Horizontal Maps at Levels of the Interpolation Scale

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Introduction

In the article [2], M. Daher defines the complex interpolation space as the holomorphic functions over the strip S however replacing the continuity of the operators over $\overline{S}$ by a condition of integrability L_p. This allows to find on certain hypotheses a minimal representation $\Gamma(x)$ for each point x of the interpolation domain in a uniformly continuous way.

The minimal function Γ allow us to consider the *horizontal maps* that arise at different levels of the scale: that is, taking $\theta_1, \theta_2 \in (0,1)$ and fixed $t \in \mathbb{R}$, we consider the horizontal map

$$\begin{aligned} A_{\theta_1} &\longrightarrow A_{\theta_2} \\ \Gamma(x)(\theta_1 + it) &\longmapsto \Gamma(x)(\theta_2 + it) \end{aligned}$$

The goal is to show that in the case of L_p spaces ($1 < p < \infty$) the horizontal maps are uniform homeomorphisms between the spheres of the interpolation spaces.

Definitions and Notations

For all $p \in [1, +\infty]$ and let $\mathcal{F}^p_\theta(\bar{A})$ be the space of functions $F : \overline{S} \to A_0 + A_1$, F holomorphic over S, such that $\tau \to F(j+it)$ is (Bochner) measurable with values in A_j ($j = 0, 1$) and

$$\|F\|^p_{\mathcal{F}^p_\theta(\bar{A})} = \int_{\mathbb{R}} \|F(it)\|^p_{A_0} d\mu_{\theta,0}(t) + \int_{\mathbb{R}} \|F(1+it)\|^p_{A_1} d\mu_{\theta,1}(t)$$

We define the space $A^p_\theta = \{F(\theta); F \in \mathcal{F}^p_\theta(\bar{A})\}$; if $a \in A^p_\theta$ put

$$\|a\|_{A^p_\theta} = \inf\{\|F\|_{\mathcal{F}^p_\theta(\bar{A})};\ F(\theta) = a\},$$

A^p_θ is a Banach space and coincides with the Complex Interpolation space A_θ.

Uniform homeomorphism between unit spheres

Let (A_0, A_1) be an interpolation couple and $\theta \in (0,1)$.

Proposition 1. *Suppose that $A_0 \cap A_1$ is dense in A_0 and in A_1 and that the spaces A_0 and A_1 are reflexive, then we have*

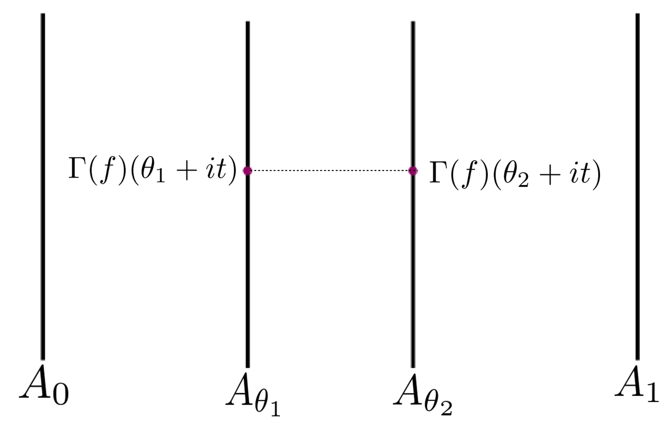
1. *If A_0 is strictly convex then for all $a \in S_{A_\theta}$ and for all $p \in (1, +\infty)$ there exists a unique $g_a \in \mathcal{F}^p_\theta(\bar{A})$ such that $g_a(\theta) = a$ and $\|g_a\|_{\mathcal{F}^p_\theta(\bar{A})} = 1$.*
2. *If A_0 and A_1 are uniformly convex then the mapping $\Gamma : S_{A_\theta} \to \mathcal{F}^p_\theta(\bar{A})$ defined by $\Gamma(a) = g_a$ is uniformly continuous ($1 < p < \infty$).*

