Singular Hyperbolicity and Sectional Lyapunov Exponents of Various Orders

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Abstract

We propose notions of singular hyperbolicity and sectional Lyapunov exponents of orders beyond the classical ones, namely, other dimensions besides the dimension 2 and the full dimension of the central subbundle of the singular hyperbolic set. It is obtained a characterization of dominated splittings, partial and singular hyperbolicity in this broad sense, by using Lyapunov exponents and the notion of infinitesimal Lyapunov functions .

1. Introduction and definitions

Let M be a connected compact finite n-dimensional manifold, $n \geq 3$, with or without boundary.
We consider a vector field X, such that X is inwardly transverse to the boundary ∂M, if $\partial M \neq \emptyset$.
The flow generated by X is denoted by $\{X_t\}$.
Let $\Lambda \subset M$ be a compact invariant subset for X.
A *dominated splitting* over a compact invariant set Λ of X is a continuous DX_t-invariant splitting $T_\Lambda M = E \oplus F$ with $E_x \neq \{0\}$, $F_x \neq \{0\}$ for every $x \in \Lambda$ and such that there are positive constants K, λ satisfying

$$\|DX_t|_{E_x}\| \cdot \|DX_{-t}|_{F_{X_t(x)}}\| < Ke^{-\lambda t}, \text{ for all } x \in \Lambda, \text{ and all } t > 0.$$

A compact invariant set Λ is said to be *partially hyperbolic* if it exhibits a dominated splitting $T_\Lambda M = E \oplus F$ such that subbundle E is uniformly contracted. In this case F is the *central subbundle* of Λ.
A compact invariant set Λ with hyperbolic singularities is said to be *singular hyperbolic* if it is partially hyperbolic and the action of the tangent cocycle expands volume along the central subbundle.

1.1 Singular hyperbolicity of various orders

Given E a vector space, we denote by $\wedge^p E$ the exterior power of order p of E, defined as follows. If $v_1, \ldots, v_n$ is a basis of E then $\wedge^p E$ is generated by

$$\{v_{i_1} \wedge \cdots \wedge v_{i_p}\}_{1 \leq i \leq n, i_j \neq i_k, j \neq k}.$$

Any linear transformation $A : E \to F$ induces a transformation $\wedge^p A : \wedge^p E \to \wedge^p F$. Moreover, $v_{i_1} \wedge \cdots \wedge v_{i_p}$ can be viewed as the p-plane generated by $\{v_{i_1}, \cdots, v_{i_p}\}$ if $i_j \neq i_k, j \neq k$.
We may define singular hyperbolicity in a broad sense, as follows.

Definition A compact invariant set Λ is **p-sectionally hyperbolic** (or **p-singular hyperbolic**) for a C^1 flow X if there exists a partially hyperbolic splitting $T_\Lambda M = E \oplus F$ such that E is uniformly contracting and the central subbundle F is p-sectionally expanding, with $2 \leq p \leq \dim(F)$.

If L_x is a p-plane, we can see it as $\widetilde{v} \in \wedge^p(F_x) \setminus \{0\}$ of norm one. Hence, to obtain the singular expansion we just need to show that for some $\lambda > 0$ and every $t > 0$ holds the following inequality

$$\| \wedge^p DX_t(x).\widetilde{v}\| > Ce^{\lambda t}.$$

2. Results

From now on, Λ is a set which admits singularities, all of them hyperbolic of saddle-type.

Recall that, if $T : Z \to Z$ is a measurable map, we say that a probability measure μ is an invariant measure of T, if $\mu(T^{-1}(A)) = \mu(A)$, for every measurable set $A \subset Z$. A subset $Y \subset Z$ has **total probability** if for every $\mu \in \mathcal{M}_X$ we have $\mu(Y) = 1$.

2.1 Singular hyperbolicity via Lyapunov functions

The following definitions are fundamental to state our results.
Given a continuous field of non-degenerate quadratic forms $\mathfrak{J}$ with constant index on the trapping region U for the flow X_t, we say that the cocycle $A_t(x)$ over X is

- $\mathfrak{J}$**-separated** if $A_t(x)(C_+(x)) \subset C_+(X_t(x))$, for all $t > 0$ and $x \in U$;
- **strictly** $\mathfrak{J}$**-separated** if $A_t(x)(C_+(x) \cup C_0(x)) \subset C_+(X_t(x))$, for all $t > 0$ and $x \in U$;
- **strictly** $\mathfrak{J}$**-monotone** if $\partial_t\big(\mathfrak{J}_{X_t(x)}(A_t(x)v)\big) \mid_{t=0} > 0$, for all $v \in T_xM \setminus \{0\}$, $t > 0$ and $x \in U$;

Thus, $\mathfrak{J}$-separation corresponds to simple cone invariance and strict $\mathfrak{J}$-separation corresponds to strict cone invariance under the action of $A_t(x)$.
A vector field X is $\mathfrak{J}$-*non-negative* on U if $\mathfrak{J}(X(x)) \geq 0$ for all $x \in U$, and $\mathfrak{J}$-*non-positive* on U if $\mathfrak{J}(X(x)) \leq 0$ for all $x \in U$.
The index $\operatorname{ind}(\mathfrak{J})$ of $\mathfrak{J}$ means that the maximal dimension of subspaces of non-positive vectors is q.

