The Decomposition Group of Plane Curves

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Abstract

In June of this year (2018), I met with professor Carolina Araujo to start my master thesis project. She challenged me to try and say something about the structure of the group of Cremona maps that fix some smooth plane curve. At that point in my education, I had taken a first course in classic algebraic geometry a first graduate course in commutative algebra and the course of algebraic curves at IMPA. With these tools, I was able to obtain some hints towards Theorem 4.2.1 but I was still far from understanding what was going on. Two months later, I came across Ivan Pan's paper [12] answering the question proposed by Carolina. After this, the goal of the thesis became to understand Pan's Theorems 1.3 and 1.4. I realized that to do this, I needed to learn some modern algebraic geometry. That is why this work has two major parts.

The first part consists of Chapter 2. This chapter is a survey of the main concepts of modern algebraic geometry that are related in some degree to Pan's paper. Very few proofs are given, and the focus lies on examples. I wrote this chapter mostly to understand these topics myself.

The second part consists of Chapters 3 and 4. In these chapters we specify the concepts discussed in Chapter 2 to the case of smooth surfaces and furthermore to the case of birational automorphisms of the complex projective plane. We finish by discussing the proofs of the theorems mentioned before.

Contents

Chapter 1

Introduction

1.1 Decomposition Groups in Number Theory

Which prime numbers can be written as a sum of two squares?

Although Fermat is usually credited with the answer, the first known proof is due to Euler. He showed that a prime p is a sum of two squares if and only if $p = 2 = 1^2 + 1^2$ or $p \equiv 1 \pmod 4$. A little bit of ring theory allowed Dedekind to give two different proofs using the algebraic properties of the Gaussian integers.

Suppose that $p = a^2 + b^2$ for some positive integers a, b. Then we can factor $p = (a + ib)(a - ib)$ in the ring $\mathbb{Z}[i]$, so that every rational prime that is expressible as a sum of two squares is reducible. Reciprocally, assume p is reducible in $\mathbb{Z}[i]$. Then $p = (a + ib)(c + id)$ with $a, b, c, d > 0$ since the units in $\mathbb{Z}[i]$ are $\{\pm 1, \pm i\}$. Then,

$$(a^2 + b^2)(c^2 + d^2) = (a + ib)(c + id)\overline{(a + ib)(c + id)} = p^2.$$

Since $\mathbb{Z}$ is a unique factorization domain, we must have that $p = a^2 + b^2$ and we have shown that the prime numbers that are sums of two squares are precisely those that become reducible in the Gaussian integers.

As the previous example shows, understanding the ramification of primes on integral extensions of the integers is an algebraic problem that has very important arithmetical applications. Let A be a Dedekind domain, K its field of fractions, L/K a finite Galois extension and B the integral closure of A in L. Denote by G the Galois group $\mathrm{Gal}(L/K)$. If $\mathfrak{p}$ is a prime ideal in A, and $\mathfrak{P}$ is a prime ideal in B above $\mathfrak{p}$, then so is $\sigma\mathfrak{P}$ for every $\sigma \in G$ since $\sigma\mathfrak{P} \cap A = \sigma(\mathfrak{P} \cap A) = \sigma\mathfrak{p} = \mathfrak{p}$. It can be shown that G acts transitively on the set of primes over $\mathfrak{p}$. Since we are interested in the ramification of $\mathfrak{p}$ in B, it is reasonable to study the group $G_{\mathfrak{P}} := \{\sigma \in G \mid \sigma\mathfrak{P} = \mathfrak{P}\}$. This group is called the *Decomposition Group* of the prime $\mathfrak{P}$, and it is the stabilizer of $\mathfrak{P}$ by the action of the Galois group G on the set of primes in B.

This group can be defined in a much more general and geometrical context through the language of Schemes. The first goal of this work is to construct the category of Schemes and Rational Maps, the setting in which the example just mentioned can be interpreted geometrically. Afterwards, we will restrict ourselves to the category of complex Algebraic Varieties and Rational Maps to study a very particular example of a decomposition group, which we introduce in the following section.

1.2 Purely Transcendental Field Extensions

Given a field k, how does the group of automorphisms of a purely transcendental extension of transcendence degree n looks like?

Given a field, some of the simplest field extensions one can consider are the finitely generated purely transcendental extensions. That is, fields $K_n := k(t_1, \ldots, t_n)$ of rational functions in the variables $t_1, \ldots, t_n$ and coefficients in k. Even though these fields are relatively "simple", their groups of automorphisms are extremely complicated. To get an idea of this, lets consider the case $n = 2$ and put $t_1 = t$ and $t_2 = u$. Several examples of automorphisms of $k(t, u)$ come to mind. For instance:

$$f(t,u) \longmapsto f(u,t), \quad g(t,u) \longmapsto g(t+u, t-u), \quad h(t,u) \longmapsto h(2t, u-1)\,.$$

In particular, for any element γ of the group $\mathrm{Aff}(k^2)$ of affine transformations of the vector space k^2, we have a field extension automorphism $f(t,u) \mapsto f\left(\gamma\left(\begin{smallmatrix} t \\ u \end{smallmatrix}\right)\right)$. Yet, not every field automorphism of $k(t, u)$ comes from an affine transformation.

Definition 1.2.1 (Standard Quadratic Transformation)**.** The *standard quadratic transformation* is the automorphism $\tau \in \mathrm{Aut}_k(k(t, u))$ given by

$$f(t,u) \mapsto f\left(\tfrac{1}{t}, \tfrac{1}{u}\right). \tag{1.1}$$

Although we postulated this question in an algebraic context, we argue that it is in fact more geometric in nature for the following reasons. Given a dominant rational map $\varphi : X \dashrightarrow Y$ between algebraic varieties, and a rational function $f \in K(Y)$, the pull-back of f by φ is a rational function on X. Furthermore, the pull-back defines an equivalence of categories between algebraic varieties over k with dominant rational maps, and finitely generated field extensions of k. Under this equivalence, we have that

$$\mathrm{Aut}_k(k(t_1, \ldots, t_n)) \cong \mathrm{Bir}(\mathbb{A}^n_k) = \mathrm{Bir}(\mathbb{P}^n_k).$$

In particular, the standard quadratic transformation 1.1 takes the form:

$$\tau : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k, \quad (x:y:z) \mapsto \left(\tfrac{1}{x} : \tfrac{1}{y} : \tfrac{1}{z}\right) \tag{1.2}$$

The study of birational automorphisms of projective spaces has been an active research area since the nineteenth century. The main objective is to understand the group of birational automorphisms of n-dimensional projective space, called the *n-Cremona group*. Things get complicated really quickly; in fact the only Cremona group that is completely understood is the 1-Cremona group $\mathrm{Bir}(\mathbb{P}^1_k) = \mathrm{PGL}(2, k)$. The equality follows from the fact that, for any rational map $\phi : X \dashrightarrow \mathbb{P}^n$, from a nonsingular variety X, the closed set of points at which ϕ is not defined has codimension ≥ 2. In dimension 2, we have the celebrated theorem of Max Noether and Guido Castelnuovo:

Theorem 1.2.1 (Noether-Castelnuovo)**.** *Every birational automorphism of the projective plane over an algebraically closed field k, is a composition of projective linear transformations and the standard quadratic transformation 1.2, that is*

$$\mathrm{Bir}(\mathbb{P}^2_k) = \langle \tau, \mathrm{PGL}(3, k) \rangle.$$

When trying to understand a group, a natural approach is to study its subgroups. Given a projective curve $C \subset \mathbb{P}^2_k$, we could consider the subgroup $\mathcal{D}(C) \subset \mathrm{Bir}(\mathbb{P}^2_k)$ of maps fixing C as a set. Lets assume that C is a smooth cubic curve, and take p, q, r three points in C not lying on the same line. Denote by ℓ_p be the line passing through q and r, and so on for ℓ_q and ℓ_r. Choosing (ℓ_p, ℓ_q, ℓ_r) as a coordinate system for $\mathbb{P}^2_k$, we have a birational automorphism

$$Q_{p,q,r} : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k, \quad (\ell_p : \ell_q : \ell_r) \mapsto (\tfrac{1}{\ell_p} : \tfrac{1}{\ell_q} : \tfrac{1}{\ell_r}). \tag{1.3}$$

We will show that $Q_{p,q,r}(C)$ is a smooth cubic isomorphic to C, and that there exists a linear change of coordinates $\varphi \in \mathrm{PGL}(3, k)$ such that $\varphi \circ Q_{p,q,r}$ fixes C. The transformations obtained in this way are called *C-generic transformations*. Assuming these results, the first observation is that $\mathcal{D}(C)$ is an infinite group since there are infinitely many C-generic transformations. Furthermore, we will show that the group $\mathcal{D}(C)$ is generated by the C-generic transformations 4.2.4, a result of Ivan Pan from 2007 [12].

The reader may guess that the $\mathcal{D}$ stands for Decomposition Group. In the following chapters it will become clear that $\mathcal{D}(C)$ is a geometrical analogue of the number theoretical decomposition group considered in the previous section. The second objective of this work is to get acquainted with some of the tools of birational geometry. We do this by studying the decomposition group in the context of birational automorphisms of the complex projective plane.

Prerequisites, Conventions and Notation

- Commutative Algebra: We assume that the reader is familiar with the concepts of a first course in commutative algebra, on the lines of [2]. This includes basic definitions from category theory. Also, we assume knowledge of basic concepts of Kähler differentials. An excellent reference for this topic is [5]. We try to maintain the notations of these references. In particular:

  Unless it is stated otherwise, all rings (usually denoted by A, B, S) will be commutative with identity 1, and all ring homomorphisms take 1 to 1. $A^{\times}$ denotes de group of units of the ring A. In integral domains, $1 \neq 0$. Ideals are usually denoted by lowercase gothic letters. An ideal $\mathfrak{p} \subset A$ is prime if $A/\mathfrak{p}$ is an integral domain, so all prime ideals are proper. $A_{\mathfrak{p}}$ denotes the localization of A over the multiplicative subset $A - \mathfrak{p}$. The same goes for modules.

  A graded ring S is a commutative ring that admits a decomposition into a direct sum of abelian groups $S = \bigoplus_{d \geq 0} S_d$ such that for any $n, m \geq 0$ we have that $S_n S_m \subset S_{n+m}$. Elements in S_d are called homogeneous of degree d. A graded S-module M is an S-module that admits a decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$ such that for any $n, m \geq 0$ we have that $S_n M_m \subset M_{n+m}$.

- Algebraic Geometry: We assume that the reader is familiar with the first chapter of Hartshorne's book [9]. We will use the well known fiber dimension theorem [13, Theorem 1.25]. Given a vector space V, $\mathbb{P}(V)$ denotes the projective space of 1-dimensional sub-spaces of V.

- Algebraic Curves: We will assume the reader is familiar with the book of Fulton [7]. For Chapters 3 and 4, we will work over the complex numbers and use that all conics are rational and that a cubic curve has at most one singular point.

Also, we mark with $\square$ the end of a proof and with $\blacksquare$ the end of an example.

Chapter 2

Preliminaries

In this chapter we give a superficial survey of the language of modern algebraic geometry, focusing on the tools that are going to be useful in the particular setting of the final chapter. Since there are several wonderful references for these topics ([6], [16],[14], [9] to name a few), we skip most of the proofs and focus on examples. We follow the lines of Hartshorne's book for most of the chapter.

2.1 Sheaves

The theory of sheaves is an abstraction of one of the fundamental philosophical lessons of Geometry, which is that geometrical objects are often better understood in terms of the functions one can define on them. The fundamental idea behind this chapter may be that given a set X some algebraic object A, the set X^A of functions $X \to A$ becomes an algebraic object itself by stealing the algebraic operations of A, introducing additional tools to understand X in a relative context. An excellent example to keep in mind when learning about sheaves is the sheaf of differentiable functions on a differentiable manifold M. It associates to each open subset $U \subset M$ the $\mathbb{R}$-algebra $C^\infty(U)$ of infinitely differentiable functions $U \to \mathbb{R}$.

Given a topological space X, consider the category $\mathfrak{Top}(X)$ whose objects are the open subsets of X and the morphisms are given exclusively by inclusion maps $i_{V,U} : V \subset U$.

Definition 2.1.1 (Category of Sheaves of Abelian Groups)**.** Let $\mathfrak{Ab}$ be the category of abelian groups and let X be a topological space. A *presheaf* of abelian groups on X is a contravariant functor $\mathcal{F} : \mathfrak{Top}(X) \to \mathfrak{Ab}$. We denote by $\mathrm{res}^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ the morphism $\mathcal{F}(i_{V,U})$. The presheaf $\mathcal{F}$ is a *sheaf* if it satisfies the following extra conditions:

(a) For every open subset $U \subset X$ and every open cover $\{V_i\}$ of U, if $s \in \mathcal{F}(U)$ is such that $\mathrm{res}^U_{V_i}(s) = 0 \in \mathcal{F}(V_i)$, then $s = 0$ in $\mathcal{F}(U)$.

(b) For every open subset $U \subset X$ and every open cover $\{V_i\}$ of U, if we have elements $s_i \in \mathcal{F}(V_i)$ for each i with the property that for every i, j the restrictions $\operatorname{res}^{U_i}_{U_i \cap U_j}(s_i) = \operatorname{res}^{U_j}_{U_i \cap U_j}(s_j)$, then there exists an element $s \in \mathcal{F}(U)$ such that $\operatorname{res}^{U}_{U_i}(s) = s_i$ for every i.

The category of presheaves of $\mathfrak{Ab}$ on X is the functor category $\mathfrak{Ab}^{\mathfrak{Top}(X)}$, where the objects are presheaves of $\mathfrak{Ab}$ on X, and the morphisms are natural transformations. The category of sheaves of $\mathfrak{Ab}$ on X, denoted by $\mathfrak{Ab}(X)$, is the subcategory of $\mathfrak{Ab}^{\mathfrak{Top}(X)}$, restricting the objects to sheaves.

We call the elements in $\mathcal{F}(U)$ sections of the sheaf $\mathcal{F}$ over the open set U. This object is often denoted by $\Gamma(U, \mathcal{F})$. Also, when there is no danger of confusion about the domain of a section $s \in \Gamma(U, \mathcal{F})$, we denote by $s|_V$ the restriction to an open subset $V \subset U$. In our case, the category $\mathfrak{Ab}$ will often be replaced by the category of commutative rings with identity and ring homomorphisms or o. For simplicity, we restrict for the rest of the section to the category $\mathfrak{Ab}$, keeping in mind that superficial modifications may be needed when considering other categories.

Definition 2.1.2 (Stalk of a Presheaf at a Point)**.** Given a presheaf $\mathcal{F}$ on X and $p \in X$ a point, we define the *stalk* of $\mathcal{F}$ at p, denoted by $\mathcal{F}_p$, to be the direct limit of the groups $\mathcal{F}(U)$ over all open subsets $U \ni p$.

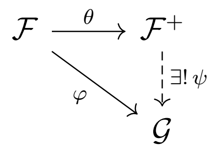
Given a point $p \in X$, the open subsets of X that contain p form a filtered category (see [2], chapter 7), so that $\mathcal{F}_p$ is a filtered direct limit and we may think of its elements as germs of sections of $\mathcal{F}$ at the point p. Given a morphism of presheaves, the universal property of the direct limit automatically gives an induced map on the stalks. We have the following useful proposition. We refer to [9, Proposition 1.1, Chapter II] for the proof.

Proposition 2.1.1. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. Then φ is an isomorphism if and only if the induced map on the stalk $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism for every p.*

Given $\theta : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves over X, we can consider the kernel presheaf, cokernel presheaf and image presheaf of θ. It is easy to show that the kernel presheaf is in fact a sheaf, but the image and cokernel presheaves are not sheaves in general. Fortunately, there is a construction that solves this problem: given a presheaf, there is always a canonical sheaf associated to it.

Proposition 2.1.2. *Given a presheaf $\mathcal{F}$, there exists a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ satisfying the following universal property:*

For every sheaf $\mathcal{G}$ and every morphism $\varphi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \psi \circ \theta$, i.e such that the following diagram commutes.



The idea of the proof is to define for each open subset $U \subset X$ the abelian group $\mathcal{F}^+(U)$ of functions $s : U \to \coprod_{p \in U} \mathcal{F}_p$ satisfying the conditions:

(a) $s(p) \in \mathcal{F}_p$ for every $p \in U$.

(b) (s is locally constant) for every $p \in U$, there exists an open neighborhood $p \in V \subset U$ and $t \in \mathcal{F}(V)$ such that for every $q \in V$, $s(q) = t_q \in \mathcal{F}_q$.

This construction will appear several times on the following pages, since many of our future examples of sheaves are born as the sheaf associated to a natural presheaf (for instance the prime spectrum sheaf discussed ahead). For a complete proof we refer to [9, Proposition-Definition 1.2, Chapter II].

Definition 2.1.3. The sheaf $\mathcal{F}^+$ given by the previous proposition is called the *sheaf associated to* $\mathcal{F}$. A *subsheaf* of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ such that for every open subset $U \subset X$, $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$ and the restriction maps of $\mathcal{F}'$ are those induced by $\mathcal{F}$.

Definition 2.1.4. Given $\theta : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves, the *kernel of* θ is the subsheaf of $\mathcal{F}$ given by the kernel presheaf. θ is called *injective* if $\ker\theta = 0$. The *image of* θ, denoted by $\operatorname{Im}\theta$, is the sheaf associated to the image presheaf. θ is called *surjective* if $\operatorname{Im}\theta = \mathcal{G}$. The *cokernel of* θ is defined as the sheaf associated to the cokernel presheaf. If $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$, the *quotient sheaf* $\mathcal{F}/\mathcal{F}'$ is defined as the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{F}'(U)$.

Definition 2.1.5 (Direct Image Sheaf)**.** Let $f : X \to Y$ be a continuous map of topological spaces and $\mathcal{F}$ a sheaf over X. The *direct image sheaf* $f_*\mathcal{F}$ on Y is defined by $f_*\mathcal{F}(V) := \mathcal{F}(f^{-1}(V))$ for every open subset $V \subset Y$.

Example 2.1.1 (Sheaf of Regular Functions of an Algebraic Variety)**.** Following Hartshorne's definitions, given $V \subset \mathbb{P}^n$ a quasi-projective variety, we have the ring $\mathcal{O}(V)$ of regular functions defined on V. It follows directly from the definitions that (fixing an ambient space $\mathbb{P}^n$), $\mathcal{O}$ is a sheaf of rings with the usual restriction of functions on $\mathbb{P}^n$. Further, the local ring of regular functions near a point p is by definition the stalk of the sheaf $\mathcal{O}$ at p. ■

2.1.1 The Prime Spectrum Sheaf

Let A be a ring (commutative with identity), and consider the set $\operatorname{Spec} A$ of all prime ideals of A. Recall that there is a "natural" topology on $\operatorname{Spec} A$, called the *Zariski* *topology*, in which the closed sets are of the form $V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supset \mathfrak{a}\}$ for some ideal $\mathfrak{a}$ in A. We define a sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$ over $\operatorname{Spec} A$ as follows:

For every prime ideal $\mathfrak{p}$ in A, we denote by $A_{\mathfrak{p}}$ the localization of A at $\mathfrak{p}$. For every open subset $U \subset \operatorname{Spec} A$ define $\mathcal{O}_{\operatorname{Spec} A}(U)$ to be the set of functions

$$s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$$

satisfying the following conditions:

(i) $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ for every $\mathfrak{p} \in U$.

(ii) For every $\mathfrak{p}$ in U, there exists an open neighborhood $\mathfrak{p} \in V \subset U$ and elements $a, f \in A$ such that for every $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f$.

We verify that $\mathcal{O}_{\operatorname{Spec} A}$ is in fact a sheaf of rings over $\operatorname{Spec} A$. First of all, condition (i) ensures that for every open subset $U \subset \operatorname{Spec} A$, $\mathcal{O}_{\operatorname{Spec} A}(U)$ is a commutative ring, since all algebraic operations are done on each local ring $A_{\mathfrak{p}}$. The multiplicative and additive identities correspond to the constant 1 and 0 functions respectively. The case $\mathcal{O}_{\operatorname{Spec} A}(\emptyset) = \{\emptyset\}$ which is the trivial ring. Given an inclusion $\iota : V \hookrightarrow U$, $\mathcal{O}_{\operatorname{Spec} A}(\iota) = \operatorname{res}_{U,V} : \mathcal{O}_{\operatorname{Spec} A}(U) \to \mathcal{O}_{\operatorname{Spec} A}(V)$ is the usual restriction of functions, which is a ring homomorphism and in the particular case of $U \subset U$, $\operatorname{res}_{U,U} = \operatorname{id}_{\mathcal{O}_{\operatorname{Spec} A}(U)}$. Finally, if $W \subset V \subset U$ are open subsets of $\operatorname{Spec} A$, then $\operatorname{res}_{U,W} = \operatorname{res}_{U,V} \circ \operatorname{res}_{V,W}$, so that $\mathcal{O}_{\operatorname{Spec} A}$ is a presheaf of rings. Finally, we check that $\mathcal{O}_{\operatorname{Spec} A}$ satisfies the sheaf conditions:

(a) Let U be an open subset of X and let $\{V_i\}$ be an open cover for U. Suppose that $s \in \mathcal{O}_{\operatorname{Spec} A}(U)$ is such that $\operatorname{res}^U_{V_i}(s) = 0$ for every i. Then take $\mathfrak{p} \in U$. $\mathfrak{p} \in V_j$ for some j, so that $s(\mathfrak{p}) = \operatorname{res}^U_{V_j}(s)(\mathfrak{p}) = 0$ in $A_{\mathfrak{p}}$. Since $\mathfrak{p} \in U$ is arbitrary, we have shown that s is the constant zero function in U, which is the additive identity of $\mathcal{O}_{\operatorname{Spec} A}(U)$.

