Generalized hyperbolicity for linear operators

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Finite dimensional case

- linear operators cannot be transitive;
- do not have density of periodic orbits;
- if it has the shadowing property, it belongs necessarily to some hyperbolic class.

Goal setting

Opposing the finite-dimensional case, the dynamics of linear operators in spaces of infinite dimension is much richer. In [BCDMP] we showed

- examples of operators in Banach spaces that are transitive, have the shadowing property, but are not hyperbolic.

In the '70, considering the sup norm in the unitary ball, it was proved by [FSW] that

- in infinite dimensional Banach spaces there are not robust transitive operators

Here, we consider a class of operators such that they are transitive in the whole Banach space and their perturbed one has "large"transitive sets.

Definitions

Definition 1. *T is* **transitive** *if for any pair of open sets U and V, there exist $n > 0$ s.t. $T^nU \cap V \neq \emptyset$*

Definition 2. *Let T be a bounded linear operator acting on a Banach space E. T is* ***GH (Generalized Hyperbolic)*** *if has bounded by below minimum norm and there exists a decomposition $E = E_T^- \oplus E_T^+$ such that*

- *$T(E_T^+) \subset E_T^+$ and $T^{-1}(E_T^-) \subset E_T^-$*
- *$T_{/E_T^+}$ and $T^{-1}_{/E_T^-}$ are contractions*

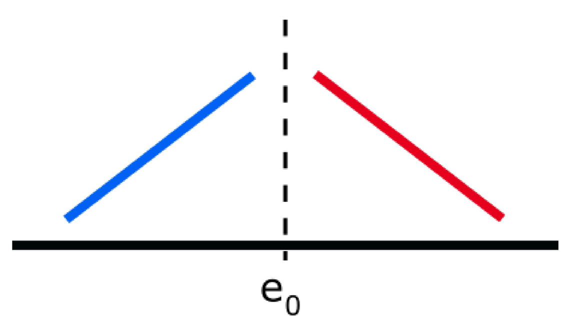
Obs.: Similar to classical hyperbolic definition, splitting s.t. vectors in one subbundle are contracted and in the other are expanded. In particular, a hyperbolic operator is a **GH** one. Difference is that for **GH** splitting is not necessarily invariant.

Definition 3. *Let* **B(T)** *be the set of points x s.t. there exist $K := K(x) > 0$ and sequences $(k_n), (m_n) \in \mathbb{N}$ s.t.,*

- *$||T^{k_n}(x)|| < K$ and*
- *$||T^{-m_n}(x)|| < K$.*

Obs.: $x \in B(T)$ does not means that the trajectory is uniformly bounded; we ask only that there are arbitrary large forward and backward iterates which are bounded by a constant that depends on the initial point.

Weighted shift: consider
$$T(e_n) = 2e_{n-1} \quad \text{if } n < 0$$
$$T(e_n) = \frac{1}{2}e_{n-1} \quad \text{if } n \geq 0$$



These invertible operators are transitive, belong to **GH** and they are not hyperbolic.

