension* is

$$\operatorname{Totdim} \mathcal{M} = \sum_{i=0}^{n} m_i.$$

Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be a n-manifold of dimension $m_0|\cdots|m_n$. Given a chart $U \subseteq M$ such that $(C_{\mathcal{M}})_{|U}$ is as in (2.1), we say that $\{e^{j_i}\}$ with $0 \leq i \leq n$ and $1 \leq j_i \leq m_i$ are *local coordinates* of $\mathcal{M}_{|U} = (U, (C_{\mathcal{M}})_{|U})$ if $\{e^{j_0}\}_{j_0=1}^{m_0}$ are local coordinates of U and $\{e^{j_i}\}_{j_i=1}^{m_i}$ form a basis of V_i for all $1 \leq i \leq n$. Therefore it is clear that

$$|e^{j_i}| = i, \quad \text{for } 0 \leq i \leq n \text{ and } 1 \leq j_i \leq m_i,$$

and that any function on $\mathcal{M}_{|U}$ can be written as a linear combination of products of $\{e^{j_i}\}_{j_i=1}^{m_i}$ times degree 0 functions.

A morphism between two graded manifolds $\Psi \in \mathrm{Hom}(\mathcal{M}, \mathcal{N})$, is a (degree preserving) morphism of ringed spaces, given by a pair

$$\begin{cases} \psi : M \to N & \text{smooth map.} \\ \psi^\sharp : C_\mathcal{N} \to \varphi_* C_\mathcal{M} & \begin{array}{l}\text{degree preserving morphism} \\ \text{of sheaf of algebras over } N.\end{array} \end{cases}$$

It is easy to check that n-manifolds with their morphisms define a category, that we denote by $\mathcal{GM}^n$.

Remark 2.1. Graded manifolds are also known in the literature as $\mathbb{N}$-manifolds, see [18, 70, 113]. They are a particular case of $\mathbb{Z}$-graded manifolds, see [39, 94] for a definition, and are different of the more common supermanifolds, that are $\mathbb{Z}_2$-graded objects, see [9, 30]. Our graded manifolds (a priori) are not supermanifolds with a compatible grading as in [108, 122] but they can be made so, see [36, Appendix B]. As it happens for supermanifolds, there are other ways of defining n-manifolds. One option is to mimic de definition of a smooth manifold by using an atlas and replace the opens of $\mathbb{R}^n$ by graded domains and gluing them. Other possible definition comes from algebraic geometry and uses the functor of points. See [39] for these definitions in the context of $\mathbb{Z}$-graded manifolds, and [9] in the context of supermanifolds.

Associated to each n-manifold $\mathcal{M} = (M, C_\mathcal{M})$ we have a tower of graded manifolds

$$M = \mathcal{M}_0 \leftarrow \mathcal{M}_1 \cdots \leftarrow \mathcal{M}_{n-1} \leftarrow \mathcal{M}_n = \mathcal{M} \tag{2.2}$$

where each $\mathcal{M}_k = (M, C_{\mathcal{M}_k})$ is a k-manifold with $C_{\mathcal{M}_k} = \langle f \in C_\mathcal{M} \mid |f| \leq k \rangle$ as a subalgebra of $C_\mathcal{M}$. In fact, if $j < k$ we have that $\mathcal{M}_j \leftarrow \mathcal{M}_k$ is an affine bundle, see [108] for a precise definition.

In order to illustrate the definition of n-manifolds we end this section with basic examples that will be used in the next chapters:

Example 2.2 (Linear n-manifolds). For classical (non-graded) manifolds we have that the most basic example of a smooth manifold of dimension k is $\mathbb{R}^k$. In fact $\mathbb{R}^k$ is the local model for all the other smooth manifolds of dimension k. We have an analogous picture in the graded world. Given a collection of non negative numbers $m_0, \cdots, m_n$ we define the graded vector space

$$\mathbf{W} = \oplus_{i=1}^n W_i \quad \text{where } W_i = \mathbb{R}^{m_i}.$$

For each dimension $m_0|\cdots|m_n$, the linear n-manifold of that dimension is defined as

$$\mathbb{R}^{m_0|\cdots|m_n} = \big(\mathbb{R}^{m_0}, C_{\mathbb{R}^{m_0|\cdots|m_n}} = C^\infty(\mathbb{R}^{m_0}) \otimes \mathrm{Sym}\, \mathbf{W}\big). \tag{2.3}$$

Example 2.3 (1-manifolds). Given any smooth manifold M, there are two canonical 1-manifolds associated to it: $(M, \Omega^\bullet(M))$ and $(M, \mathfrak{X}^\bullet(M))$, usually denoted by $T[1]M$ and $T^*[1]M$, respectively.

Let us see that they satisfy condition (2.1): Given U a chart of M, we can pick coordinates on U given by $\{x^i\}_{i=1}^m$, where $m = \dim M$. Moreover we also know that

$T^*M_{|U}$ can be trivialized and $\{dx^i\}_{i=1}^m$ form a basis of sections of $U \times \mathbb{R}^m$. Therefore $\Omega^\bullet(U) = C^\infty(U) \otimes \operatorname{Sym} \mathbb{R}^m$.

So we conclude that if U is a chart of M then $T[1]M_{|U}$ has local coordinates

$$\{x^i, \theta^i = dx^i\}_{i=1}^m \text{ with } |x^i| = 0 \text{ and } |\theta^i| = 1.$$

On the other hand, we can also define coordinates on $T^*[1]M_{|U}$ given by

$$\{x^i, p_i = \frac{\partial}{\partial x^i}\}_{i=1}^m \text{ with } |x^i| = 0 \text{ and } |p_i| = 1.$$

More generally, given a vector bundle $E^* \to M$ we could define the 1-manifold

$$E^*[1] = (M, \Gamma \bigwedge\nolimits^\bullet E).$$

On a chart U of M, where $E_{|U}$ is a trivial bundle, if we pick $\{x^i\}_{i=1}^m$ coordinates on U and $\{\xi^j\}_{j=1}^{\operatorname{rk} E}$ basis of sections of $E_{|U}$ we have that

$$\{x^i\}_{i=1}^m, \{\xi^j\}_{j=1}^{\operatorname{rk} E} \text{ with } |x^i| = 0 \text{ and } |\xi^j| = 1$$

are local coordinates of $E^*[1]$. We will see in Chapter 3 that any 1-manifold is canonically of this type.

Example 2.4 (Split n-manifolds). In the same spirit of the preceding example, if we have a non-positively graded vector bundle $\mathbf{D}^* = \oplus_{i=-n+1}^0 D_i^* \to M$ we define the graded manifold $\mathbf{D}^*[1] = (M, \Gamma \operatorname{Sym}(\mathbf{D}^*[1])^*)$. These kinds of graded manifolds are known as split n-manifolds, see [13]. We prove in Lemma 3.6 that any n-manifold is non-canonically isomorphic to one of this type.

Example 2.5 (Cartesian product of n-manifolds). Given two n-manifolds $\mathcal{M} = (M, C_\mathcal{M})$ and $\mathcal{N} = (N, C_\mathcal{N})$ we define a new n-manifold called the Cartesian product, $\mathcal{M} \times \mathcal{N} = (M \times N, C_{\mathcal{M}\times\mathcal{N}})$ where

$$C_{\mathcal{M}\times\mathcal{N}} = C_\mathcal{M} \widehat{\otimes}\, C_\mathcal{N},$$

and the hat denotes the usual completion on the product topology, see [30] for details. More concretely, on open rectangles $U \times W$ (that forms a base of the product topology), with $U \subset M$ satisfying that $(C_\mathcal{M})_{|U} = C^\infty(U) \otimes \operatorname{Sym} \mathbf{V}_1$ and $W \subset N$ satisfying that $(C_\mathcal{N})_{|W} = C^\infty(W) \otimes \operatorname{Sym} \mathbf{V}_2$, we have that

$$(C_{\mathcal{M}\times\mathcal{N}})_{|U\times W} = C^\infty(U \times W) \otimes \operatorname{Sym}(\mathbf{V}_1 \oplus \mathbf{V}_2),$$

where $\mathbf{V}_1 \oplus \mathbf{V}_2$ denotes the usual graded direct sum.

2.2 Vector fields and tangent vectors

In this section we introduce the notion of vector field on a graded manifold as derivations of the sheaf of functions. Vector fields will be fundamental for the Frobenius theorem and also to codify Poisson and related geometries into graded manifolds.

2.2.1 Vector fields

Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be an n-manifold of dimension $m_0|\cdots|m_n$. We define a *vector field of degree* k, denoted by $X \in \mathfrak{X}^{1,k}(\mathcal{M})$ and by $|X|$ its degree, as a degree k derivation of the sheaf of graded algebras $C_{\mathcal{M}}$, i.e,

$$X : C^i_{\mathcal{M}} \to C^{i+k}_{\mathcal{M}} \quad \text{such that} \quad X(fg) = X(f)g + (-1)^{|f|k} f X(g). \tag{2.4}$$

It is well known that derivations form a Lie algebra with respect to the commutator: in the present situation, since everything is graded, we define the graded commutator of vector fields by

$$[X, Y] = XY - (-1)^{|X||Y|} Y X$$

which is again a vector field. So $(\mathfrak{X}^{1,\bullet}(\mathcal{M}), [\cdot,\cdot])$ forms a graded Lie algebra.

Let us see that locally we write vector fields in terms of coordinates: Choose $U \subseteq M$ open such that $(C_{\mathcal{M}})_{|U}$ is as in (2.1) and denote by $\{e^{j_i}\}$ where $0 \leq i \leq n$ and $1 \leq j_i \leq m_i$ the coordinates of $\mathcal{M}_{|U}$. Define the vector field $\frac{\partial}{\partial e^{j_i}}$ as a derivation that on coordinates acts by

$$\frac{\partial}{\partial e^{j_i}}(e^{k_l}) = \delta_{j_i k_l}.$$

By definition, we have that $|\frac{\partial}{\partial e^{j_i}}| = -|e^{j_i}|$.

Proposition 2.6. *Let $\mathcal{M}$ be an n-manifold. Pick U a chart of $\mathcal{M}$ and $\{e^{j_i}\}$ coordinates on $\mathcal{M}_{|U}$. Then*

$$\mathfrak{X}^{1,\bullet}(\mathcal{M}_{|U}) = \langle \frac{\partial}{\partial e^{j_i}} \mid 0 \leq i \leq n, \; 1 \leq j_i \leq m_i \rangle \text{ as a } C_{\mathcal{M}_{|U}}\text{-module.}$$

Proof. Let X be a vector field on $\mathcal{M}_{|U}$. Define a new vector field

$$X' = X - \sum_{i=0}^{n} \sum_{j_i=1}^{m_i} X(e^{j_i}) \frac{\partial}{\partial e^{j_i}}$$

It is clear that X' acts by zero on the coordinates. Therefore, since any function is a linear combination of products of coordinates times degree 0 functions, we have that $X = \sum_{i=0}^{n} \sum_{j_i=1}^{m_i} X(e^{j_i}) \frac{\partial}{\partial e^{j_i}}$ as we want.

□

A direct consequence of the preceding Proposition 2.6 is that $\mathfrak{X}^{1,\bullet}(\mathcal{M})$ is generated as $C_{\mathcal{M}}$-module by non-positively graded vector fields. Therefore it is often enough to focus on them.

Remark 2.7. There is another more geometric interpretation of vector fields. For smooth manifolds we known that there is an identification between sections of the tangent bundle and vector fields, defined these as derivations of the sheaf of functions.

For graded manifold, if we introduce vector bundles as in Section A.1, then we can talk about sections of a vector bundle (but we need to consider $\mathbb{Z}$-graded manifolds). Given a graded manifold $\mathcal{M}$ we can define their tangent bundle, that we denote by $T\mathcal{M}$. In [94] it is proven that there is a natural isomorphism between $\Gamma T\mathcal{M}$ and $\mathfrak{X}^{1,\bullet}(\mathcal{M})$ as $C_{\mathcal{M}}$-modules.

Example 2.8 (The Euler vector field). Given a graded manifold $\mathcal{M} = (M, C_{\mathcal{M}})$ there is always a canonical degree 0 vector field induced by the grading. This vector field is called the *Euler vector field* and is locally defined as follows: If $\{e^{j_i}\}$ are local coordinates on $\mathcal{M}_{|U}$ then the Euler vector field has the local form

$$\mathcal{E}_u = |e^{j_i}|\ e^{j_i} \frac{\partial}{\partial e^{j_i}}.$$

The importance of the Euler vector field comes from the fact that, $\forall k \in \mathbb{N}$,

$$f \in C^k_{\mathcal{M}} \Longleftrightarrow \mathcal{E}_u(f) = kf.$$

2.2.2 Q-manifolds

One of the most important concepts for codifying Poisson and higher geometries in terms of graded manifolds is that of a Q-manifold. Here we just recall the definition and, in the next chapters, we developed this relation.

A *Q-manifold* is a pair $(\mathcal{M}, Q)$ where $\mathcal{M}$ is an n-manifold and $Q \in \mathfrak{X}^{1,1}(\mathcal{M})$ such that

$$[Q, Q] = 2Q^2 = 0. \tag{2.5}$$

Given two Q-manifolds $(\mathcal{M}, Q_1)$ and $(\mathcal{N}, Q_2)$ a *Q-morphism* is a morphism $\Psi = (\psi, \psi^\sharp) : \mathcal{M} \to \mathcal{N}$ between the graded manifolds such that

$$\psi^\sharp Q_2(f) = Q_1(\psi^\sharp f) \quad \forall f \in C_{\mathcal{N}}.$$

Therefore Q-manifolds with Q-morphisms form a category.

A direct consequence of the definition is that given a Q-manifold $(\mathcal{M} = (M, C_{\mathcal{M}}), Q)$ we have that $(C_{\mathcal{M}}, Q)$ is a differential graded algebra. This is the reason why sometimes degree 1 vector fields satisfying equation (2.5) are called *homological vector fields*, and the pair $(\mathcal{M}, Q)$ a *dg-manifold.*

Associated to a differential complex we have always a cohomology, so we define the *Q-cohomology of the Q-manifold* $(\mathcal{M}, Q)$, denoted by $H_Q(\mathcal{M})$, as the cohomology of the complex $(C_{\mathcal{M}}, Q)$.

Let us finish with some examples of Q-manifolds:

Example 2.9 (Zero Q-structure). Any graded manifold $\mathcal{M}$ is an example of a Q-manifold with $Q = 0$. In this example, Q-morphisms are the same as graded morphism. Therefore we have that the category $\mathcal{GM}^n$ is inside of the category of Q-manifolds.

Example 2.10 (Cartesian product of Q-manifolds). Following Example 2.5, the category of Q-manifolds has also a Cartesian product. Given two Q-manifolds $(\mathcal{M}, Q_1)$ and $(\mathcal{N}, Q_2)$ we have that $(\mathcal{M} \times \mathcal{N}, Q_1 \times Q_2)$ is a Q-manifold where

$$Q_1 \times Q_2(f \otimes g) = Q_1(f) \otimes g + f \otimes Q_2(g) \text{ with } f \in C_{\mathcal{M}},\ g \in C_{\mathcal{N}}.$$

Recall that a *Lie algebroid* $(A \to M, [\cdot,\cdot], \rho)$ is a vector bundle $A \to M$ together with a vector bundle map $\rho : A \to TM$, called the *anchor*, and an operation $[\cdot,\cdot] : \Gamma A \times \Gamma A \to \Gamma A$ satisfying the following compatibilities:

a). Skew-symmetry: $[a_1, a_2] = -[a_2, a_1]$,

b). Jacobi identity: $[a_1, [a_2, a_3]] = [[a_1, a_2], a_3] + [a_2, [a_1, a_3]]$,

c). Leibniz rule: $[a_1, fa_2] = \rho(a_1)(f)a_2 + f[a_1, a_2]$,

where $a_1, a_2, a_3 \in \Gamma A,\ f \in C^\infty(M)$.

Example 2.11 (Basic examples of Lie algebroids). Just to illustrate the definition we give three fundamental examples of Lie algebroids:

- If the base manifold is a point, $M = *$, then we obtain just Lie algebras, $(\mathfrak{g}, [\cdot,\cdot])$.
- The tangent bundle of any manifold $(TM \to M, [\cdot,\cdot], \mathrm{Id})$ with the bracket given by the Lie bracket of vector fields.
- Let (M, π) be a Poisson manifold. Then $(T^*M \to M, [\cdot,\cdot]_\pi, \pi^\sharp)$ is a Lie algebroid where

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\sharp\alpha}\beta - \mathcal{L}_{\pi^\sharp\beta}\alpha - d\pi(\alpha, \beta) \quad \alpha, \beta \in \Omega^1(M).$$

See e.g. [43].

Example 2.12 (Lie algebroids as degree 1 Q-manifolds). Let $(A \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid. It is known that the Lie algebroid differential $d_A : \Gamma \bigwedge^\bullet A^* \to \Gamma \bigwedge^{\bullet+1} A^*$ defined by the formula

$$\begin{aligned} d_A\alpha(a_0, \cdots, a_k) = & \sum_{i=0}^{k}(-1)^i\rho(a_i)\big(\alpha(a_0, \cdots, \widehat{a}_i, \cdots, a_k)\big) \\ & - \sum_{i<j}(-1)^{i+j}\alpha([a_i, a_j], a_0, \cdots, \widehat{a}_i, \cdots, \widehat{a}_j, \cdots, a_k) \end{aligned}$$

satisfies the following properties:

$$d_A(\alpha \wedge \beta) = d_A\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d_A\beta \quad \text{and} \quad d_A^2 = 0.$$

Therefore if we define the graded manifold $A[1] = (M, \Gamma \bigwedge^\bullet A^*)$ we have that by the first equation d_A defines a degree 1 vector field on $A[1]$ and by the second we know that the vector field satisfies equation (2.5). Therefore $(A[1], Q = d_A)$ is a Q-manifold. See also Section 3.4.1. The relation between Q-manifolds an Lie algebroids was first noticed in [120].

Example 2.13 (Tangent shifted by 1). In Example A.3 we see that given any graded manifold $\mathcal{M}$, we can define their tangent bundle $T\mathcal{M} \to \mathcal{M}$. As we explain in Section A.1 graded vector bundles can be shifted. As it happens in the non graded case, the De Rham differential defines a Q-structure for the graded manifold $T[1]\mathcal{M}$.

In coordinates, denote by $\{e^{j_i}, \theta^{j_i} = de^{j_i}\}$ the local coordinates of $T[1]\mathcal{M}$ where $\{e^{j_i}\}$ are the local coordinates on $\mathcal{M}$ with $|e^{j_i}| = i$, so $|\theta^{j_i}| = i + 1$. We can define the De Rham vector field as

$$Q_{dr} = \theta^{j_i}\frac{\partial}{\partial e^{j_i}}.$$

2.2.3 Tangent vectors at a point

Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be an n-manifold of dimension $m_0|\cdots|m_n$. Recall that we denote the stalk of $C_{\mathcal{M}}$ at $p \in M$ by $C_{\mathcal{M}_p}$. The graded algebra structure of $C_{\mathcal{M}}$ induces a graded $\mathbb{R}$-algebra structure on $C_{\mathcal{M}_p}$. Therefore we can define a *homogeneous tangent vector of degree k at a point $p \in M$* as a degree k $\mathbb{R}$-derivation of the graded $\mathbb{R}$-algebra $C_{\mathcal{M}_p}$, i.e. a linear map $v : C_{\mathcal{M}_p} \to \mathbb{R}$ satisfying:

$$v(\mathbf{fg}) = v(\mathbf{f})g^0(p) + (-1)^{k|f|} f^0(p)v(\mathbf{g}) \quad \forall \mathbf{f}, \mathbf{g} \in C_{\mathcal{M}_p},$$

where f^0 and g^0 denote the degree zero components of f and g respectively. We denote by $T_p\mathcal{M}$ the space of tangent vectors at a point $p \in M$. Since $C_{\mathcal{M}_p}$ is a graded $\mathbb{R}$-algebra, it is easy to see that $T_p\mathcal{M}$ is a graded vector space with $\dim(T_p\mathcal{M})_{-i} = m_i$ for $0 \leq i \leq n$, and zero otherwise.

Any vector field $X \in \mathfrak{X}^{1,\bullet}(\mathcal{M})$ defines a tangent vector at each point $p \in M$ by the formula:

$$X_p(\mathbf{f}) = X(f)^0(p) \qquad \forall \mathbf{f} \in C_{\mathcal{M}_p}. \tag{2.6}$$

We say that $X, Y \in \mathfrak{X}^{1,\bullet}(\mathcal{M})$ are *linearly independent* if $\forall p \in M,\ X_p$ and Y_p are linearly independent vectors. Since our graded manifolds are non-negatively graded, it follows directly from the definition that just non-positively graded vector fields can generate non-zero tangent vectors. Therefore, as opposed to smooth manifolds, we have that vector fields on a graded manifold are not determined by their tangent vectors.

Example 2.14 (Two different vector fields with the same tangent vector at all points). Consider the 1-manifold $\mathbb{R}^{1|1} = \{x, e\}$ where $|x| = 0$ and $|e| = 1$. Then we have that the vector fields

$$X = \frac{\partial}{\partial x} \quad \text{and} \quad Y = \frac{\partial}{\partial x} + e\frac{\partial}{\partial e}$$

have the same tangent vector at all points but it is clear that they are different.

Let $\mathcal{M}$ and $\mathcal{N}$ be two n-manifolds and

$$\Psi = (\psi, \psi^\sharp) : \mathcal{M} = (M, C_{\mathcal{M}}) \to \mathcal{N} = (N, C_{\mathcal{N}})$$

a morphism between them. The *differential of* Ψ *at* $p \in M$ is the linear map

$$\begin{array}{rccl} d_p\Psi : & T_p\mathcal{M} & \to & T_{\varphi(p)}\mathcal{N}, \\ & v & \longrightarrow & d_p\Psi(v)(\mathbf{g}) = v(\psi^\sharp \mathbf{g}) \quad \forall \mathbf{g} \in C_{\mathcal{N}_{\varphi(p)}}. \end{array}$$

Looking at the differential, we say that a morphism $\Psi : \mathcal{M} \to \mathcal{N}$ is:

- An *immersion* if $\forall p \in M$ $d_p\Psi$ is injective.
- A *submersion* if $\forall p \in M$ $d_p\Psi$ is surjective.
- An *embedding* if Ψ is injective immersion and φ is an embedding between smooth manifolds.

Proposition 2.15 (Local normal form of immersions). *Let $\Psi = (\psi, \psi^\sharp) : \mathcal{M} = (M, C_{\mathcal{M}}) \to \mathcal{N} = (N, C_{\mathcal{N}})$ be an immersion. Then there exist charts U around p and coordinates $\{e^{j_i}\}$ and $V = V_1 \times V_2$ around $\varphi(p)$ and coordinates $\{e^{j_i}\}$ on V_1 and $\{\widehat{e}^{l_k}\}$ on V_2 such that $\Psi_{|U} : \mathcal{M}_{|U} \to \mathcal{N}_{|V}$ has the form*

$$\psi(e^{j_0}) = (e^{j_0}, 0), \quad \psi^\sharp e^{j_i} = e^{j_i} \quad \textit{and} \quad \psi^\sharp \widehat{e}^{j_i} = 0.$$

2.3 Submanifolds

Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be a graded manifold. *A submanifold of* $\mathcal{M}$ is a graded manifold $\mathcal{N} = (N, C_{\mathcal{N}})$ together with an injective immersion $j : \mathcal{N} \to \mathcal{M}$. In addition, $(\mathcal{N}, j)$ is called *an embedded submanifold* if $j : \mathcal{N} \to \mathcal{M}$ is an embedding and is called *closed/connected submanifold* if $N \subseteq M$ is closed/connected. We say that two submanifolds $(\mathcal{N}, j)$ and $(\mathcal{N}', j')$ are *equivalent* if there exists a diffeomorphism $\Psi : \mathcal{N} \to \mathcal{N}'$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{N} \arrow[r, "j"] \arrow[dr, "\Psi"'] & \mathcal{M} \\ & \mathcal{N}' \arrow[u, "j'"'] \end{tikzcd}$$

We will make no distinction between equivalent submanifolds, and we will keep the term submanifold to refer to an equivalence class. The conclusion of some theorems in the following sections state that there exists a unique submanifold satisfying certain conditions. Uniqueness means up to equivalence, as defined above.

In differential geometry, we know that when a submanifold is closed and embedded we can recover it using the ideal defined by the functions that vanish on it. The idea of defining submanifolds using ideals of the sheaf of functions can be easily adapted to our context.

Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be a graded manifold. We say that $\mathcal{I} \subseteq C_{\mathcal{M}}$ is a subsheaf of *homogeneous ideals* if for all U open of M, $\mathcal{I}_{|U}$ is an ideal of $(C_{\mathcal{M}})_{|U}$, i.e.

$$(C_{\mathcal{M}})_{|U} \cdot \mathcal{I}_{|U} \subseteq \mathcal{I}_{|U}$$

and $\forall f \in \mathcal{I}_{|U}$ its homogeneous components also belong to $\mathcal{I}_{|U}$.

Given a subsheaf of homogeneous ideals $\mathcal{I} \subseteq C_{\mathcal{M}}$ we can define a subset of M by

$$Z(\mathcal{I}) = \{p \in M \mid \quad f(p) = 0 \ \forall f \in \mathcal{I}^0 = \mathcal{I} \cap C^0_{\mathcal{M}}\}.$$

We say that a homogeneous ideal is *regular* if $\forall p \in Z(\mathcal{I})$, $\exists\, U \subseteq M$ open around p where there are local coordinates $\{x^i, y^j\}$ of $\mathcal{M}_{|U}$ for which $\mathcal{I}_{|U} = \langle y^j = 0 \rangle$. In this case $Z(\mathcal{I})$ becomes a closed embedded submanifold of M.

Proposition 2.16. *Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be a graded manifold. There is a one to one correspondence between:*

- *Closed embedded submanifolds of $\mathcal{M}$.*
- *Subsheaves of regular homogeneous ideals of $C_{\mathcal{M}}$.*

Proof. Direct application of the local normal form for immersions. ☐

Let $(\mathcal{M}, Q)$ be a Q-manifold and $(\mathcal{N}, j)$ a closed embedded submanifold of $\mathcal{M}$ with associated ideal $\mathcal{I}$. We say that $(\mathcal{N}, j)$ is a *Q-submanifold* if

$$Q(\mathcal{I}) \subseteq \mathcal{I} \tag{2.7}$$

Example 2.17 (Graph of a morphism). Let $\Psi = (\psi, \psi^\sharp) : \mathcal{M} \to \mathcal{N}$ be a morphism between n-manifolds. We have that the graph of Ψ defines a closed embedded submanifold on $\mathcal{M} \times \mathcal{N}$ by the ideal

$$\mathcal{I}_{Gr(\Psi)} = \langle f \otimes g \in C_{\mathcal{M}} \widehat{\otimes} C_{\mathcal{N}} \mid f - \psi^\sharp g = 0 \rangle.$$

Moreover, let us see that, if $(\mathcal{M}, Q_1)$ and $(\mathcal{N}, Q_2)$ are Q-manifolds then Ψ is a Q-morphism if and only if $Gr(\Psi)$ is a Q-submanifold of $(\mathcal{M} \times \mathcal{N}, Q_1 \times Q_2)$:

$$\begin{aligned}
\psi^\sharp Q_2 g = Q_1 \psi^\sharp g \quad &\Leftrightarrow \Big(Q_1 \psi^\sharp g + \psi^\sharp g\Big) \otimes \Big(Q_2 g + g\Big) \in \mathcal{I}_{Gr(\Psi)} \\
&\Leftrightarrow Q_1 \psi^\sharp g \otimes Q_2 g + Q_1 \psi^\sharp g \otimes g + \psi^\sharp g \otimes Q_2 g + \psi^\sharp g \otimes g \in \mathcal{I}_{Gr(\Psi)} \\
&\Leftrightarrow Q_1 \psi^\sharp g \otimes g + \psi^\sharp g \otimes Q_2 g \in \mathcal{I}_{Gr(\Psi)} \\
&\Leftrightarrow Q_1 \times Q_2 (\psi^\sharp g \otimes g) \in \mathcal{I}_{Gr(\Psi)} \\
&\Leftrightarrow Q_1 \times Q_2 (\mathcal{I}_{Gr(\Psi)}) \subseteq \mathcal{I}_{Gr(\Psi)}.
\end{aligned}$$

2.4 Graded Poisson and symplectic manifolds

In this section we introduce the notions of graded Poisson and graded symplectic manifolds and prove their basic properties. Here we will define graded symplectic manifolds as graded manifolds endowed with a non-degenerate Poisson bracket, for a definition using 2-forms see Section A.3.

Consider $\mathcal{M} = (M, C_{\mathcal{M}})$ a graded manifold. We say that $\{\cdot,\cdot\}$ is a *degree k Poisson structure* on $\mathcal{M}$ if $(C_{\mathcal{M}}, \{\cdot,\cdot\})$ is a sheaf of graded Poisson algebras of degree k, i.e. $\{\cdot,\cdot\} : C^i_{\mathcal{M}} \times C^j_{\mathcal{M}} \to C^{i+j+k}_{\mathcal{M}}$ satisfies:

a). $\{f, g\} = -(-1)^{(|f|+k)(|g|+k)}\{g, f\}$.

b). $\{f, gh\} = \{f, g\}h + (-1)^{(|f|+k)|g|} g\{f, h\}$.

c). $\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{(|f|+k)(|g|+k)}\{g, \{f, h\}\}$.

Remark 2.18. Graded Poisson manifolds can be also defined in terms of bivector fields. Since we will not use it we refer the interested reader to [39, 94].

A direct consequence of the definition is that in a graded Poisson manifold $(\mathcal{M}, \{\cdot,\cdot\})$ any function $f \in C_{\mathcal{M}}$ has an associated vector field defined by the rule

$$f \rightsquigarrow X_f = \{f, \cdot\}; \tag{2.8}$$

this vector field is known as the *hamiltonian vector field of f*.

Let $X \in \mathfrak{X}^{1,i}(\mathcal{M})$ be a vector field. We say that X is a *Poisson vector field* if

$$X(\{f, g\}) = \{X(f), g\} + (-1)^{(|f|+k)i}\{f, X(g)\} \qquad \forall f, g \in C_{\mathcal{M}}. \tag{2.9}$$

A *PQ-manifold* is a triple $(\mathcal{M}, \{\cdot,\cdot\}, Q)$ where $(\mathcal{M}, \{\cdot,\cdot\})$ is a Poisson manifold equipped with $Q \in \mathfrak{X}^{1,1}(\mathcal{M})$ satisfying equations (2.5) and (2.9). In other words, a PQ-manifold is a graded manifold $\mathcal{M}$ that is graded Poisson as well as Q-manifold, and both structures are compatible.

Remark 2.19. PQ-manifolds were introduced by Schwarz in [112] in the context of supermanifolds in order to give a more geometric interpretation of the Batalin-Vilkovisky formalism. After that, the works of Ševera [113], Voronov [122] and Roytenberg [108] made the connections with Poisson and higher geometries that we emphasize here.

Let $(\mathcal{M}, \{\cdot,\cdot\})$ be a graded Poisson manifold and $\mathcal{N} \subseteq \mathcal{M}$ a closed embedded submanifold with associated ideal $\mathcal{I}$. We say that $\mathcal{N}$ is *coisotropic* if

$$\{\mathcal{I}, \mathcal{I}\} \subseteq \mathcal{I}. \tag{2.10}$$

Before defining graded symplectic manifolds we give some examples that illustrate the preceding definitions.

Example 2.20 (Lie algebroids as degree 1 Poisson manifolds). Let $(A \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid. It is known that $\Gamma \bigwedge^\bullet A$ inherits a *Gerstenhaber algebra* structure, i.e. a bracket

$$[\cdot,\cdot] : \Gamma \bigwedge^i A \times \Gamma \bigwedge^j A \to \Gamma \bigwedge^{i+j-1} A$$

satisfying properties a), b) and c) with $k = -1$. Therefore, if $A \to M$ is a Lie algebroid we have that the 1-manifold $A^*[1] = (M, \Gamma \bigwedge^\bullet A)$ inherits a Poisson structure of degree -1. See also Section 3.5.1. The relation between Lie algebroids and Gerstenhaber algebras is explained in [76].

Example 2.21 (Coisotropic submanifolds give subalgebroids). Let $(A \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid. By Example 2.20 we have that $A^*[1]$ is a Poisson manifold. Given a subbundle $B \to N$ over a closed embedded submanifold we can define a submanifold of $A^*[1]$ by $B^\circ[1]$, where "$\circ$" denotes the annihilator. The ideal associated to this submanifold is given by

$$\mathcal{I}^k = (\bigwedge^{k-1} \Gamma A) \wedge \Gamma B \ \text{ if } k \geq 1 \quad \text{and} \quad \mathcal{I}^0 = \{f \in C^\infty(M) \mid f(n) = 0 \ \forall n \in N\}.$$

Using equation (2.10) we have that $B^\circ[1]$ is a coisotropic submanifold of $A^*[1]$ if and only if $B \to N$ is a Lie subalgebroid of $(A \to M, [\cdot,\cdot], \rho)$. This example was introduced in [31].

Recall that a *Lie bialgebroid* $(A \to M, [\cdot,\cdot], \rho, \ d_{A^*})$ is a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ whose dual $(A^* \to M, [\cdot,\cdot]_{A^*}, \rho_{A^*})$ is also a Lie algebroid and d_{A^*} (see Example 2.12) is a derivation of the Gerstenhaber bracket on $\Gamma \bigwedge^\bullet A$. Lie bialgebroids were introduced in [90].

Example 2.22 (Basic examples of Lie bialgebroids). Let us present some classical examples of Lie bialgebroids:

- The trivial example: given any Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ consider the zero Lie algebroid structure on the dual. Then $(A \to M, [\cdot,\cdot], \rho, 0)$ is a Lie bialgebroid.
- Any Lie bialgebra $(\mathfrak{g}, [\cdot,\cdot], \delta)$ gives an example of a Lie bialgebroid over a point. Recall that Lie bialgebras are the infinitesimal data of Poisson-Lie groups, see e.g. [54].

- Recall from Example 2.11 that given a Poisson manifold (M, π), their cotangent bundle $(T^*M \to M, [\cdot,\cdot]_\pi, \pi^\sharp)$ is a Lie algebroid. Since the dual is also a Lie algebroid with the Lie bracket of vector fields, it remains to see that the compatibility follows and this is satisfied because we can use the Jacobi identity of the Schouten bracket and the fact that $d_\pi = [\pi, \cdot]$. Therefore $(T^*M \to M, [\cdot,\cdot]_\pi, \pi^\sharp, d)$ is a Lie bialgebroid. This example was introduced in [90].

Example 2.23 (Lie bialgebroids as degree 1 PQ-manifolds). Let $(A \to M, [\cdot,\cdot], \rho,\ d_{A^*})$ be a Lie bialgebroid. By Example 2.20 we know that $A^*[1]$ is a Poisson manifold and by Example 2.12 that it is a Q-manifold. The Lie bialgebroid compatibility condition is equivalent to equation (2.9). Therefore $(A^*[1], \{\cdot,\cdot\}, Q)$ is a degree 1 PQ-manifold. See also Section 3.6.1. The relation between Lie bialgebroids and graded geometry was explained in [122].

Let $(\mathcal{M} = (M, C_{\mathcal{M}}), \{\cdot,\cdot\})$ be a graded Poisson manifold. We say that the Poisson bracket is *non-degenerate* if $\forall p \in M$, $\exists\ U \subseteq M$ open around p and coordinates $\{e^{j_i}\}$ of $\mathcal{M}_{|U}$ such that the matrix

$$(\{e^{j_i}, e^{l_k}\}^0(q)) \quad \text{is invertible } \forall q \in U,$$

where $\{e^{j_i}, e^{l_k}\}^0(q)$ denotes the degree 0 part of the function $\{e^{j_i}, e^{l_k}\}$ evaluated at the point q.

A *graded symplectic manifold of degree n* $(\mathcal{M}, \{\cdot,\cdot\})$ is an n-manifold $\mathcal{M}$ endowed with a non-degenerate Poisson bracket of degree $-n$.

Let $(\mathcal{M}, \{\cdot,\cdot\})$ be a graded symplectic manifold and $\mathcal{N} \subset \mathcal{M}$ a closed embedded submanifold. We say that $\mathcal{N}$ is a *lagrangian submanifold* if it is coisotropic and

$$\text{Totdim } \mathcal{N} = \frac{1}{2}\text{Totdim } \mathcal{M}.$$

Remark 2.24. For a definition of graded symplectic manifolds using forms, see Section A.3. Observe that from our definition there are graded symplectic manifolds with odd total dimension. Therefore these kind of graded symplectic manifolds have no lagrangian submanifolds. There are other ways of defining lagrangian submanifolds, see Section A.3.

Example 2.25 (Cotangent bundles as degree 1 symplectic manifolds). A particular case of Example 2.20 is given by the tangent bundle of any manifold, $TM \to M$. It is a Lie algebroid with anchor the identity map and bracket given by the Lie bracket of vector fields, therefore $T^*[1]M$ is Poisson. Moreover, if we pick local coordinates of M given by $\{x^i\}$, we have that $\{x^i, \frac{\partial}{\partial x^i}\}$ are coordinates of $T^*[1]M$ that satisfies $\{x^i, \frac{\partial}{\partial x^j}\} = \delta_{ij}$ therefore $T^*[1]M$ is symplectic.

In this case for any $N \subset M$ submanifold, if we define

$$TN^\circ = \{\alpha \in T^*M | \langle \alpha, v\rangle = 0\ \forall v \in TN\}$$

we have that $(TN^\circ)[1] = (N, \bigwedge^\bullet \Gamma \frac{TM}{TN}) \subseteq T^*[1]M$ defines a lagrangian submanifold.

Chapter 3

Geometrization of graded manifolds

In this chapter we propose a new way of relating graded manifolds and classical geometric objects. Our point of view extends the correspondence between 1-manifolds and vector bundles and 2-manifolds with triples $(E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \hookrightarrow E_2)$ defined in [24]. We also relate our new geometrization with the preceding ones given in [13, 48, 70, 82, 121]. This alternative viewpoint will be fundamental to see classical Poisson and higher geometry as graded manifolds.

In the first section we describe the geometrization functor for graded manifolds while in the other we systematically describe graded objects in terms of classical data. Some of the results were already known while others are new. We provide references in each case.

Let us explain more carefully the idea behind the geometrization. Pick $\mathcal{M} = (M, C_{\mathcal{M}})$ an n-manifold. By condition (2.1) we know that $C^0_{\mathcal{M}} = C^\infty(M)$. Using now that $C_{\mathcal{M}}$ is a sheaf of graded algebras we obtain that the multiplication of functions on the graded manifold satisfies

$$C^i_{\mathcal{M}} \cdot C^j_{\mathcal{M}} \subseteq C^{i+j}_{\mathcal{M}}. \tag{3.1}$$

In particular, we get that $C^0_{\mathcal{M}} = C^\infty(M)$ acts in each $C^i_{\mathcal{M}}$. Therefore $C^i_{\mathcal{M}}$ are $C^\infty(M)$-modules that are locally free and finitely generated by condition (2.1). Hence, modulo isomorphism, there exist vector bundles $E_i \to M$ such that $C^i_{\mathcal{M}} = \Gamma E_i$.

The second observation is that on a degree n-manifold we can recover the functions of degree more than n from these of degree less or equal to n. Hence, if we define the ideal $\mathcal{I} = \langle f \in C_{\mathcal{M}} \mid |f| > n \rangle$, the shaves of graded commutative algebras $C_{\mathcal{M}}$ and $C_{\mathcal{M}}/\mathcal{I}$ have the same information. In other words, it is enough to know the first n vector bundles: $E_1, \cdots, E_n$.

Finally, the vector bundles $E_1, \cdots, E_n$ have relations among them. The multiplication of functions given by (3.1) endow the sections of the vector bundles with an algebra structure encoded on maps

$$m_{ij} : \Gamma E_i \times \Gamma E_j \to \Gamma E_{i+j}, \quad i + j \leq n.$$

that respect the $C^\infty(M)$-module structure.

Therefore, what remains to be identified are the additional conditions that this algebra bundle must satisfy in order to define a graded manifold. Or in other words, in order to satisfy condition (2.1). We give the answer in the next section and it corresponds to our definition of *admissible coalgebra bundles*.

There is a duality between the space and the functions on a space and this is why the functions form an algebra while the space is given by the dual object, in our case a coalgebra. The geometrization Theorem 3.5 is given in terms of coalgebra bundles and not of algebra bundles in order to make the functor covariant.

3.1 The category $\mathcal{CoB}^n$

Let us introduce the geometric counterpart of graded manifolds. Given a positive integer $n \in \mathbb{N}$, an *n-coalgebra bundle* is a pair $(\mathbf{E}^* \to M, \mu)$ where $\mathbf{E}^* = \oplus_{i=-n}^{-1} E_i^* \to M$ is a (negatively) graded (finite dimensional) vector bundle and $\mu : \mathbf{E}^* \to \mathbf{E}^* \otimes \mathbf{E}^*$ is a degree preserving vector bundle map.

Remark 3.1. Since our graded vector bundle is finite dimensional we can identify $\Gamma(\mathbf{E} \otimes \mathbf{E})$ with $\Gamma\mathbf{E} \otimes_{C^\infty(M)} \Gamma\mathbf{E}$ as $C^\infty(M)$-modules. Therefore if we define the induced map on sections by $m = \mu^\sharp : \Gamma(\mathbf{E} \otimes \mathbf{E}) = \Gamma\mathbf{E} \otimes_{C^\infty(M)} \Gamma\mathbf{E} \to \Gamma\mathbf{E}$, we have that $(\Gamma\mathbf{E}, m)$ defines a sheaf of graded algebras over M.

Example 3.2. Given a graded vector bundle $\mathbf{E}^* = \oplus_{i=-n}^{-1} E_i^* \to M$ define a new graded vector bundle $(\operatorname{Sym}\mathbf{E}^*)^{\leq n} \to M$ given by

$$(\operatorname{Sym}\mathbf{E}^*)^{\leq n}{}_k = \begin{cases} \operatorname{Sym}_k \mathbf{E}^* & \text{if } -n \leq k \leq -1, \\ 0 & \text{otherwise,} \end{cases}$$

where the grading of $\operatorname{Sym}_k \mathbf{E}^*$ is taken as the sum of the degrees as elements of $\mathbf{E}$. $(\operatorname{Sym}\mathbf{E}^*)^{\leq n} \to M$ has a natural coalgebra structure given by the symmetric coproduct; we denote it by $sc : (\operatorname{Sym}\mathbf{E}^*)^{\leq n} \to (\operatorname{Sym}\mathbf{E}^*)^{\leq n} \otimes (\operatorname{Sym}\mathbf{E}^*)^{\leq n}$. In other words, if we define

$$m = sc^\sharp : \Gamma\big((\operatorname{Sym}\mathbf{E}^*)^{\leq n}\big)^* \times \Gamma\big((\operatorname{Sym}\mathbf{E}^*)^{\leq n}\big)^* \to \Gamma\big((\operatorname{Sym}\mathbf{E}^*)^{\leq n}\big)^*$$

and use the identification $\Gamma\big((\operatorname{Sym}\mathbf{E}^*)^{\leq n}\big)^* = \Gamma(\operatorname{Sym}\mathbf{E})^{\leq n}$, then the algebra structure is given by

$$m(\omega, \eta) = \begin{cases} \omega \cdot \eta & \text{if } |\omega| + |\eta| \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

for $\omega, \eta \in \Gamma(\operatorname{Sym}\mathbf{E})^{\leq n}$ and where "$\cdot$" denotes the symmetric product between them.

Definition 3.3. Given an n-coalgebra bundle $(\mathbf{E}^* \to M, \mu)$ we say that it is *admissible* if there exists a graded vector bundle $\mathbf{D}^* = \oplus_{i=-n}^{-1} D_i^* \to M$ and a vector bundle isomorphism $\phi : \mathbf{E}^* \to (\operatorname{Sym}\mathbf{D}^*)^{\leq n}$ that covers the identity and $sc \circ \phi = (\phi \otimes \phi) \circ \mu$.

Remark 3.4. Observe that in our definition of coalgebras we do not require any co-associative property but if we pick an admissible one then all the usual properties must be satisfied: Co-associativity, graded co-commutative, co-unit.

For any positive integer $n \in \mathbb{N}$, the category $\mathcal{C}o\mathcal{B}^n$ has as objects the admissible n-coalgebra bundles and as morphisms $\Phi = (\varphi, \phi) : (\mathbf{E}^* \to M, \mu) \to (\mathbf{F}^* \to N, \nu)$ (degree preserving) vector bundle maps

$$\begin{array}{ccc} \mathbf{E}^* & \xrightarrow{\phi} & \mathbf{F}^* \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi} & N \end{array}$$

satisfying $\nu \circ \phi = (\phi \otimes \phi) \circ \mu$.

3.1.1 The geometrization functor

In this section we will prove the main result of this chapter:

Theorem 3.5 (Geometrization of graded manifolds)**.** *For any number $n \in \mathbb{N}^*$ the functor*

$$\begin{array}{cccl} \mathcal{F}: & \mathcal{C}o\mathcal{B}^n & \to & \mathcal{G}\mathcal{M}^n \\ & (\mathbf{E}^* \to M, \mu) & & \mathcal{F}(\mathbf{E}^* \to M, \mu) = (M, \Gamma \operatorname{Sym} \mathbf{E}/\mathcal{I}_\mu) \end{array}$$

is an equivalence of categories, where $\mathcal{I}_\mu = \langle \omega \cdot \eta = m(\omega, \eta) | \omega, \eta \in \Gamma\mathbf{E}, |\omega| + |\eta| \leq n \rangle$.

This result allows us to think of graded manifolds as vector bundles with some extra structure. Therefore we will be able to reinterpret graded objects (Q-structures, graded Poisson brackets, symplectic forms...) in terms of vector bundle data. Before we give a proof of the geometrization Theorem we need two auxiliary results:

Lemma 3.6 (Any manifold is isomorphic to a split one)**.** *Let $\mathcal{M} = (M, C_\mathcal{M})$ be an n-manifold. Then there exists a non positive vector bundle $\mathbf{D}^* = \oplus_{i=-n}^{-1} D_i^* \to M$ and an isomorphism of n-manifolds between $\mathcal{M}$ and $(M, \Gamma \operatorname{Sym} \mathbf{D})$.*

Proof. We will prove the result by induction on n.

For $n = 1$ we have that for any 1-manifold $\mathcal{M} = (M, C_\mathcal{M})$, the local condition (2.1) guarantees that $C_\mathcal{M}$ is generated as an algebra by $C^0_\mathcal{M} = C^\infty(M)$ and by $C^1_\mathcal{M}$, i.e. $C^k_\mathcal{M} = \operatorname{Sym}_k C^1_\mathcal{M}$. Since the multiplication is graded, we have that $C^1_\mathcal{M}$ is a $C^\infty(M)$-module and equation (2.1) ensures that is locally trivial. Therefore there exists a vector bundle $E \to M$ such that $C^1_\mathcal{M} \cong \Gamma E$. Hence $E^*[1] = (M, \Gamma \bigwedge^\bullet E)$ and $\mathcal{M}$ are isomorphic.

Suppose that the result is true for $n = k-1$, and let us prove for $n = k$. Consider a k-manifold $\mathcal{M} = (M, C_\mathcal{M})$. Recall that associated to any graded manifold we have the tower (2.2). By the inductive hypothesis we have that there exists a vector bundle $\mathbf{D}^* = \oplus_{i=-k+1}^{-1} D_i^*$ such that $(M, \Gamma \operatorname{Sym} \mathbf{D})$ and $\mathcal{M}_{k-1}$ are isomorphic. So $C^i_{\mathcal{M}_{k-1}} \cong \Gamma \operatorname{Sym}_i \mathbf{D}$, $\forall i \in \mathbb{N}$.

Now we have that $C_\mathcal{M}$ is generated as an algebra by $C_{\mathcal{M}_{k-1}}$ and $C^k_\mathcal{M}$. In fact, they fit in the exact sequence

$$0 \to C^k_{\mathcal{M}_{k-1}} \to C^k_\mathcal{M} \to \frac{C^k_\mathcal{M}}{C^k_{\mathcal{M}_{k-1}}} \to 0.$$

As it happens for $n = 1$ we have that the graded multiplication with degree 0 functions guarantees that $C^k_{\mathcal{M}}$ is a $C^\infty(M)$-module and equation (2.1) says that it is locally trivial. Therefore there exists a vector bundle $E \to M$ such that $C^k_{\mathcal{M}} = \Gamma E$. Moreover, the projection $\mathcal{M} \to \mathcal{M}_{k-1}$ induces the following exact sequence of vector bundles

$$0 \to \operatorname{Sym}_k \mathbf{D} \to E \to \frac{E}{\operatorname{Sym}_k \mathbf{D}} = D_k \to 0$$

Since any exact sequence of vector bundles splits, we know that $E \cong D_k \oplus \operatorname{Sym}_k \mathbf{D}$. If we define $\widehat{\mathbf{D}}^* = (\mathbf{D}^* \oplus D^*_{-k}) \to M$ we obtain that $\mathcal{M}$ and $(M, \Gamma \operatorname{Sym} \widehat{\mathbf{D}})$ are isomorphic, as we want. □

Remark 3.7. This lemma was proved in [13]. Here we give a shorter proof using the tower of affine fibrations associated to a graded manifold.

Given an n-coalgebra bundle $(\mathbf{E}^* \to M, \mu)$ we already denoted by $m = \mu^\sharp : \Gamma\mathbf{E} \times \Gamma\mathbf{E} \to \Gamma\mathbf{E}$ the dual algebra structure. Inside the algebra $(\Gamma \operatorname{Sym} \mathbf{E}, \cdot)$ we can define the following ideal:

$$\mathcal{I}_\mu = \langle \omega \cdot \eta - m(\omega, \eta) = 0, \text{ for } \omega, \eta \in \Gamma\mathbf{E} \text{ with } |\omega| + |\eta| \leq n \rangle,$$

where " $\cdot$ " denotes the symmetric product.

Remark 3.8. From the above definition it is clear that $\mathcal{I}_\mu$ is a homogeneous ideal with respect to the grading of $\operatorname{Sym}\mathbf{E}$. Therefore $\Gamma \operatorname{Sym} \mathbf{E}/\mathcal{I}_\mu$ is also a graded algebra. Moreover, for $1 \leq i \leq n$,

$$\Gamma(\operatorname{Sym} \mathbf{E}/\mathcal{I}_\mu)_i = \Gamma E_i.$$

Lemma 3.9. *Let $\mathbf{D}^* = \oplus_{i=-n}^{-1} D_i^* \to M$ be a graded vector bundle, and consider the coalgebra bundle $((\operatorname{Sym} \mathbf{D}^*)^{\leq n}, sc)$ of Example 3.2. Then*

$$\Gamma \operatorname{Sym} \left((\operatorname{Sym} \mathbf{D})^{\leq n} \right) / \mathcal{I}_{sc} = \Gamma \operatorname{Sym} \mathbf{D}$$

Proof. Let us denote the graded vector bundle $(\operatorname{Sym} \mathbf{D}^*)^{\leq n}$ by $\mathbf{E}^*$, the symmetric product on $\Gamma \operatorname{Sym} \mathbf{E}$ by " $\cdot$ " and the symmetric product on $\Gamma \operatorname{Sym} \mathbf{D}$ by " $\vee$ ". By Remark 3.8 we have that if $j \leq n$ then $\Gamma(\operatorname{Sym} \mathbf{E}/\mathcal{I}_{sc})_j = \Gamma E_j$ and since $\operatorname{Sym} \mathbf{D}^*$ is generated up to degree n we conclude that $\Gamma \operatorname{Sym} \mathbf{D}$ is included in $\Gamma(\operatorname{Sym} \mathbf{E}/\mathcal{I}_{sc})$. Moreover, $\forall\, \omega, \eta \in \Gamma\mathbf{D}$,

$$\begin{cases} m(\omega, \eta) = \omega \vee \eta & \text{if } |\omega| + |\eta| \leq n, \\ \omega \cdot \eta = \omega \vee \eta & \text{if } |\omega| + |\eta| > n. \end{cases}$$

Let us see the other inclusion. Any element on $\Gamma \operatorname{Sym} \mathbf{E}$ is of the form $e_1 \cdot - \cdot e_n$ where $e_i \in \Gamma\mathbf{E}$ and each $e_i = d_{1_i} \vee - \vee d_{j_i}$ for some $d_{k_i} \in \Gamma\mathbf{D}$ with $|d_{k_i}| \leq |d_{k+1_i}|$. Let us see that using the relations $\omega \cdot \eta = m(\omega, \eta)$ if $|\omega| + |\eta| \leq n$ we can show that $e_1 \cdot - \cdot e_n = e_1 \vee - \vee e_n$.