(b) Let U be an open subset of X and let $\{V_i\}$ be an open cover for U. Suppose that we have functions $s_i \in \mathcal{O}_{\operatorname{Spec} A}(V_i)$ is such that for every i, j, $\operatorname{res}^{V_i}_{V_i \cap V_j}(s_i) = \operatorname{res}^{V_j}_{V_i \cap V_j}(s_j)$. Just define $s = \bigcup s_i$, i.e $s(\mathfrak{p}) = s_j(\mathfrak{p})$ for any j such that $\mathfrak{p} \in V_j$. Since the s_i functions coincide on the intersections of their domains, the function s is well defined and it is clear that $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ for every $\mathfrak{p} \in U$ so that condition (i) holds. Moreover, condition (ii) also holds since for every $\mathfrak{p} \in U$, there is a neighborhood $\mathfrak{p} \in V \subset V_j \subset U$ and elements $a, f \in A$ such that for every $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = s_j(\mathfrak{q}) = a/f \in A_{\mathfrak{q}}$.

Definition 2.1.6 (Spectrum of a Ring)**.** Let A be a ring. The *spectrum* of A is the pair consisting of the topological space $\operatorname{Spec} A$ and the sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$ defined above.

As it was remarked right after Proposition 2.1.2, the sheaf constructed above could have been defined as the sheaf associated to the "constant presheaf associated to A", that is, the presheaf that to every open subset U assigns the ring A.

2.1.2 Glueing Sheaves

Two charts (U, φ) and (V, ψ) on a differentiable manifold M are said to be compatible if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism. Compatible charts over an open covering glue together to form an atlas on M. In a similar fashion, if we have compatible "sheaf-charts" over an open covering of a topological space, we can glue them together to form a global sheaf. This construction will be important ahead for defining the sheaf of differentials.

Proposition 2.1.3 (Glueing Sheaves). *Let X be a topological space, $\{U_i\}$ an open covering for X and suppose that for each i we are given a sheaf $\mathcal{F}_i$ and for each i, j we have isomorphisms $\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ satisfying the following conditions:*

(1) for each i, $\varphi_{ii} = \mathrm{id}$.

(2) for each i, j, k, $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$.

Then, there exists a unique sheaf $\mathcal{F}$ on X, together with isomorphisms $\psi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ such that for each i, j, $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$.

We give a sketch of a proof.

Proof: We start with the existence. Given $U \subset X$ open, define

$$\mathcal{F}(U) := \left\{ s = (s_i) \in \prod_i \mathcal{F}_i(U \cap U_i) \; : \; \varphi_{ij}(s_i|_{U \cap U_i \cap U_j}) = s_j|_{U \cap U_i \cap U_j} \text{ for every } i, j \right\}.$$

$\mathcal{F}(U)$ is an abelian group with zero element the zero of the product $\prod_i \mathcal{F}_i(U \cap U_i)$. If $V \subset U \subset X$ are open subsets, the restriction maps $\mathcal{F}_i(U \cap U_i) \to \mathcal{F}_i(V \cap U_i)$ induce a natural restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ by $(s_i) \mapsto (s_i|_{V \cap U_i})$, so that $\mathcal{F}$ is a presheaf.

(a) Let $U \subset X$ be open, and suppose there is an open cover $\{V_\alpha\}$ of U and some $s \in \mathcal{F}(U)$ such that $s|_{V_\alpha} = 0$ for every α. Then $s_i|_{U_i \cap V_\alpha} = 0$ for every i and every α. Since $\mathcal{F}_i$ is a sheaf and $\{V_\alpha \cap U_i\}_\alpha$ is an open cover for U_i for every i, we have that $s_i = 0$ for every i so that $s = 0$.

(b) Let $U \subset X$ be open, and suppose there is an open cover $\{V_\alpha\}$ of U and $s_\alpha = (s_{\alpha,i}) \in \mathcal{F}(V_\alpha)$ such that for every α, β, $s_\alpha|_{V_\alpha \cap V_\beta} = s_\beta|_{V_\alpha \cap V_\beta}$. This means that $s_{\alpha,i}|_{V_\alpha \cap V_\beta \cap U_i} = s_{\beta,i}|_{V_\alpha \cap V_\beta \cap U_i}$ for every i and every α, β. Since $\mathcal{F}_i$ is a sheaf, there exists a unique $s_i = \cup_\alpha s_{\alpha,i}$ for every i. It follows that $s = (s_i) \in \mathcal{F}(U)$ and $s|_{V_\alpha} = s_\alpha$ for every α.

Finally, for each i we define $\psi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ as follows. For each $U \subset X$ open subset $\psi_i(U \cap U_i) : \mathcal{F}(U \cap U_i) \to \mathcal{F}_i(U \cap U_i)$, $(s_j) \mapsto s_i$. It is clear that $\psi_i(U \cap U_i)$ is a group homomorphism and that it defines a natural transformation, so that ψ_i is a morphism of sheaves for every i. Now consider the morphism $\theta_i : \mathcal{F}_i \to \mathcal{F}|_{U_i}$ defined as follows, for every $U \subset X$ put $\theta_i(U \cap U_i)(s_i) = (\varphi_{ij}(s_i|_{U \cap U_i \cap U_j}))$. By conditions (1) and (2) this is a well defined morphism of sheaves, and it is an inverse for ψ_i, which shows that ψ_i is an isomorphism. □

2.2 Schemes

One of my commutative algebra teachers once said a phrase that stuck with me, even though I didn't understand it very well at the time. He said something like: *Just as Differential Geometry studies spaces that are locally "Calculus", Algebraic Geometry studies spaces that are locally Commutative Algebra.* He was talking about Schemes, which may seem very abstract objects but have a very clear geometrical foundation in classical algebraic geometry:

Let k be an algebraically closed field and consider the affine space $\mathbb{A}_k^n$ with the Zariski topology, and let A be the polynomial ring $k[t_1, \ldots, t_n]$. By Hilbert's Nullstellensatz, we have a bijective correspondence:

$$\begin{array}{ccc} \{\text{ Affine Varieties} \subsetneq \mathbb{A}_k^n \} & \longleftrightarrow & \{\text{ Prime Ideals of } A \} \\ \cup & & \cup \\ \{\text{ Points in } \mathbb{A}_k^n \} & \longleftrightarrow & \{\text{ Maximal Ideals of } A \} \end{array}$$

The feat of Grothendieck was to isolate the fundamental ingredients that make algebraic geometry work (varieties are locally affine, the sheaf of regular functions) and dismiss everything auxiliary (for example $A = k[t_1, \ldots, t_n]$).

Definition 2.2.1 (Category of Ringed Spaces)**.** A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting on a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A *morphism of ringed spaces* between $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ where $f : X \to Y$ is a continuous map and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ a morphism of sheaves of rings on Y.

Definition 2.2.2 (Category of Locally Ringed Spaces)**.** A ringed space $(X, \mathcal{O}_X)$ is a *locally ringed space* if for each point $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring. A *morphism of locally ringed spaces* is a morphism of ringed spaces $(f, f^\#)$ such that for every point $p \in X$, the induced map of local rings $f_p^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a local homomorphism of local rings.

A differentiable manifold M together with a the sheaf of rings C_M^∞ of differentiable functions is a ringed space. In fact, it is a locally ringed space since the evaluation map, defined on the ring of germs of M at a point, is surjective. Yet, this example won't work for us any further. We turn to the most important example of the chapter:

Example 2.2.1 (Affine Schemes)**.** As we saw on Subsection 2.1.1, given a ring A, its spectrum $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is a ringed space. In fact, it is a locally ringed space.

Proposition 2.2.1. *Let A be a ring and $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ its spectrum. Then:*

(a) For any $\mathfrak{p} \in \operatorname{Spec} A$, $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} \cong A_\mathfrak{p}$.

(b) For any element $f \in A$, the open sets $D(f) = \operatorname{Spec} A \setminus V(\langle f \rangle)$ cover $\operatorname{Spec} A$ and the ring $\mathcal{O}_{\operatorname{Spec} A}(D(f))$ is isomorphic to the localized ring A_f. In particular $\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \cong A$.

We summarize the first part of the proof here to give an idea of how to deal with affine schemes.

Proof: (a) The idea is the same as for differentiable manifolds. Consider the evaluation homomorphism $\varphi : \mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} \to A_p$, $s_\mathfrak{p} \mapsto s(\mathfrak{p})$ where s is any representative of $s_\mathfrak{p}$. Any two representatives coincide in an open neighborhood of $\mathfrak{p}$ so that φ is well defined. We show that φ is an isomorphism. For the surjectivity, take $a/f \in A_\mathfrak{p}$. Then $s : D(f) \to \coprod_{\mathfrak{q} \in D(f)} A_\mathfrak{q}$, given by $s(\mathfrak{q}) = a/f$ defines a section $s \in \mathcal{O}_{\operatorname{Spec} A}(D(f))$. Then $\varphi(s_\mathfrak{p}) = s(\mathfrak{p}) = a/f$ and φ is surjective. Now take $s, t \in \mathcal{O}_{\operatorname{Spec} A}(U)$ such that $s(\mathfrak{p}) = t(\mathfrak{p})$. By condition (ii) in the definition of $\mathcal{O}_{\operatorname{Spec} A}$, we can shrink U so that $s = a/f$ and $t = b/g$ for some $a, b, f, g \in A$ with $f, g \notin \mathfrak{p}$. Since $a/f = b/g$, there exists some $u \notin \mathfrak{p}$ such that $u(ag - bf) = 0$. Therefore $s(\mathfrak{q}) = t(\mathfrak{q})$ for every prime $\mathfrak{q}$ in $U \cap D(u) \cap D(f) \cap D(g) \ni \mathfrak{p}$. Since $s = t$ in a non-empty neighborhood of $\mathfrak{p}$, we have shown that $s_\mathfrak{p} = t_\mathfrak{p}$ so that φ is injective.

(b) [9, Proposition 2.2, Chapter II] □

Definition 2.2.3 (Affine Scheme)**.** An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ that is isomorphic as a locally ringed space to the spectrum of some ring.

Example 2.2.2 (Affine n-space)**.** Let k be a field and consider the affine scheme $\mathbf{A}_k^n := \operatorname{Spec} k[x_1, \ldots, x_n]$. It is called the *affine n-space over k.* Note that the closure of the point $\xi = \langle 0 \rangle$ is all of $\mathbf{A}_k^n$ since every prime ideal contains zero (ξ is the generic point of $\mathbf{A}_k^n$). Also, since the closed points correspond to the maximal ideals, when k is algebraically closed, Hilbert's Nullstellensatz gives a bijective correspondence between the closed points in $\mathbf{A}_k^n$ and the algebraic variety $\mathbb{A}_k^n$. ■

The notion of a generic point mentioned above is purely topological. Given a topological space X and an irreducible closed subset $Z \subset X$, a *generic point for Z* is a point $\xi \in Z$ such that $Z = \overline{\{\xi\}}$. If X is a scheme, every nonempty closed subset has a unique generic point.

Definition 2.2.4 (Category of Schemes)**.** A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ on which every point admits an open neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. A *morphism of schemes* is just a morphism of locally ringed spaces. If $(X, \mathcal{O}_X)$ is a scheme, we refer to X as the *underlying topological space* and to $\mathcal{O}_X$ as its *structure sheaf.*

It is clear from the definition that affine schemes are schemes. And just as affine schemes are the natural generalization of affine algebraic varieties, the following construction is the scheme theoretic generalization of a projective variety. It is also our first example of a scheme that is not an affine scheme.

2.2.1 Proj of a Graded Ring

For us a *graded ring* will be a ring S that admits a decomposition into a direct sum of abelian groups $S = \bigoplus_{d \geq 0} S_d$ such that for any $n, m \geq 0$ we have that $S_n S_m \subset S_{n+m}$. Elements in S_d are called homogeneous of degree d. We will assume basic facts about graded rings and graded ideals, all of which can be found in [2]. We will refer to $S_+ := \bigoplus_{d>0} S_d$ as the *irrelevant* ideal. An ideal $\mathfrak{a} \subset S$ is called *relevant* if $\mathfrak{a} \not\supset S_+$.

Let S be a graded ring. Recall that an ideal $\mathfrak{a} \subset S$ is called *homogeneous* if $\mathfrak{a} = \bigoplus_{d \geq 0} \mathfrak{a} \cap S_d$. Consider the set $\operatorname{Proj} S$ of all homogeneous relevant prime ideals of S. If $\mathfrak{a}$ is an homogeneous ideal in S, we define $V(\mathfrak{a})$ to be the set of ideals in $\operatorname{Proj} S$ containing $\mathfrak{a}$. The sets $V(\mathfrak{a})$ for homogeneous ideals $\mathfrak{a} \subset S$ define the closed sets of a topology in $\operatorname{Proj} S$, called the *Zariski Topology* of $\operatorname{Proj} S$. We consider $\operatorname{Proj} S$ as a topological space with this topology.

For each prime $\mathfrak{p} \in \operatorname{Proj} S$, let $T \subset S$ be the multiplicative set of homogeneous elements not in $\mathfrak{p}$. Then $T^{-1}S$ is a graded ring and we denote by $S_{(\mathfrak{p})} := (T^{-1}S)_0$ the elements of degree zero in this localized ring. We define a sheaf of rings $\mathcal{O}_{\operatorname{Proj} S}$ on the topological space $\operatorname{Proj} S$ as follows:

For each open subset $U \subset \operatorname{Proj} S$ define $\mathcal{O}_{\operatorname{Proj} S}(U)$ to be the set of functions

$$s : U \to \coprod_{\mathfrak{p} \in U} S_{(\mathfrak{p})}$$

satisfying the following conditions:

(i) $s(\mathfrak{p}) \in S_{(p)}$ for every $\mathfrak{p} \in U$.

(ii) For every $\mathfrak{p}$ in U, there exists an open neighborhood $\mathfrak{p} \in V \subset U$ and homogeneous elements $a, f \in S$ of the same degree such that for every $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f$ in $S_{(\mathfrak{q})}$.

Together with the usual restriction of functions, $\mathcal{O}_{\operatorname{Proj} S}$ is a sheaf of rings over $\operatorname{Proj} S$ so that $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ is a ringed space. In fact, $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ is a scheme.

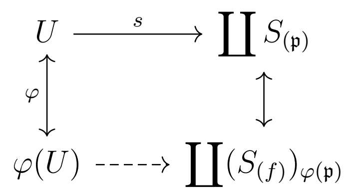
Proposition 2.2.2. *Let S be a graded ring.*

(a) $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ *is a locally ringed space. In fact, for every* $\mathfrak{p} \in \operatorname{Proj} S$, $\mathcal{O}_{\operatorname{Proj} S, \mathfrak{p}} \cong S_{(\mathfrak{p})}$.

(b) The sets $D_+(f) := \operatorname{Proj} S - V(\langle f \rangle)$ *for* $f \in S_+$ *homogeneous, form an open covering for* $\operatorname{Proj} S$. *Furthermore, for each such open subset we have an isomorphism of locally ringed spaces* $(D_+(f), \mathcal{O}|_{D_+(f)}) \cong \operatorname{Spec} S_{(f)}$, *so that* $\operatorname{Proj} S$ *is a scheme.*

Proof: (a) Once again, the idea is to define a map $\varphi : \mathcal{O}_{\operatorname{Proj} S, \mathfrak{p}} \to S_{(\mathfrak{p})}$ by evaluating any representative at $\mathfrak{p}$. This gives a well defined ring homomorphism which turns out to be an isomorphism. The proof is analogous to Proposition 2.2.1 part (a).

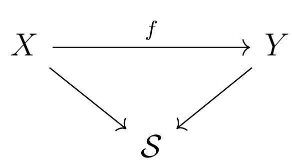
(b) The open sets $D_+(f)$ for homogeneous $f \in S_+$ cover $\operatorname{Proj} S$, since for every $\mathfrak{p} \in \operatorname{Proj} S$ there exists some $g \in S_+ \setminus \mathfrak{p}$ homogeneous, so that $\mathfrak{p} \in D_+(g)$. Fix some $f \in S_+$ homogeneous. By the properties of localization, if $\mathfrak{p} \in D_+(f)$, $\mathfrak{p}S_f$ is a prime ideal, so that $\mathfrak{p}S_f \cap S_{(f)}$ is a prime ideal and every prime ideal in $S_{(f)}$ is of this form [2, Corollary 11.18]. Therefore we have a bijective map $\varphi : D_+(f) \to \operatorname{Spec} S_{(f)}$ sending $\mathfrak{p} \mapsto \mathfrak{p}S_f \cap S_{(f)}$. Also, for any homogeneous ideal $\mathfrak{a} \subset S$, $\mathfrak{p} \supset \mathfrak{a} \iff \varphi(\mathfrak{p}) \supset \mathfrak{a}S_f \cap S_{(f)}$, so that φ is an homeomorphism. Finally, the natural ring homomorphism $S \to S_f$ induces an isomorphism $S_{(\mathfrak{p})} \cong (S_{(f)})_{\varphi(\mathfrak{p})}$ so that for every open subset $U \subset D_+(f)$ and every $s \in \Gamma(U, \mathcal{O}_{\operatorname{Proj} S}|_{D_+(f)})$ we have the commutative diagram:



Therefore, the homeomorphism φ and the natural map $S \to S_{(f)}$ induce an isomorphism of sheaves $\varphi^\# : \mathcal{O}_{\operatorname{Spec} S_{(f)}} \to \varphi_*\big(\mathcal{O}_{\operatorname{Proj} S}|_{D_+(f)}\big)$, so that $(\varphi, \varphi^\#)$ is an isomorphism of locally ringed spaces and $\operatorname{Proj} S$ is a scheme. □

Example 2.2.3 (Projective n-space)**.** Let k be a field and consider the graded ring $S = k[T_0, \dots, T_n]$. We define the projective n-space over k to be the scheme $\mathbf{P}^n_k := \operatorname{Proj} S$. The open sets $U_i := D_+(T_i)$ for $i = 0, \dots, n$ give a finite cover of $\mathbf{P}^n_k$ by affine open subsets. By the previous proposition, $(U_i, \mathcal{O}_{\mathbf{P}^n_k}|_{U_i}) \cong \operatorname{Spec} k[T_0/T_i, \dots, T_n/T_i] \cong \mathbf{A}^n_k$. When k is algebraically closed, Hilbert's Nullstellensatz gives a bijective correspondence between the closed points in $\mathbf{P}^n_k$ and the algebraic variety $\mathbb{P}^n_k$. ■

Definition 2.2.5 (Category of Schemes over a fixed Scheme)**.** Fix a scheme S. A *scheme over* S is a scheme X together with a morphism of schemes $X \to S$. If X and Y are schemes over S, an S-*morphism* $f : X \to Y$ is a morphism of schemes compatible with the given morphisms to S, i.e such that the following diagram commutes:

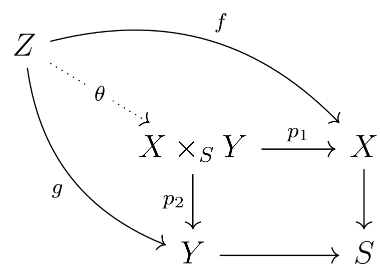


We denote by $\mathfrak{Sch}(S)$ the category of schemes over S.

An interesting consequence of the fact that the integers are an initial object in the category of rings is the fact that $\operatorname{Spec} \mathbb{Z}$ is a final object in the category of schemes, so that for every scheme X there exists a unique morphism to $\operatorname{Spec} \mathbb{Z}$, and the category of schemes coincides with $\mathfrak{Sch}(\operatorname{Spec} \mathbb{Z})$.

Let A, B be R-algebras. Then, the affine schemes $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are schemes over $S = \operatorname{Spec} R$. Consider the ring $A \otimes_R B$. By the universal property of the tensor product, to give an S-morphism $Z \to \operatorname{Spec} A \otimes_R B$ is the same as giving morphisms $Z \to X$, $Z \to Y$ which give the same morphism $Z \to S$. In other words, $\operatorname{Spec} A \otimes_R B$ is a product for X and Y in the category of schemes over S. It turns out that the category $\mathfrak{Sch}(S)$ has products for any scheme S.

Definition 2.2.6 (Fibred product of Schemes)**.** Let S be a scheme and let X and Y be schemes over S. We define the *fibred product* of X and Y over S to be a scheme $X \times_S Y$ together with morphisms $p_1 : X \times_S Y \to X$, $p_2 : X \times_S Y \to Y$ commuting with the given maps $X \to S$, $Y \to S$ such that for any other scheme Z over S with morphisms $f : Z \to X$, $g : Z \to Y$ commuting with the given morphisms to S, there exists a unique morphism $\theta : Z \to X \times_S Y$ such that $f = p_1 \circ \theta$, $g = p_2 \circ \theta$.



Theorem 2.2.1. *For any scheme S, and every X, Y schemes over S, the fibred product $X \times_S Y$ exists and it is unique up to unique isomorphism.*

We refer to [9, Theorem 3.3, Chapter II] for the proof. Given a morphism of schemes $f : X \to Y$, we can consider X as a scheme over Y and take the fibred product $X \times_Y X$. Since the identity map $X \to X$ obviously commutes with f, there exists a unique morphism $\Delta : X \to X \times_Y X$ such that $p_i \circ \Delta = \mathrm{id}_X$ for $i = 1, 2$. This morphism is called the *diagonal morphism* of f.

Definition 2.2.7. A scheme $(X, \mathcal{O}_X)$ is said to be *connected* if X is connected. It is *irreducible* if X is an irreducible topological space. If $\mathcal{O}_X(U)$ is a reduced ring, i.e it has no nonzero nilpotent elements, for every open subset $U \subset X$, we call $(X, \mathcal{O}_X)$ *reduced*. Similarly, if $\mathcal{O}_X(U)$ is an integral domain for every open subset $U \subset X$, the scheme is called *integral*. Finally, $(X, \mathcal{O}_X)$ is *noetherian* if it can be covered by finitely many open affine subsets $\operatorname{Spec} A_i$, with each A_i a noetherian ring.

Example 2.2.4 (Function field of an integral scheme)**.** Let X be an integral scheme. Let ξ be the generic point of X and consider the stalk $K(X) := \mathcal{O}_{X,\xi}$. Then $K(X)$ is a field, which we call the *function field of* X. In fact, let $U = \operatorname{Spec} A$ be an affine neighborhood of ξ. The uniqueness of ξ implies that ξ is a minimal prime in A. By proposition 2.2.1 (b), we have that $\mathcal{O}_X(U) \cong A$, so that A is an integral domain and $\xi = \langle 0 \rangle \subset A$. Therefore

$$K(X) = \varprojlim_{V \ni \xi} \mathcal{O}_X(V) = \varprojlim_{V \ni \langle 0 \rangle} \mathcal{O}_{\operatorname{Spec} A}(V) = A_{\langle 0 \rangle},$$

so that $K(X)$ is precisely the fraction field of the ring A. ■

Just as in differential geometry, the definition of a sub-geometric space is rather subtle, and it is better understood in terms of embeddings. Open subsets inherit a unique geometric structure automatically, but closed subsets require some work.

