Dynamics of May–Leonard systems with Darboux invariant

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1. Introduction

The Lotka-Volterra systems in $\mathbb{R}^3$

$$\dot{x}_i = x_i\left(\sum_{j=1}^{3} a_{ij}x_j + b_i\right) \qquad (i = 1, \ldots, 3).$$

were introduce by Lotka and Volterra in the twenties, see [4, 6]. Such differential systems have been intensively investigated due to the fact that they can be found in different phenomena in biology, ecology and chemistry, for instance.

Due to its simplicity, special attention has attracted the so-called *classical May-Leonard system* introduced by May and Leonard in [5] which describes a competition between three species using is a simple model with a rich dynamical behavior. This system can be written as

$$\begin{aligned}\dot{x} &= x(1 - x - \alpha y - \beta z),\\ \dot{y} &= y(1 - \beta x - y - \alpha z),\\ \dot{z} &= z(1 - \alpha x - \beta y - z),\end{aligned} \qquad (1)$$

where α and β are real parameters.

Dynamic aspects of system (1) was investigated by several author, according to different values of the parameters:

- In [5] the authors shown that, whenever $\alpha + \beta \neq -1$, system (1) has four singular points in $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3, x, y, z \geq 0\}$. Three of them are on the boundary of $\mathbb{R}^3_+$, $E_1 = (1, 0, 0)$, $E_2 = (0, 1, 0)$, $E_3 = (0, 0, 1)$, and the fourth one is the interior point $C = ((1+\alpha+\beta)^{-1}, (1+\alpha+\beta)^{-1}, (1+\alpha+\beta)^{-1})$. If in addition, $\alpha + \beta > 2$ and either $\alpha < 1$, or $\beta > 1$, then there is a separatrix cycle F formed by orbits connecting E_1, E_2 and E_3 on the boundary of $\mathbb{R}^3_+$, and every orbit in $\mathbb{R}^3_+$, except of the equilibrium point C has F as the ω-limit.
- It was shown in [1] that in the degenerate case $\alpha + \beta = 2$, the cycle F becomes a triangle on the invariant plane $x + y + z = 1$ and all orbits inside the triangle are closed and every orbit in the interior of $\mathbb{R}^3_+$ has one of these closed orbits as its ω-limit. In [1] the complete description of the dynamics of the May-Leonard system whenever $\alpha + \beta = 2$, or $\alpha = \beta$ is presented.

In this poster we shall study the May-Leonard systems that admit a Darboux invariant of the form $e^{st}f(x, y, z)$, where $f(x, y, z)$ is given by the product of invariant planes. We will see that the case in which there exists such an invariant in the May-Leonard model is whenever $\alpha + \beta + 1 = 0$. We consider this case and we describe the global dynamics in the compactification of $\mathbb{R}^3$ in function of α.

2. Preliminaries and main results

- An *invariant* of system (1) on an open subset U of $\mathbb{R}^3$ is a nonconstant C^1 function I in the variables x, y, z and t such that $I(x(t), y(t), z(t), t)$ is constant on all solution curves $(x(t), y(t), z(t))$ of system (1) contained in U, i.e.

$$\begin{aligned}x(1 - x - \alpha y - \beta z)\frac{\partial I}{\partial x} + y(1 - \beta x - y - \alpha z)\frac{\partial I}{\partial y}\\ + z(1 - \alpha x - \beta y - z)\frac{\partial I}{\partial z} + \frac{\partial I}{\partial t} = 0,\end{aligned} \qquad (2)$$

for all $(x, y, z) \in U$.

- On the other hand given $f \in \mathbb{C}[x, y, z]$ we say that the surface $f(x, y, z) = 0$ is an *invariant algebraic surface* of system (1) if there exists $K \in \mathbb{C}[x, y, z]$ such that

$$\begin{aligned}x(1 - x - \alpha y - \beta z)\frac{\partial f}{\partial x} + y(1 - \beta x - y - \alpha z)\frac{\partial f}{\partial y}\\ + z(1 - \alpha x - \beta y - z)\frac{\partial f}{\partial z} = Kf.\end{aligned} \qquad (3)$$

- An invariant I is called a *Darboux invariant* if it can be written in the form

$$I(x, y, z, t) = f_1^{\lambda_1} \cdots f_p^{\lambda_p} e^{s\,t},$$

where, for $i = 1, \ldots p$, $f_i = 0$ are invariant algebraic surfaces of system (1), $\lambda_i \in \mathbb{C}$, and $s \in \mathbb{R} \setminus \{0\}$.