Theorem A A compact invariant set Λ whose singularities are hyperbolic (with $\operatorname{ind} \geq \operatorname{ind}(\mathfrak{J})$) for $X \in \mathfrak{X}^1(M)$ is a p-singular hyperbolic set if, and only if, there exist a neighborhood U of Λ and a field of non-degenerate quadratic forms $\mathfrak{J}$ on U with index $1 \leq \operatorname{ind}(\mathfrak{J}) \leq n-2$ such that X is non-negative strictly $\mathfrak{J}$-separated and the spectrum of the diagonalized operator DX_t satisfies the properties:

1. $r_1^- < 1$; and
2. $\Pi_1^p r_i^+ > 1$, where $2 \leq p \leq \dim(M) - \operatorname{ind}(\mathfrak{J})$,

in a total probability subset of Λ. Moreover, if $r_i^+ \cdot r_j^+ > 1$, for all $1 \leq i, j, \leq p, i \neq j$, in a total probability set, then Λ is a 2-sectional hyperbolic set.

2.2 Lyapunov spectrum and Domination

The singular case requires domination on the singularities, once it is necessary matching the splitting.
We can get a characterization of domination property based on Lyapunov spectrum, without any other assumption on the singularities (if any). This is the content of our next result.

Theorem B Let Λ be a compact invariant set of X. Suppose that there exists a continuous invariant splitting of the tangent bundle of Λ, $T_\Lambda M = E \oplus F$. Then $T_\Lambda M = E \oplus F$ is a dominated splitting if, and only if, exists $\eta < 0$ for which

$$\liminf_{t \to +\infty} \frac{1}{t} \log |DX_t|_{E_x}| - \limsup_{t \to +\infty} \frac{1}{t} \log m(DX_t|_{F_x}) < \eta,$$

in a total probability set of Λ.

2.3 p-sectional Lyapunov exponents

Definition The **p-sectional Lyapunov exponents** (or **Lyapunov exponents of order p**) of x along F are the limits

$$\lim_{t \to +\infty} \frac{1}{t} \log \| \wedge^p DX_t(x).\widetilde{v}\|$$

whenever they exists, where $\widetilde{v} \in \wedge^p F_x - \{0\}$.
Following the corresponding result from [3, Theorem B], by changing $\| \wedge^2 DX_t(x).\widetilde{v}\|$ by $\| \wedge^p DX_t(x).\widetilde{v}\|$, we obtain, the next results.

Corollary C Let Λ be a compact invariant set of X such that every singularity in this set is hyperbolic. There exists a continuous invariant splitting of the tangent bundle, $T_\Lambda M = E \oplus F$, of Λ where:

1. the Lyapunov exponents on E are negative (or positive on F), and
2. $\liminf_{t \to +\infty} \frac{1}{t} \log |DX_t|_{E_x}| - \limsup_{t \to +\infty} \frac{1}{t} \log m(DX_t|_{F_x}) < 0$,

in a total probability set of Λ, if and only if, $T_\Lambda M = E \oplus F$ is a partially hyperbolic splitting.

Corollary D Let Λ a compact invariant set for a flow X_t such that every singularity $\sigma \in \Lambda$ is hyperbolic. Suppose that there is a continuous invariant splitting $T_\Lambda M = E \oplus F$. The set Λ is p-singular hyperbolic for the flow if, and only if, on a set of total probability in Λ,

1. $\liminf_{t \to +\infty} \frac{1}{t} \log |DX_t|_{E_x}| - \limsup_{t \to +\infty} \frac{1}{t} \log m(DX_t|_{F_x}) < 0$,
2. the Lyapunov exponents in the E direction are negative and
3. the p-sectional Lyapunov exponents in the F direction are positive .

3. Examples

Item (1) In [4] Turaev et al construct a *wild attractor* Γ (see Figure 1) of a n-dimensional vector field, $n \geq 4$, having a 3-sectional hyperbolic splitting $T_\Gamma M = E \oplus F$ where $\dim F = 3$ and $\dim E = n - 3$. Moreover, the bundle F is not (2-)sectionally expanding; see [4, p 296, formula (13)].