Definition 2.2.8 (Open subscheme - Open immersion)**.** An *open subscheme* U of a scheme X is a scheme whose topological space is an open subset of X and whose structure sheaf $\mathcal{O}_U$ is isomorphic to the restriction sheaf $\mathcal{O}_X|_U$. An *open immersion* is a morphism of schemes $f : X \to Y$ which induces an isomorphism of X with an open subscheme of Y.

Definition 2.2.9 (Closed immersion - Closed subscheme)**.** A *closed immersion* is a morphism of schemes $f : Y \to X$ such that f induces an homeomorphism of Y with a closed subspace of X and further, the induced morphism $f^{\#} : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective. A *closed subscheme* of a scheme X is an equivalence class of closed immersions, declaring two closed immersions $f : Y \to X$ and $f' : Y' \to X$ equivalent if there exists an isomorphism $i : Y' \to Y$ such that $f' = f \circ i$.

Definition 2.2.10 (Separated morphism - Separated scheme)**.** Given a morphism of schemes $f : X \to Y$, we say that X is *separated over* Y if the diagonal morphism $\Delta : X \to X \times_Y X$ is a closed immersion. In this case we call f a *separated morphism*. A scheme X is called *separated* if it is separated over $\operatorname{Spec} \mathbb{Z}$.

Example 2.2.5 (Closed subschemes of $\operatorname{Proj} S$)**.** Let S be a graded ring and let $X = \operatorname{Proj} S$. Let I be an homogeneous ideal in S and let $Y = \operatorname{Proj}(S/I)$. The projection $\eta : S \to S/I$ induces a morphism of schemes $f : Y \to X$, which is a closed immersion. In fact, η is degree preserving and surjective so that $\eta(S_+) = (S/I)_+$. Define $f : Y \to X$ by $f(\mathfrak{b}) = \eta^{-1}(\mathfrak{b})$. Since $\mathfrak{b}$ is homogeneous, so is $f(\mathfrak{b})$ and since $\mathfrak{b} \not\supset (S/I)_+ = \eta(S_+)$, $f(\mathfrak{b}) \not\supset S_+$. By correspondence, it is clear that the image of f is precisely $V(I)$, so that f is a closed immersion and the class of f gives a closed subscheme of X. In the case that S is a polynomial ring over S_0, every closed subscheme of X is of this form [9, Corollary 5.16, Chapter II]. ■

2.2.2 Algebraic Varieties and Schemes

The following discussion clarifies the assertion that the category of schemes is an enlargement of the category $\mathfrak{Var}(k)$ of quasi-projective varieties over a field k studied in classical algebraic geometry.

Given a topological space X, denote by $t(X)$ the set of nonempty irreducible closed subsets of X. Note that for a closed subset $Z \subset X$, $t(Z) \subset t(X)$ and furthermore, $t(Z_1 \cup Z_2) = t(Z_1) \cup T(Z_2)$ and $t(\bigcap Z_i) = \bigcap t(Z_i)$ for every closed subsets $Z_1, Z_2, Z_i \subset X$. Hence, we can define a topology on $t(X)$ by taking the closed sets to be of the form $t(Z)$ for $Z \subset X$ closed. Given a continuous function $f : X \to Y$, define $t(f) : t(X) \to t(Y)$ by $t(f)(Z) = \overline{f(Z)}$. Since irreduclibility is preserved by taking images and closures, $t(f)$ is well defined and $t(f)^{-1}(t(W)) = t(f^{-1}(W))$ so that t is a covariant functor of topological spaces. Further, the map $\alpha : X \to t(X)$ taking $x \mapsto \overline{\{x\}}$ is continuous and induces a bijection between $\mathfrak{Top}(X)$ and $\mathfrak{Top}(t(X))$.

Proposition 2.2.3. *Let k be an algebraically closed field. There is a naturally faithful functor $t : \mathfrak{Var}(k) \to \mathfrak{Sch}(\operatorname{Spec} k)$. Moreover, if V is a variety, V is homeomorphic to the set of closed points of $t(V)$ and the sheaf of regular functions of V is obtained by restricting the structure sheaf of $t(V)$ via this homeomorphism.*

The functor from the previous proposition is defined in the following way. Given V a variety over k and $\mathcal{O}_V$ its sheaf of regular functions, t maps $(V, \mathcal{O}_V)$ to the pair $(t(V), \alpha_*(\mathcal{O}_V))$ which is a scheme over k (see [9, Proposition 2.6, Chapter II]). It can be shown that if V, W are two varieties over k, then the natural map $\operatorname{Hom}_{\mathfrak{Var}(k)}(V, W) \to \operatorname{Hom}_{\mathfrak{Sch}(k)}(t(V), t(W))$ is bijective. This gives a natural enlargement of the category $\mathfrak{Var}(k)$ into the world of schemes:

Definition 2.2.11 (Category of Varieties over k)**.** Let k be an algebraically closed field. A scheme $(X, \mathcal{O}_X)$ over k is a *variety* if it is in the image of the functor t. Also, if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are varieties over k a *morphism of varieties* is just a morphism $f : X \to Y$ of schemes over k.

Another approach for embedding the classical theory into the framework of schemes, perhaps a more natural one considering the constructions of affine and projective schemes discussed before, is the following: Let $V \subset \mathbb{P}^n$ be a projective variety over k (in the classical sense), and consider the homogeneous coordinate ring $S(V) := k[T_0, \ldots, T_n]/I(V)$. Then, as we saw on example 2.2.5, $\operatorname{Proj} S(V)$ is a closed subscheme of $\mathbf{P}^n := \operatorname{Proj} k[T_0, \ldots, T_n]$. It turns out that $t(V) \cong \operatorname{Proj} S(V)$, so that both points of view coincide. From this point on, we will merge the classical point of view with the scheme theoretic perspective; we will consider an algebraic variety both as a scheme and a classical algebraic variety, since there is no loss of information on working on either side. In particular, $\mathbf{P}^n = \mathbb{P}^n$ and $\mathbf{A}^n = \mathbb{A}^n$.

2.2.3 Glueing Schemes

We continue with the glueing construction, now in the setting of schemes. Consider the affine varieties $U_0 = \operatorname{Spec} k[t]$ and $U_1 = \operatorname{Spec} k[u]$. By proposition 2.2.1, we have that $D(t) = \operatorname{Spec} k[t]_t = \operatorname{Spec} k[t, t^{-1}]$. We have a ring isomorphism $k[t, t^{-1}] \to k[u, u^{-1}], t \mapsto u^{-1}$ which induces a scheme isomorphism $\varphi_{0,1} : D(t) \to D(u)$. We want to define a scheme by glueing U_0 and U_1 along the isomorphism of open subsets just described. As a topological space, take $X := (U_0 \sqcup U_1)/\sim$ (where $x_1 \sim \varphi_{0,1}(x_1)$ for every $x_1 \in D(t)$) with the quotient topology. Identifying U_i with its image on X, we have an open covering $X = U_0 \cup U_1$ (with $U_0 \cap U_1 \cong D(t) \cong D(u)$) and a sheaf $\mathcal{O}_{U_i}$ on each open subset. Moreover, the isomorphism of sheaves $\varphi_{0,1}^{\#} : \mathcal{O}_{U_0}|_{U_0 \cap U_1} \to \mathcal{O}_{U_1}|_{U_0 \cap U_1}$ and its inverse satisfy the conditions of 2.1.3, so that we have the glued sheaf $\mathcal{O}_X$. It is clear that $(X, \mathcal{O}_X)$ is a scheme and the reader may guess that X is isomorphic to $\mathbb{P}^1$. In fact, if we start with $\mathbb{P}^1 = \operatorname{Proj} k[T_0, T_1]$ we have the open subsets $D_+(T_i) \cong \operatorname{Spec} k[T_j/T_i] \cong U_i$ for $\{0, 1\} = \{i, j\}$ via the ring isomorphisms $t \mapsto T_1/T_0, u \mapsto T_0/T_1$. This glueing procedure can be generalized to an arbitrary family of schemes. We have the following result:

Proposition 2.2.4 (Glueing Schemes). *Let $\{X_i\}$ be a family of schemes. Suppose that for each $i \neq j$ there is an open subset U_{ij} with its open subscheme structure, and an isomorphism of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$ such that*

(1) $\varphi_{ji} = \varphi_{ij}^{-1}$ for each i, j.

(2) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$ for every i, j, k.

Then, there exists a scheme X together with morphisms $\psi_i : X_i \to X$ for each i, such that:

(i) ψ_i is an isomorphism onto an open subscheme of X.

(ii) The open subsets $\psi_i(X_i)$ cover X.

(iii) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}.

We say that the scheme X is obtained by glueing the schemes X_i.

2.3 Rational Maps

In this section we define the category of irreducible schemes and dominant rational maps and some other essential constructions for the applications ahead. We will only consider irreducible schemes, even though most of the definitions work in a more general context. The reader will see that the definitions in [9, Chapter I, Section 4] generalize almost without change to this context. We use [15, Chapter 28, Section 28.47] and [8] as our main references.

Definition 2.3.1 (Rational Maps)**.** Let X and Y be irreducible schemes.

(a) Two morphisms of schemes $f : U \to Y$ and $g : V \to Y$, defined over open subsets $U, V \subset X$, are said to be equivalent if $f|_W = g|_W$ for some open subset $W \subset U \cap V$.

(b) A *rational map*, dented by $\phi : X \dashrightarrow Y$, is an equivalence class for the equivalence relation defined in (a).

(c) If X and Y are schemes over S, a rational map $X \dashrightarrow Y$ is called an S*-rational map* if some representative is an S-morphism.

A rational map $\phi : X \dashrightarrow Y$ is said to be *defined at* $x \in X$, if there exists a representative $f : U \to Y$ such that $x \in U$. The *domain of definition of* ϕ, denoted by $\mathrm{dom}(\phi)$, is the set of all points where ϕ is defined.

It is well known that in the setting of rational maps between algebraic varieties, there is a unique morphism $f_\phi : \mathrm{dom}(\phi) \to Y$ representing the rational map $\phi : X \dashrightarrow Y$. In general, this is no longer true. But we only need to assume that X is reduced and Y is separated to regain the result (see [15, Lemma 28.47.9]).

As in the case of classical algebraic varieties, composition of rational maps is not defined in general for the simple reason that the image of a representative of the first map may not intersect the domain of definition of the second map. Therefore, in order to be able to compose, we restrict to dominant rational maps.

Definition 2.3.2 (Dominant Morphisms, Dominant Rational Maps)**.** A morphism of schemes $f : X \to Y$ is *dominant* if $f(X)$ is dense in Y. A rational map $\phi : X \dashrightarrow Y$ is *dominant* if some representative is a dominant morphism.

In this way we obtain a category whose objects are irreducible schemes (over S) and whose morphisms are dominant S-rational maps.

Definition 2.3.3. Let X and Y be irreducible schemes over a scheme S.

(a) X and Y are S*-birational*, if they are isomorphic in the category of irreducible schemes over S and S-rational maps.

(b) We denote by $\mathrm{Bir}_S(X)$ the group of birational automorphisms of X.

Lemma 2.3.1. *Let X and Y be irreducible schemes over a base scheme S. Then X and Y are S-birational if and only if there are non empty open subsets $U \subset X$ and $V \subset Y$ which are S-isomorphic.*

We are finally ready to state the general definition of a decomposition group discussed in the introduction.

Definition 2.3.4 (Decomposition Group)**.** Let X be an irreducible scheme over S, G a subgroup of $\mathrm{Bir}_S(X)$ and Y an irreducible reduced S-subscheme of X with generic point ξ. We define the *decomposition group* of Y in G, denoted by G_Y, to be the stabilizer of ξ in G, that is:

$$G_Y := \{\, g \in G \,|\, \xi \in \mathrm{dom}(g) \cap \mathrm{dom}(g^{-1}) \text{ and } g(\xi) = \xi \,\}.$$

If $g \in G_Y$, lemma 2.3.1 and proposition 2.1.1 imply that g induces a local isomorphism of local rings $g_\xi^\# : \mathcal{O}_{X,\xi} \to \mathcal{O}_{X,\xi}$. Therefore we have an action

$$\begin{array}{ccc} G_Y \times \mathcal{O}_{X,\xi} & \longrightarrow & \mathcal{O}_{X,\xi} \\ (g,a) & \longmapsto & g_\xi^\#(a), \end{array}$$

which preserves all powers of the maximal ideal $\mathfrak{m}_\xi \subset \mathcal{O}_{X,\xi}$, so in fact G_Y acts on $\mathcal{O}_{X,\xi}/\mathfrak{m}_\xi^{i+1}$ for every $i \geq 0$. The *i-th ramification group of* Y in G is defined as:

$$G_{i,Y} := \{\, g \in G_Y \,|\, g \text{ acts trivially on } \mathcal{O}_{X,\xi}/\mathfrak{m}_\xi^{i+1} \,\}.$$

In particular, the 0-th ramification group is called the *inertia group* of Y in G.

Example 2.3.1 (Hilbert's Ramification Theory)**.** Let A be a Dedekind domain, K its field of fractions and let L/K be a finite Galois extension. Let B be the integral closure of A in L and consider the scheme $X = \mathrm{Spec}\, B$ over $S = \mathrm{Spec}\, A$. In this context, B is also a Dedekind domain (you can find a proof in [2, Corollary 24.18]). In particular, B is a domain, which implies that X is irreducible, noetherian and of dimension 1. Since the elements of B are the elements of L that are roots of monic polynomials with coefficients in A, Gal(L/K) acts on B, so that every Galois automorphism induces an automorphism of B, which in turn induces an automorphism of X. Moreover, for every $\sigma \in \mathrm{Gal}(L/K)$ and every $\mathfrak{P} \subset B$ maximal ideal, $\sigma(\mathfrak{P}) \cap A = \sigma(\mathfrak{P} \cap A) = \mathfrak{P} \cap A \in S$, so that the induced automorphism $\sigma : X \to X$ is a morphism in the category $\mathfrak{Sch}(S)$. Therefore we can think of $\mathrm{Gal}(L/K)$ as a subgroup of $\mathrm{Bir}_S(X)$ and calculate the ramification groups for every closed subscheme $Y = V(\mathfrak{P}) \cong \mathrm{Spec}\, B/\mathfrak{P}$.

Example 2.3.2 (The Cremona Group)**.** Let $X = \mathbb{P}^2_{\mathbb{C}}$, and consider the Cremona group $\mathrm{Cr}(\mathbb{P}^2_{\mathbb{C}}) := \mathrm{Bir}_{\mathbb{C}}(X)$. Let $C \subset \mathbb{P}^2$ be an irreducible projective curve. In particular C is an irreducible, reduced, closed subscheme of $\mathbb{P}^2$ and automatically we have its decomposition and ramification groups. Since the decomposition and inertia groups of an irreducible curve are the main characters of the final chapters of this thesis, we will denote them by $\mathcal{D}(C) := \mathrm{Cr}(\mathbb{P}^2_{\mathbb{C}})_C$ and $\mathcal{I}(C) := \mathrm{Cr}(\mathbb{P}^2_{\mathbb{C}})_{0,C}$ respectively.

2.4 Sheaves of Modules

The following section extends to sheaves the theory of modules over a ring. The concepts of locally free $\mathcal{O}_X$-modules, invertible sheaves and the sheaf associated to a module in Spec and Proj will be specially important in the final chapters.

Definition 2.4.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-*module* is a sheaf of abelian groups $\mathcal{F}$ on X such that for every open set $U \subset X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ module, and for every inclusion $V \subset U$ of open sets, the restriction homomorphisms $\mathcal{F}(U) \to \mathcal{F}(V)$ are compatible with the module structures via the maps $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$, i.e the following diagram commutes for every inclusion of open subsets $V \subset U$:

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

A *morphism* $\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is morphism a morphism of sheaves such that for every $U \subset X$ open, the homomorphism $\mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear.

The category of $\mathcal{O}_X$-modules is closed under all decent algebraic operations between them. Namely, the kernel, cokernel and image of a morphism of $\mathcal{O}_X$-modules is again an $\mathcal{O}_X$-module. The quotient of an $\mathcal{O}_X$-module by a sub-$\mathcal{O}_X$-module is again an $\mathcal{O}_X$-module. Also, any direct sum, direct product, direct limit or inverse limit of $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module. Another important observation is that if $\mathcal{F}$ is an $\mathcal{O}_X$-module and U is an open subset of X, then $\mathcal{F}|_U$ is an $\mathcal{O}_X|_U$-module.

The *tensor product* of two $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, denoted by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is defined as the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

An $\mathcal{O}_X$-module $\mathcal{F}$ is *free* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is *locally free* if X admits a covering of open subsets U such that $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module. The *rank* of $\mathcal{F}$ on such an open subset is the rank of $\bigoplus_i \mathcal{O}_X(U) \cong \mathcal{F}(U)$. An *invertible sheaf* over X is a locally free $\mathcal{O}_X$-module of rank 1.

Proposition 2.4.1. *Let $\mathcal{L}, \mathcal{M}$ be invertible sheaves on a ringed space X. Then $\mathcal{L} \otimes \mathcal{M}$ is also an invertible sheaf on X. Also, there exists an invertible sheaf $\mathcal{L}^{-1}$ on X such that $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$.*

Proof: Suppose that $U, V \subset X$ are open subsets such that $\mathcal{L}(U) \cong \mathcal{O}_X(U)$ and $\mathcal{M}(V) \cong \mathcal{O}_X(V)$. Then

$$\mathcal{L}(U \cap V) \otimes \mathcal{M}(U \cap V) \cong \mathcal{O}_X(U \cap V) \otimes \mathcal{O}_X(U \cap V) \cong \mathcal{O}_X(U \cap V).$$

Therefore, $\mathcal{L} \otimes \mathcal{M}$ is locally free of rank 1 on the intersection of the coverings for which $\mathcal{L}$ and $\mathcal{M}$ are trivial. For the second part, define $\mathcal{L}^{-1}$ to be $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$. Then for every U on the covering of X for which $\mathcal{L}$ is trivial, we have:

$$\begin{aligned}\operatorname{Hom}(\mathcal{L}(U), \mathcal{O}_X(U)) \otimes \mathcal{L}(U) &\cong \operatorname{Hom}(\mathcal{O}_X(U), \mathcal{O}_X(U)) \otimes \mathcal{O}_X(U) \\ &\cong \operatorname{Hom}(\mathcal{O}_X(U), \mathcal{O}_X(U)) \cong \mathcal{O}_X(U).\end{aligned}$$

□

By the previous proposition, the set of isomorphism classes of invertible sheaves of a ringed space X is a group under tensor multiplication. This group is called the *Picard group* of X, and it is denoted by $\operatorname{Pic} X$.

A *sheaf of ideals* on X is an $\mathcal{O}_X$-module $\mathcal{J}$ that is a subsheaf of $\mathcal{O}_X$. In other words, for every open subset $U \subset X$, $\mathcal{J}(U)$ is an ideal in $\mathcal{O}_X(U)$. One particularly important example of sheaves of ideals is the ideal sheaf of a closed subscheme:

Definition 2.4.2 (Ideal Sheaf of Closed Subscheme)**.** Let Y be a closed subscheme of a scheme X, and let $\iota : Y \to X$ be the inclusion morphism. The ideal sheaf $\mathcal{J}_Y$ is defined as the kernel of the morphism $\iota^\# : \mathcal{O}_X \to \iota_*\mathcal{O}_Y$.

Given a morphism of ringed spaces $f : X \to Y$ and $\mathcal{F}$ an $\mathcal{O}_X$-module, the direct image sheaf $f_*\mathcal{F}$ is naturally an $f_*\mathcal{O}_X$-module. Moreover, the morphism of sheaves of rings $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ gives $f_*\mathcal{F}$ a natural $\mathcal{O}_Y$-module structure, called the *direct image* of $\mathcal{F}$ by f. On the other hand, given $\mathcal{G}$ a $\mathcal{O}_Y$-module, then the sheaf $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module. By the universal property of inverse limits, there is a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of rings on X, so that $\mathcal{O}_X$ has a $f^{-1}\mathcal{O}_Y$-module structure and we may define the $\mathcal{O}_X$-module $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$, called the *inverse image* of $\mathcal{G}$ by f.

Definition 2.4.3 (The sheaf associated to a module on Spec)**.** Let A be a ring and M an A-module. We define $\widetilde{M}$, the *sheaf associated to* M on $\operatorname{Spec} A$ as follows:

For every open set $U \subset \operatorname{Spec} A$, let $\widetilde{M}(U)$ be the set of functions

$$s : U \to \coprod_{\mathfrak{p} \in U} M_{\mathfrak{p}}$$

satisfying the following conditions:

(i) $s(\mathfrak{p}) \in M_{\mathfrak{p}}$ for every $\mathfrak{p} \in U$.

(ii) For every $\mathfrak{p}$ in U, there exists an open neighborhood $\mathfrak{p} \in V \subset U$ and elements $m \in M$ $f \in A$ such that for every $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = m/f \in M_{\mathfrak{q}}$.

Among all the possible sheaves of modules on a scheme, the ones that are locally of the form $\widetilde{M}$ are specially important.

Definition 2.4.4 (Coherent and quasi-Coherent sheaves)**.** Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is *quasi-coherent* if there exists an open affine covering $\{U_i = \operatorname{Spec} A_i\}$ of X, such that for each i there exists an A_i-module M_i with $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$. We say that $\mathcal{F}$ is *coherent* if each M_i can be taken to be a finitely generated A_i-module.

In the final section of this chapter we will give a fundamental example of a quasi-coherent sheaf, the sheaf of differentials. The sheaf associated to a graded module is defined analogously to the sheaf associated to a graded ring:

Definition 2.4.5 (The sheaf associated to a module on Proj)**.** Let S be a graded ring and M a graded S-module. We define $\widetilde{M}$, the *sheaf associated to* M on $\operatorname{Proj} S$ as follows:

For every open set $U \subset \operatorname{Proj} S$, let $\widetilde{M}(U)$ be the set of functions

$$s : U \to \coprod_{\mathfrak{p} \in U} M_{(\mathfrak{p})}$$

satisfying the following conditions:

(i) $s(\mathfrak{p}) \in M_{(\mathfrak{p})}$ for every $\mathfrak{p} \in U$.