First notice that since $|e_i| \leq n$ we have that $|d_{1_i}| + |d_{2_i}| \leq n$. Therefore

$$e_1 \cdot (d_{1_2} \vee d_{2_2} \vee - \vee d_{j_2}) \cdot - \cdot e_n = e_1 \cdot d_{1_2} \cdot (d_{2_2} \vee - \vee d_{j_2}) \cdot - \cdot e_n$$

so it is enough to see that $(d_1 \vee - \vee d_n) \cdot d_{n+1} = d_1 \vee - \vee d_{n+1}$. For this use again the relations on $\mathcal{I}_{sc}$ combined with the symmetric product and the fact that the degree of d_i are ordered to conclude that they are equal. □

With this lemmas we can prove the geometrization of graded manifolds:

Proof of Theorem 3.5. We start by defining $\mathcal{F}$ at the level of morphisms. Consider $\Phi = (\varphi, \phi) : (\mathbf{E}^* \to M, \mu) \to (\mathbf{F}^* \to N, \nu)$ a coalgebra map. Denote by $S\Phi = (\varphi, S\phi) : \operatorname{Sym} \mathbf{E}^* \to Sym\mathbf{F}^*$ the natural extension to the symmetric products. The fact that Φ is a coalgebra map implies that $\phi^\sharp : \Gamma\mathbf{F} \to \varphi_*\Gamma\mathbf{E}$ is an algebra map, then $S\phi^\sharp\mathcal{I}_\nu \subseteq \mathcal{I}_\mu$. So there is a well defined map between $\Gamma \operatorname{Sym} \mathbf{F}/\mathcal{I}_\nu$ and $\varphi_*\Gamma \operatorname{Sym} \mathbf{E}/\mathcal{I}_\mu$ that preserves the algebra structure, denote this map by $\widehat{\phi}$. Therefore $\mathcal{F}(\Phi) = (\varphi, \widehat{\phi}) : (M, \Gamma \operatorname{Sym} \mathbf{E}/\mathcal{I}_\mu) \to (N, \Gamma \operatorname{Sym} \mathbf{F}/\mathcal{I}_\nu)$ is a map of ringed spaces such that $\widehat{\phi}$ is a morphism of sheaves of algebras as we want.

Now we show that $\mathcal{F}$ is well defined at the level of objects. Given an admissible coalgebra bundle $(\mathbf{E}^* \to M, \mu)$ we must prove that

$$\mathcal{M} = \mathcal{F}(\mathbf{E}^* \to M, \mu) = (M, C_\mathcal{M} = \Gamma \operatorname{Sym} \mathbf{E}/\mathcal{I}_\mu)$$

satisfies condition (2.1). Since $(\mathbf{E}^* \to M, \mu)$ is admissible there exists $\mathbf{D}^* = \oplus_{i=-n}^{-1} D_i^* \to M$ and an isomorphism $\Phi = (\mathrm{Id}, \phi) : ((\operatorname{Sym} \mathbf{D}^*)^{\leq n}, sc) \to (\mathbf{E}^*, \mu)$. Therefore using the previous Lemma 3.9 and that $S\Phi^\sharp$ is also an isomorphism and sends $\mathcal{I}_\mu$ to $\mathcal{I}_{sc}$, we conclude that

$$C_\mathcal{M} = \Gamma \operatorname{Sym} \mathbf{E}/\mathcal{I}_\mu \cong \Gamma \operatorname{Sym} \left((\operatorname{Sym} \mathbf{D})^{\leq n} \right) / \mathcal{I}_{sc} = \Gamma \operatorname{Sym} \mathbf{D}.$$

In order to see that $\mathcal{F}$ is a fully faithful functor, consider

$$(\mathbf{E}^* \to M, \mu), (\mathbf{F}^* \to N, \nu) \in \mathcal{C}o\mathcal{B}^n.$$

Denote by $\mathcal{M} = (M, C_\mathcal{M})$ and $\mathcal{N} = (N, C_\mathcal{N})$ their respective images under $\mathcal{F}$. Recall that by the definition of the ideals $\mathcal{I}_\mu$ and $\mathcal{I}_\nu$, we have that $C^i_\mathcal{M} = \Gamma E_i$ and $C^i_\mathcal{N} = \Gamma F_i$ for $1 \leq i \leq n$. Given two coalgebra maps $\Phi_1 = (\varphi_1, \phi_1), \Phi_2 = (\varphi_2, \phi_2) : (\mathbf{E}^*, \mu) \to (\mathbf{F}^*, \nu)$ they define two morphisms between graded manifolds $\mathcal{F}(\Phi_1) = (\varphi_1, \widehat{\phi}_1), \mathcal{F}(\Phi_2) = (\varphi_2, \widehat{\phi}_2) : \mathcal{M} \to \mathcal{N}$. We have that $\mathcal{F}(\Phi_1) = \mathcal{F}(\Phi_2)$ if and only if

$$\begin{cases} \varphi_1 = \varphi_2, \\ \widehat{\phi}^i_1 = \widehat{\phi}^i_2 : C^i_\mathcal{N} \to \varphi_{1*} C^i_\mathcal{M} \quad \text{for } 1 \leq i \leq n. \end{cases}$$

But $\widehat{\phi}^i_1 = {\phi_1^\sharp}^i : \Gamma F_i \to \varphi_{1*}\Gamma E_i$. Therefore $\Phi_1 = \Phi_2$, so $\mathcal{F}$ is faithful.

Let us see that $\mathcal{F}$ is full. Given a morphism $\Psi = (\psi, \psi^\sharp) : \mathcal{M} \to \mathcal{N}$ we have that $\psi^\sharp : C_\mathcal{N} \to \psi_* C_\mathcal{M}$ is a morphism of sheaves of graded algebras so the multiplication for degree 0 functions implies that ${\psi^\sharp}^i : C^i_\mathcal{N} = \Gamma F_i \to \psi_* C^i_\mathcal{M} = \psi_*\Gamma E_i$ is a map of modules over the map $\psi^* : C^\infty(N) \to C^\infty(M)$. Therefore there exists a vector bundle map $\phi : \mathbf{E}^* \to \mathbf{F}^*$ covering $\psi : M \to N$ such that ${\phi^\sharp}^i = {\psi^\sharp}^i$ for $1 \leq i \leq n$. Finally the fact that $\psi^\sharp$ is an algebra morphism implies that $\phi : \mathbf{E}^* \to \mathbf{F}^*$ is a coalgebra morphism.

Let us prove that $\mathcal{F}$ is essentially surjective. Given a graded manifold $\mathcal{M} = (M, C_\mathcal{M})$ by Lemma 3.6 there exists a graded vector bundle $\mathbf{D}^* \to M$ such that $\mathcal{M}$ and $(M, \Gamma \operatorname{Sym} D)$ are isomorphic. By Lemma 3.9 we have that $\mathcal{F}((\operatorname{Sym} \mathbf{D}^*)^{\leq n}, sc) = (M, \Gamma \operatorname{Sym} D)$. Therefore $\mathcal{F}$ is essentially surjective. $\square$

Corollary 3.10. *Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be an n-manifold. Choose $\mathbf{E} = \oplus_{i=1}^{n} E_i$ graded vector bundle over M, such that $C^i_{\mathcal{M}} \cong \Gamma E_i$. Then $(\mathbf{E}^* \to M, \mu)$ is an addmissible algebra bundle where $\mu^{\sharp}_{i,j} : \Gamma E_i \times \Gamma E_j \to \Gamma E_{i+j}$ is given by the multiplication of functions on the graded manifold for $i + j \leq n$.*

Proof. Given a graded manifold $\mathcal{M} = (M, C_{\mathcal{M}})$ we have that $C^0_{\mathcal{M}} = C^{\infty}(M)$ and the multiplication of functions ensures that $C^i_{\mathcal{M}}$ are $C^{\infty}(M)$-modules. In addition equation (2.1) implies that $C^i_{\mathcal{M}}$ are locally free and finitely generated. Therefore $C^i_{\mathcal{M}} = \Gamma E_i$ for some vector bundle E_i. Define $\mathbf{E}^* = \oplus_{i=-n}^{-1} E^*_{-i}$. Now see that the multiplication of functions is an operation $C^i_{\mathcal{M}} \cdot C^j_{\mathcal{M}} \subseteq C^{i+j}_{\mathcal{M}}$ that commutes with degree 0 functions therefore we have maps of $C^{\infty}(M)$-modules

$$m_{ij} : \Gamma E_i \otimes_{C^{\infty}(M)} \Gamma E_j \to \Gamma E_{i+j}.$$

Pick the maps for $i + j \leq n$. They define a morphism of $C^{\infty}(M)$-modules $m : \Gamma \mathbf{E} \otimes_{C^{\infty}(M)} \Gamma \mathbf{E} \to \Gamma \mathbf{E}$. Identifying as before $\Gamma \mathbf{E} \otimes_{C^{\infty}(M)} \Gamma \mathbf{E}$ with $\Gamma(\mathbf{E} \otimes \mathbf{E})$, we have that $\mathbf{E}^*$ has a coalgebra structure μ.

It remains to prove that $(\mathbf{E}^*, \mu)$ is admissible. By Lemma 3.6 there exists a graded vector bundle $\mathbf{D}^* = \oplus_{i=-n}^{-1} D^*_i \to M$ such that $\mathcal{M}$ is isomorphic to $(M, \Gamma \operatorname{Sym} \mathbf{D})$, meaning that there exists $\psi^{\sharp} : C_{\mathcal{M}} \to \Gamma \operatorname{Sym} \mathbf{D}$ isomorphism of sheaves of graded algebras. Therefore $\psi^{\sharp^i} : C^i_{\mathcal{M}} = \Gamma E_i \to \Gamma \operatorname{Sym}_i D$ is an isomorphism of $C^{\infty}(M)$-modules for $1 \leq i \leq n$. So there exists $\phi : (\operatorname{Sym} \mathbf{D}^*)^{\leq n} \to \mathbf{E}^*$ isomorphism of vector bundles such that, on sections, it coincides with $\psi^{\sharp}$; the fact that $\psi^{\sharp}$ is an algebra isomorphism implies that ϕ is a coalgebra isomorphism. $\square$

Let us describe the category $\mathcal{C}o\mathcal{B}^n$ for lower n:

Example 3.11 ($\mathcal{C}o\mathcal{B}^1$). Let $E^* \to M$ be a graded vector bundle just concentrated in degree -1, i.e. $\mathbf{E}^* = \oplus_i E^*_i = E^*_{-1}$. Therefore $E^* \otimes E^* = \oplus_i F_i = (F_{-2} = E^*_{-1} \otimes E^*_{-1})$. Hence, a coalgebra bundle structure on E^* is just zero. And observe that this is admissible because $(\operatorname{Sym} E^*)^{\leq 1} = E^*$. Then objects in $\mathcal{C}o\mathcal{B}^1$ can be identified with vector bundles, and morphisms with vector bundle morphisms. Hence, $\mathcal{C}o\mathcal{B}^1$ is identified with the category $\mathcal{V}ect$ of vector bundles.

So any 1-manifold $\mathcal{M} = (M, C_{\mathcal{M}})$ can be identify with $\mathcal{M} = E[1]$ where $E \to M$, is a vector bundle.

Example 3.12 ($\mathcal{C}o\mathcal{B}^2$). A 2-coalgebra bundle is the same as two vector bundles E^*_{-1} and E^*_{-2} and a map $\mu : E^*_{-2} \to E^*_{-1} \otimes E^*_{-1}$. The admissibility property is equivalent to the existence of a D^*_{-2} and an isomorphism $\phi : E^*_{-2} \to D^*_{-2} \oplus E^*_{-1} \wedge E^*_{-1}$ such that $\mu = sc \circ \phi$. This happens if and only if

$$\operatorname{Coker}(\mu) = \frac{E^*_{-1} \otimes E^*_{-1}}{E^*_{-1} \wedge E^*_{-1}}.$$

A coalgebra morphism between $(E^*_{-1} \to M, E^*_{-2} \to M, \mu^E : E^*_{-2} \to E^*_{-1} \otimes E^*_{-1})$ and $(F^*_{-1} \to N, F^*_{-2} \to N, \mu^F : F^*_{-2} \to F^*_{-1} \otimes F^*_{-1})$ is a pair of vector bundle morphisms covering the same base $(\varphi, \phi_{-1}) : (E^*_{-1} \to M) \to (F^*_{-1} \to N)$ and

$(\varphi, \phi_{-2}) : (E^*_{-2} \to M) \to (F^*_{-2} \to N)$ satisfying that the following diagram commutes:

$$
\begin{tikzcd}
E^*_{-2} \arrow[r, "\phi_{-2}"] \arrow[d, "\mu^E"] & F^*_{-2} \arrow[d, "\mu^F"] \\
E^*_{-1} \otimes E^*_{-1} \arrow[r, "\phi_{-1}\otimes\phi_{-1}"] & F^*_{-1} \otimes F^*_{-1}
\end{tikzcd}
$$

So any 2-manifold $\mathcal{M} = (M, C_{\mathcal{M}})$ can be identify with $(M, \frac{\wedge^\bullet E_1 \otimes \mathrm{Sym}^\bullet E_2}{\langle e\wedge e'\otimes 1 - 1\otimes m(e,e')\rangle})$ for some coalgebra bundle given by $(E^*_{-1}, E^*_{-2}, \mu : E^*_{-2} \to E^*_{-1} \otimes E^*_{-1})$.

Remark 3.13. In [24] the category VB2 is defined, objects are $(E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \hookrightarrow E_2)$ and morphism triples $(\varphi, \phi_1, \phi_2)$, where $(\varphi, \phi_i) : (E_i \to M) \to (F_i \to N)$ for $i = 1, 2$ are vector bundle maps satisfying

$$
\begin{tikzcd}
E_1 \wedge E_1 \arrow[r, "\phi_1\wedge\phi_1"] \arrow[d, "\psi^E"] & F_1 \wedge F_1 \arrow[d, "\psi^F"] \\
E_2 \arrow[r, "\phi_2"] & F_2
\end{tikzcd}
$$

Clearly VB2 and $\mathcal{C}o\mathcal{B}^2$ are equivalent categories. Therefore instead of working with a map with kernel we will use in degree 2 the category VB2.

Example 3.14 ($\mathcal{C}o\mathcal{B}^3$). A similar calculation as in the two previous examples tells us that admissible 3-coalgebra bundles are the same as a quintuple

$$(E^*_{-1},\ E^*_{-2},\ E^*_{-3},\ \mu_{-2} : E^*_{-2} \to E^*_{-1} \otimes E^*_{-1},\ \mu_{-3} : E_{-3} \to E^*_{-2} \otimes E^*_{-1})$$

satisfying the following conditions:

- $\mathrm{Coker}(\mu_{-2}) = \frac{E^*_{-1}\otimes E^*_{-1}}{E^*_{-1}\wedge E^*_{-1}}$.
- $\mathrm{Coker}(\mu_{-3}) = \frac{\mu_{-2}(E^*_{-2})\otimes E^*_{-1}}{\bigwedge^3 E^*_{-1}}$.
- $(\mu_{-2} \otimes \mathrm{Id}) \circ \mu_{-3} = (\mathrm{Id} \otimes \mu_{-2}) \circ \mu_{-3}$.

Clearly a coalgebra morphism is equivalent to three vector bundle maps over the same base that commute with μ_{-2} and μ_{-3} of E^* and F^*.

Will be interesting to compare this conditions with ones obtained in [18] for graded bundles instead of graded manifolds.

We finish this section with some remarks concerning functions on a graded manifold. In our description we treat the degree zero functions as functions on the body while for the other degrees they are always sections of a vector bundle. Sometimes it is convenient to treat all of them together. Hence, given a coalgebra bundle $(\mathbf{E}^* = \oplus_{i=-n}^{-1} E^*_i \to M, \mu)$ we define the *extended coalgebra bundle* as $(\mathbf{E}' \to M, \mu')$ where

$$E'^*_i = \begin{cases} E^*_i & -n \le i \le -1, \quad m'(s_i, s_j) = m(s_i, s_j) \quad s_i \in \Gamma E_i,\ s_j \in \Gamma E_j. \\ \mathbb{R}^*_M & i = 0, \quad m'(f, s_i) = f s_i \quad f \in \Gamma\mathbb{R} = C^\infty(M). \end{cases} \tag{3.2}$$

and $m' = \mu'^\sharp$.

Pick an admissible coalgebra bundle $(\mathbf{E}^* \to M, \mu)$. Since it is admissible we have that $\widetilde{\mathbf{E}}^* = \ker \mu$ is a subbundle. Moreover, one can show that $(\mathbf{E}^* \to M, \mu)$ and $((Sym\widetilde{\mathbf{E}}^*)^{\leq n} \to M, sc)$ are isomorphic coalgebra bundles. This follows from the exact sequence

$$0 \to m(\mathbf{E}, \mathbf{E}) \to \mathbf{E} \to \widetilde{\mathbf{E}} \to 0 \tag{3.3}$$

because if we pick a splitting of this sequence, we obtain a coalgebra isomorphism as desired, see Lemma 3.6 for details. The splitting also allows us to see $\Gamma\widetilde{\mathbf{E}} \subseteq \Gamma\mathbf{E}$; in this case a section $f \in \Gamma\widetilde{E}_k$ is referred to as a *pure function of degree k*.

3.1.2 Relation with other geometrizations

Before we start proving how our geometrization functor produces nice classical geometric interpretations of graded manifolds and structures over them we comment on how this is related to previous attempts to geometrize graded manifolds.

The first thing to be mentioned is that our geometrization is an equivalence of categories. Therefore graded manifolds really can be codified in classical geometric terms in contrast to the case of supermanifolds, where the Batchelor Theorem [6] shows that supermanifolds have more morphism than vector bundles.

But it was known to the experts that graded manifolds have a geometrization functor. In [13] was found an equivalence of categories between $\mathcal{V}ect$ and $\mathcal{GM}^n$. The objects on $\mathcal{V}ect$ are non-positively graded vector bundles and the image of their functor are the split n-manifolds. In other words, they convert the affine tower of fibrations (2.2) into vector bundles by choosing a zero and then prove that these vector bundles are enough to codify all the information.

Our coalgebra bundles are a bit more complicated objects but are more canonical. This implies that we do not need to make auxiliary choices to write the objects. For example, we will see that degree 2 symplectic Q-manifolds correspond to Courant algebroids in our sense; on the other hand, to link with the geometrization described in [13] one needs an auxiliary connection compatible with the pairing.

A similar geometrization for graded manifolds was also stated in [57]. They relate graded manifolds with graded polynomial superalgebras, objects that can be understood as duals of what they call graded Weil coalgebra bundles. For degree 2 they explicitly show that their geometrization is equivalent to the one given in [24].

A second geometrization functor was found in [82] for degree 2 manifolds. The geometric objects now are double vector bundles with some extra structure. We explain this other geometrization in Section B.3. Also the works [48, 70, 71, 73] developed further this functor and obtained interesting results. We will see in Chapter 7 how our geometrization combined with this one produce new interesting descriptions of higher objects.

In [121] Li-Bland's idea is extended to any degree. The geometrization functor proposed in [121] established an equivalence between degree n manifolds and n-fold vector bundles with some extra structure. Close ideas to this appear also in [18]. n-fold vector bundles are in general a bit complicated and we not know the exact relations with the geometrization proposed here. We hope this "globular" geometrization will be useful in the future.

3.2 Geometrization of submanifolds

Once we have established the geometrization for graded manifolds we can use this dictionary to express all the concepts defined in Chapter 2 in terms of the coalgebra bundles. We will start by characterizing submanifolds.

Given a coalgebra morphism $\Phi = (\varphi, \phi) : (\mathbf{E}^* \to M, \mu) \to (\mathbf{F}^* \to N, \nu)$ the fact that it intertwines the coproduct implies that $\phi(\widetilde{\mathbf{E}}^*) \subseteq \widetilde{\mathbf{F}}^*$, where $\widetilde{\mathbf{E}}^* = \ker \mu$ and $\widetilde{\mathbf{F}}^* = \ker \nu$. So we say that Φ is an *immersion* if φ is an immersion and $\phi_{|\widetilde{\mathbf{E}}^*}$ is injective. Therefore it is easy to see that the functor $\mathcal{F}$ defined on Theorem 3.5 satisfies that Φ is an immersion if and only if $\mathcal{F}(\Phi)$ is an immersion.

Given an admissible coalgebra bundle $(\mathbf{E}^* \to M, \mu)$ we have that submanifolds of $\mathcal{M} = \mathcal{F}(\mathbf{E}^* \to M, \mu)$ are in one to one correspondence with pairs $((\mathbf{F}^* \to N, \nu), j)$ where $(\mathbf{F}^* \to N, \nu)$ is an admissible coalgebra bundle and $j : (\mathbf{F}^* \to N, \nu) \to (\mathbf{E}^* \to M, \mu)$ is an immersion coalgebra bundle map. But often it is more convenient to work with the dual algebra bundle, therefore we have also the following characterization of submanifolds:

Proposition 3.15. *Let $\mathcal{M} = \mathcal{F}(\mathbf{E}^* \to M, \mu)$ be an n-manifold and denote by $(\mathbf{E} \to M, m)$ the dual algebra bundle. There is a one to one correspondence between:*

- *Submanifolds of $\mathcal{M}$.*
- *$\mathbf{F} \to N \subseteq \mathbf{E} \to M$ graded subbundle satisfying:*

$$\mathbf{F} \cap m(\mathbf{E}, \mathbf{E})_{|N} = m(\mathbf{F}, \mathbf{E}_{|N}).$$

Proof. By the preceding paragraph we know that submanifolds of $\mathcal{M}$ are in one to one correspondence with pairs $((\mathbf{H}^* \to N, \nu), j)$ where $(\mathbf{H}^* \to N, \nu)$ is an admissible coalgebra bundle and $j : (\mathbf{H}^* \to N, \nu) \to (\mathbf{E}^* \to M, \mu)$ an immersion of coalgebra bundles.

$(\Rightarrow)$
Given $((\mathbf{H}^* \to N, \nu), j)$ we have that $j^\sharp : \Gamma\mathbf{E} \to \Gamma\mathbf{H}$ is a surjective algebra map. Define $\mathbf{F} = \ker(j^\sharp_{|N})$, since it is an algebra map it is clear that $m(\mathbf{F}, \mathbf{E}_{|N}) \subseteq \mathbf{F}$. Let us see the other inclusion, given $0 \neq x \in \Gamma\big(\mathbf{F} \cap m(\mathbf{E}, \mathbf{E})_{|N}\big)$ we have that there exist $e_1, e_2 \in \Gamma\mathbf{E}$ non zero such that $x = m(e_1, e_2)$. Therefore

$$0 = j^\sharp x = j^\sharp m(e_1, e_2) = m_H(j^\sharp e_1, j^\sharp e_2)$$

since m_H is an admissible coalgebra bundle implies that or $j^\sharp e_1, j^\sharp e_2$ have some common not totally symmetric part or one of them is in $\ker j^\sharp$. Observe that the former could not happen because e_1 and e_2 having some common not totally symmetric part implies that $x = 0$. Therefore $j^\sharp e_1 = 0$ or $j^\sharp e_2 = 0$ so $x \in m(\mathbf{F}, \mathbf{E}_{|N})$ as we want.

$(\Leftarrow)$
Given $\mathbf{F} \to N$ define $\mathbf{H}^* = \mathbf{F}^\circ = \{v \in \mathbf{E}^*_{|N} | \ \langle v, f\rangle = 0 \ \forall f \in \mathbf{F}\}$, since $m(\mathbf{F}, \mathbf{E}_{|N}) \subset \mathbf{F}$ we have that $\mathbf{E}_{|N}/\mathbf{F} \cong \mathbf{H}$ inherits an algebra structure, therefore $(\mathbf{H}^*, \nu)$ is a coalgebra bundle. In consequence, the natural inclusion of $\mathbf{H}^*$ into $\mathbf{E}^*$ is an immersion and the isomorphism between $(\mathbf{E}^* \to M, \mu)$ and $(\mathrm{Sym}^{\leq n} \widetilde{\mathbf{E}}^*, sc)$ descends to $\mathbf{H}^*$. So $(\mathbf{H}^* \to N, \nu)$ is an admissible coalgebra bundle with an immersion to $(\mathbf{E}^*, \mu)$. $\square$

Let us give some concrete examples for degree 1 and 2.

Example 3.16 (Degree 1 submanifolds). In this case, since any 1-manifold is given by $E^*[1] = (M, \Gamma \bigwedge^\bullet E)$ where $E^* \to M$ is a vector bundle we have that submanifolds are given by vector subbundles $(F^* \to N) \subseteq (E^* \to M)$.

Example 3.17 (Degree 2 submanifolds). Suppose that we have $(E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \hookrightarrow E_2)$ and $\mathcal{M} = (M, C_{\mathcal{M}})$ the manifold defined by this object in VB2, see Remark 3.13. Then a submanifold of $\mathcal{M}$ is the same as two subbundles $(F_1 \to N) \subseteq E_1 \to M$ and $(F_2 \to N) \subseteq E_2 \to M$ satisfying

$$F_2 \cap \operatorname{im}(\psi)_{|N} = \psi(F_1 \wedge E_{1|N}).$$

In the degree 2 case there is also an equivalent definition that will be useful in Chapter 7.

Proposition 3.18 (see [24]). *Let $\mathcal{M}$ be a degree 2 manifold equivalent to $(E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \hookrightarrow E_2)$. Denote by $\widetilde{E}_2 = E_2 / \operatorname{im}(\psi)$. A submanifold of $\mathcal{M}$ is equivalent to $(K_1 \to N, K \to N, \phi : K \to \frac{E_{2|N}}{K_1 \wedge E_{1|N}})$ where $K_1 \subseteq E_{1|N}$, $K \subseteq \widetilde{E}_{2|N}$ are subbundles.*

Proof. By the previous Example 3.17 the only thing that we need to proof is that a subbundle $F_2 \to N$ of $E_2 \to M$ that fits on the exact sequence

$$0 \to F_2 \cap \operatorname{im}(\psi)_{|N} \to F_2 \to K \to 0$$

where $F_2 \cap \operatorname{im}(\psi)_{|N} = K_1 \wedge E_{1|N}$ is the same as K and $\phi : K \to \frac{E_{2|N}}{K_1 \wedge E_{1|N}}$ and this is straightforward. $\square$

3.3 Geometrization of vector fields

Given a vector bundle $E \to M$, recall that its *Atiyah algebroid* is a Lie algebroid, that we denote by $\mathbb{A}_E \to M$, whose sections are the infinitesimal automorphism of ΓE, i.e. pairs (D, σ) where σ is a vector field on M and $D : \Gamma E \to \Gamma E$ is $\mathbb{R}$-linear satisfying

$$D(fe) = fD(e) + \sigma(f)e \quad \forall f \in C^\infty(M),\ e \in \Gamma E.$$

Suppose now that $\mathbf{E} = \oplus_{i=k}^{l} E_i \to M$ is a graded vector bundle (for some $k, l \in \mathbb{Z}$). We can introduce its Atiyah algebroid, also denoted by $\mathbb{A}_{\mathbf{E}} \to M$, as a graded vector bundle whose sections, $(D, \sigma) \in \Gamma \mathbb{A}_{\mathbf{E}}$, satisfy

$$D(fe) = \sigma(f)e + fD(e) \quad f \in C^\infty(M),\ e \in \Gamma \mathbf{E}$$

where $\sigma \neq 0$ if and only if $(D, \sigma) \in \Gamma(\mathbb{A}_{\mathbf{E}})_0$. With this definition we have that

$$\mathbb{A}_{\mathbf{E}} = \oplus_j (\mathbb{A}_{\mathbf{E}})_j = \begin{cases} \mathbb{A}_{E_k} \times_{TM} \cdots \times_{TM} \mathbb{A}_{E_l} & \text{for } j = 0, \\ \bigoplus\limits_{b=a+j} \operatorname{Hom}(E_a, E_b) & \text{for } j \neq 0. \end{cases}$$

We can endow $\Gamma \mathbb{A}_{\mathbf{E}}$ with a Lie algebra structure where the bracket is given by the graded commutator

$$[D, D'] = D \circ D' - (-1)^{ij} D' \circ D \quad \text{for } D \in \Gamma(\mathbb{A}_{\mathbf{E}})_i,\ D' \in \Gamma(\mathbb{A}_{\mathbf{E}})_j.$$

Remark 3.19. The Atiyah algebroid of a graded vector bundle appeared before in [98], where it was used to understand representations up to homotopy of a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ on a graded vector bundle $\mathbf{E} \to M$ as an L_∞-morphism from A to $\mathbb{A}_{\mathbf{E}}$.

Suppose that $(\mathbf{E} \to M, m)$ is an algebra bundle. We say that $(D, \sigma) \in \Gamma(\mathbb{A}_{\mathbf{E}})_k$ is a *derivation of* m if

$$D(m(e_i, e_j)) = m(D(e_i), e_j) + (-1)^{ki} m(e_i, D(e_j)) \quad \text{for } e_i \in \Gamma E_i,\ e_j \in \Gamma E_j. \tag{3.4}$$

Proposition 3.20. *Let $(\mathbf{E} \to M, m)$ be an algebra bundle. Denote by $\mathbb{A}_{(\mathbf{E},m)} \subseteq \mathbb{A}_{\mathbf{E}}$ the the subspace such that $\Gamma\mathbb{A}_{(\mathbf{E},m)}$ are derivations of m. Then $\mathbb{A}_{(\mathbf{E},m)}$ is closed for the bracket.*

Proof. Straightforward computation. □

With all these ingredients we are able to state the geometric characterization of vector fields.

Theorem 3.21. *Let $(\mathbf{E}^* \to M, \mu)$ be an admissible n-coalgebra bundle and denote by $\mathcal{M} = (M, C_{\mathcal{M}}) = \mathcal{F}(\mathbf{E}^* \to M, \mu)$ the associated n-manifold. For non-positive k,*

$$\mathfrak{X}^{1,k}(\mathcal{M}) = \Gamma(\mathbb{A}_{(\mathbf{E}',m')})_k,$$

where $(\mathbf{E}', m')$ is the extended algebra bundle as defined in (3.2). Moreover the bracket of $\mathfrak{X}^{1,\bullet}(\mathcal{M})$ corresponds to the bracket of $\Gamma\mathbb{A}_{(\mathbf{E}',m')}$.

Proof. Recall that $\mathcal{F}(\mathbf{E}^* \to M, \mu) = \mathcal{M}$ implies that $C^i_{\mathcal{M}} = \Gamma E'_i$ for $0 \leq i \leq n$ and the multiplication of function on $C_{\mathcal{M}}$ becomes $m = \mu^\sharp$. Also equation (2.1) says that $C_{\mathcal{M}}$ is generated as a sheaves of algebras by the homogeneous functions of degree less or equal n. Therefore any vector field $X \in \mathfrak{X}^{1,k}(\mathcal{M})$ is completely determined by the maps

$$X^i : C^i_{\mathcal{M}} \to C^{i+k}_{\mathcal{M}} \quad \text{for } 0 \leq i \leq n.$$

If $k \leq 0$ we have that $C^i_{\mathcal{M}} = \Gamma E'_i$ for $0 \leq i \leq n$. Therefore X defines a map $D : \Gamma\mathbf{E}' \to \Gamma\mathbf{E}'$. Since X satisfies (2.4) we apply this formula to degree 0 with any other degree and obtain that $D \in \Gamma\mathbb{A}_{\mathbf{E}'}$. Finally applying (2.4) to any degree and recalling that the product on $C_{\mathcal{M}}$ corresponds to m the map D also satisfies (3.4). The converse is also true and the correspondence is one to one. □

A consequence of the preceding theorem is that $(\mathbb{A}_{(\mathbf{E}',m')})_k$ is a subundle of $(\mathbb{A}_{\mathbf{E}'})_k$. This follows from the fact that $\mathfrak{X}^{1,k}(\mathcal{M})$ is locally finitely generated as a $C^\infty(M)$-module. Also we obtain that any $X \in \mathfrak{X}^{1,0}(\mathcal{M})$ is the same as $(D^1, -, D^n)$, where $D^i \in \Gamma\mathbb{A}_{E_i}$ and have all the same symbol σ, satisfying

$$D^{i+j} m(e^i, e^j) = m(D^i e^i, e^j) + m(e^i, D^j e^j) \quad \forall e^i \in \Gamma E_i,\ e^j \in \Gamma E_j. \tag{3.5}$$

Moreover, two degree zero vector fields are linearly independent at a point if and only if their symbols are linearly independent at that point.

Proposition 3.22. *Let $(\mathbf{E}^* \to M, \mu)$ be an admissible n-coalgebra bundle and denote by $\mathcal{M} = (M, C_{\mathcal{M}}) = \mathcal{F}(\mathbf{E}^* \to M, \mu)$ the associated n-manifold. The structure of $C_{\mathcal{M}}$-module of $\mathfrak{X}^{1,\bullet}(\mathcal{M})$ is equivalent to a graded vector bundle map $\Theta : \mathbf{E}' \otimes \mathbb{A}_{(\mathbf{E}',m')} \to \mathbb{A}_{(\mathbf{E}',m')}$ that on sections is defined as*

$$\begin{array}{rccl} \Theta_{ij}: & \Gamma\Big(\mathbf{E}'_i \otimes (\mathbb{A}_{(\mathbf{E}',m')})_j\Big) & \to & \Gamma\big(\mathbb{A}_{(\mathbf{E}',m')}\big)_{i+j} \\ & e \otimes D & & \Theta_{ij}(e \otimes D) = m(e, D\cdot). \end{array}$$

Let us finish this section by expressing explicitly the vector fields in negative degrees for degree 1-manifolds and 2-manifolds.

Example 3.23 (Vector fields on 1-manifolds). Let $\mathcal{M}$ be a 1-manifold. By Example 3.11 we know that there exist a vector bundle $E \to M$ such that $\mathcal{M} = E[1]$. Then, since the coalgebra structure is zero we know by Theorem 3.21 that $\mathfrak{X}^{1,-1}(\mathcal{M}) = \Gamma E$ and $\mathfrak{X}^{1,0}(\mathcal{M}) = \Gamma \mathbb{A}_E$.

Example 3.24 (Vector fields on 2-manifolds). Let $\mathcal{M}$ be a 2-manifold, by Example 3.12 we know that exist a object in $\mathcal{C}o\mathcal{B}^2$, $(E_1^*, E_2^*, \mu : E_2^* \to E_1^* \otimes E_1^*)$ such that $\mathcal{M} = (M, \frac{\wedge^\bullet E_1 \otimes \mathrm{Sym}^\bullet E_2}{\langle e \wedge e' \otimes 1 - 1 \otimes m(e,e')\rangle})$ where $m = \mu^\sharp$. Denote by $\widetilde{E}_2^*$ the dual of the pure functions as in (3.3).

Then we have that $\mathfrak{X}^{1,-2}(\mathcal{M}) = \Gamma \widetilde{E}_2^*$, $\mathfrak{X}^{1,-1} = \Gamma F_{-1}$ where F_{-1} is the subbundle of $\mathrm{Hom}(E_2, E_1) \oplus E_1^*$ that fits on the exact sequence

$$0 \to \mathrm{Hom}(\widetilde{E}_2, E_1) \to F_{-1} \to E_1^* \to 0$$

and $\mathfrak{X}^{1,0}(\mathcal{M}) = \{(D_1, D_2) \in \Gamma\big(\mathbb{A}_{E_1} \times_{TM} \mathbb{A}_{E_2}\big) \mid D_2(m(e_1, e_2)) = m(D_1(e_1), e_2) + m(e_1, D_1(e_2)) \quad \forall e_1, e_2 \in \Gamma E_1\}$.

3.4 Geometrization of Q-manifolds

In the preceding section we saw how vector fields of non-positive degree are identified with sections of some vector bundles. But as we already said, we are also interested in Q-manifolds, i.e. in degree 1 homological vector fields. What we explore in this section is how homological vector fields are also codified in classical geometric terms. All the correspondences were already known.

For degree 1 Vaintrob proved in [120] that the category of degree 1 Q-manifolds is equivalent to the category of Lie algebroids.

For degree 2 the works [26, 48] find a description of degree 2 Q-manifold in terms of VB2 objects that we recall here. On the other hand, Li-bland introduced VB-Courant algebroids and stated in [82] that VB-Courant algebroids correspond to degree 2 Q-manifolds. This statement was clarified in the works [48, 70] and we recall them in Section B.3.2.

Finally, for arbitrary n it was suggested in [113, 124] that Q-manifolds were related with "higher Lie algebroids". In [116] the notion of an L_∞-algebroid was introduced and it stated that these objects are in correspondence with Q-manifolds; this result that was finally proved in [13].

There are two approaches to codify degree 1 vector fields. Let $(\mathcal{M} = (M, C_{\mathcal{M}}), Q)$ be a Q-manifold of degree n and suppose that we have a coalgebra bundle $(\mathbf{E}^* \to M, \mu)$ such that $C^i_{\mathcal{M}} = \Gamma E_i$ if $i \leq n$.

The first approach is that a vector field is completely determined by the image of the generators of the algebra $C_{\mathcal{M}}$. Therefore, for a degree 1 vector field it is enough to know the maps

$$Q^i : C^i_{\mathcal{M}} \to C^{i+1}_{\mathcal{M}} \quad i \in \{0, \cdots, n\}. \tag{3.6}$$

And equation (2.5) will impose some conditions on the maps. Since $C^i_{\mathcal{M}} = \Gamma E_i$ we can express this maps in terms of vector bundles but in general they are not vector bundle maps because of the rule $Q(fg) = Q(f)g + fQ(g)$ for $f \in C^\infty(M), g \in C_{\mathcal{M}}$.

The second approach is that locally, if we choose coordinates $\{e^{j_i}\}$ of $\mathcal{M}_{|U}$, a degree 1 vector field can be written as

$$Q_{|U} = f^{j_i} \frac{\partial}{\partial e^{j_i}} \quad \text{with } f^{j_i} \in C^{i+1}_{\mathcal{M}}$$

and recall that functions of degree $1, \cdots, n+1$ are completely determined by how vector fields of negative degree act on them. Therefore Q is completely determined by maps

$$Q_i : \Gamma(\mathbb{A}_{(\mathbf{E}',m')})_i \to \Gamma(\mathbb{A}_{(\mathbf{E}',m')})_{i+1} \qquad i \in \{-n, \cdots, -1\}. \tag{3.7}$$

But also these are not vector bundle maps because $Q_i(X) = [X, Q]$, where $X \in \mathfrak{X}^{1,i}(\mathcal{M})$ and the Lie bracket is not $C^\infty(M)$-linear.

Remark 3.25. Observe that with the first strategy we codify Q-manifolds in $n+1$ maps and with the second in just n maps. The dichotomy is just apparent because the vector bundles are different.

3.4.1 Degree 1: Lie algebroids

In Example 3.11 we prove that 1-manifolds are equivalent to vector bundles. Example 2.12 tells that a Lie algebroid structure on $A \to M$ induces a Q-structure on $A[1]$. Let us prove now the converse:

Theorem 3.26 (see [120])**.** *There is an equivalence of categories between Lie algebroids with Lie algebroid morphisms and degree* 1 *Q-manifolds with Q-morphisms.*

Proof. We see that given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ then $(A[1], Q = d_A)$ is a Q-manifold. Let us do the other way around. Let $(\mathcal{M}, Q)$ be a degree 1 Q-manifold then $\mathcal{M} = A[1]$ for some vector bundle $A \to M$. By Example 3.23 we have that $\mathfrak{X}^{1,-1}(A[1]) = \Gamma A$ and $\mathfrak{X}^{1,0}(A[1]) = \Gamma \mathbb{A}_A$ then a Q-manifold is completely characterized by a map $ad : \Gamma A \to \Gamma \mathbb{A}_A$ given by $ad_{a_1} = [a_1, Q]$. Define now:

$$\rho(a_1)(f) = \sigma(ad_{a_1})(f) = [[a_1, Q], f] \quad \text{and} \quad [a_1, a_2] = ad_{a_1} a_2 = [[a_1, Q], a_2]$$

where $a_1, a_2 \in \Gamma A = \mathfrak{X}^{1,-1}(A[1])$, $f \in C^\infty(M) = C^0_{A[1]}$ and $\sigma : \mathbb{A}_A \to TM$ is the symbol map.

The derived bracket formalism combined with the fact that Q satisfies equation (2.5) ensures that $(A \to M, [\cdot,\cdot], \rho)$ is a Lie algebroid. For the morphisms assertion we define Lie algebroid morphisms as Q-morphisms, so there is nothing to prove. $\square$

3.4.2 Degree 2

Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be a 2-manifold equivalent to $(E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \hookrightarrow E_2)$. Being equivalent implies that $C^1_{\mathcal{M}} = \Gamma E_1,\ C^2_{\mathcal{M}} = \Gamma E_2$; if we denote by E_3 and E_4 the vector bundles such that $C^3_{\mathcal{M}} = \Gamma E_3$ and $C^4_{\mathcal{M}} = \Gamma E_4$ then we know that

$$E_3 = \frac{\wedge^3 E_1 \oplus E_1 \otimes E_2}{\langle e_1 \wedge e_2 \wedge e_3 = e_1 \otimes \psi(e_2 \wedge e_3)\rangle},\ E_4 = \frac{\wedge^4 E_1 \oplus \operatorname{Sym}^2 E_2}{\langle e_1 \wedge e_2 \wedge e_3 \wedge e_4 = \psi(e_1 \wedge e_2)\psi(e_3 \wedge e_4)\rangle}.$$

Denote also by $p_3 : E_1 \otimes E_2 \to E_3$ and $p_4 : \operatorname{Sym}^2 E_2 \to E_4$ the natural projections.

Proposition 3.27 (see [26, 48])**.** *Let $(\mathcal{M}, Q)$ be a Q-manifold of degree 2 equivalent to $(E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \hookrightarrow E_2)$. Then the Q-structure is characterized by three maps*

$$\begin{cases} F_0 : T^*M \to E_1 \text{ Vector bundle map,} \\ F_1 : \Gamma E_1 \to \Gamma E_2, \\ F_2 : \Gamma E_2 \to \Gamma E_3, \end{cases}$$

satisfying

$$\begin{cases} F_1(fe) = \psi(F_0(df) \wedge e) + fF_1(e), \\ F_2(f\xi) = p_3(F_0(df) \otimes \xi) + fF_2(\xi), \\ F_2(\psi(e_1 \wedge e_2)) = p_3\Big(e_2 \otimes F_1(e_1) - e_1 \otimes F_1(e_2)\Big), \\ F_1 \circ F_0 = 0, \quad F_2 \circ F_1 = 0, \quad F_3 \circ F_2 = 0; \end{cases} \tag{3.8}$$

where $F_3 : \Gamma E_3 \to \Gamma E_4$ is defined on sections of $\wedge^3 E_1 \oplus E_1 \otimes E_2$ as

$$\begin{cases} F_3(e_1 \wedge e_2 \wedge e_3) = p_4\Big(F_1(e_1)\psi(e_2 \wedge e_3) - \psi(e_1 \wedge e_3)F_1(e_2) + \psi(e_1 \wedge e_2)F_1(e_3)\Big), \\ F_3(e \otimes \xi) = p_4\Big(F_1(e)\xi - e \otimes F_2(\xi)\Big), \end{cases}$$

for $f \in C^\infty(M),\ e, e_1, e_2, e_3 \in \Gamma E_1,\ \xi \in \Gamma E_2$.

Proof. If the 2-manifold $\mathcal{M}$ is equivalent to $(E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \hookrightarrow E_2)$ we have that $C^i_{\mathcal{M}} = \Gamma E_i$.

A degree 1 vector field is completely determined by the action on generators, then is determined by the three maps F_0, F_1, F_2 and the properties that satisfy comes from the fact that Q is a derivation and Q satisfies $0 = [Q, Q] = Q^2$.

Conversely, given this three maps define the vector field Q by the following formulas

$$Q(f) = F_0(df), \quad Q(e) = F_1(e), \quad Q(\xi) = F_2(\xi) \quad f \in C^\infty(M), e \in \Gamma E_1, \xi \in \Gamma E_2.$$

The fact that Q is well defined and $Q^2 = 0$ is equivalent to the properties of F_0, F_1 and F_2. □

Remark 3.28. Observe that the maps F_1 and F_2 are not linear, hence they are not vector bundle maps. But since their linearity is controlled they are in fact equivalent to vector bundles maps coming from the first jet bundle. Therefore a degree 1 vector field on a 2-manifold is also equivalent to three vector bundle maps

$$\begin{cases} \widehat{F}_0 : T^*M \to E_1; \\ \widehat{F}_1 : J^1 E_1 \to E_2, \text{ such that } \widehat{F}_1(df \otimes e) = \psi(\widehat{F}_0(df) \wedge e); \\ \widehat{F}_1 : J^1 E_2 \to E_3, \text{ such that } \widehat{F}_2(df \otimes \xi) = p_3(\widehat{F}_0(df) \otimes \xi); \end{cases}$$

satisfying $\widehat{F}_2(j^1\psi(e_1\wedge e_2)) = p_3\Big(e_2\otimes\widehat{F}_1(j^1e_1)-e_1\otimes\widehat{F}_1(j^1e_2)\Big)$ where $f \in C^\infty(M),\ e, e_1, e_2 \in \Gamma E_1,\ \xi \in \Gamma E_2$. The way of passing from one to the other is just

$$F_0 = \widehat{F}_0, \quad F_1 = \widehat{F}_1 \circ j^1, \quad F_2 = \widehat{F}_2 \circ j^1.$$

Now we state also the correspondence of degree 2 Q-manifolds in terms of double vector bundles. For a complete definitions of VB-Courant algebroids and the proof of the result see Section B.3.2.

Theorem 3.29 (See [82]). *There is a one to one correspondence between:*

$$\left\{ \begin{array}{c} \textit{Degree 2 } Q\textit{-manifolds} \\ (\mathcal{M}, Q) \end{array} \right\} \leftrightarrows \left\{ \begin{array}{c} \textit{VB-Courant algebroids} \\ ((D; A, B; M), \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], \rho) \end{array} \right\}$$

Moreover the correspondence sends Q-submanifolds to VB-Dirac structures.

We study now some relevant examples of degree 2 Q-manifolds, see also Section 7.2 for a more geometric description of this examples:

Example 3.30 (The Koszul algebra of a vector bundle). Let $E^* \to M$ be a vector bundle. We already know that $E^*[1] = (M, \Gamma \bigwedge^\bullet E)$ defines a 1-manifold. By Example 2.13 we know that $(T[1]E^*[1], Q_{dr})$ is a degree 2 Q-manifold. Therefore, by Theorem 3.29 we have associated to it a VB-Courant algebroid that is the double vector bundle

$$\begin{tikzcd} TE \oplus T^*E \arrow[r] \arrow[d] & TM \oplus E^* \arrow[d] \\ E \arrow[r] & M \end{tikzcd}$$

with the standard Courant structure.

When $E \to M$ is just a vector space V, then $C^k_{T[1]V^*[1]} = \bigoplus_{i+2j=k} \bigwedge^i V \otimes \mathrm{Sym}^j V$ and Q_{dr} is defined on generators by

$$Q_{dr}(v \otimes 1) = 1 \otimes v, \qquad Q_{dr}(1 \otimes v) = 0.$$

This is the usual Koszul complex of V as defined in [63]. In terms of the maps described in Proposition 3.27 we have $F_0 = 0, F_1 = 0 + \mathrm{Id}, F_2 = 0 + \mathrm{Id}$ and make the following diagram:

$$\begin{array}{ccccccc}
0 & & 1 & & 2 & & 3 \\
\hline
* & \longrightarrow & V & \longrightarrow & \bigwedge^2 V \oplus V & \longrightarrow & \bigwedge^3 V \oplus V \otimes V \\
* & \xrightarrow{F_0} & 0 & & & & \\
 & & u & \xrightarrow{F_1} & 0 + u & & \\
 & & & & u_1 \wedge u_2 + u & \xrightarrow{F_2} & 0 + u_1 \otimes u_2 - u_2 \otimes u_1
\end{array}$$

Example 3.31 (The Weil algebra of a Lie algebra). Following the preceding example, if the vector space V is a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ we know by Theorem 3.26 that $(\mathfrak{g}[1], d_{ce})$ is a Q-manfiold where d_{ce} is the Chevalley-Eilenberg differential. Consider now the degree 2 manifold $T[1]\mathfrak{g}[1]$. By the previous example we have that the De Rham differential Q_{dr} is a degree 1 vector field on $T[1]\mathfrak{g}[1]$. But the Cartan calculus of Section A.2 also says that $\mathcal{L}_{d_{ce}}$ is a degree 1 vector field on $T[1]\mathfrak{g}[1]$. Moreover, Proposition A.7 ensures that $[Q_{dr}, \mathcal{L}_{d_{ce}}] = 0$. Then $(T[1]\mathfrak{g}[1], Q_{brst} = Q_{dr} + \mathcal{L}_{d_{ce}})$ is a degree 2 Q-manifold.

In terms of the three maps we have that

$$F_0 = 0, \qquad F_1 = d_{ce} + \mathrm{Id}, \qquad F_2 = d_{ce} + \mathrm{Id},$$

where we just sum the differential of the previous example with the Chevalley-Eilenberg differential. In the VB-Courant algebroid language we have that $(T[1]\mathfrak{g}[1], Q_{brst})$ corresponds to

$$\begin{tikzcd} T\mathfrak{g}^* \oplus T^*\mathfrak{g}^* \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d] \\ \mathfrak{g}^* \arrow[r] & * \end{tikzcd}$$

where $T\mathfrak{g}^* \oplus T^*\mathfrak{g}^* \to \mathfrak{g}^*$ is the Courant algebroid induced by the Lie bialgebroid of the linear Poisson structure of $\mathfrak{g}^*$, see Example 2.22.

Finally, we recall that $C^\bullet_{T[1]\mathfrak{g}[1]}$ is known as the Weil algebra of $\mathfrak{g}$ and Q_{brst} is called the BRST differential. This complex plays a central role in the study of equivariant cohomology, see [63].

Example 3.32 (The Weil algebra of a Lie algebroid). If in the preceding Example 3.31 we replace the Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ by a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ everything remains true and we obtain $(T[1]A[1], Q_{brst} = Q_{dr} + \mathcal{L}_{d_A})$ and the VB-Courant algebroid

$$\begin{tikzcd} TA^* \oplus T^*A^* \arrow[r] \arrow[d] & TM \oplus A^* \arrow[d] \\ A^* \arrow[r] & M \end{tikzcd}$$

This Weil algebra of a Lie algebroid was defined in [94] and also studied in [1, 100].

3.4.3 Degree n: L_n-algebroids

An extension of the concept of a Lie algebroid was given in [116] under the name of L_∞-algebroids. We follow the notation of [13] and state the correspondence between L_n-algebroids and Q-manifolds of degree n.

