# Generalized Graph Product and its Application on Generating Solutions of the Millennium $n$-Queens Problem


## J. G. Silva[1]; O. Kolossoski[2]; L. C. Matioli[3] & E. M. R. Torrealba[4].

julianaufpi@gmail.com[1]; oliver.kolossoski@gmail.com[2]; lcmatioli@gmail.com[3]; relvis.375@gmail.com[4]

PPGM UFPR Programa de Pós-Graduação em Matemática



### Abstract

The generalized Kronecker graph product was introduced by Figueiroa-Centeno et. al. and later Baca et. al. used it for obtaining solutions of the $n$-queens problem on bigger boards from solutions of boards of lower size. In this paper we generalize the graph product and the recent results by Baca et. al., obtaining a bigger class of solutions knowing solutions on boards with lower size in advance. We also find an example where a result by Baca et. al. concerning the graph product related to modular solutions does not hold.


### Introduction

The $n$-queens problem was proposed by Bezzel in 1848 [2] for the $8 \times 8$ board consisting on placing 8 queens on it without any queen attacking each other. In 1869 Lionnet [4] generalized it for the $n$-queens problem on a $n \times n$ board. Gauss attempted to solve the original $8 \times 8$ problem in circa 1850, managing to find 72 solutions. Later, in 1874, Pauls found all the 92 solutions.

The problem of finding a single solution for the $n \times n$ board has long been solved and nowadays we have various closed formulas for known solutions. Nevertheless, the problem of finding or classifying all the solutions for an abstract given $n$ remains unsolved.

### Objectives

- In this work we study a way of obtaining solutions of boards of bigger size from solutions of boards of lower size.
- The idea is to generalize the method of [1], where Baca et. al. used the graph product $\otimes_h$ introduced in [3] to generate solutions of boards of size $mn \times mn$ from solutions of boards with sizes $m \times m$ and $n \times n$.
- We extend the result, showing that it is possible to obtain an even bigger class of solutions of bigger boards knowing solutions of smaller boards in advance.

### Notations

A graph $G = (V, E)$ is a set of vertices $V$ and a set of edges $E$, where the elements of $E$ are pairs of elements $(v_1, v_2)$ (abbreviated $v_1v_2$) of $V$. When the pairs of elements that compose edges of $E$ are ordered we say that $G$ is a digraph.

We denote the set of the solutions of the classic problem $Q_n$ and of the modular problem $M_n$. A solution $Q \in Q_n$ (resp. $M_n$) can be described as a function $\pi : \{1, \cdots, n\} \to \{1, \cdots, n\}$ such that $\pi(i) = j$ if in the placement of $Q$ on the board there is a queen on the row $i$ and on the column $j$.

The set of permutations of $\{1, \cdots, n\}$ is denoted $\Pi_n$, therefore it holds $Q_n \subset \Pi_n$ and if $n \mod 6 = \pm 1$ it holds $M_n \subset Q_n \subset \Pi_n$.

### Results

Given a family of digraphs $\Gamma$ with the same set of vertices $V$, a digraph $D$ and a function $h : E(D) \to \Gamma$, the product $D \otimes_h \Gamma$ is the digraph satisfying:

$$\begin{aligned} &V(D \otimes_h \Gamma) = V(D) \times V \\ &(ai, bj) \in E(D \otimes_h \Gamma) \Leftrightarrow ab \in E(D), \quad ij \in E(h(ab)). \end{aligned} \tag{1}$$

**Theorem 1.** *Let $D \in \Pi_m$ and consider a family $\Gamma = \{\Gamma_1, \cdots, \Gamma_m\}$ of $m$ solutions $\Gamma_k \in Q_{i_k}$ satisfying $i_k \geq 4$ and $\sum_{k=1}^{m} i_k = N$. Consider a function $h : E(D) \to \Gamma$ such that for every two given edges $ab, \bar{a}\bar{b} \in E(D)$ if the assumptions:*

$$H1) \quad s := \left| \sum_{k=1}^{m(\bar{a},\bar{b})} 2i_k + \sum_{k=1+m(\bar{a},\bar{b})}^{M(\bar{a},\bar{b})} i_k - \left( \sum_{k=1}^{m(a,b)} 2i_k + \sum_{k=1+m(a,b)}^{M(\bar{a},\bar{b})} i_k \right) \right|$$

$$H2) \quad d := \left| \left( \pm \sum_{k=a}^{b-1} i_k \right) - \left( \pm \sum_{k=\bar{a}}^{\bar{b}-1} i_k \right) \right| < max\{i_a, i_{\bar{a}}\} \tag{2}$$

*imply that:*

$$\begin{aligned} &T1) \quad [s(h(ab)) - s] \cap s(h(\bar{a}\bar{b})) = \emptyset \\ &T2) \quad [d(h(ab)) - d] \cap d(h(\bar{a}\bar{b})) = \emptyset, \end{aligned} \tag{3}$$

*then $D \widehat{\otimes}_h \Gamma \in Q_N$.*

The example of Figure 1 is a solution in $Q_{18}$ constructed as $D \otimes_h \Gamma$, with $D = (3142) \in \Pi_4$ and $\Gamma = \{(3142), (24135), (13524)\}$.