(ii) For every $\mathfrak{p}$ in U, there exists an open neighborhood $\mathfrak{p} \in V \subset U$ and elements $m \in M$ $f \in S$ homogeneous of the same degree such that for every $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = m/f \in M_{(\mathfrak{q})}$.

The only reason we defined this sheaves is to consider the following fundamental example:

Example 2.4.1 (Twisted Sheaves on Proj)**.** Let S be a graded ring, $X = \operatorname{Proj} S$ and fix $n \in \mathbb{Z}$. Consider the graded S-module obtained by shifting by n, i.e the module $S(n) := \bigoplus_{d=0}^{\infty} S_{d+n}$. The sheaf $\mathcal{O}_X(n) := \widetilde{S(n)}$, is called the *n-th twisting sheaf*. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, the sheaf $\mathcal{F}(n)$ is defined as $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$. In the case that S is generated by S_1 as an S_0-algebra, $\mathcal{O}_X(n)$ is an invertible sheaf on X for every n, and furthermore $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$ (see [9, Proposition 5.12, Chapter II]). ■

The following lemma will be useful ahead. It basically states that for coherent sheaves on a noetherian scheme, being locally free is a local property.

Lemma 2.4.1. *Let X be a noetherian scheme, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. If the stalk $\mathcal{F}_p$ is a free $\mathcal{O}_p$-module for some point $p \in X$, then there is a neighborhood U of p such that $\mathcal{F}|_U$ is free.*

2.4.1 Tensor Operations on Sheaves

Let A be a ring and M an A-module. We denote by $T^r(M) := M \otimes_A \cdots \otimes_A M$ the tensor product of M with itself M times for $r \geq 1$ and $T^0(M) := A$. The *tensor algebra of* M is defined as the direct sum $T(M) := \bigoplus_{r \geq 0} T^r(M)$. It is a non commutative graded A-algebra with $T(M)_r = T^r(M)$. Taking the quotient by the two-sided ideal generated by all expressions $m \otimes m' - m' \otimes m$ for $m, m' \in M$, we obtain the *symmetric algebra of* M denoted by $S(M) = \bigoplus_{r \geq 0} S^r(M)$. Similarly, taking the quotient of the tensor algebra by the two sided ideal generated by all expressions $m \otimes m$ for $m \in M$, we obtain the *exterior algebra* $\bigwedge M = \bigoplus_{r \geq 0} \bigwedge^r M$.

For reasons that will become evident in a few pages, we are more interested on the exterior algebra construction, so we proceed to recall some of its properties. Note that for every $x, y \in M$ we have that $x \otimes y + y \otimes x = (x + y) \otimes (x + y) - x \otimes x - y \otimes y \in \langle m \otimes m | m \in M \rangle$. This implies that $\bigwedge M$ is a skew-commutative graded A-algebra, this means that if $u \in \bigwedge^r M$ and $v \in \bigwedge^s M$, then $u \wedge v = (-1)^{rs} v \wedge u$, where "$\wedge$" denotes the multiplication in $\bigwedge M$. In the spacial case that M is free of rank n, the A-modules $T^r(M), S^r(M)$ and $\bigwedge^r M$ are free of ranks n^r, $\binom{n+r-1}{n-1}$ and $\binom{n}{r}$ respectively.

We can transport all of this definitions to the world of sheaves. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define the *tensor algebra, symmetric algebra,* and *exterior algebra of* $\mathcal{F}$ as the sheaves associated to the presheaves:

$$U \mapsto T(\mathcal{F}(U)), \quad U \mapsto S(\mathcal{F}(U)), \quad U \mapsto \bigwedge \mathcal{F}(U).$$

The resulting sheaves, denoted by $T(\mathcal{F}), S(\mathcal{F})$ and $\bigwedge \mathcal{F}$ are $\mathcal{O}_X$-algebras. Their components in degree $r \geq 0$, denoted by $T^r(\mathcal{F}), S^r(\mathcal{F})$ and $\bigwedge^r \mathcal{F}$ are $\mathcal{O}_X$-modules. Moreover, if $\mathcal{F}$ is locally free of rank n, then $T^r(\mathcal{F}), S^r(\mathcal{F})$ and $\bigwedge^r \mathcal{F}$ are locally free of ranks n^r, $\binom{n+r-1}{n-1}$ and $\binom{n}{r}$ respectively.

2.5 Divisors and Invertible Sheaves

In this chapter, we study three different concepts that in the context of smooth algebraic varieties turn out to be the same thing: Weil Divisors, Cartier Divisors and Invertible Sheaves.

2.5.1 Weil Divisors

The fundamental theorem of arithmetic implies that the group $\mathbb{Q}^*$ is the free abelian group generated by the primes. Consider two integers a and b. Then

$$a = \prod_{p|a} p^{\alpha_p} \quad \text{and} \quad b = \prod_{p|b} p^{\beta_p}.$$

Clearly, a divides b if and only if for every prime divisor p of a, we have $\alpha_p \leq \beta_p$. Given a an algebraic curve C, we can recover this divisibility relations by considering the free group generated by the points. We can unify this idea using the language of schemes, under a few necessary technical restrictions.

Definition 2.5.1. A scheme X is *regular in codimension one* if every local ring $\mathcal{O}_x$ of X of dimension one is regular.

Definition 2.5.2 (Weil Divisors)**.** Let X be a noetherian integral separated scheme which is regular in codimension one. A *prime* divisor on X, is a closed integral subscheme of codimension one. Let $\mathrm{Div}(X)$ be the free abelian group generated by the prime divisors of X. The elements of this group are called *Weil divisors*. If $D \in \mathrm{Div}(X)$, $D = \sum n_i Y_i$ is a finite sum with each $n_i \in \mathbb{Z}$ and each Y_i a prime divisor. We call D *effective* if $n_i \geq 0$ for every i.

Let X be as above and take Y a prime divisor. The underlying topological space is closed and irreducible, so there exists a unique generic point η corresponding to Y. The stalk $\mathcal{O}_{X,\eta}$ is a discrete valuation ring with fraction field $K(X)$ the function field of X (see example 2.2.4). The corresponding discrete valuation $v_Y : K(X)^\times \to \mathbb{Z}$ is called the *valuation of* Y.

Lemma 2.5.1. *Let X be as above and take $f \in K(X)^\times$. Then $v_Y(f) = 0$ for all but finitely many prime divisors Y.*

Proof: Continuing with the notation of example 2.2.4, we have that $K(X) = \mathrm{Frac}(A)$, so that $f = a/b$ for some $a, b \in A - \{0\}$. Consider the open subset $V = D(b) \subset U = \mathrm{Spec}\, A$. V is open in X, so that $Z = X - V$ is a proper closed subset of X. Since X is a noetherian topological space, so is Z, so that Z can only contain finitely many prime divisors, all the others must intersect V. Thus, it is sufficient to show that there are finitely many prime divisors Y of V such that $v_Y(f) \neq 0$. Since f is regular on V, we know that $v_Y(f) \geq 0$. Also, $v_Y(f) > 0$ if and only if $Y \subset V(\langle f \rangle A_b) \subset \mathrm{Spec}\, A_b$.

Since $f \neq 0$, this is a proper closed subset and it can only contain finitely many closed irreducible subsets of codimension 1 of V. □

By the previous lemma it makes sense to define the *divisor associated of a rational function* $f \in K(X)^\times$ by

$$\operatorname{div}(f) := \sum v_Y(f) \cdot Y,$$

where the sum ranges over all prime divisors of X. A Weil divisor D is called *principal* if there exists a rational function $f \in K(X)^\times$ such that $D = \operatorname{div}(f)$. Principal divisors form a subgroup of $\operatorname{Div}(X)$, and the quotient $\operatorname{Cl} X$ is an important invariant of the scheme X called the *divisor class group*.

Example 2.5.1 (Ideal Class Group)**.** As in the case of the decomposition group (example 2.3.1), the class group was first considered in the context of number theory. Let be a K a number field (a finite field extension of $\mathbb{Q}$) and let $O_K := \{\alpha \in K \mid P(\alpha) = 0, \text{ for some monic } P \in \mathbb{Z}[T]\}$ be its corresponding ring of integers. Consider the scheme $X = \operatorname{Spec} O_K$. Since O_K is a Dedekind Domain ([2, Theorem 24.19]) we have unique factorization of ideals into products of prime ideals and $\operatorname{Div}(X)$ is in bijective correspondence with the group of fractional ideals of O_K. $K(X) = K$ is precisely the fraction field of O_K, so that principal divisors correspond to principal fractional ideals, and taking the quotient we obtain the group $\operatorname{Cl} X$ usually denoted by Cl_K, and called the *ideal class group* of K. It turns out that this group is always finite ([11, Chapter 1, Theorem 6.3]) and its cardinality h_K is an important invariant of K, called the *class number*. Mathematicians first arrived to these constructions while trying to prove Fermat's last theorem. Gabriel Lamé gave a faulty proof in which he assumed that when if p is a prime number, ζ_p is a primitive p-th root of 1 and $K = \mathbb{Q}(\zeta_p)$, then O_K is a unique factorization domain ($h_K = 1$). Ernst Kummer already knew this was false (the first time it fails is for $p = 23$), and managed to give a proof of Fermat's last theorem for the case in which the exponent $x^p + y^p = z^p$ is a prime number not dividing h_K. Theorem 2.5.2 below, generalizes the fact that a Dedekind domain is a UFD if and only if its ideal class group is trivial. ■

Definition 2.5.3. A scheme X is called *normal* if $\mathcal{O}_{X,p}$ is an integrally closed domain for every point $p \in X$.

Let A be a noetherian domain and assume that $\operatorname{Spec} A$ is normal. Then, for every prime ideal $\mathfrak{p}$ of height one, the local ring $A_\mathfrak{p}$ is an integrally closed domain of dimension one. By commutative algebra (see [2, Theorem 23.9]), this is equivalent to $A_\mathfrak{p}$ being a regular local ring of dimension one. Therefore, $\operatorname{Spec} A$ is regular in codimension one. For a proof of the following theorem see [9, Chapter II, Proposition 6.2].

Theorem 2.5.2. *Let A be a noetherian domain. Then A is a unique factorization domain if and only if* $\operatorname{Spec} A$ *is normal and* $\operatorname{Cl}(\operatorname{Spec} A)$ *is trivial.*

The following example is specially important for the final chapters. We describe the class group of projective space, and extend notion of degree of hypersurfaces to divisors.

Example 2.5.2 (Weil Divisors on $\mathbb{P}^n$)**.** Let k be a field, $S = k[T_0, \ldots, T_n]$ and consider the scheme $\mathbb{P}^n = \operatorname{Proj} S$. Since every closed subscheme of $\mathbb{P}^n$ is of the form $\operatorname{Proj} S/I$ for some homogeneous ideal I, the prime divisors of $\mathbb{P}^n$ are precisely the hypersurfaces. For any divisor $D = \sum n_i \cdot Y_i$, define its *degree* by $\deg D = \sum n_i \deg Y_i$.

Proposition 2.5.1. *Let $H = V(T_0) \subset \mathbb{P}^n$.*

(a) For any $f \in K^\times$, $\deg(\operatorname{div}(f)) = 0$.

(b) If D is a divisor of degree d in $\mathbb{P}^n$, then $D \sim d \cdot H$.

(c) $\deg : \operatorname{Cl} \mathbb{P}^n \to \mathbb{Z}$ is a group isomorphism.

Proof: (a) Let $f \in K^\times$. Then $f = g/h$ where $g, h \in S$ are homogeneous polynomials of the same degree. By the properties of valuations, we have that $\operatorname{div}(f) = \operatorname{div}(g) - \operatorname{div}(h)$, and the degree is clearly additive. Therefore, it is enough to show that the two notions of degree agree. Take the factorization of g into irreducible polynomials $\prod g_i^{d_i}$ with $\sum d_i = d$. Then for any prime divisor Y, $v_Y(g) = \sum d_i v_Y(g_i)$ and $v_Y(g_i) = 1$ if $Y = V(g_i)$ and zero otherwise. Therefore $\operatorname{div}(g) = \sum d_i \cdot V(g_i)$ and in particular $\deg(\operatorname{div}(g)) = \sum d_i = d$.

(b) First of all, take D_0 an effective divisor. $D_0 = \sum_{i=1}^r n_i \cdot Y_i$. By Krull's principal ideal theorem, every $Y_i = V(g_i)$ for some irreducible polynomial $g_i \in S$, so that $D_0 = \operatorname{div}(g_0)$ for $g_0 = \prod g_i^{n_i}$. Now, if D is any divisor of degree d, write $D = D_1 - D_2$ with each D_i effective of degree d_i. Take $g_1, g_2 \in S$ such that $D_i = \operatorname{div}(g_i)$. Then $D - d \cdot H = \operatorname{div}(g_1) - \operatorname{div}(g_2) - \operatorname{div}(X_0^d) = \operatorname{div}(g_1/X_0^d g_2)$.

(c) It is clear from the definition that $\deg : \operatorname{Div}(\mathbb{P}^n) \to \mathbb{Z}$ is a surjective group homomorphism. By parts (a) and (b), the kernel is precisely the subgroup of principal divisors, so that $\operatorname{Cl} \mathbb{P}^n \cong \mathbb{Z}$. □

Prime divisors are "big" closed subsets, so intuitively speaking, removing "small" closed subsets from X should not affect the group of Weil divisors. The following proposition makes this intuition rigorous. We refer to [9, Proposition 6.5, Chapter 2] for the proof.

Proposition 2.5.2. *Let X be a noetherian integral separated scheme which is regular in codimension one, and let F be a proper closed subset of X. Denote $U = X - F$. Then:*

(a) There is a surjective homomorphism $\operatorname{Cl} X \to \operatorname{Cl}(X - F)$ given by $\sum n_i \cdot [Y_i] \mapsto \sum n_i \cdot [Y_i \cap U]$.

(b) If $\operatorname{codim}(F, X) \geq 2$, then the homomorphism in (a) in an isomorphism.

(c) If F is irreducible of codimension 1, then there is an exact sequence

$$\mathbb{Z} \to \operatorname{Cl} X \to \operatorname{Cl}(X - F) \to 0,$$

where the first map is given by $1 \mapsto [F]$.

2.5.2 Cartier Divisors

Lets consider the case of a non singular variety X. In this context, for every prime divisor $Y \subset X$ and every closed point $x \in X$ there is an open neighborhood of x in which Y is defined by a local equation g. For an arbitrary divisor $D = \sum n_i \cdot Y_i$, let g_i be the local equation for Y_i in U. Then, defining $f = \prod g_i^{n_i} \in K(X)$, we have that $D|_U = \text{div}(f)$, i.e, Weil divisors are locally principal. Since the closed points are dense, we can find a finite cover $\{U_i\}$ and rational functions $\{f_i\}$ such that $D|_{U_i} = \text{div}(f_i)$. Since the divisors $\text{div}(f_i)$ and $\text{div}(f_j)$ are forced to coincide on the intersection $U_i \cap U_j$, the functions $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X)$ are nowhere zero (invertible) there for every i, j. A Cartier divisor is a generalization of these observations. Here we use the sheaf theory terminology introduced in the first section of this chapter.

Definition 2.5.4 (Sheaf of total quotient rings)**.** Let X be a scheme. For each open subset $U \subset X$ let $S(U)$ be the multiplicative set of elements of $\mathcal{O}_X(U)$ whose images in $\mathcal{O}_x$ are not zero divisors for every $x \in U$. Then, the assignment $U \mapsto S(U)^{-1}\mathcal{O}_X(U)$ is a presheaf whose associated sheaf $\mathcal{K}$ we call the *sheaf of total quotient rings* of $\mathcal{O}_X$.

Given a sheaf of rings $\mathcal{F}$ on a topological space X, we denote by $\mathcal{F}^\times$ the sheaf of multiplicative groups that assigns to each open subset $U \subset X$ the group of units $\mathcal{F}(U)^\times$.

Definition 2.5.5 (Cartier Divisors)**.** A *Cartier divisor* on a scheme X is a global section of the sheaf $\mathcal{K}^\times/\mathcal{O}_X^\times$. A Cartier divisor is *principal* if it belongs to the image of the natural map $\Gamma(X, \mathcal{K}^\times) \to \Gamma(X, \mathcal{K}^\times/\mathcal{O}_X^\times)$. Two Cartier divisors are said to be *linearly equivalent* if their difference is principal.

Note that a Cartier divisor on a scheme X can be completely described by giving an open cover $\{U_i\}$ of X and for each i, a section $f_i \in \Gamma(U_i, \mathcal{K}^\times)$ such that for every pair of indices i, j we have that $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^\times)$. We have the following proposition [9, Proposition 6.11, Chapter II].

Proposition 2.5.3. *Let X be an integral, separated noetherian scheme, all of whose local rings are unique factorization domains. Then, the group* $\text{Div}(X)$ *of Weil divisors is isomorphic to the group* $\Gamma(X, \mathcal{K}^\times/\mathcal{O}_X^\times)$ *of Cartier divisors. Furthermore, the principal Weil divisors correspond to principal Cartier divisors under this isomorphism.*

Continuing the discussion of the beginning of the section, we already have that for a smooth projective variety X every Weil divisor is Cartier. Note that for integral schemes, the sheaf of total quotient rings is just the constant sheaf corresponding to the function field $K(X) = K$. Take a Cartier divisor D given by $\{(U_i, f_i)\}$ for U_i an open cover of X and $f_i \in \Gamma(U_i, \mathcal{K}^\times) = K^\times$. We define a Weil divisor $\tilde{D}$ as follows: given a prime divisor Y define its coefficient by $c_Y := v_Y(f_i)$, where i is any index for which $Y \cap U_i \neq \emptyset$, and put $\tilde{D} = \sum c_Y \cdot Y$. The coefficient c_Y is well defined since if $U_j \cap Y \neq \emptyset$ as well, f_i/f_j is invertible in $U_i \cap U_j$ and $0 = v_Y(f_i/f_j) = v_Y(f_i) - v_Y(f_j)$.

Moreover, since X is a noetherian topological space, it is quasi-compact and we can take $\{U_i\}$ to be a finite cover. The verification that this definition is independent of the open cover follows by the same argument of compatibility on the refined covering.

Example 2.5.3 (A Weil Divisor that is not Cartier)**.** For this example, we will need that every integrally closed noetherian domain is the intersection of the localizations over all primes of height one (see [2, Theorem 23.17]).

We study the affine quadric cone at the origin, its singular point. Let k be some field, $A := k[X, Y, Z]/\langle XY - Z^2\rangle$ and $V = \operatorname{Spec} A$. Denote by x, y, z the classes of the variables X, Y, Z in A and consider the closed subscheme $P = V(\mathfrak{p})$, where $\mathfrak{p} = \langle y, z\rangle$. Since $\langle Y, Z\rangle \supset \langle XY - Z^2\rangle$ is a prime ideal of height 2 in $k[X, Y, Z]$, $\mathfrak{p}$ is prime of height 1 by correspondence and P is a prime divisor in V. Consider the local ring $\mathcal{O}_{X,P} = A_{\mathfrak{p}}$. The maximal ideal $\mathfrak{p}A_{\mathfrak{p}} = \langle z\rangle$ since in $A_{\mathfrak{p}}$, $y = x^{-1}z^2$. Furthermore, the last equation implies that $v_P(y) = 2$. If $\mathfrak{q}$ is a prime ideal containing y, then $z^2 = xy \in \mathfrak{q}$ so that $z \in \mathfrak{q}$ and $\mathfrak{q} \supset \mathfrak{p}$. Thus, P is the only prime divisor for which the valuation is not zero and we have that $\operatorname{div}(y) = 2 \cdot P$. By proposition 2.5.2 we have an exact sequence

$$\mathbb{Z} \longrightarrow \operatorname{Cl}(V) \longrightarrow \operatorname{Cl}(V - P) \longrightarrow 0\,,$$

where the first morphism is given by $1 \mapsto [P]$. Since $V - P = \operatorname{Spec} A_{\mathfrak{p}}$ and $A_{\mathfrak{p}} = A_y = k[y, y^{-1}, z]$ is a UFD, theorem 2.5.2 implies that $\operatorname{Cl}(V - P) = 0$, so that $[P]$ generates $\operatorname{Cl}(V)$ and $2[P] = 0$. If we show that P is not principal, we would have that $\operatorname{Cl}(V) \cong \mathbb{Z}/2\mathbb{Z}$. Taking $f = xy$ in the following lemma, we have that A is integrally closed:

Lemma 2.5.3. *Let k be a field of characteristic $\neq 2$, $f \in k[t_1, \ldots, t_n]$ a square-free non constant polynomial. Then the ring $A = k[t_1, \ldots, t_n, z]/\langle z^2 - f\rangle$ is integrally closed.*

Proof: Let L the field $k(t_1, \ldots, t_n)$. Since f is square-free, the polynomial $z^2 - f$ is irreducible, and $L(z)/L$ is a Galois extension of degree 2 ($z = \sqrt{f}$). We calculate the integral closure of $L(z)$ in $k[t_1, \ldots, t_n]$. Take $\alpha = g + zh \in L(z)$ integral over $k[t_1, \ldots, t_n]$. The minimal polynomial of α is

$$m(T) = (T - \alpha)(T - \bar{\alpha}) = T^2 - 2gT + (g^2 - fh^2) \in L[T],$$

and since α is integral over the polynomial ring, m divides some monic polynomial $p(T) \in k[t_1, \ldots, t_n][T]$. Gauss' lemma implies that m has coefficients in $k[t_1, \ldots, t_n]$. Thus $g, h \in k[t_1, \ldots, t_n]$ and $\alpha \in A$. Since every element of A is integral over $k[t_1, \ldots, t_n]$, we have shown that A is the integral closure of $L(z)$ in $k[t_1, \ldots, t_n]$, so in particular A is integrally closed in this field. Since $\operatorname{Frac}(A) = L(z)$, we are done. □

Suppose that the prime divisor P is principal. Then there exits $f \in K(V) = \operatorname{Frac}(A)$ such that $\operatorname{div}(f) = P$. Since $v_P(f) = 1$, $f \in A_{\mathfrak{p}}$ generates the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$. If $\mathfrak{q}$ is any other prime ideal of height one, it corresponds to a prime divisor Q of V so that $v_Q(f) = 0$ and $f \in A_{\mathfrak{q}}$. By the lemma, A is integrally closed and therefore

$f \in \bigcap_{\mathrm{ht}(\mathfrak{q})=1} A_{\mathfrak{q}} = A$. Thus, to show that P is not principal, it is enough to show that $\mathfrak{p}$ is not principal.

