Networks with gradient and Hamiltonian structures

Miriam Manoel

ICMC, Universidade de São Paulo

miriam@icmc.usp.br

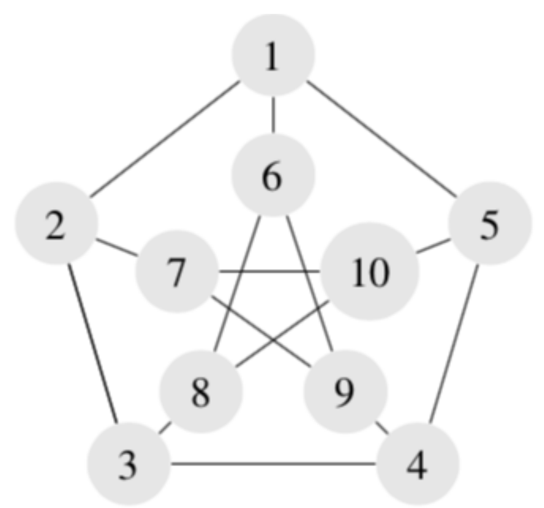
Introduction

- **In what mathematical grounds can we model brain cells that are connected in an excitatory-inhibitory synapsis?**
- **How to predict possible synchronization patterns among those neurons?**

Here we consider a dynamical system of neurons which in reality has a very complex behavior. Our approach generalizes the mathematical model given in [3], but we do not claim a specific result which would explain the real neuron dynamics. Instead, we explore the maths behind it, which we believe has potential for many applications in related topics.

In general, one big system formed by the coupling of "smaller" dynamical systems (cells) can schematically be given by a graph, with each vertex representing an individual cell and an edge between two vertices representing the coupling of the two cells. It is an interesting fact that when the dynamics is ruled by a system of autonomous differential equations, the vector field must respect the topology of the graph – the so-called admissible vector field for that graph [5].

Example - the Petersen graph



Admissible gradient vector field

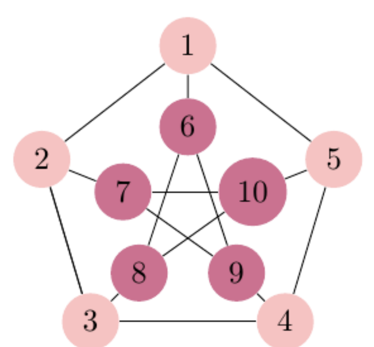
$$\dot{x}_i = f(x_i, \overline{x_j, x_k, x_\ell})$$

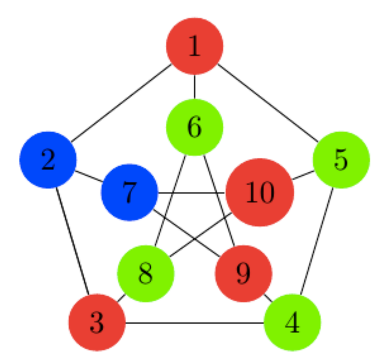
where j, k, ℓ are the cells connected to the cell i, for $1 \le i, j, k, \ell \le 10$.

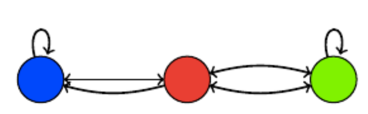
Adjacency matrix

$$\begin{pmatrix} 0&1&0&0&1&1&0&0&0&0\\ 1&0&1&0&0&0&1&0&0&0\\ 0&1&0&1&0&0&0&1&0&0\\ 0&0&1&0&1&0&0&0&1&0\\ 1&0&0&1&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&1&1&0\\ 0&1&0&0&0&0&0&0&1&1\\ 0&0&1&0&0&1&0&0&0&1\\ 0&0&0&1&0&1&1&0&0&0\\ 0&0&0&0&1&0&1&1&0&0 \end{pmatrix}$$

Two possible synchrony patterns – quotient graphs







$$\begin{pmatrix} 2 & 1\\ 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2\\ 2 & 1 & 0\\ 2 & 0 & 1 \end{pmatrix}$$

$$\dot{x}_1 = f(x_1, x_1, x_1, x_6)$$
$$\dot{x}_6 = f(x_6, x_6, x_6, x_1)$$

$$\dot{x}_1 = f(x_1, x_2, x_5, x_5)$$
$$\dot{x}_2 = f(x_2, x_2, x_1, x_1)$$
$$\dot{x}_5 = f(x_5, x_5, x_1, x_1)$$

Gradient and Hamiltonian structures

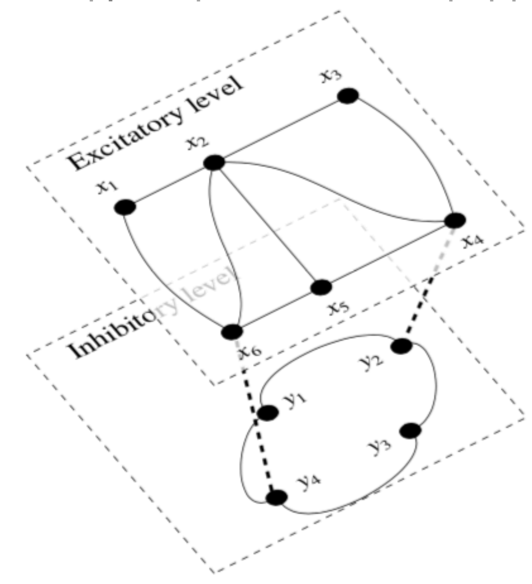
Theorem 1 (4) *A necessary and sufficient condition for a network to admit Gradient or Hamiltonian admissible vector fields is that it defines an undirected graph.*