A L_n-algebroid is a non positively graded vector bundle $\mathbf{A} = \bigoplus_{i=0}^{n-1} A_{-i} \to M$ together with a bundle map $\rho : A_0 \to TM$ and graded antisymmetric brackets $l_i : \Gamma(\mathbf{A} \times \cdots^{i} \times \mathbf{A}) \to \Gamma\mathbf{A}$ with $i \in \{1, \cdots, n+1\}$ of degree $2 - i$ such that:

- For any $r \geq 1$
$$\sum_{i+j=r+1} \sum_{\sigma \in Sh(i,j-1)} (-1)^{i(j-1)} Ksgn(\sigma) l_j(l_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), x_{\sigma(i+1)}, \cdots, x_{\sigma(r)}) = 0$$
where $Ksgn(\sigma)$ is the signature of the permutation multiply by the Koszul sign.

- For $i \neq 2$ $\quad l_i$ are $C^\infty(M)$-linear.
- l_2 is $C^\infty(M)$-linear if both arguments are not in ΓA_0 and satisfies:

$$\begin{cases} l_2(x_0, fx) = fl_2(x_0, x) + \rho(x_0)(f)x \\ l_2(fx_0, x) = fl_2(x_0, x) \end{cases} \qquad \forall x_o \in \Gamma A_0,\ x \in \Gamma \mathbf{A},\ f \in C^\infty(M)$$

This definition involves shuffles for the higher Jacobi identities and is complicated to deal with. But for some particular cases, as we will see in Chapter 5, everything can be made explicit.

In Example 2.12 combined with Theorem 3.26 we showed that a vector bundle $A \to M$ has a Lie algebroid structure if and only if $\Gamma \bigwedge^\bullet A^*$ inherits a differential complex structure. In [13] the same is proven for L_∞-algebroids, we give now the precise statement. Recall that given any n-manifold $\mathcal{M}$ by Lemma 3.6 there exist $\mathbf{D}^* = \oplus_{i=-n}^{-1} D_i^* \to M$ such that $\mathcal{M}$ and $(M, \Gamma \operatorname{Sym} \mathbf{D})$ are isomorphic. Using this we have:

Theorem 3.33 (See [13]). *Let $(\mathbf{A} \to M, l_i, \rho)$ be a L_n-algebroid. Then the n-manifold $(M, \Gamma \operatorname{Sym}(\mathbf{A}[1])^*)$ is a Q-manifold. Moreover, given any degree n Q-manifold $(\mathcal{M}, Q)$ and an isomorphism with $(M, \Gamma \operatorname{Sym} \mathbf{D})$ for some graded vector bundle $\mathbf{D}^* \to M$ we have that $\mathbf{D}^*[-1] \to M$ inherits a L_n-algebroid structure.*

Remark 3.34. Sometimes L_n-algebroids are called split L_n-algebroids and Q-manifolds are called L_n-algebroids. We believe that the terminology is not the appropriate one and it is better to talk just about L_n-algebroids and Q-manifolds, and the relation between them is given by a splitting of the graded manifolds.

It is clear from the definition that L_1-algebroids are just Lie algebroids. Let us explicit the definition for the next case:

Example 3.35 (L_2-algebroids). An L_2-algebroid is a the same as the following data $(\mathbf{A} \to M, \rho, \partial, [\cdot,\cdot], \Psi, [\cdot,\cdot,\cdot])$ where:

- $\mathbf{A} = A_{-1} \oplus A_0$ is a graded vector bundle over M.
- $\rho : A_0 \to TM$ is the anchor.
- The l_1 is given by a vector bundle map $\partial : A_{-1} \to A_0$.
- The l_2 has two pieces the one given by a bracket $[\cdot,\cdot]_2 : \Gamma(A_0 \wedge A_0) \to \Gamma A_0$ and another given by an A_0-connection on A_{-1}, $\Psi : \Gamma A_0 \times \Gamma A_{-1} \to \Gamma A_{-1}$.
- The l_3 is given by a $C^\infty(M)$-linear operation $[\cdot,\cdot,\cdot]_3 : \Gamma \bigwedge^3 A_0 \to \Gamma A_{-1}$.

satisfying the following Jacobi-like identities:

$$\begin{cases} \rho \circ \partial = 0 \\ [a_1, \partial\xi]_2 = \partial \Psi_{a_1} \xi \\ [[a_1, a_2]_2, a_3]_2 + c.p. = -\partial([a_1, a_2, a_3]_3) \\ \Psi_{[a_1,a_2]_2}\xi - \Psi_{a_1}\Psi_{a_2}\xi + \Psi_{a_2}\Psi_{a_1}\xi = -[\partial\xi, a_1, a_2]_3 \\ [[a_1, a_2]_2, a_3, a_4]_3 + c.p. = \Psi_{a_1}[a_2, a_3, a_4]_3 + c.p. \end{cases}$$

for $a_1, a_2, a_3, a_4 \in \Gamma A_0,\ \xi \in \Gamma A_{-1}$.

By the preceding Theorem 3.33 we have that once we choose a splitting, degree 2 Q-manifolds become L_2-algebroids. Finally, if we also take into account Theorem 3.29 we obtain the following diagram

$$\begin{tikzcd}
\text{Deg. 2 Q-man.} \arrow[rr, Leftrightarrow, "\text{Equivalent}"'] \arrow[dr, "\text{Splitting}"'] & & \text{VB-Courant alg.} \arrow[dl, "\text{Decomposition}"] \\
& L_2\text{-algebroids} &
\end{tikzcd}$$

An equivalent definition of L_2-agebroids can be found in [73], where L_2-algebroids are related directly with VB-Courant algebroids.

Example 3.36. In the previous Example 3.35 if we take $M = *$ then we obtain the definition of a L_2-algebra. It was proved in [3] that L_2-algebras are equivalent to *semistrict Lie* 2 *algebras*, i.e. groupoid objects in the category of Lie algebras. In fact, if $[\cdot,\cdot,\cdot] \equiv 0$ then the L_2-algebra is equivalent to a *strict Lie* 2 *algebra* and the description that we give in Example 3.35 corresponds to their *differential crossed module.*

For more examples of L_n-algebroids see also Section 5.2.3.

3.5 Geometrization of graded Poisson manifolds

In this section we will translate the information given by a Poisson bracket on a graded manifold into geometric objects on vector bundles. The way to proceed is the following one:

Let $(\mathcal{M}, C_{\mathcal{M}})$ be an n-manifold equivalent to the coalgebra bundle $(\mathbf{E}^* \to M, \mu)$. Then a Poisson bracket is, by the Leibniz rule, completely determined by the operations $\{\cdot,\cdot\} : C^i_{\mathcal{M}} \times C^j_{\mathcal{M}} \to C^{i+j+k}_{\mathcal{M}}$ with $i, j \in \{0, \cdots, n\}$ and since $C^i_{\mathcal{M}} = \Gamma E_i$ for $i \in \{0, \cdots, n\}$ we have that if $k \leq 0$ the operations determine some operations $\{\cdot,\cdot\} : \Gamma E_i \times \Gamma E_j \to \Gamma E_{i+j+k}$.

3.5.1 Degree 1: Lie algebroids

In Example 3.11 we saw that 1-manifolds are equivalent to vector bundles. Example 2.20 tells that a Lie algebroid structure on $A \to M$ induces a Poisson structure on $A^*[1]$. Let us prove now that, in fact we have an equivalence.

Theorem 3.37 (See e.g. [34])**.** *There is an equivalence of categories between Lie algebroids with Lie algebroid comorphism and* 1*-manifolds with degree* -1 *Poisson brackets with Poisson morphism. Moreover Lie subalgebroids correspond to coisotropic submanifolds.*

Proof. Let $A \to M$ be a vector bundle. A Gerstenhaber bracket on $\Gamma \bigwedge^\bullet A$ is the same as a degree -1 Poisson bracket on $A^*[1]$. Therefore since a Gerstenhaber bracket on $\Gamma \bigwedge^\bullet A$ is the same as a Lie algebroid on $A \to M$, we are done.

For the assertion on morphisms, observe that there is another, maybe more conceptual, way of expressing this correspondence. It is well known that given a vector bundle $A \to M$ there is a one to one correspondence between Lie algebroid structures on $A \to M$ and linear poisson structures on $A^* \to M$, moreover a comorphism is a vector bundle morphism that is a Poisson map. Therefore degree -1 Poisson structures on $A^*[1]$ are in one to one correspondence with linear Poisson structures on $A^* \to M$, and with this point of view the assertion on the morphism is trivial. The moreover part follows almost by definition. $\square$

Corollary 3.38. *Let $(A[1] = (M, \Gamma \bigwedge^\bullet A^*), \{\cdot,\cdot\})$ be a symplectic 1-manifold. Then $A[1] = T^*[1]M$.*

Proof. The non-degeneracy condition says that the map $\{\cdot,\cdot\} : C^1_{A[1]} \times C^0_{A[1]} \to C^0_{A[1]}$ must be non-degenerate. This is equivalent to saying that

$$\rho : C^1_{A[1]} = \Gamma A^* \to \mathfrak{X}(M) = \Gamma TM$$

is an isomorphism, therefore $A[1] = T^*[1]M$. $\square$

3.5.2 Degree 2

In this section we give a geometric characterization of degree 2 Poisson manifolds endowed with a degree -2 Poisson bracket.

Proposition 3.39 (see [26])**.** *Consider a degree 2 manifold $\mathcal{M} = (M, C_{\mathcal{M}})$ equivalent to $(E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \hookrightarrow E_2)$. Then $(\mathcal{M}, \{\cdot,\cdot\})$ is Poisson of degree -2 if and only if*

- *a). $E_1 \to M$ has a symmetric pairing $\langle\cdot,\cdot\rangle : \Gamma E_1 \times \Gamma E_1 \to C^\infty(M)$.*
- *b). $(E_2 \to M, [\cdot,\cdot], \rho)$ is a Lie algebroid.*
- *c). E_1 carries a flat E_2-connection $\nabla : \Gamma E_2 \times \Gamma E_1 \to \Gamma E_1$.*

Such that

$$\begin{cases} [\xi, \psi(e_1 \wedge e_2)] = \psi\Big(\nabla_\xi e_1 \wedge e_2 + e_1 \wedge \nabla_\xi e_2\Big), \\ \rho(\xi)\langle e_1, e_2\rangle = \langle \nabla_\xi e_1, e_2\rangle + \langle e_1, \nabla_\xi e_2\rangle, \\ \nabla_{\psi(e_1\wedge e_2)} e_3 = \langle e_2, e_3\rangle e_1 - \langle e_1, e_3\rangle e_2. \end{cases} \tag{3.9}$$

for $\xi \in \Gamma E_2,\ e_1, e_2, e_3 \in \Gamma E_1$.

Proof. The proof is just a translation of the operations given by the Poisson bracket and the use of the Jacobi and Leibniz identities. $\square$

Recall that given a vector bundle $E \to M$ we denote its Atiyah algebroid by $\mathbb{A}_E \to M$, whose sections are the derivative endomorphisms of ΓE. If our vector bundle has a pairing $(E \to M, \langle\cdot,\cdot\rangle)$ we denote by $\mathbb{A}_{(E,\langle\cdot,\cdot\rangle)}$ the subbundle of the Atiyah algebroid such that

$$\Gamma \mathbb{A}_{(E,\langle\cdot,\cdot\rangle)} = \{(D,\sigma) \in \Gamma \mathbb{A}_E \mid \sigma\langle e_1, e_2\rangle = \langle D(e_1), e_2\rangle + \langle e_1, D(e_2)\rangle \ e_1, e_2 \in \Gamma E\ \}$$

Corollary 3.40 (See [108])**.** *There is a one to one correspondence between degree 2 symplectic manifolds and vector bundles endowed with a non-degenerate pairing.*

Proof. A degree 2 symplectic manifold is the same as a degree 2-manifold with a degree -2 non-degenerate Poisson bracket. So by the preceding Proposition 3.39 we have that $E_1 \to M$ has a pairing and the fact that the Poisson bracket is non-degenerate implies that the pairing is non-degenerate.

Conversely given a $(E \to M, \langle \cdot, \cdot \rangle)$ a vector bundle with a non degenerate bracket we know that the Atiyah algebroid preserving metric fits in the exact sequence

$$0 \to \bigwedge^2 E \xrightarrow{\psi} \mathbb{A}_{(E,\langle \cdot,\cdot \rangle)} \to TM \to 0$$

Therefore $(E \to M, \mathbb{A}_{(E,\langle \cdot,\cdot \rangle)} \to M, \psi : \bigwedge^2 E \to \mathbb{A}_{(E,\langle \cdot,\cdot \rangle)})$ defines a 2-manifold. Moreover, the paring on $E \to M$, the lie algebroid structure of the Atiya algebroid and the action of $\mathbb{A}_{(E,\langle \cdot,\cdot \rangle)})$ in E by derivations satisfy the hypothesis of Proposition 3.39. Therefore we have a degree -2 Poisson manifold and is easy to see that since the pairing is non-degenerate the Poisson bracket is also non-degenerate. □

Before to give examples of degree 2 Poisson manifolds we state the relation between degree 2 Poisson manifolds and VB-algebroids. The definitions and proof of this statement we do it in Section B.3.3.

Proposition 3.41 (see [48, 70]). *There is a one to one correspondence between:*

$$\left\{ \begin{array}{c} \textit{Degree 2 Poisson manifolds} \\ (\mathcal{M}, \{\cdot,\cdot\}) \end{array} \right\} \leftrightarrows \left\{ \begin{array}{c} \textit{Metric VB-algebroids} \\ ((D; A, B; M), \langle \cdot,\cdot \rangle, [\cdot,\cdot], \rho) \end{array} \right\}$$

Example 3.42 (Tangent lift of a degree 1 Poisson manifold). We will see this example with more details in Section 7.2. Let $A \to M$ be a vector bundle and consider the 1-manifold $A^*[1] = (M, \Gamma \bigwedge^\bullet A)$. Suppose that $(A^*[1], \{\cdot,\cdot\}_{-1})$ has a Poisson structure of degree -1. Then the graded manifold $T[1]A^*[1]$ carries a Poisson structure of degree -2 given by the tangent lift that we will denote $\{\cdot,\cdot\}_{-2}$.

Let U be a open of M such that $A_{|U}$ is trivial and consider $\{x^i\}$ coordinates on U and $\{a^j\}$ a base of sections of $A_{|U}$. Recall from Example A.3 that locally we can write $T[1]A^*[1]_{|U}$ in the following coordinates

$$\{x^i, a^j, dx^i, da^j\} \text{ where } |x^i| = 0, |a^j| = 1, |dx^i| = 1, |da^j| = 2.$$

In these coordinates the degree -2 Poisson bracket has the following expression

$$\{dx^i, a^j\} = -\{x^i, da^j\} = \{x^i, a^j\}_{-1} \quad \text{and} \quad \{da^j, da^k\} = d\{a^j, a^k\}_{-1}.$$

Therefore, given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ we produced a degree 2 Poisson manifold, $(T[1]A^*[1], \{\cdot,\cdot\})$.

Remark 3.43. In [73] was proved that once we choose an splitting of the degree 2 manifold, a degree 2 Poisson manifold became a Lie algebroid with a *self-dual* 2-term representation up to homotopy. This can be seen also as a refinement of the relation between VB-algebroids and 2-term representation up to homotopy given in [59].

3.6 Geometrization of PQ-manifolds

In this section we mix the two previous ones and give a geometric description of PQ-manifolds and their coisotropic Q-submanifolds. This geometric characterization is crucial for Chapter 7.

3.6.1 Degree 1: Lie bialgebroids

In Section 2.4 we saw that a Lie bialgebroid $(A \to M, [\cdot,\cdot], \rho,\ d_{A^*})$ is a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ whose dual $(A^* \to M, [\cdot,\cdot]_{A^*}, \rho_{A^*})$ is also a Lie algebroid and d_A (see definition in Example 2.12) is a derivation of the Gerstenhaber bracket on $\Gamma \bigwedge^\bullet A^*$, i.e. satisfies

$$d_A[\alpha,\beta]_{A^*} = [d_A\alpha,\beta]_{A^*} + (-1)^{|\alpha|-1}[\alpha, d_A\beta]_{A^*} \quad \alpha,\beta \in \Gamma \bigwedge^\bullet A^*. \tag{3.10}$$

Given a Lie bialgebroid $(A \to M, [\cdot,\cdot], \rho, d_{A^*})$ and $B \to N$ a subbundle of $A \to M$ we say that $B \to N$ *is a Lie subbialgebroid* if $B \to N$ is a subalgebroid of $A \to M$ and $B^\circ \to N$ is a subalgebroid of $A^* \to M$.

We can prove now the following correspondence:

Theorem 3.44 (see [122]). *There is a correspondence between:*

$$\left\{ \begin{array}{c} \textit{Lie bialgebroids} \\ (A \to M, [\cdot,\cdot], \rho, d_{A^*}) \end{array} \right\} \rightleftarrows \left\{ \begin{array}{c} \textit{Degree 1 PQ-manifolds} \\ (\mathcal{M}, \{\cdot,\cdot\}, Q) \end{array} \right\}$$

Moreover there is a correspondence between Lie subbialgebroids and coisotropic Q-submanifolds.

Proof. In Section 3.4.1 we saw that $(A[1], Q)$ is a Q-manifold if and only if $(A \to M, [\cdot,\cdot], \rho)$ is a Lie algebroid and in Section 3.5.1 that $(A[1], \{\cdot,\cdot\})$ is Poisson manifold of degree 1 if and only if $(A^* \to M, [\cdot,\cdot], \rho)$ is a Lie algebroid. Therefore just remains to see that the equations (3.10) is equivalent to equation (2.9) and this is obvious.

For the moreover part, observe that a submanifold is the same as a vector subbundle $B \to N$ of $A^* \to M$. The fact that is coisotropic implies that $B \to N$ is a subalgebroid of $A^* \to M$ and the fact that is a Q-submanifold implies that $B^\circ \to N$ is a subalgebroid of $A \to M$. Therefore coisotropic Q-submanifolds correspond to Lie subbialgebroids. $\square$

Corollary 3.45 (See [113]). *There is a one to one correspondence between:*

$$\left\{ \begin{array}{c} \textit{Poisson manifolds} \\ (M,\pi) \end{array} \right\} \rightleftarrows \left\{ \begin{array}{c} \textit{Degree 1 symplectic Q-manifolds} \\ (\mathcal{M}, \{\cdot,\cdot\}, Q = X_\theta) \end{array} \right\}$$

Moreover there is a correspondence between coisotropic submanifolds of (M,π) and lagrangian Q-submanifolds.

Proof. In Corollary 3.38 we saw that when the bracket is symplectic then the 1-manifold is symplectomorphic to $T^*[1]M$ with the Poisson bracket given by the Schouten bracket on multivector fields. In Proposition A.10 we saw that any degree 1 symplectic vector field is hamiltonian, i.e. exist $\theta \in C^2_{T^*[1]M} = \mathfrak{X}^{2,0}(M)$ such that $Q = X_\theta$ and equation (2.5) become $\{\theta,\theta\} = 0$. Therefore we have the equivalence.

The moreover part is a classical result that can be found in [31]. $\square$

3.6.2 Degree 2

We will study here the degree 2 case that will be fundamental for Chapter 7. In Remark 3.13 we saw that 2-manifolds $\mathcal{M} = (M, C_M)$ are equivalent to triples $(E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \hookrightarrow E_2)$; consider

$$E_3 = \frac{\wedge^3 E_1 \oplus E_1 \otimes E_2}{\langle e_1 \wedge e_2 \wedge e_3 = e_1 \otimes \psi(e_2 \wedge e_3)\rangle}, \tag{3.11}$$

and the natural projection $p_3 : E_1 \otimes E_2 \to E_3$.

In Proposition 3.39 we saw that a Poisson bracket of degree -2 on $\mathcal{M}$ amounts to $(E_2, [\cdot,\cdot], \rho)$ being a Lie algebroid and E_1 carriyng a pairing $\langle\cdot,\cdot\rangle$ and a flat E_2-connection ∇ satisfying equations (3.9).

Proposition 3.46. *Let $(\mathcal{M}, \{\cdot,\cdot\})$ be a degree 2 Poisson manifold equivalent to $(E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \hookrightarrow E_2, \langle\cdot,\cdot\rangle, \nabla, [\cdot,\cdot], \rho)$ as before. Define the following operations*

$$\begin{cases} \mathcal{L}_f g = 0, & \mathcal{L}_f e_2 = 0, & \mathcal{L}_f \xi_2 = -\rho(\xi_2)(f), \\ \mathcal{L}_{e_1} g = 0, & \mathcal{L}_{e_1} e_2 = \langle e_1, e_2\rangle, & \mathcal{L}_{e_1}\xi_2 = -\nabla_{\xi_2} e_1 \\ \mathcal{L}_{\xi_1} g = \rho(\xi_1)(g), & \mathcal{L}_{\xi_1} e_2 = \nabla_{\xi_1} e_2, & \mathcal{L}_{\xi_1}\xi_2 = [\xi_1, \xi_2]. \end{cases}$$

for $f, g \in C^\infty(M)$, $e_1, e_2 \in \Gamma E_1$, $\xi_1, \xi_2 \in \Gamma E_2$ and extend them to the algebra $\Gamma(\bigwedge^\bullet E_1 \otimes \mathrm{Sym}^\bullet E_2)$ as graded derivations of the wedge, tensor and symmetric product. Then they produce natural operation on the quotient bundle E_3 defined by equation (3.11).

Proof. Observe that since the operations are given by the Poisson bracket and the Poisson bracket acts on functions it is clear that we have an operation on ΓE_3. If we want really a proof the only thing we need to check is that

$$\mathcal{L}_A e_1 \otimes \psi(e_2 \wedge e_3) = \mathcal{L}_A(e_1 \wedge e_2 \wedge e_3)$$

where $A \in \{f, e, \xi\}$ and this follows from the compatibility equations (3.9). $\square$

Theorem 3.47 (See [26])**.** *There is a correspondence between:*

- *$(\mathcal{M}, \{\cdot,\cdot\}, Q)$ degree 2 PQ-manifolds.*
- *Data $(E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \hookrightarrow E_2, \langle\cdot,\cdot\rangle, \nabla, [\cdot,\cdot], \rho, F_0, F_1, F_2)$ satisfying:*
  - *a). $(E_2 \to M, [\cdot,\cdot], \rho)$ is a Lie algebroid and E_1 has a pairing $\langle\cdot,\cdot\rangle$ and a flat E_2-connection ∇ satisfying equations (3.9).*
  - *b). $F_0 : T^*M \to E_1$ is a vector bundle map and $F_1 : \Gamma E_1 \to \Gamma E_2$ and $F_2 : \Gamma E_2 \to \Gamma E_3$ satisfy (3.8).*
  - *c). The following compatibilities for $f \in C^\infty(M)$, $e, e' \in \Gamma E_1$, $\xi, \xi' \in \Gamma E_2$:*

$$\begin{cases} \langle F_0 f, e\rangle = \rho(F_1 e)(f), \\ F_0\rho(\xi)(f) = \nabla_\xi F_0 f - \mathcal{L}_f F_2 \xi, \\ F_0\langle e, e'\rangle = \nabla_{F_1 e} e' + \nabla_{F_1 e'} e, \\ F_1(\nabla_\xi e) = [\xi, F_1 e] + \mathcal{L}_e F_2 \xi, \\ F_2[\xi, \xi'] = \mathcal{L}_\xi F_2 \xi' - \mathcal{L}_{\xi'} F_2 \xi. \end{cases} \tag{3.12}$$

Proof. By Propositions 3.27 and 3.39 we have that a Q-structure and a Poisson structure are equivalent to all that structures, so it just remains to prove that the vector field is Poisson, i.e. that satisfying equation (2.9), is equivalent to the equations (3.12). This assertion is quite obvious because by Leibniz and using that Q is a vector field we just need that the equation (2.9) is satisfied by the generators of $C_{\mathcal{M}}$ and these are the five equations of the statement. □

For the relation between degree 2 PQ-manifolds and LA-Courant algebroids see Section 7.1.1.

Let us now focus on the description of the coisotropic Q-submanifolds. Recall from Section 3.2 that if our 2-manifold is equivalent to $(E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \to E_2)$ we show in Example 3.17 that a submanifold is given by $(K_1 \to N, K_2 \to N)$ where $K_1 \to N$ subbundle of $E_1 \to M$ and $K_2 \to N$ subbundle of $E_2 \to M$ and satisfies

$$K_2 \cap \operatorname{im}(\psi)_{|N} = \psi(K_1 \wedge E_{1|N}).$$

Theorem 3.48. *Let $\mathcal{M}$ be a degree 2 PQ-manifold equivalent to the data $(E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \hookrightarrow E_2, \langle\cdot,\cdot\rangle, \nabla, [\cdot,\cdot], \rho, F_0, F_1, F_2)$ satisfying the hypothesis of Theorem 3.47.*

There is a one to one correspondence between:

- *$\mathcal{N} \subseteq (\mathcal{M}, \{\cdot,\cdot\}, Q)$ coisotropic Q-submanifold.*
- *Triples (N, K_1, K_2) where $N \subseteq M$ submanifold, $K_1 \subseteq E_{1|N}$ subbundle and $K_2 \subseteq E_{2|N}$ subbundle, satisfying:*

$$K_2 \cap \operatorname{im}(\psi)_{|N} = \psi(K_1 \wedge E_{1|N}), \tag{3.13}$$

$$\begin{cases} F_0(TN^\circ) \subseteq E_1, \\ F_1(\Gamma K_1) \subseteq \Gamma K_2, \\ F_2(\Gamma K_2) \subseteq p_3\Big(\Gamma(K_1 \otimes E_2 \oplus E_1 \otimes K_2)\Big), \end{cases} \tag{3.14}$$

$$\begin{cases} \langle K_1, K_1\rangle = 0, \\ K_2 \to N \text{ subalgebroid of } E_2 \to M, \\ \nabla_{\Gamma K_2}\Gamma K_1 \subseteq \Gamma K_1. \end{cases} \tag{3.15}$$

Proof. For simplicity, suppose that $\mathcal{N}$ is a closed embedded submanifold (otherwise we work locally). First, since $\mathcal{N}$ is closed and embedded we know by Proposition 2.16 that $\mathcal{N}$ is equivalent to a sheaf of ideals $\mathcal{I}$. Second, by the geometric characterization of submanifolds we know that these are equivalent to $K_1 \to N$ subbundle of $E_1 \to M$ and $K_2 \to N$ subbundle of $E_2 \to M$ satisfying equation (3.13). Moreover we have that

$$\begin{cases} \mathcal{I}^0 = Z(N) = \{f \in C^\infty(M) \mid f(n) = 0 \ \forall n \in N\}, \\ \mathcal{I}^1 = \{s \in \Gamma E_1 \mid s(n) \in K_1 \ \forall n \in N\}, \\ \mathcal{I}^2 = \{s \in \Gamma E_2 \mid s(n) \in K_2 \ \forall n \in N\}, \\ \mathcal{I}^3 = \{s \in \Gamma E_3 \mid s(n) \in p_3(K_1 \otimes E_2 \oplus E_1 \otimes K_2) \ \forall n \in N\}. \end{cases}$$

which follow directly from the characterization of submanifolds.

Hence $\mathcal{N}$ is a Q-submanifold if and only if satisfies that $Q(\mathcal{I}) \subseteq \mathcal{I}$ and this is equivalent to say that $(K_1 \to N, K_2 \to N)$ satisfy equations (3.14).

Finally, $\mathcal{N}$ is coisotropic if and only if $\{\mathcal{I}, \mathcal{I}\} \subseteq \mathcal{I}$ and this is equivalent to saying that $(K_1 \to N, K_2 \to N)$ satisfy the equations (3.15). □

Before to end this section we study degree 2 symplectic Q-manifolds. In order to do that we need the definition of a Courant algebroid.

A *Courant algebroid* $(E \to M, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!], a)$ is a vector bundle $E \to M$ endowed with a non degenerate pairing $\langle \cdot, \cdot \rangle : \Gamma E \times \Gamma E \to C^\infty(M)$, a bracket $[\![\cdot, \cdot]\!] : \Gamma E \times \Gamma E \to \Gamma E$ and a vector bundle map $a : E \to M$ called the anchor satisfying the following properties:

- $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$,
- $[\![e_1, fe_2]\!] = a(e_1)(f)e_2 + f[\![e_1, e_2]\!]$,
- $a(e_1)\langle e_2, e_3 \rangle = \langle [\![e_1, e_2]\!], e_3 \rangle + \langle e_2, [\![e_1, e_3]\!] \rangle$,
- $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = a^{\flat *} d\langle e_1, e_2 \rangle$,

where $e_1, e_2, e_3 \in \Gamma E$, $f \in C^\infty(M)$ and $a^{\flat *} = \langle , \rangle^\flat \circ a^* : T^*M \to E$.

A *Dirac structure on a Courant algebroid* $(E \to M, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!], a)$ is a subbundle $(L \to N) \subseteq (E \to M)$ satisfying:

- $L = L^\perp = \{a \in E_{|N} \mid \langle a, l \rangle = 0 \ \forall \ l \in L\}$,
- $\rho(L) \subseteq TN$,
- $[\![\Gamma L, \Gamma L]\!] \subseteq \Gamma L$.

Remark 3.49. Sometimes this kind of Dirac structures are called Dirac structures with support, see [27]. But since we always treated this case we just call them Dirac structures.

Theorem 3.50 (see [108, 113])**.** *There is a one to one correspondence between:*

$$\left\{ \begin{array}{c} \textit{Courant algebroids} \\ (E \to M, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!], \rho) \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{c} \textit{Degree 2 symplectic } Q\textit{-manifolds} \\ (\mathcal{M}, \{\cdot, \cdot\}, Q) \end{array} \right\}$$

Moreover this correspondence sends Dirac structures to lagrangian Q-submanifolds.

We will see in Example 7.7 how Courant algebroids produce the geometric data of a degree 2 PQ-manifold.

Chapter 4

The Frobenius theorem for graded manifolds

In this chapter we will prove the Frobenius theorem for graded manifolds. In classical geometry given a manifold M and a regular distribution $D \subseteq TM$ over it, the Frobenius theorem states that a distribution is involutive, i.e. closed for the Lie bracket, if and only if locally it is spanned by coordinate vector fields. Here we give an analogous result for graded manifolds.

The classical Frobenius theorem implies that the distribution has integral submanifolds, i.e. $\forall\, p \in M$ there exists $S \subseteq M$ passing through p such that $TS = D_{|S}$. Here we also explore integral submanifolds for distributions over graded manifolds explaining the saddle points.

Apart from being a fundamental result in differential geometry, we are also interested in the Frobenius theorem because is a key step for proving the integration of graded Lie algebroid morphisms to graded groupoid morphisms, i.e. a graded version of the Lie II theorem for groupoids.

4.1 Distributions on graded manifolds

Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be a graded manifold of degree n and dimension $m_0|\cdots|m_n$. Recall that $\mathfrak{X}^{1,\bullet}(\mathcal{M})$ denotes the sheaf of $C_{\mathcal{M}}$-modules of vector fields over $\mathcal{M}$. We say that a subsheaf $\mathcal{A} \subseteq \mathfrak{X}^{1,\bullet}(\mathcal{M})$ is *a graded subsheaf of $C_{\mathcal{M}}$-modules* if

$$\forall\, X \in \mathcal{A} \text{ and } \ f \in C_{\mathcal{M}} \ \Rightarrow fX \in \mathcal{A}$$

and, for $X \in \mathcal{A}$, the homogeneous components of X also belong to $\mathcal{A}$.

A *distribution* is a graded subsheaf of $C_{\mathcal{M}}$-modules $\mathcal{D} \subseteq \mathfrak{X}^{1,\bullet}(\mathcal{M})$ such that $\forall p \in M$, $\exists\, U$ open around p and vector fields $X^{j_i} \in \mathcal{D}_{|U}$, satisfying

a). $\{X^{j_i}_q\}$ are linearly independent $\forall q \in U$.
b). $\mathcal{D}_{|U} = \langle X^{j_i} \rangle$ as a $(C_{\mathcal{M}})_{|U}$-module.

We say that a distribution $\mathcal{D}$ is *involutive* if it is closed under the Lie bracket on $\mathfrak{X}^{1,\bullet}(\mathcal{M})$.

Observe that since $\mathcal{D}$ is a graded subsheaf, then we can always find X^{j_i} locally generating the distribution that are homogeneous. Therefore we assume that locally we have *homogeneous generators*. On the other hand our condition of the generators being linearly independent implies that on the connected components the number of generators is fixed. This implies that for each $0 \le i \le n$ there exists $d_i \in \mathbb{N}$ with $d_i \le m_i$ such that $\mathcal{D}_{|U} = \langle X^{1_0}, \cdots X^{d_0}, \cdots X^{1_n}, \cdots X^{d_n} \rangle$. These numbers define the *rank of the distribution* and we will denoted by $\mathrm{rk}(\mathcal{D}) = d_0|\cdots|d_n$.

Also by the linear independence of the local generators of $\mathcal{D}$ we have that for each point $p \in M$, $\mathcal{D}$ defines a subspace of the graded vector space $T_p\mathcal{M}$ that we denote by $\mathcal{D}_p$. In particular $\mathcal{D}_p^0$, the degree zero component of $\mathcal{D}_p$ is a subspace of T_pM. It is easy to see that $\cup_{p\in M}\mathcal{D}_p^0$ defines a regular distribution on M that we denote by D.

Remark 4.1. Such distributions on smooth manifolds are often called *regular distributions* but since they are the only type of distributions that we will study we just called them distributions.

Example 4.2 (Distributions are not determined by their tangent vectors). Recall from Example 2.14 that vector fields on a graded manifold are not determined by their tangent vectors, see equation (2.6). Therefore a distribution $\mathcal{D}$ is not determined by the collection of their tangent vectors at all the points. More concretely, and following the notation of Example 2.14, on $\mathbb{R}^{1|1} = \{x, e\}$ with $|x| = 0$ and $|e| = 1$ we have that

$$\mathcal{D}^1 = \langle \frac{\partial}{\partial x} \rangle \quad \text{and} \quad \mathcal{D}^2 = \langle \frac{\partial}{\partial x} + e\frac{\partial}{\partial e} \rangle$$

are two different distributions that have the same tangent vectors at all points.

Example 4.3 (The tangent vectors of a distributions do not determine their involutivity). In $\mathbb{R}^{0|1|1} = \{e, p\}$ with $|e| = 1$ and $|p| = 2$ define the following distributions:

$$\mathcal{D}^3 = \langle \frac{\partial}{\partial e} \rangle \quad \text{and} \quad \mathcal{D}^4 = \langle \frac{\partial}{\partial e} + e\frac{\partial}{\partial p} \rangle.$$

Then clearly these two distributions have the same tangent vectors at all points, because the vector field that generat them have the same tangent vector at all points. But $\mathcal{D}^3$ is involutive while $\mathcal{D}^4$ is not.

4.2 The Frobenius theorem

In this section we will give a proof for the graded version of the Frobenius theorem. The result is as follows:

Theorem 4.4 (Frobenius Theorem for graded manifolds)**.** *Let $\mathcal{M}$ be an n-manifold and $\mathcal{D}$ an involutive distribution on $\mathcal{M}$ of rank $d_0|\cdots|d_n$. For each $p \in M$ there exists $U \subseteq M$ open around p and coordinates $\{e^{j_i}\}$ on $\mathcal{M}_{|U}$ such that*

$$\mathcal{D}_{|U} = \langle \frac{\partial}{\partial e^{1_0}}, \cdots, \frac{\partial}{\partial e^{d_0}}, \cdots, \frac{\partial}{\partial e^{1_n}}, \cdots, \frac{\partial}{\partial e^{d_n}} \rangle.$$

Before we start with the proof we will make some remarks:

Remark 4.5 (Supermanifold case). This statement is the usual one proved for supermanifolds, see e.g. [9, 30, 46, 49]. In [30, 46, 49] the definition of distribution is different. In this point we follow [9] that is closer to differential geometry.

Remark 4.6 (On local normal forms for vector fields I). Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be a graded manifold and $X \in \mathfrak{X}^{1,\bullet}(\mathcal{M})$. Then we know that $\mathcal{D} = \langle X \rangle$ defines a distribution if and only if $X_p \neq 0$, $\forall p \in M$. Suppose that this condition is satisfied; Theorem 4.4 says that if $[X, X] = 0$ then around $p \in M$, $X = \frac{\partial}{\partial e}$ where e is a coordinate. Therefore the local Frobenius theorem can be seen as a result on local normal forms for vector fields. This kind of local normal form for vector fields on supermanifolds was obtained in [110].

Remark 4.7 (On local normal forms for vector fields II). Observe that our condition of the vector field having non zero tangent vector at all points implies that if $X \in \mathfrak{X}^{1,1}(\mathcal{M})$ then $\mathcal{D} = \langle X \rangle$ is never a distribution in our sense. But in [119] a local normal form for degree 1 vector fields with $[X, X] = 0$, i.e. for Q-structures was described. In fact, when the graded manifold is a 1-manifold, this result was the first proof of the local splitting theorem for Lie algebroids. We believe that further results in this way can also give the splitting for Courant algebroids or for L_∞-algebroids as obtained in [12].

The proof of Theorem 4.4 is divided in two steps: we first prove the result when a distribution has no degree 0 vector fields, and after that we add the degree zero part.

Lemma 4.8. *Let $\mathcal{M}$ be a n-manifold and $\mathcal{D}$ an involutive distribution on $\mathcal{M}$ with* $\mathrm{rk}(\mathcal{D}) = 0|d_1|\cdots|d_n$*. For each $p \in M$, there exists U open around p and coordinates $\{e^{j_i}\}$ on $\mathcal{M}_{|U}$ such that*

$$\mathcal{D}_{|U} = \langle \frac{\partial}{\partial e^{1_1}}, \cdots, \frac{\partial}{\partial e^{d_1}}, \cdots, \frac{\partial}{\partial e^{1_n}}, \cdots, \frac{\partial}{\partial e^{d_n}} \rangle.$$

Proof. Suppose that $\mathcal{M}$ is of dimension $m_0|\cdots|m_n$. By the definition of distribution of rank $\mathrm{rk}(\mathcal{D}) = 0|d_1|\cdots|d_n$ we have that for each $p \in M$, there exists $U \subset M$ open around p and vector fields $\{X^{j_i}\} \in \mathcal{D}_{|U}$ satisfying:

a). $X^{j_i} \in \mathfrak{X}^{1,-i}(\mathcal{M}_{|U})$ for all $1 \leq j_i \leq d_i$ with $1 \leq i \leq n$.

b). $\{X^{j_i}_q\}$ are linearly independent $\forall q \in U$.

c). $\mathcal{D}_{|U} = \langle X^{j_i} \rangle$.

We also can assume that $(C_{\mathcal{M}})_{|U} \cong C^\infty(U) \otimes \mathrm{Sym}\,\mathbf{V}$ otherwise we just shrink U.

Consider coordinates $\{\widehat{e}^{b_a}\}$ of $\mathcal{M}_{|U}$ with $0 \leq a \leq n$ and $1 \leq b_a \leq m_a$. In terms of these coordinates the vector fields are written as

$$X^{j_i} = \sum_{a=i}^{n} \sum_{b_a=1}^{m_a} X^{j_i}(\widehat{e}^{b_a}) \frac{\partial}{\partial \widehat{e}^{b_a}} \quad \text{where } X^{j_i}(\widehat{e}^{b_a}) \in (C^{a-i}_{\mathcal{M}})_{|U}.$$

Let us proceed by induction on i. For $i = n$, we have that

$$X^{j_n} = \sum_{b_n=1}^{m_n} X^{j_n}(\widehat{e}^{b_n}) \frac{\partial}{\partial \widehat{e}^{b_n}}, \quad \text{where } X^{j_n}(\widehat{e}^{b_n}) \in (C^0_{\mathcal{M}})_{|U} = C^\infty(U).$$

Since $\{X_q^{1_n}, \cdots, X_q^{d_n}\}$ are linearly independent $\forall q \in U$ then the family of matrices $A = (X^{j_n}(\widehat{e}^{b_n}))_{jb}(q)$ has maximal rank $\forall q \in U$. Consider the change of coordinates

$$\begin{cases} e^{k_l} = \widehat{e}^{k_l} & \text{if } l \neq n, \\ e^{k_n} = D_{kl}\widehat{e}^{l_n}, \end{cases}$$

where $AD = \mathrm{Id}_{d_n \times m_n}$. It is easy to see that $X^{j_n} = \frac{\partial}{\partial e^{j_n}}$, as we want.

Now suppose that the result holds for $i = k+1$. We will prove that it also holds for $i = k$. By the inductive hypothesis we have coordinates $\{\widehat{e}^{b_a}\}$ of $\mathcal{M}_{|U}$ and vector fields $\{X^{j_i}\} \in \mathcal{D}_{|U}$ satisfying the same hypothesis as before, and in addition

$$X^{j_i} = \frac{\partial}{\partial \widehat{e}^{j_i}} \quad \text{for } 1 \leq j_i \leq d_i \ \text{ with } k+1 \leq i \leq n.$$

Since $\{X_q^{j_k}\}$ are also linearly independent $\forall\ q \in U$, we also can make the same change of coordinates as before, and one can show that this does not change the vector fields of degree less than $-k$. Also by linear combinations with the vector fields of degree less than $-k$ on the distribution, we can assume that X^{j_k} has the form

$$X^{j_k} = \frac{\partial}{\partial \widehat{e}^{j_k}} + \sum_{r=r_0}^{n} \sum_{s_r=d_r+1}^{m_r} X^{j_k}(\widehat{e}^{s_r}) \frac{\partial}{\partial \widehat{e}^{s_r}} \quad \text{for } 1 \leq j_k \leq d_k$$

for some $r_0 \in \{k+1, \cdots n\}$. Define the change of coordinates:

$$\Psi = \begin{cases} e^{j_i} = \widehat{e}^{j_i}, & i \neq r_0. \\ e^{s_{r_0}} = \widehat{e}^{s_{r_0}} - \frac{1}{d_k+1} \sum\limits_{l_k=1}^{d_k} \frac{1}{deg(\widehat{e}^{l_k})+1} \widehat{e}^{l_k} X^{l_k}(\widehat{e}^{s_{r_0}}), & 1 \leq s_{r_0} \leq m_{r_0}. \end{cases}$$

Let us prove that

$$\begin{cases} \Psi_* \dfrac{\partial}{\partial \widehat{e}^{b_a}} = \dfrac{\partial}{\partial e^{b_a}} & \text{for } k+1 \leq a \leq n,\ 1 \leq b_a \leq d_a \\ \Psi_* X^{j_k} = \dfrac{\partial}{\partial e^{j_k}} + \sum\limits_{r=r_0+1}^{n} \sum\limits_{s_r=d_r+1}^{m_r} \Psi \circ X^{j_k} \circ \Psi^{-1}(e^{s_r}) \dfrac{\partial}{\partial e^{s_r}} & \text{for } 1 \leq j_k \leq d_k. \end{cases}$$

Using that $\mathcal{D}$ is an involutive distribution we obtain that for all $k+1 \leq a \leq n,\ 1 \leq b_a \leq d_a$, we have

$$[\frac{\partial}{\partial \widehat{e}^{b_a}}, X^{j_k}] = \sum_{r=r_0}^{n} \sum_{s_r=d_r+1}^{m_r} \frac{\partial}{\partial \widehat{e}^{b_a}} \Big(X^{j_k}(\widehat{e}^{s_r})\Big) \frac{\partial}{\partial \widehat{e}^{s_r}} \in \mathcal{D}_{|U} \Leftrightarrow \frac{\partial \Big(X^{j_k}(\widehat{e}^{s_r})\Big)}{\partial \widehat{e}^{b_a}} = 0.$$

Therefore $\Psi_* \frac{\partial}{\partial \widehat{e}^{b_a}} = \frac{\partial}{\partial e^{b_a}}$ for $k+1 \leq a \leq n,\ 1 \leq b_a \leq d_a$.

On the one hand it follows directly from the change of variables that, for $r \geq r_0$,

$$\Psi_* \frac{\partial}{\partial \widehat{e}^{s_r}} = \frac{\partial}{\partial e^{s_r}} \quad \forall 1 \leq s_r \leq m_r.$$

On the other hand, since $1 \leq k < r_0$, the involutivity condition $[X^{j_k}, X^{l_k}] \in \mathcal{D}_{|U}$ implies that

$$(-1)^{k^2}\frac{\partial\Big(X^{l_k}(\widehat{e}^{s_{r_0}})\Big)}{\partial\widehat{e}^{j_k}} = \frac{\partial\Big(X^{j_k}(\widehat{e}^{s_{r_0}})\Big)}{\partial\widehat{e}^{l_k}} \quad \forall\ d_{r_0}+1 \le s_{r_0} \le m_{r_0}.$$

As a consequence, we obtain that

$$\frac{\partial}{\partial\widehat{e}^{j_k}} = \frac{\partial}{\partial e^{j_k}} - \sum_{s_{r_0}=d_{r_0}+1}^{m_{r_0}} \tfrac{1}{d_k+1}\Big(X^{j_k}(\widehat{e}^{s_{r_0}}) + \sum_{l_k=1}^{d_k} \tfrac{1}{deg(\widehat{e}^{l_k})+1}\widehat{e}^{l_k}(-1)^{k^2}\frac{\partial X^{l_k}(\widehat{e}^{s_{r_0}})}{\partial\widehat{e}^{j_k}}\Big)\frac{\partial}{\partial e^{s_{r_0}}}$$

$$= \frac{\partial}{\partial e^{j_k}} - \sum_{s_{r_0}=d_{r_0}+1}^{m_{r_0}} X^{j_k}(\widehat{e}^{s_{r_0}})\frac{\partial}{\partial e^{s_{r_0}}},$$

which implies that $\Psi_* X^{j_k} = \frac{\partial}{\partial e^{j_k}} + \sum_{r=r_0+1}^{n}\sum_{s_r=d_r+1}^{m_r} \Psi\circ X^{j_k}\circ\Psi^{-1}(e^{s_r})\frac{\partial}{\partial e^{s_r}}$. Therefore after a maximum of $n-k+1$ changes of variables we have proved the inductive case. $\square$

proof of Theorem 4.4. By the definition of distribution of rank $d_0|\cdots|d_n$ we have that for each $p\in M$ there exists $U\subseteq M$ open around p and linearly independent vector fields $\{X^{j_i}\}$ with $0\le i\le n$ and $1\le j_i\le d_i$ such that $\mathcal{D}_{|U} = \langle X^{j_i}\rangle$. Denote by $\mathcal{D}^{<0}_{|U} = \langle X^{j_i}\ |1\le i\le n, 1\le j_i\le d_i\rangle$ and by $\mathcal{D}^0 = \langle X^{j_0}\ |1\le j_0\le d_0\rangle$. Clearly $\mathcal{D}_{|U} = \mathcal{D}^0_{|U}\oplus\mathcal{D}^{<0}_{|U}$.

Therefore by the previous lemma there exist coordinates $\{\widehat{e}^{b_a}\}$ in $\mathcal{M}_{|U}$, shrinking U if necessary, such that

$$\mathcal{D}_{|U} = \mathcal{D}^0_{|U}\oplus\mathcal{D}^{<0}_{|U} = \langle X^{j_0}\rangle\oplus\langle\frac{\partial}{\partial\widehat{e}^{1_1}},\cdots,\frac{\partial}{\partial\widehat{e}^{d_1}},\cdots,\frac{\partial}{\partial\widehat{e}^{1_n}},\cdots,\frac{\partial}{\partial\widehat{e}^{d_n}}\rangle.$$

Now by Theorem 3.21 and equation (3.5) we have that

$$X^{j_0} = (\Theta^{j_0}_1,\cdots,\Theta^{j_0}_n) \quad \text{where} \quad \Theta^{j_0}_i\in\Gamma\mathbb{A}_{E_i} \tag{4.1}$$

all with the same symbol $Y^{j_0}\in\mathfrak{X}(M)$ and satisfying equation (3.5). Moreover the fact that X^{j_0} are linearly independent implies that $\{Y^{j_0}\}$ are also linearly independent, so they span the distribution $D_{|U}\subseteq TM_{|U}$. Therefore we can think that there exists ∇^i partial D-connection on E_i satisfying

$$\nabla^{i+j}_X m(s^i,s^j) = m(\nabla^i_X s^i, s^j) + m(s^i,\nabla^j_X s^j) \quad \forall X\in\Gamma F,\ s^i\in\Gamma E^i,\ s^j\in\Gamma E^j. \tag{4.2}$$

Also for each $1\le i\le n$ we can define

$$G^*_i = \langle\frac{\partial}{\partial\widehat{e}^{j_i}}|1\le j_i\le d_i\rangle\subseteq(\widetilde{E}^*_i)_{|U}.$$

The involutivity property of $\mathcal{D}$ implies the following conditions:

a). D is an involutive distribution on U.
b). By the previous condition, it makes sense to talk on the curvature of ∇^i, and one can check that ∇^i are flat connections.
c). The dual connection of ∇^i preserves G^*_i.

Observe that by linear combination we can assume that $\nabla^i_X(\widehat{e}^{j_i}) = 0$ for any $1 \leq j_i \leq d_i$. Therefore by the flatness condition and the fact that ∇^i dual preserves G_i^*, we conclude that we can complete $\{\widehat{e}^{j_i}\}$ to a basis of flat sections of E_i.

Finally, using the Frobenius theorem for $D_{|U}$ we can find a basis of U such that $D_{|U} = \langle \frac{\partial}{\partial e^{1_0}}, \cdots, \frac{\partial}{\partial e^{d_0}} \rangle$.

As a result, we have found a basis of coordinate of $\mathcal{M}_{|U}$ for which $\mathcal{D}_{|U}$ is spanned by the coordinates vector fields. □

In differential geometry it is well known that there is a one to one correspondence between involutive distributions and foliations. We now re-state Theorem 4.4 in a way that makes clear that locally we have a foliated chart, see [102] for the classical picture.

Corollary 4.9. *Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be an n-manifold of dimension $m_0|\cdots|m_n$ and $\mathcal{D}$ an involutive distribution of rank $d_0|\cdots|d_n$. Then $\forall p \in M$, $\exists$ U open around p and a diffeomorphism*

$$\Psi : \mathbb{R}^{d_0|\cdots|d_n} \times \mathbb{R}^{m_0-d_0|\cdots|m_n-d_n} \to \mathcal{M}_{|U}$$

such that $\Psi_*(T\mathbb{R}^{d_0|\cdots|d_n} \times 0) = \mathcal{D}_{|U}$.

4.3 Integral submanifolds

Let M be a smooth manifold and $D \subseteq TM$ a regular involutive distribution. Classically, the condition that $D_{|U} = \langle \frac{\partial}{\partial x^i} \rangle$ for some coordinates $\{x^i\}$ of M, immediately implies the existence of integral submanifolds passing through any point, i.e. there exists $S \hookrightarrow M$ such that $TS = D_{|S}$.

The naive notion of integral submanifold in the graded case is that there exists $(\mathcal{L}, j)$ submanifold of $\mathcal{M}$ satisfying that

$$dj_p(T_p\mathcal{L}) = \mathcal{D}_{j(p)} \quad \forall p \in L, \tag{4.3}$$

where L is the body of $\mathcal{L}$. But as we saw in Example 4.3 the involutivity condition for distributions does not follow just by knowing the tangent vectors, therefore this condition (4.3) is not enough to characterize involutivity. So we will refine the notion.

Let $\mathcal{M}$ be an n-manifold of dimension $m_0|\cdots|m_n$ and $\mathcal{D} \subseteq \mathfrak{X}^{1,\bullet}(\mathcal{M})$ a distribution of $\mathrm{rk}(\mathcal{D}) = d_0|\cdots|d_n$. We say that a submanifold $(\mathcal{L}, j)$ of $\mathcal{M}$ is an *integral submanifold of* $\mathcal{D}$ if $\forall\, p \in L$ there exists $U \subseteq L$ open neighbourhood of p, $V \subseteq \mathbb{R}^{m_0-d_0}$ open around 0 and a morphism $\Psi : \mathcal{L}_{|U} \times \mathcal{V} \to \mathcal{M}$ where $\mathcal{V} = V \times \mathbb{R}^{0|m_1-d_1|\cdots|m_n-d_n}$ satisfying:

a). Ψ is an embedding.

b). $\Psi \circ i_1 = j_{|U}$, where $i_1 : \mathcal{L}_{|U} \to \mathcal{L}_{|U} \times \mathcal{V}$ is the natural inclusion in the first factor, for which $i_1 : L \to L \times V, \quad i_1(l) = (l, 0)$.

c). $\Psi_*(T\mathcal{L}_{|U} \times 0) = \mathcal{D}_{|\psi(U \times V)}$.