Consider the maximal ideal $\mathfrak{m} = \langle x, y, z \rangle$ in A. The quotient $\mathfrak{m}/\mathfrak{m}^2$ is a three dimensional vector space over k with basis $\{\bar{x}, \bar{y}, \bar{z}\}$. The image of $\mathfrak{p}$ in this quotient is $\mathrm{Span}_k(\bar{y}, \bar{z})$ so that $\mathfrak{p}$ cannot be principal. More than that, the ideal $\mathfrak{p}A_{\mathfrak{m}}$ is not principal, so the divisor P is not locally principal and therefore it cannot be a Cartier Divisor. ■

The final link that closes the circle between Weil divisors, Cartier divisors and invertible sheaves is given by the following definition/proposition [9, Proposition 6.13, Chapter II].

Definition 2.5.6 (Sheaf associated to a divisor)**.** Let D be a Cartier divisor on a scheme X, represented by $\{(U_i, f_i)\}$. We define $\mathcal{L}(D)$ to be the sub-$\mathcal{O}_X$-module of $\mathcal{K}$ generated by f_i^{-1} on each U_i. Since f_i/f_j is invertible on $U_i \cap U_j$, this is well defined. $\mathcal{L}(D)$ is a subsheaf of $\mathcal{K}$ called the *sheaf associated to* D.

Proposition 2.5.4. *Let X be a scheme. Then:*

(a) For any Cartier divisor D, $\mathcal{L}(D)$ is an invertible sheaf on X and the map $D \mapsto \mathcal{L}(D)$ gives a bijective correspondence between Cartier divisors on X and invertible subsheaves of $\mathcal{K}$.

(b) $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$.

(c) $D_1 \sim D_2$ if and only if $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$ as invertible sheaves.

Just to get our feet on the ground again, consider the case of $X = \mathbb{P}^n = \mathrm{Proj}\, k[T_0, \ldots, T_n]$ and the Weil divisor $D = d \cdot H$ where H is the hyperplane $V(T_0)$. The associated Cartier divisor is given by the compatible system $\{(\mathbb{P}^n, t_0^d)\}$ where $t_0 \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ is the global section induced by T_0. Then $\mathcal{L}(D)$ is the sub-$\mathcal{O}_{\mathbb{P}^n}$-module of $\mathcal{K}$ generated by t_0^{-d}. Since $\mathcal{K}$ is the constant sheaf corresponding to the function field $K(\mathbb{P}^n) = k[T_0, \ldots, T_n]_{(0)}$, it follows that $\mathcal{L}(D) = \mathcal{O}_{\mathbb{P}^n}(d)$.

2.6 Linear Systems

For this section, X will be a nonsingular projective variety over an algebraically closed field k. In this context, Weil and Cartier divisors are equivalent and furthermore $\operatorname{Cl} X \cong \operatorname{Pic} X$.

Lemma 2.6.1. *Let X be a projective variety over k. For any invertible sheaf $\mathcal{L}$ on X, the k-vector space $\Gamma(X, \mathcal{L})$ is finite dimensional.*

We have already seen that given a divisor D on X, we have the invertible sheaf $\mathcal{L}(D)$ associated to it. Now, take $\mathcal{L}$ an invertible sheaf on X and pick a global section $s \in \Gamma(X, \mathcal{L})$. For an open subset $U \subset X$ over which $\mathcal{L}$ is trivial, let $\varphi_U : \mathcal{L}|_U \to \mathcal{O}_U$ be an isomorphism, so that $\varphi_U(s) \in \Gamma(U, \mathcal{O}_U)$. As U ranges over a covering of X, the collection $\{(U, \varphi_U(s))\}$ determines a well defined effective Cartier divisor $(s)_0$, which we call the *divisor of zeros of s*.

Proposition 2.6.1. *Let X be a nonsingular variety over an algebraically closed field k and let D_0 be a divisor on X. Then:*

(a) For each nonzero $s \in \Gamma(X, \mathcal{L}(D_0))$, the divisor of zeros $(s)_0$ is an effective divisor and $D_0 \sim (s)_0$.

(b) Every effective divisor linearly equivalent to D_0 is the principal divisor of a nonzero global section of $\mathcal{L}(D_0)$.

(c) Two sections $s, t \in \Gamma(X, \mathcal{L}(D_0))$ have the same divisor of zeros if and only if there exists some $\lambda \in k^\times$ such that $t = \lambda s$.

Definition 2.6.1 (Complete Linear System)**.** A *complete linear system* on X is the set of all effective divisors that are linearly equivalent to some given divisor D_0, and it is denoted by $|D_0|$.

Note that proposition 2.6.1 gives a bijection between $|D_0|$ and $\mathbb{P}(\Gamma(X, \mathcal{L}(D_0)))$, so that $|D_0|$ has a structure of the set of closed points of a projective space over k.

Definition 2.6.2 (Linear System)**.** A linear subspace $\Lambda \subset |D_0|$ is called a *linear system*. Thus $\Lambda = \mathbb{P}(V)$, where V is the sub-vector space of $\Gamma(X, \mathcal{L}(D_0))$ given by $V = \{s \in \Gamma(X, \mathcal{L}(D_0)) \,|\, (s)_0 \in \Lambda\} \cup \{0\}$. The *dimension* of the linear system is its dimension as a linear projective variety.

Definition 2.6.3 (Base Points, Fixed Components, General Elements)**.** A closed point $p \in X$ is a *base point* of the linear system Λ if $p \in D$ for every $D \in \Lambda$. A prime divisor P is a *fixed component* of Λ if every divisor in Λ contains P. A linear system without fixed components is called a *mobile linear system*. We say that a *general* or *generic* element of Λ has a property P, if elements in a non-empty Zariski open subset of Λ have the property P.

Example 2.6.1 (Conics in $\mathbb{P}^2$ through three points)**.** Consider the surface $X = \mathbb{P}^2 = \operatorname{Proj} k[T_0, T_1, T_2]$. Consider the divisor $2 \cdot H$ where H is the line $T_0 = 0$. By proposition 2.5.1, the complete linear system $|2 \cdot H|$ is the $\mathbb{P}^5$ of conics in $\mathbb{P}^2$, i.e,

$$\{a_0T_0^2 + a_1T_0T_1 + a_2T_0T_2 + a_3T_1^2 + a_4T_1T_2 + a_5T_2^2 \,|\, (a_0 : a_1 : a_2 : a_3 : a_4 : a_5) \in \mathbb{P}^6\}.$$

As we discussed earlier, $\mathcal{L}(2 \cdot H) = \mathcal{O}_{\mathbb{P}^2}(2)$ and that $V = \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ is the six dimensional vector space generated by the global sections of degree two homogeneous polynomials on t_0, t_1, t_2, so $|2 \cdot H| \cong \mathbb{P}(V)$. Fix the three coordinate points $\mathfrak{p} = \langle T_1, T_2 \rangle$, $\mathfrak{q} = \langle T_0, T_2 \rangle$, $\mathfrak{r} = \langle T_0, T_1 \rangle$ and consider the linear subspace $\Lambda \subset |2 \cdot H|$ of all conics passing through these three points. Note that Λ is a two dimensional linear system (it is given by $a_0 = a_3 = a_5 = 0$) with no fixed part, and by definition, the base points of Λ are $\mathfrak{p}, \mathfrak{q}$ and $\mathfrak{r}$. Consider the vector subspace of $V_\Lambda \subset V$ corresponding to Λ. Note that $V_\Lambda = \operatorname{Span}_k\{t_1t_2, t_0t_2, t_0t_1\}$. The ordered basis t_1t_2, t_0t_2, t_0t_1 induces an automorphism of the function field $K(\mathbb{P}^2) = k[T_0, T_1, T_2]_{(0)}$:

$$\begin{aligned} q : k[T_0, T_1, T_2]_{(0)} &\longrightarrow k[T_0, T_1, T_2]_{(0)} \\ T_0 &\longmapsto T_1T_2, \\ T_1 &\longmapsto T_0T_2, \\ T_2 &\longmapsto T_0T_1, \end{aligned}$$

is in fact an automorphism, so it induces a rational map $\tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ called the standard quadratic transformation with $\operatorname{dom}(\tau) = \mathbb{P}^2 - \{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}\}$. Thus, for each choice of an ordered basis B of the vector space V_Λ we have a rational map τ_B. Reciprocally, if we start with the rational map τ described above, we can consider the pullback of the linear system $|H|$ by τ and recover Λ.

2.7 Differentials

In Differential geometry, differential forms are defined as sections of the dual tangent bundle. We will work in the opposite direction. The idea is to use the algebraic construction of the module $\Omega_{B/A}$ of Kähler differentials of a ring extension B/A, and use the sheaf associated to a module on the prime spectrum (definition 2.4.3) and the glueing of sheaves to construct the quasi-coherent sheaf of differentials. A good reference for reviewing the theory of Kälher differentials is [5].

Let A be a ring and B an A-algebra. Then $V = \operatorname{Spec} B$ is a scheme over $U = \operatorname{Spec} A$, so we may consider the scheme $V \times_U V$. As we mentioned before, $V \times_U V = \operatorname{Spec}(B \otimes_A B)$. Let $(\Omega_{B/A}, \mathrm{d})$ be the B-module of relative differentials of B/A. Recall that $\Omega_{B/A} \cong I/I^2$, where I is the kernel of the diagonal homomorphism $\delta : B \otimes_A B \to B$, $b \otimes b' \mapsto bb'$ and $\mathrm{d} : B \to \Omega_{B/A}$ is given by $b \mapsto (b \otimes 1 - 1 \otimes b) + I^2$.

Define $\Omega_{V/U}$ as the $\mathcal{O}_V$-module associated to the module $\Omega_{B/A}$. Since $\mathcal{O}_V(V) \cong B$ and $\Omega_{V/U}(V) \cong \Omega_{B/A}$, the derivation $\mathrm{d} : B \to \Omega_{B/A}$ induces a map $\mathrm{d} : \mathcal{O}_V \to \Omega_{V/U}$ of sheaves of abelian groups, which is a derivation of the local rings at each prime in V.

Note that the diagonal morphism $\Delta : V \to V \times_U V$ is precisely the morphism induced by the diagonal homomorphism δ. Since δ is surjective, we have that $\Delta(V) = V(I)$, so in particular $\Delta(V)$ is closed in $V \times_U V$ and the sheaf of ideals $\mathcal{J}$ of $\Delta(V)$ in $V \times_U V$ corresponds to $\tilde{I}$.

Definition 2.7.1 (Sheaf of Relative Differentials)**.** Let X be a scheme over Y, with morphism $f : X \to Y$. For each affine open subset U of Y, there is an open affine subset V of X such that $f(V) \subset U$, and when U ranges over a covering of Y, such open subsets V cover X. The sheaf of $\mathcal{O}_X$-modules given by glueing the sheaves $\Omega_{V/U}$ constructed above.

We are interested in the special case of X being a smooth projective variety over an algebraically closed field k. In this setting, $Y = \operatorname{Spec} k$ is just a point, so any affine covering of X is an admissible covering for defining $\Omega_{X/k}$. In order to understand the properties of $\Omega_{X/k}$ we will need a couple of algebraic results. For the proofs, we refer to [5, Proposition 16.9], [5, Corollary 16.13] and [9, Theorem 8.8, Chapter II] respectively.

Proposition 2.7.1 (Localization of Differentials)**.** *Let A be a ring, B an A-algebra and $S \subset B$ a multiplicative system. Then $\Omega_{S^{-1}B/A} \cong S^{-1}\Omega_{B/A}$.*

Proposition 2.7.2. *Let B be a local ring containing a field k which is isomorphic to the quotient field $B/\mathfrak{m}$. Then, the map $D : \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \otimes_B k$ mapping $\bar{b} \mapsto db \otimes 1$ is an isomorphism.*

Theorem 2.7.1. *Let B be a local ring containing a field k isomorphic to its residue field. Furthermore, assume that k is perfect, and that B is a localization of a finitely generated k-algebra. Then $\Omega_{B/k}$ is a free B-module of rank equal to $\dim B$ if and only if B is a regular local ring.*

Example 2.7.1 (Affine Space)**.** Let k be a field, $B = k[x_1, \ldots, x_n]$ and let $X = \operatorname{Spec} A$. Then, $\Omega_{X/k} = (\Omega_{A/k})^{\sim}$ by definition. Since $\Omega_{A/k} = A\,\mathrm{d}x_1 \oplus \cdots \oplus A\,\mathrm{d}x_n$, for each prime ideal $\mathfrak{p} \subset A$ we have that $(\Omega_{A/k})_{\mathfrak{p}} = \Omega_{A_{\mathfrak{p}}/k} \cong (\mathfrak{m}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^2) \otimes_A k$, so that $\Omega_{A_{\mathfrak{p}}/k}$ and $\mathfrak{m}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^2$ are isomorphic as k-vector spaces and we obtain a generalization of the classical definition of differential one forms as sections of the cotangent bundle which are locally linear combinations of differentials of regular functions, with regular functions coefficients (see [13, Chapter 3, Section 5]). ■

Example 2.7.2 (Nonsingular Varieties)**.** From the previous example it is clear that $\Omega_{\mathbb{A}^n/k}$ is free of rank n. More generally, if X is a nonsingular variety over an algebraically closed field k of dimension n, then $\Omega_{X/k}$ is locally free of rank n. To see this, take $p \in X$ a closed point. Let $U = \operatorname{Spec} A$ be an open affine neighborhood of p and let $\mathfrak{p}$ be the maximal ideal in A corresponding to p. Denote by $B = A_{\mathfrak{p}}$. Then $(\Omega_{X/k})_p \cong \Omega_{B/k}$ by definition and proposition 2.7.1. By theorem 2.7.1, we have that $\Omega_{B/k}$ is free of dimension $\dim B = n$. By lemma 2.4.1, there is a neighborhood V of p such that $\mathcal{F}|_V$ is free of rank n. Since the closed points are dense in X, we have that X can be covered by open subsets V for which $\mathcal{F}|_V$ is a free $\mathcal{O}_X|_V$-module of rank n, i.e $\mathcal{F}$ is locally free of rank n. ■

It turns out that for an irreducible separated scheme of finite type X over an algebraically closed field k, $\Omega_{X/k}$ is locally free of rank $n = \dim X$ if and only if X is a nonsingular variety over k (see [9, Theorem 8.15]).

Definition 2.7.2 (Canonical Sheaf and Geometric Genus)**.** Let X be a nonsingular variety of dimension n over an algebraically closed field k. The *canonical sheaf of* X is defined as the n-th exterior power of the sheaf of differentials $\omega_X := \bigwedge^n \Omega_{X/k}$. The *geometric genus* of X is defined to be $p_g = \dim_k \Gamma(X, \omega_X)$.

By the discussion of section 2.4.1, the canonical sheaf is an invertible sheaf. Also, the k-vector space $\Gamma(X, \omega_X)$ is finite dimensional (see [9, Theorem 5.19, Chapter II]), so that p_g is always a non negative integer.

Chapter 3

Surfaces and Rationality

In this chapter we introduce some fundamental concepts from intersection theory on surfaces, and discuss the important relation between linear systems on surfaces and rational maps to projective space. Afterwards, we apply the results from the previous chapter to prove a beautiful criterion of rationality in terms of numerical properties of linear systems.

3.1 Surfaces

We follow the notation of [9, Chapter V]. By a *surface* X, we mean a nonsingular projective surface over an algebraically closed field k. A *curve* C on X will be an effective divisor, and by a *point* we mean a closed point, unless otherwise specified.

Theorem 3.1.1. *There exists a unique symmetric bilinear form*

$$\begin{array}{ccc} \mathrm{Div}(X)\times\mathrm{Div}(X) & \longrightarrow & \mathbb{Z}\,, \\ (C,D) & \longmapsto & C\cdot D, \end{array}$$

such that:

(1) If C and D are nonsingular curves meeting transversally, then $C\cdot D = \#(C\cap D)$.

(2) Given $C, D \in \mathrm{Div}(X)$, the intersection number $C\cdot D$ depends only on the linear equivalence classes of C and D.

For a proof we refer to [9, Theorem 1.1, Chapter V]. This bilinear form is called the *intersection form* of the surface X. If C is any divisor on X, we call $C\cdot C$ the *self-intersection number* of C, and denote it by C^2. If C and D are curves without common irreducible components, and $p \in C\cap D$, we define the *intersection multiplicity* $(C\cdot D)_p$ to be the length of the ring $\mathcal{O}_{X,p}/\langle f, g\rangle$, where f and g are local equations of C and D at p. The following proposition gives a useful way of calculating the intersection of two curves. We refer to [9, Proposition 1.4, Chapter 2] for a proof.

Proposition 3.1.1. *If C, D are curves on X having no common irreducible component, then*

$$C \cdot D = \sum_{p \in C \cap D} (C \cdot D)_p \, .$$

Example 3.1.1 (Bezout's Theorem)**.** Let $X = \mathbb{P}^2$. Then $\operatorname{Pic} X \cong \mathbb{Z}$ and we may take the class of a line ℓ as a generator. Let $\ell' \neq \ell$ be another line. Then $\ell \sim \ell'$ and $\ell^2 = \ell \cdot \ell' = \sum_{p \in \ell \cap \ell'} (\ell \cap \ell')_p = (\ell \cap \ell')_{\ell \cap \ell'} = 1$. This determines completely the intersection form on $\mathbb{P}^2$: taking curves C, D of degrees n and m respectively, we have that $C \sim n\ell$ and $D \sim m\ell$, so that $C \cdot D = (n\ell) \cdot (m\ell) = nm(\ell^2) = nm$. ■

Example 3.1.2 (The Canonical Divisor)**.** Let $\Omega_{X/k}$ be the sheaf of differentials of X/k and let $\omega_X = \bigwedge^2 \Omega_{X/k}$ be the canonical sheaf of X. Any divisor K in the linear equivalence class corresponding to ω_X is called a *canonical divisor*, and it is usually denoted K_X.

Let us calculate the canonical divisor of $\mathbb{P}^2$. We know that $\omega_{\mathbb{P}^2}$ is an invertible sheaf on $\mathbb{P}^2$, and invertible sheaves there are of the form $\mathcal{O}_{\mathbb{P}^2}(n)$. So we want to find n such that $\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(n)$. Writing $\mathbb{P}^2 = \operatorname{Proj} k[T_0, T_1, T_2]$, we have that $\mathbb{P}^2$ is covered by the open affine subsets $U_i := D_+(T_i) \cong \operatorname{Spec} k[t_{l/i}, t_{j/i}]$ for indexes $\{0,1,2\} = \{l, j, i\}$ with $t_{j/1} = T_j/T_i$. Consider the section $\omega_0 = \mathrm{d}t_{1/0} \wedge \mathrm{d}t_{2/0} \in \Gamma(U_0, \omega_{\mathbb{P}^2})$. On the intersection $U_0 \cap U_1$, the sections $t_{1/0}, t_{2/0}, t_{0/1}, t_{2/1} \in \Gamma(U_0 \cap U_1, \mathcal{O}_{\mathbb{P}^2})$ satisfy $t_{1/0} t_{0/1} = 1$ and $t_{2/0} t_{0/1} = t_{2/1}$. This equations imply that

$$\mathrm{d}t_{1/0} = -\frac{1}{t_{0/1}^2}\, \mathrm{d}t_{0/1} \text{ and } \mathrm{d}t_{2/0} = \frac{1}{t_{1/0}} \left(\mathrm{d}t_{2/1} - t_{2/0} \mathrm{d}t_{0/1} \right), \text{ so } \omega_0 = -\frac{1}{t_{0/1}^3} \mathrm{d}t_{0/1} \wedge \mathrm{d}t_{2/1}.$$

The same calculation for the intersection $U_0 \cap U_2$ yields

$$\omega_0 = \frac{1}{t_{0/2}^3} \mathrm{d}t_{0/2} \wedge \mathrm{d}t_{1/2}.$$

Thus, the global rational section defined by ω_0 has a pole of order three at $H_0 = V(T_0)$, so that $K_{\mathbb{P}^2} = -3H_0$ and $\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$. Generalizing this calculation to higher dimensions gives $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$. ■

3.1.1 The blowup of a surface at a point

The reader familiar with the theory of algebraic curves knows that the blowup construction plays a fundamental role in the theory of resolution of singularities. In the case of surfaces, the blowup at a point also plays a central role. In fact, every birational map between surfaces is given by a composition of blowups and blowdowns (see [3, Theorem II.11] and its corollary), and this fundamental fact allowed the Italian school to classify all surfaces up to birational equivalence by the beginning of the 20th century. We avoid the more general construction of the blowup of a coherent sheaf of ideals and use local coordinates and the glueing construction instead.

Fix the point p in X and take x, y local coordinates at p. Choose $U = \operatorname{Spec} A \subset X$ an open affine neighborhood of p such that x and y are regular functions on U and the maximal ideal $\mathfrak{m}$ in A corresponding to p is generated by x and y. This means that p is the only point in both curves $x = 0$ and $y = 0$ in U. Take a complex projective line $\mathbb{P}^1$ with homogeneous coordinates t, u. Recall that $\mathbb{P}^1 = U_0 \cup U_1$, with $U_i = \operatorname{Spec} \mathbb{C}[z_i]$ for $i = 0, 1$ and $z_0 = u/t$, $z_1 = t/u$.