Theorem 2 (1) *An m-vertex network Q is a quotient network of an n-vertex undirected network G by a balanced equivalence relation on the set of vertices of G, with $m < n$, if and only if the entries of the adjacency matrix $A_Q = [q_{ij}]_{m\times m}$ of Q satisfy the following: there are positive integers $k_1, \ldots, k_m$ summing n and such that $k_i q_{ij} = k_j q_{ji}$.*

Theorem 3 (1) *Let Q be an m-vertex quotient network of an n-vertex undirected network G, with $m < n$, associated with a balanced equivalence relation $\bowtie$ on the vertices of G. The quotient network Q is undirected if, and only if, for each pair of connected vertices in Q, the corresponding $\bowtie$-classes on the set of vertices of G have the same cardinality.*

Excitatory-inhibitory synapses of neurons

In simple terms, there are two types of neurons in the brain, namely those that increase activity in other cells (excitatory neurons) and those that decrease activity (inhibitory neurons) [2]. In addition, this interaction and the dynamical behavior of a gradient vector field is related to artificial learning mec [illegible] l neural networks.



The general system of ordinary differential equations defined on $\mathbb{R}^n \times \mathbb{R}^m$ is given by

$$(\dot{x}, \dot{y}) = -\nabla f(x, y) + J\nabla h(x, y), \qquad (1)$$

where J is the skew-symmetric transformation with matricial form

$$J = \begin{pmatrix} 0 & A_H\\ -A_H^t & 0 \end{pmatrix}.$$

generating function for such network is $g = f + h$, where

$$f(x, y) = \sum_{i \le k}^{n} a_{ik}^{G_1} \beta_{G_1}(x_i, x_k) + \sum_{j \le \ell}^{m} a_{j\ell}^{G_2} \beta_{G_2}(y_j, y_\ell) + \sum_{i=1}^{n} \alpha_1(x_i) + \sum_{j=1}^{m} \gamma_1(y_j)$$

and

$$h(x, y) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^{H} \beta_H(x_i, y_j) + \sum_{i=1}^{n} \alpha_2(x_i) + \sum_{j=1}^{m} \gamma_2(y_j),$$

with the conditions of invariance under permutation of variables on the two coupling functions β_{G_1} and β_{G_2} as given in the two theorems.

It follows that the excitation-inhibition configuration is realized in such model if the synapses between groups are sufficiently stronger than interior synapses. The graph structure of this network implies that β_{G_1} and β_{G_2} are invariant under permutation of variables [4], so the generating function is of the form

$$g(x, y) = a(x_1x_2 + x_1x_6 + x_2x_3 + x_2x_4 + x_2x_5 + x_2x_6 + x_3x_4 + x_4x_5 + x_5x_6) +$$
$$+ b(y_1y_2 + y_1y_4 + y_2y_3 + y_3y_4) + \sum_{i=1}^{6} \alpha x_i^2 + \sum_{j=1}^{4} \beta y_j^2 + c(x_4y_2 + x_6y_4) + o^2(x, y),$$

for some constants a, b, c, α, β. The excitation-inhibition configuration is realized in such model if, for example,

$$\alpha = \beta = 1, \ a, b, c \ll 1.$$

Patterns of synchrony. These are derived from the Theorem 3 above, namely, from balanced equivalence relations on the sets of vertices whose equivalence classes have the same cardinality. As a direct consequence, in the network in the picture, the excitatory neurons and inhibitory neurons synchronyze in pairs: $x_1 = x_3, \ x_4 = x_6$ and $y_1 = y_3, \ \ y_2 = y_4$.

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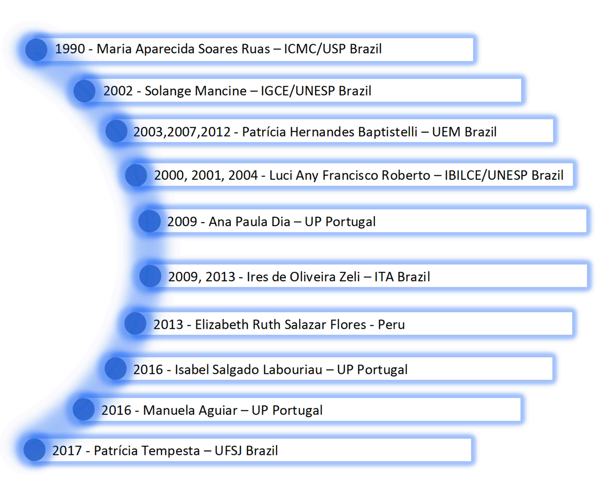

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