Robustness

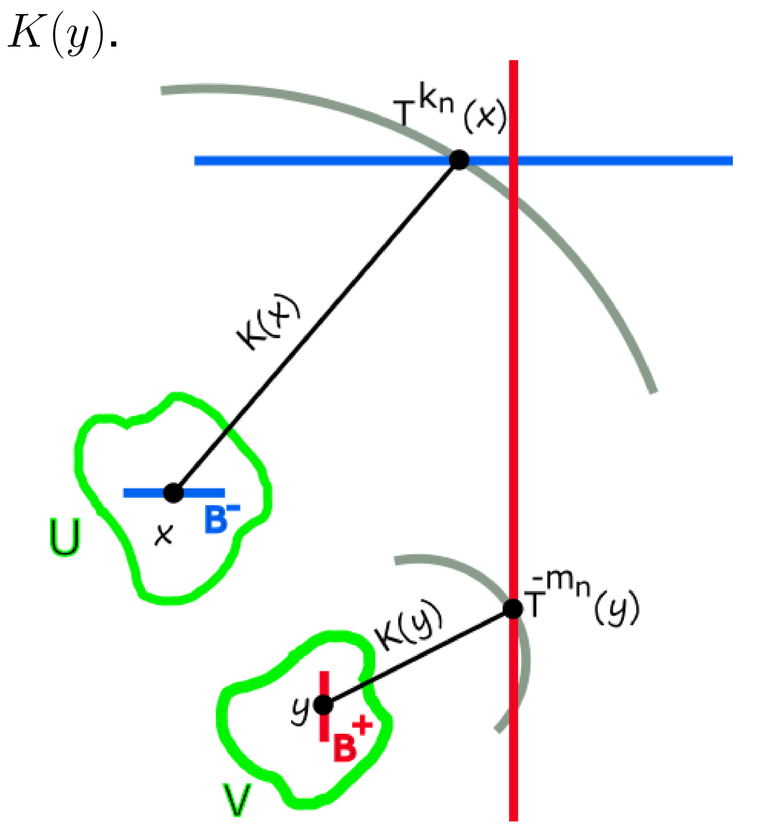
Theorem 1. **GH** *is an open property.*

Transitivity for GH operators

Theorem 2. $T \in$ **GH** $\Rightarrow$ $T|_{\overline{B(T)}}$ *transitive.*

Proof. $T \in$ **GH** $\Rightarrow$ $T^k(B_r^-(x)) \supset B^-_{\lambda^{-k}r}(T^kx)$
and $T^{-m}(B_r^+(x)) \supset B^+_{\lambda^{-m}r}(T^{-m}x)$, where $\lambda < 1$.

Let k_n and m_n from definition 3 s.t. $||T^{k_n}(x)|| < K(x)$ and $||T^{-m_n}(y)|| < K(y)$.



Then for n large enough $B^-_{\lambda^{-k_n}r}(T^{k_n}x) \cap B^+_{\lambda^{-m_n}r}(T^{-m_n}y) \neq \emptyset$.

Question: How large is $B(T)$?

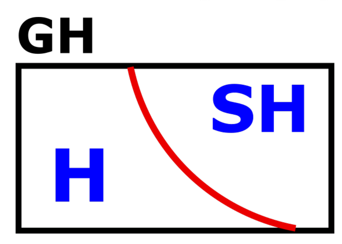
- If T is hyperbolic, then $B(T) = \{0\}$.
- But there exist **GH** operators s.t. $\overline{B(T)}$ is the entire Banach Space. E.g.: Weighted Shifts.
- $T \in$ **GH** $\Rightarrow$ $\overline{B(T)} = \Omega(T)$

Theorem 3. $T \in$ **GH** *and* $S \sim T$ $\Rightarrow$ $S|_{\overline{B(S)}}$ *transitive.*

T is called **SHIFTED HYPERBOLIC** if
$$T^{-1}(E^+) \cap E^-$$
is a non-trivial subspace.

Theorem

If T is **GH** then it is either HYPERBOLIC or SHIFTED HYPERBOLIC.



T Shifted Hyperbolic $\Rightarrow$ $\overline{B(T)}$ infinite dimensional subspace.

Corollary

- Robust transitivity on the Non-Wandering set

$$(T_{|\overline{B(T)} = \Omega(T)} \text{ is RT}).$$

AN OPEN SET OF LINEAR OPERATORS SUCH THAT THE NON-WANDERING IS AN INFINITE DIMENSIONAL ROBUST TRANSITIVE SUBSPACE

Final conclusion

There is **robust transitivity** in "large" sets, even if it is impossible to have robust transitivity in the whole space.

This is a joint work with B. Gollobit and E. Pujals

Referências

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[FSW] P.A. Fillmore, J.G. Stampfli, J.P. Williams; On the essential numerical range, the essential spectrum, and a problem of Halmos. *Acta Sci. Math. (Szeged)* 33 (1972), 179-192.

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