Let $\overline{U}$ be the subvariety of $U \times \mathbb{P}^1$ defined by the equation $ux - ty = 0$. Then we have that $U \times \mathbb{P}^1 = (U \times U_0) \cup (U \times U_1)$. Define $V_0 := \overline{U} \cap (U \times U_0)$ and $V_1 := \overline{U} \cap (U \times U_1)$. Then V_0 and V_1 are defined respectively by the equations

$$V_0 = \{z_0 x - y = 0\} \subset U \times U_0, \qquad \text{and} \qquad V_1 = \{x - z_1 y = 0\} \subset U \times U_1.$$

From the covering $\overline{U} = V_0 \cup V_1$ and the equations above we have that $\overline{U}$ is smooth. Consider the morphism $\pi : \overline{U} \to U$ given by restricting the projection $U \times \mathbb{P}^1 \to U$. Note that $\pi^{-1}(p) = \{p\} \times \mathbb{P}^1 =: E$. Furthermore, the morphisms

$$\begin{aligned} U - \{x = 0\} &\longrightarrow V_0, \quad (x, y) \longmapsto ((x, y), (1 : y/x)), \\ U - \{y = 0\} &\longrightarrow V_1, \quad (x, y) \longmapsto ((x, y), (x/y : 1)), \end{aligned}$$

patch together to give an inverse of $\pi|_{\overline{U} - E} : \overline{U} - E \to U - \{p\}$. Thus, the morphism $\pi : \overline{U} \to U$ has the desired property, but we are not done since $\overline{U}$ may not be projective. To solve this problem we will use proposition 2.2.4. Consider the varieties $\overline{U}$ and $X - \{p\}$ with their respective open subsets $\overline{U} - E$ and $U - \{p\}$. Since this two open subsets are isomorphic, we can glue this varieties together over these open subsets to obtain a scheme $\operatorname{Bl}_p(X)$ which is covered by open subsets isomorphic to $X - \{p\}$ and $\overline{U}$. This implies that every point away from E is smooth. Also, since $E \subset V_0 \cup V_1 = \overline{U}$ with equations $x = 0$ in V_0 and $y = 0$ in V_1, E is smooth at all of its points. Thus, $\operatorname{Bl}_p(X)$ is a smooth projective surface.

Call $\pi_p : \operatorname{Bl}_p(X) \to X$ the morphism obtained by glueing $\pi : \overline{U} \to X$ and $i : X - \{p\} \hookrightarrow X$. It follows from the construction that

$$\pi_p^{-1}(p) = \{p\} \times \mathbb{P}^1 = E, \tag{3.1}$$

$$\pi_p^{-1}(X - \{p\}) = \operatorname{Bl}_p(X) - E \cong X - \{p\}. \tag{3.2}$$

Definition 3.1.1 (Blowup of a Surface at a point)**.** The pair $(\operatorname{Bl}_p(X), \pi_p)$ is called the *blowup of X at p*, and E_p is called the *exceptional curve*, or *exceptional divisor* of the blowup.

The following proposition shows that the blowup of a surface at a point is well defined, i.e that it is independent (up to isomorphism) of the choice of coordinates near p.

Proposition 3.1.2. *Let $\epsilon : X' \to X$ be a second blowup of p on X obtained from an open neighborhood U' and coordinates x', y' on it. Then, there is a unique X-isomorphism $\varphi : \mathrm{Bl}_p(X) \to X'$.*

$$\begin{array}{ccc} \mathrm{Bl}_p(X) & \xrightarrow{\sim} & X' \\ & {\scriptstyle \pi_p} \searrow \quad \swarrow {\scriptstyle \epsilon} & \\ & X & \end{array}$$

Proof: We continue with the same notation for the construction of $\mathrm{Bl}_p(X)$. Let $V \subset U \cap U'$ be an affine neighborhood of p such that we can write

$$x' = a_{oo}x + a_{01}y, \qquad y' = a_{10}x + a_{11}y$$

for $a_{ij} \in \Gamma(V, \mathcal{O}_X)$ with $\det(a_{ij}(q)) \neq 0$ for every $q \in V$. This is possible since both x, y and x', y' are generators of the maximal ideal corresponding to p in $\mathcal{O}_X(V)$. Then, the morphism

$$\begin{aligned} V \times \mathbb{P}^1 &\longrightarrow V \times \mathbb{P}^1, \\ (q, (t : u)) &\longmapsto (q, (a_{00}t + a_{01}u : a_{10}t + a_{11}u)). \end{aligned}$$

is a V-isomorphism that restricts to a V-isomorphism $\psi : \pi_p^{-1}(V) \to \epsilon^{-1}(V)$. Define φ by extending ψ to $\mathrm{Bl}_p(X)$ using the birational map $(\epsilon|_{X'-E'})^{-1} \circ (\pi_p|_{\mathrm{Bl}_p(X)-E})$. For the uniqueness of φ, it is enough to note that any other isomorphism φ' coincides with φ on the open (dense) subset $\mathrm{Bl}_p(X) - E$. $\square$

From now on we will omit the subscript p to ease the notation. We want to understand what happens to a curve lying on our surface X after blowing up one of its points. From the construction of $\mathrm{Bl}_p(X)$, it is clear that we have a correspondence between points in E and tangent directions at p (see figure 3.1). Thus, we are interested in the number of tangent directions of a curve at a point.

Definition 3.1.2. Multiplicity of a curve at a point. Let C be a curve on X, and let f be a local equation for C at the point p. The *multiplicity* of C at p, denoted by $m_p(C)$, is the largest integer m such that $f \in \mathfrak{m}_p^m \subset \mathcal{O}_{X,p}$.

Given a curve $C \subset X$ passing through p, the closure of $\pi^{-1}(C - \{p\})$ in $\mathrm{Bl}_p(X)$ is an irreducible curve $\widetilde{C}$, which we call the *strict transform* of C. See figure 3.1 [1].

Lemma 3.1.2. *Let C be a curve, p a point of multiplicity m in C, and $\pi : \mathrm{Bl}_p(X) \to X$ the blowup at p. Then*

$$\pi^* C = \widetilde{C} + mE.$$

[1] I thank professor Eduardo Casas-Alvero for sharing with me this image from his wonderful book *Singularities of Plane Curves* [4]

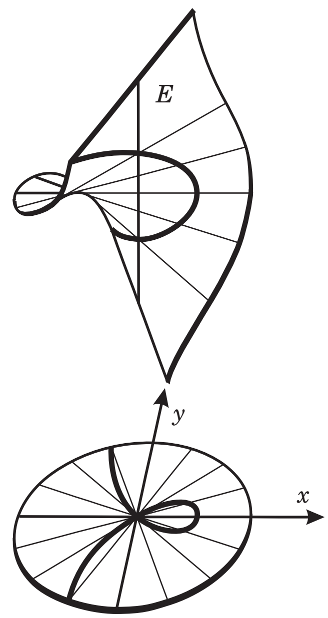


Figure 3.1: The strict transform of a curve after blowing up a node.

Proof: We carry on with the notation from the construction of $\mathrm{Bl}_p(X)$. Let f be a local equation of C on p. By definition, $f \in \mathfrak{m}_p^m - \mathfrak{m}_p^{m+1}$. Since $\mathfrak{m}_p = \langle x, y\rangle$, $f + \mathfrak{m}_p^{m+1} \in \mathfrak{m}_p^m/\mathfrak{m}_p^{m+1} \cong \mathrm{Span}_{\mathbb{C}}(x^m, x^{m-1}y, \ldots, y^m)$, we can write $f = f_m + g$ where f_m is a nonzero homogeneous polynomial of degree m in x and y. Also, $f_m \in \mathfrak{m}_p^m - \mathfrak{m}_p^{m+1}$ and $g \in \mathfrak{m}_p^{m+1}$. Shrinking U if necessary, we have that $f = f_m + g$ is a local equation for C in U. Consider the open affine subset V of $U \times \mathbb{P}^1$ defined by $t \neq 0$. On the intersection $\mathrm{Bl}_p(X) \cap V$ we have that $y = ux$, so $\pi^* f = f_m(x, xu) + g(x, xu) = x^m(f_m(1, u) + xg(1, u))$. Since x is a local equation for E and $f_m(1, u)$ is zero at only finitely many points on E, we have the result. □

The following proposition gives a characterization of the Picard group of the blowup, which completely determines the intersection form on it.

Proposition 3.1.3. *Let X be a surface, p a point on X, and $\pi : \mathrm{Bl}_p(X) \to X$ the blowup of X at p. Then:*

(a) $\mathrm{Pic}\,\mathrm{Bl}_p(X) = \pi^*\mathrm{Pic}\, X \oplus \mathbb{Z} \cdot [E]$.

(b) $(\pi^* D) \cdot (\pi^* D') = D \cdot D'$, *for every* $D, D' \in \mathrm{Div}X$.

(c) $E \cdot (\pi^* D) = 0$ *for every* $D \in \mathrm{Div}X$.

(d) $E^2 = -1$.

(e) $K_{\mathrm{Bl}_p(X)} = \pi^* K_X + E$.

Proof: (b) We can replace D and D' by linearly equivalent divisors such that $p \notin \mathrm{Supp}(D), \mathrm{Supp}(D')$ (see [13, Theorem 3.1, Section 3]). Since $\mathrm{Bl}_p(X) - E \cong X - \{p\}$ via π, the result follows from theorems 3.1.1 (2) and 3.1.1.

(c) Just as in (b), replacing D by a linearly equivalent divisor not containing p yields the result.

(d) Choose a curve C passing through p with multiplicity 1 (for example the closure of the curve $x = 0$ in U). The strict transform $\widetilde{C}$ meets E transversely at one point, corresponding to the tangent direction of p at C, therefore $\widetilde{C} \cdot E = 1$. By lemma 3.1.2 we have that $E = \pi^*C - \widetilde{C}$, so that $E^2 = (\pi^*C - \widetilde{C}) \cdot E = (\pi^*C) \cdot E - \widetilde{C} \cdot E = -1$.

(a) By 2.5.2 we know that $\mathrm{Pic}\, X \cong \mathrm{Pic}\,(X - \{p\})$. Since $X - \{p\}$ is isomorphic to $\mathrm{Bl}_p(X) - E$, we have that $\mathrm{Pic}\, X \cong \mathrm{Pic}\,(\mathrm{Bl}_p(X) - E)$ and again by 2.5.2 gives an exact sequence

$$\mathbb{Z} \to \mathrm{Pic}\, \mathrm{Bl}_p(X) \to \mathrm{Pic}\, X \to 0,$$

where the first morphism is given by $1 \mapsto 1 \cdot [E]$. Since for every $n \in \mathbb{Z}$ different from zero we have by (d) that $(nE) \cdot (nE) = -n^2 \neq 0$, the first morphism in the sequence is injective. On the other hand, the map $\pi^* : \mathrm{Pic}\, X \to \mathrm{Pic}\, \mathrm{Bl}_p(X)$ splits this sequence, so that $\mathrm{Pic}\, \mathrm{Bl}_p(X) \cong \mathrm{Pic}\, X \oplus \mathbb{Z} \cdot [E]$ as we wanted to show.

(e) We already know that $K_{\mathrm{Bl}_p(X)} = \pi^*K_X + \ell \cdot E$ for some $\ell \in \mathbb{Z}$. To see that $\ell = 1$, take an open affine neighborhood U and local equations x, y at p as in the construction of the blowup. Consider the section $\omega = \mathrm{d}x \wedge \mathrm{d}y \in \omega_X(U)$. Since in U we have that $\mathrm{Bl}_p(X)|_{\pi^{-1}(U)} = V(ty - ux) \subset U \times \mathbb{P}^1$, the regular function π^*y is locally of the form ux, so that $\pi^*\omega = \mathrm{d}(\pi^*x) \wedge \mathrm{d}(\pi^*y) = \mathrm{d}x \wedge (x\mathrm{d}u + u\mathrm{d}x) = x\, \mathrm{d}x \wedge \mathrm{d}u$ on some neighborhood of a point in E, and $\ell = 1$. □

Definition 3.1.3 (-1 Curve)**.** A curve C on a smooth surface X is said to be a -1-*curve* if $C \cong \mathbb{P}^1$ and $C^2 = -1$.

By the previous proposition, if a surface X' is the blowup of a smooth surface, then it contains a -1-curve, namely the exceptional divisor of the blowup. The following theorem gives the converse to this fact:

Theorem 3.1.3 (Castelnuovo's Contractibility Theorem)**.** *Let X' be a surface, and $C \subset X'$ a -1-curve. Then, there exists a surface X and a point $p \in X$ such that X' is isomorphic to the blowup of X at p. Furthermore, under this isomorphism, C corresponds to the exceptional divisor.*

3.2 Linear Systems and Rational Maps

Recall that in example 2.6.1 we saw that the linear system of conics through three distinct points gave rise to a birational transformation of the plane, and reciprocally that this map determined the linear system. More generally, given a surface X we have the following bijective correspondence of sets:

$$\left\{ \begin{array}{c} \text{Rational maps } \phi : X \dashrightarrow \mathbb{P}^n \text{ with } \phi(X) \\ \text{not contained in any hyperplane.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} n\text{-dimensional linear systems} \\ \Lambda \text{ on } X, \text{ without fixed curves.} \end{array} \right\}$$

This correspondence is a particular instance of a much more general phenomenon (see [9, Theorem 7.1, Chapter II]). We take a more geometrical approach that will be enough for the purposes of this work:

To the rational map $\phi : X \dashrightarrow \mathbb{P}^n$, assign the linear system $\Lambda = \phi^*|H|$, where $|H|$ is the linear system of hyperplanes in $\mathbb{P}^n$. Since $\phi(X)$ is not contained in any hyperplane, it is clear that Λ is n-dimensional. Also, since there are finitely many points at which ϕ is not defined, Λ is mobile. Conversely, given a mobile n-dimensional linear system Λ on X, denote by $\Lambda^* \cong \mathbb{P}^n$ the projective dual of Λ. Define ϕ_Λ by sending $x \in X$ to the line in Λ of divisors passing through x. By construction, ϕ_Λ is defined at a point $x \in X$ (not necessarily closed) if and only if x is not a base point of Λ.

Let Λ be a mobile linear system on a smooth surface X. We use the notation ϕ_Λ to make explicit that Λ is the linear system corresponding to ϕ_Λ. We define the *self intersection number of* Λ, denoted by Λ^2, to be the intersection number of two general members of Λ. When ϕ_Λ is a morphism, $\Lambda^2 = \deg(\phi_\Lambda) \cdot \deg(\phi_\Lambda(X))$. Similarly, the *multiplicity* of a the linear system Λ at a point p is the multiplicity of a general element there, we denote it by m_p. We will see that when Λ has base points, the self intersection number keeps track of their multiplicities.

Definition 3.2.1 (Infinitesimal Neighborhoods)**.** Let p be a point in a surface X and let E be the exceptional divisor after blowing up p on X. We will call E the *first infinitesimal neighborhood of* p. For $i > 0$, an *i-th infinitesimal neighborhood of* p, is defined by induction as the collection of points on a first infinitesimal neighborhood of a point in the $(i-1)$-th infinitesimal neighborhood of p. The points that are on a i-th infinitesimal neighborhood of p, for some $i > 0$, are called points *infinitely near* to p. To distinguish the points of X with the infinitely near to X, we will call the points of X *proper*. This notion yields a natural partial ordering of the points proper or infinitely near to X, just put $p \prec q$ if q is infinitely near to p.

The previous definition extends the notion of base point of a linear system introduced in definition 2.6.3 since the points infinitely near to a base point of a linear system also carry important geometrical information about the linear system. We illustrate this fact with an example.

Example 3.2.1. Take the rational map

$$\tau_2 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (x : y : z) \mapsto (x^2 : xy : y^2 + xz). \tag{3.3}$$

Thinking of τ_2 as a rational map between classical algebraic varieties, we see that the map is defined at every point different from $p_1 = (0 : 0 : 1)$. Furthermore, every point in U_0 can be written as $(1 : y : -y^2 + z)$. Then $\tau_2(1 : y : -y^2 + z) = (1 : y : z)$ so that $\tau_2(\mathbb{P}^2) \supset U_0$ and τ_2 is dominant. In particular, $\tau_2(\mathbb{P}^2)$ is not contained in any hyperplane of $\mathbb{P}^2$ and the associate linear system $\Lambda := \tau_2^*|H|$ is 2-dimensional without fixed curves. By definition,

$$\Lambda = \{\lambda_1 x^2 + \lambda_2 xy + \lambda_3(y^2 + xz) \mid (\lambda_1 : \lambda_2 : \lambda_3) \in \mathbb{P}^2\}, \tag{3.4}$$

and the base points of Λ correspond to the points where τ_2 is not defined. Clearly, p_1 is the only (proper) base point of Λ.

Take C a general element of Λ_2. C is given by an irreducible polynomial of the form $F = ax^2 + bxy + c(y^2 + xz)$. From this equation it is easy to see that if $c \neq 0$, $x = 0$ is the tangent line of C at p_1. Furthermore, this tangent line at p_1 is shared by every divisor of Λ in the open subset given by $\lambda_3 \neq 0$. This shows that a general element of Λ has $x = 0$ as a tangent line at p_1.

Blowing up the point p_1, we obtain the point

$$p_2 = (p_1, (0 : 1)) \in E_{p_1} \subset \mathrm{Bl}_{p_1}(\mathbb{P}^2) = \{((x : y : z), (t : u)) \mid ux = ty\} \subset \mathbb{P}^2 \times \mathbb{P}^1$$

corresponding to the tangent line $x = 0$. Call $X_1 = \mathrm{Bl}_{p_1}(\mathbb{P}^2)$. We have the following commutative diagram:

$$\begin{array}{ccc} p_2 \in X_1 & & \\ {\scriptstyle \pi}\downarrow & \searrow & \\ p_1 \in X & \overset{}{\underset{\tau_2}{\dashrightarrow}} & \mathbb{P}^n \end{array}$$

The composite map $\tau_2 \circ \pi$ is a rational map to $\mathbb{P}^2$ not containing any hyperplane, so that we may consider its associated linear system Λ_1. A general element on Λ_1 is given by the strict transform of a general element C of Λ. On the open affine subset $U_2 \times U_1 \cap X_1$ of p_2 (here $U_2 = D_+(z) \subset \mathbb{P}^2$ and $U_1 = D_+(u) \subset \mathbb{P}^1$), the equation of the strict transform $\widetilde{C}$ of C is of the form $at^2y + bty + c(t + y)$. Thus, the linear system Λ' has a shared tangent at p_2 given by $t + y = 0$. The point $p_3 \in E_{p_2} \subset \mathrm{Bl}_{p_2}(X_1)$ corresponding to this tangent direction is a point in the first infinitesimal neighborhood of p_2 and in second infinitesimal neighborhood of p_1. Calling $X_2 = \mathrm{Bl}_{p_2}(X_1)$, we have the following diagram:

$$
\begin{array}{ccc}
p_3 \in X_2 & & \\
\downarrow & \searrow & \\
p_2 \in X_1 & & \\
\downarrow & \searrow & \\
p_1 \in X & \overset{}{\underset{\tau_2}{\dashrightarrow}} & \mathbb{P}^n
\end{array}
\qquad (p_1 \prec p_2 \prec p_3)
$$

Example 3.2.1 will work as blueprint for the discussion ahead. Let $p \in X$ be a base point of the n-dimensional mobile linear system Λ. Consider the blowup $\pi : X' \to X$ of p at X. The composition $\phi_\Lambda \circ \pi : X' \to X$ is a rational map to $\mathbb{P}^n$ whose image is not contained in any hyperplane, thus we can consider its associated linear system Λ'. We have the following commutative diagram:

$$
\begin{array}{ccc}
E \subset X' & & \\
\pi \downarrow & \overset{\phi_{\Lambda'}}{\searrow} & \\
p \in X & \underset{\phi_\Lambda}{\dashrightarrow} & \mathbb{P}^n
\end{array}
$$

If C is a general member of Λ, then we see from the construction of the blowup that its strict transform $\widetilde{C}$ intersects the exceptional divisor E at the points corresponding to the tangent directions of C at p. Therefore the base points of Λ' infinitely near to p correspond to the tangent directions at p that are shared by the elements of Λ. Let us consider the behaviour of the self intersection number of a linear system after blowing up one of its base points.

Lemma 3.2.1. *Let Λ be an n-dimensional mobile linear system on a smooth surface X, and let p be a proper base point of Λ. Let $X' = \mathrm{Bl}_p(X)$ and Λ' be the linear system associated to $\pi \circ \phi_\Lambda$. Then,*

$$
(\Lambda')^2 = \Lambda^2 - m_p^2 \quad \text{and} \quad \Lambda' \cdot K_{X'} = \Lambda \cdot K_X + m_p\,.
$$

Proof: By lemma 3.1.2, we have that $\Lambda' = \pi^*\Lambda - m_p E$. Then, proposition 3.1.3 implies:

$$
\begin{aligned}
(\Lambda')^2 = (\pi^*\Lambda - m_p E) \cdot (\pi^*\Lambda - m_p E) &= (\pi^*\Lambda)^2 - 2m_p E \cdot (\pi^*\Lambda) + m_p^2 E^2 \\
&= \Lambda^2 - m_p^2\,.
\end{aligned}
$$

On the other hand, we have by proposition 3.1.3 (c) that $K_{X'} = \pi^* K_X + E$, so that:

$$
\begin{aligned}
\Lambda' \cdot K_{X'} &= (\pi^*\Lambda - m_p E) \cdot (\pi^* K_X + E) \\
&= (\pi^*\Lambda) \cdot (\pi^* K_X) + (\pi^*\Lambda) \cdot E - m_p E \cdot (\pi^* K_X) - m_p E^2 = \Lambda \cdot K_X + m_p\,.
\end{aligned}
$$

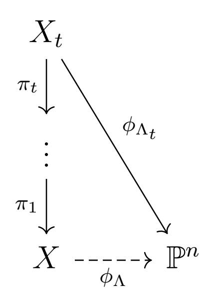
And this completes the proof of the lemma. □

An important corollary of the fact that the self-intersection number drops with each blowup over a base point, is that there are finitely many base points infinitely near to Λ, call them $B_\phi = B_\Lambda = \{p_1, \dots, p_t\}$. An *admissible ordering* of the base points of Λ is simply a total order that refines the natural order $\prec$ introduced in definition 3.2.1. This means that $p_i \preceq p_j$ implies that $i \leq j$. Therefore, it makes sense to successively blowup the base points according to this ordering. For the sake of clarity, we fix the following notation for $i = 1, \dots, t$.