Theorem 1. *The following holds for system* (1) *with either* $\alpha + \beta \neq 2$ *or* $\alpha \neq \beta$

(a) It has a Darboux invariant of the form $I(x, y, z, t) = e^{st}f(x, y, z)$*, where* $f(x, y, z)$ *is given by the product of invariant planes if and only if* $\alpha + \beta + 1 = 0$.

(b) It is invariant under the symmetry $(x, y, z) \to (y, z, x)$.

(c) The ω*-limit of any of its orbits in* $\mathbb{R}^3$ *is contained in* Ω *union with its boundary at infinity in the Poincaré compactification in* $\mathbb{R}^3$ *(see [2] for details), where*

$$\begin{aligned}\Omega = &\{(x, y, z) \in \mathbb{R}^3 : x = 0\} \cup\\ &\{(x, y, z) \in \mathbb{R}^3 : y = 0\} \cup \{(x, y, z) \in \mathbb{R}^3 : z = 0\}.\end{aligned}$$

From now on, we consider system (1) restricted to $\beta = -1 - \alpha$ with $\alpha \in \mathbb{R}$, that is

$$\begin{aligned}\dot{x} &= x(1 - x - \alpha y + (1 + \alpha)z),\\ \dot{y} &= y(1 + (1 + \alpha)x - y - \alpha z),\\ \dot{z} &= z(1 - \alpha x + (1 + \alpha)y - z).\end{aligned} \qquad (4)$$

In order to describe de global dynamics of system (4) we consider its dynamics in each one of the ith-octants, O_i, for $i = \{1, 2, .., 8\}$.

The global dynamics of system (4) in O_1 (the positive octant) is given in the following theorem. In that theorem, we denote by O_1^+ the interior of O_1, i.e.,

$$O_1^+ = \{(x, y, z) : x > 0, y > 0, z > 0\}.$$

Theorem 2. *The following statements hold for system* (4) *restricted to* O_1 *for* $\alpha \in (-\infty, 1)$*:*

(a) The phase portraits in the Poincaré disc of system (4) *restricted to the invariant planes are topologically equivalent to one of the phase portraits of Figure 1.*

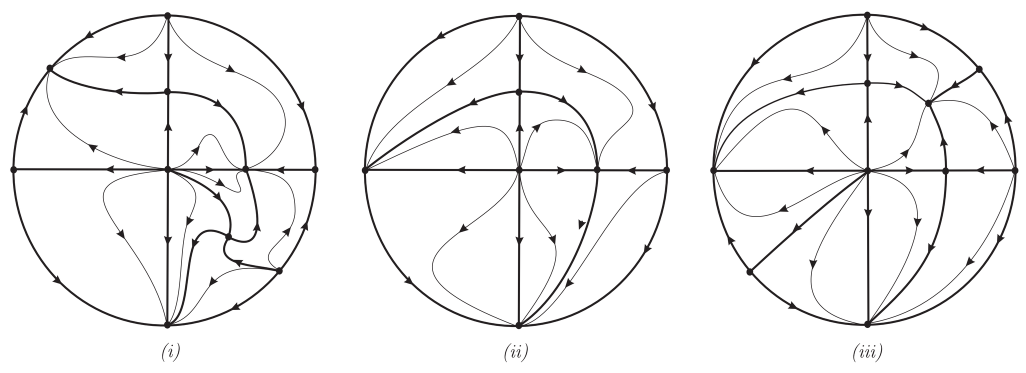

Figure 1: *Phase portrait of system* (4) *on the Poincaré disc restricted to the invariant planes* $x = 0$, $y = 0$ *or* $z = 0$ *when: (i)* $\alpha < -2$, *or* $\alpha > 1$*; (ii)* $\alpha = -2$, *or* $\alpha = 1$*; (iii)* $-2 < \alpha < 1$.