Observe that our definition of integral submanifolds is more than just a submanifold: it is a submanifold plus some information in the "normal directions".

Remark 4.10 (Maps between graded manifolds)*.* The extra information that an integral submanifold carries can be summarized by saying that the submanifold is immersed with a *map between graded manifolds* in the sense of [39, 109].

A clear consequence of our definition is that if $(\mathcal{L}, j)$ is an integral submanifold of $\mathcal{D}$ then $\forall\ p \in L$ equation (4.3) is satisfied.

Remark 4.11*.* There are some works on supermanifolds where integral submanifolds of a distribution are defined just by condition (4.3). We known that this is not enough and create some pathologies as different distributions with the same integral submanifolds or even worse, non involutive distributions with integral submanifolds passing through all the points.

Chapter 5

Lie algebroids through graded manifolds

In this chapter we will study two aspects of Lie algebroids where graded manifolds can help. The first one is understanding homology theory for Lie algebroids. The second concerns semi-direct products with 2-term representations up to homotopy. The goal of this chapter is to clarify some points on the constructions and combine different points of view.

5.1 Homology for Lie algebroids

In this section we will discus the problem of defining homology for Lie algebroids. The approach that we follow does not extend the singular homology theory for manifolds that, in some sense, we believe must be the most powerful one. Our point of view is based on an extension of the Poisson Homology, as defined in [19]. The main ideas of this theory of homology for Lie algebroids were given in [55, 56, 67, 79, 129].

5.1.1 Left and right representations

Here we introduce the concept of left and right representations of a Lie algebroid by using connections as in [86] and define cohomology (resp. homology) with values on a left (resp. right) representation. Let $(A \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid and consider a vector bundle over the same base, $E \to M$. We say that (E, ∇) is a *left representation* if $\nabla : \Gamma A \times \Gamma E \to \Gamma E$ satisfies

$$L1)\ \nabla_a fe = f\nabla_a e + \rho(a)(f)e, \quad L2)\ \nabla_{fa} e = f\nabla_a e \ \text{ and } \ L3)\ \nabla_{[a,b]} = [\nabla_a, \nabla_b].$$

$\forall a, b \in \Gamma A,\ e \in \Gamma E,\ f \in C^\infty(M)$.

On the other hand we say that (E, Υ) is a *right representation* if $\Upsilon : \Gamma E \times \Gamma A \to \Gamma E$ satisfies

$$R1)\ \Upsilon_a fe = f\Upsilon_a e - \rho(a)(f)e, \quad R2)\ \Upsilon_{fa} e = \Upsilon_a fe \ \text{ and } \ R3)\ \Upsilon_{[a,b]} = -[\Upsilon_a, \Upsilon_b].$$

$\forall a, b \in \Gamma A,\ e \in \Gamma E,\ f \in C^\infty(M)$.

Remark 5.1. Observe that given a left representation (E,∇), if we consider $-\nabla$ then it satisfies $R1)$ and $R3)$ but not $R2)$. Therefore, it is not true that a left representation induces a right representation on the same vector bundle (as it happens for example for Lie algebras). Our notion of right representations comes from algebraic geometry and was taken from [68].

Examples 5.2. We will give some examples of left and right representations for any Lie algebroid $(A\to M,[\cdot,\cdot],\rho)$:

a). $(\mathbb{R}_M,\rho)$ is a left representation with action

$$\begin{array}{rcl} \rho: & \Gamma A\times C^\infty(M)\to & C^\infty(M)\\ & (a,\ f) & \rho(a)(f)\end{array}$$

b). $(\bigwedge^{top}A^*,\mathcal{L})$ is a right representation with action

$$\begin{array}{rcl} \mathcal{L}: & \Gamma A\times\Gamma\bigwedge^{top}A^*\to & \Gamma\bigwedge^{top}A^*\\ & (a,\quad\omega) & \mathcal{L}_a\omega=-L_a\omega\end{array}$$

c). Let $B\to M$ be another Lie algebroid and $\phi:A\to B$ a Lie algebroid morphism over the identity. Consider (E,∇) a left representation of B. Then (E,∇_ϕ) is a left representation of A. Analogously, if (E,Υ) is a right representation of B then (E,Υ_ϕ) is a right representation of A.

d). For any Lie algebroid the anchor $\rho:A\to TM$ is a Lie algebroid morphism. Then $(\bigwedge^{top}T^*M,\mathcal{L}_\rho)$ is a right representation of A.

e). Let (E,∇) and (F,∇') be two left representations. Then $(E\otimes F,\nabla\otimes\nabla')$ is a left representation.

f). Let (E,∇) be a left representation and (F,Υ) a right representation. Then $(E\otimes F,\nabla\otimes\Upsilon)$ is a right representation.

g). Consider (E,Υ) and (F,Υ') two right representations. Then we obtain that $\big(\mathrm{Hom}(E,F),\ \mathrm{Hom}(\Upsilon,\Upsilon')\big)$ is a left representation.

h). Combining items b), d) and g) we have that any Lie algebroid has a left representation on $\big(\mathrm{Hom}(\bigwedge^{top}A^*,\bigwedge^{top}T^*M),\mathrm{Hom}(\mathcal{L},\mathcal{L}_\rho)\big)$.

Summarizing, by the preceding examples we obtained that any Lie algebroid $(A\to M,[\cdot,\cdot],\rho)$ carries four natural line bundle representations: two left representations $(\mathbb{R}_M,\rho)$ and $\big(\mathrm{Hom}(\bigwedge^{top}A^*,\bigwedge^{top}T^*M),\ \mathrm{Hom}(\mathcal{L},\mathcal{L}_\rho)\big)$, and two right representations $(\bigwedge^{top}A^*,\mathcal{L})$ and $(\bigwedge^{top}T^*M,\mathcal{L}_\rho)$.

Remark 5.3. The left line representation $\big(\mathrm{Hom}(\bigwedge^{top}A^*,\bigwedge^{top}T^*M),\mathrm{Hom}(\mathcal{L},\mathcal{L}_\rho)\big)$ can be understood as the determinant of the adjoint representation up to homotopy as we will define in Example 5.23. This representation was defined for the first time in [55] and used to compute the modular class of a Lie algebroid.

Now we introduce the concepts of cohomology and homology with values on a representation.

Let $(A\to M,[\cdot,\cdot],\rho)$ be a Lie algebroid and (E,∇) a left representation. Then we can define a differential complex associated to it: $(\mathrm{Hom}(\bigwedge^\bullet A,E),d^\nabla)$, where $d^\nabla:\mathrm{Hom}(\bigwedge^k A,E)\to\mathrm{Hom}(\bigwedge^{k+1}A,E)$ is defined by the formula

$$d^\nabla\omega(a_0,\cdots,a_k)=\sum_i(-1)^i\nabla_{a_i}\omega(a_0,\cdots,a_k)+\sum_{i<j}(-1)^{i+j}\omega([a_i,a_j],a_0,\cdots,a_k).$$

The *cohomology of the Lie algebroid* $(A \to M, [\cdot,\cdot], \rho)$ *with values in the left representation* (E, ∇) is defined as the cohomology of this complex and will be denoted by $H^*(A; E, \nabla)$, this was introduced in [86].

Analogously, given a right representation (E, Υ) of $(A \to M, [\cdot,\cdot], \rho)$, we can define another differential complex $(\bigwedge^\bullet A \otimes E, \partial_\Upsilon)$ where $\partial_\Upsilon : \bigwedge^k A \otimes E \to \bigwedge^{k-1} A \otimes E$ is defined by the formula:

$$\begin{aligned}\partial_\Upsilon(a_0 \wedge \cdots \wedge a_{k-1} \otimes e) = \;& \sum_i (-1)^i a_0 \wedge \cdots \wedge a_{k-1} \otimes \Upsilon_{a_i} e \\ & + \sum_{i<j} (-1)^{i+j} [a_i, a_j] \wedge a_0 \wedge \cdots \wedge a_{k-1} \otimes e\end{aligned}$$

In this case, we define the *homology of the Lie algebroid with values on the right representation* (E, Υ) as the homology of the above complex, and we denote it by $H_*(A; E, \Upsilon)$, see also [68].

The good definition of the operators d^∇ and ∂_Υ, as well as the facts that they are derivations of the wedge product and squares to zero are equivalent to the axioms of being a left or right representation.

We end with a result that relates left and right representations and explains why we do not need to take care of right representations:

Theorem 5.4 (See [68]). *Given a Lie algebroid* $(A \to M, [\cdot,\cdot], \rho)$*, there is a one to one correspondence between*

$$\begin{array}{ccc} \{\textit{Left representations}\} & \leftrightharpoons & \{\textit{Right representations}\} \\ (E, \nabla) & \rightsquigarrow & (E \otimes \bigwedge^{top} A^*, \nabla \otimes \mathcal{L}) \\ (\mathrm{Hom}(\bigwedge^{top} A^*, F), \mathrm{Hom}(\Upsilon, \mathcal{L})) & \dashleftarrow & (F, \Upsilon) \end{array}$$

Moreover, the correspondence gives an isomorphism between

$$H^*(A; E, \nabla) = H_{top-*}(A; E \otimes \bigwedge^{top} A^*, \nabla \otimes \mathcal{L}).$$

Proof. The proof easily follows from the Tensor-Hom adjunction. $\square$

5.1.2 Divergence and generating operators

By the previous section we have that a Lie algebroid has a canonical cohomology because it has a left representation on the trivial bundle $\mathbb{R}_M$. Then, in order to have also a canonical homology we need to study the right representations on the trivial bundle $\mathbb{R}_M$. For that we introduce the concept of a divergence for a Lie algebroid. This concept mimics de usual divergence operator of calculus, that produces a function for any given vector field.

A *divergence for the Lie algebroid* $(A \to M, [\cdot,\cdot], \rho)$ is a map $D : \Gamma A \to C^\infty(M)$ satisfying

$$D(fa) = fD(a) + \rho(a)(f) \quad \text{and} \quad D([a,b]) = \rho(a)(D(b)) - \rho(b)(D(a))$$

$\forall\ f \in C^\infty(M),\ a, b \in \Gamma A.$

Remark 5.5. In the work [56] divergence operators are referred as Calabi-Yau structures. This notion comes from derived algebraic geometry and is an active topic of research now, see e.g. [44, 75].

Proposition 5.6. *A divergence for a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ is the same as a right representation of $(A \to M, [\cdot,\cdot], \rho)$ on the vector bundle $\mathbb{R}_M$.*

Proof. Given a divergence $D : \Gamma A \to C^\infty(M)$ define $\Upsilon : \Gamma A \times C^\infty(M) \to C^\infty(M)$ by the formula

$$\Upsilon_a f = -fD(a) - \rho(a)(f)$$

Conversely, given $\Upsilon : \Gamma A \times C^\infty(M) \to C^\infty(M)$ a right representation, define a divergence $D : \Gamma A \to C^\infty(M)$ by

$$D(a) = -\Upsilon_a 1$$

Since the constructions are inverse to each other, we have the result. □

Recall from Example 2.20 that given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ we have that $(\Gamma \bigwedge^\bullet A, [\cdot,\cdot])$ has the structure of a Gerstenhaber algebra. In [67], Huebschmann defines a *generating operator for the Gerstenhaber algebra* as a map $\partial : \bigwedge^\bullet \Gamma A \to \bigwedge^{\bullet-1} \Gamma A$ satisfying:

$$[\omega,\tau] = (-1)^{|\omega|}\big(\partial(\omega\wedge\tau) - \partial(\omega)\wedge\tau - (-1)^{|\omega|}\omega\wedge\partial(\tau)\big) \text{ and } \partial^2 = 0.$$

An immediate consequence of the definition is the following result:

Proposition 5.7. *Let $(A \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid. There is a one to one correspondence between generating operators and right representations on the vector bundle $\mathbb{R}_M$.*

Proof. Given a generating operator $\partial : \bigwedge^\bullet \Gamma A \to \bigwedge^{\bullet-1} \Gamma A$ we have that $-\partial^0 : \Gamma A \to C^\infty(M)$ is a divergence, and by the previous proposition we have a right representation on $\mathbb{R}_M$. Conversely, given a right representation $(\mathbb{R}_M, \Upsilon)$ we have that ∂_Υ is a generating operator. □

In consequence, just putting together all the partial results that we already proved we obtain the following:

Theorem 5.8 (see [56])**.** *Given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$, there is a one to one correspondence between:*

a). Divergence operators $D : \Gamma A \to C^\infty(M)$.
b). Right representations $(\mathbb{R}_M, \Upsilon)$.
c). Left representations $(\bigwedge^{top} A, \nabla)$.
d). Generating operators $\partial : \bigwedge^\bullet \Gamma A \to \bigwedge^{\bullet-1} \Gamma A$.

Remark 5.9. It seems to us that Theorem 5.8 was discovered many times independently, sometimes just covering some of the items. The result in this form can be found in [56] but also important contributions appear in [7, 67, 129].

In conclusion, we know that given a Lie algebroid all the objects in Theorem 5.8 give a homology for the Lie algebroid. But it is well known that, for an arbitrary Lie algebroid, there is no canonical choice of any of them. Therefore, in contrast with cohomology theory, we have that Lie algebroids has no canonical homology.

5.1.3 Graded manifold approach

As we saw in Section 3.5.1, we can use degree 1 manifolds to codify Lie algebroids. What we do next is to give a graded perspective to the objects defined before as divergence and generating operators. Through these other "glasses" we will explain why the Lie algebroid associated to a Poisson manifold has a canonical homology in contrast with the general case.

In our context we define a *Batalin-Vilkovisky manifold or BV-manifold* as a triple $(E[1], \{\cdot,\cdot\}, \Delta)$, where $(E[1], \{\cdot,\cdot\})$ is a degree 1 Poisson manifold with Poisson bracket of degree -1 and $\Delta : C^i_{E[1]} \to C^{i-1}_{E[1]}$ is a second order differential operator satisfying:

- $\Delta^2 = 0$.
- $\{f, g\} = (-1)^{|f|}\big(\Delta(fg) - \Delta(f)g - (-1)^{|f|} f\Delta(g)\big)$

The operator Δ is known as the *BV-Laplacian or odd Laplacian.*

Remark 5.10. BV-manifolds were introduced in [5] in the context of quantization of gauge theories and they normally are infinite dimensional $\mathbb{Z}$-graded manifolds.

Just by comparing the preceding definition with the one of a Gerstenhaber algebra and a generating operator we obtain the following result:

Theorem 5.11. *Let $A \to M$ be a vector bundle. There is a one to one correspondence between:*

a). Lie algebroid structure with a divergence operator (or any of the equivalences given in Theorem 5.8) on the the vector bundle $A \to M$.

b). Gerstenhaber algebras on ΓA with a generating operator.

c). BV-structures on the graded manifold $A^*[1]$.

Now we can introduce the concept of divergence for a graded manifold just replacing usual manifolds by graded manifolds. Consider $\mathcal{M}$ a graded manifold. A *superdivergence for* $\mathcal{M}$ is a map $\mathcal{D} : \mathfrak{X}^{1,k}(\mathcal{M}) \to C^k_{\mathcal{M}}$ satisfying:

- $\mathcal{D}(fX) = f\mathcal{D}(X) + (-1)^{|f||X|}X(f)$.
- $\mathcal{D}([X, Y]) = X(\mathcal{D}(Y)) - (-1)^{|X||Y|}Y(\mathcal{D}(X))$.

We start by characterizing superdivergence on the manifold $E[1]$ in classical terms.

Proposition 5.12. *A superdivergence $\mathcal{D}$ on $E[1]$ is equivalent to a divergence $\widetilde{D}$ on the Atiyah algebroid $\mathbb{A}_E$ such that $\widetilde{D}_{|\,\mathrm{End}(E)} = -tr$.*

Proof. Given a superdivergence $\mathcal{D} : \mathfrak{X}^{1,\bullet}(E[1]) \to C^\bullet_{E[1]}$ we know that $\mathcal{D}(fX) = f\mathcal{D}(X) + (-1)^{|f||X|}X(f)$ so by Proposition 2.6 it is enough to know what happens for degree 0 and -1 vector fields. But since $\mathcal{D}$ preserves degree and all the functions of degree -1 are zero we have that the superdivergence is completely determined by the degree zero. Now by Example 3.23 we know that $\mathfrak{X}^{1,0}(E[1]) = \Gamma\mathbb{A}_E$ therefore $\mathcal{D}$ is equivalent to a map

$$\widetilde{D} = \mathcal{D}^0 : \Gamma\mathbb{A}_E \to C^\infty(M).$$

The conditions that it defines a superdivergence are just telling us that $\widetilde{D}$ is a divergence for the Atiyah algebroid. Finally, if we apply the first condition to a degree 1 function and a degree -1 vector field we obtain that $\widetilde{D}_{|\,\mathrm{End}(E)} = -tr$, as we want. $\square$

Now we relate superdivergence on the graded Poisson manifold $(E[1], \{\cdot,\cdot\})$ with BV-Laplacian operators.

Proposition 5.13 (see [79]). *Let $(E[1], \{\cdot,\cdot\})$ be a graded Poisson manifold with degree -1 Poisson bracket and $\mathcal{D}$ a superdivergence on $E[1]$. Then the operator*

$$\Delta(f) = (-1)^{|f|}\frac{1}{2}\mathcal{D}(X_f) = (-1)^{|f|}\frac{1}{2}\mathcal{D}(\{f,\cdot\})$$

is a BV-Laplacian

Instead of proving this result as a computation using the properties of the Poisson bracket and superdivergence as it was done in [79], we will use the characterization of superdivergences of Proposition 5.12 and prove the following:

Proposition 5.14. *Let $(A \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid and $\widetilde{D}$ a divergence operator of the Atiyah algebroid $\mathbb{A}_A$ satisfying that $\widetilde{D}_{|\,\mathrm{End}(A)} = -tr$. If we define $\mathrm{ad} : \Gamma A \to \Gamma \mathbb{A}_A$ as $a \to \mathrm{ad}_a = [a,\cdot]$ then $D = (\frac{1}{2}\widetilde{D}) \circ \mathrm{ad}$,*

$$\begin{tikzcd} \Gamma A \arrow[r, "\mathrm{ad}"] \arrow[dr, "D"'] & \Gamma \mathbb{A}_A \arrow[d, "\frac{1}{2}\widetilde{D}"] \\ & C^\infty(M) \end{tikzcd}$$

is a divergence operator for the Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$.

Proof. Let us check that D as defined above is a divergence operator:

$$\begin{aligned} D(fa) &= \tfrac{1}{2}\widetilde{D}(\mathrm{ad}_{fa}) = \tfrac{1}{2}\widetilde{D}(f\,\mathrm{ad}_a - a \otimes \rho^* df) \\ &= \tfrac{1}{2} f \widetilde{D}(\mathrm{ad}_a) + \tfrac{1}{2}\rho(a)(f) + \tfrac{1}{2} tr(a \otimes \rho^* df) = fD(a) + \rho(a)(f) \\ D([a,b]) &= \tfrac{1}{2}\widetilde{D}(\mathrm{ad}_{[a,b]}) = \tfrac{1}{2}\widetilde{D}([\mathrm{ad}_a, \mathrm{ad}_b]) = \tfrac{1}{2}\big(\rho(a)\widetilde{D}(\mathrm{ad}_b) - \rho(b)\widetilde{D}(\mathrm{ad}_a)\big) \\ &= \rho(a)D(b) - \rho(b)D(a) \quad \square \end{aligned}$$

The previous proposition leads to a natural question: Given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$, can we obtain all the divergence operators of $(A \to M, [\cdot,\cdot], \rho)$ as the composition of the adjoint with a divergence operator of the Atiyah algebroid?

The answer to this question is no and the counterexamples are given by Lie algebras:

Example 5.15 (Divergence operator for a Lie algebra). Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra. Since in a Lie algebra the anchor map is zero a divergence operator is equivalent to a linear map $D : \mathfrak{g} \to \mathbb{R}$ satisfying $D([a,b]) = 0 \quad \forall a, b \in \mathfrak{g}$. Therefore divergence operators for $(\mathfrak{g}, [\cdot,\cdot])$ are the same as elements of $(\frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]})^*$.

On the other hand, the Atiyah algebroid of a vector bundle is just $\mathrm{End}(\mathfrak{g})$ hence the only divergence operator obtained in this way is $-\frac{1}{2}tr \circ \mathrm{ad} : \mathfrak{g} \to \mathbb{R}$.

Recall that our goal is to study BV-Laplacians of the graded Poisson 1-manifold $(E[1], \{\cdot,\cdot\})$ and we saw that given a superdivergence we produce a BV-Laplacian. What we show now is how to obtain superdivergence using Berezinians.

Berezinians are the odd analogs of volume forms, and in classical geometry it is well known that given a volume form we construct an associated divergence. In the case of 1-manifolds the procedure is similar.

Consider the 1-manifold $E[1]$ with the projection to the body $p : E[1] \to M$. Over M we can consider the vector bundle $\mathrm{Hom}(\bigwedge^{top} E^*, \bigwedge^{top} T^*M)$. Doing the pull-back along p we obtain a graded vector bundle over $E[1]$. This graded bundle is known as *the Berezinian line bundle* and is denoted by $Ber(E[1]) = p^*\,\mathrm{Hom}(\bigwedge^{top} E^*, \bigwedge^{top} T^*M)$. The sections of the Berezinian line bundle are called Berezinians. Then we have

$$\Gamma(E[1], Ber(E[1])) = \Gamma\Big(M, \mathrm{Hom}(\bigwedge^{top} E^*, \bigwedge^{top} T^*M)\Big) \otimes_{C^\infty(M)} \Gamma \bigwedge E^*$$

The importance of this line bundle is that we can define an integration map

$$\begin{array}{rccc} \int_{\mathcal{M}} : & \Gamma Ber(E[1]) & \to & \mathbb{R} \\ & \mu & & \int_{\mathcal{M}} \mu \end{array}$$

defined in the following way: If we think of Berezinians as pairs $\nu \otimes f$, where $\nu = \xi \otimes \omega \in \Gamma\big(\bigwedge^{top} E \otimes \bigwedge^{top} T^*M\big)$ and $f \in \Gamma \bigwedge E^* = C_{E[1]}$, then $\int_{\mathcal{M}} \nu \otimes f = \int_M \langle \nu, f\rangle$ where the operation $\langle \nu, f\rangle = \langle \xi, f^{top}\rangle \omega$.

The other ingredient that we need is just to recall that vector fields act on Berezinians by "Lie derivatives":

$$\begin{array}{rccc} \mathcal{L} : & \mathfrak{X}^{1,\bullet}(E[1]) \times \Gamma Ber(E[1]) & \to & \Gamma Ber(E[1]) \\ & (X, \mu) & \rightsquigarrow & \mathcal{L}_X \mu \end{array} \tag{5.1}$$

Since the Berezinian is a line bundle, as it happens in classical geometry, we have that given a Berezinian on $E[1]$ then $\forall X \in \mathfrak{X}^{1,k}(E[1]), \quad \exists f \in C^k_{E[1]}$ such that

$$\mathcal{L}_X \mu = (-1)^{|X|} f \mu. \tag{5.2}$$

therefore associated to any Berezinian we have a map $\mathcal{D}^\mu : \mathfrak{X}^{1,k}(E[1]) \to C^k_{E[1]}$.

Proposition 5.16 (see [79]). *For any Berezinian μ the map $\mathcal{D}^\mu : \mathfrak{X}^{1,\bullet}(E[1]) \to C^\bullet_{E[1]}$ is a superdivergence.*

Proof. This assertion just follows from the properties of the Lie derivative acting on the Berezinian line bundle. □

Up to now we obtain that given a Berezinian on the manifold $A[1]$ we construct a superdivergence and if in addition $A[1]$ has a Poisson bracket of degree -1 then the superdivergence induce a BV-laplacian and therefore the associated Lie algebroid has a generating operator and hence a homology.

The importance of Berezinians is manifest in the following fact:

Proposition 5.17. *The manifold $T[1]M$ has a canonical Berezinian.*

Proof. Let us think Berezinians as pairs $\nu \otimes f$, where $f \in C_{T[1]M} = \Omega(M)$. Then

$$\int_{T[1]M} \nu \otimes f = \int_M f^{top}.$$

$\square$

Therefore, as a direct consequence of all these results we see that Lie algebroids defined over the vector bundle $T^*M \to M$ have a canonical homology.

Theorem 5.18 (see [79])**.** *Let (M, π) be a Poisson manifold. Then the Lie algebroid $(T^*M, [\cdot,\cdot]_\pi, \pi^\sharp)$ has a canonical homology operator ∂_π defined by the formula*

$$\partial_\pi = [d, i_\pi]$$

The operator ∂_π is known as the *Koszul-Brylinski* operator. It was defined by Koszul in the symplectic case and extended by Brylinsky to any Poisson manifold in [19]. One of the remarkable facts on Poisson homology is that it does not satisfy the Hodge decomposition, see e.g. [93]. For this reason other homologies and cohomologies have been proposed [118].

5.1.4 The Lie bialgebroid case

We study briefly the Lie bialgebroid case, i.e. degree 1 PQ-manifold (for the equivalence see Theorem 3.44). As before, the most important case is also given by the Lie algebroid of a Poisson manifold.

Proposition 5.19. *Let $(A^*[1], \{\cdot,\cdot\}, Q)$ be a PQ-manifold endowed with a superdivergence operator $\mathcal{D}$. Denote by Δ the BV-Laplacian as defined in Proposition 5.13. Then:*

a). $Q(\mathcal{D}(Q)) = 0,$

b). $[Q, \Delta] = \frac{1}{2}\{\mathcal{D}(Q), \cdot\},$

where $[\cdot,\cdot]$ denotes the bracket of differential operators of the algebra $C_{A^*[1]}$.

Proof. For the first identity we have that

$$0 = \mathcal{D}([Q, Q]) = Q\mathcal{D}(Q) - (-1)^1 Q\mathcal{D}(Q) = 2Q\mathcal{D}(Q),$$

while the second follows from the fact that $\forall f \in C_{A^*[1]}$ we have:

$$\begin{aligned}
[Q, \Delta](f) = &\; Q\Delta(f) + \Delta Q(f) = (-1)^{|f|}\tfrac{1}{2}Q(\mathcal{D}(X_f)) + (-1)^{|f|+1}\tfrac{1}{2}\mathcal{D}(X_{Q(f)}) \\
= &\; (-1)^{|f|}\tfrac{1}{2}Q(\mathcal{D}(X_f)) + (-1)^{|f|+1}\tfrac{1}{2}\mathcal{D}([Q, X_f]) \\
= &\; -(-1)^{|f|+1}(-1)^{|f|-1}\tfrac{1}{2}X_f\big(\mathcal{D}(Q)\big) = \tfrac{1}{2}\{\mathcal{D}(Q), f\}. \quad \square
\end{aligned}$$

The first equation in Proposition 5.19 is telling us that $\mathcal{D}(Q)$ defines a cohomology class on the first cohomology group of the Lie algebroid associated to $(A^*[1], Q)$. These class is known as the *modular class*, and was introduced in [55]. In fact it is suggested in this article that the modular class is the divergence of the Q-structure. Since we have that, given any two superdivergences $\mathcal{D}$ and $\mathcal{D}'$, there exists $f \in C^\infty(M)$ such that

$$\mathcal{D}(X) - \mathcal{D}'(X) = X(f) \qquad \forall X \in \mathfrak{X}^{1,\bullet}(A^*[1])$$

we obtain that modular class do not depend on the superdivergence that we chose to define it. Hence it is an invariant of the Lie algebroid $(A^*[1], Q)$.

As we already said, one important case of Lie bialgebroid is given by the Lie algebroid of a Poisson manifold (M, π). We know that $(TM \to M, [\cdot,\cdot], d_\pi)$ and $(T^*M \to M, [\cdot,\cdot]_\pi, d)$ are Lie bialgebroids. These Lie bialgebroids correspond to the PQ-manifolds $(T^*[1]M, \{\cdot,\cdot\}_s, Q_\pi)$ and $(T[1]M, \{\cdot,\cdot\}_\pi, Q_{dr})$ respectively.

On the one hand, since the 1-manifold $T[1]M$ has a canonical Berezinian we have a superdivergence and a BV-Laplacian.

Corollary 5.20 (See [19]). *Let (M, π) be a Poisson manifold. Then the Koszul-Brylinsky operator ∂_π satisfies*

$$[\partial_\pi, d] = 0.$$

Proof. It follows from the fact that the divergence of Q_{dr} is zero. □

On the other hand, given a volume form on M, $\Lambda \in \Omega^{top}(M)$, we can define a divergence for the tangent Lie algebroid (and also a Berezinian for $T^*[1]M$). The homology operator for De Rham in this case is given by $\partial^\Lambda = - *^{-1} d*$, where

$$\begin{array}{cccl} *: & \mathfrak{X}^k(M) & \to & \Omega^{top-k}(M) \\ & \Pi & & *(\Pi) = \langle \Pi, \Lambda \rangle \end{array}$$

Corollary 5.21 (See [79]). *Let (M, π) be a Poisson manifold and $\Lambda \in \Omega^{top}(M)$ a volume form. Then the Poisson differential satisfies*

$$[\partial^\Lambda, d_\pi] = \frac{1}{2}[\partial^\Lambda(\pi), \cdot] = \frac{1}{2}\mathcal{L}_{\partial^\Lambda(\pi)}.$$

The vector field $\partial^\Lambda(\pi)$ is known as the *modular vector field associated to the volum* Λ and the class of that vector field in the Poisson cohomology is the modular class defined above.

5.1.5 The derived category

Given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ and $(E, \nabla), (F, \nabla')$ two left representations, we say that they are *quasi-isomorphic* if there exists $\Phi : \bigwedge^\bullet A^* \otimes E \to \bigwedge^\bullet A^* \otimes F$ chain map such that $H(\Phi) : H^*(A; E) \to H^*(A; F)$ is an isomorphism. So, the question that we want to answer now is: Given a Lie algebroid, how many non isomorphic homologies can it support? Or in other words, how many non quasi-isomorphic left representations does the vector bundle $\bigwedge^{top} A$ admit?

We start by answering an easier question: How many left representations does the vector bundle $\bigwedge^{top} A$ admit?

Suppose that we have one $(\bigwedge^{top} A, \nabla)$; in this case any other linear connection differs from ∇ by a section of $\alpha \in \Gamma A^*$ given by the formula:

$$\nabla'_a \omega = \nabla_a \omega + \langle \alpha, a \rangle\, \omega \qquad \forall a \in \Gamma A,\ \omega \in \Gamma \bigwedge^{top} A$$

Now observe that ∇' is flat if and only if

$$\begin{aligned} \nabla'_{[a,b]} = [\nabla'_a, \nabla'_b] \ &\Leftrightarrow\ \langle \alpha, [a,b] \rangle = \rho(b)(\langle \alpha, a \rangle)\langle \alpha, b \rangle - \rho(a)(\langle \alpha, b \rangle)\langle \alpha, a \rangle \\ &\Leftrightarrow\ d_A \alpha = 0 \end{aligned}$$

Then the number of representations on $\bigwedge^{top} A$ is given by $\Omega^1_{cl}(A)$. As a direct consequence we obtain also a formula for the differentials:

$$d^{\nabla'}\eta = d^{\nabla}\eta + \alpha \wedge \eta \quad \forall \eta \in \Gamma(\bigwedge^{\bullet} A^* \otimes \bigwedge^{top} A)$$

Suppose now that α is exact, i.e. $\alpha = df$ for some $f \in C^\infty(M)$. Then we could consider

$$\begin{array}{rccc} \Phi: & (\Gamma(\bigwedge^{\bullet} A^* \otimes \bigwedge^{top} A), d^{\nabla'}) & \to & (\Gamma(\bigwedge^{\bullet} A^* \otimes \bigwedge^{top} A), d^{\nabla}) \\ & \eta & & \Phi(\eta) = e^f \eta \end{array} \tag{5.3}$$

that is a chain map ($d^{\nabla} \circ \Phi = \Phi \circ d^{\nabla'}$). Since clearly it is an isomorphism then Φ is a quasi-isomorphism between $(\bigwedge^{top} A, \nabla')$ and $(\bigwedge^{top} A, \nabla)$.

Therefore we have the following result:

Theorem 5.22 (see [129]). *Let $(A \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid. The number of homologies for the Lie algebroid is controlled by $H^1(A)$.*

5.2 Semi-direct products with 2-term representations up to homotopy

In order to understand the adjoint and coadjoint representations of a Lie algebroid, Abad–Crainic and Gracia-Saz–Mehta introduced the notion of 2-term representation up to homotopy, see [1, 59]. In the literature, we find two different viewpoints to the semi-direct product of a Lie algebroid with a 2-term representation up to homotopy: The first one leads to VB-algebroids, see [59]; the second produces an L_2-algebroid or, more generally, an L_k-algebroid, see [116].

The goal of this section is to explain the relation between these two viewpoints. We will do that by expressing both in terms of graded Q-manifolds, concluding that they coincide up to "splittings". The relation is schematically illustrated in the following diagram:

$$\begin{array}{ccccc} \left\{\begin{array}{c}\text{VB-algebroid}\\ \text{over } A \to M\end{array}\right\} & \xrightarrow[F_1]{[1]} & \left\{\begin{array}{c}\text{Deg.1 Q-bundle}\\ \text{over } A[1]\end{array}\right\} & \xrightarrow[F_2]{[k]} & \left\{\begin{array}{c}\text{Deg. k Q-bundle}\\ \text{over } A[1]\end{array}\right\} \\ S_1 \downarrow \text{Decomposition} & & & & S_2 \downarrow \text{Splitting} \\ \left\{\begin{array}{c}\text{2-term rep. up}\\ \text{to homotopy}\end{array}\right\} & \xrightarrow[G_1]{\ltimes[k]} & \left\{\begin{array}{c}L_k\text{-algebroid}\\ \text{extension of } A\end{array}\right\} & \xrightarrow[G_2]{[1]} & \left\{\begin{array}{c}\text{Split deg. } k\\ \text{Q-bundle over } A[1]\end{array}\right\} \end{array} \tag{5.4}$$

Morally, the diagram says that the functor "[1]" (arrows "F_1" and "G_2"), that goes from the category of L_k-algebroids to the category of Q-manifolds, commutes with the functor "[k]" (arrows "F_2" and "G_1"), which shifts by k 2-term representations up to homotopy or the fibres of a Q-bundle.

The arrows "S_1" and "F_1" are explained in [59], while the arrow "G_1" for $k = 2$ is constructed in [116], and the next arrow "G_2" is from [13]. We will briefly review these constructions in this section and explain the remaining arrows. This also must be compared with [98] where Mehta construct an arrow going from Q-bundles over $A[1]$ to n-term representations up to homotopy.

5.2.1 2-term representations up to homotopy

Following [1], given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ a *2-term representation up to homotopy* is defined by:

a). Two vector bundles E_0, E_1 over M and a vector bundle map $\partial : E_0 \to E_1$ over the identity.

b). An $A-$connection on each vector bundle, ∇^0 and ∇^1, satisfying $\partial\circ\nabla^0 = \nabla^1\circ\partial$.

c). A two form $K \in \Gamma\left(\bigwedge^2 A^* \otimes \mathrm{Hom}(E_1, E_0)\right)$ such that:

$$F^{\nabla^0} = K \circ \partial, \quad F^{\nabla^1} = \partial \circ K \quad \text{and} \quad d^{\nabla}(K) = 0,$$

where $F^{\nabla^0}, F^{\nabla^1}$ denote the curvature of the respective connection.

With this definition and the choice of an auxiliary connection on the vector bundle $A \to M$, Abad and Crainic were able to make sense of the adjoint and coadjoint representations up to homotopy of a Lie algebroid as follows.

Example 5.23 (The adjoint representation up to homotopy). Let $(A \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid and $\nabla : \mathfrak{X}(M) \times \Gamma A \to \Gamma A$ any connection on $A \to M$. Define the adjoint representation up to homotopy by:

a). The vector bundles $E_0 = A,\ E_1 = TM$, and $\partial = \rho$.

b). The connections:

$$\begin{array}{ll} \nabla^0_a b = [a, b] + \nabla_{\rho(b)} a, & a, b \in \Gamma A, \\ \nabla^1_a X = [\rho(a), X] + \rho(\nabla_X a,) & a \in \Gamma A,\ X \in \mathfrak{X}(M). \end{array}$$

c). The two form:

$$K(a, b)(X) = \nabla_X[a, b] - [\nabla_X a, b] - [a, \nabla_X b] + \nabla_{\nabla^1_a X} b - \nabla_{\nabla^1_b X} a$$

$a, b, \in \Gamma A,\ X \in \mathfrak{X}(M)$.

We denote by D^{ad} the degree 1 differential operator associated to the adjoint representation up to homotopy.

Example 5.24 (The coadjoint representation up to homotopy). We define it as the dual representation of the adjoint:

a). The vector bundles $E_0 = T^*M$, $E_1 = A^*$, and $\partial = \rho^*$.

b). The connections:

$$\begin{aligned}
\langle \nabla^1_a \beta, b\rangle &= \rho(a)\langle \beta, b\rangle - \langle \beta, \nabla^0_a b\rangle, && a, b \in \Gamma A,\ \beta \in \Gamma A^*;\\
\langle \nabla^0_a \tau, X\rangle &= \rho(a)\langle \tau, X\rangle - \langle \tau, \nabla^1_a X\rangle, && a \in \Gamma A,\ X \in \mathfrak{X}(M),\ \tau \in \Omega^1(M);
\end{aligned}$$

where the connections on the right are the ones coming from the adjoint representation.

c). The two form $K^* = K$, Since $\mathrm{Hom}(TM, A) = \mathrm{Hom}(A^*, T^*M)$.

We denote by D^{ad^*} the differential operator associated with the coadjoint representation.

Observe that these definitions of the adjoint and coadjoint representations up to homotopy depend on the choice of an auxiliary connection on $A \to M$. Therefore, they are not unique but it was proved in [1] that they are all quasi-isomorphic.

5.2.2 VB-algebroids and Q-bundles

Once we know what a 2-term representation up to homotopy is, we introduce the first approach to semi-direct products.

Recall that a *VB-algebroid* is a double vector bundle, $D \to E$ over $A \to M$, equipped with a Lie algebroid structure on $D \to E$ compatible with the vector bundle structure of $D \to A$, see Appendix B for precise definition. In this case, $A \to M$ inherits a natural Lie algebroid structure.

Given a double vector bundle recall that we have the core bundle, $C \to M$, and the fat bundle, $\widehat{A} \to M$ that fits in the exact sequence

$$\mathrm{Hom}(E, C) \to \widehat{A} \to A \tag{5.5}$$

Any 2-term representation up to homotopy of $(A \to M, [\cdot,\cdot], \rho)$ defines a VB-algebroid on $A \oplus C \oplus E \to E$ over $A \to M$. A Theorem of [59] says that, given a VB-algebroid any decomposition of the double vector bundle (or what is the same, any splitting of (5.5)) defines a 2-term representation up to homotopy of $A \to M$ on $E_0 = C$, $E_1 = E$, see also Apendix B. This explains the arrow "S_1" in (5.4).

Example 5.25. Given any Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$, the tangent and cotangent prolongations, $(TA; A, TM; M)$ and $(T^*A; A, A^*; M)$ as defined in Examples B.3 and B.6, are examples of VB-algebroids. In both cases the fat bundle is the jet prolongation bundle of A, $\widehat{A} = J^1A$, and a splitting of (5.5) is the same as a connection on $A \to M$. The representations up to homotopy obtained from the tangent and cotangent prolongations are the adjoint and coadjoint representations as defined in Examples 5.23 and 5.24, see also [59].

The next step is to describe VB-algebroids in terms of graded manifolds. For that, we need the following definition.

Let $p : \mathcal{E} \to \mathcal{M}$ be a vector bundle in the category of graded manifolds, see Section A.1. We say that it is a *Q-bundle* if $(\mathcal{E}, Q)$ is a Q-manifold and the vector field Q is p-projectable, i.e. there exists $\widehat{Q} \in \mathfrak{X}^{1,1}(\mathcal{M})$ such that the following diagram commutes:

$$\begin{array}{ccc} T[1]\mathcal{E} & \xrightarrow{dp} & T[1]\mathcal{M} \\ Q\uparrow & & \uparrow\widehat{Q} \\ \mathcal{E} & \xrightarrow{p} & \mathcal{M} \end{array}$$

As proved in [59] a double vector bundle D is a VB-algebroid if and only if $p : D[1]_E \to A[1]$ is a Q-bundle and Q is linear on the fibres of $p : D[1]_E \to A[1]$. Here the subscript on the shifting indicates the base of the vector bundle for which we are shifting the fibre coordinates. This gives to "F_1" in (5.4).

Remark 5.26. The idea that Q-bundles over $A[1]$ with linear vector fields were representations of the Lie algebroid $(A[1], \widehat{Q})$ appears in [120] and was developed in [98].

Let us define the arrow "F_2" of (5.4). Suppose that $p : \mathcal{E} \to \mathcal{M}$ is a vector bundle in the category of graded manifolds. We can define a new graded manifold $\mathcal{E}[k]_{\mathcal{M}}$, that also defines a vector bundle over $\mathcal{M}$, where the fibre coordinates are shifted by k.

If $p : \mathcal{E} \to \mathcal{M}$ is a Q-bundle with Q linear on the fibres, then $p : \mathcal{E}[k]_{\mathcal{M}} \to \mathcal{M}$ is also a Q-bundle. This follows from the fact that since $Q \in \mathfrak{X}^{1,1}(\mathcal{E})$ is p-projectable and linear on the fibres, locally it can be written as

$$Q = f^i(x)\frac{\partial}{\partial x^i} + g^l_j(x)e^j\frac{\partial}{\partial e_l} \quad \text{with } |f^i| - |x^i| = 1 \quad |g^l_j| + |e^j| - |e^l| = 1,$$

where $f^i(x), g^l_j(x) \in C_{\mathcal{M}}$, $\{x^i\}$ are local coordinates of $\mathcal{M}$ and $\{e^j\}$ fibre coordinates of $\mathcal{E}$. So once we shift the degree of the coordinates e^j by k, the same expression defines a vector field on $\mathcal{E}[k]_{\mathcal{M}}$ that is p-projectable, linear on the fibres, of degree 1 and squares to zero.

Corollary 5.27. *Let $D \to E$ be a VB-algebroid over $A \to M$. For any $k \in \mathbb{N}^*$, the graded manifolds $p : D[1]_E[k]_{A[1]} \to A[1]$ define Q-bundles.*

Examples 5.28. Given any Lie algebroid $(A \to M, \rho, [\cdot,\cdot])$ the tangent and cotangent prolongations have natural structures of VB-algebroids. So we can consider the manifolds $TA[1]_{TM}[k]_{A[1]} = T[k]A[1]$ and $T^*A[1]_{A^*}[k]_{A[1]} = T^*[k+1]A[1]$. By the previous result we obtain that

$$p : (T[k]A[1], \mathcal{L}_Q) \to (A[1], Q) \quad \text{and} \quad q : (T^*[k+1]A[1], \mathcal{L}_Q = X_\theta) \to (A[1], Q)$$

are Q-bundles. As we already mentioned in Example 5.25, these Q-manifolds are related to the adjoint and coadjoint representation up to homotopy of the Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$.

Before finishing this section, we give a definition of the graded manifolds $D[1]_E[k]_{A[1]}$ in terms of vector bundles. In the case of the cotangent prolongation it coincides with the one given for $T^*[k+1]A[1]$ in Proposition 6.13.

Proposition 5.29. *Given a double vector bundle $D \to E$ over $A \to M$, define for any $k \in \mathbb{N}^*$ the degree $k+1$ manifold $D[1]_E[k]_{A[1]} = (M, C_{D[1]_E[k]_{A[1]}})$. Then*

- *For $k = 1$, $C_{D[1]_{E[1]_{A[1]}}}$ is generated by:*

$$C^i_{D[1]_{E[1]_{A[1]}}} = \begin{cases} C^\infty(M), & i = 0, \\ \Gamma(A^* \oplus E^*), & i = 1, \\ \Gamma(\wedge^2 A^* \oplus \widehat{C}^* \oplus \wedge^2 E^*), & i = 2. \end{cases}$$

- *For $k > 1$, $C_{D[1]_{E[k]_{A[1]}}}$ is generated by:*

$$C^i_{D[1]_{E[k]_{A[1]}}} = \begin{cases} C^\infty(M), & i = 0, \\ \Gamma(\wedge^i A^*), & i < k, \\ \Gamma(E^* \oplus \wedge^k A^*), & i = k, \\ \Gamma(\widehat{C}^* \oplus \wedge^{k+1} A^*), & i = k+1. \end{cases}$$

where $\Gamma\widehat{C}^* = \Gamma_l(D^\uparrow, C^*)$ and $D^\uparrow \to C^*$ denotes the dual double vector bundle over $A \to M$.

5.2.3 L_∞-algebroids and Q-manifolds

The second approach to the semi-direct product of a Lie algebroid with a representation up to homotopy uses the concept of L_∞-algebroids as defined in Section 3.4.3.

Let $(A \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid and $(E_0 \to E_1, \partial, \nabla^0, \nabla^1, K)$ a representation up to homotopy. In [116], it is shown that $\mathbf{A} = (A_0 = A \oplus E_1) \oplus (A_{-1} = E_0)$ inherits a L_2-algebroid structure with brackets given by

$$\begin{cases} \rho = \rho \circ p_A, \quad l_1 = \partial, \\ l_2(a+e, b+e') = [a,b] + \nabla^1_a e' - \nabla^1_b e, \quad l_2(a+e, \xi) = \nabla^0_a \xi, \\ l_3(a+e, b+e', c+e'') = -K(a,b)e'' + K(a,c)e' - K(b,c)e, \end{cases}$$

where $a, b, c \in \Gamma A$, $e, e', e'' \in \Gamma E_1$, $\xi \in \Gamma E_0$ and $p_A : A \oplus E_1 \to A$. We denote this L_2-algebroid by $A \ltimes (E_0 \to E_1)[1]$. Moreover, $A \ltimes (E_0 \to E_1)[1]$ is a Lie extension of $A \to M$ in the sense of [116]. Formally we are shifting the representation up to homotopy by 1 and after that taking the semi-direct product, which explains our notation.

We now discuss L_k-algebroids concentrated in 3 degrees.

Example 5.30. Fix $k > 3$ and consider $\mathbf{A} = \oplus_i A_i$ a graded vector bundle where $A_i = 0$ if $i \notin \{0, -k+1, -k+2\}$. An L_k-algebroid structure on $\mathbf{A}$ is the same as:

- The anchor $\rho : A_0 \to TM$.
- The l_1 given by a bundle map $\partial : A_{-k+1} \to A_{-k+2}$.
- The l_2 given by:
  - A bracket $[\cdot,\cdot] : \Gamma A_0 \times \Gamma A_0 \to \Gamma A_0$.
  - A_0-connections $\Phi : \Gamma A_0 \times \Gamma A_{-k+2} \to \Gamma A_{-k+2}$ and $\Psi : \Gamma A_0 \times \Gamma A_{-k+1} \to \Gamma A_{-k+1}$.
- The l_3 given by $[\cdot,\cdot,\cdot]_3 : \Gamma A_0 \times \Gamma A_0 \times \Gamma A_{-k+2} \to \Gamma A_{-k+1}$ that is $C^\infty(M)$-linear.

- The l_k given by $[\cdot, \cdots, \cdot]_k : \Gamma A_0 \times \cdots \times \Gamma A_0 \to \Gamma A_{-k+2}$ that is $C^\infty(M)$-linear.
- The l_{k+1} given by $[\cdot, \cdots, \cdot]_{k+1} : \Gamma A_0 \times \cdots \times \Gamma A_0 \to \Gamma A_{-k+1}$ that is $C^\infty(M)$-linear.

satisfying the following Jacobi-like identities:

$$\begin{cases} 0 = [a_1, [a_2, a_3]_2]_2 + c.p. \\ \Phi_a(\partial(X)) = \partial(\Psi_a X). \\ [a_1, a_2, \partial(X)]_3 = F^\Psi(a_1, a_2)X. \\ \partial([a_1, a_2, \xi]) = F^\Phi(a_1, a_2)\xi. \\ 0 = ([[a_1, a_2]_2, a_3, \xi]_3 + c.p.) + ([a_1, a_2, \Phi_{a_3}\xi]_3 + c.p.) + (\Psi_{a_1}[a_2, a_3, \xi]_3 + c.p.). \\ 0 = \partial([a_1, \cdots, a_{k+1}]_{k+1} + c.p.) + ([[a_1, a_2]_2, a_3, \cdots, a_{k+1}]_k + c.p.) + \\ \qquad +(\Phi_{a_1}[a_2, \cdots, a_{k+1}]_k + c.p.). \\ 0 = (\Psi_{a_1}[a_2, \cdots, a_{k+2}]_{k+1} + c.p.) + ([[a_1, a_2]_2, a_3, \cdots, a_{k+2}]_{k+1} + c.p.) + \\ \qquad +([a_1, a_2, [a_3, \cdots, a_{k+2}]_k]_3 + c.p.). \end{cases} \tag{5.6}$$

where $a_1, \cdots, a_{k+2} \in \Gamma A_0,\ \xi \in \Gamma A_{-k+2},\ X \in \Gamma A_{-k+1}$.

The case $k = 3$ is more complicated but can be trated similarly.

Proposition 5.31. *Let $(A \to M, [\cdot, \cdot], \rho)$ be a Lie algebroid and consider $(E_0 \to E_1, \partial, \nabla^0, \nabla^1, K)$ a representation up to homotopy. Then, for any $k > 2$ the graded vector bundle*

$$\mathbf{A} = \begin{cases} A_0 = A, \\ A_{-k+1} = E_0, \\ A_{-k+2} = E_1. \end{cases}$$

inherits an L_k-algebroid structure with brackets given by

$$\begin{cases} \rho = \rho, \quad \partial = \partial, \\ [\cdot, \cdot]_2 = [\cdot, \cdot], \quad \Phi = \nabla^1, \quad \Psi = \nabla^0, \\ [a, b, e]_3 = K(a, b)e, \quad [\cdot, -, \cdot]_k = 0, \quad [\cdot, -, \cdot]_{k+1} = 0. \end{cases}$$

We denote this L_k-algebroid by $A \ltimes (E_0 \to E_1)[k-1]$. Moreover, $A \ltimes (E_0 \to E_1)[k-1]$ are Lie extensions of $A \to M$.

Proof. Observe that all the objects have the right skew-symmetry and $C^\infty(M)$ linearity properties. Let us check the Jacobi-like identities (5.6). The first one is satisfied because it is the Jacobi identity for the Lie bracket on $A \to M$. The following four are equivalent to $(E_0 \to E_1, \partial, \nabla^0, \nabla^1, K)$ being a representation up to homotopy of $A \to M$. Finally, since the k and the $k + 1$ brackets are zero the last two equations are trivially satisfied.

The last assertion is straightforward. $\square$

Up to now, given a Lie algebroid and a representation up to homotopy we constructed, for any $k \in \mathbb{N}^*$, an L_{k+1}-algebroid that we denoted by $A \ltimes (E_0 \to E_1)[k]$ and is an extension of the Lie algebroid $(A \to M, [\cdot, \cdot], \rho)$ in the sense of [116]. This corresponds to the arrow "G_1" in (5.4).