- $X_0 = X$, and $X_i := \mathrm{Bl}_{p_i}(X_{i-1})$.
- $\pi_i := \pi_{p_i} : X_i \to X_{i-1}$.
- $\phi_0 = \phi_\Lambda$ and $\phi_i := \phi_{i-1} \circ \pi_i$.
- Λ_i the linear system corresponding to ϕ_i.

Then we arrive at a smooth surface X_t and a base point free linear system Λ_t such that $\phi_t = \phi_\Lambda \circ \pi_1 \circ \cdots \circ \pi_t : X_t \to \mathbb{P}^n$. This procedure is called *resolving the indeterminacy* of the linear system Λ.

Proposition 3.2.1 (Resolution of Indeterminacy)**.** *Let $\phi_\Lambda : X \dashrightarrow \mathbb{P}^n$ a rational map from a surface X. Then there exists a chain of blowups $X_t \to X_{t-1} \to \cdots \to X_1 \to X$ such that the composite rational map $\phi_t : X_t \dashrightarrow \mathbb{P}^n$ is a morphism.*



The multiplicity of an infinitely near base point is defined as its multiplicity on its corresponding linear system Λ_i. By resolving the indeterminacy of the linear system Λ defined by a birational map, we get the following result.

Lemma 3.2.2. *Suppose that $\phi_\Lambda : X \dashrightarrow \mathbb{P}^n$ is a birational map, and denote by Y the image of X under this map. Then*

$$\Lambda^2 - \sum_{p \in B_\Lambda} m_p^2 = \deg(Y) \quad \text{and} \quad K_X \cdot \Lambda + \sum_{p \in B_\Lambda} m_p = \mathcal{H}_Y \cdot K_Y \, ,$$

where B_Λ is the set of all the base points of Λ, including the infinitely near ones, and $\mathcal{H}_Y$ denotes the restriction to Y of the hyperplane class in $\mathbb{P}^n$.

Proof: Let $p_1, \dots, p_t$ be an admissible ordering of the base points of Λ. Note that since ϕ_Λ is birational, the morphism ϕ_t is an isomorphism and therefore has degree one, so that $(\Lambda_t)^2 = \deg(Y)$. Iterating the first equation in lemma 3.2.1 we obtain:

$$\Lambda^2 = \Lambda_1^2 + m_1^2 = \Lambda_2^2 + m_2^2 + m_1^2 = \cdots = \Lambda_t^2 + \sum_{i=1}^{t} m_i^2 = \deg(Y) + \sum_{p \in B_\Lambda} m_p^2 .$$

Recall from the discussion at the beginning of the section that the linear system $\Lambda_t = \phi_t^* |H|$. Therefore, iterating the second equation in lemma 3.2.1 we get:

$$\mathcal{H}_Y \cdot K_Y = \Lambda_t \cdot K_{X_t} = \Lambda_{t-1} \cdot K_{X_{t-1}} + m_t = \cdots = \Lambda \cdot K_X + \sum_{p \in B_\Lambda} m_p,$$

as we wanted to show. □

We are now ready to prove the main result of this chapter. It is a nice criterion for rationality of a smooth surface in terms of linear systems. It gives two important numerical conditions that prove to be very useful to study birational automorphisms of the plane.

Theorem 3.2.3. *Let X be a smooth surface. Then X is rational over k if and only if X admits a mobile linear system Λ of dimension 2 defined over k satisfying:*

(a) $\Lambda^2 - \sum_{p \in B_\Lambda} m_p^2 = 1$,

(b) $K_X \cdot \Lambda + \sum_{p \in B_\Lambda} m_p = -3$.

Proof: Assume that X is rational over k, and let $\phi_\Lambda : X \dashrightarrow \mathbb{P}^2$ be a birational map with associated mobile linear system Λ. As we saw at the beginning of the section, the dimension of Λ is two. Conditions (a) and (b) follow from lemma 3.2.2:

$$\Lambda^2 - \sum_{p \in B_\Lambda} m_p^2 = \deg(\mathbb{P}^2) = 1, \quad K_X \cdot \Lambda + \sum_{p \in B_\Lambda} m_p = |H| \cdot K_{\mathbb{P}^2} = |H| \cdot |-3H| = -3.$$

Conversely, suppose Λ is a 2-dimensional mobile linear system satisfying conditions (a) and (b), and let $\phi_\Lambda : X \dashrightarrow \mathbb{P}^2$ be its corresponding rational map. Resolving the indeterminacy of Λ we obtain a surface $\widehat{X}$ and a morphism $\phi_{\widehat{\Lambda}} : \widehat{X} \to \mathbb{P}^2$ as in proposition 3.2.1. Note that $\widehat{\Lambda}$ is two dimensional and that

$$(\widehat{\Lambda})^2 = \Lambda^2 - \sum m_p^2 = 1, \text{ and } \widehat{\Lambda} \cdot K_{\widehat{X}} = \Lambda \cdot K_X + \sum m_p = -3 .$$

Since X and $\widehat{X}$ are birational, it is enough to show that the map $\phi_{\widehat{\Lambda}}$ is birational. To check surjectivity, suppose by contradiction that $\phi_{\widehat{\Lambda}}(\widehat{X})$ is a plane curve $C \subset \mathbb{P}^2$. Then every member of $\widehat{\Lambda}$ is a union of fibers of $\widehat{X} \to C$, so the self intersection number of $\widehat{\Lambda}$ should be zero, contradiction. Finally, since $1 = (\widehat{\Lambda})^2 = \deg(\phi_{\widehat{\Lambda}})$, we have that $\phi_{\widehat{\Lambda}}$ is birational, so the surface X is rational. □

Chapter 4

Birational Automorphisms of the Plane

In this final chapter, we apply the results of surfaces previously discussed to the setting of birational automorphisms of $\mathbb{P}^2_{\mathbb{C}}$. We apply the tools developed in the previous chapter to classify all quadratic birational automorphisms. Afterwards, we use this classification and other results developed in the chapter to prove two recent theorems on the structure of decomposition groups of plane curves found in [12].

4.1 Birational Automorphisms of the Plane

In this section we restrict our focus to the smooth surface $X = \mathbb{P}^2$ over the field of complex numbers. We will apply the results from the previous section to this case and study some of the special properties of the group $\mathrm{Bir}(\mathbb{P}^2)$. We follow [7], [10] and [1, Chapter 2].

Definition 4.1.1 (Cremona Map)**.** A *plane Cremona map* is a birational map of the complex projective plane $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. The group $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is called the *Cremona Group*.

Once coordinates x, y, z in $\mathbb{P}^2$ are fixed, ϕ is given by three homogeneous polynomials $F, G, H \in \mathbb{C}[x, y, z]$ of the same degree d with no common factor:

$$\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2\,, \qquad (x : y : z) \longmapsto (F(x, y, z) : G(x, y, z) : H(x, y, z))\,.$$

Recall that the degree of ϕ is defined to be d, the common degree of F, G and H. We denote by Λ the mobile linear system associated to ϕ, but we omit the subscript used in the previous section. Recall that B_ϕ is the set of all base points of the linear system $\phi^*|H| = \Lambda$ (both proper and infinitely near). When we consider a fixed Cremona map ϕ, we will denote the sets B_ϕ and $B_{\phi^{-1}}$ by K and L respectively.

We start by applying theorem 3.2.3 to the particular case of Cremona maps.

Proposition 4.1.1 (Equations of Condition)**.** *Let* $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *be a Cremona map of degree* d*. Then*

$$\sum_{p \in B_\phi} m_p^2 = d^2 - 1, \tag{4.1}$$

$$\sum_{p \in B_\phi} m_p = 3d - 3. \tag{4.2}$$

Proof: Since $\deg(\phi) = d$, $\Lambda^2 = d^2$ and $K_{\mathbb{P}^2} \cdot \Lambda = -3d$ so theorem 3.2.3 implies the result. □

Equations 4.1 and 4.2 will be called the *first* and *second equations of condition* respectively. Another important corollary of theorem 3.2.3 in the context of birational automorphisms of the plane in Noether's Inequality:

Proposition 4.1.2 (Noether's Inequality)**.** *Let* $d, m_1 \geq m_2 \geq \cdots \geq m_t \geq 0$ *be integers satisfying the equations of condition. Then*

$$m_1 + m_2 + m_3 \geq d + 1\,. \tag{4.3}$$

Proof: First of all, note that the first equation of condition 4.1 implies that $m_i \leq d-1$ for every i. Now consider the equations of condition:

$$m_1^2 + m_2^2 + m_3^2 + \cdots + m_t^2 = d^2 - 1\,, \qquad (4.1)$$
$$m_1 + m_2 + m_3 + \cdots + m_t = 3(d-1)\,. \qquad (4.1)$$

Computing (4.1) $- m_3$(4.2) we obtain

$$m_1(m_1 - m_3) + m_2(m_2 - m_3) - \sum_{i=4}^{t} m_i(m_3 - m_i) = d^2 - 1 - 3m_3(d-1)\,,$$

which can be re-written as

$$\begin{aligned}&(d-1)(m_1 + m_2 + m_3 - (d+1))\\ &\qquad = (m_1 - m_3)(d - 1 - m_1) + (m_2 - m_3)(d - 1 - m_2) + \sum_{i=4}^{t} m_i(m_3 - m_i)\,.\end{aligned}$$

Note that $m_i \leq d-1$ for all i. Therefore the right hand side of the equation is non negative, we have the result. Furthermore, we see that $m_1 + m_2 + m_3 = d + 1$ if and only if either $m_1 = m_i$ for all i, or if $m_1 = d-1$ and $m_i = 1$ for all $i \geq 2/$ □

Lemma 4.1.1. *Given a plane Cremona map* ϕ*, there exists an admissible total ordering on* $K = B_\phi$*, so that the multiplicities of the points in* K *form a non-increasing sequence.*

Proof: Let $\mu_1 > \cdots > \mu_s$ be an ordering of the different positive values appearing as the multiplicities of the points in K. For $1 \leq i \leq s$, let $K_i := \{p \in K \,|\, m_p = \mu_i\} \subset K$. Fix an admissible total ordering on each subset K_i, and then take the points in K_i preceding the points in K_{i+1}. We have a total ordering " $\leq$ " on K, we have to check that this ordering is admissible. Indeed, if $p, q \in K$ and $p \prec q$, then lemma 3.1.2 implies that $m_p \geq m_q$, so that $p \leq q$. If $m_p = m_q = \mu_i$ then $p \leq q$ due to the fact that on each K_i the order is admissible. If $\mu_p > \mu_q$, then $p \leq q$ according to the ordering of the subsets K_i. □

Definition 4.1.2 (Characteristic of a Cremona map)**.** Let $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a Cremona map of degree d, and let $K = \{p_1, \ldots, p_t\}$ be an admissible ordering of its base points. The *characteristic* of ϕ is the vector $(d; m_1, m_2, \ldots, m_t)$. According to lemma 4.1, the admissible ordering on K will always be such that $m_1 \geq m_2 \geq \cdots \geq m_t$.

Example 4.1.1 (The standard quadratic transformation)**.** Consider the projective plane $\mathbb{P}^2$ with projective coordinates x, y, z. Fix the coordinate points $p_1 = (0:0:1)$, $p_2 = (0:1:0)$ and $p_3 = (1:0:0)$ and consider the linear system of conics in $\mathbb{P}^2$ passing through these points. The set $\{yz, xz, xy\}$ is an ordered basis for this linear system, so it induces a rational map

$$\tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (x:y:z) \mapsto (yz:xz:xy),$$

for which $K = \{p_1, p_2, p_3\}$. τ is called the *standard quadratic transformation*, and its characteristic is $(2;1,1,1)$. We denote the lines $V(z), V(y), V(x)$ by L_1, L_2 and L_3 respectively and refer to them as the *exceptional lines*. Observe that each exceptional line contains only the base points with different index, i.e $L_1 \cap L_2 = \{p_3\}$, $L_2 \cap L_3 = \{p_1\}$ and $L_3 \cap L_1 = \{p_2\}$.

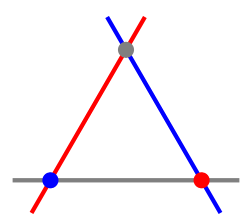


Figure 4.1: Exceptional lines and base points.

Lemma 4.1.2. *τ is a birational involution of $\mathbb{P}^2$ satisfying $\tau(L_i) = \{p_i\}$ for $i = 1, 2, 3$.*

Proof: It is well defined since $\tau(\lambda x : \lambda y : \lambda z) = (\lambda^2 yz : \lambda^2 xz : \lambda^2 xy) = \tau(x:y:z)$ for any $\lambda \in k^\times$ and $yz = xz = xy = 0$ implies $(x:y:z) \in \{p_1, p_2, p_3\}$. Take a point p in L_1 different from p_2 and p_3, say $p = (a:b:0)$ with $ab \neq 0$. Then $\tau(p) =$

$(0:0:ab) = p_1$. By the symmetry of τ we have that the quadratic transformation collapses each exceptional line onto the fundamental point with corresponding index. To calculate the image of τ we just need to check what happens to the points in the set $\mathbb{P}^2 \setminus (L_1 \cup L_2 \cup L_3) = \mathbb{P}^2 \setminus V(xyz) =: U$. First of all, it is clear from the definition that $\tau(U) \subset U$. Note that if $(x:y:z) \in U$, then $\lambda = 1/xyz \in k^\times$ and $\tau(x:y:z) = \tau(\lambda x : \lambda y : \lambda z) = (1/x : 1/y : 1/z)$. Then τ is an involution and $U = \tau\tau(U) \subset \tau(U)$. Therefore we have that

- $\tau(\mathbb{P}^2) = U \cup \{p_1, p_2, p_3\}$, in particular Q is dominant.
- $\tau|_U \colon U \to U$ is an isomorphism with $(\tau|_U) = (\tau|_U)^{-1}$.

□

The standard quadratic transformation is not the only quadratic transformation modulo projective equivalence. It turns out that there are three structurally different types of quadratic Cremona maps.

Proposition 4.1.3 (Classification of Quadratic Cremona Maps)**.** *Up to a linear change of coordinates, the following are the only quadratic Cremona maps:*

1. $\tau : (x:y:z) \longmapsto (yz : xz : xy)$,
2. $\tau_1 : (x:y:z) \longmapsto (xz : yz : x^2)$,
3. $\tau_2 : (x:y:z) \longmapsto (x^2 : xy : y^2 + xz)$.

Proof: Let ϕ be a quadratic Cremona map with base points $p_1, \dots, p_t$ and multiplicities $m_1 \geq m_2 \geq \cdots \geq m_t \geq 1$. By the second equation of condition 4.2, $m_1 + \cdots + m_t = 3$ so that $t \leq 3$. By the first equation of condition 4.1 (and the fact that 3 is not a square nor a sum of two squares), we have that $t = 3$ so that ϕ has exactly three base points p_1, p_2, p_3 of multiplicities $m_1 = m_2 = m_3 = 1$ so that the characteristic of every quadratic transformation is $(2; 1, 1, 1)$. There are three types of quadratic maps according to the number of proper base points:

(0) ϕ has three proper base points.

(1) ϕ has two proper base points p_1, p_2. The third base point p_3 in the first infinitesimal neighborhood of p_1.

(2) ϕ has just one proper base point p_1, the second base point p_2 is in the first infinitesimal neighborhood of p_1 and the third base point p_3 in in the first infinitesimal neighborhood of p_2.

We will show that these cases correspond to the maps τ, τ_1 and τ_2 respectively. Note that the case of a single proper base point p_1 and two base points p_2, p_3 on the first infinitesimal neighborhood of p_1 is impossible, since this would mean that a general element in the linear system of conics Λ_ϕ has two different tangent directions (corresponding to p_2 and p_3) at p_1.

(Type 0) Let $p_1, p_2, p_3 \in \mathbb{P}^2$ be the three proper base points. Since Λ is a linear subsystem of $|2H|$, these points are not on the same line, so we may apply a linear change of coordinates and assume that they are the coordinate points. In this case, $\Lambda = \phi^*|H| = \{\lambda_1 yz + \lambda_2 xz + \lambda_3 xy \,|\, (\lambda_1 : \lambda_2 : \lambda_3) \in \mathbb{P}^2\}$ and $\phi = (F : G : H)$ for F, G, H a basis of $\mathrm{Span}_{\mathbb{C}}\{yz, xz, xy\}$. Let $\varphi \in \mathrm{PGL}(3, \mathbb{C})$ be the change of coordinates induced by the change of basis $F \mapsto yz, G \mapsto xz, H \mapsto xy$. Then $\tau = \varphi \circ \phi$, as we wanted to show.

(Type 1) In this case, the linear system Λ_1 is the linear subsystem of $|2H|$ of conics passing through p_1 and p_2 with a common tangent at p_1 corresponding to the point $p_3 \in \mathrm{Bl}_{p_1}(\mathbb{P}^2)$. Composing with a linear change of coordinates mapping p_1 to $(0 : 1 : 0)$, p_2 to $(0 : 0 : 1)$ and the line corresponding to p_3 to the line $\ell = z$, we may assume that the linear system of ϕ is given by $\Lambda_1 = \{\lambda_1 xz + \lambda_2 yz + \lambda_3 x^2 \,|\, (\lambda_1 : \lambda_2 : \lambda_3) \in \mathbb{P}^2\}$. Therefore, composing with a projective linear change of coordinates induced by a change of basis, we have that $\phi = \tau_1$.

(Type 2) We have that $p_1 \prec p_2 \prec p_3$. By composing with suitable changes of coordinates we hay assume that $p_1 = (0 : 0 : 1)$ and that that the shared tangent of Λ at p_1 is the line $x = 0$. Thus, taking $X_1 = \mathrm{Bl}_{p_1}(\mathbb{P}^2) = V(ux - ty) \subset \mathbb{P}^2 \times \mathbb{P}^1$ with coordinates (x, y, z) on $\mathbb{P}^2$ and (t, u) on $\mathbb{P}^1$, the point $p_2 = (p_1, (0 : 1))$. Note that point p_3 corresponds to a shared tangent at p_2 of the strict transform of a general element of Λ. Let $ax^2 + bxy + cy^2 + dxz$ be the equation of a curve $C \in \Lambda$. After blowing up p_1, the strict transform $\widetilde{C}$ has a local equation in the open subset $U_2 \times U_1 = (z \neq 0) \times (u \neq 0)$ of the form $ayt^2 + byt + cy + dt$. Therefore, the line $cy + dt = 0$ is the shared tangent at p_2 corresponding to p_2. This implies that if F is the equation of a general curve in Λ, we have that $(\partial_x^2 F : \partial_y \partial_z F) = (c : d) \in \mathbb{P}^1$. Thus we have two possibilities; $c \neq 0$ and $d \neq 0$. Composing the linear system Λ with the projective changes of variables $y \mapsto (d/c)^{1/2} y$ and $z \mapsto (c/d)z$ respectively, we obtain the result. $\square$

Given a Cremona map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, we can resolve the indeterminacy of ϕ to obtain the commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{\Pi_K}\swarrow & & \searrow^{\Pi_L} \\ \mathbb{P}^2 & \overset{\phi}{\dashrightarrow} & \mathbb{P}^2. \end{array}$$

Using induction and the universal property of the blowup at a point, it can be shown that the maps Π_K and Π_L are the composition of an isomorphism with the blowups of K and L. This factorization of birational maps is not particular of the surface $\mathbb{P}^2$, every birational map between smooth surfaces is a composition of a finite number of blowups and blowdowns like this. The interested reader find a proof of this fact in [1].

The Cremona maps τ, τ_1 and τ_2 play a central role in the proof of the final theorem 4.2.4. Before advancing to the final chapter, we resolve the indeterminacy of these transformations.

- (The standard quadratic transformation τ)

  By example 4.1.1, we know that τ is an involution so that $K = L$. Moreover, the three base points are the coordinate points and each fundamental line is contracted to the base point not contained on it. Therefore we have figure 4.2.

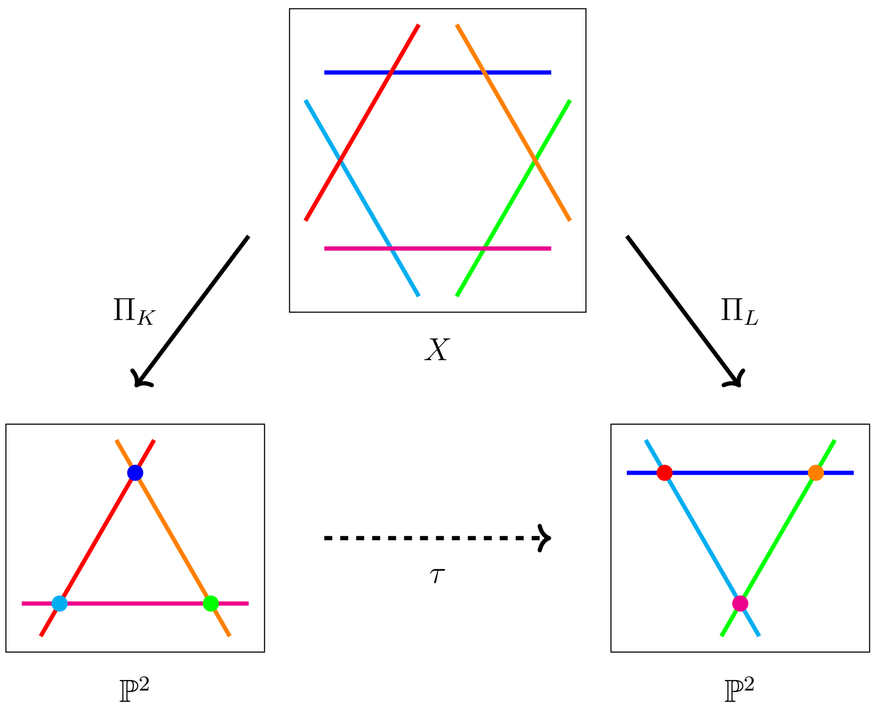


Figure 4.2: Factorization of τ.

- (Type 1 quadratic transformation τ_1)

  Recall $\tau_1 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is given by $(x : y : z) \mapsto (xz : yz : x^2)$. As we saw in the proof of proposition 4.1.3, τ_1 has $p_1 = (0 : 1 : 0)$ and $p_2 = (0 : 0 : 1)$ as proper points, and the point p_3 infinitely near to p_1 corresponds to the line $z = 0$. It is easy to see from the formula that the line $z = 0$ is mapped to the point $q_2 = (0 : 0 : 1)$, and the line $x = 0$ is mapped to $q_1 = (0 : 1 : 0)$. Furthermore, computing τ_1^2 we obtain that τ_1 is an involution in the open subset $\mathbb{P}^2 - V(x) \cup V(z)$. Thus $K = L$ and we have figure 4.3.