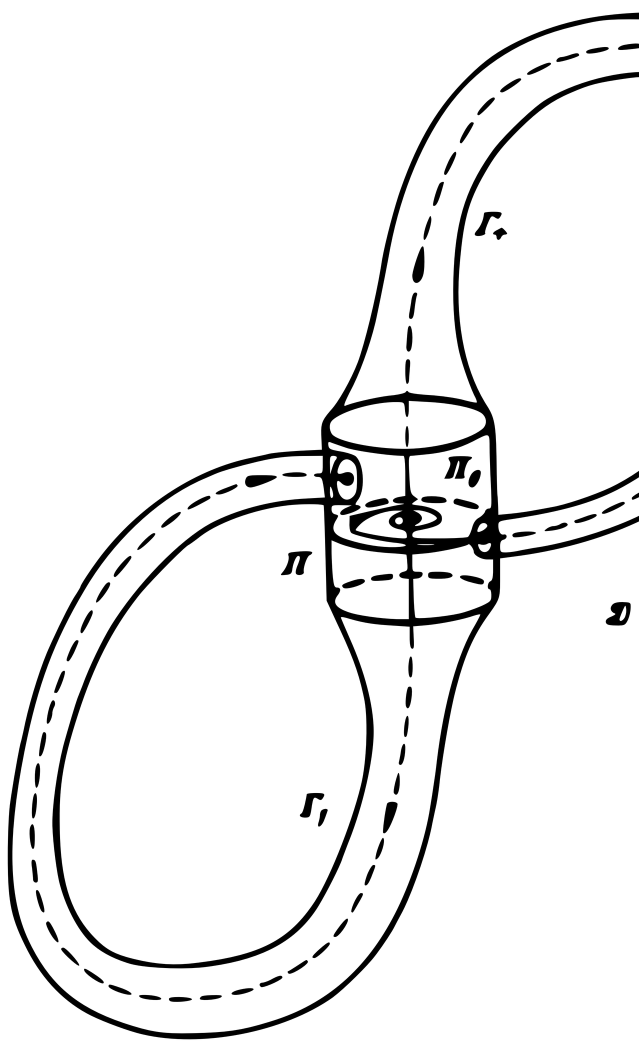

Figure 1: *Example of Turaev-Shil'nikov's wild strange attractor.*

Item (2)**:** The geometric Lorenz attractor is an emblematic example of singular hyperbolic set. Note that, in dimension 3, both definitions of singular and (2-)sectional hyperbolicity coincides.

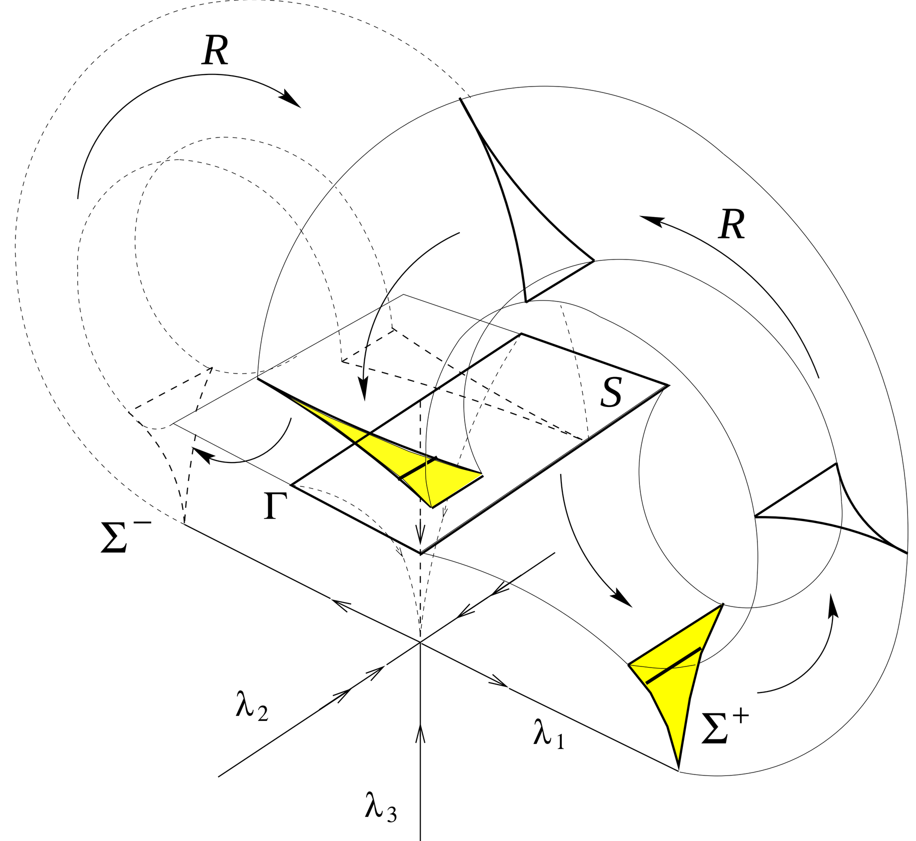

Figure 2: *The geometric Lorenz flow.*

4. Thanks

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