The arrow "G_2" in (5.4) corresponds to Theorem 3.33, and gives an equivalence of categories between L_k-algebroids and Q-manifolds. Therefore given a Lie algebroid

$(A \to M, [\cdot,\cdot], \rho)$ and a representation up to homotopy $(E_0 \to E_1, \partial, \nabla^0, \nabla^1, K)$, for any $k \in \mathbb{N}^*$, $\big(A \ltimes (E_0 \to E_1)[k]\big)[1]$ is a Q-manifold. In fact, since $A \ltimes (E_0 \to E_1)[k]$ is a Lie extension of $A \to M$, then $\big(A \ltimes (E_0 \to E_1)[k]\big)[1]$ is a Q-bundle over $(A[1], Q)$. The only thing that remains to be proven is that these graded Q-bundles are the split version of the graded Q-bundles obtained in Corollary 5.27.

Let $D \to E$ over $A \to M$ be a VB-algebroid with core $C \to M$ and consider $S : A \to \widehat{A}$ a splitting of the exact sequence (5.5). As we already mentioned, $A \to M$ is a Lie algebroid and $(C \to E, \partial, \nabla^0, \nabla^1, K)$ is a representation up to homotopy of $A \to M$, that depends on the splitting $S : A \to \widehat{A}$.

Theorem 5.32. *For any $k \in \mathbb{N}^*$, the splitting $S : A \to \widehat{A}$ induces a Q-manifold isomorphism*

$$(D[1]_E[k]_{A[1]}, Q) \cong \big((A \ltimes (C \to E)[k])[1], Q\big).$$

Proof. First recall that, by the theory of double vector bundles, given a splitting of the exact sequence (5.5) we obtain a decomposition of the double vector bundle $D \to E$ over $A \to M$. A decomposition of the double vector bundle produces a decomposition of the dual double vector bundle $D^{\uparrow} \to C^*$ over $A \to M$. Finally, this decomposition induce a splitting of the sequence

$$\mathrm{Hom}(A, E) \to \widehat{C^*} \to C^*. \tag{5.7}$$

Therefore, given $S : A \to \widehat{A}$ splitting of (5.5) we obtain $S' : C^* \to \widehat{C^*}$ splitting of (5.7).

Using Proposition 5.29 we see that the map $S' : C^* \to \widehat{C^*}$ induces the following isomorphism between the graded manifolds $D[1]_E[k]_{A[1]} = (M, C_{D[1]_E[k]_{A[1]}})$ and $(A \ltimes (C \to E)[k])[1] = (M, C_{(A\ltimes(C\to E)[k])[1]})$:

$$\begin{cases} \mathrm{Id} : M \to M; & \\ \Psi^i : C^i_{(A\ltimes(C\to E)[k])[1]} \to C^i_{D[1]_E[k]_{A[1]}}, & \Psi^i = \mathrm{Id} \ \text{ for } i < k; \\ \Psi^k : C^k_{(A\ltimes(C\to E)[k])[1]} \to C^k_{D[1]_E[k]_{A[1]}}, & \Psi^k(c + \alpha\cdot\xi + \tau) = S'(c) + m(\alpha, \xi) + \tau; \end{cases}$$

where $c \in \Gamma C^*$, $\alpha \in \Gamma A^*, \xi \in \Gamma E$ and $\tau \in \Gamma \bigwedge^k A^*$.

One can see that this rules defines a graded isomorphism and going along the proof of the main Theorem in [13] it is easy to conclude that this map also commutes the Q-structures. $\square$

Chapter 6

The geometry of graded cotangent bundles

This chapter is devoted to the study of the graded cotangent bundles. Graded cotangent bundles are the prototypical examples of graded symplectic manifolds. In addition, we also study their possible Q-structures.

In the first section we provide the definition of graded vector bundles in terms of sheaves, study their non-degenerate Poisson bracket and in some cases their possible Q-structures. Finally, we prove that all the odd symplectic manifolds are isomorphic to cotangent bundles.

The other sections are devoted to the study of the simplest case of graded cotangent bundles, i.e. the manifolds $T^*[k]A[1]$. We give the geometrization of this manifolds as well as all their structures as: Q-structures, lagrangian submanifolds, splittings... This allows us to give a precise relation between this manifolds and higher Courant algebroids structures on $A \oplus \bigwedge^{k-1} A^*$.

6.1 Graded cotangent bundle of a graded manifold

In this section we will consider $\mathcal{M} = (M, C_{\mathcal{M}})$ being an n-manifold of dimension $m_0|\cdots|m_n$. Here we study the geometry of its cotangent bundles. Since we are considering the cotangent bundle as a dual for the tangent bundle, then $T^*\mathcal{M}$ is not a positively graded manifold because it has coordinates concentrated in degrees $-n$ to n. Therefore, to avoid this problem, for an n-manifold $\mathcal{M}$ we will consider the manifolds $T^*[k]\mathcal{M}$ with $k \geq n$ as the objects of our study.

If $k > n$ we can define the shifted graded cotangent bundle as follows: given an n-manifold $\mathcal{M} = (M, C_{\mathcal{M}})$ we define the functions on $T^*[k]\mathcal{M} = (M, C_{T^*[k]\mathcal{M}})$ as

$$C^l_{T^*[k]\mathcal{M}} = \bigoplus_{i \cdot k + j = l} \mathfrak{X}^{i,j}(\mathcal{M}), \tag{6.1}$$

We can also give an expression in local coordinates: Take U a chart of $\mathcal{M}$ and consider coordinates $\{e^{j_i}\}$ of $\mathcal{M}_{|U}$ where $0 \leq i \leq n$ and $1 \leq j_i \leq m_i$. Then

$$\{e^{j_i}, \frac{\partial}{\partial e^{j_i}}\} \quad \text{with} \quad |e^{j_i}| = i, \ |\frac{\partial}{\partial e^{j_i}}| = k - i. \tag{6.2}$$

The problem with this definition in the case $k = n$ is that $\frac{\partial}{\partial e^{j_n}}$ are coordinates of degree 0 but we add them just polynomially. Therefore to solve this problem we will just make a completion on these coordinates in the following way. Let be $(\mathbf{E}^* \to M, \mu)$ a coalgebra bundle such that $\mathcal{F}(\mathbf{E}^* \to M, \mu) = \mathcal{M}$, where $\mathcal{F}$ denotes the geometrization functor of Theorem 3.5. Recall that we denote by $\widetilde{\mathbf{E}}^* = \ker \mu$ and call $p : \widetilde{E}^*_{-n} \to M$ the vector bundle projection. Then we can define the graded cotangent bundle as

$$T^*[n]\mathcal{M} = (\widetilde{E}^*_{-n}, p^* C_{T^*[n]\mathcal{M}}) \tag{6.3}$$

The only difference between the first approach and the second at the level of functions is just that in degree 0 we have a completion of some polynomial coordinates. Because of that, if we do not consider explicit degree zero functions, we can follow to think that functions on $T^*[n]\mathcal{M}$ are given by equation (6.1) and for sure the coordinates formulas of (6.2) are valid.

The other aspect of classical cotangent bundles is that they are symplectic manifolds. Now we will define the symplectic structure on the graded cotangent bundles by giving their non-degenerate Poisson bracket.

Let $\mathcal{M}$ be an n-manifold and $k \geq n$. The manifolds $T^*[k]\mathcal{M}$ are symplectic with a Poisson bracket of degree $-k$ on functions given by the graded version of the Schouten bracket of multi-vector fields for the manifold $\mathcal{M}$. By (6.1) the functions on $T^*[k]\mathcal{M}$ are graded symmetric powers of vector fields on $\mathcal{M}$. We define the graded version of the Schouten bracket by the following rules:

$$\{X, f\} = X(f), \quad \text{and} \quad \{X, Y\} = [X, Y], \tag{6.4}$$

for $f \in C^{\bullet}_{\mathcal{M}} = \mathfrak{X}^{0,\bullet}(\mathcal{M})$, $X, Y \in \mathfrak{X}^{1,\bullet}(\mathcal{M})$ and extended by graded skew symmetry and as a derivation with respect to the graded symmetric product. With this definition, the Schouten bracket satisfies the graded analogues of Leibniz and Jacobi identities. If we shift the vector field coordinates by k then it is clear that it defines a Poisson bracket of degree $-k$. It remains to see that it is non-degenerate. Notice that if $U \subset M$, has coordinates $\{e^{j_i}\}$ of $\mathcal{M}_{|U}$ then by equation (6.2) we have that $\{e^{j_i}, \frac{\partial}{\partial e^{j_i}}\}$ defines local coordinates for $T^*[k]\mathcal{M}$. It is clear that these coordinates are a Darboux base.

6.1.1 $T^*[n+1]\mathcal{M}$ and homotopy Poisson structures

Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be an n-manifold. The first case that we can study is the geometry of the cotangent bundles $T^*[n+1]\mathcal{M}$. We will see that symplectic Q-structures give rise to objects known as homotopy Poisson structures, see [97].

Observe that $(T^*[n+1]\mathcal{M}, \{\cdot,\cdot\})$ is a symplectic manifold with a Poisson bracket of degree $-n-1$. Therefore, functions of degree $n+2$ have hamiltonian vector fields of degree 1. Moreover, by Proposition A.10 we have that any degree 1 symplectic vector field of $T^*[n+1]\mathcal{M}$ must be hamiltonian for some function of degree $n+2$. In fact, we have the following result:

Proposition 6.1. *Given an n-manifold $\mathcal{M} = (M, C_{\mathcal{M}})$, the symplectic Q-structures of the manifold $(T^*[n+1]\mathcal{M}\{\cdot,\cdot\})$ are equivalent to functions $\theta \in C^{n+2}_{T^*[n+1]\mathcal{M}}$ satisfying*

$$\{\theta, \theta\} = 0.$$

Proof. Given $Q \in \mathfrak{X}^{1,1}(T^*[n+1]\mathcal{M})$, by Proposition A.10 we know that $Q = X_\theta = \{\theta, \cdot\}$ and now using Proposition A.8 we have that $[Q,Q] = 0 \Leftrightarrow \{\theta, \theta\} = 0$. □

This proposition motivates the following definition. A *homotopy Poisson manifold of degree n* is an n-manifold $\mathcal{M}$ together with $\theta \in C^{n+2}_{T^*[n+1]\mathcal{M}}$ satisfying

$$\{\theta, \theta\} = 0,$$

or in other words, together with a PQ-manifold structure on $T^*[n+1]\mathcal{M}$.

Remark 6.2. The definition in this context was taken from [97], but the idea of homotopy Poisson algebras has also appeared in [35, 123]. See also [17] for the relation of homotopy Poisson algebras and L_∞-algebras.

Let us explain the name of homotopy Poisson. For that we need to take a look at the degree $n+2$ functions of $T^*[n+1]\mathcal{M}$:

$$C^{n+2}_{T^*[n+1]\mathcal{M}} = \bigoplus_{i(n+1)+j=n+2} \mathfrak{X}^{i,j}(\mathcal{M}) = \bigoplus_{i=0}^{n+2} \mathfrak{X}^{i,\ n+2-i(n+1)}(\mathcal{M}).$$

Expanding the sum, we obtain

$$C^{n+2}_{T^*[n+1]\mathcal{M}} = C^{n+2}_{\mathcal{M}} \oplus \mathfrak{X}^{1,1}(\mathcal{M}) \oplus \mathfrak{X}^{2,-n}(\mathcal{M}) \oplus \mathfrak{X}^{3,-2n-1}(\mathcal{M}) \oplus \cdots \oplus \mathfrak{X}^{n+2,(n+2)(-n)}(\mathcal{M}).$$

Then let us see examples:

Example 6.3 (Poisson manifolds)*.* Suppose that $\theta = \pi_2$ where $\pi_2 \in \mathfrak{X}^{2,-n}(\mathcal{M})$. Then the condition

$$\{\theta, \theta\} = 0 \Leftrightarrow [\pi_2, \pi_2] = 0$$

so we obtain that $\mathcal{M}$ is a Poisson manifold with a Poisson bracket of degree $-n$.

Example 6.4 (PQ-manifolds)*.* Suppose that $\theta = Q + \pi_2$ where $Q \in \mathfrak{X}^{1,1}(\mathcal{M})$ and $\pi_2 \in \mathfrak{X}^{2,-n}(\mathcal{M})$. Then the condition

$$\{\theta, \theta\} = 0 \Leftrightarrow \begin{cases} [Q,Q] = 0, \\ [Q, \pi_2] = 0, \\ [\pi_2, \pi_2] = 0. \end{cases}$$

so we obtain that $\mathcal{M}$ is a degree n PQ-manifold.

Example 6.5 (First homotopy term)*.* Suppose that $\theta = Q + \pi_2 + \pi_3$ where $Q \in \mathfrak{X}^{1,1}(\mathcal{M})$, $\pi_2 \in \mathfrak{X}^{2,-n}(\mathcal{M})$ and $\pi_3 \in \mathfrak{X}^{3,-2n-1}(\mathcal{M})$. Then the condition

$$\{\theta, \theta\} = 0 \Leftrightarrow \begin{cases} [Q,Q] = 0, \quad [Q, \pi_2] = 0, \\ [\pi_2, \pi_2] + 2\,[Q, \pi_3] = 0, \\ [\pi_2, \pi_3] = 0, \quad [\pi_3, \pi_3] = 0. \end{cases}$$

Therefore we obtain that our bracket is no longer Poisson but the defect is controlled by the vector field and the three bracket.

We end the study of homotopy Poisson manifolds of degree n by describing the lower n cases:

Example 6.6 (n=0, Poisson manifolds). There is not much to say here: if $\mathcal{M} = M$ then $C^2_{T^*[1]M} = \mathfrak{X}^2(M)$ therefore $\theta = \pi$ and the equation is just that the bivector field is Poisson.

Example 6.7 (n=1, Lie quasi-bialgebroid and quasi-Lie bialgebroids). Suppose that $\mathcal{M}$ is a degree 1 manifold. Then $\mathcal{M} = A[1]$ and we need to study the degree 3 functions on $T^*[2]A[1]$. In this case we have that

$$C^3_{T^*[2]A[1]} = C^3_{A[1]} \oplus \mathfrak{X}^{1,1}(A[1]) \oplus \mathfrak{X}^{2,-2}(A[1]) \oplus \mathfrak{X}^{3,-3}(A[1]).$$

This case was considered in detail by Kosmann-Schwarzbach, see e.g. [78], and it gives rise to Lie-quasi bialgebroids and quasi-Lie bialgebroids on the vector bundle $A \to M$.

6.1.2 $T^*[2n+1]\mathcal{M}$ and odd degree symplectic manifolds

It also happens that for the cotangent bundles $T^*[2n+1]\mathcal{M}$, where $\mathcal{M}$ is an n-manifold, we can also give an explicit description of all their symplectic Q-structures just in terms of structures over $\mathcal{M}$.

Theorem 6.8. *Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be an n-manifold. The symplectic Q-structures of $T^*[2n+1]\mathcal{M}$ are equivalent to triples (H, Q, π) satisfying*

$$[Q,Q] + 2\,[\pi, H] = 0, \qquad Q(H) = 0, \qquad [\pi,\pi] = 0,$$

where $H \in C^{2n+2}_{\mathcal{M}}$, $Q \in \mathfrak{X}^{1,1}(\mathcal{M})$ and $\pi \in \mathfrak{X}^{2,-2n}(\mathcal{M})$, and $[\cdot,\cdot]$ denotes the Schouten bracket of multivector fields on $\mathcal{M}$.

Proof. By Proposition A.10 we have that any degree 1 symplectic vector field is hamiltonian. So there exists $\theta \in C^{2n+2}_{T^*[2n+1]\mathcal{M}}$ such that $Q = X_\theta = \{\theta, \cdot\}$.

By Proposition A.8 we have that $0 = [Q,Q] = [X_\theta, X_\theta] = X_{\{\theta,\theta\}}$. Therefore the symplectic Q structures of $T^*[2n+1]\mathcal{M}$ are equivalent to $\theta \in C^{2n+2}_{T^*[2n+1]\mathcal{M}}$ such that $\{\theta,\theta\} = 0$.

The functions on $T^*[2n+1]\mathcal{M}$ are given by equation (6.1). Hence we can see that

$$C^{2n+2}_{T^*[2n+1]\mathcal{M}} = C^{2n+2}_{\mathcal{M}} \oplus \mathfrak{X}^{1,1}(\mathcal{M}) \oplus \mathfrak{X}^{2,-2n}(\mathcal{M})$$

Therefore the function decomposes as a sum $\theta = H + Q + \pi$ and the equation $\{\theta,\theta\} = 0$ becomes the three equations in the statement involving the Schouten bracket of $\mathcal{M}$. $\square$

These kind of graded cotangent bundles are important because all the odd symplectic manifolds are isomorphic to one of this. In order to state the result, recall from (2.2) that given an n-manifold $\mathcal{M}$ we have a tower of affine fibrations associated to it and $\mathcal{M}_i = (M, C_{\mathcal{M}_i} = \langle f \in C_{\mathcal{M}} |\ |f| \leq i \rangle)$. Then we have the following characterization result that must be understood as a graded analogue of the classification theorem for odd symplectic supermanifolds given in [111].

Theorem 6.9. *Let $(\mathcal{M}, \omega)$ be a symplectic n-manifold with $n = 2k+1$. Then $\mathcal{M}$ and $T^*[2k+1]\mathcal{M}_k$ are symplectomorphic.*

Proof. Given $\mathcal{M}$ we can consider the associated tower of fibrations given by equation (2.2). In particular we can consider the affine bundle $p : \mathcal{M} \to \mathcal{M}_k$, where $\mathcal{M}_k = (M, C_{\mathcal{M}_k})$ and $C_{\mathcal{M}_k} = \langle f \in C_{\mathcal{M}} | \, |f| \leq k \rangle$.

Recall that $T^*[2k+1]\mathcal{M}_k = (M, C_{T^*[2k+1]\mathcal{M}_k})$ where

$$C^l_{T^*[2k+1]\mathcal{M}_k} = \bigoplus_{i(2k+1)+j=l} \mathfrak{X}^{i,j}(\mathcal{M}_k).$$

Moreover $C_{T^*[2k+1]\mathcal{M}_k}$ is generated by $C^i_{\mathcal{M}_k}$ with $i \in \{0, \cdots, k\}$ and by $\mathfrak{X}^{1,j}(\mathcal{M}_k)$ with $j \in \{-k, \cdots, 0\}$ but $X \in \mathfrak{X}^{1,j}(\mathcal{M}_k)$ has degree $j + 2k + 1$ as a function in $C_{T^*[2k+1]\mathcal{M}_k}$.

Since $p : \mathcal{M} \to \mathcal{M}_k$ is an affine bundle, we can choose a zero section and convert it into a vector bundle. In other words, by Lemma 3.6 we can find $\mathcal{I}_{\mathcal{M}_k} \subseteq C_{\mathcal{M}}$ regular homogeneous ideal such that

$$C_{\mathcal{M}} = \mathcal{I}_{\mathcal{M}_k} \oplus C_{\mathcal{M}_k}.$$

This can be done by choosing a splitting of the graded manifold $\mathcal{M}$, but in fact we only need a splitting of all the degrees greater than k.

By dimensional reasons, since the Poisson bracket has degree $-2k-1$, we know that all the coordinates of degree less or equal k must be paired with coordinates of degree bigger than k and vice-versa. Therefore

$$\{\mathcal{I}_{\mathcal{M}_k}, \mathcal{I}_{\mathcal{M}_k}\} \subseteq \mathcal{I}_{\mathcal{M}_k} \text{ and } \operatorname{Totdim} \mathcal{M}_k = \frac{1}{2} \operatorname{Totdim} \mathcal{M}$$

so we can see $\mathcal{M}_k$ as a lagrangian submanifold of $\mathcal{M}$.

Using the symplectic form and the splitting we define the following morphism of graded manifolds $\Psi = (\psi, \psi^\sharp) : T^*[2k+1]\mathcal{M}_k = (M, C_{T^*[2k+1]\mathcal{M}_k}) \to \mathcal{M} = (M, C_{\mathcal{M}})$ given by

$$\begin{cases} \psi = \mathrm{Id} : M \to M, \\ \begin{array}{rccl} \psi^\sharp : & C_{\mathcal{M}} = \mathcal{I}_{\mathcal{M}_k} \oplus C_{\mathcal{M}_k} & \to & C_{T^*[2k+1]\mathcal{M}_k} \\ & f \in C_{\mathcal{M}_k} & & \psi^\sharp(f) = f, \\ & g \in \mathcal{I}_{\mathcal{M}_k} & & \psi^\sharp(g) = X_{g|C_{\mathcal{M}_k}}. \end{array} \end{cases}$$

Observe that Ψ is well defined because the pairing has degree $-2k-1$ and $\mathcal{I}_{\mathcal{M}_k}$ is an ideal closed for the bracket.

Finally, choosing Darboux coordinates we can see that Ψ is locally a diffeomorphism and since there is no topology because the odd coordinates are just polynomial then it is globally a diffeomorphism. Moreover, the fact that Ψ is a sympectomorphism is straightforward. □

Remark 6.10. As it happens in the supermanifold case, see [111], the preceding theorem must be seen as an application of the Weinstein tubular neighbourhood for Lagrangian submanifolds, but now the result is global because the odd coordinates are just polynomial and therefore do not have topology.

In conclusion, we obtain that the odd symplectic manifolds are always isomorphic (non-canonically) to shifted cotangent bundles and we can classify their symplectic Q-structures in terms of structures coming from the base of the shifted cotangent bundle. This result is relevant in the AKSZ construction, see Section 6.4 and [2, 101, 109].

We finish this part with some examples of this classification:

Example 6.11 (1-symplectic Q-manifolds). By the preceding theorem we know that any 1-symplectic Q-manifold $(\mathcal{M}, \{\cdot,\cdot\}, \widehat{Q})$ is isomorphic (and this is the only case where the isomorphism is canonical) to $(T^*[1]M_0, \{\cdot,\cdot\}, \widehat{Q} = \{Q + H + \pi, \cdot\})$ where $Q \in \mathfrak{X}^{1,1}(M_0) = 0,\ H \in C^2(M_0) = 0$ and $\pi \in \mathfrak{X}^{2,0}(M_0)$, that is, a bivector field on M_0.

Therefore we obtain that $(\mathcal{M}, \{\cdot,\cdot\}, \widehat{Q}) = (T^*M_0, \{\cdot,\cdot\}, \pi)$ satisfying $[\pi, \pi] = 0$. Hence (M_0, π) is a Poisson manifold. In conclusion degree 1-symplectic Q-manifolds are the same as Poisson manifolds.

The example of 3-symplectic Q-manifolds is given in Theorem 6.29.

Example 6.12 (Case $\pi = 0$). Let us suppose that we have $(T^*[2n+1]\mathcal{M}, \{\cdot,\cdot\}, \widehat{Q} = \{Q + H, \cdot\})$ where $\mathcal{M}$ is an n-manifold. Then by the previous theorem we obtain that $Q \in \mathfrak{X}^{1,1}(\mathcal{M})$ with $[Q, Q] = 0$ and $H \in C^{2n+2}_{\mathcal{M}}$ with $Q(H) = 0$. In conclusion we have that given a Q-manifold and a cohomology class we can define an AKSZ theory with target the Q-manifold twisted by the cohomology class. Recall that in Section 3.4.3 we see that degree n Q-manifolds are equivalent to L_n-algebroids. Therefore any L_n-algebroid has an AKSZ theory and in some sense, for odd degree they are all (modulo the twist by π).

6.2 Graded cotangent bundles of 1-manifolds

Consider a vector bundle $A \to M$. As we already saw we can define a degree 1-manifold by $A[1] = (M, \Gamma \bigwedge^{\bullet} A^*)$. By the preceding Section we can define the graded manifolds $T^*[k]A[1],\ \forall k \in \mathbb{N}^*$.

The case $k = 1$ is the special, because new degree 0 coordinates are introduced. In this case we take a completion of the polynomial coordinates of degree 0 and obtain a degree 1 manifold isomorphic to $T^*[1]A^*$.

For $k \geq 2$, when no new degree 0 coordinates are added, $T^*[k]A[1] = (M, C_{T^*[k]A[1]})$ is a degree k manifold with

$$C^l_{T^*[k]A[1]} = \bigoplus_{i\cdot k + j = l} \mathfrak{X}^{i,j}(A[1]), \tag{6.5}$$

In local coordinates $T^*[k]A[1]$ has the following expression: Take U a trivialization for the bundle $A \to M$ and consider local coordinates x^i on U and α^j fibre coordinates for $A_{|U}$ (that we identify conveniently with sections of A^*), and denote by a^j the dual coordinates for $A^*_{|U}$. The set

$$\{x^i, \alpha^j, a_j = \frac{\partial}{\partial \alpha^j}, \frac{\partial}{\partial x^i}\} \text{ with } |x^i| = 0, |\alpha^j| = 1, |a_j| = k-1 \text{ and } |\frac{\partial}{\partial x^i}| = k$$

are local coordinates for $T^*[k]A[1]$.

6.2.1 Classical description of the manifolds $T^*[k]A[1]$

Although we have a nice definition of the manifolds $T^*[k]A[1]$ in terms of sheaves over M we also want to characterize them in terms of algebra bundles, because this point of view allows us to work with classical vector bundles.

Proposition 6.13. *Let $A \to M$ be a vector bundle. Then:*

- *For $k = 1$: $T^*[1]A[1] = T^*[1]A^* = (A^*, \mathfrak{X}^\bullet(A^*))$. So $C_{T^*[1]A[1]}$ is generated by $C^\infty(A^*)$ and ΓTA^*.*
- *For $k = 2$: the algebra bundle corresponding to $T^*[2]A[1]$ is $(\mathbf{E} = \oplus_{i=1}^2 E_i \to M, m)$, where*

$$E_i = \begin{cases} A \oplus A^* & i = 1, \qquad m : \Gamma E_1 \otimes \Gamma E_1 \to \Gamma E_2 \\ \mathbb{A}_{(A\oplus A^*, \langle,\rangle)} & i = 2. \qquad m(e_1, e_2) = e_1 \wedge e_2 \end{cases} \tag{6.6}$$

  where $\mathbb{A}_{(A\oplus A^*,\langle,\rangle)}$ denotes the Atiyah algebroid of derivations that preserve the natural pairing on $A \oplus A^*$ and $e_1 \wedge e_2$ is seen as an orthogonal endomorphism of $A \oplus A^*$. Therefore the generators of $C_{T^*[2]A[1]}$ are $C^0_{T^*[2]A[1]} = C^\infty(M)$ and $C^i_{T^*[2]A[1]} = \Gamma E_i$ for $i = 1, 2$.
- *For $k > 2$: the algebra bundle corresponding to $T^*[k]A[1]$ is $(\mathbf{E} = \oplus_{i=1}^k E_i \to M, m)$, where*

$$E_i = \begin{cases} \bigwedge^i A^* \quad 1 \le i < k-1, & m : \Gamma E_i \otimes \Gamma E_j \to \Gamma E_{i+j} \\ A \oplus \bigwedge^{k-1} A^* \; i = k-1, & m(\alpha_i, \alpha_j) = \alpha_i \wedge \alpha_j, \; i + j \le k, \; j \ne k-1 \\ \mathbb{A}_A \oplus \bigwedge^k A^* \quad i = k. & m(\alpha_1, a + \alpha_{k-1}) = \alpha_1 \otimes a + \alpha_1 \wedge \alpha_{k-1} \end{cases} \tag{6.7}$$

  where $\alpha_1 \otimes a$ is seen as an endomorphism inside $\mathbb{A}_A$. Therefore the generators of $C_{T^*[k]A[1]}$ are $C^0_{T^*[k]A[1]} = C^\infty(M)$ and $C^i_{T^*[k]A[1]} = \Gamma E_i$ for $i = 1, \cdots, k$.

Proof. For the case $k = 1$ we already mentioned that we consider a completion of multi-vector fields over $A[1]$ shifted by 1, so $T^*[1]A[1]$ is isomorphic to $T^*[1]A^*$.

For $k = 2$ equation (6.5) combined with the geometric characterization of vector fields on a 1-manifold given in Example 3.23 tell us that:

- $C^0_{T^*[2]A[1]} = C^0_{A[1]} = C^\infty(M)$.
- $C^1_{T^*[2]A[1]} = C^1_{A[1]} \oplus \mathfrak{X}^{1,-1}(A[1]) = \Gamma A^* \oplus \Gamma A = \Gamma E_1$.
- $C^2_{T^*[2]A[1]} = C^2_{A[1]} \oplus \mathfrak{X}^{1,0}(A[1]) \oplus \mathfrak{X}^{2,-2}(A[1]) = \Gamma(\bigwedge^2 A^* \oplus \mathbb{A}_A \oplus \bigwedge^2 A) = \Gamma\mathbb{A}_{(A\oplus A^*,\langle,\rangle)} = \Gamma E_2$.

The case $k > 2$ is analogous: equation (6.5) and Example 3.23 tell us that

- $C^0_{T^*[k]A[1]} = C^0_{A[1]} = C^\infty(M)$.
- For $1 \le i < k-1$, $\quad C^i_{T^*[k]A[1]} = C^i_{A[1]} = \Gamma \bigwedge^i A^* = \Gamma E_i$.
- $C^{k-1}_{T^*[k]A[1]} = \mathfrak{X}^{1,-1}(A[1]) \oplus C^{k-1}_{A[1]} = \Gamma(A \oplus \bigwedge^{k-1} A^*) = \Gamma E_{k-1}$.
- $C^k_{T^*[k]A[1]} = \mathfrak{X}^{1,0}(A[1]) \oplus C^k_{A[1]} = \Gamma(\mathbb{A}_A \oplus \bigwedge^k A^*) = \Gamma E_k$.

and the algebra structure of $C_{T^*[k]A[1]}$ given by the multiplication of functions clearly gives the map m. $\square$

Remark 6.14. Observe that (6.6) is an admissible algebra bundle because if we define the graded vector bundle $\mathbf{A} = A_1 \oplus A_2$, where $A_1 = A \oplus A^*$, and $A_2 = T^*M$, any connection on $A \oplus A^*$ that preserves the pairing induces an algebra isomorphism between (6.6) and $((Sym\mathbf{A})^{\leq 2}, \cdot)$. Also (6.7) is an admissible algebra bundle because in this case any connection on A induces an isomorphism between (6.7) and $((Sym\mathbf{A})^{\leq k}, \cdot)$ where

$$\mathbf{A} = \bigoplus_i A_i = \begin{cases} A^* & i = 1, \\ A & i = k-1, \\ TM & i = k, \\ 0 & \text{otherwise.} \end{cases}$$

From now on we focus on the case $k > 2$. By the previous proposition, the case $k = 1$ is just a vector bundle and it is well known what happens there. The case $k = 2$ was described by Roytenberg in [108] and corresponds to the standard structures on $A \oplus A^*$. Our results should be understood as a generalization of this case.

The following step is to obtain a classical description for the submanifolds of $T^*[k]A[1]$. For this we use Proposition 3.15 and obtain a characterization of the submanifolds in terms of the algebra bundles that we introduced in the previous proposition.

Proposition 6.15. *For $k > 2$ there is a 1-1 correspondence between:*

- *Submanifolds of $T^*[k]A[1]$*
- *Quadruples (N, D, L, K) where $N \subseteq M$ is a submanifold and D, L and K are three vector bundles over N satisfying*

$$D \subseteq A^*_{|N}, \quad L \subseteq \left(A \oplus \bigwedge^{k-1} A^*\right)_{|N} \text{ and } K \subseteq \left(\mathbb{A}_A \oplus \bigwedge^{k} A^*\right)_{|N}, \tag{6.8}$$

$$L \cap \bigwedge^{k-1} A^*_{|N} = D \wedge \bigwedge^{k-2} A^*_{|N}, \tag{6.9}$$

$$K \cap (\mathrm{End}(A) \oplus \bigwedge^{k} A^*)_{|N} = D \otimes A_{|N} \oplus L \wedge A^*_{|N}. \tag{6.10}$$

Proof. By Proposition 3.15 we have that submanifolds are the same as a submanifold $N \subseteq M$ and $\mathbf{F} = \oplus_{i=1}^k F_i \to N$ vector subundle of (6.7) satisfying

$$\mathbf{F} \cap m(\mathbf{E}, \mathbf{E})_{|N} = m(\mathbf{F}, \mathbf{E}_{|N}). \tag{6.11}$$

In our case this means that:

a). For $i = 1$: $F_1 = D \subseteq A^*_{|N}$ is vector subundle.

b). For $1 < i < k-1$: $F_i = F_i \cap \wedge^i A^*_{|N} = D \cap \wedge^{i-1} A^*_{|N}$, so they are completely determined.

c). For $i = k-1$: $F_{k-1} = L \subseteq (A \oplus \bigwedge^{k-1} A^*)_{|N}$ and equation (6.11) becomes (6.9).

d). For $i = k:\ F_k = K \subseteq (\mathbb{A}_A \oplus \bigwedge^k A^*)_{|N}$ and equation (6.11) becomes (6.10).

$\square$

Remark 6.16. Denote by $p_1 : A \oplus \bigwedge^{k-1} A^* \to A$ the projection. One of the consequences of equation (6.9) is that $p_1(L) \subseteq A$ defines a vector subundle. This is not the case for $k = 2$, and this property makes the submanifolds here much more rigid, as we will see.

Given a submanifold $\mathcal{N} \subseteq T^*[k]A[1]$ equivalent to (N, D, L, K) denote by $\widehat{F}$ the image of the projection of K onto TM, that is a regular distribution. Then

$$\operatorname{Totdim}\mathcal{N} = \dim N + rk(A^*) - rk(D) + rk(A) - rk(p_1(L)) + dim M - rk(\widehat{F}). \quad (6.12)$$

Recall that given a graded manifold $\mathcal{M} = (M, C_{\mathcal{M}})$, closed embedded submanifolds are in correspondence with sheaves of regular homogeneous ideals $\mathcal{I} \subseteq C_{\mathcal{M}}$. A direct consequence of the preceding proposition is that given a quadruple (N, D, L, K) satisfying the hypothesis of Proposition 6.15, if $N \subseteq M$ is closed and embedded then the ideal $\mathcal{I} \subseteq C_{T^*[k]A[1]}$ associated to the submanifold can be expressed in each degree as:

	0	1	2		$k-2$	$k-1$	k
$C_{T^*[k]A[1]}$	$C^\infty(M)$	A^*	$\bigwedge^2 A^*$	$\cdots$	$\bigwedge^{k-2} A^*$	$A \oplus \bigwedge^{k-1} A^*$	$\mathbb{A}_A \oplus \bigwedge^k A^*$
$\mathcal{I}$	$Z(N)$	$\widehat{\Gamma} D$	$\widehat{\Gamma}(D \wedge A^*_{\mid N})$		$\widehat{\Gamma}(D \wedge \bigwedge^{k-3} A^*_{\mid N})$	$\widehat{\Gamma} L$	$\widehat{\Gamma} K$

where $Z(N) = \{f \in C^\infty(M) \mid f(n) = 0\ \forall n \in N\}$ and for any subbundle $(H \to N) \subseteq (G \to M)$ we denote

$$\widehat{\Gamma}(H) = \{s \in \Gamma G \mid s(n) \in H\ \forall n \in N\}.$$

6.2.2 The symplectic structure

Another aspect of the graded cotangent bundles is that they are graded symplectic manifolds. As mentioned in the introduction the symplectic structure of $T^*[k]A[1]$ will be responsible for the pairing on $A \oplus \bigwedge^{k-1} A^*$, as it happens when $k = 2$.

Proposition 6.17. *For $k > 2$, the manifold $T^*[k]A[1]$ is equivalent to (E, m) given in (6.7). The Poisson bracket on $T^*[k]A[1]$ is determined by the following operations on $E = \oplus E_i \to M$:*

$$\begin{array}{ll}
\langle \cdot,\cdot \rangle : \Gamma E_{k-1} \times \Gamma E_1 \to C^\infty(M), & \langle a + \omega, \alpha \rangle = \alpha(a), \\
\langle \cdot,\cdot \rangle : \Gamma E_{k-1} \times \Gamma E_{k-1} \to \Gamma E_{k-2}, & \langle a_1 + \omega_1, a_2 + \omega_2 \rangle = i_{a_1}\omega_2 + i_{a_2}\omega_1, \\
\cdot : \Gamma E_k \times C^\infty(M) \to C^\infty(M), & (D,\tau) \cdot f = X(f), \\
\Psi : \Gamma E_k \times \Gamma E_1 \to \Gamma E_1, & \Psi_{(D,\tau)}\alpha = D(\alpha), \\
\Upsilon : \Gamma E_k \times \Gamma E_{k-1} \to \Gamma E_{k-1}, & \Upsilon_{(D,\tau)}(a + \omega) = D(a + \omega) - i_a \tau, \\
[\cdot,\cdot] : \Gamma E_k \times \Gamma E_k \to \Gamma E_k, & [(D_1,\tau_1),(D_2,\tau_2)] = ([D_1, D_2], D_1(\tau_2) - D_2(\tau_1)).
\end{array}$$

Proof. Use that the Poisson bracket is defined by equations (6.4) and recall how vector fields act on functions and on other vector fields. $\square$

Having described the symplectic structure, we now apply Proposition 6.15 to coisotropic and lagrangian submanifolds. We have special interest in describing the lagrangian ones because for $T^*[2]A[1]$ they are in correspondence with almost Dirac structures of $(A \oplus A^*, \langle\cdot,\cdot\rangle)$. In the degree 2 case the coisotropic and lagrangian submanifolds of an arbitrary degree 2 symplectic manifold where described in [24]. Our results must be compared with that ones.

Theorem 6.18 (Coisotropic submanifolds)**.** *For $k > 2$, there is a 1-1 correspondence between:*

- *Coisotropic submanifolds of $T^*[k]A[1]$.*
- *Data $(N, D, L, \widehat{F}, \nabla)$, where:*

  a). $N \subseteq M$ is a submanifold.
  b). D and L are vector bundles over N satisfying $D \subseteq A^*_{|N}$ and $L \subseteq (A \oplus \bigwedge^{k-1} A^*)_{|N}$ such that condition (6.9) follows, $D \subseteq p_1(L)^\circ$ and $\langle L, L\rangle \subseteq D \wedge \bigwedge^{k-3} A^*_{|N}$.
  c). $\widehat{F} \subseteq TN$ is a regular and involutive distribution.
  d). ∇ is flat partial $\widehat{F}$-connection in the vector bundle $\frac{p_1(L)^\circ}{D}$.

Proof. By Proposition 6.15 submanifolds are the same as (N, D, L, K) satisfying equations (6.8),(6.9) and (6.10). Associated to the quadruple we have an ideal $\mathcal{I} \subseteq C_{T^*[k]A[1]}$ given by $\mathcal{I}^0 = Z(N)$, $\mathcal{I}^1 = \widehat{\Gamma} D$, $\mathcal{I}^{k-1} = \widehat{\Gamma} L$ and $\mathcal{I}^k = \widehat{\Gamma} K$. Denote the image of the projection of K into $\mathbb{A}_A$ by $\widetilde{F}$ and into TM by $\widehat{F}$. Since D, L and K satisfy (6.10) we know that $\widetilde{F}$ and $\widehat{F}$ are vector bundles.

Using the preceding description of the symplectic bracket in classical terms, the coisotropic condition (2.10) becomes the following equations:

$$
\begin{array}{lcl}
\{\mathcal{I}^0, \mathcal{I}^k\} \subseteq \mathcal{I}^k & \Leftrightarrow & \widehat{F} \subset TN. \\
\{\mathcal{I}^1, \mathcal{I}^{k-1}\} \subseteq \mathcal{I}^0 & \Leftrightarrow & D \subseteq p_1(L)^\circ. \\
\{\mathcal{I}^1, \mathcal{I}^k\} \subseteq \mathcal{I}^1 & \Leftrightarrow & \widetilde{F} \text{ preserves } D. \\
\{\mathcal{I}^{k-1}, \mathcal{I}^{k-1}\} \subseteq \mathcal{I}^{k-2} & \Leftrightarrow & \langle L, L\rangle \subseteq D \wedge \bigwedge^{k-3} A^*_{|N}. \\
\{\mathcal{I}^{k-1}, \mathcal{I}^k\} \subseteq \mathcal{I}^{k-1} & \Leftrightarrow & \left\{ \begin{array}{l} \widetilde{F} \text{ preserves } L. \\ \text{The projection into the second factor of } K \\ \ \text{is inside } L \wedge A^*. \end{array} \right. \\
\{\mathcal{I}^k, \mathcal{I}^k\} \subseteq \mathcal{I}^k & \Leftrightarrow & K \text{ is involutive so } \widetilde{F} \text{ and } \widehat{F} \text{ are involutive.}
\end{array}
$$

Since the second projection of K is inside $L \wedge A^*$ and K satisfies equation (6.10) and preserves L and D, we know that it is completely characterized by $\widehat{F}$ and a partial $\widehat{F}-$connection on the vector bundle $\frac{p_1(L)^\circ}{D}$. Moreover this connection must be flat since $\widetilde{F}$ is involutive. □

Corollary 6.19. *(Lagrangian submanifolds) For $k > 2$, a lagrangian submanifold of $T^*[k]A[1]$ is the same as a submanifold $N \subseteq M$ and a vector bundle $L \to N$ such that $L \subseteq (A \oplus \bigwedge^{k-1} A^*)_{|N}$ and satisfying*

$$p_1(L) \subseteq A \text{ is a subbundle,} \tag{6.13}$$
$$L \cap \bigwedge^{k-1} A^*_{|N} = p_1(L)^\circ \wedge \bigwedge^{k-2} A^*_{|N}, \tag{6.14}$$
$$\langle L, L\rangle \subseteq p_1(L)^\circ \wedge \bigwedge^{k-3} A^*_{|N}. \tag{6.15}$$

Proof. Since Totdim $T^*[k]A[1]$ is even a lagrangian submanifold is a coisotropic one that has total dimension half of the total dimension of the manifold. Using the chacterization of coisotropic submanifolds of the Theorem 6.18 and formula (6.12) we obtain that a coisotropic submanifold given by $(N, D, L, \widehat{F}, \nabla)$ has total dimension equal to 1/2 Totdim $T^*[k]A[1]$ if and only if $D = p_1(L)^\circ$ and $\widehat{F} = TN$. As a consequence, ∇ is defined on a zero bundle so the only information that we need is a vector bundle $L \to N$ satisfying (6.13), (6.14) and (6.15). □

The equations (6.13), (6.14) and (6.15) that define a lagrangian submanifold have also the following geometric interpretation:

Corollary 6.20. *For $k > 2$, there is a 1-1 correspondence between:*

- *Lagrangian submanifolds of $T^*[k]A[1]$.*
- *Pairs $(E \to N, \Omega)$ where $(E \to N) \subseteq (A_{|N} \to N)$ is a subbundle and $\Omega \in \Gamma \bigwedge^k E^*$.*

Proof. Given a subbundle

$$\begin{array}{ccc} E & \xrightarrow{j} & A \\ \downarrow & & \downarrow \\ N & \xrightarrow{\widehat{j}} & M \end{array}$$

the inclusion map induces a map $j^* : \Gamma A^* \to \Gamma E^*$ that extends to a map $j^* : \Gamma \bigwedge^i A^* \to \Gamma \bigwedge^i E^*$ for all $i \in \mathbb{N}$. Recall that $\frac{\bigwedge^{k-1} A^*_{|N}}{E^\circ \wedge \bigwedge^{k-2} A^*_{|N}} \cong \bigwedge^{k-1} E^*$ and the following diagram commutes:

$$\begin{array}{ccc} \Gamma \bigwedge^{k-1} A^*_{|N} & \xrightarrow{j^*} & \Gamma \bigwedge^{k-1} E^* \\ i_e \downarrow & & \downarrow i_e \\ \Gamma \bigwedge^{k-2} A^*_{|N} & \xrightarrow{j^*} & \Gamma \bigwedge^{k-2} E^* \end{array}$$

where i_e is the contraction with $e \in \Gamma E$.

Given a pair $(E \to N, \Omega)$, define

$$L = \{e + w \in \Gamma(A_{|N} \oplus \bigwedge^{k-1} A^*_{|N}) |\ e \in \Gamma E \text{ and } i_e \Omega = j^* w\}.$$

Clearly equation (6.14) is satisfied because $p_1(L) = E$ and $L \cap \bigwedge^{k-1} A^*_{|N} = \ker j^* = E^\circ \wedge \bigwedge^{k-2} A^*_{|N}$ as we wish. Let us check now (6.15):

$$j^* \langle e + w, e' + w' \rangle = j^*(i_e w' + i_{e'} w) = i_e j^* w' + i_{e'} j^* w = i_e i_{e'} \Omega + i_{e'} i_e \Omega = 0$$

$\forall e + w, e' + w' \in \Gamma L$ and therefore we obtain that (6.15) is also satisfied.

Given a lagrangian submanifold $(L \to N) \subseteq (A \oplus \bigwedge^{k-1} A^*)_{|N}$ define $E = p_1(L)$, which is a subbundle by hypothesis. The fact that L satisfies equation (6.14) means that it fits into the exact sequence

$$\begin{array}{ccccc} \bigwedge^{k-1} A^* & \to & A \oplus \bigwedge^{k-1} A^* & \to & A \\ E^\circ \wedge \bigwedge^{k-2} A^*_{|N} & \to & L & \to & E \end{array}$$

so there exists a map $\varphi : E \to \frac{\bigwedge^{k-1} A^*_{|N}}{E^\circ \wedge \bigwedge^{k-2} A^*_{|N}} \cong \bigwedge^{k-1} E^*$ such that

$$L = \{e + w \in (A \oplus \bigwedge^{k-1} A^*)_{|N} |\ e \in E, \quad \varphi(e) = j^* w\}.$$

Since L also satisfies equation (6.15), it follows that

$$0 = j^*\langle e + w, e + w'\rangle = j^*(i_e w' + i_{e'} w) = i_e j^* w' + i_{e'} j^* w = i_e \varphi(e') + i_{e'} \varphi(e)$$

$\forall e + w,\ e' + w' \in \Gamma L$ and this is equivalent to $\varphi = \Omega^\sharp$ for some $\Omega \in \Gamma \bigwedge^k E^*$. □

Remark 6.21. As a consequence of Corollary 6.20 we see that the condition $p_1(L)$ has constant rank is a bit restrictive. For some examples it is interesting to allow the case when $p_1(L)$ changes rank at points on the base. Observe that in this case the other equations also make sense. We call *weak lagrangian submanifold* a subbundle $L \to N$ over a submanifold of M such that $L \subseteq (A \oplus \bigwedge^{k-1} A^*)_{|N}$ and (6.14) and (6.15) hold.

For these, the characterization of the Corollary 6.20 is just true pointwise. In terms of sheaves of ideals, we are allowing non regular ideals in a particular way, but they are still closed for the Poisson bracket. When M is just a point, the notion of weak lagrangians and lagrangians coincide.

Let us compare our definition with previous ones that have appeared in the literature, in particular the one given by Hagiwara in [64] when $A = TM$ and the one given by Wade in [125].

Definition 6.22 (see [64]). An *almost Nambu-Dirac structure* in $TM \oplus \bigwedge^{k-1} T^*M$ is a subbundle $L \subseteq TM \oplus \bigwedge^{k-1} T^*M$ over M satisfying

$$\bigwedge^{k-1} p_1(L) = pr_2(L^\circ); \tag{6.16}$$
$$(i'_a w + i_a w')_{|\bigwedge^{k-2} p_1(L)} = 0, \quad \forall a + w, a' + w' \in \Gamma L; \tag{6.17}$$

where $pr_2 : T^*M \oplus \bigwedge^{k-1} TM \to \bigwedge^{k-1} TM$ is the projection. In addition, we say that L is regular if $p_1(L) \subseteq TM$ is a subbundle.

Theorem 6.23. *There is a one to one correspondence between*

$$\left\{ \begin{array}{c} \textit{Regular almost Nambu-Dirac structures} \\ \textit{in } \ TM \oplus \bigwedge^{k-1} T^*M \end{array} \right\} \leftrightharpoons \left\{ \begin{array}{c} \textit{Lagrangian submanifolds} \\ \textit{of } \ T^*[k]T[1]M \textit{ with body } M \end{array} \right\}$$

More generally, almost Nambu-Dirac structures in $TM \oplus \bigwedge^{k-1} T^*M$ are in correspondence with weak lagrangian submanifolds of $T^*[k]T[1]M$ with body M.

Proof. Using Corollary 6.19 we just need to prove that equations (6.14) and (6.15) are equivalent to (6.16) and (6.17). Clearly (6.15) is the same as (6.17). Now we check that the annhilator of (6.14) gives (6.16):

$$\left(p_1(L)^\circ \wedge \bigwedge^{k-2} A^*_{|N}\right)^\circ = \bigwedge^{k-1} p_1(L)$$

and

$$\left(L\cap\bigwedge^{k-1}A^*_{|N}\right)^{\circ}=pr_2\left(L^{\circ}\oplus(\bigwedge^{k-1}A^*_{|N})^{\circ}\right)=pr_2(L^{\circ}\oplus A^*_{|N})=pr_2(L^{\circ}).$$

It is clear that if the almost Nambu-Dirac structure is regular we obtain a lagrangian submanifold otherwise we just have a weak lagrangian submanifold. □

Definition 6.24 (see [125]). An *almost Dirac structure of order* k on $A\to M$ is a subbundle $L\subseteq A\oplus\bigwedge^{k-1}A^*$ over M satisfying that there exists $E\subseteq A$ subbundle over M and $\Omega\in\Gamma\bigwedge^{k}E^*$ such that

$$\forall p\in M\quad L_p=\{e+w\in(A\oplus\bigwedge^{k-1}A^*)_p|\ e\in E_p,\ i_e\Omega_p=w_{|\bigwedge^{k-1}E}\},\quad(6.18)$$

$$\forall Z_1,\cdots,Z_{k-1}\in\Gamma\bigwedge^{k-1}E\quad\exists\ e\in\Gamma E\text{ such that }i_{Z_1}\Omega\wedge\cdots i_{Z_{k-1}}\Omega=i_e\Omega.\quad(6.19)$$

Using Corollary 6.20 we see that an almost Dirac structure of order k as defined by Wade is the same as a lagrangian submanifold of $T[k]A[1]$ with body M that in addition satisfies equation (6.19). Therefore the Wade almost Dirac structures of order k are a particular case of lagrangian submanifolds of $T^*[k]A[1]$.

In the literature, apart from the definitions of Hagiwara and Wade there are at least four more approaches. Some of them are defined for an arbitrary vector bundle and some of them just for TM. By chronological order: In [14] Bonelli and Zabzine define almost generalized Dirac structures as maximal isotropic subbundles of $TM\oplus\bigwedge^{k-1}T^*M$, in [130] Zambon notices that the geometry of $TM\oplus\bigwedge^{k-1}T^*M$ is related to the one coming from $T^*[k]T[1]M$ but use different equations and call them higher Dirac structures. The work of Bi and Sheng [11], using the name of $(p,k)-$Dirac structure, gives a definition that includes the Nambu-Dirac structure of Hagiwara as the case $(k-1,k-2)$ and the higher Dirac of Zambon as the case $(k-1,0)$. Finally in [28] Bursztyn, Martinez and Rubio give another possible definition based on Zambon keeping the terminology, but with different requirements.

We believe that this Supergeometric approach sheds light on the Hagiwara definition, justifying his equations and putting them into the general framework of lagrangian submanifolds inside graded symplectic manifolds.

Example 6.25 (Conormal bundles). Given any submanifold of $A[1]$, i.e. $(B\to N)\subseteq(A\to M)$ subbundle, that we denote by $B[1]\subseteq A[1]$, the conormal bundle shifted by k, $N^*[k]B[1]\subseteq T^*[k]A[1]$, defines a lagrangian submanifold. The associated vector bundle is given by

$$L=B\oplus B^{\circ}\wedge\bigwedge^{k-2}A^*_{|N}.$$

The associated pair as in Corollary 6.20 is $(B\to N,0)$. In the case where $A=TM$ this is codifying regular distributions. Observe that in this case weak lagrangians are not encoding singular distributions because L itself will be singular.

