On summing and nuclear types of operators and their related results.

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Abstract

Summing and Nuclear type linear operators, investigated by A. Grothendieck in the 1950's, turned out to be the germs of several important classes of operators between Banach spaces. These classes play a central role in the theory of operator ideals systematized by A. Pietsch in the 1970's [5]. Among such classes one can find τ-summing and σ-nuclear operators (cf., e.g., [5, Chapter 23]).

Introduction

From now on, the $*$ character and such will stand for varied possibilities of symbols involving summing type norms, as well as the # character and such will relate to nuclear type norms.

Summing type operators share definitions which always look something like: Let $\ldots \leq q \leq p \leq \ldots$. If there is a constant $\mathsf{C} \geq 0$ such that

$$\left(\sum_{j=1}^{m} |\bar{*}_j(S(x_j))\tilde{*}_j|^p\right)^{1/p} \leq \mathsf{C} \sup_{\substack{\hat{*} \in B_{\widehat{**}} \\ x' \in B_{E'} \\ \ddot{*} \in B_{\ddot{**}}}} \left(\sum_{j=1}^{m} |\bar{*}_j(\hat{*})x'(x_j)\ddot{*}(\tilde{*}_j)|^q\right)^{1/q}, \forall m \in \mathbb{N},$$

S is said to be $*(p;q)$-*summing*, where its norm $\|S\|_{*(p;q)}$ is the infimum of all such constants C. In that case, we write $S \in \mathcal{L}_{*(p;q)}(E;F)$.

Besides sharing similar definitions, these operators share similar kinds of domination theorems - those stand for the case when $q = p'$ is the conjugate for p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$:

Let $\ldots \leq p \leq \ldots$. An operator $S \in \mathcal{L}(E;F)$ is $*(p)$-summing if there exist a constant $\mathsf{B} > 0$ and a regular Borel probability measure μ on $B_{\overline{**}'} \times B_{E'} \times B_{\widetilde{**}'}$ endowed with the product of their respective topologies topologies, such that, for all $x \in E$

$$|\bar{*}((S(x))\tilde{*}| \leq \mathsf{B}\left(\int_{B_{\overline{**}'} \times B_{E'} \times B_{\widetilde{**}'}} |\bar{*}'(\bar{*})x'(x)\tilde{*}'(\tilde{*})|^p d\mu(\bar{*}', x', \tilde{*}')\right)^{1/p}.$$

In [3], Botelho, Pellegrino and Rueda proved there exists a unified domination theorem, for all summing type operators.

Now we turn to nuclear type operators, which always have representations that look like $A = \sum_{j=1}^{\infty} \lambda_j x'_j \otimes y_j$, as long as $\sum_{j=1}^{\infty} \lambda_j x'_j(x) y_j$ converges in F for all $x \in E$, and provided the sequences $(\lambda_j)_{j=1}^{\infty} \subset \mathbb{K}^{\mathbb{N}}$, $(x'_j)_{j=1}^{\infty} \subset E'^{\mathbb{N}}$, and $(y_j)_{j=1}^{\infty} \subset F^{\mathbb{N}}$ satisfy some condition $\#$, which depends on some p, q, r.

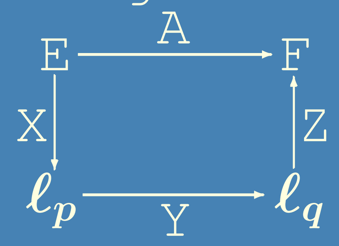
The norms defined for these operators are always an infimum taken over all possible representations for that operator, and look something like

$$\|A\|_{\#nuc} := \inf_{\substack{\text{reprs} \\ of\, A}} \left\{ \|(\lambda_j)_{j=1}^{\infty}\|_{\widehat{\#}'} \sup_{\substack{x \in B_E \\ y' \in B_{F'}}} \left(\sum_{j=1}^{\infty} |x'_j(x) y'(y_j)|^{\widehat{\#}}\right)^{\frac{1}{\widehat{\#}}} \right\}.$$

Then, we write $A \in \mathcal{L}_{\#nuc}(E;F)$, and say A is $\#$-nuclear.

While summing type operators present domination theorems, nuclear type operators present factorization theorems. However, unlike domination theorems, which all look alike, there are two kinds of factorization theorems:

An operator $A \in \mathcal{L}(E;F)$ is #-nuclear if and only if there exists a commutative diagram



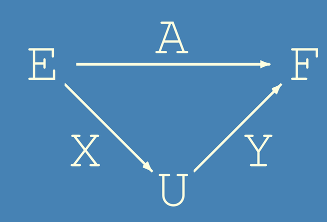
where X, Y and Z satisfy # properties. In this case,

$$\|A\|_{\#nuc} = \inf \|Z\|.\|Y\|.\|X\|,$$

taking the infimum over all possible factorizations.

or

An operator $A \in \mathcal{L}(E;F)$ is #-nuclear if and only if there exists a commutative diagram



where X, Y and U satisfy # properties. In this case,

$$\|A\|_{\#nuc} = \inf \|Y\|.\|X\|,$$

where the infimum is taken over all possible factorizations.

So it remains to be seen wether there exists a unified factorization theorem for all nuclear type operators.

More Results and Goal

But why do we talk about two such different sorts of operators? Well, because for each summing type operator there corresponds a nuclear type one, related through the *Borel Transform*:

$$\begin{array}{rccc} \mathcal{B}\colon & [\mathcal{L}_{\#nuc}(E;F), \|\cdot\|_{\#(p)}]' & \longrightarrow & \mathcal{L}_{*(p)}(E';F') \\ & \varphi & \mapsto & \mathcal{B}(\varphi) \end{array}$$

given by

$$\begin{array}{rccc} \mathcal{B}(\varphi)\colon & E' & \longrightarrow & F' \\ & x' & \mapsto & \mathcal{B}(\varphi)(x') \end{array} \quad \text{with} \quad \begin{array}{rccc} \mathcal{B}(\varphi)(x')\colon & F & \longrightarrow & \mathbb{K} \\ & y & \mapsto & \varphi(x' \otimes y) \end{array}$$

which is an isomertric isomorphism.

Most Borel Transforms establish a duality relation between a pair of summing and nuclear type operators, *regardless of the spaces* E and F. However, when it comes to τ-summing and σ-nuclear operators, the target space F has to be a reflexive one. In [1] and [2], Botelho and Mujica made efforts to dismiss the reflexive condition, with no success, so far.

So it remains to be seen:

(i) if a unified factorization for nuclear operators exists;

(ii) is it possible to withdraw the reflexive condition on F on the Borel transform for τ-summing and σ-nuclear operators?

References

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