(b) The phase portraits of system (4) *at the infinity of* O_1 *are topologically equivalent to one of the phase portraits given in Figure 2. More precisely,*

(b.1) for $\alpha \leq -2$ *the boundary of the infinity of* O_1 *is a heteroclinic cycle formed by three equilibrium points coming from the ones located at the end of the three positive half-axes of coordinates, and three orbits connecting these equilibria, each one coming from the orbit at the end of every plane of coordinates. In the interior of the infinity of* O_1 *there is an attractor whose orbits fill completely this interior.*

(b.2) For $\alpha \in (-2, 1)$ *the boundary of the infinity of* O_1 *is a graph formed by six equilibrium points all of them located at the positive half-axes of coordinates. Three of them are at the end of the axes and the other three are between them. Moreover, there are six orbits connecting these equilibria. Each of them coming from the orbit at the end of every plane of coordinates. In the interior of the infinity of* O_1 *there is an attractor of all the orbits coming from the equilibria at the end of every plane of coordinates.*

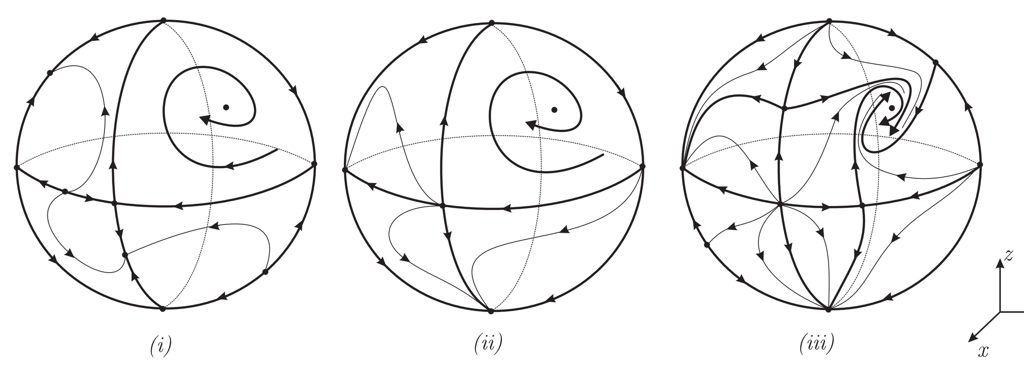

Figure 2: *Phase portrait of system* (4) *on the Poincaré sphere when: (i)* $\alpha < -2$, *or* $\alpha > 1$*; (ii)* $\alpha = -2$, *or* $\alpha = 1$*; (iii)* $-2 < \alpha < 1$.

(c) The phase portrait of system (4) *on* O_1 *is giving in Figure 3 (i) when* $\alpha \leq 2$ *and Figure 3 (ii) when* $\alpha \in (-2, 1)$*. Namely,*

(c.1) for $\alpha \leq -2$*, there exists a separatrix cycle* F *formed by orbits connecting the finite singular points on the boundary of* O_1 *and every orbit on* O_1*, except the origin, has* F *as its* ω*-limit. The* $\alpha-$*limit set of the orbits on* $O_1^+ \setminus F$ *is formed by four equilibrium points, the origin and the three equilibria located at the end of the positive half-axes of coordinates.*

(c.2) for $\alpha \in (-2, 1)$*, there exists a graph* G *formed by orbits connecting the finite singular points on the boundary of* O_1*, except the origin. The* $\omega-$*limit set of every orbit on* O_1*, except the origin, is one of the vertices of* G*. The* $\alpha-$*limit set of the orbits on* $O_1^+ \setminus G$ *is formed by seven equilibrium points, the origin and the equilibria located at the end of the invariant planes.*

