Obstructions to extensions of semilattices of groups by groups

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Abstract

In this work, we give an interpretation of the third partial cohomology group (defined in [1]) with values in an abelian semilattice of groups in terms of extensions of a semilattice of groups by a group. For this end, we define a partial abstract kernel inspired on the classic case of group cohomology and Lausch's case of inverse semigroup cohomology.

Introduction

Any admissible extension

$$E : A \xrightarrow{i} U \xrightarrow{j} G$$

of a semilattice of groups A by a group G induces a twisted partial action of G on A, that is, a pair $\Theta = (\theta, w)$, in which θ is a family of isomorphism between ideals of A ($\theta_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g$), and w is a family of invertible multipliers ($w_{g,h} \in \mathcal{U}(\mathcal{M}(\mathcal{D}_g\mathcal{D}_{gh}))$, satisfying certain properties. It thus induces a partial representation $\psi : G \to \varsigma(A)$, called an **abstract kernel** of the extension, that maps each $g \in G$ to the congruence class $[\theta_g]$ in the inverse monoid $\varsigma(A) = \Sigma(A)/\sim$. More generally, by an **abstract kernel** we mean a triple (A, G, ψ), where A is a semilattice of groups, G is a group, and $\psi : G \to \varsigma(A)$ is a partial representation. The general problem of these extensions is that of constructing all extensions E to a given abstract kernel (A, G, ψ); that is, of constructing all inverse semigroups U with a monomorphism $i : A \to U$ and an epimorphism $j : U \to G$ such that $i(A) = j^{-1}(1)$, with the induced abstract kernel $\psi : G \to \varsigma(A)$.

Description of extensions

Any admissible extension $A \to U \to G$ is equivalent to $A \to A *_\Theta G \to G$, for a twisted partial action of G on A, in which the crossed product $A *_\Theta G$ is the inverse semigroup

$$\{a\delta_g \mid a \in \mathcal{D}_g\}$$

with multiplication $a\delta_g \cdot b\delta_h := \theta_g(\theta_g^{-1}(a)b)w_{g,h}\delta_{gh}$, and inverse $(a\delta_g)^{-1} := w_{g^{-1},g}^{-1}\theta_{g^{-1}}(a^{-1})\delta_{g^{-1}}$.

Theorem 1. ▶ *Given a twisted partial action $\Theta = (\theta, w)$ of a group G on a semilattice of groups A, the crossed product $A *_\Theta G$ is an admissible extension of A by G. (This is [2, Proposition 5.15].)*

▶ *Any admissible extension of the abstract kernel (A, G, ψ) is equivalent to that of a crossed product $A *_\Theta G$, for some twisted partial action Θ of G on A. (This is [2, Theorem 6.12].)*

The converse

Now suppose that only the abstract kernel (A, G, ψ) is given. For each element of G, we choose a representative $\theta_g \in \psi(g)$, and we have for θ:

$$(\theta_g \circ \theta_h)(a) = \mu(w_{g,h}) \circ \theta_{gh}(a), \forall a \in \mathcal{D}_{h^{-1}}\mathcal{D}_{h^{-1}g^{-1}}. \quad (1)$$

Using associativity in $\Sigma(A)$, we compute $\theta_g \circ \theta_h \circ \theta_k$ in two different ways and get:

$$\mu(w_{g,h}w_{gh,k}) = \mu(w_{h,k}^{\theta_g}w_{g,hk}) \text{ on } \mathcal{D}_g\mathcal{D}_{gh}\mathcal{D}_{ghk}.$$

So we define

$$\beta(g, h, k) := w_{gh,k}^{-1}w_{g,h}^{-1}w_{h,k}^{\theta_g}w_{g,hk},$$

where all the multipliers are restricted to the common domain $\mathcal{D}_g\mathcal{D}_{gh}\mathcal{D}_{ghk}$. This also means that the central multiplier $\beta(g, h, k)$ satisfies

$$w_{h,k}^{\theta_g}w_{g,hk} = \beta(g, h, k)w_{g,h}w_{gh,k}. \quad (2)$$

We call this β an **obstruction** to the extension, which can be shown to be a partial 3-cochain (in the sense of [3]) whose cohomology class is independent of the choices of θ and w, and this cohomology class is the desired element of $H^3(G, C(A))$.

Theorem 2. *In any abstract kernel (A, G, ψ), the centre $C(A)$ is regarded as a partial G-module with $\tilde{\theta} = \{\tilde{\theta}_g : C(\mathcal{D}_{g^{-1}}) \to C(\mathcal{D}_g)\}_{g \in G}$, for any choice of representatives $\theta_g \in \psi(g)$ and each $\tilde{\theta}_g$ is the restriction of θ_g to its centre. Taking the cohomology class of any of its obstructions we have a well defined element $\text{Obs}(A, G, \psi) \in H^3(G, C(A))$.*

The abstract kernel (A, G, ψ) admits an extension if and only if $\text{Obs}(A, G, \psi)$ is trivial.

To complete the study of the extension problem we have the following result:

Theorem 3. *If the abstract kernel (A, G, ψ) has an admissible extension, then the set of equivalence classes of admissible extensions is in one-to-one correspondence with the set $H^2(G, C(A))$.*

Observation

This study considers non-unital twisted partial actions. When the twisted partial action is unital, that is, when the ideals $\mathcal{D}_g$ are generated by idempotents, the inverse monoid $\Sigma(A)$ turns to the inverse semigroup (we lose the identity element!) of relatively invertible isomorphisms $\text{end}\, A$, the congruence $\sim$ turns to the kernel normal system of conjugations $\text{in}\, A$, thus the quotient $\varsigma(A)$ turns to the quotient $\text{end}\, A/\text{in}\, A$; also the partial representation $\psi : G \to \varsigma(A)$ turns to the partial representation $\psi : G \to \text{end}\, A/\text{in}\, A$.

What this means is that for a unital twisted partial action, our Theorems regard the groups $H^3(G, C(\tilde{A}))$ and $H^2(G, C(\tilde{A}))$, in which $\tilde{A}$ is a monoid contained in A.

References

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