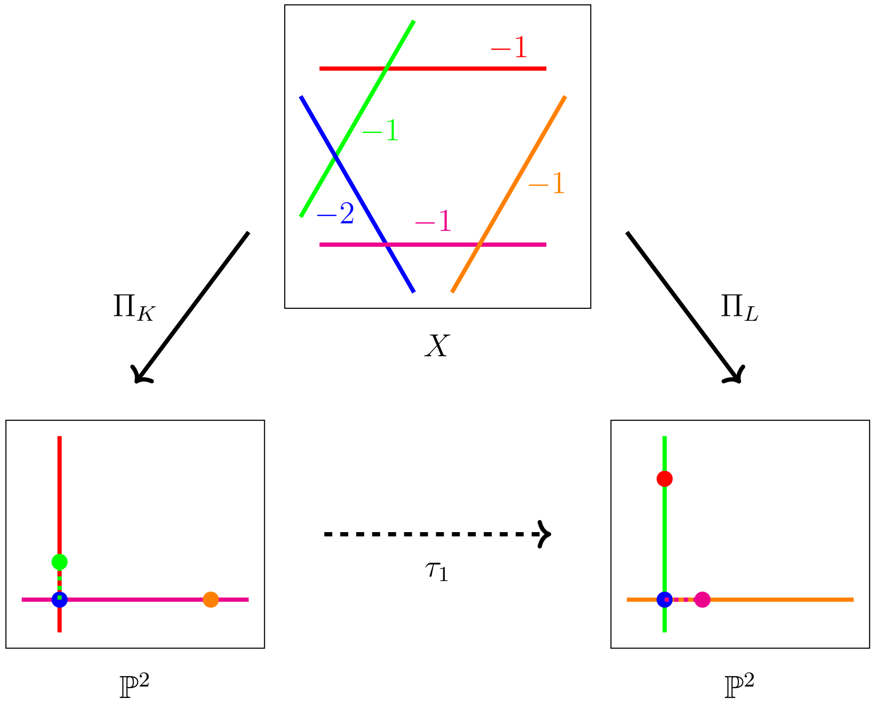


Figure 4.3: Factorization of τ_1.

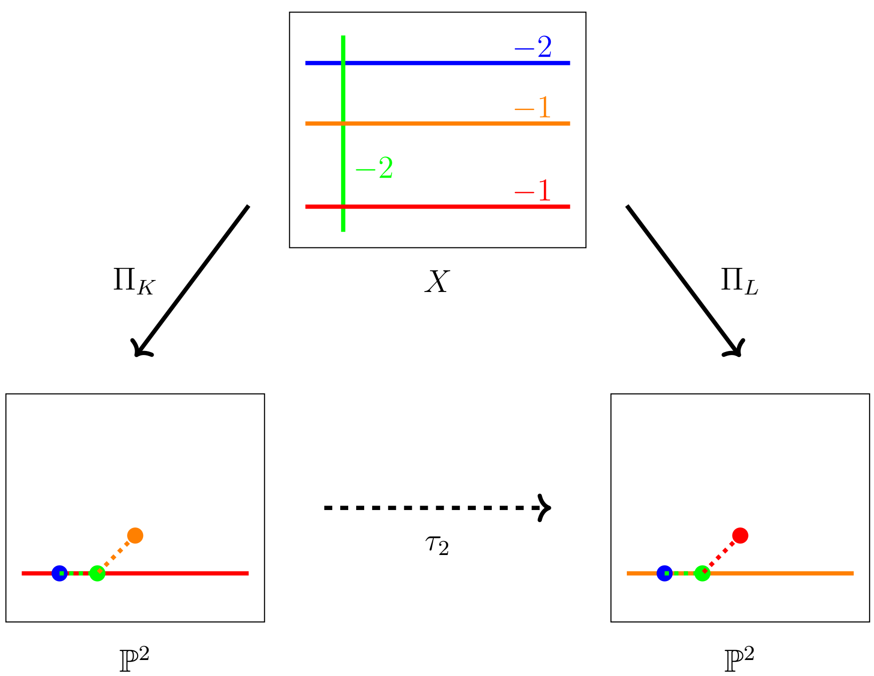


Figure 4.4: Factorization of τ_2.

- (Type 2 quadratic transformation τ_2)

  We have seen in example 3.2.1 that $\tau_2 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, (x:y:z) \mapsto (x^2 : xy : y^2 + xz)$ has a unique proper base point $p_1 = (0:0:1)$, and two infinitely near base points p_2 and p_3 satisfying $p_1 \prec p_2 \prec p_3$. The point p_2 corresponds to the shared tangent line $x = 0$ at p_1 of the linear system $\tau_2^*|H|$. Similarly, the point p_3 corresponds to the shared tangent line $y + t = 0$ at p_2 of the linear system $(\tau_2 \circ \pi_{p_1})^*|H|$. Furthermore, $\tau_2^{-1}(x:y:z) = (x^2 : xy : -y^2 + xz)$. In fact:

  $$\tau_2(x^2 : xy : y^2 + xz) = (x^4 : x^3 y : -(xy)^2 + x^2(y^2 + xz)) = (x^4 : x^3 y : x^3 z).$$

  Therefore τ_2 is injective on the open subset $\mathbb{P}^2 - V(x)$ and it is easy to see that τ_2 and τ_2^{-1} are projectively equivalent. Blowing up the point p_2, one sees that the point p_3 does not lie on the strict transforms of either E_1 or $L : x = 0$.

Lemma 4.1.3. *Every quadratic Cremona map can be written as a composition of standard quadratic transformations.*

Proof: By proposition 4.1.3, every quadratic Cremona map is projectively equivalent to either τ, τ_1 or τ_2. Therefore, it suffices to show that τ_1 and τ_2 can be written as composition of standard quadratic transformations.

We show that τ_1 is the composition of two standard quadratic Cremona maps. Take the point $q = (1:0:1)$. This point belongs to the open subset $xz \neq 0$ over which τ_1 is an involution. Consider the following standard quadratic transformation obtained by choosing a basis of the linear system through $p_1 = (0:1:0)$, $p_2 = (0:0:1)$ and q:

$$Q_{p_1,p_2,q} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (x:y:z) \mapsto (x^2 + xz : xy : yz).$$

Calculating the composition $\rho = Q_{p_1,p_2,q} \circ \tau_1$, we get that

$$\rho : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (x:y:z) \mapsto (x^2 + xz : yz : xy),$$

so that ρ is a standard quadratic transformation. Then $\rho^{-1} = \tau_1 \circ Q^{-1}_{p_1,p_2,q}$ is also standard quadratic and $\tau_1 = \rho^{-1} \circ Q_{p_1,p_2,q}$ as we wanted to show.

We show that τ_2 is the composition of two type 1 quadratic Cremona maps. Let $p_1 = (0:0:1), p_2, p_3$ be the base points of τ_2. Recall that τ_2 is injective on the open subset $x \neq 0$. Take the point $r = (1:0:0)$ and consider the quadratic transformation

$$Q_1 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (x:y:z) \mapsto (xy : xz : y^2).$$

It is clear that Q_1 is a type 1 quadratic transformation with base points $\{p_1, r, p_2\}$, p_1 and r proper, and p_2 infinitely near to p_1. Computing the composition $\psi = Q_1 \circ \tau_2^{-1}$, we get that

$$\psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (x:y:z) \mapsto (xy : -y^2 + xz : y^2),$$

which is also a type 1 quadratic transformation. Therefore $\tau_2 = \psi^{-1} \circ Q_1$ is composition of two type 1 quadratic Cremona maps. ☐

4.2 The Decomposition and Inertia Groups

As in the previous section we work over the field of complex numbers. We will present some results about the structure of the decomposition group of birational transformations of the projective plane over a smooth curve, as in example 2.3.2, we consider the following group:

Definition 4.2.1 (Decomposition and Inertia Groups)**.** The *decomposition group* of a plane curve C is the subgroup of the Cremona group consisting of the birational transformations of the plane that fix C as a set, that is

$$\mathcal{D}(C) := \{g \in \mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2) \mid g(C) = C\}.$$

The *inertia group* of C is the subgroup of the decomposition group of C given by the birational transformations of the plane that fix C pointwise, that is

$$\mathcal{I}(C) := \{g \in \mathcal{D}(C) \mid g(p) = p \text{ for a general point } p \in C\}.$$

We will only consider smooth curves in this chapter. In this case, the groups $\mathcal{D}(C)$ and $\mathcal{I}(C)$ coincide with $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)_C$ and $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)_{0,C}$, defined in 2.3.4.

Example 4.2.1 (Smooth curves fixed by τ)**.** Recall the notation used in example 4.1.1. Let $C \subset \mathbb{P}^2$ be any irreducible curve. $C = V(F)$ for some irreducible homogeneous polynomial $F \in \mathbb{C}[x, y, z]$ with $\deg(F) = n$. Suppose that $\tau \in \mathcal{D}(C)$, i.e $\tau(C) = C$. As we showed before, the domain of definition of τ is $\mathbb{P}^2 - K$ where $K = B_\tau = \{p_1, p_2, p_3\}$ are the coordinate points. By definition, $\tau(C) = \overline{\tau(C - K)} = \overline{\tau(C \cap U)}$, and since $\tau^2 = \mathrm{id}$ on U:

$$\tau(C \cap U) = \{\tau(p) \,|\, F(p) = 0\} = \{q \,|\, (F \circ \tau)(q) = 0\}.$$

Thus, we are led to consider the homogeneous polynomial $F^\tau = F(yz, xz, xy)$ of degree $2n$. Let $m_i := m_i(C)$ denote the multiplicity of F at p_i for $i = 1, 2, 3$. We may write

$$F = F_{m_1}(x, y)z^{n-m_1} + \cdots + F_n(x, y),$$

where $F_j \in \mathbb{C}[x, y]$ is homogeneous of degree j for $j = m_1, \ldots, n$. Composing with τ,

$$\begin{aligned}
F^\tau &= F_{m_1}(yz, xz)(xy)^{n-m_1} + \cdots + F_n(yz, xz) \\
&= z^{m_1}F_{m_1}(y, x)(xy)^{n-m_1} + \cdots + z^n F_n(y, x) \\
&= z^{m_1}(F_{m_1}(y, x)(xy)^{n-m_1} + \cdots + z^{n-m_1}F_n(y, x)),
\end{aligned}$$

so by symmetry, we deduce that $F^\tau = x^{m_3}y^{m_2}z^{m_1}F'$ where $F' \in \mathbb{C}[x, y, z]$ is homogeneous of degree $2n - (m_1 + m_2 + m_3)$. Also, from $(F^\tau)^\tau = (xyz)^n F$ follows that F' is irreducible and $(F')' = F$. Since $V(F') \supset \tau(C \cup U)$, we have that $\tau(C) = V(F')$. This implies that $F = \lambda F'$ for some $\lambda \in \mathbb{C}^\times$. In particular, $n = \deg(F) = \deg(F') = 2n - (m_1 + m_2 + m_3)$. Thus $n = m_1 + m_2 + m_3$. Assuming C to be smooth, necessarily $m_1 = m_2 = m_3 = 1$ so that $\deg(C) = 3$ and $B_\tau \subset C$. ■

We have that if a smooth curve C is fixed by the standard quadratic transformation, then $\deg C = 3$ and $B_\tau \subset C$. Since every Cremona map is given by composition of projective changes of coordinates with τ (theorem 1.2.1 in the introduction), one might guess that the same is true for any smooth curve fixed by a non trivial Cremona map.

Theorem 4.2.1 ([12], theorem 1.3)**.** *Let $C \subset \mathbb{P}^2$ be an irreducible non rational smooth curve, and suppose there exists $\phi \in \mathcal{D}(C) \setminus \mathrm{PGL}(3)$. Then $\deg(C) = 3$ and $B_\phi \subset C$.*

Proof:

Let $d = \deg(\phi)$ and take L a line in $\mathbb{P}^2$ intersecting C transversely and such that $L \cap C$ is contained in the open set where ϕ^{-1} is injective, and let $D := \phi^* L$. By Castelnuovo's contractibility theorem 3.1.3, the curve C is not contracted by ϕ^{-1} (otherwise C would be rational). Then

$$\begin{aligned} d \cdot \deg(C) &= D \cdot C && \text{(Bezout's Theorem)} \\ &= \deg(C) + \sum_{p \in B_\phi \cap C} m_p && \text{(since } C \text{ is smooth)} \\ &\leq \deg(C) + \sum_{p \in B_\phi} m_p \\ &= \deg(C) + 3(d-1)\,, && \text{(Equation 4.2)} \end{aligned}$$

so we have that $\deg(C) \leq 3$ with equality if and only if $B_\phi \subset C$. Since C is non rational, $\deg(C) = 3$ and we have the result. □

Example 4.2.2 (The elliptic curve $y^2 = x^3 - x$)**.** Consider the curve $C = V(F) \subset \mathbb{P}^2$ with $F(x,y,z) = y^2 z - x^3 - xz^2 \in \mathbb{C}[x,y,z]$. Take the points $p = (0:0:1)$, $q = (0:1:0)$ and $r = (1:0:1)$. It is easy to see that these points lie in C and that they are not colinear. The linear system of conics through these points is

$$\Lambda = \{\lambda_1(x^2 - xz) + \lambda_2 xy + \lambda_3 yz \,|\, (\lambda_1 : \lambda_2 : \lambda_3) \in \mathbb{P}^2\}.$$

Arbitrarily choosing an ordered basis of the corresponding vector space, we get a standard quadratic transformation with base points p, q and r. We choose

$$Q = Q_{p,q,r} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (x:y:z) \mapsto (yz : x(x-z) : xy). \tag{4.4}$$

The inverse of this map is

$$Q^{-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (x:y:z) \mapsto (yz : z(z-x) : xy). \tag{4.5}$$

Computing $F^{Q^{-1}}(x,y,z) = F(yz, z(z-x), xy)$, we get $F^{Q^{-1}} = yz(x-z)F'$, where $F' = y^2(z+x) - xz(z-x)$. Therefore, as in example 4.2.1, we have that $C' = Q(C) = V(F')$. Finally, the curve C' is projectively equivalent to C via the change of variables $\varphi = (z - x : 2y : z + x)$. Therefore, the Cremona map $\varphi \circ Q \in \mathcal{D}(C)$. ■

Before proving the final theorem, we work through two important auxiliary lemmas. We want to generalize the explicit calculations done in example 4.2.2 to arbitrary smooth cubic curves. Let C be a smooth cubic curve and let $p, q, r \in C$ be three points not all on the same line. We fix $Q_{p,q,r}$ a standard quadratic transformation determined by a choice of an ordered basis of the linear system of conics passing through these points.

Lemma 4.2.2. *$Q_{p,q,r}(C) \subset \mathbb{P}^2$ is a smooth cubic curve isomorphic to C.*

Proof: Denote by U the open set given by the complement of the triangle pqr and consider the morphism $f = Q_{p,q,r}|_U$. By the proof of lemma 4.1.2 we know that $f : U \to U$ is an isomorphism, so that $Q_{p,q,r}(C)$ is not a point and therefore the restriction $Q_{p,q,r}|_C : C \dashrightarrow Q_{p,q,r}(C)$ is birational. Since C is smooth, $Q_{p,q,r}|_C$ is represented by a surjective morphism. To show that this is an isomorphism, it is enough to show that $Q_{p,q,r}(C)$ is smooth. By example 4.2.1, we know that $C' := Q_{p,q,r}(C)$ has degree $6 - (m_p(C) + m_q(C) + m_r(C)) = 3$, so that C' is smooth. □

Lemma 4.2.3. *There exists $\varphi \in \mathrm{PGL}(3)$ such that $\varphi \circ Q_{p,q,r} \in \mathcal{D}(C)$.*

Proof: Being non singular cubic curves, C and $Q_{p,q,r}(C)$ are elliptic curves. Since they are isomorphic, they have the same j-invariant ([9, Chapter IV, Theorem 4.1]) and therefore there are linear automorphisms mapping C to $E_\lambda : y^2z = x(x-z)(x-\lambda z)$ for some $\lambda \neq 0, 1$, and $Q_{p,q,r}(C)$ to E_μ with $\mu \in \{\lambda, 1/\lambda, 1-\lambda, 1/(1-\lambda), \lambda/(\lambda-1), (\lambda-1)/\lambda\}$. Since E_λ and E_μ are linearly isomorphic, we have the result. □

Definition 4.2.2 (C-generic Quadratic Transformation)**.** A *C-generic quadratic transformation* for a non singular cubic curve C is a Cremona map of the form $\varphi \circ Q_{p,q,r} \in \mathcal{D}(C)$ as in the previous lemma.

With this definition, we are ready to state the final theorem of this work, also due to Ivan Pan [12, Theorem 1.4].

Theorem 4.2.4. *If $C \subset \mathbb{P}^2$ is a smooth cubic, then $\mathcal{D}(C)$ is generated by the C-generic quadratic transformations.*

C-generic quadratic transformations seem to play the role in $\mathcal{D}(C)$ of the standard quadratic transformation in $\mathrm{Bir}(\mathbb{P}^2)$, but there is a fundamental difference: a single C-generic quadratic transformation is not enough to generate all of $\mathcal{D}(C)$ modulo $\mathrm{PGL}(3)_C$.

To see this, let $\mathcal{E} := \{E_\lambda \mid \lambda \neq 0, 1\} \subset \mathbb{P}^9$ where $E_\lambda = y^2z - x(x-z)(x-\lambda z)$. Consider the morphism

$$\Psi : \mathcal{E} \times \mathrm{PGL}(3) \to U \subset \mathbb{P}^9, \quad (E_\lambda, \varphi) \mapsto \varphi(E_\lambda),$$

where $\mathbb{P}^9$ is the linear system $|3H|$ and U is the open subset of smooth curves. Since every smooth cubic is projectively equivalent to some E_λ, $\text{Im}(\Psi) = U$. Since $\dim(\mathcal{E}) = 1$, $\dim(\text{PGL}(3)) = 8$ and $\dim(U) = 9$, fiber dimension theorem implies that $\#\text{PGL}(3)_{E_\lambda} \leq \#\Psi^{-1}(E_\lambda) < \infty$. Since for any smooth cubic the group $\text{PGL}(3)_C$ is conjugate to $\text{PGL}(3)_{E_\lambda}$ for some λ, this group is finite and therefore a single C-generic quadratic transformation is not enough to generate $\mathcal{D}(C)$ modulo $\text{PGL}(3)_C$.

Combining the lemmas 4.1.3, 4.2.2 and 4.2.3 we have the following result:

Lemma 4.2.5. *Let $C \subset \mathbb{P}^2$ be a smooth cubic and $Q \in \mathcal{D}(C)$ a quadratic Cremona map. Then Q is the composition of C-generic quadratic transformations.*

As the reader may have noticed from the examples, the composition of Cremona maps is highly unpredictable. Given ϕ_1 and ϕ_2 Cremona maps, several factors determine the properties of their composition; for instance the coincidences between the base points of ϕ^{-1} and ϕ_2. Despite its intrinsic complexity, this problem is now completely solved in the following sense: given the characteristics of both ϕ_1, ϕ_2 and their inverses, and the coincidences between base points and base points lying in contracted curves, it is possible to determine the characteristic of the composition and its inverse. We refer the interested reader to [1, Chapter 4] for a complete exposition of this topic.

For the proof of theorem 4.2.4, we are interested in the composition of an arbitrary Cremona map ϕ with a quadratic transformation. We have the following proposition

Proposition 4.2.1. *Let $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a plane Cremona map of degree n. Let $p_1, \ldots, p_t$ be an admissible ordering of its base points and m_i the corresponding multiplicity of p_i for $i = 1, \ldots, t$. Assume that p_1, p_2 and p_3 are the base points of a quadratic Cremona map $Q : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Then, the composite map $\phi \circ Q^{-1}$ has degree $2n - (m_1 + m_2 + m_3)$.*

When p_1, p_2, p_3 are proper, Q is a standard quadratic, and the proof follows from example 4.2.1. In fact, we can compose with a projective change of coordinates so that the base points are the coordinate points and Q corresponds to τ. Let $\phi = (F, G, H)$ with $F, G, H \in \mathbb{C}[x, y, z]$ homogeneous of degree n without common components. Then $\phi \circ \tau = (F', G', H')$ and

$$\deg(\phi \circ \tau) = \deg(F') = \deg(G') = \deg(H') = 2n - (m_1 + m_2 + m_3).$$

Unfortunately, the same approach doesn't seem to work for other base point configurations. The proof in [1, Proposition 4.2.5] uses some technical constructions that where left out of this work.

We are ready to prove theorem 4.2.4, i.e, that for a non rational smooth cubic $C \subset \mathbb{P}^2$, the group $\mathcal{D}(C)$ is generated by the C-generic quadratic transformations.

Proof: (Theorem 4.2.4)

Let $\phi \in \mathcal{D}(C)$. The proof is by induction on $n = \deg(\phi)$. The base case $n = 2$ is precisely lemma 4.2.5. Suppose that $n > 2$ and that the result holds for any $\psi \in \mathcal{D}(C)$ with $\deg(\psi) < n$.

Let $p_1 \leq \cdots \leq p_t$ be an admissible ordering as in lemma 4.1 so that $m_1 \geq m_2 \geq m_3$ are the maximal multiplicities. Since every smooth cubic is non-rational, we have by theorem 4.2.1 that $B_\phi \subset C$, and by the smoothness of C, the configuration of the points p_1, p_2 and p_3 is such that there exists a quadratic transformation $Q : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with these base points. Furthermore, by the lemmas 4.1.3 and 4.2.3, we may take $Q \in \mathcal{D}(C)$. Then,

$$\begin{aligned} \deg(\phi \circ Q^{-1}) &= 2n - (m_1 + m_2 + m_3) && \text{(Proposition 4.2.1)} \\ &\leq n - 1\,, && \text{(Noether's Inequality 4.3)} \end{aligned}$$

and the result follows by the induction hypothesis and lemma 4.2.5. □

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