Example 6.26 (k-forms). Using Corollary 6.20 we know that given any $\omega\in\Gamma\bigwedge^{k}A^*$ we obtain a lagrangian submanifold by considering the pair $(A\to M,\omega)$. In fact, these lagrangian submanifolds correspond to the ones satisfying

$$L\cap\bigwedge^{k-1}A^*=0.$$

Definition 6.27. Let $A \to M$ be a vector bundle and $\Pi \in \Gamma \bigwedge^k A$. We say that Π is *decomposable* if $\forall p \in M$ where $\Pi_p \neq 0$ there exists an open neighbourhood U where $A_{|U}$ is trivial and basis of sections of A, $\{a_1, \cdots, a_k\}$ such that

$$\Pi_{|U} = f(x) a_1 \wedge \cdots \wedge a_k \qquad f(x) \in C^\infty(M).$$

Example 6.28 (Decomposable k-multi-vector field). Now we study the other extreme case. Suppose that we have a weak lagrangian submanifold over M with

$$L \cap A = 0.$$

Then we know that $L = graph(\Lambda)$ for some $\Lambda : S \to A$ with $S \subseteq \bigwedge^{k-1} A^*$ and satisfies

$$\ker(\Lambda) = \operatorname{im}(\Lambda)^\circ \wedge \bigwedge^{k-2} A^*,$$
$$i_{\Lambda(\omega)}\omega' + i_{\Lambda(\omega')}\omega \in \operatorname{im}(\Lambda)^\circ \wedge \bigwedge^{k-3} A^*.$$

If we impose that $S = \bigwedge^{k-1} A^*$ on the one hand the first equation says that $\Lambda = \Pi^\sharp$ for some $\Pi \in \Gamma \bigwedge^k A$. On the other hand, it was seen by Hagiwara on [64] that the second equation is equivalent to Π being decomposable. In general, decomposable tensors give rise to weak lagrangians, the condition that $p_1(L)$ is a subbundle implies that the tensor is zero or never vanishes, so we can dualize and obtain a k-form. This example is the main reason why we also want to consider weak lagrangian submanifolds.

6.3 The symplectic Q-structures of $T^*[k]A[1]$

In the previous sections we established a relation between the symplectic structure on the manifolds $T^*[k]A[1]$ and the natural pairing on $A \oplus \bigwedge^{k-1} A^*$. For $k = 2$, it was proved by Ševera and Roytenberg that the Courant brackets on $A \oplus A^*$ correspond to the Q-structures on $T^*[2]A[1]$ that preserve the symplectic structure. Following this idea, we study the symplectic Q-structures on $T^*[k]A[1]$ and construct the corresponding brackets on $A \oplus \bigwedge^{k-1} A^*$. Finally we define higher Dirac structures on $A \oplus \bigwedge^{k-1} A^*$ as the Lagrangian Q-submanifolds of $T^*[k]A[1]$, these are called Λ-structures in [113].

6.3.1 Q-structures and twists

As it was defined in Section A.3 a symplectic Q-structure on $T^*[k]A[1]$ is a vector field $Q \in \mathfrak{X}^{1,1}(T^*[k]A[1])$ that satisfies equations (2.5) and (2.9). For $k \geq 2$, we can use Proposition A.10 and conclude that $Q = \{\theta, \cdot\}$ for some $\theta \in C^{k+1}_{T^*[k]A[1]}$. In addition, we have the following equivalence for condition (2.5):

$$0 = [Q, Q] = 2Q^2 = 2\{\theta, \{\theta, \cdot\}\} \Leftrightarrow \{\theta, \theta\} = 0, \tag{6.20}$$

which follows from the graded Jacobi identity of the symplectic bracket. This last equation is known as the classical master equation. Our goal is to express $\theta \in C^{k+1}_{T^*[k]A[1]}$ satisfying (6.20) in terms of classical geometric objects.

The case $k = 2$ is what we consider in Example 6.7. As a consequence of the results that we will prove in this section, we will see that $k = 2$ is the most flexible case, their higher versions being much more rigid. The next result deals with $k = 3$, see [61, 69]:

Theorem 6.29. *Symplectic Q-structures on $T^*[3]A[1]$ are equivalent to $(A \to M, [\cdot,\cdot], \rho, \langle\cdot,\cdot\rangle, H)$ where $[\cdot,\cdot] : \Gamma(A \wedge A) \to \Gamma A$, $\rho : A \to TM$ vector bundle map covering the identity, $\langle\cdot,\cdot\rangle$ is a pairing on the vector bundle A^* and $H \in \Gamma \bigwedge^4 A^*$ satisfying the following conditions:*

$$\begin{cases} [a, fb] = f\,[a, b] + \mathcal{L}_{\rho(a)} fb & a, b, c \in \Gamma A, \\ [a, [b, c]] + c.p. = \flat(i_a i_b i_c H) & f \in C^\infty(M), \\ \langle \mathcal{L}_a \omega, \tau\rangle + \langle \omega, \mathcal{L}_a \tau\rangle = \rho(a)\langle \omega, \tau\rangle & \omega, \tau \in \Gamma A^*, \\ d_A H = 0, \end{cases}$$

where $\flat : A^* \to A$ is the morphism induced by the pairing on A^* and $d_A : \Gamma \bigwedge^j A^* \to \Gamma \bigwedge^{j+1} A^*$ is the derivation associated with the pair $[\cdot,\cdot]$ and ρ, as defined in Example 2.12.

Proof. This follows from Theorem 6.8. $\square$

In particular, if the pairing is zero we have a honest Lie algebroid on $A \to M$ with $H \in \Gamma \bigwedge^4 A^*$ satisfying $d_A H = 0$. On the other hand, when $H = 0$ we obtain a Lie algebroid on $A \to M$ with an ad-invariant paring on $A^* \to M$. The case when the three structures are nontrivial is analogous to the case $k = 2$ and should be thought of as a quasi-algebroid. When $A \to M$ is just a vector space these structures were described in [32]. For other examples see [61, 69].

Theorem 6.30. *For $k > 3$, symplectic Q-structures on $T^*[k]A[1]$ are equivalent to a Lie algebroid structure on $A \to M$ and $H \in \Gamma \bigwedge^{k+1} A^*$ with $d_A H = 0$.*

Proof. As in the previous theorem, we must study the functions of degree $k + 1$ satisfying the classical master equation. Since $k > 3$, we have that

$$C^{k+1}_{T^*[k]A[1]} = \mathfrak{X}^{1,1}(A[1]) \oplus C^{k+1}_{A[1]}$$

by (6.5), so any $k + 1$ function can be written as $\theta + H$, where $\theta \in \mathfrak{X}^{1,1}(A[1])$ and $H \in C^{k+1}_{A[1]} = \Gamma \bigwedge^{k+1} A^*$; the classical master equation is equivalent to

$$\begin{cases} \{\theta, \theta\} = 0, \\ \{\theta, H\} = 0. \end{cases}$$

By Theorem 3.26, θ defines a Lie algebroid on $A \to M$. The other equation says that $d_A H = 0$, where d_A is the Lie algebroid differential on $\Gamma \bigwedge^\bullet A^*$. $\square$

From now on, we use the notation $\theta_H = \theta + H$ and $X_{\theta_H} = \{\theta_H, \cdot\}$ for the corresponding hamiltonian vector field.

6.3.2 Q-cohomology

If we have a Q-manifold $(\mathcal{M}, Q)$, the fact that Q has degree 1 and $Q^2 = 0$ implies that $(C_{\mathcal{M}}, Q)$ becomes a differential complex. Therefore, we have an associated cohomology that we denote by $H_Q(\mathcal{M})$. Observe that, in contrast with other well known cohomologies, it is usual that $H^n_Q(\mathcal{M}) \neq 0 \quad \forall n \in \mathbb{N}$. For example, if $Q = 0$ and $\mathcal{M}$ has degree 2 then $H^n_Q(\mathcal{M}) = C^n_{\mathcal{M}} \neq 0$. For the standard Courant algebroid $(T^*[2]T[1]M, \{\cdot,\cdot\}, \theta)$ this cohomology is isomorphic to the de Rham cohomology of M.

Also it is well known that exact Courant algebroids are classified by their Ševera classes $[H] \in H^3(M)$. We will extend this result for $k > 3$.

Definition 6.31. For any $k \geq 2$ and any element $B \in \Gamma \bigwedge^k A^* = C^k_{A[1]} \subset C^k_{T^*[k]A[1]}$, let $\tau^B : T^*[k]A[1] \to T^*[k]A[1]$ be the graded automorphism given by the exponential of the hamiltonian vector field of B, i.e. $\tau^B = Id - \{B, \cdot\}$, or more concretely:

$$\tau^B = \begin{cases} \tau^B = Id : M \to M, \\ \tau^B : C_{T^*[k]A[1]} \to C_{T^*[k]A[1]}, \quad \tau^B(f + X) = f + X + X(B). \end{cases}$$

for $f \in C_{A[1]}, \;\; X \in \mathfrak{X}^{1,\bullet}(A[1])$.

The symmetries τ^B were introduced in [62] and also defined in [10, 14, 15].

Proposition 6.32. *The diffeomorphisms $\tau^B : T^*[k]A[1] \to T^*[k]A[1]$ are symplectomorphisms and send X_{θ_H} to $X_{\theta_{H+d_A B}}$.*

Proof. A diffeomorphism is a symplectomorphism if and only if preserves the Poisson bracket and we have that $\forall\; f, g \in C_{T^*[k]A[1]}$

$$\begin{aligned} \{\tau^B f, \tau^B g\} = &\;\; \{f - \{B, f\}, g - \{B, g\}\} = \{f, g\} - \{f, \{B, g\}\} - \{\{B, f\}, g\} \\ = &\;\; \{f, g\} - \{B, \{f, g\}\} \end{aligned}$$

where we use the graded Jacobi identity and the fact that $\{\{B, f\}, \{B, g\}\} = 0 \; \forall f, g \in C_{T^*[k]A[1]}$. Therefore τ^B are symplectomorphisms. For the second assertion, since the maps are symplectomorphisms, it is enough to notice that

$$\tau^B(\theta + H) = \theta + H + \theta(B) = \theta + H + d_A(B). \qquad \square$$

As a consequence of Proposition 6.32, we have that if $(A \to M, [\cdot,\cdot], \rho)$ is a Lie algebroid and $H, H' \in \Gamma \bigwedge^{k+1} A^*$ are such that $d_A H = d_A H' = 0$ with $H - H' = d_A B$ for some $B \in \Gamma \bigwedge^k A^*$, so $(T^*[k]A[1], X_{\theta_H})$ and $(T^*[k]A[1], X_{\theta_{H'}})$ have the same Q-cohomology.

Corollary 6.33. *Let $(A \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid and denote by $H^\bullet(A)$ its Lie algebroid cohomology. For $k > 3$, the Q-structures on $T^*[k]A[1]$ compatible with the symplectic structure for which $p : T^*[k]A[1] \to A[1]$ is a Q-morphism are parametrized up to equivalence by $H^{k+1}(A)$.*

6.3.3 Brackets and higher Dirac structures

Given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ and $H \in \Gamma \bigwedge^{k+1} A^*$ with $d_A H = 0$, consider the symplectic Q-manifold $(T^*[k]A[1], \{\cdot,\cdot\}, \theta_H)$. We can use the derived bracket formalism with respect to the non-degenerate Poisson bracket and θ_H to obtain a bracket on sections of $A \oplus \bigwedge^{k-1} A^*$ by the following formula:

$$\{\{\cdot,\theta_H\},\cdot\} : C^{k-1}_{T^*[k]A[1]} \times C^{k-1}_{T^*[k]A[1]} \to C^{k-1}_{T^*[k]A[1]},$$

where we use formulas (6.5) and (6.7) to identify

$$C^{k-1}_{T^*[k]A[1]} = \mathfrak{X}^{1,-1}(A[1]) \oplus C^{k-1}_{A[1]} = \Gamma(A \oplus \bigwedge^{k-1} A^*).$$

Therefore, the above formula defines a bracket $[\![\cdot,\cdot]\!]_H : \Gamma(A \oplus \bigwedge^{k-1} A^*) \times \Gamma(A \oplus \bigwedge^{k-1} A^*) \to \Gamma(A \oplus \bigwedge^{k-1} A^*)$ that in classical terms is just

$$\begin{aligned} [\![a+\omega, b+\eta]\!]_H &= \{\{\widehat{a}+\omega, \theta+H\}, \widehat{b}+\eta\} = \{[\widehat{a},\theta] + \widehat{a}(H) + d_A\omega, \widehat{b}+\eta\} \\ &= \left[[\widehat{a},\theta], \widehat{b}\right] + [\widehat{a},\theta](\eta) - \widehat{b}(\widehat{a}(H)) - \widehat{b}(d_A\omega) \\ &= [a,b] + \mathcal{L}_a\eta - i_b d_A\omega - i_b i_a H, \end{aligned} \tag{6.21}$$

where $a, b \in \Gamma A$, $\omega, \eta \in \Gamma \bigwedge^{k-1} A^*$, and $\widehat{a}, \widehat{b} \in \mathfrak{X}^{1,-1}(A[1]) = \Gamma A$ denote the vector fields induced by a and b, respectively.

Remark 6.34. For $A = TM$, these brackets have been previously considered e.g. in [62, 64, 65]. These works also explain the H-twist and the symmetries τ^B. The connection with the graded manifolds $T^*[k]T[1]M$ was noticed in [15, 130].

Definition 6.35. Let $(A \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid and $H \in \Gamma \bigwedge^{k+1} A^*$ with $d_A H = 0$. For $k > 2$, a *higher Dirac structure* on $(A \oplus \bigwedge^{k-1} A^*, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!]_H, \rho)$ is a vector subbundle $L \to N$ of $(A \oplus \bigwedge^{k-1} A^*)_{|N}$ over a submanifold $N \subseteq M$ satisfying equations (6.13), (6.14), (6.15) and

$$\rho(L) \subset TN \quad \text{and} \quad [\![\Gamma L, \Gamma L]\!]_H \subseteq \Gamma L.$$

N is called the body or "support" of the higher Dirac structure (See e.g. [27] for Dirac structures with support in the ordinary sense).

Theorem 6.36. *Let $(A \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid and $H \in \Gamma \bigwedge^{k+1} A^*$ with $d_A H = 0$. For $k > 2$, there is a one to one correspondence:*

$$\left\{ \begin{array}{c} \textit{Higher Dirac structures} \\ \textit{on } (A \oplus \bigwedge^{k-1} A^*, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!]_H, \rho) \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{c} \textit{Lagrangian } Q\textit{-submanifolds} \\ \textit{of } (T^*[k]A[1], \{\cdot,\cdot\}, X_{\theta_H}) \end{array} \right\}$$

Proof. By Corollary 6.19 the only thing that remains to prove is that $\rho(L) \subset TN$ and $[\![\Gamma L, \Gamma L]\!]_H \subseteq \Gamma L$ if and only if $Q = X_{\theta_H}$ is tangent to the lagrangian submanifold. Denote by $\mathcal{I}$ the sheaf of ideals that defines the lagrangian submanifold. Recall that on a lagrangian submanifold

$$f \in \mathcal{I}^i \Leftrightarrow X_f(\mathcal{I}) \subseteq \mathcal{I} \quad \forall i \in \mathbb{N}^*.$$

Therefore

$$\begin{aligned} Q(\mathcal{I}) \subseteq \mathcal{I} \;&\Leftrightarrow\; X_{Q(f)}(\mathcal{I}) \subseteq \mathcal{I} \quad \forall f \in \mathcal{I} \\ &\Leftrightarrow\; \{\{f,\theta_H\}, g\} \in \mathcal{I} \quad \forall f, g \in \mathcal{I} \\ &\Leftrightarrow\; \rho(L) \subseteq TN \text{ and } [\![\Gamma L, \Gamma L]\!]_H \subseteq \Gamma L. \end{aligned}$$

□

Corollary 6.37. *Given a lagrangian submanifold of* $(T^*[k]A[1], \{\cdot,\cdot\}, \theta_H)$ *equivalent to the pair* $(E \to N, \Omega)$ *then* X_{θ_H} *is tangent to it if and only if* $E \to N$ *is a subalgebroid of* $A \to M$ *and* $d_E\Omega = j^*H$.

Proof. Denote by $j : E \to A$ the inclusion and recall that

$$L = \{e + w \in \Gamma(A \oplus \bigwedge^{k-1} A^*)_{|N} \mid e \in \Gamma E \text{ and } i_e\Omega = j^*w\}$$

with $\rho(E) \subseteq TN$ and $[\![\Gamma L, \Gamma L]\!]_H \subseteq \Gamma L$. The second condition implies that, $\forall e + w,\ e' + w' \in \Gamma L$,

$$[\![e + w, e' + w']\!]_H = [e, e'] + \mathcal{L}_e w' - i_{e'} d_A w - i_{e'} i_e H \in \Gamma L,$$

so $\rho(E) \subseteq TN$ and, $\forall e, e' \in \Gamma E$, $[e, e'] \in \Gamma E$ if and only if $E \to N$ is a Lie subalgebroid. In particular this condition implies that $j^* d_A = d_E j^*$, so j^* is a chain map and therefore

$$i_{[e,e']}\Omega = j^*(\mathcal{L}_e w' - i_{e'} d_A w - i_{e'} i_e H) \Leftrightarrow d_E\Omega = j^*H. \qquad \square$$

Example 6.38. Consider the zero section $N^*[k]A[1]$, that by definition is a Q-lagrangian inside $(T^*[k]A[1], \{\cdot,\cdot\}, X_\theta)$. Given $\omega \in \Gamma \bigwedge^k A^*$ with $d\omega = H$ we could take the symplectomorphism τ^ω that sends X_θ to X_{θ_H} and $N^*[k]A[1]$ to $L = graph(\omega)$. Therefore L is a Q-lagrangian submanifold inside $(T^*[k]A[1], \{\cdot,\cdot\}, X_{\theta_H})$.

Recall that decomposable tensors, see Example 6.28, correspond to weak lagrangian submanifolds. To finish this section, we analyze their involutivity condition.

Definition 6.39. Let $A \to M$ be a Lie algebroid and $H \in \Gamma \bigwedge^{k+1} A^*$ with $d_A H = 0$. Consider $\Pi \in \Gamma \bigwedge^k A$ a decomposable tensor. We say that it defines an $H-$*twisted Nambu structure* if

$$(\mathcal{L}_{\Pi(w)}\Pi)(w') = -\Pi\big(i_{\Pi(w')} d_A w + i_{\Pi(w')} i_{\Pi(w)} H\big) \qquad \forall w, w' \in \Gamma \bigwedge^k A^*. \tag{6.22}$$

In [125] Wade introduced this equation for general Lie algebroids and here we extend it with the H-twist. Wade also observed that the decomposability of the tensor does not follow from equation (6.22), so we must impose it in order to obtain a weak lagrangian submanifold.

Example 6.40 (H-twisted Nambu tensor). Let $L = graph(\Pi)$ where $\Pi \in \Gamma \bigwedge^k A$ is a decomposable tensor. We claim that L is a higher Dirac structure on $(A \oplus \bigwedge^{k-1} A^*, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!]_H)$ if and only if Π is an $H-$twisted Nambu structure. The proof follows from the following computation for all $w, w' \in \Gamma \bigwedge^{k-1} A^*$:

$$\begin{aligned}
[\Pi(w), \Pi(w')] &= \mathcal{L}_{\Pi(w)}(\Pi(w')) \\
&= \big(\mathcal{L}_{\Pi(w)}\Pi\big) w' + \Pi\left(\mathcal{L}_{\Pi(w)} w'\right) \\
&= -\Pi\big(i_{\Pi(w')} d_A w + i_{\Pi(w')} i_{\Pi(w)} H\big) + \Pi\left(\mathcal{L}_{\Pi(w)} w'\right) \\
&= \Pi\left(\mathcal{L}_{\Pi(w)} w' - i_{\Pi(w')} d_A w - i_{\Pi(w')} i_{\Pi(w)} H\right).
\end{aligned}$$

6.3.4 Relation with the coadjoint representation up to homotopy

Observe that given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ and a connection $\nabla : \mathfrak{X}(M) \times \Gamma A \to \Gamma A$ we have defined in Example 5.24 the coadjoint representation up to homotopy of the Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$. If we use Theorem 5.32 we obtain that the connection give an isomorphism between the graded Q-manifolds

$$(T^*[k+1]A[1], X_\theta) \cong \big((A \ltimes (A^* \to T^*M)[k])[1], Q\big).$$

Therefore this give an identification between the graded cotangent bundles and the coadjoint representation up to homotopy. But since we already classify all the Q-structures on the cotangent bundles we want also to make sense in terms of semidirec products of the twist version. In the case of the coadjoint representation up to homotopy we can incorporate the H-twist appearing in Section 6.3 in the following way:

Corollary 6.41. *Let $(A \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid and $H \in \Gamma \bigwedge^{k+1} A^*$. Given a connection on $A \to M$ consider the coadjoint representation up to homotopy as defined in Example 5.24. For any $k > 2$ the graded vector bundle*

$$\mathbf{A} = \begin{cases} A_0 = A, \\ A_{-k+1} = T^*M, \\ A_{-k+2} = A^*. \end{cases}$$

inherits an L_k-algebroid structure with brackets

$$\begin{cases} \rho = \rho, \quad \partial = \rho^*, \\ [\cdot,\cdot]_2 = [\cdot,\cdot], \quad \Phi = \nabla^1, \quad \Psi = \nabla^0, \\ [a,b,e]_3 = K(a,b)e, \\ [a_1,\cdots,a_k]_k = i_{a_k}\cdots i_{a_1}H, \\ [a_1,\cdots,a_{k+1}]_{k+1} = d(i_{a_{k+1}}\cdots i_{a_1}H) - \sum_{i=1}^{k+1}(-1)^{i+1}\langle D(a_i), i_{a_{k+1}}\cdots \widehat{i_{a_i}}\cdots i_{a_1}H\rangle. \end{cases}$$

We denote these L_k-algebroids by $A \ltimes_H (A^* \to T^*M)[k-1]$.

Proof. By Proposition 5.31 it remains to prove that the k and the $k+1$ brackets satisfy the Jacobi like identities. Given $H \in \Gamma \bigwedge^{k+1} A^*$ consider $H \otimes 1$ in the coadjoint complex. The last two equations on (5.6) are equivalent to $D^{ad^*}(H \otimes 1) = 0$ and if we remember the compatibility between D^{ad^*} and d_A we obtain

$$D^{ad^*}(H \otimes 1) = d_A H \otimes 1 + (-1)^{k+1} H \otimes D^{ad^*}(1) = d_A H,$$

so the k and $k+1$ brackets satisfy the Jacobi like identities iff $d_A H = 0$. $\square$

Corollary 6.42. *For any $k \in \mathbb{N}^*$, any connection on $A \to M$ induces a Q-manifold isomorphism*

$$(T^*[k+1]A[1], X_{\theta_H}) \cong \big((A \ltimes_H (A^* \to T^*M)[k])[1], Q\big).$$

6.4 AKSZ σ-models and integrating objects

6.4.1 AKSZ σ-models

In physics, a gauge theory is a field theory with a lagrangian action functional which is invariant under some symmetries. If the symmetries are not given by a Lie algebra, Batalin and Vilkovisky propose a way to enlarge the space of fields using supermanifolds that allows to compute some physically interesting quantities of the system.

In the celebrated paper [2] Alexandrov, Kontsevich, Schwarz and Zaboronsky show how to produce a BV theory with space of superfields the infinite dimensional $\mathbb{Z}$-graded manifold $Maps(T[1]\Sigma, \mathcal{M})$, where $\mathcal{M}$ is a symplectic Q-manifold of degree k and Σ a $k+1$ dimensional smooth manifold, see [101] for more details. The gauge theories that are constructed in this way are known as AKSZ σ-models and examples of them include the A and B models, the Poisson σ-model, Chern-Simons theory among others.

As a consequence, we have that our manifolds $(T^*[k]A[1], \{\cdot,\cdot\}, X_{\theta_H})$ can be considered as targets of AKSZ σ-models with source $k+1$ dimensional manifolds. For some cases these gauge theories were already considered: when $A \to M$ is just a Lie algebra, it is called BF theory and has been extensively studied in [32, 33, 38]. When $A = TM$ and $k = 2$, it corresponds to the standard Courant algebroid and this particular example was called the open topological membrane and studied in [66]. The general $k = 2$ fits in the Courant σ-model as described in [109], but not many explicit computations have been done.

Following [36], we obtain that for any manifold Σ, possibly with boundary, of dimension $k+1$ the space $Maps(T[1]\Sigma, T^*[k]A[1])$ defines a BV-BFV theory using the AKSZ construction, see [36, section 6]. Now, what we impose on the boundary is that the image of any boundary component takes values on a lagrangian Q-submanifold, or what is equivalent by Theorem 6.36 on a higher Dirac structure. For more details about this see [36, Section 3.7].

In forthcoming works, we plan to develop the full theory for this kind of σ-models and compute perturbative topological invariants, associated to Lie algebroids, for manifolds of any dimension using the techniques of [37].

Remark 6.43. Comparing to other cases, note that the Topological open k-brane of [66] has a different target manifold. For example when $k = 3$, the target is the manifold $T^*[3]T^*[2]T[1]M$, which is bigger than $T^*[3]T[1]M$. Also the Nambu σ-model proposed in [15] has a different target manifold, in this case $T^*[k]\big((\bigwedge^{k-1} T)[k-1]T[1]M\big)$. In the AKSZ σ-model proposed here the Nambu geometry appears naturally as boundary contributions not in the action, as in [15].

6.4.2 Speculations on the integration of Q-manifolds

The works of Ševera [114] and [115] propose that degree k Q-manifolds integrate to Lie k-groupoids. Lie k-groupoids are understood here as simplicial manifolds that satisfy the Kan condition for any $l > k$.

The manifolds $(T^*[k]A[1], \{\cdot,\cdot\}, \theta_H)$ are examples of Q-manifolds and we can apply Ševera's procedure to integrate them. In particular, for $k = 2$ we are discussing the integration of the double of a Lie algebroid seen as a Courant algebroid; this integration is not completely understood yet but some particular cases have been treated in the literature, see [83, 99, 116]. The integrations of lagrangians Q-submanifolds of $(T^*[2]T[1]M, \{\cdot,\cdot\}, \theta_H)$ include the integrations of Dirac structures to lagrangian subgroupoids as shown in [99], that are expected to correspond to presymplectic groupoids as in [25]. For $k > 2$ what happens should be similar to this case.

The interesting consequence is that now we have a place where Nambu structures must be integrated. As Poisson structures integrate to symplectic groupoids that corresponds to lagrangian subgroupoids inside symplectic 2-groupoids, Nambu structures should integrate to lagrangian k-1-subgroupoid of symplectic k-groupoids. Since Nambu structures just define weak lagrangians the lagrangian k-1-subgroupoid could be singular.

Remark 6.44. Another approach to integration in our particular case could be using the fact that we have a semidirect product. We could try to integrate first the algebroid and after that a cocycle given by the representation; these techniques were used in [116] but then it is unknown how to obtain the symplectic structure.

Remark 6.45. Recently it was proposed by Laurent-Gengoux and Wagemann that Leibniz algebroids will be integrated to Lie rackoids [80]. Nambu-Dirac structures are examples of Leibniz algebroids, see [64]. So it would be interesting to compare, if possible, this rackoid integration with the one proposed here.

Chapter 7

IM-Dirac structures

Courant algebroids and their Dirac structures have been an active area of research in Poisson an related geometries in recent years. The work [20] introduced the concept of *multiplicative Dirac structures*, that roughly speaking are Dirac structures on Lie groupoids satisfying a compatibility condition. In [103] it was shown that multiplicative Dirac structures codify many important objects in Poisson geometry such as symplectic groupoids, Poisson groupoids, foliated groupoids...

More recently, the works [72, 81, 82, 104] have proposed new viewpoints to these objects relating them to many interesting geometries. In particular, in [82] it is proposed a relation between Courant groupoids and multiplicative Dirac structures with degree 2 symplectic Q-groupoids and lagrangian Q-subgroupoids.

The question of giving an infinitesimal description of multiplicative Dirac structure has been addressed in [72, 103]. Here we follow the idea that such infinitesimal object corresponds to coisotropic Q-submanifolds of a degree 2 PQ-manifold and use the geometrization of Chapter 3 to give a new infinitesimal description. We call the resulting object an *IM-Dirac structure*.

We also explain how given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$, Lie bialgebroids [90], IM 2-forms [21] and IM-distributions [51, 74] fit in our picture. In the case of the standard Courant algebroid over a groupoid we in fact prove an integration result, see Theorem 7.20, establishing a correspondence between IM-Dirac structures on Lie algebroids and multiplicative Dirac structures on Lie groupoids.

Finally, in the last section of this chapter we offer more details on how symplectic Q-groupoids fit in this framework.

7.1 Degenerate Courant algebroids and IM-Dirac structures

In Chaper 3 we saw the geometrization of degree 2 PQ-manifolds and their coisotropic Q-submanifolds. Since they are the central object of this chapter we will give them a name.

A *degenerate Courant algebroid* is a collection of data $\mathfrak{D}_{ca} = (E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \hookrightarrow E_2, \langle\cdot,\cdot\rangle, \nabla, [\cdot,\cdot], \rho, F_0, F_1, F_2)$ such that

- $E_1 \to M,\ E_2 \to M$ are vector bundles and $\psi : E_1 \wedge E_1 \hookrightarrow E_2$ is an injective vector bundle map.

- $(E_2 \to M, [\cdot,\cdot], \rho)$ is a Lie algebroid.
- $E_1 \to M$ carries a (possibly degenerate) symmetric pairing $\langle\cdot,\cdot\rangle : \Gamma E_1 \times \Gamma E_1 \to C^\infty(M)$ and a flat E_2-connection $\nabla : \Gamma E_2 \times \Gamma E_1 \to \Gamma E_1$.
- $F_0 : T^*M \to E_1$ is a vector bundle map.
- $F_1 : \Gamma E_1 \to \Gamma E_2$ and $F_2 : \Gamma E_2 \to \Gamma E_3$, where E_3 is the vector bundle defined by equation (3.11).

and they satisfy the following equations:

$$A)\begin{cases} 1.)\ [\xi, \psi(e_1 \wedge e_2)] = \psi\Big(\nabla_\xi e_1 \wedge e_2 + e_1 \wedge \nabla_\xi e_2\Big), \\ 2.)\ \rho(\xi)\langle e_1, e_2\rangle = \langle \nabla_\xi e_1, e_2\rangle + \langle e_1, \nabla_\xi e_2\rangle, \\ 3.)\ \nabla_{\psi(e_1\wedge e_2)} e_3 = \langle e_2, e_3\rangle e_1 - \langle e_1, e_3\rangle e_2; \end{cases}$$

$$B)\begin{cases} 4.)\ F_1(fe_1) = \psi(F_0(df) \wedge e_1) + fF_1(e_1), \\ 5.)\ F_2(f\xi) = p_3(F_0(df) \otimes \xi) + fF_2(\xi), \\ 6.)\ F_2(\psi(e_1 \wedge e_2)) = p_3\Big(e_2 \otimes F_1(e_1) - e_1 \otimes F_1(e_2)\Big), \\ 7.)\ F_1 \circ F_0 = 0, \quad 8.)\ F_2 \circ F_1 = 0, \quad 9.)\ F_3 \circ F_2 = 0; \end{cases} \tag{7.1}$$

$$C)\begin{cases} 10.)\ \langle F_0 f, e_1\rangle = \rho(F_1 e_1)(f), \\ 11.)\ F_0\rho(\xi)(f) = \nabla_\xi F_0 f - \mathcal{L}_f F_2\xi, \\ 12.)\ F_0\langle e_1, e_2\rangle = \nabla_{F_1 e_1} e_2 + \nabla_{F_1 e_2} e_1, \\ 13.)\ F_1(\nabla_\xi e_1) = [\xi, F_1 e_1] + \mathcal{L}_{e_1} F_2 \xi, \\ 14.)\ F_2[\xi, \xi'] = \mathcal{L}_\xi F_2 \xi' - \mathcal{L}_{\xi'} F_2 \xi; \end{cases}$$

where $f \in C^\infty(M)$, $e_1, e_2, e_3 \in \Gamma E_1$, $\xi, \xi' \in \Gamma E_2$, the Lie derivative is defined by Proposition 3.46 and $F_3 : \Gamma E_3 \to \Gamma E_4$ is given as in Proposition 3.27.

An *IM-Dirac structure on a degenerate Courant algebroid* given by $\mathfrak{D}_{ca}$ is a triple $(N, K_1 \to N, K_2 \to N)$, where $N \subseteq M$ is a submanifold, $K_1 \subseteq E_{1|N}$ is a subbundle and $K_2 \subseteq E_{2|N}$ is a subbundle satisfying:

$$\begin{cases} I) & K_2 \cap \mathrm{im}(\psi)_{|N} = \psi(K_1 \wedge E_{1|N}), \\ II) & \langle K_1, K_1\rangle = 0, \\ III) & K_2 \text{ is a subalgebroid of } E_2, \\ IV) & \nabla_{\Gamma K_2}(\Gamma K_1) \subseteq \Gamma K_1, \\ V) & F_0(TN^\circ) \subseteq E_1, \\ VI) & F_1(\Gamma K_1) \subseteq \Gamma K_2, \\ VII) & F_2(\Gamma K_2) \subseteq \Gamma p_3(K_1 \otimes E_{2|N} \oplus E_{1|N} \otimes K_2), \end{cases} \tag{7.2}$$

where $p_3 : E_1 \otimes E_2 \to E_3$ is the projection.

Therefore what we proved in Section 3.6.2 (see Theorem 3.47 and 3.48) can be reformulated in the following way:

Theorem 7.1. *There is a one to one correspondence between:*

$$\left\{ \begin{array}{c} \textit{Degenerate Courant algebroids} \\ \mathfrak{D}_{ca} \end{array} \right\} \leftrightarrows \left\{ \begin{array}{c} \textit{Degree 2 PQ-manifolds} \\ (\mathcal{M}, \{\cdot,\cdot\}, Q) \end{array} \right\}$$

Moreover this correspondence sends IM-Dirac structures to coisotropic Q-submanifolds.

In other words, degenerate Courant algebroids are geometric versions of degree 2 PQ-manifolds, while IM-Dirac structures geometrize coisotropic Q-submanifolds. The terminology IM-Dirac structure (where IM stands for "Infinitesimally Multiplicative") will be explained in the next section.

Remark 7.2. In [71] the terminology of degenerate Courant algebroid is used for "Courant algebroids" where the metric can be degenerate, see also Remark 7.12 for the connection between these two notions.

In Proposition 3.18 we saw that for 2-manifolds there is a second way of defining submanifolds. What we do next is to characterize IM-Dirac structures using this second approach. For that, we need the following auxiliary results:

Lemma 7.3. *Consider* $(E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \hookrightarrow E_2)$, *where* $(E_2 \to M, [\cdot,\cdot], \rho)$ *is a Lie algebroid and* E_1 *has a paring* $\langle\cdot,\cdot\rangle$ *and a flat* E_2*-connection* ∇ *satisfying equations 1), 2) and 3) of* (7.1). *Then* $\mathrm{im}(\psi)$ *is an ideal inside* $E_2 \to M$.

Proof. It follows directly from the fact that the bracket satisfies

$$[\xi, \psi(e_1 \wedge e_2)] = \psi(\nabla_\xi e_1 \wedge e_2 + e_1 \wedge \nabla_\xi e_2), \quad \rho(\psi(e_1 \wedge_2)) = 0.$$

□

Lemma 7.4. *Let* $(E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \hookrightarrow E_2)$, *where* $(E_2 \to M, [\cdot,\cdot], \rho)$ *is a Lie algebroid and* E_1 *has a paring* $\langle\cdot,\cdot\rangle$ *and a flat* E_2*-connection* ∇ *satisfying equations 1), 2) and 3) of* (7.1). *Consider* $K_1 \to N$ *subbundle of* $E_1 \to M$. *Suppose that* $\langle K_1, K_1\rangle = 0$ *and define*

$$O_{K_1,N} = \{\xi \in \Gamma E_{2|N} \mid \rho(\xi) \in \Gamma TN,\ \nabla_\xi(\Gamma K_1) \subseteq \Gamma K_1\}.$$

Then $O_{K_1,N} \to N$ *is a subalgebroid of* $(E_2 \to M, [\cdot,\cdot], \rho)$; *moreover* $K_1 \wedge E_{1|N} \to N$ *is an ideal inside* $O_{K_1,N}$.

Proof. Since ∇ is a flat connection and $\rho([\xi_1, \xi_2]) = [\rho(\xi_1), \rho(\xi_2)]$ then clearly we have that $O_{K_1,N}$ is a Lie subalgebroid of $E_2 \to M$. Remains to see that $K_1 \wedge E_{1|N} \to N$ is an ideal inside $O_{K_1,N}$. Observe that $K_1 \wedge E_1 \subseteq E_1 \wedge E_1$ and we know that $\rho(E_1 \wedge E_1) = 0$ by Lemma 7.3. Finally

$$\nabla_{\psi(k\wedge e)} k' = \langle e, k'\rangle k - \langle k, k'\rangle e = \langle e, k'\rangle k \in \Gamma K_1.$$

□

Corollary 7.5. *Let* $\mathfrak{D}_{ca}$ *be a degenerate Courant algebroid. An IM-Dirac structure is equivalent to* $(K_1 \to N, K \to N, \phi : K \to \frac{E_{2|N}}{K_1 \wedge E_{1|N}})$, *where* $K_1 \subseteq E_1$, $K \subseteq \widetilde{E}_2 = \frac{E_2}{\mathrm{im}(\psi)}$ *are subbundles such that:*

$$\begin{cases} K \to N \text{ is a subalgebroid of } \widetilde{E}_2 \to M, \quad \langle K_1, K_1\rangle = 0, \\ \phi : K \to \frac{O_{K_1,N}}{K_1 \wedge E_{1|N}} \text{ is a Lie algebroid morphism,} \\ F_0(TN^\circ) \subseteq E_1, \quad \pi(F_1(K_1)) \subseteq \mathrm{im}(\phi), \quad F_2(K_2) \subseteq p_3(K_1 \otimes E_2 \oplus E_1 \otimes K_2), \end{cases}$$

where $p_3 : E_1 \otimes E_2 \to E_3$ *is the natural projection,* $\pi : E_2 \to \frac{E_2}{K_1 \wedge E_1}$ *and* $K_2 = \pi^{-1}(\mathrm{im}(\phi))$, *and* $O_{K_1,N}$ *is defined as in the previous lemma.*

Proof. By Lemma 7.3 we have that $\widetilde{E}_2 = E_2/\operatorname{im}(\psi)$ is a Lie algebroid therefore using the definition of an IM-Dirac structure the only thing that we need to prove is that K_2 being a subalgebroid is equivalent to K being a subalgebroid and ϕ an algebroid morphism.

By Lemma 7.4 ϕ is well defined because K_1 is isotropic. Hence it is clear that K_2 is a Lie algebroid if and only if K is a Lie algebroid and ϕ is a Lie algebroid morphism. □

Example 7.6 (Degenerate Courant algebroid over a point). Suppose that our degenerate Courant algebroid has $M = *$. Then the two vector bundles become vector spaces V_1 and V_2; moreover we can identify $V_2 = \bigwedge^2 V_1 \oplus \widetilde{V}_2$ for some other vector space $\widetilde{V}_2$.

With this we obtain the following structures:

- $\widetilde{V}_2^* \to V_1^*$ is an L_2-algebra as in Example 3.36.
- $V_1 \to \widetilde{V}_2$ is also an L_2-algebra with $\partial = 0$ and $[\cdot,\cdot,\cdot] \equiv 0$.
- Both structures are compatible.

These structures are known as *Lie* 2 *bialgebras*, see [4, 42], and are the infinitesimal counterpart of *quasi-Poisson* 2*-groups*, see [42].

Example 7.7 (Courant algebroids as degenerate Courant algebroids). Here we will describe how given a Courant algebroid $(E \to M, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], a)$ we can produce a degenerate Courant algebroid. Recall that we need $(E_1 \to M, E_2 \to M, \psi : E_1 \wedge E_1 \hookrightarrow E_2, \langle\cdot,\cdot\rangle, \nabla, [\cdot,\cdot], \rho, F_0, F_1, F_2)$. The correspondence is as follows:

a). $M = M,\ E_1 = E$ and $E_2 = \mathbb{A}_{(E,\langle\cdot,\cdot\rangle)}$.

b). $\psi : E_1 \wedge E_1 \to E_2$ is given by the natural inclusion $\mathfrak{so}(E) \to \mathbb{A}_{(E,\langle\cdot,\cdot\rangle)}$.

c). The pairing on E_1 is given by the pairing on the Courant algebroid $\langle\cdot,\cdot\rangle$.

d). The E_2-connection on E_1 is given by $\nabla : \Gamma\mathbb{A}_{(E,\langle\cdot,\cdot\rangle)} \times \Gamma E \to \Gamma E, \quad \nabla_{(D,\sigma)} e = D(e)$.

e). $(E_2 \to M, [\cdot,\cdot], \rho)$ is the usual Lie algebroid structure of the Atiyah algebroid.

f). The vector bundle map $F_0 = a^{\flat *}$, and the map $F_1 : \Gamma E \to \Gamma\mathbb{A}_{(E,\langle\cdot,\cdot\rangle)}, \quad F_1(e) = [\![e,\cdot]\!]$.

g). Finally, the map $F_2 : \Gamma\mathbb{A}_{(E,\langle\cdot,\cdot\rangle)} \to \Gamma E_3$, is given by the following formula

$$\begin{aligned} F_2(D,\sigma)(e_0,e_1,e_2) &= a(e_0)\langle D(e_1), e_2\rangle - a(e_1)\langle D(e_0), e_2\rangle + a(e_2)\langle D(e_0), e_1\rangle \\ &\quad -\langle D([\![e_0,e_1]\!]), e_2\rangle - \langle D(e_1), [\![e_0,e_2]\!]\rangle + \langle D(e_0), [\![e_1,e_2]\!]\rangle, \end{aligned}$$

where $e_0, e_1, e_2 \in \Gamma E_1 = \Gamma E$.

This example justifies the terminology "degenerate" Courant algebroid: recall that by Theorem 3.50 we obtain a correspondence between degree 2 symplectic Q-manifolds and Couran algebroids. Therefore if we "degenerate" the Poisson bracket we obtain a degenerate Courant algebroid.

7.1.1 LA-Courant algebroids and morphic Dirac structures

It was explained in [82] that degree 2 PQ-manifolds have also an interpretation in terms of double vector bundles. We explain this other geometrization in Section B.3. Here we recall and explain how the interaction with the previous geometrization gives interesting new descriptions.

Let us start by introducing the main definitions, see also Section B.3. Let $((D; A, B; M), \langle\cdot,\cdot\rangle)$ be a metric double vector bundle, meaning that the double vector bundle map $\langle,\rangle^b : (D; A, B; M) \to (D^{\uparrow}; A, C^*; M)$ is an isomorphism. Then we can define the following objects:

- We say that $(D; A, B; M)$ is a *VB-Courant algebroid* (see [82]) if $(D \to A, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], a)$ is a Courant algebroid satisfying:
  - $a : (D; A, B; M) \to (TA; A, TM; M)$ is a double vector bundle morphism.
  - $[\![\Gamma_l(D, A), \Gamma_l(D, A)]\!] \subseteq \Gamma_l(D, A)$, $[\![\Gamma_l(D, A), \Gamma_c(D)]\!] \subseteq \Gamma_c(D)$ and $[\![\Gamma_c(D), \Gamma_c(D)]\!] = 0$.
- We say that $(D; A, B; M)$ is a *metric VB-algebroid* (see [48, 70]) if $(D \to B, [\cdot,\cdot], \rho)$ is a Lie algebroid satisfying:
  - $\rho : (D; A, B; M) \to (TB; TM, B; M)$ is a double vector bundle morphism.
  - $[\Gamma_l(D, B), \Gamma_l(D, B)] \subseteq \Gamma_l(D, B)$, $[\Gamma_l(D, B), \Gamma_c(D)] \subseteq \Gamma_c(D)$ and $[\Gamma_c(D), \Gamma_c(D)] = 0$.
  - $\langle,\rangle^b : (D; A, B; M) \to (D^{\uparrow}; A, C^*; M)$ is a Lie algebroid isomorphism.
- We say that $(D; A, B; M)$ is an *LA-Courant algebroid* (see [82]) if
  - $((D; A, B; M), \langle\cdot,\cdot\rangle)$ is a VB-Courant algebroid.
  - $((D; A, B; M), \langle\cdot,\cdot\rangle)$ is a metric VB-algebroid.
  - Both structures are compatible.

Let $(D; A, B; M)$ be an LA-Courant algebroid and $(L; K, F; N)$ a double vector subbundle of $(D; A, B; M)$. We say that $(L; K, F; N)$ is a *morphic Dirac structure of* $((D; A, B; M), \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], a, [\cdot,\cdot], \rho)$ if

- $L \to K$ is a Dirac structure of $D \to A$.
- $L \to F$ is a Lie subalgebroid of $D \to B$.

Remark 7.8. The compatibility between the VB-Courant algebroid structure and the metric VB-algebroid was given in [82] as a compatibility between triple vector bundles that most of the time is difficult to check. In [73], Jotz gave a different version of the compatibility by looking into the split case, i.e. a compatibility between Lie 2-algebroids and 2-term representations up to homotopy. Here we will assume that the compatibility is by definition the degree 1 vector field being Poisson.

Remark 7.9. Morphic Dirac structures were introduced in [104], in [71, 82] they were called LA-Dirac structures. We prefer to keep the terminology of [104] to emphasize the connection with multiplicative Dirac structures.

Theorem 7.10 (see [82]). *There is a one to one correspondence between:*

$$\left\{ \begin{array}{c} \textit{LA-Courant algebroids} \\ ((D;A,B;M),\langle\cdot,\cdot\rangle,[\![\cdot,\cdot]\!],a,[\cdot,\cdot],\rho) \end{array} \right\} \leftrightarrows \left\{ \begin{array}{c} \textit{Degree 2 PQ-manifolds} \\ (\mathcal{M},\{\cdot,\cdot\},Q) \end{array} \right\}$$

Moreover this correspondence sends morphic Dirac structures to coisotropic Q-submanifolds.

Proof. By Theorem 3.29 we know that there is a correspondence between degree 2 Q-manifolds and VB-Courant algebroids. In Theorem B.26 together with Proposition B.25 we re-prove the correspondence between metric VB-algebroids and degree 2 Poisson manifolds (see [48, 70]). Hence as we are considering the compatibility between the structures as the vector field Q being Poisson we obtain a correspondence between degree 2 PQ-manifolds and LA-Courant algebroids.

For the moreover part observe that Theorem 3.29 also establish a correspondence between Q-submanifolds and VB-Dirac structures. Therefore it just remains to see that the submanifold is coisotropic if and only if the double vector subbundle associated to it is a Lie subalgebroid.

As it happens for Theorem B.26 there is a conceptual way of proving this just by seeing that something is a subalgebroid if and only if the annihilator is coisotropic and coisotropic goes to coisotropic.

The more hands on proof is as follows: by Proposition 3.39 a degree 2 Poisson manifold is the same as $(E_1 \to M, E_2 \to M, \langle\cdot,\cdot\rangle, \nabla, [\cdot,\cdot], \rho)$ satisfying equations A) of (7.1). Theorem 3.48 says that a coisotropic submanifold is the same as $(K_1 \to N, K_2 \to N)$ satisfying that K_2 is a subalgebroid of E_2, K_1 is isotropic for the pairing and $\nabla_{\Gamma K_2}\Gamma K_1 \subseteq \Gamma K_1$. Hence we want to see that $(L;K,K_1^\circ;N)$ is a VB-subalgebroid of $(D;\widetilde{E}_2,E_1^*;M)$ if and only if $(K_1 \to N, K_2 \to N)$ is a coisotropic submanifold. But this follows from the fact that $\Gamma_l(L,K_1^\circ) = K_2$ and $\Gamma_c(L) = K_1$, hence equations (B.18) means that $(K_1 \to N, K_2 \to N)$ is coisotropic if and only if $(L;K,K_1^\circ;N)$ is a VB-subalgebroid. □

Hence just by putting together Theorems 7.1 and 7.10 we have the following

Corollary 7.11. *There is a one to one correspondence between LA-Courant algebroids and degenerate Courant algebroids. Moreover this correspondence sends morphic Dirac structures to IM-Dirac structures.*

Summarizing, what we proved up to now can be sketched in the following diagram:

$$\left\{ \begin{array}{c} \text{LA-Courant algbds.} \\ \text{Morphic Dirac} \end{array} \right\} \leftrightarrows \left\{ \begin{array}{c} \text{Degree 2 PQ-manifolds} \\ \text{Coisotropic Q-subman.} \end{array} \right\} \leftrightarrows \left\{ \begin{array}{c} \text{Degen. Courant algbds.} \\ \text{IM-Dirac} \end{array} \right\}$$

Remark 7.12. It is shown in [71] that the core of an LA-Courant algebroid inherits an anchor a bracket and a pairing $(C \to M, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], \rho)$ that satisfies the axioms of a Courant algebroid except that the pairing is degenerate. In order to obtain a formula for the bracket, they pick a particular decomposition and proves that the formula does not depend on the decomposition. From our viewpoint this structure can be read more easily: Given an LA-Courant algebroid we have a Degenerate Courant algebroid $\mathfrak{D}_{ca}$, the core can be identify with $E_1 \to M$, the pairing is exactly the pairing in E_1 and the bracket and anchor are defined by the following formulas

$$[\![e, e']\!] = \nabla_{F_1(e)} e', \qquad \rho(e) = \rho(F_1(e)) \tag{7.3}$$

for $e, e' \in \Gamma E_1$. In terms of the degree 2 PQ-manifold the bracket is written as the derived bracket $[\![e, e']\!] = \{Q(e), e'\}$ where $e, e' \in C^1_{\mathcal{M}}$.

From this point of view, it is clear that the pairing, the anchor and this bracket are just codifying part of the information associated with the degenerate Courant algebroid $\mathfrak{D}_{ca}$. Therefore they are not enough to characterize LA-Courant algebroids.

7.2 Example: $T[1]A^*[1]$

We start defining the degree 2 manifold $T[1]A^*[1] = (M, C_{T[1]A^*[1]})$. Recall from Section A.2 that given a graded manifold we can define the bigraded complex of differential forms over it. Denote by $\Omega^{\bullet,\bullet}(A^*[1])$ the forms of $A^*[1]$. By the way we define $\Omega^{\bullet,\bullet}(A^*[1])$ it is clear that there exists a natural isomorphism with $\Omega^{\bullet,\bullet}(A^*)$, where here the first degree is the De Rham degree and the second the linearity with respect to the vector bundle structure of $A^* \to M$.