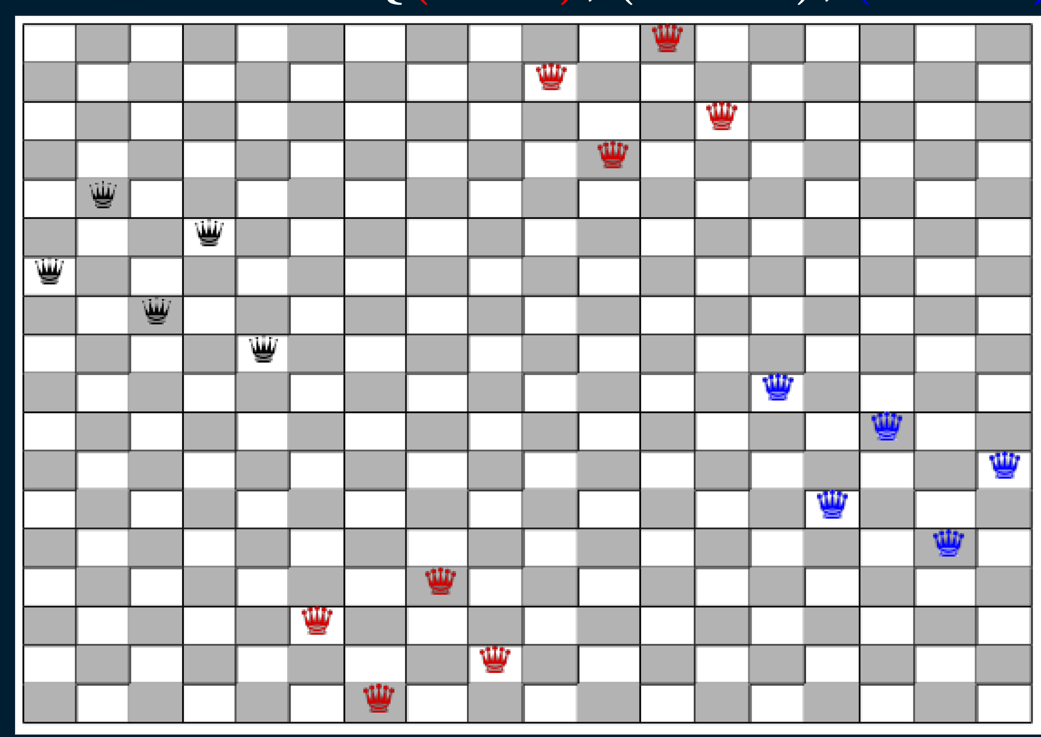

**Figure 1:** Solution in $Q_{18}$ generated by the generalized $\otimes_h$.

**Proposition 2.** *If $D \in M_m$, $\Gamma$ is a family of solutions of $M_n$ and for every pair of edges $(ai, bj), (\bar{a}\bar{i}, \bar{b}\bar{j}) \in E(D \otimes_h \Gamma)$ we have*

$$i \pm j \neq \bar{i} \pm \bar{j} \mod n,$$

*for every edges $ij, \bar{i}\bar{j} \in \Gamma$, given a pair $(ai, bj)$ and $(\bar{a}\bar{i}, \bar{b}\bar{j})$ of edges of $D \otimes_h \Gamma$, then $D \otimes_h \Gamma$ is a modular solution of $M_{mn}$.*

### Conclusion

- In this paper we have generalized the graph $\otimes_h$-product for attaining composite solutions of graphs via adjacency matrix even when the block matrices corresponding to subboards are not of the same size.
- We obtained necessary conditions for constructing such composite solutions for the classical problem in Theorem 1
- We concluded that in a previous result by Baca et. al (namely Theorem 3.1) the authors used a hasty argument which does not necessarily hold and must have been added as an additional hypothesis, as stated in Proposition 2.
- We have extended the theory about the class of composite solutions further.


### References

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