Theorem. *Let (A_0, A_1) be an interpolation couple and $\theta_1, \theta_2 \in (0,1)$ with $A_0 \cap A_1$ dense in A_0 and in A_1. If A_0 and A_1 are uniformly convex, then the mapping $U : S_{A_{\theta_1}} \to S_{A_{\theta_2}}$ given by $a \mapsto \Gamma(a)(\theta_2)$ is a uniform homeomorphism.*

Horizontal Maps

Looking at the minimal functions given by Proposition 1, we can consider the *horizontal maps* that arise at different levels of the scale: that is, taking $\theta_1, \theta_2 \in (0,1)$, and fixed $t \in \mathbb{R}$, consider the map

$$\begin{aligned} A_{\theta_1} &\longrightarrow A_{\theta_2} \\ \Gamma(f)(\theta_1 + it) &\longmapsto \Gamma(f)(\theta_2 + it) \end{aligned}$$



We will show that in the case of L_p spaces, the horizontal maps are a uniform homeomorphism between the spheres of interpolation spaces.

Consider $A_0 = L_{p_0}(0,1)$ and $A_1 = L_{p_1}(0,1)$, with $1 < p_0 < p_1 < +\infty$. Let $\theta_1, \theta_2 \in (0,1)$, and define $1 < p_{\theta_1}, p_{\theta_2} < \infty$ such that

$$\frac{1}{p_{\theta_j}} = \frac{1-\theta_j}{p_0} + \frac{\theta_j}{p_1},\ j = 1, 2.$$

Denote by M the Mazur map

$$\begin{aligned} M : S_{L_{p_{\theta_1}}} &\longrightarrow S_{L_{p_{\theta_2}}} \\ f &\longmapsto \operatorname{sgn}(f)|f|^{p_{\theta_1}/p_{\theta_2}} \end{aligned}$$

If $\Gamma : S_{\theta_1} \to \mathcal{F}^p_{\theta_1}(\bar{A})$ is the mapping given by Proposition 1, we have that,

$$S_{\theta_1} = \{\Gamma(g)(\theta_1 + it) : g \in S_{\theta_1}\},\ \forall t \in \mathbb{R} \tag{1}$$

$$S_{\theta_2} = \{\Gamma(g)(\theta_2 + it) : g \in S_{\theta_1}\},\ \forall t \in \mathbb{R} \tag{2}$$

Thus, for each $t \in \mathbb{R}$, we can define the function,

$$\begin{aligned} \Phi : S_{\theta_1} &\longrightarrow S_{\theta_2} \\ f = \Gamma(g)(\theta_1 + it) &\longmapsto \Gamma(g)(\theta_2 + it) \end{aligned}$$

Let us show that $\Phi(f) = M(f)$ for all $f \in S_{\theta_1}$. Note that, if $f = \Gamma(g)(\theta_1 + it)$, then

$$\operatorname{sgn}(f) = \operatorname{sgn}(\Gamma(g)(\theta_1 + it)) = \operatorname{sgn}(g)|g|^{p_{\theta_1}\left(\frac{-it}{p_0} + \frac{it}{p_1}\right)}$$

and

$$|f| = |\Gamma(g)(\theta_1 + it)| = |g|$$

So we have

$$\begin{aligned} M(f) &= M(\Gamma(g)(\theta_1 + it)) = \operatorname{sgn}(f)|f|^{p_{\theta_1}/p_{\theta_2}} \\ &= \operatorname{sgn}(g)|g|^{p_{\theta_1}\left(\frac{-it}{p_0} + \frac{it}{p_1}\right)}|g|^{p_{\theta_1}/p_{\theta_2}} = \Gamma(g)(\theta_2 + it) = \Phi(f) \end{aligned}$$

Therefore, $\Phi = M$.

Referências

[1] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer-Verlag 223, Berlin, Heidelberg, New York, 1976.

[2] M. Daher, *Homèomorphismes uniformes entre les sphères unité des espaces d'interpolation*, Canad. Math. Bull., Vol 38 (3), 1995, pp. 286-294.

[3] J. Garnett, *Bounded Analytic Functions*, Graduate Texts in Mathematics 236, Springer, 2007.

[4] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Vol. 2, Ergeb. Math. Grenzgeb. 97, Springer Verlag, 1979.

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