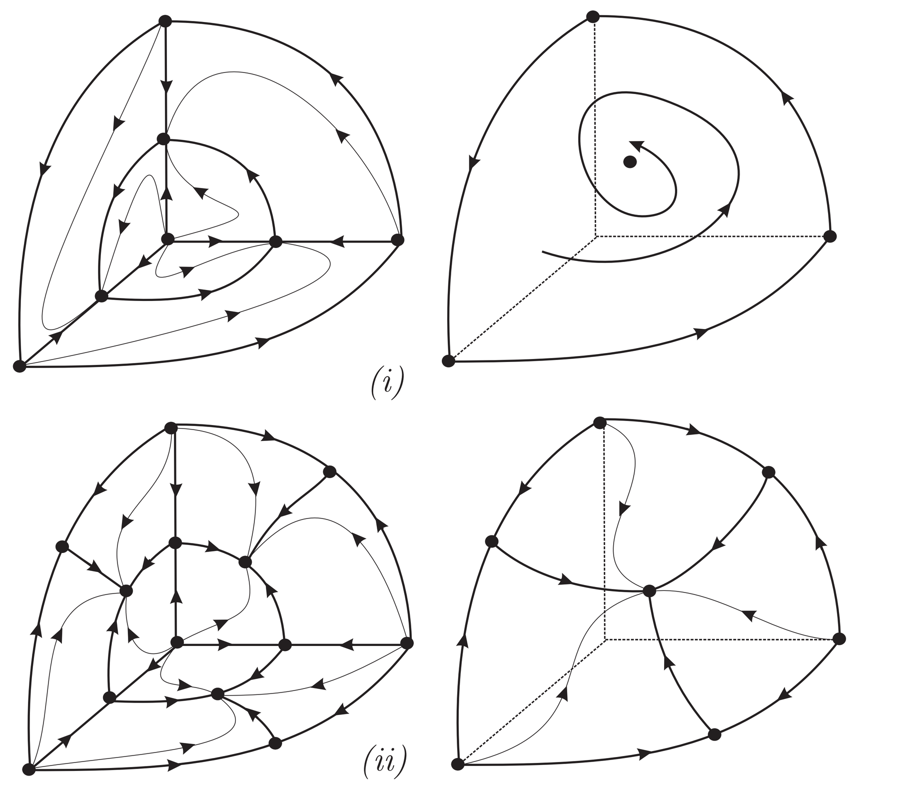

Figure 3: *Phase portrait of system* (4) *in the first octant* O_1*: (i) for* $\alpha \leq -2$ *or* $\alpha \geq 1$ *and (ii) for* $-2 < \alpha < 1$.

3. Remarks and conclusions

- Since Theorem 2 provides the ω-limit and the α-limit of all orbits inside O_1 (which is the octant where system (4) has biological meaning), in that theorem we are determining all the initial and final evolution of the three species considered by system (4) according to the values of the parameter α.
- The statements in Theorem 2 also hold for $\alpha \geq 1$. Indeed, the dynamics for $\alpha > 1$ is the same as the one for $\alpha < -2$ reversing the time and the dynamics for $\alpha = 1$ is the same of the one for $\alpha = -2$ also reversing the time.

Using the same tools we get the global dynamics in each Octant of sphere. It follows from Theorem 1 (b) that the dynamics in O_6 and O_8 are the same as the one in O_1.

- In order to reach information about the dynamics of system (4) in each octante we apply the Poincaré compactification to get the finite and infinite singular points in the sphere. Then jointing the local behavior of the solutions near of the singular points (finite and infinite) with the behavior of the solutions in the invariant planes and the existence of a Darboux invariant.
- The existence of a Darboux invariant of system (1) provides information about the ω– and α–limit sets of all orbits of system (1).

Proposition 3 (see [3] for the 2-dimensional case)**.** *Let* $\mathbb{S}^2$ *be the infinity of the Poincaré sphere and* $I(x, y, z, t) = f(x, y, z)e^{st}$ *be a Darboux invariant of system* (1)*. Let also* $p \in \mathbb{R}^3$ *and* $\phi_p(t)$ *be the solution of system* (1) *with maximal interval* (α_p, ω_p) *such that* $\phi_p(0) = p$*. If* $\omega_p = \infty$ *then* $\omega(p) \subset \{f(x, y, z) = 0\} \cup \mathbb{S}^2$ *and if* $\alpha_p = -\infty$ *then* $\alpha(p) \subset \{f(x, y, z) = 0\} \cup \mathbb{S}^2$.

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