We define the degree 2 manifold $T[1]A^*[1] = (M, C_{T[1]A^*[1]})$ by

$$C^k_{T[1]A^*[1]} = \bigoplus_{i+j=k} \Omega^{i,j}(A^*[1]) = \bigoplus_{i+j=k} \Omega^{i,j}(A^*).$$

It is well known, see e.g. [59], that $\Omega^{\bullet,\bullet}(A^*)$ is generated as an algebra by

$$\Omega^{0,0}(A^*) = C^\infty(M), \quad \Omega^{1,0}(A^*) = \Gamma T^*M, \quad \Omega^{0,1}(A^*) = \Gamma A \quad \text{and} \quad \Omega^{1,1}(A^*) = \Gamma J^1 A,$$

where $J^1A \to M$ denotes the first jet bundle of the vector bundle A. An important fact of the first jet bundle is that it fits into the exact sequence

$$0 \to \mathrm{Hom}(TM, A) \xrightarrow{\iota} J^1 A \to A \to 0. \tag{7.4}$$

Now clearly we have the following identities for functions on $T[1]A^*[1]$:

$$\begin{cases} C^0_{T[1]A^*[1]} = C^\infty(M), \\ C^1_{T[1]A^*[1]} = \Omega^{1,0}(A^*) \oplus \Omega^{0,1}(A^*) = \Gamma(T^*M \oplus A), \\ C^2_{T[1]A^*[1]} = \Omega^{2,0}(A^*) \oplus \Omega^{1,1}(A^*) \oplus \Omega^{0,2}(A^*) = \Gamma\left(\bigwedge^2 T^*M \oplus J^1A \oplus \bigwedge^2 A\right) \end{cases}$$

Therefore it follows that the geometrization of the manifold $T[1]A^*[1]$ in terms of algebra bundles is just given by

$$\begin{cases} E_1 = A \oplus T^*M, \\ E_2 = \bigwedge^2 A \oplus J^1 A \oplus \bigwedge^2 T^*M, \\ \psi : E_1 \wedge E_1 \to E_2, \quad \psi = \mathrm{Id} \oplus \iota \oplus \mathrm{Id}, \end{cases} \tag{7.5}$$

where $E_1 \wedge E_1 = \bigwedge^2 A \oplus A \otimes T^*M \oplus \bigwedge^2 T^*M$ and ι is the map given in (7.4).

Since our manifold is a tangent bundle shifted by 1, we see in Example 2.13 that it has a natural Q-structure given by the De Rham vector field Q_{dr}. By Proposition 3.27 it is equivalent to three maps $F_0 : T^*M \to E_1$, $F_1 : \Gamma E_1 \to \Gamma E_2$ and $F_2 : \Gamma E_2 \to \Gamma E_3$ where E_3 is the vector bundle defined by equation (3.11).

Since E_3 is a quotient, we will define F_2 upstairs and see that it descends to the quotient. In this case, the three maps equivalent to the De Rham vector field are given by:

$$\begin{cases} F_0 : T^*M \to A \oplus T^*M, \quad F_0(\omega) = 0 + \omega, \\ F_1 : A \oplus T^*M \to \bigwedge^2 A \oplus J^1A \oplus \bigwedge^2 T^*M, \quad F_1(a+\omega) = 0 + j^1a + d\omega, \\ F_2 : E_2 \to E_3, \quad F_2(a \wedge b + j^1a' + \eta) = p_3(j^1a \wedge b - a \wedge j^1b + d\eta), \end{cases} \tag{7.6}$$

where $a, b, a' \in \Gamma A$, $\omega \in \Omega^1(M)$, $\eta \in \Omega^2(M)$.

Suppose that $(A \to M, [\cdot,\cdot], \rho)$ is a Lie algebroid. Then we already see that the 1-manifold $A^*[1]$ inherits a graded Poisson bracket of degree -1. The tangent lift of the Poisson structure on $A^*[1]$ induces a degree -2 Poisson bracket on $T[1]A^*[1]$, that we gave in Example 3.42 using coordinates. Moreover, since it is a tangent lift, we have that the De Rham vector field is Poisson. Therefore, given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ we have an associated degree 2 PQ-manifold given by $(T[1]A^*[1], \{\cdot,\cdot\}, Q_{dr})$.

Let us now describe the Poisson bracket in more geometric terms. The degree -2 Poisson bracket of $T[1]A^*[1]$ is equivalent to the following structures:

$$\begin{cases} \text{- A pairing in } E_1 : \langle a+\omega, b+\eta\rangle = i_{\rho(a)}\eta + i_{\rho(b)}\omega, \\ \text{- A bracket in } E_2 : \ [j^1a, j^1b] = j^1[a,b], \\ \text{- The anchor: } \rho : E_2 \to TM, \ \rho(a\wedge b + j^1a' + \eta) = \rho(a'), \\ \text{- An } E_2\text{-connection on } E_1 : \nabla_{j^1a} b + \omega = [a,b] + \mathcal{L}_{\rho(a)}\omega, \end{cases} \tag{7.7}$$

where all the other possible combinations are determined by the equations A) in (7.1).

It is easy to see that the maps defined by (7.6) and the structures defined by (7.7) satisfy the compatibility equations C) in (7.1) as we want. Therefore given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ we have that

$$(A \oplus T^*M \to M, \left(\bigwedge^2 A \oplus J^1A \oplus \bigwedge^2 T^*M\right) \to M, \psi, \langle\cdot,\cdot\rangle, \nabla, [\cdot,\cdot], \rho, \mathrm{Id}, F_1, F_2)$$

defines a degenerate Courant algebroid that we refer to as the *standard degenerate Courant algebroid associated to* $(A \to M, [\cdot,\cdot], \rho)$ and it will be denoted by $\mathfrak{D}_{ca}(A)$.

In order to complete the picture we also describe the LA-Courant algebroid that corresponds to the degree 2 PQ-manifold $(T[1]A^*[1], \{\cdot,\cdot\}, Q_{dr})$.

As an immediate consequence of Theorem B.15 we obtain that the graded manifold $T[1]A^*[1]$ corresponds to the metric double vector bundle

$$\begin{array}{ccc} TA \oplus T^*A & \longrightarrow & TM \oplus A^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$$

where the metric is given by the natural pairing between the tangent and cotangent bundles of $A \to M$.

We know by Theorem 3.29 that the De Rham vector field must produce a VB-Courant algebroid structure on the double vector bundle $(TA \oplus T^*A; A, TM \oplus A^*; M)$ and this VB-Courant structure is just the standard structure on the tangent plus cotangent of a manifold. It was proved in [82] that this is, in fact, a VB-Courant algebroid.

Finally, the Poisson structure of $T[1]A^*[1]$ given by the Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ must induce a VB-algebroid structure on $(TA \oplus T^*A; A, TM \oplus A^*; M)$. But it is well known that given a Lie algebroid then the tangent and cotangent prolongations have a VB-algebroid structure, see e.g. [59]. Therefore their sum have also a VB-algebroid structure and by the duality between tangent and cotangent bundles it is easy to see that it respects the metric. Hence this is the LA-Courant algebroid structure of $(TA \oplus T^*A; A, TM \oplus A^*; M)$.

Remark 7.13. The structures that we defined by equations (7.7) are a skew-symmetric version of the fat algebroids and their fat representations as defined in [59]. The derived bracket defined by equation (7.3) in this case is the well known bracket

$$[\![a+\omega, b+\eta]\!] = [a,b] + \mathcal{L}_{\rho(a)}\eta - i_{\rho(b)}d\omega$$

where $a, b \in \Gamma A,\ \omega, \eta \in \Omega^1(M)$, see also [71].

Summarizing, given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ we have associated a degree 2 PQ-manifold $(T[1]A^*[1], \{\cdot,\cdot\}, Q_{dr})$. This graded manifold corresponds to the standard degenerate Courant algebroid $\mathfrak{D}_{ca}(A)$ and also to the LA-Courant algebroid $(TA \oplus T^*A; A, TM \oplus A^*; M)$. Therefore both structures are related and moreover IM-Dirac structures on $\mathfrak{D}_{ca}(A)$ correspond to morphic Dirac strucuter on $(TA \oplus T^*A; A, TM \oplus A^*; M)$.

Now we give examples of IM-Dirac structures and their corresponding morphic Dirac structures. For that recall that an IM-Dirac structure of $\mathfrak{D}_{ca}(A)$ is given by a triple (N, K_1, K_2) where

$$\begin{cases} N \subseteq M, \\ K_1 \subseteq (A \oplus T^*M)_{|N}, \\ K_2 \subseteq \left(\bigwedge^2 A \oplus J^1 A \oplus \bigwedge^2 T^*M\right)_{|N}. \end{cases}$$

satisfying equations (7.2) while a morphic Dirac structure of $(TA \oplus T^*A; A, TM \oplus A^*; M)$ is the same as

$$(L; K, F, N) \subseteq (TA \oplus T^*A; A, TM \oplus A^*; M)$$

double vector subbundle such that $L \to K$ is a Dirac structure of $TA \oplus T^*A$ and $L \to F$ is a subalgebroid of $TA \oplus T^*A \to TM \oplus A^*$.

Example 7.14 (2-differentials). Given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$, a 2-differential is a map $\delta : \Gamma \bigwedge^\bullet A \to \Gamma \bigwedge^{\bullet+1} A$ satisfying

$$\delta(a \wedge b) = \delta(a) \wedge b + (-1)^{|a|} a \wedge \delta(b), \quad \delta^2 = 0, \quad \delta([a,b]) = [\delta(a), b] + (-1)^{|a|-1}[a, \delta(b)]$$

where $[\cdot,\cdot]$ denotes the Gerstenhaber bracket of the Lie algebroid. Equivalently, $(A \to M, [\cdot,\cdot], \rho, \delta)$ is a Lie bialgebroid. These objects were introduced in [90] and shown in [92] to be the infinitesimal counterparts of Poisson groupoids.

In graded terms, δ is a degree 1 Poisson vector field on the graded manifold $(A^*[1], \{\cdot,\cdot\})$ that is integrable, i.e. $\delta^2 = 0$. Hence we can seen it as a graded manifold morphism

$$\delta : A^*[1] \to T[1]A^*[1].$$

For smooth manifolds, it is well known that given a Poisson manifold (M, π) and a vector field $X \in \mathfrak{X}(M)$, then X is a Poisson vector field if and only if $\mathrm{im}(X) \subseteq TM$ defines a coisotropic submanifold for the tangent Poisson structure on TM, see [91].

We claim that, in our case the same holds; moreover the condition $\delta^2 = 0$ is equivalent to Q_{dr} being tangent to $\mathrm{im}(\delta) \subseteq T[1]A^*[1]$.

More concretely, given $\delta : \bigwedge^\bullet \Gamma A \to \bigwedge^{\bullet+1} \Gamma A$ we define:

$$\begin{cases} N = M, \\ K_1 = graph(\delta_0) = \{\delta_0(f) + df \mid f \in C^\infty(M)\}, \\ K_2 = (K_1 \wedge E_1) \oplus \{\delta_1(a) + j^1(a) \mid a \in \Gamma A\}. \end{cases}$$

Let us check that this is an IM-Dirac structure of $\mathfrak{D}_{ca}(A)$, i.e. that equations (7.2) hold:

$$\begin{array}{rcl} K_2 \cap \mathrm{im}(\psi) = \psi(K_1 \wedge E_1) & \Leftrightarrow & \text{By construction}, \\ K_1 \text{ isotropic} & \Leftrightarrow & 0 = [\delta_0(f), g] - [f, \delta_0(g)], \\ K_2 \text{ subalgebroid} & \Leftrightarrow & \delta_1([a,b]) = [\delta_1(a), b] + [a, \delta_1(b)], \\ \nabla_{\Gamma K_2}\Gamma K_1 \subseteq \Gamma K_1 & \Leftrightarrow & \delta_0([a,f]) = [\delta_1(a), f] + [a, \delta_0(f)], \\ F_0(0) \subset K_1 & \Leftrightarrow & \text{Trivial in this case}, \\ F_1(\Gamma K_1) \subseteq \Gamma K_2 & \Leftrightarrow & \delta_1 \circ \delta_0 = 0, \\ F_2(\Gamma K_2) \subseteq \Gamma(K_1 \wedge E_2 \otimes E_1 \wedge K_2) & \Leftrightarrow & \delta_2 \circ \delta_1 = 0. \end{array}$$

Finally, we will describe the morphic Dirac structure on $(TA \oplus T^*A; A, TM \oplus A^*; M)$ associated with a 2-differential.

Since a 2-differential makes $(A \to M, [\cdot,\cdot], \rho, \delta)$ into a Lie bialgebroid, $(A^* \to M, [\cdot,\cdot]_\delta, \rho^*)$ is also a Lie algebroid and this implies that (A^*, π_δ) is a linear Poisson manifold. Therefore this Poisson structure defines a Dirac structure on $TA \oplus T^*A$; in fact, since it is linear Poisson, this Dirac structure is a double vector bundle over $graph(\rho^*)$. Hence $(graph(\pi_\delta); A, graph(\rho^*); M)$ defines a VB-Dirac structure. Finally recall from [90] that π_δ defines a Lie bialgebroid if and only if $\pi_\delta^\sharp : TA \to T^*A$ is a Lie algebroid morphism. So we have that $(graph(\pi_\delta); A, graph(\rho^*); M)$ is a morphic Dirac structure.

Example 7.15 (IM 2-forms). Given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$, an *IM 2-form* is given by two vector bundle maps $\mu : A \to T^*M$ and $\nu : A \to \bigwedge^2 T^*M$ satisfying

$$\begin{cases} i_{\rho(a)}\mu(b) = -i_{\rho(b)}\mu(a), \\ \mu([a,b]) = \mathcal{L}_{\rho(a)}\mu(b) - i_{\rho(b)}d\mu(a) - i_{\rho(b)}\nu(a), \\ \nu([a,b]) = \mathcal{L}_{\rho(a)}\nu(b) - i_{\rho(b)}d\nu(a). \end{cases}$$

These objects were defined in [21], where it was show that IM 2-forms are the infinitesimal counterparts of multiplicative 2-forms on a Lie groupoid. Moreover, they show that the 2-form is closed if and only if the map ν is zero.

Given (μ, ν) we define:

$$\begin{cases} N = M, \\ K_1 = graph(\mu) = \{a + \mu(a) \mid a \in \Gamma A\}, \\ K_2 = (K_1 \wedge E_1) \oplus \{j^1 a + d\mu(a) + \nu(a) \mid a \in \Gamma A\}. \end{cases}$$

Let us check that it is an IM-Dirac structure of $\mathfrak{D}_{ca}(A)$, i.e. that equations (7.2) hold:

$$\begin{array}{rcl} K_2 \cap \mathrm{im}(\psi) = \psi(K_1 \wedge E_1) & \Leftrightarrow & \text{By construction,} \\ K_1 \text{ isotropic} & \Leftrightarrow & i_{\rho(a)}\mu(b) = -i_{\rho(b)}\mu(a), \\ \nabla_{\Gamma K_2}\Gamma K_1 \subseteq \Gamma K_1 & \Leftrightarrow & \mu([a,b]) = \mathcal{L}_{\rho(a)}\mu(b) - i_{\rho(b)}d\mu(a) - i_{\rho(b)}\nu(a), \\ K_2 \text{ subalgebroid} & \Leftrightarrow & \nu([a,b]) = \mathcal{L}_{\rho(a)}\nu(b) - i_{\rho(b)}d\nu(a), \\ F_0(0) \subset K_1 & \Leftrightarrow & \text{Trivial in this case,} \\ F_1(\Gamma K_1) \subseteq \Gamma K_2 & \Leftrightarrow & \nu = 0, \\ F_2(\Gamma K_2) \subseteq \Gamma(K_1 \wedge E_2 \otimes E_1 \wedge K_2) & \Leftrightarrow & \text{Trivialy satisfied because } d^2 = 0. \end{array}$$

Therefore we see that an IM 2-form defines an IM-Dirac structure if and only if the 2-form is closed.

It was also shown in [21] that these two maps (μ, ν) correspond to $\omega \in \Omega^{2,1}(A)$, a linear two form in A. Therefore the graph of ω defines a Dirac structure in $TA \oplus T^*A$ if and only if $d\omega = 0$. In fact, since ω is linear we have that $(graph(\omega); A, graph(\mu^*); M)$ defines a double vector subbundle of $(TA \oplus T^*A; A, TM \oplus A^*; M)$ that is a morphic Dirac structure if and only if ω is closed and $\omega^\sharp : TA \to T^*A$ is an algebroid morphism over μ^*.

Example 7.16 (Spencer operator). Given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ it was shown in [47] that there is a correspondence between involutive IM-distributions covering TM and triples (K, ∇, T), where $K \subseteq A$ is subbundle, ∇ is a flat connection on A/K and $T : \wedge^2(A/K) \to (A/K)$ satisfying

$$p([a,b]) = \nabla_{\rho(a)}p(b) - \nabla_{\rho(b)}p(a) + T(p(a), p(b)), \quad \nabla_X T(e, e') = T(\nabla_X e, e') + T(e, \nabla_X e'),$$

for $a, b \in \Gamma A,\ X \in \mathfrak{X}(M),\ e, e' \in \Gamma(A/K)$ and where $p : A \to (A/K)$ denotes the natural projection. These data define a *Spencer operator relative to* p as defined in [47].

Given (K, ∇, T), define:

$$\begin{cases} N = M, \\ K_1 = K, \\ K_2 = (E_1 \wedge K_1) \oplus \{j^1 a + D^{up}(a) | a \in \Gamma A\}, \end{cases}$$

where $D^{up} : \Gamma A \to \Gamma(T^*M \otimes A)$ is the operator associated to any connection ∇^{up} satisfying $p(\nabla^{up}_X a) = \nabla_X p(a) \quad \forall X \in \mathfrak{X}(M), a \in \Gamma A$. Let us check that it is an IM-Dirac structure of $\mathfrak{D}_{ca}(A)$, i.e. that equations (7.2) hold:

$$\begin{array}{rcl} K_2 \cap \mathrm{im}(\psi) = \psi(K_1 \wedge E_1) & \Leftrightarrow & \text{By construction,} \\ K_1 \text{ isotropic} & \Leftrightarrow & \text{Trivial in this case,} \\ \nabla_{\Gamma K_2}\Gamma K_1 \subseteq \Gamma K_1 & \Leftrightarrow & \nabla_X T(e,e') = T(\nabla_X e, e') + T(e, \nabla_X e'), \\ K_2 \text{ subalgebroid} & \Leftrightarrow & p([a,b]) = \nabla_{\rho(a)}p(b) - \nabla_{\rho(b)}p(a) + T(p(a), p(b)), \\ F_0(0) \subset K_1 & \Leftrightarrow & \text{Trivial in this case,} \\ F_1(\Gamma K_1) \subseteq \Gamma K_2 & \Leftrightarrow & D^{up} \text{ covers } \nabla, \\ F_2(\Gamma K_2) \subseteq K_1 \wedge E_2 \otimes E_1 \wedge K_2 & \Leftrightarrow & \nabla \text{ flat.} \end{array}$$

Finally, if we denote the involutive IM-distribution by Δ we have that since it covers TM then $(\Delta; A, TM; M) \subseteq (TA; A, TM; M)$ is a VB-subalgebroid in both directions. Therefore it is clear that the morphic Dirac structure associated to it is given by $(\Delta \oplus \Delta^\circ; A, TM \oplus K^\circ; M)$.

Remark 7.17. This Example 7.16 can be extended to general involutive IM-distributions (not necessarily covering TM) and the data found in [51] with the equations appearing in [74] characterizing the involutivity give rise to an IM-Dirac structure similar to the one given in this example.

7.2.1 Integration to multiplicative Dirac structures

Given a Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$ we studied the LA-Courant algebroid $(TA \oplus T^*A; A, TM \oplus A^*; M)$ and the standard degenerate Courant algbroid $\mathfrak{D}_{ca}(A)$.

Suppose that our Lie algebroid integrates to a Lie groupoid $G \rightrightarrows M$. Then we can define their tangent and cotangent VB-groupoids,

$$\begin{tikzcd} TG \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & TM \arrow[d] \\ G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} \qquad \begin{tikzcd} T^*G \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & A^* \arrow[d] \\ G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

as we did in Section B.2 and finally take their sum

$$\begin{tikzcd} TG \oplus T^*G \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & TM \oplus A^* \arrow[d] \\ G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} \tag{7.8}$$

This is a VB-groupoid that is also a Courant algebroid. In fact, it is the prototypical example of CA-groupoids as proposed in [96] and studied in [82].

A *multiplicative Dirac structure* in $(TG \oplus T^*G \to G) \rightrightarrows (TM \oplus A^* \to M)$ is a VB-subgroupoid

$$\begin{tikzcd} H \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F \arrow[d] & \subseteq & TG \oplus T^*G \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & TM \oplus A^* \arrow[d] \\ D \arrow[r, shift left] \arrow[r, shift right] & N & \subseteq & G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

such that $H \to D$ is also a Dirac structure of $TG \oplus T^*G \to G$, see [104].

Remark 7.18. Multiplicative Dirac structures were defined for the first time in [20]. Since then, many related works have appeared, see e.g. [72, 81, 82, 104].

With the notion of multiplicative Dirac structure it is proven in [104] the following result.

Proposition 7.19 (see [104]). *Let $G \rightrightarrows M$ be a source simply connected Lie groupoid with Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$. The following holds:*

a). Given a multiplicative Dirac structure $(H \to D) \rightrightarrows (F \to N)$ then $(A_H; A_D, F; N)$ is a morphic Dirac structure of $(TA \oplus T^*A; A, TM \oplus A^*; M)$, where A_H and A_D are the Lie algebroids of $H \rightrightarrows F$ and $D \rightrightarrows N$ respectively.

b). Given a morphic Dirac structure $(L; K, F; N)$ of $(TA \oplus T^*A; A, TM \oplus A^*; M)$ such that $K \to N$ integrates to an embedded subgroupoid of $G \rightrightarrows M$ then $(L; K, F; N)$ integrates to a multiplicative Dirac structure.

Moreover, these two constructions are inverses of one another.

Given a Lie algebroid $(A \to M, [\cdot, \cdot], \rho)$, denote by $\pi : \bigwedge^2 A \oplus J^1 A \oplus \bigwedge^2 T^*M \to A$ the natural projection $\pi(a \wedge b + j^1 a' + \eta) = a'$.

Theorem 7.20. *Let $G \rightrightarrows M$ be a source simply connected Lie groupoid with Lie algebroid $(A \to M, [\cdot, \cdot], \rho)$. The following holds:*

a). Given a multiplicative Dirac structure, there is an associated IM-Dirac structure on $\mathfrak{D}_{ca}(A)$.

b). Given an IM-Dirac structure (N, K_1, K_2) on $\mathfrak{D}_{ca}(A)$ such that $\pi(K_2) \to N$ integrates to an embedded subgroupoid of $G \rightrightarrows M$ then the IM-Dirac structure integrates to a multiplicative Dirac structure.

Moreover, these two constructions are inverses of one another.

Proof. For the first part observe that given a multiplicative Dirac structure, Proposition 7.19 constructs a morphic Dirac structure on $(TA \oplus T^*A; A, TM \oplus A^*; M)$ and using Corollary 7.11 we obtain an IM-Dirac structure on $\mathfrak{D}_{ca}(A)$.

Let us prove the second part now. If (N, K_1, K_2) defines an IM-Dirac structure of $\mathfrak{D}_{ca}(A)$ we saw in Corollary 7.5 that $\pi(K_2) \to N$ is a Lie subalgebroid of $(A \to M, [\cdot, \cdot], \rho)$.

Given an IM-Dirac structure of $\mathfrak{D}_{ca}(A)$ we use Corollary 7.11 to obtain a morphic Dirac structure of $(TA \oplus T^*A; A, TM \oplus A^*; M)$. Going along the proof one shows that the Lie algebroid on the base of the morphic Dirac structure is precisely $\pi(K_2) \to N$. Therefore if $\pi(K_2) \to N$ integrates to an embedded subgroupoid of $G \rightrightarrows M$ we can use Proposition 7.19 and we have that our IM-Dirac structure integrates to a multiplicative Dirac structure.

Finally, the moreover part follows also from Proposition 7.19 and the fact that the correspondence between IM-Dirac and morphic Dirac structures is one to one. $\square$

Remark 7.21. The paper [72] studies Dirac groupoids are introduced and gives an inifinitesimal object that is called a *Dirac bialgebroid*. In [71] it is proven that, in the case of the standard Courant algebroid over a groupoid, Dirac bialgebroids correspond to morphic Dirac structures on the LA-Courant algebroid $(TA \oplus T^*A; A, TM \oplus A^*; M)$. Therefore IM-Dirac structures on $\mathfrak{D}_{ca}(A)$ are also equivalent to Dirac bialgebroids on the standard Courant algebroid.

7.3 Graded symplectic Q-groupoids

In the previous Section 7.2.1 we showed how, given a Lie groupoid $G \rightrightarrows M$ with Lie algebroid $(A \to M, [\cdot, \cdot], \rho)$, an IM-Dirac structure on $\mathfrak{D}_{ca}(A)$ integrates to a multiplicative Dirac structures on $TG \oplus T^*G \rightrightarrows TM \oplus A^*$. In this section, we will further explain this correspondence by introducing graded symplectic Q-groupoids.

On the one hand, symplectic groupoids were introduced by Weinstein in [127] as a possible path to quantize Poisson manifolds. Since then, the concept became central in the study of Poisson geometry. On the other hand, graded Q-groupoids were defined by Mehta in [94, 96] in order to provide a better understanding of LA-groupoids as well as generalizing them. Here we mix the two concepts in order to relate multiplicative Dirac structures and coisotropic Q-submanifolds.

The idea of graded symplectic Q-groupoids is not new, see e.g. [96], where an example of a degree 2 symplectic Q-groupoid is given. In [103] the relation between graded symplectic Q-groupoids and Courant groupoids was stated without proof. Finally, in [82] Courant groupoids were systematically studied and the author suggests a correspondence between degree 2 symplectic Q-groupoids and Courant groupoids.

A symplectic groupoid $(G \rightrightarrows M, \omega)$ is a Lie groupoid $G \rightrightarrows M$ such that (G, ω) is a symplectic manifold and in addition the symplectic form satisfies the following cocycle condition:

$$m^*\omega = p_1^*\omega + p_2^*\omega,$$

where $p_1, p_2 : \mathcal{G}^{(2)} \to \mathcal{G}$ are the first and the second projection respectively.

If $(G \rightrightarrows M, \omega)$ is a symplectic groupoid then the basic properties, see [43], that we want to emphasize here are:

a). M inherits a Poisson structure, that we denote by π, such that $t : G \to M$ is a Poisson map.

b). The Lie algebroid of the groupoid $(A \to M, [\cdot,\cdot], \rho)$ is canonically isomorphic with $(T^*M \to M, [\cdot,\cdot]_\pi, \pi^\sharp)$.

c). If $L \rightrightarrows C$ is a lagrangian subgroupoid of $G \rightrightarrows M$ then $C \subseteq M$ defines a coisotropic submanifold of (M, π). Moreover its Lie algebroid $A_L \to C$ is isomorphic with $(TC)^\circ \to C$, see [31].

Hence this shows that Poisson manifolds can be considered as the infinitesimal description of symplectic groupoids and their coisotropic submanifolds encode the infinitesimal description of their lagrangian subgroupoids.

We can introduce now the same concepts in the graded world: A *degree n symplectic Q-groupoid* $(\mathcal{G} \rightrightarrows \mathcal{M}, \omega, Q)$ is a graded groupoid $\mathcal{G} \rightrightarrows \mathcal{M}$ such that $(\mathcal{G}, \omega, Q)$ is a symplectic Q-manifold and ω and Q are multiplicative, i.e.

- Q is a symplectic vector field, i.e. $\mathcal{L}_Q\omega = 0$.
- ω is multiplicative, i.e. $m^*\omega = pr_1^*\omega + pr_2^*\omega$.
- Q is multiplicative, i.e. $Q : \mathcal{G} \to T[1]\mathcal{G}$ is a groupoid morphism.

Given a graded symplectic Q-groupoid $(\mathcal{G} \rightrightarrows \mathcal{M}, \omega, Q)$ a *lagrangian Q-subgroupoid* is a subgroupoid $\mathcal{L} \rightrightarrows \mathcal{C}$ of $\mathcal{G} \rightrightarrows \mathcal{M}$ such that $\mathcal{L}$ is a lagrangian Q-submanifold of $(\mathcal{G}, \omega, Q)$.

Remark 7.22. See Section A.3 for the definition of graded symplectic manifolds using differential forms and Section A.4 for the definition of a graded groupoid

Now, following the heuristics of the classical case we expect the following results to hold. Given a n-symplectic Q-groupoid $(\mathcal{G} \rightrightarrows \mathcal{M}, \omega, Q)$ then, if we denote by $X : \mathcal{M} \to T[1]\mathcal{M}$ the vector field on the base induced by Q, we have that:

a). $(\mathcal{M}, X)$ becomes a PQ-manifold.
b). The Q-algebroid of the groupoid is canonically isomorphic with $(T^*[n]\mathcal{M} \to \mathcal{M}, [\cdot,\cdot]_\pi, \pi^\sharp, \mathcal{L}_X)$.
c). If $\mathcal{L} \rightrightarrows \mathcal{C}$ is a lagrangian Q-subgroupoid of $\mathcal{G} \rightrightarrows \mathcal{M}$ then $\mathcal{C} \subseteq \mathcal{M}$ defines a coisotropic Q-submanifold of $(\mathcal{M}, \{\cdot,\cdot\}, X)$. Moreover the Q-algebroid of the lagrangian Q-groupoid is canonically isomorphic with the conormal bundle.

So we will have the following principle: coisotropic Q-submanifolds of a PQ-manifolds are the infinitesimal information of lagrangian Q-subgroupoids.

The expected correspondence of symplectic Q-groupoids and their lagrangian Q-subgroupoids with classical objects is the following:

<table>
<tr><th>Degree</th><th>Symplectic Q-groupoid</th><th>PQ-manifold</th><th>Lagrangian Q-subgroupoid</th><th>Coisotropic Q-submanifold</th></tr>
<tr><td>0</td><td>Symplectic groupoid</td><td>Poisson manifold</td><td>Lagrangian subgroupoid</td><td>Coisotropic submanifold</td></tr>
<tr><td>1</td><td>Poisson groupoid</td><td>Lie bialgebroid</td><td>Coisotropic subgroupoid</td><td>Lie subbialgebroid</td></tr>
<tr><td rowspan="2">2</td><td rowspan="2">Courant groupoid</td><td>Degenerate Courant algebroid</td><td rowspan="2">Multiplicative Dirac structure</td><td>IM-Dirac structure</td></tr>
<tr><td>LA-Courant algebroid</td><td>Morphic Dirac structure</td></tr>
</table>

Example 7.23. Consider $G \rightrightarrows M$ a Lie groupoid with Lie algebroid $(A \to M, [\cdot,\cdot], \rho)$. Then, as we already mentioned in Section 7.2.1, we have the Courant groupoid

$$\begin{tikzcd} TG \oplus T^*G \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & TM \oplus A^* \arrow[d] \\ G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

The degree 2 symplectic Q-groupoid associated to it is

$$(T[1]T^*[1]G \rightrightarrows T[1]A^*[1], \omega_{can}, Q_{dr})$$

where ω_{can} denotes the canonical symplectic structure on $T^*[2]T[1]G \cong T[1]T^*[1]G$.

For this case, we already proved that the graded manifold of the base is a degree 2 PQ-manifold, the lagrangian Q-subgroupoids correspond to multiplicative Dirac structures and their infinitesimal information is codified by the coisotropic Q-submanifolds of the base. Moreover, we also were able to integrate the coisotropic Q-submanifolds to lagrangian Q-subgroupoids. This was the example suggested in [96].

Appendix A

More on graded manifolds

In this Appendix we include other aspects of graded manifolds that are also used in the thesis but are not in the core of our work. This include: vector bundles in the category of graded manifolds as defined in [94], differential forms and Cartan calculus for graded manifolds (we follow the supermanifold convention, see [49]), graded symplectic manifolds as in [39] and graded groupoids and algebroids as introduced in [96, 95].

A.1 Vector bundles

Here we introduce vector bundles in the category $\mathcal{GM}^n$. We briefly recall the basic notions and fix notation. This topic is developed in full detail in [94].

A *vector bundle in the category* $\mathcal{GM}^n$, $\pi : \mathcal{E} \to \mathcal{M}$, is given by two n-manifolds $\mathcal{E} = (E, C_{\mathcal{E}})$ and $\mathcal{M} = (M, C_{\mathcal{M}})$ and a submersion $\pi = (p, \pi^\sharp) : \mathcal{E} \to \mathcal{M}$ satisfying that there exists a cover $\{U_\alpha\}_\alpha$ of M, such that

$$\mathcal{E}_{|p^{-1}(U_\alpha)} \cong \mathcal{M}_{|U_\alpha} \times \mathbb{R}^{k_0|\cdots|k_n},$$

and the transition functions between different opens are linear in the second factor. The numbers $k_0|\cdots|k_n$ are referred to as the *rank of* $\pi : \mathcal{E} \to \mathcal{M}$.

By our definition, it is clear that we can define a vector bundle just giving the trivialization over each open and gluing them using the transition functions as it happens in differential geometry.

A natural operation that we can do with graded vector bundles is *shifting*. Let $\mathcal{M}$ be a graded manifold and consider the trivial bundle of rank $k_0|\cdots|k_n$ over it:

$$pr_1 : \mathcal{M} \times \mathbb{R}^{k_0|\cdots|k_n} \to \mathcal{M}.$$

Given $j \in \mathbb{N}$ we can define a trivial graded vector bundle in the category $\mathcal{GM}^{n+j}$ just by

$$pr_1 : \mathcal{M} \times \mathbb{R}^{0|\cdots|0|k_0|\cdots|k_n} \to \mathcal{M}.$$

In general, given a vector bundle $\pi : \mathcal{E} \to \mathcal{M}$ and a number $j \in \mathbb{N}$ we define a new vector bundle $\pi[j] : \mathcal{E}[j] \to \mathcal{M}$ by shifting the fibre coordinates by j as we illustrated in the trivial bundle.

For a graded vector bundle $\mathcal{E} \to \mathcal{M}$, we use the notation $\mathcal{E}[j]_{\mathcal{M}}$ to make the base of the shift explicit.

Let $\pi : \mathcal{E} \to \mathcal{M}$ be a vector bundle. A *homogeneous section of degree* $-j$ is a morphism $\Psi : \mathcal{M} \to \mathcal{E}[j]$ such that $\pi[j] \circ \Psi = \mathrm{Id}_{\mathcal{M}}$.

Remark A.1. Many natural operations of graded vector bundles (shifting by any integer, dual vector bundle, taking sections...) go out of the world of non-negatively graded vector bundles. The right place to consider it is in the category of $\mathbb{Z}$-graded manifolds, see [94] for details.

We finish this quick review on graded vector bundles by giving some examples:

Example A.2. The 1-manifolds $E^*[1] = (M, \Gamma \bigwedge E)$ defined in Example 2.3 can be understood as zero vector bundle shifted by 1 and are vector bundles in the category $\mathcal{GM}^1$ with base M understood as a 1-manifold without degree 1 coordinates.

Example A.3 (The tangent bundle of a graded manifold). Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be an n-manifold of dimension $m_0|\cdots|m_n$. Then $\pi = (p, \pi^\sharp) : T\mathcal{M} = (TM, C_{T\mathcal{M}}) \to \mathcal{M} = (M, C_{\mathcal{M}})$ is a graded vector bundle. Let us define the manifold $T\mathcal{M}$ locally. Pick $U \subset M$ such that $(C_{\mathcal{M}})_{|U}$ is as in equation (2.1) and consider coordinates $\{e^{j_i}\}$ with $0 \leq i \leq n$ and $1 \leq j_i \leq m_i$. Then $\{e^{j_i}, \theta^{j_i} = de^{j_i}\}$ are coordinates of $(C_{T\mathcal{M}})_{p^{-1}(U)}$ where $|\theta^{j_i}| = |e^{j_i}|$. As it happens in differential geometry, the way a change of coordinates in e^{j_i} induces a change of coordinates for de^{j_i} proves that $T\mathcal{M}$ is a graded vector bundle with base $\mathcal{M}$.

Example A.4 (The cotangent bundle of a graded manifold). Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be a n-manifold of dimension $m_0|\cdots|m_n$. If we want to have some duality between the tangent and cotangent bundles we have that $T^*\mathcal{M}$ is not a non negatively graded manifold unless $\mathcal{M}$ be a degree 0 manifold. For that reason, cotangent bundles must be shifted (if we want to consider them as n-manifolds). See Chapter 6 for details.

A.2 Differential forms and Cartan calculus

In this section we introduce differential forms for graded manifolds. Our convention for differential forms is that they have a natural bi-degree and follows the commutation rules defined in [49], as opposite to the conventions used in [39, 94] where they supercommute with respect to the total degree.

Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be an n-manifold. We define a *differential k-form of degree j*, denoted by $\omega \in \Omega^{k,j}(\mathcal{M})$ and by $|\omega| = j$ its degree, as a map

$$\omega : \mathfrak{X}^{1,i_1}(\mathcal{M}) \times \cdots \times \mathfrak{X}^{1,i_k}(\mathcal{M}) \to C^{i_1+\cdots+i_k+j}_{\mathcal{M}}$$

satisfying:

- $\omega(\cdots, X, Y, \cdots) = (-1)^{|X||Y|}\omega(\cdots, Y, X, \cdots)$.
- $\omega(fX_1, \cdots, X_k) = (-1)^{|f||\omega|} f\omega(X_1, \cdots, X_k)$.

As it happens in the non graded case we can also define the wedge product. Suppose that $\omega \in \Omega^{i,j}(\mathcal{M})$ and $\eta \in \Omega^{k,l}(\mathcal{M})$. Then we define a new form $\omega \wedge \eta \in$

$\Omega^{i+k,j+l}(\mathcal{M})$ by the formula

$$\omega \wedge \eta(X_1, \cdots, X_{i+k}) = \sum_{\sigma \in Sh(i,k)} Ksgn(\sigma)\omega(X_{\sigma(1)}, \cdots, X_{\sigma(i)})\eta(X_{\sigma(i+1)}, \cdots, X_{\sigma(i+k)})$$

where $X_1, \cdots, X_{i+k} \in \mathfrak{X}^{1,\bullet}(\mathcal{M})$ and $Ksgn(\sigma)$ denotes the signature of the permutation multiplied by the Koszul sign. Observe that, with our sign conventions, we have that

$$\omega \wedge \eta = (-1)^{ik+jl}\eta \wedge \omega.$$

The rule for the wedge product implies that $(\Omega^{\bullet,\bullet}(\mathcal{M}), \wedge)$ is generated by $\Omega^{1,\bullet}(\mathcal{M})$ as a sheaf of algebras over $C_{\mathcal{M}}$.

As it happens for vector fields, see Remark 2.7, 1-forms can be identified with sections of the cotangent bundle:

Proposition A.5 (see [94]). *Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be an n-manifold. There is a degree preserving natural identification between $\Gamma T^*\mathcal{M}$ and $\Omega^{1,\bullet}(\mathcal{M})$ as $C_{\mathcal{M}}$-modules.*

Remark A.6. In [94] (pseudo-differential) forms on $\mathcal{M}$ are identified with functions of $T\mathcal{M}$. The disadvantage of our sign convention is that we can not make this identification directly because our objects have a double grading. But modulo signs we can. Therefore we make no distinction between the Cartan calculus on forms and the Cartan calculus on function in $T\mathcal{M}$.

Here we recall the basics of the Cartan calculus on graded manifolds. As in classical geometry we can define the contraction, Lie derivative and De Rham differential operators for a graded manifold and prove the usual formulas with some extra signs. In order to know the appropriated sign just recall that whenever two symbols are transposed, the sign of those symbols pop up.

Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be a graded manifold and consider $X \in \mathfrak{X}^{1,r}(\mathcal{M})$ a vector field. The contraction with respect to this vector field is defined by the following rule:

$$\begin{aligned} &i_X : \Omega^{k,l}(\mathcal{M}) \to \Omega^{k-1,l+r}(\mathcal{M}) \\ &(i_X\omega)(X_2, \cdots, X_{k-1}) = (-1)^{|\omega||X|}\omega(X, X_2, \cdots, X_k). \end{aligned}$$

It is a derivation of the wedge product, i.e.

$$i_X(\omega \wedge \eta) = i_X\omega \wedge \eta + (-1)^{|\omega||X|}\omega \wedge i_X\eta.$$

Therefore we obtain that i_X is a derivation of the algebra $(\Omega^{\bullet,\bullet}(\mathcal{M}), \wedge)$ of bidegree $(-1, |X|)$. We also define the Lie derivative with respect to a vector field as

$$\mathcal{L}_X f = X(f), \qquad \mathcal{L}_X Y = [X, Y],$$

and extend it to forms by the following formula

$$\mathcal{L}_X i_{X_1} \cdots i_{X_k}\omega = (-1)^{r\sum_{a=1}^{k}|X_a|} i_{X_1} \cdots i_{X_k}\mathcal{L}_X\omega + \sum_{j=1}^{i}(-1)^{r\sum_{a=1}^{j-1}|X_a|} i_{X_1} \cdots i_{[X,X_j]} \cdots i_{X_k}\omega \tag{A.1}$$

By definition, this operation is a derivation of the wedge product and we obtain that $\mathcal{L}_X$ has bidegree $(0,k)$. Finally the De Rham differential can be defined by the usual Cartan formula

$$\begin{aligned} i_{X_0}\cdots i_{X_k} d\omega = & \sum_{i=0}^{k}(-1)^{i+|X_i|\sum\limits_{a=0}^{i-1}|X_a|}\mathcal{L}_{X_i} i_{X_0}\cdots\widehat{i_{X_i}}\cdots i_{X_k}\omega \\ & +\sum_{i<j}(-1)^{i+1+|X_i|\sum\limits_{a=i+1}^{j-1}|X_a|} i_{X_0}\cdots\widehat{i_{X_i}}\cdots i_{[X_i,X_j]}\cdots i_{X_k}\omega \end{aligned}$$

The formula says that it is a derivation of the wedge product and that it has bidegree $(1,0)$.

With all these formulas we are able to prove the usual Cartan magic formulas.

Proposition A.7. *Let $\mathcal{M}$ be a graded manifold. The following formulas hold:*

$$\begin{aligned} &[d,d]=2d^2=0, \quad \mathcal{L}_X=[i_X,d], \quad \mathcal{L}_{[X,Y]}=[\mathcal{L}_X,\mathcal{L}_Y], \\ &[\mathcal{L}_X,d]=0, \quad i_{[X,Y]}=[\mathcal{L}_X,i_Y], \quad [i_X,i_Y]=0, \end{aligned}$$

where the bracket denotes the graded commutator with bidegree, i.e. for example $[i_X,i_Y]=i_X\circ i_Y-(-1)^{(-1)(-1)+|X||Y|}i_Y\circ i_X$.

A.3 Graded symplectic manifolds (2-form definition)

Let $\mathcal{M}=(M,C_{\mathcal{M}})$ be an n-manifold. We say that the pair $(\mathcal{M},\omega)$ is a *graded symplectic n-manifold* if the following is true:

- The 2-form is of degree n, i.e. $\omega\in\Omega^{2,n}(\mathcal{M})$.
- The 2-form is closed, i.e. $d\omega=0$.
- The map $\omega^\sharp: T\mathcal{M}\to T^*[n]\mathcal{M}$ is a vector bundle isomorphism over the identity.

A direct consequence of the third property is that for symplectic n-manifolds we can also assign to each function a vector field. Given $f\in C_{\mathcal{M}}$, *the hamiltonian vector field of f* is defined as the unique vector field satisfying

$$i_X\omega=df$$

and it will be denoted also by X_f. Observe that if $f\in C^k_{\mathcal{M}}$ then we have that $|X_f|=k-n$. Moreover, we can define a bracket on functions by the formula

$$\{f,g\}=i_{X_f}i_{X_g}\omega. \tag{A.2}$$

This clearly satisfies $\{\cdot,\cdot\}: C^i_{\mathcal{M}}\times C^j_{\mathcal{M}}\to C^{i+j-n}_{\mathcal{M}}$.

Proposition A.8. *Let $(\mathcal{M},\omega)$ be a symplectic n-manifold. Then the following equalities hold:*

$$\mathcal{L}_{X_f}g=\{f,g\}, \qquad \mathcal{L}_{X_f}\omega=0 \quad \textit{and} \quad X_{\{f,g\}}=[X_f,X_g]$$

Moreover $\{\cdot,\cdot\}$ is a Poisson bracket of degree $-n$.

Proof. We use the formulas of Proposition A.7 and formula (A.1):

$$\mathcal{L}_{X_f} g = (i_{X_f} d + d i_{X_f}) g = i_{X_f} dg = i_{X_f} i_{X_g} \omega = \{f, g\}.$$

$$\mathcal{L}_{X_f} \omega = d i_{X_f} \omega + i_{X_f} d\omega = 0.$$

$$i_{[X_f, X_g]} \omega = [\mathcal{L}_{X_f}, i_{X_g}] \omega = \mathcal{L}_{X_f} i_{X_g} \omega = \mathcal{L}_{X_f} dg = d\mathcal{L}_{X_f} g = d\{f, g\} = i_{X_{\{f,g\}}} \omega.$$

Let us see that the bracket is Poisson. Since $0 = [i_X, i_Y] = i_X i_Y - (-1)^{1+|X||Y|} i_Y i_X$, we obtain that $\{\cdot, \cdot\}$ is skew-symmetric. The graded Leibniz identity:

$$\begin{aligned} \{f, gh\} = & \; i_{X_f} i_{X_{gh}} \omega = i_{X_f} d(gh) = i_{X_f} dg \; h + i_{X_f} (-1)^{|g|} g \; dh \\ = & \; \{f, g\} h + (-1)^{|g||X_f|} g \; i_{X_f} dh = \{f, g\} h + (-1)^{|g|(|f|-n)} g \{f, h\}. \end{aligned}$$

The graded Jacobi identity:

$$\begin{aligned} \{\{f, g\}, h\} + (-1)^{(|f|-n)(|g|-n)} \{g, \{f, h\}\} = & \; i_{X_{\{f,g\}}} i_{X_h} \omega + (-1)^{|X_f||X_g|} i_{X_g} i_{X_{\{f,g\}}} \omega \\ = & \; i_{[X_f, X_g]} i_{X_h} \omega + (-1)^{|X_f||X_g|} i_{X_g} i_{[X_f, X_g]} \omega \\ = & \; \mathcal{L}_{X_f} i_{X_g} i_{X_h} \omega = \mathcal{L}_{X_f} \{g, h\} \\ = & \; \{f, \{g, h\}\}. \end{aligned}$$

$\square$

Let $(\mathcal{M}, \omega)$ be a symplectic n-manifold. We say that a vector field $X \in \mathfrak{X}^{1,k}(\mathcal{M})$ is *symplectic* if $\mathcal{L}_X \omega = 0$.

Proposition A.9. *Let $(\mathcal{M}, \omega)$ be a symplectic n-manifold. A vector field $X \in \mathfrak{X}^{1,k}(\mathcal{M})$ is symplectic if and only if it is Poisson for $\{\cdot, \cdot\}$.*

Proof. Suppose that $X \in \mathfrak{X}^{1,k}(\mathcal{M})$ is a symplectic vector field. Then using formula (A.1) we obtain

$$X\{f, g\} = \mathcal{L}_X i_{X_f} i_{X_g} \omega = i_{[X, X_f]} i_{X_g} \omega + (-1)^{|X||X_f|} i_{X_f} i_{[X, X_g]} \omega.$$

Just remains to see that if it is symplectic then $[X, X_f] = X_{X(f)}$. But this is obvious because

$$i_{[X, X_f]} \omega = \mathcal{L}_X i_{X_f} \omega = \mathcal{L}_X df = dX(f) = i_{X_{X(f)}} \omega.$$

The other way around. Suppose that X is Poisson. Then

$$X\{f, g\} = \{X(f), g\} + (-1)^{|X|(|f|-n)} \{f, X(g)\} \Longleftrightarrow [X, X_f] = X_{X(f)}.$$

Therefore using the same formula but the other way around we obtain that

$$(-1)^{|X|(|X_f|+|X_g|)} i_{X_f} i_{X_g} \mathcal{L}_X \omega = -\mathcal{L}_X i_{X_f} i_{X_g} \omega + i_{X_{X(f)}} i_{X_g} \omega + (-1)^{|X||X_f|} i_{X_f} i_{X_{X(g)}} \omega.$$

$\square$

There are two important results due to Roytenberg that are just true in the context on graded n-manifolds. The first one says that symplectic n-manifolds are exact and the second that for almost all degrees symplectic vector fields are always hamiltonian.

Proposition A.10 (see [108]). *Let $(\mathcal{M}, \omega)$ be a graded symplectic n-manifold. The following is true:*

a). If $n \geq 1$ then ω is exact. Moreover $\omega = d(\frac{1}{n} i_{\mathcal{E}_u}\omega)$, where $\mathcal{E}_u$ is the Euler vector field of $\mathcal{M}$ as defined in Example 2.8.

b). Let $X \in \mathfrak{X}^{1,l}(\mathcal{M})$ be such that $\mathcal{L}_X\omega = 0$. If $l + n \neq 0$ then X is hamiltonian. Moreover $i_X\omega = d(\frac{1}{l+n} i_{\mathcal{E}_u} i_X\omega)$.

Proof. Both formulas are straightforward:

$$n\ \omega = \mathcal{L}_{\mathcal{E}_u}\omega = d i_{\mathcal{E}_u}\omega \Rightarrow \omega = d(\frac{1}{n} i_{\mathcal{E}_u}\omega).$$

$$d i_{\mathcal{E}_u} i_X\omega = \mathcal{L}_{\mathcal{E}_u} i_X\omega - i_{\mathcal{E}_u} d i_X\omega = (l+n) i_X\omega \Rightarrow i_X\omega = d(\frac{1}{n+l} i_{\mathcal{E}_u} i_X\omega).$$

□

Recall that in Section 2.4 we have introduced symplectic n-manifolds as n-manifold $\mathcal{M}$ endowed with a non-degenerate Poisson bracket of degree $-n$. What we state next is that both definitions coincide:

Theorem A.11. *Let $\mathcal{M}$ be an n-manifold. There is a one to one correspondence between symplectic forms and non-degenerate Poisson brackets of degree $-n$.*

Remark A.12 (On lagrangian submanifolds). Recall that we have defined a lagrangian submanifold as a submanifold that has half the total dimension of the manifold and it is coisotropic. Our definition seems a bit arbitrary but one can show that is equivalent to the other usual definitions. For example defining lagrangian submanifold as $\omega^\sharp(T\mathcal{N}) = N^*[n]\mathcal{N}$ or as $T^\omega\mathcal{N} = T\mathcal{N}$.

A.4 Graded groupoids and graded algebroids

In Poisson geometry the notions of Lie groupoid and Lie algebroid have become central. Here we introduce graded groupoids and algebroids, i.e. groupoids and algebroids in the category of graded manifolds. These objects were defined by Mehta in [94, 95, 96].

Roughly speaking, an *n-graded groupoid* is a groupoid object in the category $\mathcal{GM}^n$. More concretely an n-graded groupoid $\mathcal{G} \rightrightarrows \mathcal{M}$ consists of two n-manifolds $\mathcal{G}$ and $\mathcal{M}$ and five morphisms: Two surjective submersions called *source* and *target* $s, t : \mathcal{G} \to \mathcal{M}$, the *multiplication* $m : \mathcal{G}_s \times_t \mathcal{G} \to \mathcal{G}$, the *unit* $u : \mathcal{M} \to \mathcal{G}$ and the *inversion* $i : \mathcal{G} \to \mathcal{G}$, satisfying the following commutative diagrams:

$$\begin{tikzcd} \mathcal{G}^{(3)} \arrow[r, "\mathrm{Id}\times m"] \arrow[d, "m\times\mathrm{Id}"'] & \mathcal{G}^{(2)} \arrow[d, "m"] \\ \mathcal{G}^{(2)} \arrow[r, "m"'] & \mathcal{G} \end{tikzcd} \quad \begin{tikzcd} \mathcal{G} \arrow[r, "\mathrm{Id}\times(u\circ s)"] \arrow[dr, "\mathrm{Id}"'] & \mathcal{G}^{(2)} \arrow[d, "m"] \\ & \mathcal{G} \end{tikzcd} \quad \begin{tikzcd} \mathcal{G} \arrow[r, "(u\circ t)\times\mathrm{Id}"] \arrow[dr, "\mathrm{Id}"'] & \mathcal{G}^{(2)} \arrow[d, "m"] \\ & \mathcal{G} \end{tikzcd} \quad \begin{tikzcd} \mathcal{G} \arrow[r, "i\times\mathrm{Id}"] \arrow[dr, "\mathrm{Id}"'] & \mathcal{G}^{(2)} \arrow[d, "m"] \\ & \mathcal{G} \end{tikzcd} \quad \begin{tikzcd} \mathcal{G} \arrow[r, "\mathrm{Id}\times i"] \arrow[dr, "\mathrm{Id}"'] & \mathcal{G}^{(2)} \arrow[d, "m"] \\ & \mathcal{G} \end{tikzcd}$$

where $\mathcal{G}^{(k)} = \mathcal{G}_s \times_t \cdots_s \times_t \mathcal{G}$. For a detailed definition of the fibre product in the category of graded manifolds see [94].

Since any morphism of n-manifolds has an associated smooth map between the bodies, it is clear that if $\mathcal{G} = (G, C_{\mathcal{G}}) \rightrightarrows \mathcal{M} = (M, C_{\mathcal{M}})$ is a graded groupoid then $G \rightrightarrows M$ is a Lie groupoid with structural maps given by the graded maps restricted to the body.

A *groupoid morphism between $\mathcal{G} \rightrightarrows \mathcal{M}$ and $\mathcal{H} \rightrightarrows \mathcal{N}$* is a pair of morphisms $\Psi : \mathcal{G} \to \mathcal{H}$ and $\Phi : \mathcal{M} \to \mathcal{N}$ for which the following diagram commutes:

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\Psi"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & \mathcal{H} \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ \mathcal{M} \arrow[r, "\Phi"] & \mathcal{N} \end{tikzcd}$$

and Ψ also commutes with the multiplications.

Remark A.13. n-graded groupoids must not be confused with n-groupoids, see [132]. These last objects are simplicial manifolds satisfying a Kan condition.

The basic examples of graded groupoids are:

Example A.14. Recall from Example 2.3 that vector bundles give examples of 1-manifolds. Therefore if we have a groupoid in the category of vector bundles we will have an example of a graded groupoid. A VB-groupoid is a groupoid in the category of vector bundles, see for example [22, 96]. Given a VB-groupoid

$$\begin{tikzcd} H \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & E \arrow[d] \\ G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

we denote by $H[1] \rightrightarrows E[1]$ the associated 1-graded groupoid.

Example A.15. As it happens for Lie groupoids, we have that for any n-graded groupoid $\mathcal{G} \rightrightarrows \mathcal{M}$ their tangent bundle shifted by a positive integer k is a $n+k$-graded groupoid that we denote by $T[k]\mathcal{G} \rightrightarrows T[k]\mathcal{M}$, where the structural maps are the tangent shifted by k of the original n-graded groupoid.

We finish this section defining multiplicative vector fields and forms. Let us start with vector fields. Let $\mathcal{G} \rightrightarrows \mathcal{M}$ be a graded groupoid and $Y \in \mathfrak{X}^{1,i}(\mathcal{G})$ be a vector field in $\mathcal{G}$. We say that Y is a *multiplicative vector field* if there exists $X \in \mathfrak{X}^{1,i}(\mathcal{M})$ such that

$$\begin{tikzcd} \mathcal{G} \arrow[r, "Y"] \arrow[d, shift right] \arrow[d, shift left] & T[i]\mathcal{G} \arrow[d, shift right] \arrow[d, shift left] \\ \mathcal{M} \arrow[r, "X"] & T[i]\mathcal{M} \end{tikzcd}$$

is a groupoid morphism.

Remark A.16. Mehta defined in [94] a multiplicative vector field as a vector field for which there exists another vector field on $\mathcal{G}^{(2)}$ such that they are related by the multiplication and the two projections. As it happens in the classical case both definitions are equivalent.

A *graded Q-groupoid* $(\mathcal{G} \rightrightarrows \mathcal{M}, Q)$ is a graded groupoid $\mathcal{G} \rightrightarrows \mathcal{M}$ such that $(\mathcal{G}, Q)$ is a Q-manifold and Q is a multiplicative vector field.

Example A.17 (The tangent groupoid). Let $\mathcal{G} \rightrightarrows \mathcal{M}$ be a graded groupoid. We see in Example 2.13 that the tangent shifted by 1 of any graded manifold is a Q-manifold; that means that $(T[1]\mathcal{G}, Q_{dr})$ and $(T[1]\mathcal{M}, Q_{dr})$ are Q-manifolds. Moreover, Example A.15 says that $(T[1]\mathcal{G} \rightrightarrows T[1]\mathcal{M})$ is a graded groupoid. Therefore $(T[1]\mathcal{G} \rightrightarrows T[1]\mathcal{M}, Q_{dr})$ is an example of a Q-groupoid. The multiplicativity of Q_{dr} follows from the fact that Q_{dr} commutes with pull-backs.

Let $\mathcal{G} \rightrightarrows \mathcal{M}$ be a graded groupoid and consider $\Omega^{\bullet,\bullet}(\mathcal{G})$ the space of forms on the space of arrows of the groupoid. Given $\omega \in \Omega^{i,j}(\mathcal{G})$ we say that ω is a *multiplicative form* if

$$m^\sharp \omega = p_1^\sharp \omega + p_2^\sharp \omega$$

where $(m, m^\sharp) : \mathcal{G}^{(2)} \to \mathcal{G}$ is the multiplication of the groupoid and $(p_1, p_1^\sharp), (p_2, p_2^\sharp) : \mathcal{G}^{(2)} \to \mathcal{G}$ are the natural first and second projections.

Graded algebroids

An *n-graded algebroid* $(\mathcal{A} \to \mathcal{M}, \Upsilon, [\cdot,\cdot])$ is a vector bundle $\mathcal{A} \to \mathcal{M}$ where $\mathcal{A}$ and $\mathcal{M}$ are n-manifolds endowed with a vector bundle map $\Upsilon : \mathcal{A} \to T\mathcal{M}$ and a bracket $[\cdot,\cdot] : \Gamma\mathcal{A} \times \Gamma\mathcal{A} \to \Gamma\mathcal{A}$ satisfying:

- $[a_1, a_2] = -(-1)^{|a_1||a_2|}[a_1, a_2]$.
- $[a_1, [a_2, a_3]] = [[a_1, a_2], a_3] + (-1)^{|a_1||a_2|}[a_2, [a_1, a_3]]$.
- $[a_1, f a_2] = \Upsilon(a_1)(f)a_2 + (-1)^{|a_1||f|} f[a_1, a_2]$.

for $a_1, a_2, a_3 \in \Gamma\mathcal{A}$ and $f \in C_\mathcal{M}$.

As it happens for n-graded groupoids, the body of an n-graded algebroid inherits the structure of a classical Lie algebroid that we denote by $(A \to M, [\cdot,\cdot], \rho)$.

We have already seen in Example 2.12 that if $(A \to M, [\cdot,\cdot], \rho)$ is a Lie algebroid then the graded manifold $A[1]$ is a Q-manifold. This result was extended to the context of graded algebroids in the following way:

Theorem A.18 (Mehta [95]). *Let $\mathcal{A} \to \mathcal{M}$ be a vector bundle between n-graded manifolds. $\mathcal{A} \to \mathcal{M}$ is an n-algebroid if and only if $\mathcal{A}[1] \to \mathcal{M}$ is a Q-bundle.*

Remark A.19. Recall that a Q-bundle $(\mathcal{A} \to \mathcal{M}, Q)$ is a vector bundle $\pi : \mathcal{A} \to \mathcal{M}$ where $(\mathcal{A}, Q)$ is a Q-manifold and Q is π-projectable and linear on the fibres of π. Therefore Theorem A.18 extends Theorem 3.26 to the world of graded algebroids.

We define *n-graded algebroid morphism* as a vector bundle morphism

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\widehat{\Psi}"] \arrow[d] & \mathcal{B} \arrow[d] \\ \mathcal{M} \arrow[r, "\Phi"] & \mathcal{N} \end{tikzcd}$$

such that $\Psi[1] : \mathcal{A}[1] \to \mathcal{B}[1]$ is a Q-morphism.

One of the main results of [94] is the Lie functor for graded groupoids that we recall here.

Theorem A.20 (Lie functor)**.** *Let $\mathcal{G} \rightrightarrows \mathcal{M}$ be an n-graded groupoid. Then the vector bundle $\mathcal{A}_{\mathcal{G}} = u^* \ker(Ts) \to \mathcal{M}$ is an n-algebroid.*

Analogous to the graded groupoid case one can also define graded Q-algebroids. A *graded Q-algebroid* $(\mathcal{A} \to \mathcal{M}, \widehat{Q})$ is a graded algebroid $(\mathcal{A} \to \mathcal{M}, [\cdot,\cdot], \Upsilon)$ such that $(\mathcal{A}, \widehat{Q})$ is a Q-manifold satisfying that there exists $X \in \mathfrak{X}^{1,1}(\mathcal{M})$ such that

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\widehat{Q}"] \arrow[d] & T[1]\mathcal{A} \arrow[d] \\ \mathcal{M} \arrow[r, "X"] & T[1]\mathcal{M} \end{tikzcd}$$

is an algebroid morphism. In [94] the following Lie functor was proven:

Proposition A.21 (See [94])**.** *Let $\mathcal{G} \rightrightarrows \mathcal{M}$ be a graded groupoid with associated graded algebroid $\mathcal{A}_{\mathcal{G}} \to \mathcal{M}$. If $(\mathcal{G} \rightrightarrows \mathcal{M}, Q)$ is a graded Q-groupoid then there exists $\widehat{Q} \in \mathfrak{X}^{1,1}(\mathcal{A}_{\mathcal{G}})$ such that $(\mathcal{A}_{\mathcal{G}} \to \mathcal{M}, \widehat{Q})$ is a graded Q-algebroid.*

Remark A.22. We believe that the graded Frobenius theorem of Chapter 4 will help us to integrate morphisms between graded algebroids to morphism between graded groupoids, see [90] for the classical statement. In particular, this will allow to integrate Q-algebroids to Q-groupoids.

Appendix B

Double vector bundles and 2-manifolds

In this appendix we summarize the facts on double vector bundles that are needed for some Chapters of this thesis. Double vector bundles were introduced in [106] and extensively used by Mackenzie and collaborators; the standard reference for this topic is [88].

B.1 Basic properties

A *double vector bundle* is a vector bundle in the $\mathcal{V}ect$ category, or more explicitly, a square

$$\begin{array}{ccc} D & \xrightarrow{p_B} & B \\ \big\downarrow{\scriptstyle q_A} & & \big\downarrow{\scriptstyle q} \\ A & \xrightarrow{p} & M \end{array} \tag{B.1}$$

in which D has two different vector bundle structures with base A and B, that are also a vector bundles over a common base M, and the projection $q_A : D \to A$ and the sum $+_A : D \times_A D \to D$ are vector bundles morphism over $q : B \to M$ and $+ : B \times_M B \to B$

Remark B.1. The structural maps of the vector bundle $D \to B$ are denoted by $+_B : D \times_B D \to D, p_B : D \to B$ and $\widetilde{0}^B : B \to D$ and we will denote by $\Gamma(D, B)$ the set of sections of $D \to B$. For the vector bundle $D \to A$ the structural maps are denoted $+_A : D \times_A D \to D, q_A : D \to A$ and $\widetilde{0}^A : A \to D$ and the sections will be denoted by $\Gamma(D, A)$.

Remark B.2. The most clear and symmetric definition of a double vector bundle was given in [58] where it is defined as a manifold D with two commuting Euler-like vector fields.

We refer to A and B as the *side bundles* and to M as the *base*. We will denote double vector bundles by $(D; A, B; M)$.

Example B.3. The most important example of a double vector bundles is given by the tangent prolongation of a vector bundle. More concretely, let $E \to M$ be a vector bundle. It is clear that $TE \to E$ is a vector bundle with the natural structure given by the tangent bundle of any manifold. But now we can apply the tangent functor to all the structural maps of the vector bundle $E \to M$ and we obtain that $TE \to TM$ is also a vector bundle. One can easily see that

$$\begin{tikzcd} TE \arrow[r] \arrow[d] & TM \arrow[d] \\ E \arrow[r] & M \end{tikzcd}$$

defines a double vector bundle, referred to as the tangent prolongation.

Let $(D; A, B; M)$ and $(H; E, F; N)$ be two double vector bundles. A *morphism* $(\Phi; \varphi_A, \varphi_B; \varphi) : (D; A, B; M) \to (H; E, F; N)$ is a cube

$$\begin{tikzcd}[row sep=small, column sep=small] & B \arrow[rr, "\varphi_B"] \arrow[dd] & & F \arrow[dd] \\ D \arrow[ur] \arrow[rr, "\Phi"] \arrow[dd] & & H \arrow[ur] \arrow[dd] & \\ & M \arrow[rr, "\varphi"] & & N \\ A \arrow[ur] \arrow[rr, "\varphi_A"] & & E \arrow[ur] & \end{tikzcd} \tag{B.2}$$

where all the faces are vector bundle morphism. Clearly, double vector bundles with their morphism form a category.

Given a double vector bundle $(D; A, B; M)$ we can exchange the vertical and the horizontal structure, producing a new double vector bundle called the *flip*, denoted by $(D^f; B, A; M)$. Notice that as sets D and D^f are equal but as double vector bundles they are different.

On a double vector bundle $(D; A, B; M)$, apart from D and the side bundles A and B, there are some other important bundles appearing on it. One of them is the *core* bundle

$$C = \ker(q_A) \cap \ker(p_B)$$

that is a vector bundle over M because the two vector bundle structures of D commute. Observe that given $c \in \Gamma C$ we can define a section of $D \to A$ and also a section of $D \to B$ by the formulas:

$$\widetilde{c} = i \circ c \circ p +_A \widetilde{0}^B \in \Gamma(D, A) \qquad \widetilde{c} = i \circ c \circ q +_B \widetilde{0}^A \in \Gamma(D, B) \tag{B.3}$$

where $i : C \to D$ is the inclusion map. We will denote by $\Gamma_c(D)$ the image of the maps given by (B.3) inside of $\Gamma(D, A)$ and $\Gamma(D, B)$.

Remark B.4. Let $(D; A, B; M)$ and $(H; E, F; N)$ be double vector bundles with cores C and K respectively. Consider $(\Phi; \varphi_A, \varphi_B; \varphi) : (D; A, B; M) \to (H; E, F; N)$ a double vector bundle morphism. Then it induces a vector bundle morphism

$$\begin{tikzcd} C \arrow[r, "\varphi_C"] \arrow[d] & K \arrow[d] \\ M \arrow[r, "\varphi"] & N \end{tikzcd} \tag{B.4}$$

between the core bundles.

One of the properties of the core bundle is that there exists a double vector bundle exact sequence:

$$0 \to \begin{array}{ccc} C & \to & M \\ \downarrow & & \downarrow \\ M & \to & M \end{array} \to \begin{array}{ccc} D & \to & B \\ \downarrow & & \downarrow \\ A & \to & M \end{array} \to \begin{array}{ccc} A\oplus B & \to & B \\ \downarrow & & \downarrow \\ A & \to & M \end{array} \to 0 \tag{B.5}$$

Since splittins of this sequence always exist, we have that a double vector bundle $(D; A, B; M)$ with core C is always isomorphic to $(A \oplus C \oplus B; A, B; M)$. The choice of an isomorphism is called a *decomposition*.

An important aspect of double vector bundles is their duality theory. Since D is a vector bundle over two different manifolds we could take duals with respect to both structures. We denote the horizontal dual by $D^{\to} \to B$ and the vertical dual by $D^{\uparrow} \to A$.

Proposition B.5. *Let $(D; A, B; M)$ be a double vector bundle with core C. Then $(D^{\to}; C^*, B; M)$ and $(D^{\uparrow}; A, C^*; M)$ are also double vector bundles. Moreover $(D^{\to\uparrow}; C^*, A; M)$ and $(D^{\uparrow f}; A, C^*; M)$ are isomorphic as double vector bundles and also $(D^{\uparrow\to}; B, C^*; M)$ and $(D^{\to f}; C^*, B; M)$.*

Therefore double vector bundles (up to flip) appear in triples: themselves and their two duals. This means that the structures defined on one of them must have incarnations on the other two duals.

Example B.6. In Example B.3 we see that given a vector bundle $E \to M$ we have that $(TE; E, TM; M)$ is a double vector bundle. It is well known that in this case we can identify their core with the vector bundle itself, i.e. $C = E$. Then the duals of the tangent prolongation are

$$\begin{array}{ccc} T^*E & \longrightarrow & E^* \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} TE^{\to} & \longrightarrow & TM \\ \downarrow & & \downarrow \\ E^* & \longrightarrow & M \end{array}$$

known as the cotangent prolongation and the tangent prolongation dual respectively. The tangent prolongation dual is canonically isomorphic with the tangent prolongation of the dual vector bundle and this isomorphism is known as the internalization isomorphism

$$I : (TE^{\to}; E^*, TM; M) \to (TE^*; E^*, TM; M),$$

see [88] for more details.

In our work the duality between vector bundles and their modules of sections plays an important role. Therefore we want to analyse with more details the sections of D, that in this case give rise to two different spaces $\Gamma(D, A)$ and $\Gamma(D, B)$. As we see in equation (B.3) both have a subsheaf given by $\Gamma_c(D)$. Also it is easy to see that since A and B are vector bundles over M we have that $C^\infty(M)$ is inside of $C^\infty(A)$ and $C^\infty(B)$, therefore $\Gamma(D, A)$ and $\Gamma(D, B)$ are also $C^\infty(M)$-modules.

We say that a section $S \in \Gamma(D, B)$ is a *linear section* if

$$\begin{array}{ccc} D & \xleftarrow[S]{} & B \\ \downarrow & & \downarrow \\ A & \xleftarrow[s]{} & M \end{array} \tag{B.6}$$

is a vector bundle map over some section $s \in \Gamma A$. We denote the linear sections by $\Gamma_l(D, B)$. With an analogous definition we have $\Gamma_l(D, A)$.

It is clear that if $f \in C^\infty(M)$ and (S, s) is a linear section then so is (fS, fs). Therefore linear sections are also $C^\infty(M)$-modules. From the exact sequence (B.5) it follows that $\Gamma_l(D, A)$ and $\Gamma_l(D, B)$ are locally finitely generated and trivial $C^\infty(M)$-modules. Hence there exist $\widehat{A} \to M$ and $\widehat{B} \to M$ vector bundles, such that $\Gamma\widehat{A} = \Gamma_l(D, B)$ and $\Gamma\widehat{B} = \Gamma_l(D, A)$ as $C^\infty(M)$-modules. These vector bundles are called *the fat bundles* and are unique up to isomorphism.

Proposition B.7. *Given a double vector bundle $(D; A, B; M)$ with core C we have that $\Gamma(D, A)$ is generated by $\Gamma_c(D)$ and $\Gamma_l(D, A)$ as $C^\infty(A)$-module and $\Gamma(D, B)$ is generated by $\Gamma_c(D)$ and $\Gamma_l(D, B)$ as $C^\infty(B)$-module. Moreover the fat bundles fit in the exact sequence:*

$$0 \to \mathrm{Hom}(B, C) \to \widehat{A} \to A \to 0 \quad \textit{and} \quad 0 \to \mathrm{Hom}(A, C) \to \widehat{B} \to B \to 0.$$

Recall that on a vector bundle $E \to M$ there is a duality between sections of E and linear functions of E^*. The *linear functions of $E^* \to M$* can be defined using the *Euler vector field* that locally is defined as

$$\mathcal{E} = e^i \frac{\partial}{\partial e^i} \tag{B.7}$$

where e^i are the fibre coordinates of $E \to M$.

Let us explore this duality for a double vector bundle $(D; A, B; M)$. Since D is a vector bundle in two different ways we have two different Euler vector fields, $\mathcal{E}_A, \mathcal{E}_B \in \mathfrak{X}(D)$; moreover they commute i.e. $[\mathcal{E}_A, \mathcal{E}_B] = 0$. Given $f \in C^\infty(D)$ we say that it has bidegree p, q if and only if

$$\mathcal{E}_A(f) = pf, \qquad \mathcal{E}_B(f) = qf. \tag{B.8}$$

We denote these functions by $C^{p,q}(D)$. It is clear that $C^{0,0}(D) = C^\infty(M)$ and that the product of functions satisfies $C^{i,j}(D) \times C^{k,l}(D) \subset C^{i+k,j+l}(D)$.

Proposition B.8. *Let $(D; A, B; M)$ be a double vector bundle. Then there are natural identifications of $C^\infty(M)$-modules between: $C^{1,1}(D^\to) \cong \Gamma_l(D, A)$, $C^{1,1}(D^\uparrow) \cong \Gamma_l(D, B)$ and $C^{1,0}(D^\to) = C^{0,1}(D^\uparrow) \cong \Gamma_c D$.*

Observe that for a vector bundle it is enough to know the constant and linear functions to recover the vector bundle. We conclude this section defining a new category that is equivalent to double vector bundles, as first defined in [41]. The objects in this category are three vector bundles over the same base $A \to M, B \to M$ and $\Omega \to M$ and a surjective map

$$\lambda : \Omega \twoheadrightarrow A \otimes B \tag{B.9}$$

and morphisms between $\lambda : \Omega \twoheadrightarrow A \otimes B$ and $\lambda' : \Omega' \to A' \otimes B'$ are given by $(\Phi; \varphi_A, \varphi_B; \varphi)$ where the first three are vector bundle maps over $\varphi : M \to M'$

satisfying

$$
\begin{tikzcd}
\Omega \arrow[rr, two heads] \arrow[dr] \arrow[dd, "\Phi"] & & A \otimes B \arrow[dl] \arrow[dd, "\varphi_A \otimes \varphi_B"] \\
& M \arrow[dd, "\varphi"] & \\
\Omega' \arrow[rr, two heads] \arrow[dr] & & A' \otimes B' \arrow[dl] \\
& M' &
\end{tikzcd}
\tag{B.10}
$$

Proposition B.9 (see [41])**.** *There exists an equivalence of categories between double vector bundles and the category defined above given by $\lambda : \Omega \twoheadrightarrow A \otimes B$ goes to $(D(\Omega); A, B; M)$ where $D(\Omega) = \{(\omega, a, b) \in \Omega \times_M A \times_M B \mid \lambda(\omega) = a \otimes b\}$ and $(\Phi; \varphi_A, \varphi_B; \varphi)$ goes to $(D(\Phi); \varphi_A, \varphi_B; \varphi)$. Moreover Ω^* and $\widehat{C^*}$ are isomorphic vector bundles over M, where the vector bundle $\widehat{C^*}$ satisfies that $\Gamma\widehat{C^*} \cong \Gamma_l(D(\Omega)^{\rightarrow}, B)$.*

B.2 VB-groupoids and VB-algebroids

Roughly speaking, VB-groupoids and VB-algebroids are groupoids or algebroids in the category of vector bundles. Since the works of Mackenzie [16, 87] they emerge as natural objects in Poisson geometry connecting different aspects of the theory, see e.g. [22, 52, 59, 60, 88, 90, 100].

A *VB-groupoid*

$$
\begin{tikzcd}
H \arrow[r, shift left, "\widetilde{t}"] \arrow[r, shift right, "\widetilde{s}"'] \arrow[d] & E \arrow[d] \\
G \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & M
\end{tikzcd}
\tag{B.11}
$$

is given by two groupoids $(H \rightrightarrows E, \widetilde{s}, \widetilde{t}, \widetilde{m}, \widetilde{u}, \widetilde{i})$ and $(G \rightrightarrows M, s, t, m, u, i)$ such that $H \to G$ and $E \to M$ are vector bundles satisfying the following conditions:

- $(\widetilde{s}, s)$ and $(\widetilde{t}, t)$ are vector bundle morphisms.
- $H^{(2)} \to G^{(2)}$ is a vector bundle.
- $(\widetilde{m}, m)$ is a vector bundle morphism.

In [60] are compared many equivalent ways of defining VB-groupoids.

On the other hand we can also define their infinitesimal counterpart. A *VB-algebroid* is a double vector bundle $(D; A, B; M)$ such that the vector bundle $(D \to B, [\cdot, \cdot], \rho)$ is a Lie algebroid satisfying:

- $\rho : (D; A, B; M) \to (TB; TM, B; M)$ is a double vector bundle morphism.
- $[\Gamma_l(D, B), \Gamma_l(D, B)] \subseteq \Gamma_l(D, B)$.
- $[\Gamma_l(D, B), \Gamma_c(D)] \subseteq \Gamma_c(D)$.
- $[\Gamma_c(D), \Gamma_c(D)] = 0$.

Other definitions can be found in [59]. With this definition it is not apparent but it happens that the vector bundle $A \to M$ inherits also the structure of a Lie algebroid. As one may expect, there is a Lie functor relating VB-groupoids and VB-algebroids and also an integration theory, see [22].

One of the main results of [59] relates 2-term representations up to homotopy and VB-algebroids:

Theorem B.10 (see [59]). *Let $(D; A, B; M)$ be a double vector bundle with core C. If $(D; A, B; M)$ is a VB-algebroid then any decomposition induces a 2-term representation up to homotopy of $(A \to M, [\cdot,\cdot], \rho)$ on the graded vector bundle $C \oplus B$.*

We already mentioned that double vector bundles have an interesting theory of duality. We will comment now on the duality for VB-algebroids.

Given a double vector bundle $(D; A, E; M)$ we say that it is *double linear Poisson* if (D, π) is a Poisson manifold such that

$$\mathcal{L}_{\mathcal{E}_A}\pi = -\pi \qquad \text{and} \qquad \mathcal{L}_{\mathcal{E}_B}\pi = -\pi \tag{B.12}$$

where $\mathcal{E}_A$ and $\mathcal{E}_B$ denote the Euler vector fields of D.

Proposition B.11 (see [89]). *Let $(D; A, B; M)$ be a double vector bundle with core C. The following are equivalent:*

- *$(D; A, B; M)$ is a VB-algebroid.*
- *$(D^{\to}, C^*, B; M)$ is double linear Poisson.*
- *$(D^{\uparrow}; A, C^*; M)$ is a VB-algebroid.*

The duality theory for VB-algebroids makes clear that VB-algebroids appear in pairs. Since VB-algebroids are the infinitesimal counterpart of VB-groupoids, that means that VB-groupoids must also appear in pairs. We just enumerate here the results, see [60] for a detailed explanation of the phenomenon.

Let $H \rightrightarrows E$ be a VB-groupoid over the Lie groupoid $G \rightrightarrows M$. We can define a new vector bundle over M known as the *core* and denoted by C such that

$$\begin{array}{ccc} H^* & \rightrightarrows & C^* \\ \downarrow & & \downarrow \\ G & \rightrightarrows & M \end{array}$$

is also a VB-groupoid.

Finally this dual VB-groupoid fits in a nice cube in the cotangent prolongation that we will explain now. Recall from Example B.3 that any vector bundle defines a tangent double vector bundle and since the tangent is also a VB-groupoid we can find the following cube

$$\begin{tikzcd}
TH \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] \arrow[dd] & & TE \arrow[dr] \arrow[dd] & \\
& H \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd] & & E \arrow[dd] \\
TG \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] & & TM \arrow[dr] & \\
& G \arrow[rr, shift left] \arrow[rr, shift right] & & M
\end{tikzcd}$$

Now as we see in Example B.6 the cotangent prolongation is one of the duals of the tangent prolongation. And it is well known that the cotangent of a groupoid forms a VB-groupoid (that in fact is symplectic) with base the dual of the Lie algebroid. Therefore we have the following picture:

$$\begin{tikzcd}
T^*H \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] \arrow[dd] & & A_H^* \arrow[dr] \arrow[dd] & \\
& H \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd] & & E \arrow[dd] \\
H^* \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] & & C^* \arrow[dr] & \\
& G \arrow[rr, shift left] \arrow[rr, shift right] & & M
\end{tikzcd} \tag{B.13}$$

Moreover, if $(L \to K) \rightrightarrows (F \to N)$ is a VB-subgroupoid of $(H \to G) \rightrightarrows (E \to M)$ with core C_L, then the conormal bundle

$$\begin{tikzcd}
(TL)^\circ \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] \arrow[dd] & & (A_L)^\circ \arrow[dr] \arrow[dd] & \\
& L \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd] & & F \arrow[dd] \\
L^\circ \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] & & (C_L)^\circ \arrow[dr] & \\
& K \arrow[rr, shift left] \arrow[rr, shift right] & & N
\end{tikzcd} \tag{B.14}$$

is naturally a subgrupoid of the cotangent cube B.13.

These examples give the first examples of groupoids in the category of double vector bundles.

B.3 2-manifolds as double vector bundles

We already proved that the category VB2 is equivalent to $\mathcal{C}o\mathcal{B}^2$ and therefore also to $\mathcal{GM}^2$, statement that originally appeared in [24]. Now we will give another interpretation of the category $\mathcal{GM}^2$ in terms of double vector bundles. This was proved independently by [48, 70, 82] and was extended to other degrees by [121].

B.3.1 Another geometric description of 2-manifolds

An *involutive double vector bundle* is a double vector bundle $(D; E, E; M)$ with two equal sides endowed with a double vector bundle morphism

$$J = (J; -\mathrm{Id}, \mathrm{Id}; \mathrm{Id}) : (D; E, E; M) \to (D^f; E, E; M)$$

such that the morphism induced on the cores is the identity. If the sides are clear we will denote involutive double vector bundles just by (D, J). Given two involutive double vector bundles $(D, J), (D', J')$ a morphism between them is a double vector bundle morphism Φ that intertwines the involutions, i.e.

$$J' \circ \Phi = \Phi^f \circ J$$

Therefore, involutive double vector bundles with morphisms between them form a category.

Remark B.12. A direct consequence of the definition is that J is an isomorphism of double vector bundles between D and its flip.

As we said in the previous section, double vector bundles have a rich theory of duality so let us introduce another incarnation of involutive double vector bundles.

Let $(D; A, B; M)$ be a double vector bundle. We say that it is a *metric double vector bundle* if $D \to A$ has a linear (non-degenerate) metric, i.e. $\langle \cdot, \cdot \rangle : \Gamma D \times \Gamma D \to C^\infty(A)$, satisfying

$$\begin{cases} \langle \Gamma_l(D, A), \Gamma_l(D, A) \rangle \subseteq C^1(A), \\ \langle \Gamma_l(D, A), \Gamma_c D \rangle \subseteq C^0(A) = C^\infty(M), \\ \langle \Gamma_c D, \Gamma_c D \rangle = 0. \end{cases} \tag{B.15}$$

Proposition B.13 (see [70])**.** *Let $(D; A, B; M)$ be a double vector bundle with core C. Any double vector bundle isomorphism between $(D; A, B; M)$ and $(D^\uparrow; A, C^*; M)$ induces a metric on $(D; A, B; M)$ and, conversely, given a metric, the map $\langle , \rangle^b$: $(D; A, B; M) \to (D^\uparrow; A, C^*; M)$ is a double vector bundle isomorphism.*

From the proposition, one clearly sees that $B \cong C^*$. Now we can prove that involutive double vector bundles and metric double vector bundles are dual objects:

Proposition B.14 (see [48, 70])**.** *Let $(D; A, B; M)$ be a double vector bundle. Then $(D; A, B; M)$ is a metric double vector bundle if and only if $(D^\rightarrow; B, B; M)$ is involutive.*

Proof. Given a metric double vector bundle $(D; A, B; M)$ we know by the previous Proposition B.13 that $\langle , \rangle^b : (D; A, B; M) \to (D^\uparrow; A, C^*; M)$ is a double vector bundle isomorphism. Then the horizontal dual has the two sides equal, i.e. $(D^\rightarrow; B, B; M)$. Moreover, taking duals we have that D is isomorphic to $D^\uparrow$ if and only if $D^\rightarrow$ is isomorphic to $D^{\uparrow\rightarrow}$ that, by Proposition B.5, is isomorphic to $D^{\rightarrow f}$. Then we define J as the composition of this isomorphism. Checking that it induces Id on the core and one side and $-\mathrm{Id}$ on the other side is easy.

Conversely, suppose that $(D^\rightarrow, J)$ is involutive. Then we have that $D^\rightarrow$ is isomorphic to $(D^\rightarrow)^f$ and by Proposition B.5 we have that $D^\rightarrow$ and $D^{\uparrow\rightarrow}$ are isomorphic. Hence taking horizontal duals we conclude that D is isomorphic to $D^\uparrow$ and by the previous Proposition B.13 metric. $\square$

Theorem B.15 (see [48, 70])**.** *The category $\mathcal{GM}^2$ and the category of involutive double vector bundles are equivalent.*

Proof. We will prove that the category VB2 is equivalent to involutive double vector bundles. Let $(E_1, E_2, \psi : E_2 \to E_1 \wedge E_1)$ be an object in VB2. Define $\Omega = E_2 \oplus Sym^2 E$ and

$$\begin{array}{cccc} J: & \Omega & \to & \Omega \\ & e + a \cdot b & & J(e + a \cdot b) = e - a \cdot b \end{array}$$

By Proposition B.9 we have a double vector bundle $(D(\Omega); E_1, E_1; M)$ and J defines a double vector bundle morphism between $D(\Omega)$ and $D(\Omega)^f$ that by definition induce Id on the core and the horizontal side and $-\mathrm{Id}$ on the vertical side. Therefore $(D(\Omega), J)$ is involutive.

Now let $((D; E, E; M), J)$ be an involutive double vector bundle. By Proposition B.9 we obtain a sequence $\Omega \twoheadrightarrow E \otimes E$ and a map $J : \Omega \to \Omega$. Define $E_1 = E$ and $E_2 = \{a \in \Omega \mid J(a) = a\}$. Then $(E_1, E_2, \psi : E_2 \twoheadrightarrow E_1 \wedge E_1)$ is what we want. At the level of morphism the correspondence is left to the reader. □

In order to make more clear the above correspondence, let us introduce a definition. Given an involutive double vector bundle (D, J) we say that a function $f \in C^\infty(D)$ is *involutive* if $J^* f = f$. We denote the set of involutive functions by $C^\infty(D)^J$.

Proposition B.16. *Let $((D; E^*, E^*; M), J)$ be an involutive double vector bundle. Denote by $\mathcal{M} = (M, C_{\mathcal{M}})$ the 2-manifold equivalent to it. Then*

- $C^0_{\mathcal{M}} = C^\infty(M) = C^{0,0}(D) = C^{0,0}(D) \cap C^\infty(D)^J$.
- $C^1_{\mathcal{M}} = \Gamma E = C^{1,0}(D) = C^{0,1}(D) = C^{0,1}(D) \cap C^\infty(D)^J$.
- $C^2_{\mathcal{M}} = C^{1,1}(D) \cap C^\infty(D)^J$.

Let (D, J) be an involutive double vector bundle. An *involutive double vector subbundle* is another involutive double vector bundle (H, J') with an injective double vector bundle morphism that commutes with the involutions.

The following result is straightforward:

Proposition B.17. *Let $\mathcal{M}$ be a degree 2 manifold equivalent to (D, J). Then there is a one to one correspondence between submanifolds of $\mathcal{M}$ and involutive double vector subbundles of (D, J).*

Recall that given a vector bundle with metric $(E \to M, \langle\cdot,\cdot\rangle)$, a subbundle $F \to N$ is called *self-orthogonal* if $F = F^\perp = \{v \in E \mid \langle v, w\rangle = 0\ \forall w \in F\}$.

Remark B.18. In the literature usually these subbundles are called lagrangians, but since we deal with symplectic manifold we prefer to avoid this terminology.

Proposition B.19. *Let $((D; E, E; M), J)$ be an involutive double vector bundle with core C. There is a one to one correspondence between involutive double vector subbundle of (D, J) and self-orthogonal double vector subbundles of $(D^{\to}, \langle\cdot,\cdot\rangle)$. Moreover the correspondence sends $((H; F, F; N), J')$ with core K to $(H^\circ; K^\circ, F; N)$.*

Proof. Let $(H; F, F; N)$ with core K be an double vector subbundle of the involutive double vector bundle $((D; E, E, M), J)$ with core C. By Proposition B.14 we know that $((D \to; C^*, E, M), \langle\cdot,\cdot\rangle)$ is a metric double vector bundles. Moreover, if we denote by $H^\circ = \{v \in D^{\to}{}_{|F} | \langle v, h\rangle = 0 \forall h \in H\}$ and by $K^\circ = \{c \in C^*_{|N} | \langle c, k\rangle = 0 \forall k \in K\}$ then we have the following exact sequence of vector bundles

$$\begin{tikzcd}
0 \arrow[r] & H^\circ \arrow[r] \arrow[d] & D^{\to}_{|N} \arrow[r] \arrow[d] & H^{\to} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K^\circ \arrow[r] & C^*_{|N} \arrow[r] & K^* \arrow[r] & 0
\end{tikzcd}$$

Using Proposition B.14 we have that $((H; F, F; N), J)$ is involutive if and only if $((H^{\to}; K^*, F; N), \langle\cdot,\cdot\rangle)$ is metric and from the above exact sequence we obtain that $((H^{\to}; K^*, F; N), \langle\cdot,\cdot\rangle)$ is metric if and only if $(H^\circ; K^\circ, F; N)$ is a self-orthogonal double vector subbundle of $((D^{\to}; C^*, E; M), \langle\cdot,\cdot\rangle)$. □

B.3.2 Degree 2 Q-manifolds: VB-Courant algebroids

Now we can use Section B.3.1 and codify degree 2 Q-manifolds in terms of metric double vector bundles. Let start by identifying vector fields on a degree 2 manifold with objects on the associated involutive double vector bundle.

Proposition B.20. *Let $((D; E^*, E^*; M), J)$ be an involutive double vector bundle with core C^*. Denote by $\mathcal{M} = (M, C_{\mathcal{M}})$ the 2-manifold equivalent to it. Then*

$$\mathfrak{X}^{1,-2}(\mathcal{M}) = C^{1,0}(D^{\to}) \quad \textit{and} \quad \mathfrak{X}^{1,-1}(\mathcal{M}) = \Gamma_l(D^{\to}, C).$$

Moreover, the metric on $(D^{\to}; C, E^*; M)$ is computed in the following equations

$$\langle \Gamma_l(D^{\to}, C), \Gamma_l(D^{\to}, C)\rangle = [\xi_1, \xi_2] \textit{ and } \langle \Gamma_l(D^{\to}, C), \Gamma_c D^{\to}\rangle = \xi_1(e)$$

where $\xi_1, \xi_2 \in \Gamma_l(D^{\to}, C) = \mathfrak{X}^{1,-1}(\mathcal{M})$, $e \in \Gamma E = \Gamma_c(D^{\to}) = C^{1,0}(D) = C^1_{\mathcal{M}}$.

A *VB-Courant algebroid*, see [82], is a metric double vector bundle $(D; A, B; M)$ such that $(D \to A, \langle\cdot,\cdot\rangle, \rho, [\![\cdot,\cdot]\!])$ is a Courant algebroid, with the same metric, satisfying:

- The anchor $\rho : (D; A, B; M) \to (TA; A, TM; M)$ is a double vector bundle morphism.
- The bracket , $[\![\cdot,\cdot]\!] : \Gamma(D, A) \times \Gamma(D, A) \to \Gamma(D, A)$ satisfy:
  - a). $[\![\Gamma_l(D, A), \Gamma_l(D, A)]\!] \subset \Gamma_l(D, A)$.
  - b). $[\![\Gamma_l(D, A), \Gamma_c D]\!] \subset \Gamma_c D$.
  - c). $[\![\Gamma_c D, \Gamma_c D]\!] = 0$.

A VB-Dirac structure is a double vector subbundle $(L; E, F; N) \subseteq (D; A, B; M)$ such that $L \to E$ is a Dirac structure of $D \to A$. Recall that a subbundle $L \to E$ is a Dirac structure of $(D \to A, \langle\cdot,\cdot\rangle, \rho, [\![\cdot,\cdot]\!])$ if and only if

$$L = L^\perp, \quad \rho(L) \subseteq TN, \quad [\![\Gamma L, \Gamma L]\!] \subseteq \Gamma L.$$

Remark B.21. Sometimes such Dirac structures are called Dirac structures with support in N, see [27], but we just refer to Dirac structures.

What we want to prove next is the following geometric characterization of degree 2 Q-manifolds in terms of double vector bundles:

Theorem B.22 (See [82]). *Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be a degree 2 manifold equivalent to the involutive double vector bundle $((D; E, E; M), J)$ with core C. Then $\mathcal{M}$ is a Q-manifold if and only if $((D^{\to}; C^*, E; M), \langle\cdot,\cdot\rangle)$ is a VB-Courant algebroid.*

Moreover the correspondence sends Q-submanifolds to VB-Dirac structures.

But for that we first need an auxiliary lemma. Consider an n-manifold $\mathcal{M}$ and $(\mathcal{N}, j)$ a closed embedded submanifold of $\mathcal{M}$ with associated ideal $\mathcal{I}$. Define the *Lie normalizer of* $\mathcal{I}$ as

$$N_{\mathcal{I}} = \{X \in \mathfrak{X}(\mathcal{M}) \mid X(\mathcal{I}) \subset \mathcal{I}\}.$$

Lemma B.23. *Let $(\mathcal{M}, Q)$ be a Q-manifold and $(\mathcal{N}, j)$ a closed embedded submanifold of $\mathcal{M}$ with associated ideal $\mathcal{I}$. Then $(\mathcal{N}, j)$ is a Q-submanifold if and only if*

$$\forall X \in N_{\mathcal{I}}, \quad [Q, X] \in N_{\mathcal{I}}.$$

Proof. Observe that one of the implications is trivial. For the other we will prove the contrapositive. Since vector fields are local operators we can do the proof locally taking coordinates.

Suppose that $\mathcal{M}$ has coordinates $\{e^i\}_{i=1}^n$ (it is not necessary to know the degree for this proof) and $\mathcal{I} = \langle e^1 = \cdots = e^k = 0\rangle$. Therefore we obtain that

$$N_{\mathcal{I}} = \langle \frac{\partial}{\partial e_{k+1}}, \cdots, \frac{\partial}{\partial e_n} \rangle_{C_{\mathcal{M}}}$$

Suppose now that $Q(\mathcal{I}) \nsubseteq \mathcal{I}$. That means that there exists $j \in \{1, \cdots, k\}$ such that $Q = f\frac{\partial}{\partial e^j} + Q'$ where $0 \neq f \in C_{\mathcal{M}}$ with $\frac{\partial}{\partial e^i}(f) = 0$ $\forall i \in \{1, \cdots, k\}$ and $\frac{\partial}{\partial e^j}$ is not in Q'.

Since $f \neq 0$ and $\frac{\partial}{\partial e^i} f = 0 \forall i \in \{1, \cdots, k\}$ implies that there exists $l \in \{k+1, \cdots, n\}$ such that $\frac{\partial}{\partial e^l} f \neq 0$. Hence $\frac{\partial}{\partial e^l} \in N_{\mathcal{I}}$ and $[Q, X] \notin N_{\mathcal{I}}$ as we want. $\square$

Proof of Theorem B.22. By Proposition B.14 we know that $(D^{\to}; C^*, E; M)$ is a metric double vector bundle. Hence it remains to see that when $\mathcal{M}$ is a Q-manifold $(D^{\to} \to C^*, \langle\cdot,\cdot\rangle)$ has a bracket and anchor that makes it into a VB-Courant algebroid. Recall also tha by Propositions B.16, B.20 and B.8 we see that the linear and core section of $(D^{\to}; C^*, E; M)$ can be expressed as:

$$\Gamma_l(D^{\to}, C^*) = \mathfrak{X}^{1,-1}(\mathcal{M}), \quad \Gamma_c(D^{\to}) = C^1_{\mathcal{M}}, \quad C^{1,0}(TC^*) = \Gamma C = \mathfrak{X}^{1,-2}(\mathcal{M}). \tag{B.16}$$

$(\Rightarrow)$

Let us suppose that $(\mathcal{M}, Q)$ is a Q-manifold. Define the anchor and the bracket by the following equalities

$$\begin{cases} [\![\phi_1, \phi_2]\!] = \big[[\phi_1, Q], \phi_2\big], \quad [\![\phi_1, e]\!] = [\phi_1, Q](e), \\ \rho(e)(f) = 0, \quad \rho(e)(\psi) = [Q, \psi](e) \\ \rho(\phi)(f) = [\phi, Q](f), \quad \rho(\phi)(\psi) = \big[[\phi, Q], \psi\big], \end{cases} \tag{B.17}$$

where $\phi_1, \phi_2 \in \Gamma_l(D^{\to}, C^*)$, $e \in \Gamma_c(D^{\to})$, $\psi \in C^{1,0}(TC^*)$ and $f \in C^\infty(M)$.

Verifying that the bracket and the anchor satisfy the Courant axioms is just an easy calculation using the derived bracket and the fact that $[Q, Q] = 0$.

$(\Leftarrow)$

Let suppose that $(D^{\to}; C^*, E; M), \langle\cdot,\cdot\rangle, \rho, [\![\cdot,\cdot]\!])$ is a VB-Courant algebroid. In order to define a Q in $\mathcal{M}$ is enough to define two maps

$$Q_2 : \mathfrak{X}^{1,-2}(\mathcal{M}) \to \mathfrak{X}^{1,-1}(\mathcal{M}) \quad \text{and} \quad Q_1 : \mathfrak{X}^{1,-1}(\mathcal{M}) \to \mathfrak{X}^{1,0}(\mathcal{M})$$

and then we recover Q by the formula $[Q, \psi] = Q_2(\psi)$, $[Q, \phi] = Q_1(\phi)$. Recall also that in order to define a vector field it is enough to know how they act on the generators of the graded manifold. Hence we define

$$Q_2(\psi)(e) = \rho(e)(\psi), \quad Q_1(\phi)(f) = \rho(\phi)(f), \quad Q_1(\phi)(e) = [\![\phi, e]\!],$$

where $f \in C^\infty(M)$, $e \in C^1_{\mathcal{M}}$, $\psi \in \mathfrak{X}^{1,-2}(\mathcal{M})$, $\phi \in \mathfrak{X}^{1,-1}(\mathcal{M})$ and a priory, we not know how to express

$$Q_2(\psi)(\xi), \quad Q_1(\phi)(\xi),$$

where $\xi \in C^2_{\mathcal{M}}$, because degree 2 two functions are not easy to express in terms of the double vector bundle. But recall that in order to know a degree 1 or 2 function on a graded manifold it is enough to know all their derivatives by vector fields of negative degree.

Hence in order to know $Q_2(\psi)(\xi) \in C^1_{\mathcal{M}}$ it is enough to know $\phi\big(Q_2(\psi)(\xi)\big)$ and using the Jacobi identity of the bracket and the first inequality we obtain that it is enough to know

$$\phi\big(Q_2(\psi)\big) = \rho(\phi)(\psi).$$

With a similar deduction we obtain that in order to know $Q_1(\phi)(e)$ it is enough to know

$$\phi'\big(Q_1(\phi)\big) = [\![\phi', \phi]\!], \quad \text{and} \quad \psi\big(Q_1(\phi)\big) = -\rho(\phi)(\psi).$$

As before, the identity $[Q, Q] = 0$ follows from the Courant axioms and the Jacobi identity of the Lie bracket of vector fields.

For the moreover part notice that in Proposition B.19 we prove that submanifolds of $\mathcal{M}$ are in one to one correspondence with self-orthogonal subbundles of $(D; C^*, E; M)$. Therefore it is enough to see that Q is tangent to the submanifold if and only if it is Dirac.

Suppose by simplicity that our manifold is closed and embedded, otherwise we work locally. By Proposition B.23 we have that Q is tangent to the submanifold if only if $\forall X \in \mathcal{N}_{\mathcal{I}}$ then $[Q, X] \in N_{\mathcal{I}}$, where $\mathcal{I}$ denotes the associated vanishing ideal of the submanifold.

By the equalities B.16 we have that $\mathcal{N}_{\mathcal{I}}$ is generated in negative degrees by the linear sections of the submanifold and the sections of the base. Finally, equations B.17 are equivalent to $\mathcal{L}_Q(\mathcal{N}_{\mathcal{I}}) \subseteq \mathcal{N}_{\mathcal{I}}$. □

Remark B.24. The proof in [82] is maybe more conceptual than this one but there it is not easy to see that the Courant axioms really correspond to $[Q, Q] = 0$. Our proof emphasizes that point. On the other hand, the proof appearing in the works [70, 73] is done by choosing a splitting of the graded manifold and a decomposition of the double vector bundle and comparing the results; we explain this in Example 3.35.

B.3.3 Degree 2 Poisson manifolds: Involutive double linear Poisson

At this point we can also characterize degree 2 Poisson manifolds in terms of involutive double vector bundles.

We say that $((D; E, E; M), J, \pi)$ is an *involutive double linear Poisson* if (D, J) is an involutive double vector bundle, (D, π) is double linear Poisson (i.e. π is a Poisson bivector field on D and $\mathcal{L}_{\mathcal{E}_E}\pi = -\pi$ for both Euler vector fields on D) and $J^*\pi = \pi$.

In Section B.2 we stated that the dual of a double linear Poisson is a VB-algebroid and Proposition B.14 stated that the dual of an involutive double vector bundle is a metric double vector bundle. Therefore we can make the following definition: a *metric VB-algebroid* is a double vector bundle $(D; A, B; M)$ such that

- $(D; A, B; M)$ is metric, i.e. $D \to A$ carries a linear metric $\langle\cdot,\cdot\rangle$ satisfying equations B.15.
- $(D; A, B; M)$ is a VB-algebroid, see Section B.2 for definition.
- $\langle,\rangle^b : (D; A, B; M) \to (D^{\uparrow}; A, B; M)$ is a VB-algebroid isomorphism.

As a direct consequence of the definition we have:

Proposition B.25. *Let $((D; E^*, E^*; M), J)$ be an involutive double vector bundle with core C^*. Then $((D; E^*, E^*; M), J, \pi)$ is involutive double linear Poisson if and only if $((D^{\to}; C, E^*; M), \langle\cdot,\cdot\rangle)$ is a metric VB-algebroid.*

We state the geometrization of degree 2 Poisson manifolds in terms of double vector bundles:

Theorem B.26 (see [48, 70]). *Let $\mathcal{M} = (M, C_{\mathcal{M}})$ be a 2-manifold and consider an involutive double vector bundle equivalent to it $((D; E^*, E^*; M), J)$. Then $\mathcal{M}$ is Poisson with a Poisson bracket of degree -2 if and only if $((D; E^*, E^*; M), J)$ is involutive double linear Poisson.*

Proof. The conceptual way of proving this result is just by seeing that since π is double linear Poisson then it has bidegree $(-1, -1)$ and the way of passing from double vector bundles to graded manifolds is by taking as functions the total complex. Then it defines a Poisson structure of degree -2 and vice-versa.

But let us give a more hands on proof also. We will see that $(E_1 \to M, E_2 \to M, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \rho, \nabla)$ as in Proposition 3.39 defines a Poisson manifold if and only if the metric double vector bundle $(D; \widetilde{E}_2, E_1^*; M)$ is a metric VB-algebroid.

With this definition we obtain the following identifications:

$$\Gamma_l(D, E_1^*) = \Gamma E_2 \oplus \Gamma \operatorname{Sym}^2 E_1, \quad \Gamma_c(D) = \Gamma E_1.$$

Denote the bracket on the VB-algebroid by $[\cdot,\cdot]_{VB}$ and the anchor by $a : (D; \widetilde{E}_2, E_1^*; M) \to (TE_1^*; TM, E_1^*; M)$. Therefore we state the following formulas:

$$\begin{cases} [\xi + s, \xi' + s']_{VB} = [\xi, \xi'], \quad [\xi, e] = \nabla_\xi e. \\ a(\xi)(e) = \nabla_\xi e, \quad a(\xi)(f) = \rho(\xi)(f), \quad a(e)(e') = \langle e, e'\rangle, \end{cases} \tag{B.18}$$

where $s, s' \in \Gamma \operatorname{Sym}^2 E_1, \xi, \xi' \in \Gamma E_2, e, e' \in \Gamma E_1, f \in C^\infty(M)$. Then the equations 3.9 guarantee that the structure is a metric VB-algebroid and vice-versa. Given a metric VB-algebroid the structures defined in the above way define a degree 2 Poisson manifold. $\square$

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